Math 231: Final Exam A - Sequence 111111 **Fall 2016**

- Please use the scantron for multiple choice. Since you have test version A, please code the SEQUENCE NUMBER on the scantron as 111111 (all 1's).
- No calculators allowed.
- No partial credit on multiple choice.
- For short answer questions, you must show work for full and partial credit. For short answer questions, all work to be graded needs to go on the test.
- Give exact values instead of decimal approximations unless otherwise specified.
- Sign the honor pledge below after completing the exam.

First and last name ....Key..

PID ....

Instructor (circle one):

Linda Green Elizabeth McLaughlin Lev Rozansky David Rose

Recitation TA (circle one):

Marc Besson Gonzalo Cazes-Nasitiqui Robert Hunt

Samuel Jeralds Claire Kiers Logan Tathan

Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature: ....

1. (2 pts) True or False: If the velocity of a particle is given by $v(t) = 9 - t^2$, then $\int_0^5 v(t)\,dt$ represents the total distance traveled by the particle between time $t = 0$ and $t = 5$.

   A. True

   B. False

2. (2 pts) True or False: If $\lim_{t\to 0} f(x) = 5$, then $\lim_{t\to 0} \frac{f(x)}{x^2}$ must be ∞.

   A. True

   B. False

3. (4 pts) For what value of a is the following function continuous?

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x < 3 \\ ax^2 + x - 6 & \text{for } x \geq 3 \end{cases}$$

   A. $a = \frac{1}{3}$

   B. $a = 1$

   C. $a = 3$

   D. $a = 6$

   E. There is no value of a that will make this function continuous.

$$\lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x\to 3^-} x+3 = 6$$

$$\lim_{x\to 3^+} ax^2+x-6 = a\cdot 9 + 3 - 6 = 9a-3$$

Set $9a - 3 = 6 \Rightarrow 9a = 9 \Rightarrow a = 1$

4. (4 pts) Find $\lim_{x\to 0} \frac{\tan(4x)}{x + \sin(2x)}$ $\quad \frac{0}{0}$

   A. $-\frac{1}{3}$

   B. 0

   C. $\frac{1}{3}$

   D. $\frac{4}{3}$

   E. DNE

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\sec^2(\cancel{4x}) \cdot 4}{1 + \cos(\cancel{2x}) \cdot 2}$$

$$= \frac{4}{1+2} = \frac{4}{3}$$

5. (4 pts) Find $\lim_{x\to 1^-} \frac{4x^2+x-5}{|x-1|}$

for $x<1$ $\quad x-1<0$

so $|x-1| = -(x-1)$

A. -9
B. -3
C. 0
D. 9
E. ∞

$$= \lim_{x\to 1^-} \frac{4x^2+x-5}{-(x-1)} = \lim_{x\to 1^-} \frac{(4x+5)(x-1)}{-(x-1)}$$

$$= \lim_{x\to 1^-} \frac{(4x+5)}{(-1)} = -9$$

6. (4 pts) Find $\lim_{x\to\infty} \frac{\sqrt{4x^2+x-5}}{x}$

A. 0
B. 1
C. 2
D. 4
E. ∞

$$= \lim_{x\to\infty} \frac{\sqrt{x^2(4+\frac{1}{x}-\frac{5}{x^2})}}{x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{4+\frac{1}{x}-\frac{5}{x^2}}}{x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{4+\cancel{\tfrac{1}{x}}^{\to 0}-\cancel{\tfrac{5}{x^2}}^{\to 0}}}{\cancel{x}} \quad \text{since } x>0$$

$$= 2$$

7. (4 pts) Suppose that $h(x) = \sqrt{f(x)} \cdot \ln(g(x))$. Use the fact that $f(5) = 4$, $f'(5) = 16$, $g(5) = 2$, $g'(5) = 4$ to find $h'(5)$.

A. $1 + 4\ln 2$
B. $4 + 4\ln 2$
C. $4 + \frac{1}{4}\ln 2$
D. $8 + \ln 2$
E. $8\ln 2$

$$h'(x) = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) \cdot \ln(g(x)) + \sqrt{f(x)} \cdot \frac{1}{g(x)} \cdot g'(x)$$

$$h'(5) = \frac{1}{2}(4)^{-1/2} \cdot 16 \cdot \ln(2) + \sqrt{4} \cdot \frac{1}{2} \cdot 4$$

$$= 4\ln(2) + 4$$

8. (4 pts) What is the absolute MINIMUM value of $f(x) = (x-2)^3 + 100$ on the interval $[-2, 4]$?

A. 0
B. 12
C. 36
D. 48
E. 100

$$f'(x) = 3(x-2)^2 = 0 \Rightarrow x = 2$$

x	$f(x)$
-2	$(-4)^3 + 100 = 36$
2	$0^3 + 100 = 100$
4	$2^3 + 100 = 108$

9. (4 pts) Find the slope of the tangent line to $\frac{x}{y^2} + \frac{x^2}{8} = 3$ at the point $(4, 2)$.

A. $-\frac{1}{4}$

B. $-\frac{3}{8}$

C. $\frac{5}{4}$

D. $\frac{4}{3}$

E. $\frac{9}{4}$

$$\frac{y^2 \cdot 1 - x \cdot 2y\frac{dy}{dx}}{y^4} + \frac{2x}{8} = 0$$

$$\frac{y^2 - 2xy\frac{dy}{dx}}{y^4} = -\frac{x}{4} \Rightarrow y^2 - 2xy\frac{dy}{dx} = \frac{-xy^4}{4}$$

$$\Rightarrow -2xy\frac{dy}{dx} = \frac{-xy^4}{4} - y^2 \Rightarrow \frac{dy}{dx} = \frac{-\frac{xy^4}{4} - y^2}{-2xy}$$

$$\left.\frac{dy}{dx}\right|_{(4,2)} = \frac{-16-4}{-16} = \frac{-20}{-16} = \frac{5}{4}$$

10. (4 pts) A function of the form $f(x) = \frac{a}{x^2 + bx}$ has a local extreme point (max or min point) at $(2, 3)$. Find the value of b.

A. $b = -4$

B. $b = -2$

C. $b = 0$

D. $b = 2$

E. Cannot be determined from this information.

$$f'(x) = \frac{(x^2+bx)\cdot 0 - a(2x+b)}{(x^2+bx)^2} = 0$$

$$\Rightarrow -a(2x+b) = 0 \Rightarrow 2x+b=0$$

$$\Rightarrow x = -\frac{b}{2} = 2$$

$$\Rightarrow b = -4$$

11. (4 pts) Which of the following limits represents $f'(3)$ if $f(x) = \ln(x + 4)$?

1) $\lim_{h\to 0} \frac{\ln(3 + h) - \ln(3)}{h}$

2) $\lim_{h\to 0} \frac{\ln(7 + h) - \ln(7)}{h}$

3) $\lim_{x\to 3} \frac{\ln(x + 4) - \ln(7)}{x - 3}$

A. 1, 2

B. 1, 3

C. 2, 3

D. All of them.

E. None of them.

12. (4 pts) If $g(x) = \int_{\pi/2}^{x} \sqrt{\sin(t) + 5}\, dt$, find $g'(2\pi)$.

A. $\sqrt{5}$

B. $\sqrt{5} - \sqrt{6}$

C. $\frac{1}{2}$

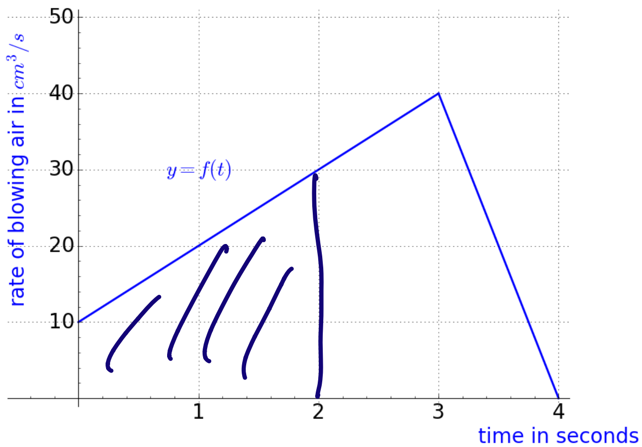
D. $\frac{1}{2\sqrt{5}}$

E. $\frac{1}{2\sqrt{6}}$

$g'(x) = \sqrt{\sin(x)+5}$

$g'(2\pi) = \sqrt{\sin(2\pi)+5} = \sqrt{5}$

13. (7 pts) Josie is blowing up a balloon. The **rate** at which she blows air into the balloon at time t is $f(t)$ cm^3 per second, graphed below. When $t = 0$, the balloon is empty. How many cm^3 of air are in the balloon at time $t = 2$ seconds?



$4 \cdot 10 = 40$

A. 10

B. 20

C. 30

D. 40

E. Cannot be determined from this information.

14. (4 pts) Which integral is equal to $\int_0^1 x^2 \sqrt{x^2+3}\, dx$? Hint: use u-substitution.

A. $\int_0^1 (u-3)\sqrt{u}\, du$

B. $\int_3^4 (u-3)\sqrt{u}\, du$

C. $\frac{1}{2}\int_0^1 \sqrt{u-3}\sqrt{u}\, du$

D. $\frac{1}{2}\int_3^4 \sqrt{u-3}\sqrt{u}\, du$

E. $\int_3^4 u^2\sqrt{u}\, du$

$u = x^2+3$ $\quad x^2 = u-3$

$du = 2x\,dx$ $\quad x = \sqrt{u-3}$ since $x>0$

$\frac{1}{2}du = x\,dx$

$x=0 \Rightarrow u=3$

$x=1 \Rightarrow u=4$

$\int_0^1 x\sqrt{x^2+3}\, x\,dx = \int_3^4 \sqrt{u-3}\sqrt{u}\,\frac{1}{2}du$

15. (4 pts) Find y' if $y = x^{\arctan(x)}$.

A. $y' = \arctan(x)x^{\arctan(x)-1}$

B. $y' = \dfrac{\arctan(x)}{x} + \dfrac{\ln(x)}{1+x^2}$

C. $y' = x^{\arctan(x)}\ln(x)$

D. $y' = x^{\arctan(x)}\left(\arctan(x) + \dfrac{x}{1+x^2}\right)$

E. $y' = x^{\arctan(x)}\left(\dfrac{\arctan(x)}{x} + \dfrac{\ln(x)}{1+x^2}\right)$

$\ln y = \ln x^{\arctan(x)}$

$\ln y = \arctan(x)\ln x$

$\frac{1}{y}y' = \frac{1}{1+x^2}\ln x + \arctan(x)\cdot\frac{1}{x}$

$y' = y\left[\frac{\ln x}{1+x^2} + \frac{\arctan(x)}{x}\right]$

$y' = x^{\arctan(x)}\left[\frac{\ln x}{1+x^2} + \frac{\arctan(x)}{x}\right]$

16. (7 pts) A large piece of ice in the shape of a perfect cube is melting. Its volume is decreasing at a rate of 60 cm^3 per minute. Find the rate at which its surface area is decreasing when its side length is 10 cm.

$V = x^3$ (1 pt)

$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$ (1 pt)

$-60 = 3 \cdot 10^2 \cdot \frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{-60}{300} = -\frac{1}{5}$ (2 pts)

$A = 6x^2$ (1 pt)

$\frac{dA}{dt} = 12x \frac{dx}{dt}$ (1 pt)

$\frac{dA}{dt} = 12 \cdot 10 \cdot (-\frac{1}{5})$

$= \frac{-120}{5} = -24$ (1 pt)

OR

$V = s^3$ (1 pt) $A = 6s^2$ (1 pt) $= 6(V^{1/3})^2 = 6V^{2/3}$ (1 pt)

$\frac{dA}{dt} = 6 \cdot \frac{2}{3} V^{-1/3} \frac{dV}{dt}$ (2 pts) $= \frac{4}{V^{1/3}} \frac{dV}{dt} = \frac{4}{(10^3)^{1/3}} \cdot (-30)$

$= \frac{-120}{10} = -12$ (1 pt)

Answer: 24 cm^2/min

17. (8 pts) Consider the function $f(x) = 1 + \sqrt{x}$.

(a) Find the linear approximation for $f(x)$ at $a = 1$.

$f'(x) = \frac{1}{2} x^{-1/2}$

$f'(1) = \frac{1}{2} 1^{-1/2} = \frac{1}{2}$

$f(1) = 1 + \sqrt{1} = 2$

$L(x) = f(a) + f'(a)(x - a)$

2 pts slope
2 pts intercept or correct values of constants in alternate form

no partial credit within these 2 pt chunks

$L(x) = \boxed{2 + \frac{1}{2}(x - 1) \quad \text{OR } L(x) = \frac{1}{2}x + 3/2}$

(b) In order to use this linear approximation to approximate the number $1 + \sqrt{0.7}$, what value of x would you plug into your answer from (a)?

$x = \boxed{0.7}$

1 pt

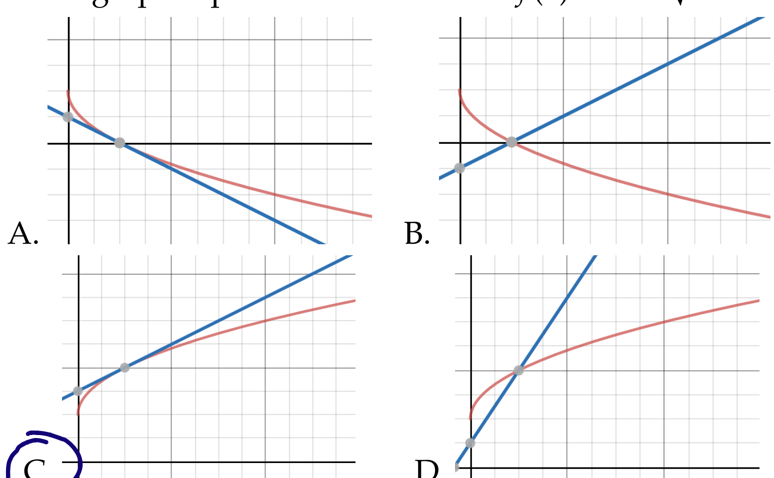
(c) Approximate $1 + \sqrt{0.7}$

$L(0.7) = \frac{1}{2} \cdot 0.7 + \frac{3}{2} = 0.35 + 1.5 = 1.85$

Answer: $\boxed{1.85}$

1 pt

(d) Which graph represents the function $f(x) = 1 + \sqrt{x}$ and its linearization?



A. B. C. D.

1 pt

(e) Is your estimate in part (c) an overestimate or an underestimate of the actual value of of $1 + \sqrt{0.95}$?

A. Overestimate

B. Underestimate

1 pt

18. (8 pts) A particle is moving with the given data. Find a formula for the position of the particle. Note: $a(t)$ is acceleration, $v(t)$ is velocity, and $s(t)$ is position.

$$a(t) = \sin(t) + 3\cos(t), s(0) = 0, v(0) = 2$$

$v(t) = -\cos(t) + 3\sin(t) + C$

$2 = v(0) = -\cos(0) + 3\sin(0) + C \Rightarrow 2 = -1 + C \Rightarrow C = 3$

$v(t) = -\cos(t) + 3\sin(t) + 3$

$s(t) = -\sin(t) - 3\cos(t) + 3t + D$

$0 = s(0) = -\sin(0) - 3\cos(0) + 3\cdot 0 + D \Rightarrow 0 = -3 + D \Rightarrow D = 3$

$s(t) = -\sin(t) - 3\cos(t) + 3t + 3$

-1 for sign error

2 pts

-1 for sign error

2 pts

-1 for incorrect value of constant

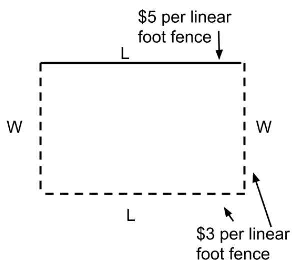
2 pts

-1 for incorrect value of v constant

2 pts

$s(t) =$ $\boxed{-\sin(t) - 3\cos(t) + 3t + 3}$

19. (7 pts) The town of Chapel Hill plans to build a park, that will be fenced in with two types of fencing as shown. The fencing costs \$3 per linear foot for the cheaper fence (dotted lines) and \$5 per linear foot for the more expensive fence (solid line). Assuming the town can spend no more than \$4800 on fencing, what dimensions should be used to build a park with the largest possible area? Use calculus in your solution.



$\text{Cost} = 3w + 3L + 3w + 5L$

$= 6w + 8L = 4800$ 1 pt

$A = w \cdot L$ 1 pt

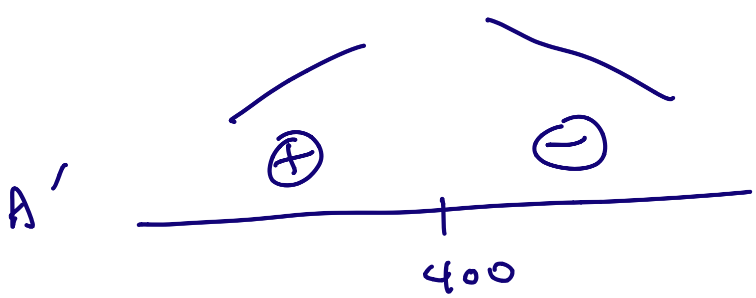
$L = \dfrac{4800 - 6w}{8} = 600 - \frac{3}{4}w$

1 pt $A = w(600 - \frac{3}{4}w) = 600w - \frac{3}{4}w^2$

1 pt $A' = 600 - \frac{3}{2}w = 0 \Rightarrow 600 = \frac{3}{2}w$

$\Rightarrow w = \frac{1200}{3} = 400$

$\Rightarrow L = 600 - \frac{3}{4} \cdot 400 = 300$



1 pt OR

$A'' = -\frac{3}{2} < 0$

concave down

$L = \boxed{300}$ 1 pt $W = \boxed{400}$ 1 pt

20. (7 pts) Compute $\int \frac{5^{\sqrt{x}}}{\sqrt{x}}\,dx$

1 pt u

1 pt du

$u = \sqrt{x}$

$du = \frac{1}{2} x^{-1/2} dx = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} dx$

$\Rightarrow \quad 2\,du = \frac{1}{\sqrt{x}} dx$

$= \int 5^u \cdot 2\,du$

2 pts (-1 if forget factor of 2, or forget to write du or both)

$= \frac{5^u \cdot 2}{\ln(5)} + C$

1 pt anti-deriv

$= \frac{2 \cdot 5^{\sqrt{x}}}{\ln(5)} + C$

1 pt +C

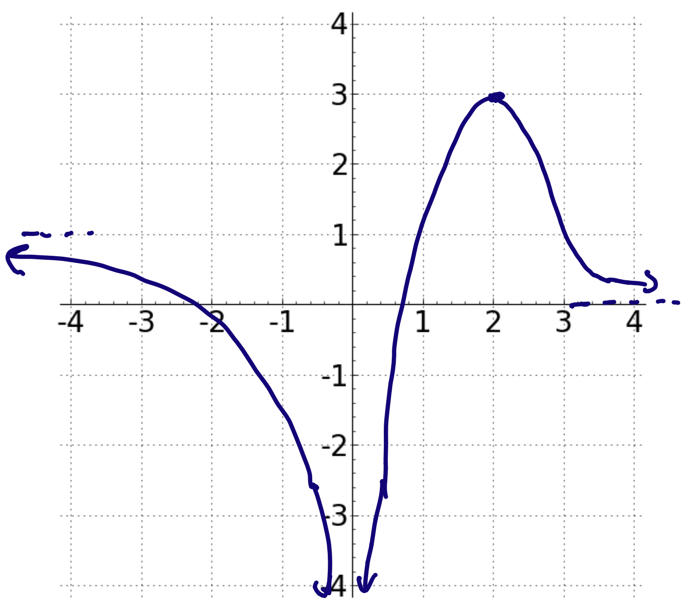
1 pt substitute back in for x

Answer: $\frac{2}{\ln(5)} \cdot 5^{\sqrt{x}} + C$

21. (7 pts) Sketch the graph of a function defined on $(-\infty, \infty)$ with exactly one discontinuity that satisfies:

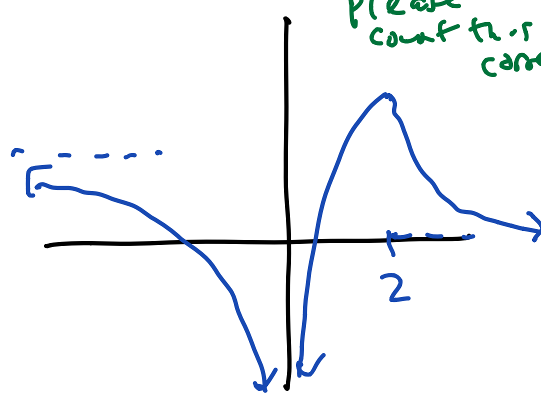
- $f(2) = 3$
- $\lim_{x\to-\infty} f(x) = 1$ and $\lim_{x\to\infty} f(x) = 0$
- $\lim_{x\to 0} f(x) = -\infty$
- $f'(x) < 0$ for $x < 0$ and $x > 2$ and $f'(x) > 0$ for $0 < x < 2$
- $f''(x) < 0$ for $x < 0$ and $0 < x < 3$ and $f''(x) > 0$ for $x > \cancel{2}\,3$

this was supposed to say $x > 3$



Alternative answer if last statement is interpreted as $0 < x < 3$ & $x > 3$

please also count this correct



1 pt horiz asymptote on left side

1 pt horiz asymptote on right side

1 pt $f(2) = 3$

2 pts correct shape on left side (can give partial credit)

2 pts correct shape on right side (can give partial credit)

-1 if more than one discontinuity, or graph is not a function