Math 3 Variable Manipulation Part 7
Absolute Value & Inequalities

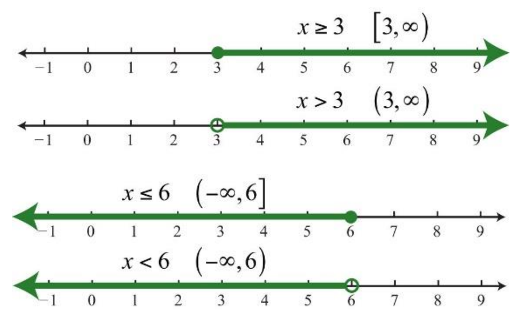
MATH 1 REVIEW
SOLVING AN ABSOLUTE VALUE EQUATION
Absolute value is a measure of distance; how far a number is from zero. In practice, the absolute value of any number is always a positive number; it can never be negative. To solve an equation that includes absolute value signs, think about the two different cases-one where what is inside the absolute value sign equals a positive number, and one where it equals a negative number.

Example: Solve $|x – 12| = 3$
Solution: Think of it as two equations: $x - 12 = 3$ or $x - 12 = -3$
Solving for x, will yield the two solutions: $x = 15$ or 9

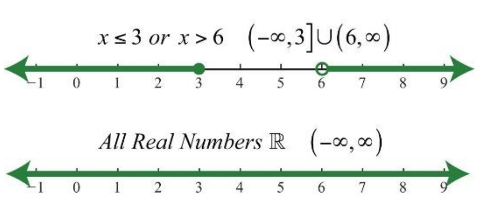
To solve more complex problems, use various algebraic manipulation in conjunction with absolute value manipulation. First get the absolute value by iteself on one side of the equation and the rest on the other side of the equals sign. Then make two equations, one positive and one negative. Then solve both equations to get both possible answers.

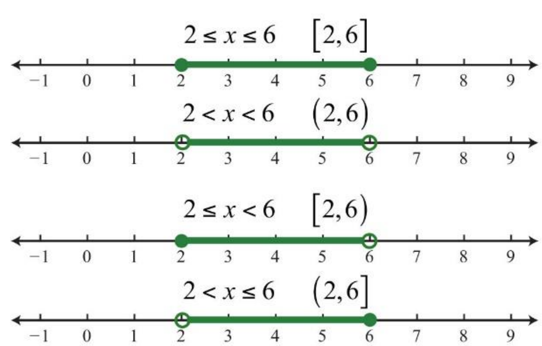
Example: $5|3 + 7m| + 1 = 51$
Solution: First, subtract 1 from both sides and then divide by 5 to get: $|3 + 7m| = 50/5 = 10$
Write two equations: $3 + 7m = 10$ and $3 + 7m = 10$
Subtract three from both sides and then divide by 7 for each of the equations to get: $m = 1$ and $m = -13/7$

INEQUALITY AND INTERVAL NOTATIONS
Two common ways of expressing solutions to an inequality are by graphing them on a number line and using interval notation. To express the solution graphically draw a circle around the starting number on a number line. If that number is also equal to that value, then fill in the circle, if it is not equal than leave empty. Then shade in all the values that are solutions to the inequality with an arrow representing that the solutions continue to infinity. Interval notation is textual and uses specific notation as follows:



COMPOUND INEQUALITIES
A compound inequality is actually two or more inequalities in one statement joined by the word "and" or by the word "or." Compound inequalities with the logical "or" require that either condition must be satisfied. Therefore, the solution set of this type of compound inequality consists of all the elements of the solution sets of each inequality. When we join these individual solution sets it is called the union, denoted ∪.





SOLVING AN INEQUALITY

Use the following properties when solving inequalities.

Properties of Inequalities (a, b and c are real numbers)	**Example**
Addition Property of Inequality Adding the same number to each side of an equation produces an equivalent inequality.	If $a > b$ then $a + c > b + c$
Subtraction Property of Inequality Subtracting the same number from each side of an equation produces an equivalent inequality.	If $a < b$ then $a - c > b - c$
Multiplication Property of Inequality Multiplying each side of the inequality by the same *positive* number produces an equivalent inequality. Multiplying each side of the inequality by the same *negative* number produces an equivalent inequality when the inequality sign is *reversed*.	If $a \ge b$ and $c > 0$ then $ac \ge bc$ If $a \ge b$ and $c < 0$ then $ac \le bc$
Division Property of Inequality Dividing each side of the inequality by the same *positive* number produces an equivalent inequality. Dividing each side of the inequality by the same *negative* number produces an equivalent inequality when the inequality sign is *reversed*.	If $a \le b$ and $c > 0$ then $\frac{a}{c} \le \frac{b}{c}$ If $a \le b$ and $c < 0$ then $\frac{a}{c} \ge \frac{b}{c}$

Solving inequalities works just the same as solving equations--do the same thing to both sides, until the variable you are solving for is isolated--with one exception: When you **multiply or divide both sides by a negative number**, you must **reverse the inequality sign**. This may seem strange, but you can prove that it is true by trying out values. It can also be explained by thinking about the inequality and absolute values. If a number $x > 5$ then $-x < 5$ (not greater than). The signs must be switched to make it correct.

Example: Solve the compound inequality $4x - 3 > 9$ or $-2x > 2$. Write the answer in interval notation.

Solution: Solve each inequality separately.

$4x - 3 > 9$	or
$4x > 12$	$-2x > 2$
$x > 3$	$x < -1$ (remember to switch direction of inequality)

Since this is an "or" compound inequality, it is a union of the solutions

$(-\infty, -1) \cup (3, \infty)$

Note that $x < -1$. If we did not switch the direction of the inequality, it would state $x > -1$. Let's try it out. 0 is a number greater than -1. If we substitute 0 into the equation $-2x > 2$, we would get that $0 > 2$, which is not true. However, the correct answer is $x < -1$. If we substitute -2 (which is less than -1), into the equation, we get $-4 < 2$ which is true.

Example: Solve the compound inequality $5v + 10 \le -4v - 17 < 9 \ -2v$. Write the answer in interval notation.

Solution: Write the compound inequality as two inequalities and Solve each inequality separately.

$5v + 10 \le -4v - 17$	$-4v - 17 < 9 - 2v$
$5v + 4v \le -17 - 10$	$-4v + 2v < 9 + 17$
$9v \le -27$	$-2v < 26$
$v \le -3$	$v > -13$

Put the compound inequality back together. This is an "and" compound inequality so it is the intersection of the two solutions.

$-13 < v \le -3$ so $(-13, -3)$ in interval notation.

ABSOLUTE VALUE (LESS THAN)

Inequalities can also contain absolute values. There are two forms, less than and greater than with absolute value. In general, if an inequality is in the form: $|a| < b$, then the solution will be in the form: $-b < a < b$ for any argument a.

Example: For which values of x will this inequality be true? $|2x - 1| < 5$.

Solution: The argument, $2x - 1$, will fall between -5 and 5:

$-5 < 2x - 1 < 5$.

$-5 + 1 < 2x < 5 + 1$

$-4 < 2x < 6$

$-2 < x < 3$

ABSOLUTE VALUE (GREATER THAN)

If the inequality is in the form $|a| > b$ (and $b > 0$), then $a > b$ or $a < -b$.

Example: $|a| > 3$

For which values of a will this be true?

Solution: Geometrically, $a > 3$ or $a < -3$.

Sample Questions:

1. $5 + 8|-10n - 2| = 101$
2. $\frac{1}{3}|6x + 5| = 7$
3. Graph and give the interval notation equivalent: $-3 < x \leq 10$
4. Graph and give the interval notation equivalent: $x < 5$ and $x \geq -3$
5. Graph and give the interval notation equivalent: $x < 2$ or $x \geq 9$
6. $7v - 5 \geq 65$ or $-3v - 2 \geq -2$
7. $5x - (x + 2) < -5(1 + x) + 3$
8. $8x + 8 \geq -64$ and $-7 - 8x \geq -79$

9. Solve this inequality for x: $|3x - 5| < 10$.

10. For which values of x will this be true? $|x + 2| > 7$.

11. If 5 times a number x is subtracted from 15, the result is negative. Which of the following gives the possible value(s) for x?
    a. All $x < 3$
    b. All $x > 3$
    c. 10 only
    d. 3 only
    e. 0 only
12. Which of the following is the set of all real numbers x such that $x - 3 < x - 5$?
    a. The empty set
    b. The set containing only zero
    c. The set containing all nonnegative real numbers
    d. The set containing all negative real numbers
    e. The set containing all real numbers
13. Which of the following intervals contains the solution to the equation $x - 2 = \frac{2x+5}{3}$?
    a. $-6 < x < 11$
    b. $11 \leq x < 15$
    c. $6 < x \leq 10$
    d. $-5 < x \leq -3$
    e. $-11 \leq x \leq -2$
14. Passes to Renaissance Faire cost $9 when purchased online and $12 when purchased in person. The group sponsoring the fair would like to make at least $4,000 from sales of passes. If 240 passes were sold online, what is the minimum number of tickets that must be sold in person in order for the group to meet its goal?

15. What is the largest integer value of t that satisfies the inequality $\frac{24}{30} > \frac{t}{24}$?

16. How many different integer values of a satisfy the inequality $\frac{1}{11} < \frac{2}{a} < \frac{1}{8}$?

17. It costs *a* dollars for an adult ticket to a reggae concert and s dollars for a student ticket. The difference between the cost of 12 adult tickets and 18 student tickets is $36. Which of the following equations represents this relationship between *a* and s?
    a. $\frac{12a}{18s} = 36$
    b. $216as = 36$
    c. $|12a - 18s| = 36$
    d. $|12a + 18s| = 36$
    e. $|18a - 12s| = 36$

MATH 2 LEVEL

ABSOLUTE VALUES IN QUADRATICS

Quadratic problems can contain absolute values. Use similar techniques that you learned in Math 1 to solve. First, use variable manipulation to get the absolute value alone on one side of the equation. Then write two equations, one positive and one negative. Then solve. Because the variable is squared, you will most likely take the square root of both sides of the equation to get the value of x. This means that for each equation, there will be two solutions: one positive and one negative. We often write it as $\pm$ the number. That means there are generally four solutions to quadratic absolute value problems.

Example: Solve $| x^2 - 4x - 5 | = 7$

Solution: First, since the absolute value is alone in one side of the equation write two equations, one positive and one negative

$x^2 - 4x - 5 = 7$ and $x^2 - 4x - 5 = -7$

$x^2 - 4x - 12 = 0$ $\qquad$ $x^2 - 4x + 2 = 0$

$(x - 6)(x + 2) = 0$ $\qquad$ $x = \frac{4 \pm \sqrt{4^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$

$x = 6, x = -2$ $\qquad$ So Solutions are: 6, -2, $2+\sqrt{2}$ and $2-\sqrt{2}$

Example: Given the equation $|y^2 - 11| - 2 = 0$, what are solutions but NOT a rational number?

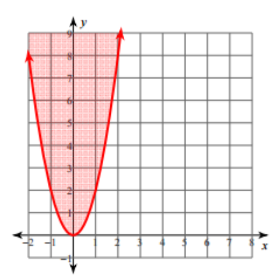
Solution: To find the solutions, solve the equation for y. First isolate the absolute value expression $|y^2 - 11| = 2$. Remember that you will have to create two different equations as you remove the absolute value sign: $y^2 - 11 = 2$ and $y^2 - 11 = -2$. Solve each of these equations to find that $y^2 = 13$ and $y^2 = 9$ so $y = \pm 3$ and $y = \pm\sqrt{13}$. The irrational solutions are $\pm\sqrt{13}$.

QUADRATIC INEQUALITIES

Quadratic problems can also contain inequalities that can be solved algebraically and graphed to visually understand the equation. Quadratics are parabolic in nature so the solution is greater or less than the line of the parabola which means that the inside of the parabola is a solution (and shaded on a graph) or the outside of the parabola is a solution (and thus shaded on a graph).

Example: $y \geq 2x^2$

Solution: The graph of the solution is a parabola at point (0, 0) opening up and since y is greater than the line $2x^2$, the region inside the parabola is shaded.



To solve for a quadratic inequality algebraically manipulate the equation to get 0 on one side of the inequality and find the zeros by factoring or using the quadratic equation. Then use sample points all around the inequality to "test" if that point on the equation is true or untrue for that point. It is also valuable to check graphically.

Steps for Solving a Quadratic Inequality

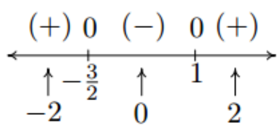
1. Rewrite the inequality, if necessary, as a quadratic function $f(x)$ on one side of the inequality and 0 on the other.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a real number, called a **test value**, in each of the intervals determined in step 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.
5. Choose the intervals which correspond to the correct sign to solve the inequality.

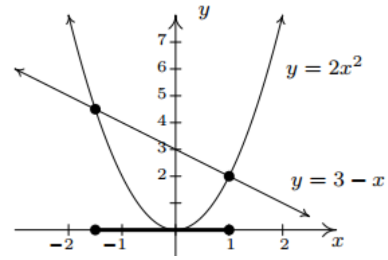
Example: $2x^2 \le 3 - x$

Solution: To solve $2x^2 \le 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \le 0$. Then find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for x.

Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -3/2$ or $x = 1$. Place these values on the number line with 0 above them and choose test values in the intervals $(-\infty, -3/2)$, $(-3/2, 1)$ and $(1, \infty)$. For the interval $(-\infty, -3/2)$, we choose $x = -2$; for $(-3/2, 1)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place (+) above $(-\infty, -3/2)$; $f(0) = -3 < 0$, so (−) goes above the interval $(-3/2), 1$; and, $f(2) = 7$, which means (+) is placed above $(1, \infty)$. Since we are solving $2x^2 + x - 3 \le 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a (−). Checking the sign diagram, we see this is $(-3/2, 1)$. We know $2x^2 + x - 3 = 0$ when $x = -3/2$ and $x = 1$, so our final answer is $-3/2, 1$.

To verify our solution graphically, we refer to the original inequality, $2x^2 \le 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$). The graphs of g and h are given on the right with the sign chart on the left. 3



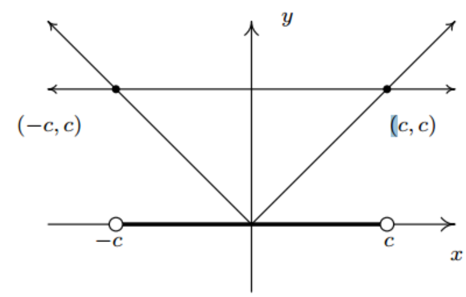


INEQUALITIES WITH ABSOLUTE VALUE AND QUADRATICS

Theorem 2.4. Inequalities Involving the Absolute Value: Let c be a real number.

- For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- For $c > 0$, $|x| \le c$ is equivalent to $-c \le x \le c$.
- For $c \le 0$, $|x| < c$ has no solution, and for $c < 0$, $|x| \le c$ has no solution.
- For $c \ge 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- For $c \ge 0$, $|x| \ge c$ is equivalent to $x \le -c$ or $x \ge c$.
- For $c < 0$, $|x| > c$ and $|x| \ge c$ are true for all real numbers.

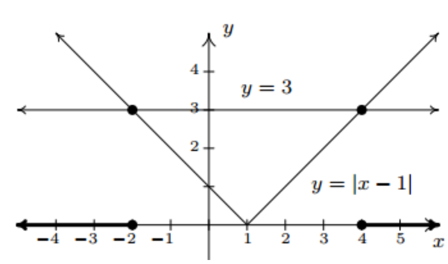
If $c > 0$, the graph of $y = c$ is a horizontal line which lies above the x-axis through $(0, c)$. To solve $|x| < c$, we are looking for the x values where the graph of $y = |x|$ is below the graph of $y = c$. We know that the graphs intersect when $|x| = c$, which, we know happens when $x = c$ or $x = -c$. We see that the graph of $y = |x|$ is below $y = c$ for x between $-c$ and c, and hence we get $|x| < c$ is equivalent to $-c < x < c$.



Example: Solve $|x - 1| \ge 3$ analytically; check your answers graphically.

Solution: $|x - 1| \ge 3$ is equivalent to $x - 1 \le -3$ or $x - 1 \ge 3$. Solving, we get $x \le -2$ or $x \ge 4$, which, in interval notation is $(-\infty, -2] \cup [4, \infty)$.

We see that the graph of $y = |x - 1|$ is above the horizontal line $y = 3$ for $x < -2$ and $x > 4$ hence this is where $|x - 1| > 3$. The two graphs intersect when $x = -2$ and $x = 4$, so we have graphical confirmation of our analytic solution.



INEQUALITIES AND FUNCTIONS

Linear Example: Let *f(x) = 2x − 1* and *g(x) = 5*

a. Solve *f(x) = g(x).*

b. Solve *f(x) < g(x).*

c. Solve *f(x) > g(x).*

d. Graph *y = f(x)* and *y = g(x)* on the same set of axes and interpret your solutions to parts 1 through 3 above. Solution.

Solution:

a. To solve *f(x) = g(x),* we replace *f(x)* with *2x − 1* and *g(x)* with *5*

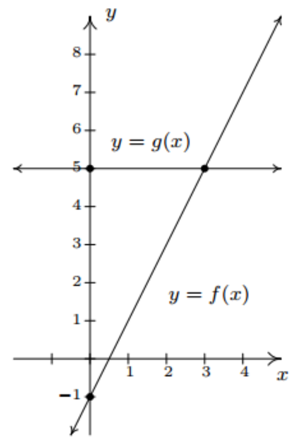
2x − 1 = 5

Solving for x, we get x = 3.

b. The inequality *f(x) < g(x)* is equivalent to 2x − 1 < 5. Solving gives x < 3 or (−∞, 3).

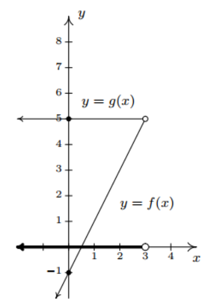
c. To find where *f(x) > g(x),* we solve 2x − 1 > 5. We get x > 3, or (3, ∞).

d. To graph *y = f(x),* we graph y = 2x − 1, which is a line with a y-intercept of (0, −1) and a slope of 2. The graph of *y = g(x)* is y = 5 which is a horizontal line through (0, 5).

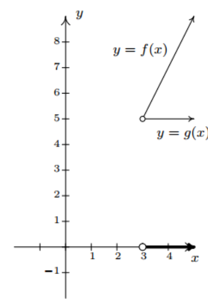


To see the connection between the graph and the Algebra, we recall that the point (a, b) is on the graph of *f* if and only if *f(a) = b*. In other words, a generic point on the graph of *y = f(x)* is *(x, f(x)),* and a generic point on the graph of *y = g(x)* is *(x, g(x)).* When we seek solutions to *f(x) = g(x),* we are looking for *x* values whose *y* values on the graphs of *f* and *g* are the same.

In part a, we found *x = 3* is the solution to *f(x) = g(x).* Sure enough, *f(3) = 5* and *g(3) = 5* so that the point (3, 5) is on both graphs. In other words, the graphs of *f* and *g* intersect at (3, 5). In part b, we set *f(x) < g(x)* and solved to find x < 3. For x < 3, the point *(x, f(x))* is below *(x, g(x))* since the y values on the graph of *f* are less than the y values on the graph of *g* there. Analogously, in part c, we solved *f(x) > g(x)* and found x > 3. For x > 3, note that the graph of *f* is above the graph of *g*, since the y values on the graph of *f* are greater than the y values on the graph of *g* for those values of x.



$f(x) < g(x)$ on $(-\infty, 3)$



$f(x) > g(x)$ on $(3, \infty)$

Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions.

- The solutions to $f(x) = g(x)$ are the x values where the graphs of $y = f(x)$ and $y = g(x)$ intersect.
- The solution to $f(x) < g(x)$ is the set of x values where the graph of $y = f(x)$ is *below* the graph of $y = g(x)$.
- The solution to $f(x) > g(x)$ is the set of x values where the graph of $y = f(x)$ *above* the graph of $y = g(x)$.

Quadratic Example: See the graphs of *f* and *g*. (The graph of y = g(x) is bolded.) Use these graphs to answer the following questions.

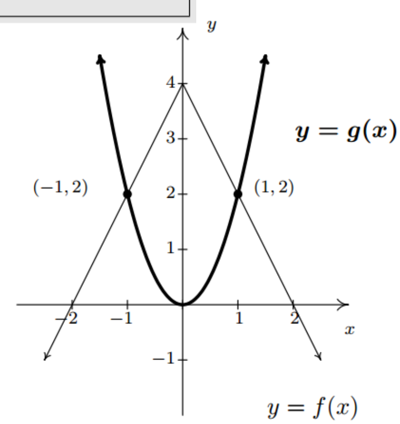
a. Solve *f(x) = g(x)*

b. Solve *f(x) < g(x)*

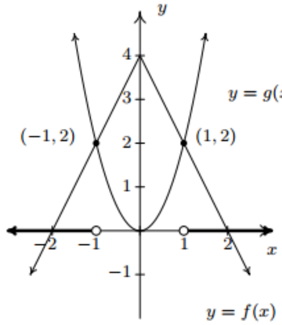
c. Solve *f(x) ≥ g(x)*

Solution:

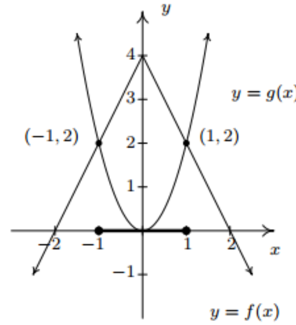
a. To solve *f(x) = g(x),* we look for where the graphs of f and g intersect. These appear to be at the points (−1, 2) and (1, 2), so our solutions to *f(x) = g(x)* are x = −1 and x = 1.



b. To solve *f(x) < g(x),* we look for where the graph of *f* is below the graph of *g*. This appears to happen for the x values less than −1 and greater than 1. Our solution is (−∞, −1)∪(1, ∞).
c. To solve *f(x) ≥ g(x),* we look for solutions to *f(x) = g(x)* as well as *f(x) > g(x).* We solved the former equation and found x = ±1. To solve *f(x) > g(x),* we look for where the graph of f is above the graph of g. This appears to happen between x = −1 and x = 1, on the interval (−1, 1). Hence, our solution to *f(x) ≥ g(x)* is [−1, 1].

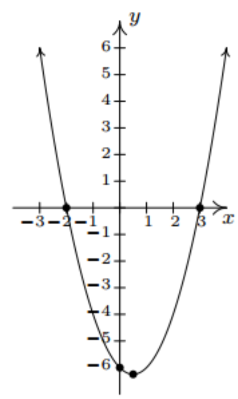


$f(x) < g(x)$



$f(x) \geq g(x)$

Example: If we consider *f(x) =* $x^2 - x - 6$ and *g(x) = 0,*
Solution: Solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of x for which the graph of *y = f(x) =* $x^2 - x - 6$ (the parabola) is below the graph of *y = g(x) = 0* (the x-axis). We've provided the graph again for reference.



$y = x^2 - x - 6$

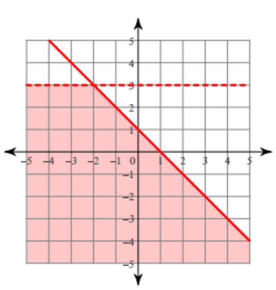
We can see that the graph of *f* does dip below the x-axis between its two x-intercepts. The zeros of *f* are x = −2 and x = 3 in this case and they divide the domain (the x-axis) into three intervals: (−∞, −2), (−2, 3) and (3, ∞). For every number in (−∞, −2), the graph of f is above the x-axis; in other words, *f(x) > 0* for all x in (−∞, −2). Similarly, *f(x) < 0* for all x in (−2, 3), and *f(x) > 0* for all x in (3, ∞). We can schematically represent this with the sign diagram below.

Here, the (+) above a portion of the number line indicates *f(x) > 0* for those values of x; the (−) indicates *f(x) < 0* there. The numbers labeled on the number line are the zeros of *f*, so we place 0 above them. We see at once that the solution to f(x) < 0 is (−2, 3).

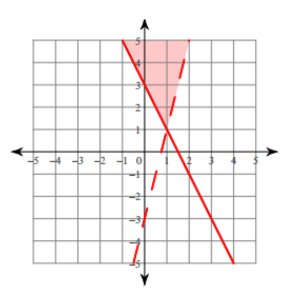
SYSTEM OF INEQUALITIES

A system of linear inequalities in two variables consists of at least two linear inequalities in the same variables. The solution of a linear inequality is the ordered pair that is a solution to all inequalities in the system and the graph of the linear inequality is the graph of all solutions of the system.

Example: Sketch the solution to the system of inequalities y < 3 and y ≤ −x + 1
Solution:



Example: Sketch the solution to the system of inequalities y > 4x − 3 and y ≥ −2x + 3
Solution:



Sample Questions

18. Solve $| 6n^2 - 18n - 18 | = 6$

19. $x^2 - 2x > 1$

20. $y \leq 4x^2 - 32x + 62$

21. $2 < |x - 1| \leq 5$

22. Sketch the solution to the system of inequalities $y \geq -5x + 3$ and $y > -2$

23. If $x^2 - 3 \leq 13$, what is the greatest real value that x can have?

24. Considering all values of a and b for which a + b is at most 9, a is at least 2, and b is at least -2, what is the minimum value of b – a?

25. For all values x, y, and z, if $x \leq y$ and $y \leq z$, which of the following CANNOT be true?

    I. $x = z$
    II. $x > z$
    III. $x < z$

a. I only
b. II only
c. III only
d. I and II only
e. I, II, and III

26. If X, Y and Z are real numbers, and XYZ = 1, then which of the following conditions must be true?
    a. XZ = 1/Y
    b. X, Y, and Z > 0
    c. Either X = 1, Y = 1, or Z = 1
    d. Either X = 0, Y = 0, or Z = 0
    e. Either X < 1, Y < 1, or Z < 1
27. If $y \neq z$, what are the real values of x that make the following inequality true?

$$\frac{xy-xz}{3y-3z} < 0$$

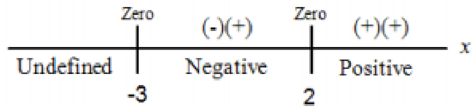
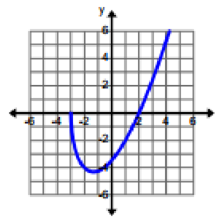
    a. All negative real numbers
    b. All positive real numbers
    c. $-\frac{1}{3}$ only
    d. $\frac{1}{3}$ only
    e. 3 only
28. If x and y are any real numbers such that 0 < x < 2 < y, which of these must be true?
    a. x < (xy)/2 < y
    b. 0 < xy < 2x
    c. x < xy < 2
    d. 0 < xy < 2
    e. xy < y
29. If x and y are real numbers such that x > 1 and y < -1, then which of the following inequalities must be true?
    a. $\frac{x}{y} > 1$
    b. $|x|^2 > |y|$
    c. $\frac{x}{3} - 5 > \frac{y}{3} - 5$
    d. $x^2 + 1 > y^2 + 1$
    e. $x^{-2} > y^{-2}$
30. Which of the following expressions has a positive value for all x and y such that x > 0 and y < 0?
    a. y – x
    b. x + y
    c. x^3y
    d. $\frac{x^2}{y}$
    e. $\frac{x}{y^2}$
31. If a < b, then |a – b| is equivalent to which of the following?
    a. a + b
    b. –(a + b)
    c. $\sqrt{a-b}$
    d. a – b
    e. –(a – b)
32. What are the values of a and b, if any, where - a|b + 4| > 0?
    a. a > 0 and b ≠ -4
    b. a > 0 and b ≠ 4
    c. a < 0 and b ≥ -4
    d. a < 0 and b ≠ -4
    e. a < 0 and b ≤ -4

MATH 3 LEVEL

SOLVING ONE VARIABLE INEQUALITIES

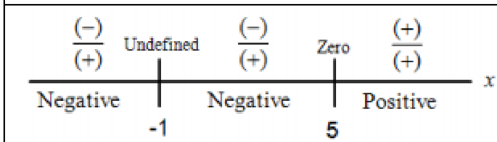
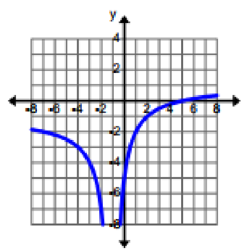
Example: Solve: $(x-2)\sqrt{x+3} \geq 0$

Solution:

$(x-2)\sqrt{x+3} \geq 0$	The square root makes it so that the expression on the left side of the equation is undefined if $x < -3$. The left side of the equation is zero when $x = -3$ and $x = 2$.
	Create a sign chart, using the zeros, to determine where the expression is positive or equal to zero.
The solution is $\{-3\} \cup [2, \infty)$.	 The graph confirms the solution.

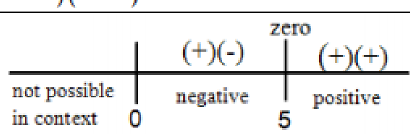
Example: Solve: $\frac{x-5}{|x+1|} \leq 0$

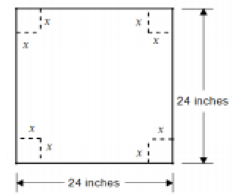
Solution:

$\frac{x-5}{\lvert x+1 \rvert} \leq 0$	The expression on the left is undefined when $x = -1$. The expression is zero when $x = 5$.
	Create a sign chart, using where the function is zero or undefined, to determine where the expression is negative or equal to zero.
The solution is $(-\infty, -1) \cup (-1, 5]$.	 The graph confirms the solution.

Example: The length of a rectangle is five more than the twice the width. If the area is at least 75 square centimeters, what are the possible values for the width?

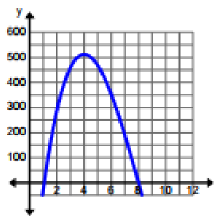
Solution:

$A = lw$ $l = 2w + 5$ $w(2w+5) \geq 75$	Write an inequality statement with the known information.
$w(2w+5) \geq 75$ $2w^2 + 5w - 75 \geq 0$ $(2w+15)(w-5) \geq 0$	Gather all the terms on one side of the inequality.
	Create a sign chart.
The width must be greater than or equal to 5 cm or $[5, \infty)$.	



Example: A packaging company is designing a new open-topped box with a volume of at least 512 in^3. The box is to be made from a piece of cardboard measuring 24 inches by 24 inches by cutting identical squares from the corners and turning up the sides. Describe the possible lengths of the sides of the removed squares.

Solution:

$512 \geq x(24-2x)(24-2x)$	The volume of a rectangular prism is $V = lwh$. The height is x. The width and length are $24-2x$. The length of the side of the square being cut out must be $0 < x < 12$.
$512 \geq x(24-2x)(24-2x)$ $512 \geq 4x^3 - 96x^2 + 576x$ $0 \geq 4x^3 - 96x^2 + 576x - 512$	Expand the equation and get all the terms on the same side so that the expression is compared to zero.
	Use technology to graph the function and find the zeros.
The volume of the box will equal or exceed 512 in^3 if the removed square has a side length on the interval $[1.072, 8]$.	

Sample Questions:

33. $x^2 + x - 12 \geq 0$

34. $x^2 + 3x \geq 4$

35. $x^3 + 2x^2 - 15x < 0$

36. $x^3 - 6x^2 \leq 7x$

37. $\frac{x}{x+3} \geq 0$

38. $\frac{x+2}{x^2-9} \leq 0$

39. $x|x - 2| > 0$

40. $(3x - 4)\sqrt{2x+1} \geq 0$

41. A diver leaps into the air at 20 feet per second from a diving board that is 12 feet above the water. For how many seconds is the diver at least 10 feet above the water?

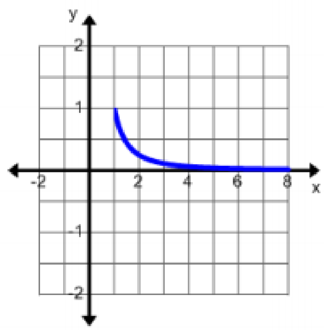
42. An open box is made from a rectangular piece of cardboard measuring 11 inches by 14 inches by cutting identical squares from the corners and turning up the sides. Describe the possible lengths of the sides of the removed squares if the volume of the open box is not to exceed 132 cubic inches.

43. A new drink company is packaging their new cola in 1-liter (1000 cm^3) cylindrical cans. Find the radius of the cans if the cans have a surface area that is less than 750 cm^2.

WRITING AND SOLVING INEQUALITIES IN TWO VARIABLES

Example: Given the sequence 1, 1/4, 1/9, 1/16, . . . write and graph the rational equation that models the relationship between the term in the sequence and its value.

Solution:

<table>
<tr><td>$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$</td><td></td></tr>
<tr><td>
<table>
<tr><th>Term</th><th>Value</th></tr>
<tr><td>1</td><td>$1 = \frac{1}{1}$</td></tr>
<tr><td>2</td><td>$\frac{1}{2} = \frac{1}{2^2}$</td></tr>
<tr><td>3</td><td>$\frac{1}{9} = \frac{1}{3^2}$</td></tr>
<tr><td>4</td><td>$\frac{1}{16} = \frac{1}{4^2}$</td></tr>
</table>
</td><td>The general term is $\frac{1}{n^2}$.</td></tr>
<tr><td>$f(x) = \frac{1}{x^2}$ when $x \geq 1$ and an integer.

</td><td>Note: this is a graph of the rational equation, not the graph of the sequence.</td></tr>
</table>

Example: All-a-Shirt budgets \$6000 to restock 200 shirts. T-shirts sell for \$12, polo for \$24, and rugby for \$36. You need to buy twice as many rugby shirts as polo shirts. If you buy all three types of shirts, how many of each type should you buy?

Solution:

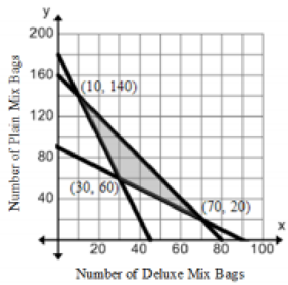
<table>
<tr><td>Relationship of polo to rugby: $z = 2y$
Total number of shirts: $x + y + z = 200$
Total cost: $12x + 24y + 36z = 6000$</td><td>Write the equations. Let x represent the number of t-shirts, y represent the number of polo shirts, and z represent the number of rugby shirts.</td></tr>
<tr><td>$x + y + 2y = 200$
$x + 3y = 200$

$12x + 24y + 36(2y) = 6000$
$12x + 24y + 72y = 6000$
$12x + 96y = 6000$</td><td>Use the relationship of the polo to rugby shirts to rewrite the equations in terms of two variables.</td></tr>
<tr><td>$x + 3y = 200$
$12x + 96y = 6000$

$x = 20,\ y = 60,\ z = 120$</td><td>Solve the system of equations using the method of your choice.</td></tr>
<tr><td colspan="2">You should order 20 t-shirts, 60 polo shirts, and 120 rugby shirts.</td></tr>
</table>

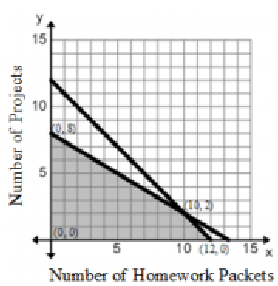
Example: The Sweet Tooth Candy Shoppe is purchasing a candy mix with two types of chocolate: dark chocolate and white chocolate. They need at least 180 pounds of dark chocolate and 90 pounds of white. Their supplier has two mixes for them to buy. The deluxe mix costs $10.00 a bag and has 4 pounds of dark and 1 pound of white. The plain mix costs $5.00 a pound of each. The Sweet Tooth Candy Shoppe can pay at most $800 for the chocolate. How many bags of each can be purchased? Use a graph to help you decide.

Solution:

$4x+y\ge 180$ $x\ge 0$ $x+y\ge 90$ $y\ge 0$ $10x+5y\le 800$	Let x represent the number of deluxe bags and y represent the number of plain bags. Write all of the constraint equations.
	Graph to find the solution area.
$10(10)+5(140)\le 800$ $800\le 800$ $10(70)+5(20)\le 800$ $800\le 800$ $10(30)+5(60)\le 800$ $600\le 800$	Any point in the shaded area is a solution. However, to minimize the cost check all three of the intersection points in $10x+5y$ to see which one is less than $800.
Buying 30 bags of the deluxe mixture and 60 bags of the plain mixture will give you the required pounds of chocolate for the least amount of money.	

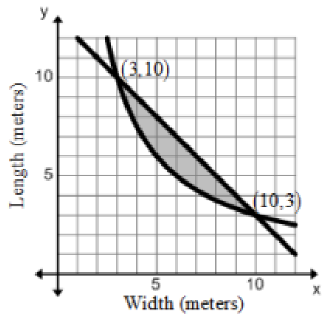
Example: For his math grade, Carter can do extra homework packets for 70 points each or math projects for 80 points each. He estimates that each homework packet will take 9 hours and each project will take 15 hours and that he will have at most 120 hours to spend. He may turn in a total of no more than 12 packets or projects. How many of each should he complete in order to receive the highest score?

Solution:

$x+y\le 12$ $x\ge 0$ $9x+15y\le 120$ $y\ge 0$ $70x+80y=\text{maximum}$	Let x represent the number of homework packets and y represent the number of projects. Write all of the constraint equations.
	Graph to find the solution area.
$70(0)+80(0)=0$ $70(12)+80(0)=840$ $70(10)+80(2)=860$ $70(0)+80(8)=640$	Any point in the shaded region is a solution. To find the highest score, substitute the intersection points into the expression $70x+80y$ to see which one is the greatest.
Completing 10 homework packets and 2 projects will maximize Carter's grade.	

Example: The perimeter of a rectangle is at most 26 meters. Its area is at least 30 square meters. What are the possible dimensions of the rectangle?

Solution:

$2(x+y) \le 26$ $xy \ge 30$ $x \ge 0$ $y \ge 0$	Write the constraints.
	Graph to find the solution area.
Any point in the shaded region is a solution. For example, $(5,7)$ is a solution because the perimeter would be $2(5+7)=24$ meters which is less than 26 meters. The area would be $5(7)=35$ square meters which is greater than 30 square meters.	

Sample Questions:

For the sequence, write and graph the rational equation that models the relationship between the term in the sequence and its value.

44. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

45. $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \ldots$

Write the constraints for each situation and graph the solution area.

46. A manufacturer produces the following two items: backpacks and messenger bags. Each item requires processing in each of two departments. The cutting department has 60 hours available and finishing department has 42 hours available each week for production. To manufacture a backpack requires 4 hours in cutting and 3 hours in finishing while a messenger bag requires 3 hours in cutting and 2 hours in finishing. Profits on the items are $10 and $7 respectively. If all the bags can be sold, how many of each should be made to maximize profits?

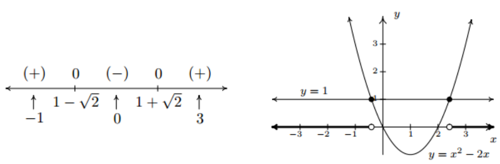
47. Kathy owns a car and a moped. She has at most 12 gallons of gasoline to be used between the car and the moped. The car's tank holds at most 18 gallons and the moped's 3 gallons. The mileage for the car is 20 mpg. The mileage for the moped is 100 mpg. How many gallons of gasoline should each vehicle use if Kathy wants to travel as far as possible? What is the maximum number of miles?

48. Bob's Furniture produces chairs and sofas. The chairs require 20 feet of wood, 1 pound of foam rubber, and 2 square yards of fabric. The sofas require 100 feet of wood, 50 pounds of foam rubber, and 20 square yards of fabric. The company has 1900 feet of wood, 500 pounds of foam rubber, and 240 square yards of fabric. The chairs can be sold for $80 and the sofas for $1,200. How many of each should be produced to maximize the income?

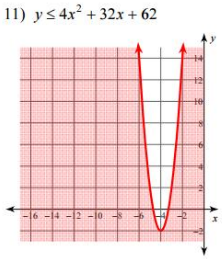
Answer Key

1. .
2. .
3. .
4.
5. .
6. .
7.
8. .
9. -5/3 < x < 5
10. x > 5 or x < −9
11. All x > 3 (B)
12. No solution or The empty set (A)
13. 11 ≤ x < 15 (B)
14. 154
15. 19
16. 5 different integers
17. C
18. -1, 4, 3 $\frac{3\pm\sqrt{3}}{2}$
19. $x^2 - 2x > 1$

2. Once again, we re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the quadratic formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$ and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$. To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x values where the graph of g is above the graph of h. As before we present the graphs on the right and the sign chart on the left.

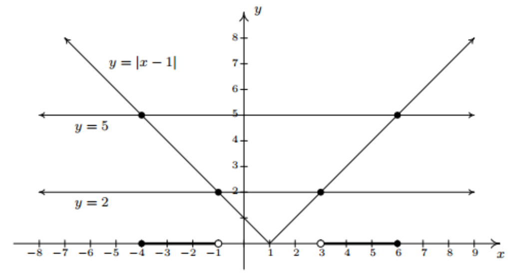


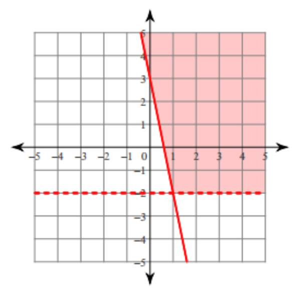
11) $y \le 4x^2 + 32x + 62$

20. 

21.

3. Rewriting the compound inequality $2 < |x-1| \le 5$ as '$2 < |x-1|$ and $|x-1| \le 5$' allows us to solve each piece using Theorem 2.4. The first inequality, $2 < |x-1|$ can be re-written as $|x-1| > 2$ so $x - 1 < -2$ or $x - 1 > 2$. We get $x < -1$ or $x > 3$. Our solution to the first inequality is then $(-\infty, -1) \cup (3, \infty)$. For $|x-1| \le 5$, we combine results in Theorems 2.1 and 2.4 to get $-5 \le x - 1 \le 5$ so that $-4 \le x \le 6$, or $[-4, 6]$. Our solution to $2 < |x-1| \le 5$ is comprised of values of x which satisfy both parts of the inequality, so we take intersection[1] of $(-\infty, -1) \cup (3, \infty)$ and $[-4, 6]$ to get $[-4, -1) \cup (3, 6]$. Graphically, we see that the graph of $y = |x-1|$ is 'between' the horizontal lines $y = 2$ and $y = 5$ for x values between -4 and -1 as well as those between 3 and 6. Including the x values where $y = |x-1|$ and $y = 5$ intersect, we get



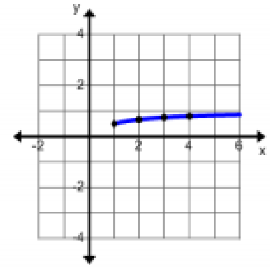
22. 

23. 4
24. -13
25. II only (B)
26. XZ = 1/Y (A)
27. All negative real numbers (A)
28. x < (xy)/2 < y (A)
29. $\frac{x}{3} - 5 > \frac{y}{3} - 5$ (C)
30. $\frac{x}{y2}$ (E)
31. −(a – b) (E)
32. a < 0 and b ≠ -4 (D)
33. (-∞, −4] ∪ [3, ∞)
34. (-∞, −4] ∪ [1, ∞)
35. (-∞, −5) ∪ (0, 3)
36. (-∞, −1] ∪ [0, 7]
37. (-∞, −3) ∪ [0, ∞)
38. (-∞, −3) ∪ [−2, 3)
39. (0, 2) ∪ (2, ∞)
40. [4/3, ∞)
41. 0, 1.343) seconds
42. (0, 1.5] ∪ [2.628, 7) inches

43. (2.863, 9.209) centimeters

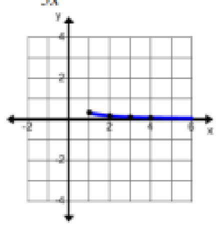
44. .

$$f(x)=\frac{x}{x+1} \text{ for } x\geq 1$$



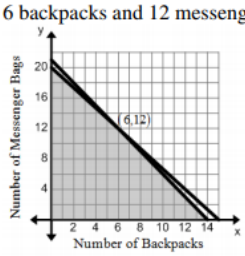
45.

$$f(x)=\frac{1}{3x} \text{ for } x\geq 1$$

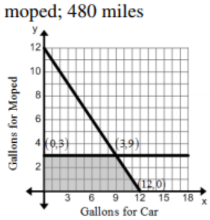


46.

6 backpacks and 12 messenger bags



47. 9 gallons for the car, 3 gallons for the moped; 480 miles



48. 25 chairs and 9 sofas