Math 302: Solutions to Homework

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Abstract

Below are detailed solutions to the homework problems from Math 302 Complex Analysis (Williams College, Fall 2010, Professor Steven J. Miller, sjm1@williams.edu). The course homepage is

`http://www.williams.edu/Mathematics/sjmiller/public_html/302`

and the textbook is *Complex Analysis* by Stein and Shakarchi (ISBN13: 978-0-691-11385-2). Note to students: it's nice to include the statement of the problems, but I leave that up to you. **I am only skimming the solutions. I will occasionally add some comments or mention alternate solutions. If you find an error in these notes, let me know for extra credit.**

Contents

1 Homework #1: Yuzhong (Jeff) Meng and Liyang Zhang

Due by 10am Friday, September 17: Chapter 1: Page 24: #1abcd, #3, #13.

Problem: Chapter 1: #1: Describe geometrically the sets of points z in the complex plane defined by the following relations: (a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$; (b) $1/z = \overline{z}$; (c) $\mathrm{Re}(z) = 3$; (d) $\mathrm{Re}(z) > c$ (resp., $\geq c$) where $c \in \mathbb{R}$.

Solution: (a) When $z_1 \neq z_2$, this is the line that perpendicularly bisects the line segment from z_1 to z_2. When $z_1 = z_2$, this is the entire complex plane.
(b)

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}. \tag{1.1}$$

So

$$\frac{1}{z} = \overline{z} \Leftrightarrow \frac{\overline{z}}{|z|^2} = \overline{z} \Leftrightarrow |z| = 1. \tag{1.2}$$

This is the unit circle in $\mathbb{C}$.
(c) This is the vertical line $x = 3$.
(d) This is the open half-plane to the right of the vertical line $x = c$ (or the closed half-plane if it is $\geq$).

Problem: Chapter 1: #3: With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

Solution: Notice that

$$\omega = se^{i\varphi} = se^{i(\varphi + 2\pi m)}, m \in \mathbb{Z}. \tag{1.3}$$

It's worth spending a moment or two thinking what is the best choice for our generic integer. Clearly n is a bad choice as it is already used in the problem; as we often use t for the imaginary part, that is out too. The most natural is to use m (though k would be another fine choice); at all costs do not use i!

Based on this relationship, we have

$$z^n = se^{i(\varphi+2\pi m)}. \tag{1.4}$$

So,

$$z = se^{\frac{i(\varphi+2\pi m)}{n}}. \tag{1.5}$$

Thus, we will have n unique solutions since each choice of $m \in \{0, 1, \ldots, n-1\}$ yields a different solution so long as $s \neq 0$. Note that $m = n$ yields the same solution as $m = 0$; in general, if two choices of m differ by n then they yield the same solution, and thus it suffices to look at the n specified values of m. If $s = 0$, then we have only 1 solution.

Problem: Chapter 1: #13: Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:
(a) Re(f) is constant;
(b) Im(f) is constant;
(c) $|f|$ is constant;
one can conclude that f is constant.

Solution: Let $f(z) = f(x, y) = u(x, y) + iv(x, y)$, where $z = x + iy$.
(a) Since Re$(f) = constant$,

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0. \tag{1.6}$$

By the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0. \tag{1.7}$$

Thus, in Ω,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0. \tag{1.8}$$

Thus $f(z)$ is constant.

(b) Since Im(f) = $constant$,

$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0. \tag{1.9}$$

By the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0. \tag{1.10}$$

Thus in Ω,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0. \tag{1.11}$$

Thus f is constant.

(c) We first give a mostly correct argument; the reader should pay attention to find the difficulty. Since $|f| = \sqrt{u^2 + v^2}$ is constant,

$$\begin{cases} 0 = \frac{\partial (u^2+v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}. \\ 0 = \frac{\partial (u^2+v^2)}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \end{cases} \tag{1.12}$$

Plug in the Cauchy-Riemann equations and we get

$$u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0. \tag{1.13}$$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0. \tag{1.14}$$

$$(1.14) \Rightarrow \frac{\partial v}{\partial x} = \frac{v}{u}\frac{\partial v}{\partial y}. \tag{1.15}$$

Plug (1.15) into (1.13) and we get

$$\frac{u^2 + v^2}{u}\frac{\partial v}{\partial y} = 0. \tag{1.16}$$

So $u^2 + v^2 = 0$ or $\frac{\partial v}{\partial y} = 0$.

If $u^2 + v^2 = 0$, then, since u, v are real, $u = v = 0$, and thus $f = 0$ which is constant.

Thus we may assume $u^2 + v^2$ equals a non-zero constant, and we may divide by it. We multiply both sides by u and find $\frac{\partial v}{\partial y} = 0$, then by (1.15), $\frac{\partial v}{\partial x} = 0$, and by Cauchy-Riemann, $\frac{\partial u}{\partial x} = 0$.

$$f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0. \tag{1.17}$$

Thus f is constant.

Why is the above only mostly a proof? The problem is we have a division by u, and need to make sure everything is well-defined. Specifically, we need to know that u is never zero. We do have $f' = 0$ except at points where $u = 0$, but we would need to investigate that a bit more.

Let's return to

$$\begin{cases} 0 = \frac{\partial(u^2+v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}. \\ 0 = \frac{\partial(u^2+v^2)}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \end{cases} \tag{1.18}$$

Plug in the Cauchy-Riemann equations and we get

$$\begin{aligned} u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} &= 0 \\ -u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{1.19}$$

We multiply the first equation u and the second by v, and obtain

$$\begin{aligned} u^2\frac{\partial v}{\partial y} + uv\frac{\partial v}{\partial x} &= 0 \\ -uv\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{1.20}$$

Adding the two yields

$$u^2\frac{\partial v}{\partial y} + v^2\frac{\partial v}{\partial y} = 0, \tag{1.21}$$

or equivalently

$$(u^2 + v^2)\frac{\partial v}{\partial y} = 0. \tag{1.22}$$

We now argue in a similar manner as before, except now we don't have the annoying u in the denominator. If $u^2 + v^2 = 0$ then $u = v = 0$, else we can divide by $u^2 + v^2$ and find $\partial v/\partial y = 0$. Arguing along these lines finishes the proof. $\square$

One additional remark: we can trivially pass from results on partials with respect to v to those with respect to u by noting that if $f = u + iv$ has constant magnitude, so too does $g = if = -v + iu$, which essentially switches the roles of u and v. Though this isn't needed for this problem, arguments such as this can be very useful.

It's worth mentioning that (a) and (b) follow immediately from (c). For example, assume we know the real part of f is constant. Consider

$$g(z) \ = \ \exp(f(z)) \ = \ \exp(u(x,y))\exp(iv(x,y)).$$

As $|g(z)| = \exp(u(x,y))$, we see that the real part of f being constant implies the function g has constant magnitude. By part (c) this implies that g is constant, which then implies that f is constant.

2 Homework #2: Nick Arnosti and Thomas Crawford

Due in my mailbox by 10am Friday, September 24: Chapter 1: Page 24: #16abc, #24, #25ab. Chapter 2: (#1) We proved Goursat's theorem for triangles. Assume instead we know it holds for any rectangle; prove it holds for any triangle. (#2) Let γ be the closed curve that is the unit circle centered at the origin, oriented counter-clockwise. Compute $\oint_\gamma f(z)dz$ where $f(z)$ is complex conjugation (so $f(x+iy) = x - iy$). Repeat the problem for $\oint_\gamma f(z)^n dz$ for any integer n (positive or negative), and compare this answer to the results for $\oint_\gamma z^n dz$; is your answer surprising? (#3) Prove that the four triangles in the subdivision in the proof of Goursat's theorem are all similar to the original triangle. (#4) In the proof of Goursat's theorem we assumed that f was complex differentiable (ie, holomorphic). Would the result still hold if we only assumed f was continuous? If not, where does our proof break down?

Problem: If γ is a curve in $\mathbb{C}$, show that $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$.

Parameterize γ by $z = g(t)$ for t in $[a, b]$, and define $w(t) = g(a + b - t)$. Then $w(t)$ is a parameterization of $-\gamma$ on the interval $[a, b]$ (note that $w(a) = g(b), w(b) = g(a)$). Additionally, $w'(t) = -g'(a + b - t)$. It follows that

$$\int_{-\gamma} f(z)dz = \int_a^b f(w(t))w'(t)dt = -\int_a^b f(g(a + b - t))g'(a + b - t)dt.$$

Making the substitution $u = a + b - t$, we get that

$$\begin{aligned} -\int_{t=a}^{b} f(g(a+b-t))g'(a+b-t)dt &= \int_{u=b}^{a} f(g(u))g'(u)du \\ &= -\int_{u=a}^{b} f(g(u))g'(u)du. \quad (2.1) \end{aligned}$$

But

$$-\int_{u=a}^{b} f(g(u))g'(u)du = -\int_\gamma f(z)dz,$$

which proves the claim.

Problem: If γ is a circle centered at the origin, find $\int_\gamma z^n dz$.

We start by parameterizing γ by $z = re^{i\theta}, 0 \leq \theta < 2\pi$, so $dz = ire^{i\theta}d\theta$. Then

$$\int_\gamma z^n dz = \int_0^{2\pi} r^n e^{in\theta}(ire^{i\theta})d\theta = ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta.$$

If $n = -1$, this is $ir^0 \int_0^{2\pi} d\theta = 2\pi i$. Otherwise, we get

$$ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta = \frac{r^{n+1}}{n+1}e^{i(n+1)}\theta\Big|_0^{2\pi} = 0.$$

Problem: If γ is a circle not containing the origin, find $\int_\gamma z^n dz$.

If $n \neq -1$, the function $f(z) = z^n$ has a primitive (namely $\frac{z^{n+1}}{n+1}$), so by Theorem 3.3 in Chapter 1 of our book, $\int_\gamma f(z)dz = 0$.

If $n = -1$, we parameterize γ by $z = z_0 + re^{i\theta}, 0 \leq \theta < 2\pi$, so $dz = ire^{i\theta}d\theta$.Then

$$\int_\gamma \frac{1}{z}dz = \int_0^{2\pi} \frac{ire^{i\theta}}{z_0 + re^{i\theta}}d\theta = \frac{ir}{z_0}\int_0^{2\pi} \frac{e^{i\theta}}{1 + \frac{r}{z_0}e^{i\theta}}d\theta.$$

Note that because our circle does not contain the origin, $|z_0| > r$, so $|\frac{r}{z_0}e^{i\theta}| < 1$. Thus, we can write this expression as a geometric series:

$$\frac{ir}{z_0}\int_0^{2\pi} \frac{e^{i\theta}}{1 + \frac{r}{z_0}e^{i\theta}}d\theta = \frac{ir}{z_0}\int_0^{2\pi} e^{i\theta}\sum_{k=0}^{\infty}(\frac{-r}{z_0}e^{i\theta})^k d\theta.$$

Interchanging the sum and the integral, we see that this is just

$$-i\sum_{k=0}^{\infty}(\frac{-r}{z_0})^{k+1}\int_0^{2\pi} e^{i(k+1)\theta}d\theta = -\sum_{k=0}^{\infty}(\frac{-r}{z_0})^{k+1}\frac{e^{i(k+1)\theta}}{k+1}\bigg|_0^{2\pi} d\theta = 0.$$

Why may we interchange? We can justify the interchange due to the fact that the sum of the absolute values converges.

Problem: If γ is the unit circle centered at the origin, find $\int_\gamma \bar{z}^n dz$.

We start by parameterizing γ by $z = e^{i\theta}, 0 \leq \theta < 2\pi$, so $\bar{z} = e^{-i\theta}$ and $dz = ie^{i\theta}d\theta$. Then

$$\int_\gamma \bar{z}^n dz = \int_0^{2\pi} e^{-in\theta}(ie^{i\theta})d\theta = i\int_0^{2\pi} e^{-i(n-1)\theta}d\theta.$$

If $n = 1$, this is $i\int_0^{2\pi} d\theta = 2\pi i$. Otherwise, we get

$$i\int_0^{2\pi} e^{-i(n-1)\theta} d\theta = \frac{e^{i(1-n)\theta}}{1-n}\Big|_0^{2\pi} = 0.$$

Note that instead of doing the algebra, we could have observed that on the unit circle $\bar{z} = z^{-1}$, so $\int_\gamma \bar{z}^n dz = \int_\gamma z^{-n} dz$. Applying our work from Problem 2, we get the answer above.

Problem: Where in the proof of Goursat's theorem do we use the fact that the function f is holomorphic? Is it sufficient to know that f is continuous?

Start by recapping the main ideas behind the proof. We began by continually splitting our triangle T into smaller triangles. These triangles converge to a point in the limit, and we called this point z_0. We then established the bound

$$\Big|\int_T f(z)dz\Big| \leq 4^n \Big|\int_{T^{(n)}} f(z)dz\Big|.$$

Our goal was to show that this quantity tends to zero as $z \to z_0$.

To do this, we Taylor expanded $f(z)$ around the point $z_0 : f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$. Note that $(z - z_0)$ divides $\psi(z)$, so $\psi(z) \to 0$ as $z \to z_0$.

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq \left|\int_{T^{(n)}} f(z_0) + f'(z_0)(z - z_0)dz\right| + \int_{T^{(n)}} |\psi(z)(z - z_0)|dz$$

The first integrand in this sum has a primitive, so the value of this integral is zero. Let $M_n = \max_{z \text{ on } T^{(n)}} |\psi(z)|$. Then $|\psi(z)| \leq M_n$, and $z - z_0 \leq \text{diam}(T^{(n)})$. Hence, the value of the second integral is at most $\text{perim}(T^{(n)}) \cdot \text{diam}(T^{(n)}) \cdot M_n$.

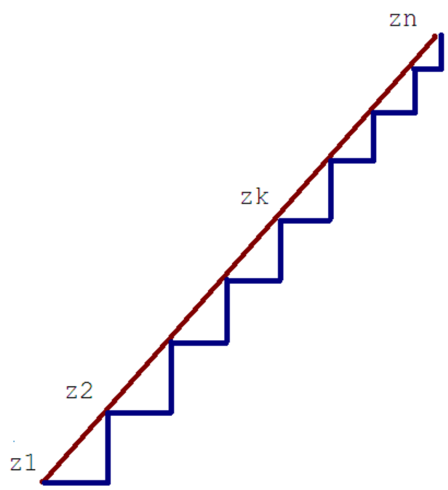
Since the perimeter and diameter of $T^{(n)}$ both decay at a rate of 2^{-n}, we establish the bound that $\left|\int_{T^{(n)}} f(z)dz\right| \leq 4^{-n} CM_n$ for some constant C. Hence, CM_n is an upper-bound for $\left|\int_T f(z)dz\right|$, and since $\psi(z) \to 0$ as $z \to z_0$, $M_n \to 0$ as desired.

Now let us see what happens if we don't know that f is differentiable. Using only continuity, we can approximate $f(z)$ by $f(z_0) + \psi(z)(z - z_0)$. Defining M_n as before, we can still bound our integral by CM_n. We want to say that M_n tends to 0, but $\lim_{z\to z_0} \psi(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$, which may not exist if f is not differentiable (and certainly may not tend to zero). Thus, this approach fails.

We could also try the expression $f(z) = f(z_0) + \psi(z)$, and then $\psi(z) \to 0$ as $z \to z_0$. Unfortunately, without the factor of $(z - z_0)$, our bound on $|\int_{T^{(n)}} f(z)dz|$ will simply be perim$(T^{(n)}) \cdot M_n = 2^{-n}CM_n$. Thus, our bound for $|\int_{T^{(n)}} f(z)dz|$ is $4^n 2^{-n}CM_n = 2^n CM_n$. Even though M_n tends to 0, the factor of 2^n may overwhelm it, so this approach fails. From these attempts, it seems that knowing that f was differentiable was a fairly important step in the proof.
Problem: Prove Goursat's theorem for triangles using only the fact that it holds for rectangles.

Note that it suffices to prove that the integral along any right triangle is zero, since any triangle can be divided into two right triangles by dropping an altitude.

Given a right triangle ABC, by drawing a series of rectangles inside the triangle, we can reduce the desired integral to the integral along a series of n congruent triangles similar to ABC, each of which border the original hypotenuse (as shown in the figure).



Since f is continuous on the original triangle ABC (a compact set) we know that f is uniformly continuous on the region of interest.

Thus, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any two points x, y in ABC with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$. If h is the length of the hypotenuse of ABC, choose n large enough so that the diameter of each small triangle, h/n, is less than δ. Then for any triangle T_k and any point z_k on that triangle write $f(z) = f(z_k) + \psi(z)$, so that

$$\int_{T_k} f(z)dz = \int_{T_k} f(z_k) + \psi(z)dz = \int_{T_k} f(z_k)dz + \int_{T_k} \psi(z)dz$$

Since $f(z_k)$ is a constant, it has a primitive, so the first integral is zero. Meanwhile, since any point on triangle T_k is within h/n of z_k, and we chose n to be such that $h/n < \delta$, we know that $|\psi(z)| = |f(z) - f(z_k)| < \varepsilon$. Thus, $|\int_{T_k} \psi(z)dz| <$ perim$(T_k) \cdot \varepsilon$. But perim$(T_k) < 3h/n$, so the integral of $f(z)$ along triangle T_k is at most $3h\varepsilon/n$. Summing over all n triangles, we see that the integral of $f(z)$ along the entire curve is at most $3h\varepsilon$. Since this technique works for arbitrarily small ε, this implies that the integral of f along any right triangle is zero, proving the claim.

3 Homework #3: Carlos Dominguez, Carson Eisenach, David Gold

HW: Due in my mailbox by 10am Friday, October 1 (even if this is Mountain Day): Chapter 2, Page 64: #1, #8. Also do: Chapter 2: (#1) In the proof of Liouville's theorem we assumed f was bounded. Is it possible to remove that assumption? In other words, is it enough to assume that $|f(z)| < g(z)$ for some real-valued, non-decreasing function g? If yes, how fast can we let f grow? (#2) a) Find all z where the function $f(z) = 1/(1+z^4)$ is not holomorphic; b) Let a, b, c, and d be integers such that $ad - bc = 1$. Find all z where the function $g(z) = (az+b)/(cz+d)$ is not holomorphic. (#3) Compute the power series expansion of $f(z) = 1/(1-z)$ about the point $z = 1/2$ (it might help to do the next problem first, or to write $1 - z$ as $1/2 - (z - 1/2)$). (#4) Do Chapter 1, Page 29, #18.

1. Let γ_1 denote the straight line along the real line from 0 to R, γ_2 denote the eighth of a circle from R to $Re^{i\frac{\pi}{4}}$, and γ_3 denote the line from $Re^{i\frac{\pi}{4}}$ to 0. Then by Cauchy's theorem,

$$\int_{\gamma_1+\gamma_2+\gamma_3} e^{-z^2}\, dz = 0.$$

We can calculate

$$
\begin{aligned}
-\int_{\gamma_3} e^{-z^2}\,dz &= \int_0^R e^{-(e^{i\pi/4}t)^2} e^{i\pi/4}\,dt \\
&= e^{i\pi/4}\int_0^R e^{-it^2}\,dt \\
&= e^{i\pi/4}\int_0^R \cos(-t^2)dt + i\sin(-t^2)\,dt \\
&= e^{i\pi/4}\int_0^R \cos(t^2)dt - i\sin(t^2)\,dt
\end{aligned}
$$

So we can calculate the Fresnel integrals by calculating $\int_{\gamma_1+\gamma_2} e^{-z^2}\,dz$, taking $R\to\infty$, dividing by $e^{i\pi/4}$, and looking at the real and negative imaginary parts. First we show the integral over γ_2 goes to zero:

$$
\begin{aligned}
\left|\int_{\gamma_2} e^{-z^2}\,dz\right| &= \left|\int_0^{\pi/4} e^{-R^2e^{2i\theta}} iRe^{i\theta}\,d\theta\right| \\
&\le R\int_0^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta \\
&= R\int_0^{\pi/4-1/R\log R} e^{-R^2\cos 2\theta}\,d\theta + R\int_{\pi/4-1/R\log R}^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta \\
&\le R\left(\frac{\pi}{4}-\frac{1}{R\log R}\right)e^{-R^2\cos\left(\frac{\pi}{2}-\frac{2}{R\log R}\right)} + R\cdot\frac{1}{R\log R} \\
&\le \frac{\pi}{4}Re^{-R^2\sin\left(\frac{2}{R\log R}\right)} + \frac{1}{\log R}
\end{aligned}
$$

The $\frac{1}{\log R}$ term goes to zero as R goes to infinity. So we need to show that the first term goes to zero. Note that $\sin x \ge x/2$ for positive x sufficiently close to 0, since $\sin 0 = 0$ and $\frac{d}{dx}\sin x \ge 1/2$ for sufficiently small x. So for sufficiently large R the first term is less than or equal to

$$
\frac{\pi}{4}Re^{-R^2\cdot\frac{1}{R\log R}} = \frac{\pi}{4}e^{\log R - \frac{R}{\log R}},
$$

which goes to zero as R goes to infinity. So, as $R\to\infty$, the contribution from γ_2 goes to zero. And we know that as $R\to\infty$, $\int_0^R e^{-x^2}\,dx = \sqrt{\pi}/2$.

So, finally,

$$\int_0^\infty \cos(t^2)dt - i\sin(t^2)\,dt = \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{2}/2+i\sqrt{2}/2}$$
$$= \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4}i$$

as desired.

8. Since $x \in \mathbb{R}$, f is holomorphic in an open circle of radius ϵ centered at x, 0 $< \epsilon < 1$. And by Cauchy's inequality,

$$|f^{(n)}(x)| \leq \frac{n!||f||_C}{R^n}$$

Case 1: $\eta \geq 0$. For some $0 < \epsilon < 1$,

$$|z| \leq |x+\epsilon|$$

thus,

$$|f(z)| \leq A(1+|x+\epsilon|)^\eta \leq A(1+\epsilon+|x|)^\eta$$

by both the given and the triangle inequality. And in Cauchy's inequality R is just ϵ. So by combining results from above

$$\begin{aligned} |f^{(n)}(x)| &\leq \frac{n!||f||_C}{\epsilon^n} \\ &\leq \frac{An!}{\epsilon^n}(1+\epsilon+|x|)^\eta \\ &\leq \frac{An!}{\epsilon^n}(1+\epsilon+|x|+\epsilon|x|)^\eta \\ &\leq \frac{An!}{\epsilon^n}(1+\epsilon)^\eta(1+|x|)^\eta. \qquad (3.1) \end{aligned}$$

Now let

$$A_n = \frac{A(n!)}{\epsilon^n}(1+\epsilon)^\eta$$

thus,

$$|f^{(n)}(x)| \leq A_n(1+|x|)^\eta.$$

Case 2: $\eta < 0$. For some $0 < \epsilon < 1$,

$$\epsilon \geq |x-z| \geq |x|-|z|$$

by the reverse triangle inequality. When we rearrange the inequality we see that

$$|z| \geq |x| - |\epsilon| = |x| + \epsilon$$

Since η is negative, our goal is to minimize (1+|z|) in order to get an upper bound. Now, by combining our result above with the Cauchy inequality we get that:

$$\begin{aligned} |f^{(n)}(x)| &\leq \frac{n!||f||_C}{\epsilon^n} \leq \frac{An!}{\epsilon^n}(1 - \epsilon + |x|)^\eta \\ &\leq \frac{An!}{\epsilon^n}(1 - \epsilon + |x| - \epsilon|x|)^\eta \\ &\leq \frac{An!}{\epsilon^n}(1-\epsilon)^\eta(1+|x|)^\eta. \end{aligned} \tag{3.2}$$

Now let

$$A_n = \frac{A(n!)}{\epsilon^n}(1-\epsilon)^\eta$$

thus,

$$|f^{(n)}(x)| \leq A_n(1+|x|)^\eta.$$

q.e.d.

1. In the proof of Liouville's theorem, we had that

$$|f'(z_0)| \leq \frac{B}{R}$$

where B was an upper bound for f. It only matters that B is an upper bound for f in a disc of radius R about z_0, however. Let B_R be the smallest upper bound for f in a disc of radius R about z_0. Liouville's theorem still holds if $B_R \to \infty$ as long as $B_R/R \to 0$ for every choice of z_0. Alternatively, we just need f to grow slower than linear; say $|f(z)|$ is less than $C|z|^{1-\epsilon}$ or $C|z|/\log|z|$ or anything like this (for those who have seen little-oh notation, $f(z) = o(z)$ suffices).

2. (a) f is holomorphic wherever its derivative exists:

$$f'(z) = -\frac{4z^3}{1+z^4}$$

That is, whenever $z^4 \neq -1$. This gives $z = e^{i\pi/4}$, $e^{3i\pi/4}$, $e^{5i\pi/4}$, and $e^{7i\pi/4}$, or $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, and $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

(b) The $ad - bc = 1$ condition prevents g from being a mostly-constant function with an undefined value at $z = -d/c$. (That is, if $ad - bc = 0$, then $a/c = b/d$, and so the function would simply collapse to the value of a/c.) So

$$g'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

The function is then not holomorphic at $z = -d/c$.

3. Just use the geometric series formula:

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{1/2 - (z - 1/2)} \\ &= \frac{2}{1 - 2(z-1/2)} \\ &= \sum_{n=0}^{\infty} 2^{n+1}(z-1/2)^n.\end{aligned}$$

4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\begin{aligned}f(z) &= \sum_{n=0}^{\infty} a_n (z_0 + (z - z_0))^n \\ &= \sum_{n=0}^{\infty} a_n \left[\sum_{m=0}^{n} \binom{n}{m} (z-z_0)^m z_0^{n-m}\right] \\ &= \sum_{m=0}^{\infty} (z-z_0)^m \left(\sum_{n=m}^{\infty} a_n \binom{n}{m} z_0^{n-m}\right).\end{aligned}$$

The inner sum converges by the root test:

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{a_n \binom{n}{m}} &= \frac{1}{R} \lim_{n\to\infty} \sqrt[n]{\binom{n}{m}} \\ &= \frac{1}{R}\end{aligned}$$

where R is the radius of convergence of the original power series for f and second limit is evaluated by noting $1 \le \sqrt[n]{\binom{n}{m}} \le n^{m/n}$ and $\lim_{n\to\infty} n^{m/n} =$

1. Since the inner sum has the same radius of convergence as the original sum, z_0 still lies in the disc of convergence in the inner sum; hence all the coefficients of $z - z_0$ converge, and f has a power series expansion about z_0.

4 Homework #4: Pham, Jensen, Koloğlu

HW: Due in my mailbox by 10am Friday, October 8 (even if this is Mountain Day): Chapter 3, Page 103: #1, #2, #5 (this is related to the Fourier transform of the Cauchy density), #15d, #17a (hard). Additional: Let $f(z) = \sum_{n=-5}^{\infty} a_n z^n$ and $g(z) = \sum_{m=-2}^{\infty} b_m z^m$ be the Laurent expansions for two functions holomorphic everywhere except possibly at $z = 0$. a) Find the residues of $f(z)$ and $g(z)$ at $z = 0$; b) Find the residue of $f(z) + g(z) atz = 0$; c) Find the residue of $f(z)g(z)$ at $z = 0$; d) Find the residue of $f(z)/g(z)$ at $z = 0$.

4.1 Chapter 3, Exercise 1

Exercise 4.1. *Using Euler's formula $\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$, show that the complex zeros of $\sin \pi z$ are exactly the integers, and that they are each of order 1. Calulate the residue of $\frac{1}{\sin \pi z}$ at $z = n \in \mathbb{Z}$.*

Solution: To show that the complex zeros of $\sin \pi z$ are exactly the integers, we will show that $\frac{e^{i\pi z_0} - e^{-i\pi z_0}}{2i} = 0$ if and only if $z_0 \in \mathbb{Z}$.

First prove the forward direction. We see that $\frac{e^{i\pi z_0} - e^{-i\pi z_0}}{2i} = 0$ gives

$$e^{i\pi z_0} = e^{-i\pi z_0}. \tag{4.1}$$

Since $z_0 = x + iy$ with $x, y \in \mathbb{R}$,

$$e^{i\pi x} e^{-\pi y} = e^{-i\pi x} e^{\pi y}. \tag{4.2}$$

For complex numbers to be equivalent, their magnitudes must be the same. Thus,

$$e^{-\pi y} = e^{\pi y}. \tag{4.3}$$

This implies that $-\pi y = \pi y$, so $y = 0$. The angles corresponding to Equation 4.2 must be congruent modulo 2π as well. Thus,

$$\pi x \equiv -\pi x \bmod 2\pi, \tag{4.4}$$

which means $\pi x \equiv 0$ or π. So we have

$$2\pi x \mod 2\pi \equiv 0, \tag{4.5}$$

which implies that x is an integer. Thus $x \in \mathbb{Z}$. Since $y = 0$, we have $z_0 = x$, implying $z_0 \in \mathbb{Z}$.

To prove the backward direction, consider $z_0 \in \mathbb{Z}$ for z_0 even,

$$\begin{aligned} \sin \pi z_0 &= \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \\ &= \frac{1-1}{2i} = 0. \end{aligned} \tag{4.6}$$

Similarly for z_0 odd,

$$\begin{aligned} \sin \pi z_0 &= \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \\ &= \frac{-1+1}{2i} = 0. \end{aligned} \tag{4.7}$$

Thus $\sin \pi z_0 = 0$ if and only if $z_0 \in \mathbb{Z}$. So the zeros of $\sin \pi z$ are exactly the integers.

Next we must show that each zero has order 1. We refer to Theorem 1.1 in Stein and Shakarchi.

Theorem 4.2. *Suppose that f is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then there exists a neighborhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that $f(z) = (z - z_0)^n g(z)$ for all $z \in U$.*

Since $\sin \pi z$ is analytic, take its Taylor series about z_0. We add zero to write z as $z - z_0 + z_0$. Using properties of the sine function, we claim

$$\sin \pi z \;=\; \sin \pi(z+z_0-z_0) \;=\; \sin \pi(z-z_0) \cos \pi z_0 + \cos \pi(z-z_0) \sin \pi z_0. \tag{4.8}$$

Note this statement does require proof, but will follow from standard properties of the exponential function (or from analytic continuation). The reason some work needs to be done is that $z - z_0$ need not be real, but the relation above does hold when z is real. What we are trying to do is understand the behavior of the function near z_0 from knowledge near 0 (as $z - z_0$ is close to zero). This is a common trick, but of course what makes this tractable is that we have the angle addition formula for sine.

When z_0 is an integer, we always have $\sin \pi z_0 = 0$. If z_0 is odd then $\cos \pi z_0$ is -1 while if z_0 is even it is 1. Thus for odd z_0,

$$\sin \pi z = -\frac{\pi}{1!}(z - z_0)^1 + \frac{\pi^3}{3!}(z - z_0)^3 - \frac{\pi^5}{5!}(z - z_0)^5 + \cdots \tag{4.9}$$

and for even z_0,

$$\sin \pi z = \frac{\pi}{1!}(z - z_0)^1 - \frac{\pi^3}{3!}(z - z_0)^3 + \frac{\pi^5}{5!}(z - z_0)^5 - \cdots . \tag{4.10}$$

We thus see that all zeros are simple.

We now turn to finding the residue at $z = n$ for $1/\sin \pi z$. From our Taylor expansion above, we have

$$\frac{1}{\sin \pi z} = \frac{1}{\sin \pi(z-n) \cos \pi n} = \frac{1}{\cos \pi n} \frac{1}{\sin \pi(z-n)}. \tag{4.11}$$

The problem is now solved by using the Taylor expansion of sine and the geometric series. We have $\cos \pi n = (-1)^n$, so

$$\begin{aligned} \frac{1}{\sin \pi z} &= (-1)^n \frac{1}{(z-z_0) - \frac{1}{3!}(z-z_0)^3 + \cdots} \\ &= \frac{(-1)^n}{z-z_0} \frac{1}{1 - \left(\frac{1}{3!}(z-z_0)^2 + \cdots\right)} \\ &= \frac{(-1)^n}{z-z_0} \left(1 + \left(\frac{1}{3!}(z-z_0)^2 + \cdots\right) + \left(\frac{1}{3!}(z-z_0)^2 + \cdots\right)^2 + \cdots\right). \end{aligned} \tag{4.12}$$

Note that each term in parentheses in the last line is divisible by $(z - z_0)^2$, and thus *none* of these will contribute to the residue, which is simply $(-1)^n$.

4.2 Chapter 3, Exercise 2

Exercise 4.3. *Evaluate the integral*

$$\int_{--\infty}^{\infty} \frac{dx}{1 + x^4}.$$

Solution: Consider the function $f(z) = \frac{1}{1+z^4}$. This function has poles at

$$\begin{aligned} 1/f(z) &= 0 \\ 1 + z^4 &= 0 \\ z &= e^{i\left(\frac{\pi}{4}+n\frac{\pi}{2}\right)}. \end{aligned} \tag{4.13}$$

Consider the contour of the semicircle in the upper half plane of radius R, denoted γ. Denote the part of the contour along the real line γ_1 and the part along the arc γ_2. Note that two of the poles of $f(z)$ lie inside this contour. Thus by Cauchy's residue theorem,

$$\frac{1}{2\pi i}\oint_\gamma f dz = \mathrm{Res}_f(e^{i\pi/4}) + \mathrm{Res}_f(e^{i3\pi/4}). \tag{4.14}$$

To find the residues, write

$$f(z) = \frac{1}{1+z^4} = \left(\frac{1}{z-e^{i\frac{\pi}{4}}}\right)\left(\frac{1}{z-e^{i\frac{3\pi}{4}}}\right)\left(\frac{1}{z-e^{i\frac{5\pi}{4}}}\right)\left(\frac{1}{z-e^{i\frac{7\pi}{4}}}\right).$$

Thus

$$\begin{aligned} \mathrm{Res}_f(e^{i\pi/4}) &= \left(\frac{1}{e^{i\frac{\pi}{4}}-e^{i\frac{3\pi}{4}}}\right)\left(\frac{1}{e^{i\frac{\pi}{4}}-e^{i\frac{5\pi}{4}}}\right)\left(\frac{1}{e^{i\frac{\pi}{4}}-e^{i\frac{7\pi}{4}}}\right) \\ &= e^{-i\frac{3\pi}{4}}\left(\frac{1}{1-i}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1+i}\right) \\ &= -\frac{1+i}{4\sqrt{2}} \end{aligned} \tag{4.15}$$

and similarly

$$\begin{aligned} \mathrm{Res}_f(e^{i\frac{3\pi}{4}}) &= e^{-i\frac{9\pi}{4}}\left(\frac{1}{1+i}\right)\left(\frac{1}{1-i}\right)\left(\frac{1}{2}\right) \\ &= \frac{1-i}{4\sqrt{2}} \end{aligned} \tag{4.16}$$

Thus we have

$$\begin{aligned} \frac{1}{2\pi i}\oint_\gamma f dz &= -\frac{1+i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}} \\ &= -\frac{i}{2\sqrt{2}} \\ \oint_\gamma f dz &= \frac{\pi}{\sqrt{2}}. \end{aligned} \tag{4.17}$$

Now, note that

$$\oint_{\gamma} f dz = \oint_{\gamma_1+\gamma_2} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz. \tag{4.18}$$

Observe that

$$\int_{\gamma_1} f dz = \int_{-R}^{R} \frac{1}{1+x^4} dx$$

and that

$$\begin{aligned} \int_{\gamma_2} f dz &= \int_{-R}^{R} \frac{1}{1+z^4} dz \\ \left| \int_{\gamma_2} f dz \right| &= \left| \int_{-R}^{R} \frac{1}{1+z^4} dx \right| \\ &\leq \max_{z \in \gamma_2} \left| \frac{1}{1+z^4} \right| \pi R \\ &= \frac{1}{R^4 - 1} \pi R. \end{aligned} \tag{4.19}$$

Thus

$$\lim_{R\to\infty} \left| \int_{\gamma_2} f dz \right| \leq \lim_{R\to\infty} \frac{\pi R}{R^4-1} = 0. \tag{4.20}$$

Hence, as $R \to \infty$, $\int_{\gamma_2} f dz \to 0$. Therefore as $R \to \infty$ we get our final result;

$$\begin{aligned} \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4} dx + \lim_{R\to\infty} \int_{\gamma_2} f dz &= \frac{\pi}{\sqrt{2}} \\ \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}. \end{aligned} \tag{4.21}$$

4.3 Chapter 3, Exercise 5

Exercise 4.4. *Use contour integration to show that* $\int_{-\infty}^{+\infty} \frac{e^{-2\pi i x \xi}}{(1+x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}$ *for all* ξ *real.*

Solution: Let $f(z) = \frac{e^{-2\pi i z \xi}}{(1+z^2)^2} = \frac{e^{-2\pi i z \xi}}{(z+i)^2(z-i)^2}$. We see that $f(z)$ has poles of order 2 at $z = \pm i$. Thus

$$\text{res}_{z_0} f(z) = \lim_{z\to z_0} \frac{d}{dz}(z-z_0)^2 f(z). \tag{4.22}$$

Alternatively, we could write our function as

$$f(z) \;=\; \frac{g(z)}{(z-z_0)^2}, \tag{4.23}$$

and then we need only compute the coefficient of the $z - z_0$ term of g.

Now consider the residue at $z_0 = i$:

$$\begin{aligned} \mathrm{res}_{z_0=i} f(z) &= \lim_{z\to i} \frac{d}{dz}(e^{-2\pi i z\xi}(z+i)^{-2}) \\ &= \lim_{z\to i}(-2\pi i\xi e^{-2\pi i z\xi}(z+i)^{-2} - 2e^{-2\pi i z\xi}(z+i)^{-3}) \\ &= \frac{1}{2}\pi i\xi e^{2\pi\xi} - \frac{1}{4} i e^{2\pi\xi}. \end{aligned} \tag{4.24}$$

For $z_0 = -i$, we have:

$$\begin{aligned} \mathrm{res}_{z_0=-i} f(z) &= \lim_{z\to i} \frac{d}{dz}(e^{-2\pi i z\xi}(z-i)^{-2}) \\ &= \lim_{z\to -i}(-2\pi i\xi e^{-2\pi i z\xi}(z-i)^{-2} - 2e^{-2\pi i z\xi}(z-i)^{-3}) \\ &= \frac{1}{2}\pi i\xi e^{-2\pi\xi} + \frac{1}{4} i e^{-2\pi\xi}. \end{aligned} \tag{4.25}$$

Now let us first consider the case when $\xi < 0$. We will use the contour γ of a semicircle oriented counterclockwise in the upper half-plane with radius R. Call the portion of γ along the real line γ_1 and the arc portion γ_2. Note that there is a pole inside γ at $z_0 = i$. By the residue formula, we have that

$$\int_\gamma f(z)dz = 2\pi i\left(\frac{1}{2}\pi i\xi e^{2\pi\xi} - \frac{1}{4} i e^{2\pi\xi}\right) = -\pi^2\xi e^{2\pi\xi} + \frac{1}{2}\pi e^{2\pi\xi}. \tag{4.26}$$

We also know that

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{R\to\infty}\int_{\gamma_1} f(z)dz. \tag{4.27}$$

Along γ_2, $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\theta$, where $z = R\cos\theta + iR\sin\theta$. Thus

$$\int_{\gamma_2} f(z)dz = \int_0^\pi \frac{e^{-2\pi i\xi Re^{i\theta}} iRe^{i\theta}}{(1-R^2e^{i2\theta})^2}d\theta. \tag{4.28}$$

Then it follows that

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)dz\right| &\leq \int_0^\pi \left|\frac{e^{-2\pi i\xi R\cos\theta}e^{2\pi\xi R\sin\theta}iRe^{i\theta}}{(1-R^2e^{i2\theta})^2}\right| d\theta \\
&\leq \int_0^\pi \left|\frac{Re^{-2\pi|\xi|R\sin\theta}}{(1-R^2)^2}\right| d\theta \\
&\leq \int_0^\pi \frac{R}{(1-R^2)^2} d\theta = \frac{\pi R}{(R^2-1)^2}. \qquad (4.29)
\end{aligned}$$

Taking the limit as R goes to infinity, we have

$$\lim_{R\to\infty}\left|\int_{\gamma_2} f(z)dz\right| \leq \lim_{R\to\infty}\frac{\pi R}{(R^2-1)^2} = 0. \qquad (4.30)$$

Thus

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)dz = 0. \qquad (4.31)$$

So $\lim_{R\to\infty}\int_\gamma f(z) = \lim_{R\to\infty}\int_{\gamma_1} f(z)$. It thus follows from Equation 4.26 that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2}dx &= -\pi^2\xi e^{2\pi\xi} + \frac{1}{2}\pi e^{2\pi\xi} \\
&= \frac{\pi}{2}\left(1+2\pi|\xi|\right)e^{-2\pi|\xi|} \qquad (4.32)
\end{aligned}$$

Now consider $\xi \geq 0$. We will use the contour γ of a semicircle oriented counterclockwise in the lower half-plane with radius R. Call the portion of γ along the real line γ_1 and the arc portion γ_2. Note that there is a pole inside γ at $z_0 = -i$. By the residue formula, we have that

$$\int_\gamma f(z)dz = 2\pi i\left(\frac{1}{2}\pi i\xi e^{-2\pi\xi} + \frac{1}{4}ie^{-2\pi\xi}\right) = -\pi^2\xi e^{-2\pi\xi} - \frac{1}{2}\pi e^{-2\pi\xi}. \qquad (4.33)$$

Also note that,

$$\int_{-\infty}^{+\infty} f(x)dx = -\lim_{R\to\infty}\int_{\gamma_1} f(z)dz. \qquad (4.34)$$

Along γ_2, $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\theta$, where $z = R\cos\theta + iR\sin\theta$. Thus,

$$\int_{\gamma_2} f(z)dz = \int_{-\pi}^{0} \frac{e^{-2\pi i\xi R\cos\theta}e^{2\pi\xi R\sin\theta}iRe^{i\theta}}{(1-R^2e^{i2\theta})^2}d\theta. \qquad (4.35)$$

Accordingly,

$$\begin{aligned}\left|\int_{\gamma_2} f(z)dz\right| &\leq \int_{\gamma_2} |f(z)|dz \\ &\leq \int_{-\pi}^{0} \left|\frac{Re^{2\pi|\xi|R\sin\theta}}{(1-R^2e^{i2\theta})^2}\right| d\theta \\ &\leq \int_{-\pi}^{0} \left|\frac{R}{(1-R^2)^2}\right| d\theta \\ &= \frac{\pi R}{(1-R^2)^2} \end{aligned} \tag{4.36}$$

Taking the limit as R goes to infinity, we have

$$\lim_{R\to\infty}\left|\int_{\gamma_2} f(z)dz\right| \leq \lim_{R\to\infty}\frac{\pi R}{(R^2-1)^2} = 0. \tag{4.37}$$

And thus,

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)dz = 0. \tag{4.38}$$

So $\lim_{R\to\infty}\int_{\gamma} f(z) = \lim_{R\to\infty}\int_{\gamma_1} f(z)$. It thus follows from Equation 4.33 that

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{e^{-2\pi i x\xi}}{(1+x^2)^2}dx &= -\left(-\pi^2\xi e^{-2\pi\xi} - \frac{1}{2}\pi e^{-2\pi\xi}\right) \\ &= \frac{\pi}{2}\left(1+2\pi|\xi|\right)e^{-2\pi|\xi|}\end{aligned} \tag{4.39}$$

Thus for all ξ real,

$$\int_{-\infty}^{+\infty}\frac{e^{-2\pi i x\xi}}{(1+x^2)^2}dx = \frac{\pi}{2}\left(1+2\pi|\xi|\right)e^{-2\pi|\xi|} \tag{4.40}$$

4.4 Chapter 3 Exercise 15d

[1] For any entire function f, let's consider the function $e^{f(x)}$. It is an entire function and furthermore we have the real part of f is bounded so:

$$|e^f| = |e^{u+iv}| = |e^u| \leq \infty$$

[1] Hint from Professor Miller

Hence e^f is bounded and therefore, by Louisville's Theorem, e^f is constant. It then follows that f is constant .

Alternatively, we could argue as follows. We are told the real part of f is bounded. Let's assume that the real part is always at most $B-1$ in absolute value. Then if we consider $g(z) = 1/(B - f(z))$ we have $|g(z)| \leq 1$. To see this, note the real part of $B - f(z)$ is at least 1. We again have constructed a bounded, entire function, and again by Liouville's theorem we can conclude g (and hence f) is constant.

4.5 Additional Problem 1

[2] Let:

$$f(z) = \sum_{n=-5}^{\infty} a_n\, z^n$$

$$g(z) = \sum_{m=-2}^{\infty} b_m\, z^m$$

1. We have:

$$\mathrm{res}_0 f = a_{-1}$$

$$\mathrm{res}_0 g = b_{-1}$$

2. We have

$$f(z) + g(z) = \sum_{n=-5}^{-3} a_n z^n \;+\; \sum_{n=-2}^{\infty} (a_n + b_n) z^n$$

   So $\mathrm{res}_0(f+g) = a_{-1} + b_{-1}$.

3. We have $-1 = -5+4 = -4+3 = -3+2 = -2+1 = -1+0 = 0-1 = 1-2$ so:

$$\mathrm{res}_0(f\,g) = a_{-5}\, b_4 + a_{-4}\, b_3 + a_{-3}\, b_2 + a_{-2}\, b_1 + a_{-1}\, b_0 + a_0\, b_{-1} + a_1\, b_{-2}$$

[2]Hint from Professor Miller

4. We have (assuming $b_2 \neq 0$):

$$\begin{aligned} \frac{f(z)}{g(z)} &= \frac{\sum_{n=-5}^{\infty} a_n\, z^n}{\sum_{m=-2}^{\infty} b_m\, z^m} \\ &= \frac{1}{z^3} \frac{\sum_{n=-2}^{\infty} a_{n-3}\, z^n}{\sum_{m=-2}^{\infty} b_m\, z^m} \\ &= \frac{1}{b_{-2} z} \; \frac{\sum_{n=-2}^{\infty} a_{n-3}\, z^n}{1 - (-\frac{1}{b_{-2}} \sum_{m=1}^{\infty} b_{m-2}\, z^m)}. \end{aligned} \tag{4.41}$$

As $z \to 0$ the final quantity in parentheses tends to zero, and thus we can expand using the geometric series formula. We only care about the constant term of this fraction, as it is multiplied by $1/b_{-2}z$ and thus only the constant term contributes to the pole. This is a very useful observation. It means that, when we expand with the geometric series, we can drop many terms, as we only need to keep terms that contribute to the constant term. Remember, we are not trying to find the Taylor expansion of this function, but rather just one particular term. We can thus write:

$$\begin{aligned} \frac{f(z)}{g(z)} &= \frac{1}{b_{-2} z} \, (\sum_{n=-2}^{\infty} a_{n-3}\, z^n) \sum_{k=0}^{\infty} \left(-\frac{1}{b_{-2}} \sum_{m=1}^{\infty} b_{m-2}\, z^m) \right)^k \\ &= \frac{1}{b_{-2} z} \Bigg[(a_{-5} z^{-2}) \left(\frac{-1}{b_{-2}} (b_0 z^2 + \cdots) + \frac{1}{b_{-2}^2} (b_{-1} z^2 + \cdots) + \cdots \right) \\ &\quad + (a_{-4} z^{-1}) \left(\frac{-1}{b_{-2}} (b_{-1} z^1 + \cdots) + \cdots \right) + (a_{-3} z^0)\,(1 + \cdots) + \cdots . \Bigg] \end{aligned} \tag{4.42}$$

So:

$$\mathrm{res}_0(\frac{f}{g}) = \frac{1}{b_{-2}} \left[a_{-5}(-\frac{b_0}{b_{-2}} + \frac{b_{-1}}{b_{-2}^2}) + a_{-4}(-\frac{b_{-1}}{b_{-2}}) + a_{-3} \right]$$

.

4.6 Chapter 3 Exercise 17a, Extra Creadit

Exercise 4.5. *Let f be non-constant and holomorphic in an open set containing the closed unit disc. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of f contains the unit disc.*

Solution: Suppose $f(z)$ does not have a zero in the unit disc, $\mathbb{D}$. Then $1/f(z)$ is holomorphic in $\mathbb{D}$. Note that since $|f(z)| = 1$ whenever $|z| = 1$, $|1/f(z)| = 1/|f(z)| = 1$ whenever $|z| = 1$ as well. But $f(z)$ is holomorphic in $\mathbb{D}$, implying $|f(z)| \leq 1$ in $\mathbb{D}$ by the maximum modulus principle since $|f(z)| = 1$ on the boundary of $\mathbb{D}$. We find $1 \leq |f(z)| \leq 1$ in the unit disk, which implies that our function is constant as its modulus is constant (from an earlier exercise), contradicting the assumption that f is not constant!

Let $w_0 \in \mathbb{D}$. Consider the constant function $g(z) = -w_0$. On the unit circle, $|f(z)| = 1 > |w_0| = |g(z)$ for all $|z| = 1$. Thus by Rouché's theorem, $f(z)$ and $f(z)+g(z)$ have the same number of zeroes inside the unit circle (ie, in $\mathbb{D}$). But we have shown that $f(z)$ has at least one zero, thus for some z_w, $0 = f(z_w)+g(z_w) = f(z_w) - w_0$. Thus for all $w_0 \in \mathbb{D}$, there exists z_w such that $f(z_w) = w_0$. Thus the image of $f(z)$ contains the unit disc. $\square$

5 Homework #5: Pegado, Vu

HW: Due in my mailbox by 10am Friday, October 22 (as there is no class on Tuesday): Chapter 5: Page 155: #6, #7, #9 (extra credit: what is the combinatorial significance of this problem?). Chapter 3: Page 104: #10. Additional Problems: (1) Find all poles of the function $f(z) = 1/(1-z^2)^4$ and find the residues at the poles. (2) Consider the map $f(z) = (z-i)/(z+i)$. Show that this is a 1-to-1 and onto map from the upper half plane (all $z = x + iy$ with $y > 0$) to the unit disk. (3) Calculate the Weierstrass product for $\cos(\pi z)$ (this is also problem #10b in Chapter 5, and the answer is listed there), and for $\tan(\pi z)$.

6. *Prove Wallis's product formula*

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdots \frac{2m \cdot 2m}{(2m-1)\cdot(2m+1)} \cdots.$$

[Hint: Use the product formula for $\sin z$ *at* $z = \pi/2$*.]*

6. We know (from p. 142) the product formula for the sine function is

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Let $z = 1/2$. Then,

$$\frac{\sin(\pi/2)}{\pi} = \frac{1}{2}\prod_{n=1}^{\infty}\left(1 - \frac{(1/2)^2}{n^2}\right).$$

Using $\sin(\pi/2) = 1$, we simplify this equation:

$$\frac{1}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) \tag{5.1}$$

$$\frac{1}{\pi} = \frac{1}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n)^2}\right) \tag{5.2}$$

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(\frac{(2n)^2 - 1}{(2n)^2}\right) \tag{5.3}$$

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(\frac{(2n+1)(2n-1)}{(2n)^2}\right). \tag{5.4}$$

$$\tag{5.5}$$

But this implies that

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{(2n)^2}{(2n+1)(2n-1)}\right),$$

proving the identity.

7. *Establish the following properties of infinite products.*

(a) *Show that if* $\Sigma|a_n|^2$ *converges, and* $a_n \neq -1$*, then the product* $\prod(1 + a_n)$ *converges to a non-zero limit if and only if* Σa_n *converges.*

(b) *Find an example of a sequence of complex numbers* $\{a_n\}$ *such that* Σa_n *converges but* $\prod(1 + a_n)$ *diverges.*

(c) *Also find an example such that* $\prod(1 + a_n)$ *converges and* Σa_n *diverges.*

7. a) Let $\sum |a_n|^2$ converge with $a_1 \neq -1$.

$(\Leftarrow)$ First assume $\sum a_n$ converges to a nonzero limit. Without loss of generality we may assume that each a_n satisfies $|a_n| \leq 1/2$; this is clearly true in the limit (as the sum converges, the summands must tend to zero). We assume this to facilitate expanding with logarithms. Consider the product $\prod(1+a_n)$. Taking logs, we see $\log\left(\prod(1+a_n)\right) = \sum \log(1+a_n)$. Setting $x = -a_n$ and using the Taylor expansion

$$\log(1+x) \ = \ -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots,$$

we see that

$$\log\left(\prod(1+a_n)\right) = \sum\left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right).$$

In general, notice that

$$\sum_{k=2}^{\infty} -|x|^k \le \left|-\frac{x^2}{2} + \frac{x^3}{3} - \cdots\right| \le \sum_{k=2}^{\infty} |x|^k,$$

or

$$-|x|^2(1+|x|+|x|^2+\dots) \le \left|-\frac{x^2}{2} + \frac{x^3}{3} - \cdots\right| \le |x|^2(1+|x|+|x|^2+\dots).$$

If a sum $\sum x$ converges to a nonzero limit, we know that $|x|$ converges to zero; thus we may assume (without changing convergence) that $|x| \le \frac{1}{2}$. Thus using the geometric expansion, we see that $1 + |x| + |x|^2 + \cdots = \frac{1}{1-|x|}$. Because $|x| \le \frac{1}{2}$, we have that $\frac{1}{1-|x|} \le 2$. Hence we have that

$$-2|x|^2 \le \left|-\frac{x^2}{2} + \frac{x^3}{3} - \cdots\right| \le 2|x|^2.$$

Recall that we were looking at $\log\left(\prod(1+a_n)\right) = \sum\left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right)$. Since $\sum a_n$ converges, we know eventually we must have $|a_n| < 1/2$, so we can assume $|a_n| < 1/2$ without changing convergence, and thus use the simplification involving the geometric series expansion developed in the previous paragraph. Thus we write

$$\begin{aligned} \log\left(\prod(1+a_n)\right) &= \sum\left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right) \\ &\le \sum\left(a_n + 2|a_n|^2\right) \\ &= \sum a_n + 2\sum |a_n|^2. \end{aligned} \tag{5.6}$$

A QUICK WORD OF WARNING. THE ABOVE EQUATION, AND THE ONES BELOW, ARE A LITTLE ODD. REMEMBER THAT OUR SEQUENCE NEED NOT BE JUST REAL NUMBERS. AS SUCH, WE MUST BE CAREFUL WITH THE DEFINITION OF ABSOLUTE VALUE. WE ABUSE NOTATION A BIT – WHEN WE WRITE $a \leq b + c$, THIS MEANS THE DESIRED RELATION IS TRUE UP TO A LINEAR RESCALING. REALLY WHAT WE MEAN IS $a = b$ UP TO AN ERROR AT MOST $|c|$. WE REALLY SHOULD WRITE THINGS LIKE $|a - b| \leq c$, BUT IN A HOPEFULLY OBVIOUS ABUSE OF NOTATION....

Since by assumption both $\sum a_n$ and $\sum |a_n|^2$ converge, we must have that $\sum a_n + 2\sum |a_n|^2$ is finite, call it L. Thus $\log\big(\prod(1+a_n)\big) \leq L$, so $\prod(1+a_n) \leq e^L$, which is again finite. Thus the product converges.

($\Rightarrow$) Next assume $\prod(1+a_n)$ converges to a nonzero limit. Since $\prod(1+a_n)$ is converging to a nonzero limit, the terms in the product must be converging to 1, so we must have $|a_n|$ approaching zero and we can assume $|a_n| < 1/2$ without affecting convergence.

We now write:

$$\begin{aligned} \log\big(\prod(1+a_n)\big) &= \sum\big(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\big) \\ &\geq \sum\big(a_n - \frac{|a_n|^2}{2} - \frac{|a_n|^3}{3} - \cdots\big) \\ &\geq \sum\big(a_n - |a_n|^2 - |a_n|^3 - \cdots\big). \end{aligned} \tag{5.7}$$

As before, we substitute in using the geometric series expansion:

$$\begin{aligned} \log\big(\prod(1+a_n)\big) &\geq \sum\big(a_n - |a_n|^2 - |a_n|^3 - \cdots\big) \\ &= \sum\big(a_n - |a_n|^2(1 + |a_n| + |a_n|^2 + \cdots)\big) \\ &\geq \sum\big(a_n - 2|a_n|^2\big) \\ &= \sum a_n - 2\sum |a_n|^2. \end{aligned} \tag{5.8}$$

Thus we see that $\log\big(\prod(1+a_n)\big) + 2\sum |a_n|^2 \geq \sum a_n$. Since $\prod(1+a_n)$ and $\sum |a_n|^2$ converge, we must have that $\log\big(\prod(1+a_n)\big) + 2\sum |a_n|^2$ are both finite.

Thus our sum $\sum a_n$ is bounded by finite terms, and so the sum must also be finite itself. Hence the sum $\sum a_n$ must converge to a finite limit.

b) Let $\{a_n\} = \{\frac{i}{\sqrt{1}}, \frac{-1}{\sqrt{1}}, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, \dots\}$. The sum $\sum a_n$ converges by the alternating series test, since the absolute value of the terms approaches zero (one can show this by showing that first the odd terms tend to zero in absolute value and then that the even terms do as well).

Consider now the product $\prod\left(1 + a_n\right)$. For an arbitrary integer N, look at the $2N$-th partial product:

$$\begin{aligned}\prod_{n=1}^{2N}\left(1+a_n\right) &= \left(1+\frac{i}{\sqrt{1}}\right)\left(1-\frac{i}{\sqrt{1}}\right)\cdots\left(1+\frac{i}{\sqrt{2N}}\right)\left(1-\frac{i}{\sqrt{2N}}\right)\\ &= \left(1-\frac{i^2}{\sqrt{1^2}}\right)\cdots\left(1-\frac{i^2}{\sqrt{(2N)^2}}\right)\\ &= \left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{2N}\right)\\ &= \left(\frac{2}{1}\right)\cdots\left(\frac{2N+1}{2N}\right)\\ &= 2N+1.\end{aligned} \tag{5.9}$$

Thus when we evaluate at an even term $2N$, we see that

$$\lim_{2N\to\infty}\prod_{n=1}^{2N}(1+a_n) = \lim_{2N\to\infty}(2N+1) = \infty,$$

so the product diverges. Hence the product diverges at even terms and thus cannot converge in general.

c) For a trivial example, let $\{a_n\} = \{1, -1, 1, -1, \dots\}$. The sum $\sum a_n$ does not converge because the limit of the Nth partial sum as N tends to infinity does not converge; it alternates between 0 and 1. However, the product will clearly converge:

$$\prod a_n = (1+1)(1-1)(1+1)(1-1)\cdots = (1)(0)(1)(0)\cdots = 0.$$

For an example in which the sum diverges but the product converges to a nonzero limit, consider the sequence $\{a_n | a_{2n-1} = 1/\sqrt{n}, a_{2n} = -1/(1+\sqrt{n})\}_{n=1}^{\infty}$. Grouping the pairs $2n$ and $2n-1$ together, we see that

$$\begin{aligned}\sum_{m=1}^{\infty} a_m &= \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{1+\sqrt{n}}\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}.\end{aligned}$$

We'll show that this series diverges. Notice that for every n,

$$\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty}\frac{1}{2n}$$

and since the series on the RHS diverges, by comparison test, so does the series on the LHS. So $\sum a_n$ diverges. However, grouping again the even and odd pair terms, for even N, we have

$$\begin{aligned}\prod_{m=1}^{N}(1+a_m) &= \prod_{n=1}^{N/2}(1+\frac{1}{\sqrt{n}})(1-\frac{1}{\sqrt{n}+1})\\ &= \prod_{n=1}^{N/2}(1+\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}+1} - \frac{1}{\sqrt{n}+n})\\ &= \prod_{n=1}^{N/2}(1-\frac{-\sqrt{n}+\sqrt{n}+1-1}{\sqrt{n}+n})\\ &= \prod_{n=1}^{N/2} 1 = 1\end{aligned}$$

and for odd N,

$$\prod_{m=1}^{N}(1+a_m) = (1+\frac{1}{\sqrt{N}})$$

which converges to 1 as $N \to \infty$. Thus,

$$\prod_{n=1}^{\infty}(1+a_n) = 1.$$

Hence $\{a_n\}$ is the desired sequence. □

9. *Prove that if* $|z| < 1$, *then*

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \prod_{k=0}^{\infty}\left(1+z^{2^k}\right) = \frac{1}{1-z}.$$

9. Consider the product $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots$. Suppose we tried to multiply this product out: to get one term, we would need to choose either the 1 or the power of z in each term to multiply by. For example, one term we could get out is simply z, where we would choose the z in the first term and the 1 in every succeeding term; another way to say this is to write $z = z \times 1 \times 1 \times \cdots$. To write out the entire product, we would have to make sure we evaluated every possible choice of ones and powers of z.

But this isn't so bad if we think of choosing terms as counting in binary. In binary counting, a number is written entirely in terms of 0s and 1s. For any given number, each digit represented a choice between the digit 0 and the digit 1. If we think of selecting the power of z in a term as picking 1 for a given digit in binary counting, and selecting the 1 in a term as picking 0 for a given digit in binary, we can identify a bijective correspondence between integers written in binary and products from our term (with the exception that $000000000\cdots = 1$ in our product). For example, the binary number $101 = \cdots 000101 = 2^2 \times 1 + 2^1 \times 0 + 2^0 \times 1 = 5$, and if choose the terms $(z)(1)(z^4)(1)(1)\cdots$, we see that we get the product z^5.

To evaluate our product we must sum over all such possible choices. Since all possible binary numbers together yield precisely the nonnegative integers, this bijective correspondence importantly tells us that the sum over all such products will be the sum over all nonnegative powers of z, or $1+z+z^2+z^3+\ldots$. Thus we have $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = 1+z+z^2+z^3+\ldots$. Since $|z| < 1$, we can use the geometric expansion of z to write $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \frac{1}{1-z}$, as desired.

Significance for combinatorics: notice the way in which our solution invokes combinatorics (such as seeing how many ways we can choose our terms to make a product).

Alternatively, we can truncate the product and multiply by $1-z$. Note that $(1-z)(1+z) = (1-z^2)$, then $(1-z^2)(1+z^2) = (1-z^4)$, and so

$$(1-z)(1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^k}) \;=\; 1 - z^{2^{k+1}};$$

as $|z| < 1$ the latter tends to 1, and thus

$$(1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^k}) = \frac{1}{1-z} - \frac{z^{2^{k+1}}}{z-1} \to \frac{1}{1-z}.$$

Chapter 3

10. *Show that if $a > 0$, then*

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx = \frac{\pi}{2a}\log a.$$

[Hint: Use the contour in Figure 10.]

10. We will first find the residue at ia and then integrate over the given contour. Let $f(z) = \frac{\log z}{z^2+a^2}$, where we take the branch cut of the logarithm along $-ib$ for all $b \in [0, \infty)$. Furthermore, ia is a zero of order 1. Finding the residue at ia, we have

$$\begin{aligned}
\text{res}_{ia} f &= \lim_{z\to ia}(z-ia)\frac{\log z}{z^2+a^2} \\
&= \lim_{z\to ia}\left(\frac{\log z}{z+ia}\right) \\
&= \frac{\log ia}{2ia} \\
&= \frac{\log a}{2ia} + \frac{\pi}{2a}
\end{aligned}$$

Label the contours from the portion on the positive real axis γ_1, the larger arc γ_2, the portion on the negative real axis γ_3, and the smaller arc γ_4. Choose $\epsilon < \min\{a, 1\}a$, $R > \max\{a, 1\}$. Parametrize γ_1 with $z(t) = t$ from ϵ to R, γ_2 with $z(t) = Re^{it}$ from 0 to π, γ_3 with $z(t) = t$ from $-R$ to $-\epsilon$, and γ_4 with $z(t) = \epsilon e^{it}$

from π to 0. Integrating over the γ_2 and taking absolute values, we have

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{\log z}{z^2+a^2}dz\right| &= \left|\int_0^\pi \frac{\log Re^{it}}{(Re^{it})^2+a^2}Rie^{it}dt\right| \\
&\leq \int_0^\pi \left|\frac{\log Re^{it}}{R^2e^{2it}+a^2}Rie^{it}\right| dt \\
&= \int_0^\pi \left|\frac{\log Re^{it}}{R^2e^{2it}+a^2}\right| Rdt \\
&= \int_0^\pi \left|\frac{\log R+it}{R^2e^{2it}+a^2}\right| Rdt \\
&\leq \int_0^\pi \frac{\log R+|it|}{|R^2e^{2it}|+|a^2|}Rdt \\
&= \int_0^\pi \frac{\log R+t}{|Re^{2it}|+\frac{|a^2|}{R}}dt \\
&\leq \int_0^\pi \frac{\log R+t}{R+\frac{|a^2|}{R}}dt \\
&\leq \pi\frac{\log R+\pi}{R+\frac{|a^2|}{R}}
\end{aligned}$$

since $t, \log R > 0$. Since $R \to \infty$, $\log R + \pi, R + \frac{|a^2|}{R} \to \infty$, by L'Hopital,

$$\begin{aligned}
\lim_{R\to\infty} \frac{\log R+\pi}{R+\frac{|a^2|}{R}} &= \lim_{R\to\infty} \frac{1/R}{1-\frac{|a^2|}{R^2}} \\
&= \lim_{R\to\infty} \frac{1}{R-\frac{|a^2|}{R}} \\
&= 0.
\end{aligned}$$

Thus, as $R \to \infty$, the contribution along γ_2 vanishes to 0. Similarly, for γ_4, we have

$$\begin{aligned}
\left|\int_{\gamma_4} \frac{\log z}{z^2+a^2}dz\right| &= \left|\int_{\pi}^{0} \frac{\log \epsilon e^{it}}{(\epsilon e^{it})^2+a^2}\epsilon e^{it}dt\right| \\
&\le \int_{\pi}^{0} \left|\frac{\log \epsilon e^{it}}{\epsilon^2 e^{2it}+a^2}\epsilon e^{it}\right| dt \\
&= \int_{\pi}^{0} \left|\frac{\log \epsilon e^{it}}{\epsilon^2 e^{2it}+a^2}\right| \epsilon dt \\
&= \int_{\pi}^{0} \left|\frac{-\log \epsilon + it}{\epsilon^2 e^{2it}+a^2}\right| \epsilon dt \\
&\le \int_{\pi}^{0} \frac{-\log \epsilon + |it|}{|\epsilon^2 e^{2it}|+|a^2|}\epsilon dt \\
&\le \int_{\pi}^{0} \frac{-\log \epsilon + t}{|\epsilon e^{2it}|+\frac{|a^2|}{\epsilon}}dt \\
&\le \int_{\pi}^{0} \frac{-\log \epsilon + t}{\epsilon+\frac{|a^2|}{\epsilon}}dt \\
&\le \pi\frac{-\log \epsilon + \pi}{\epsilon+\frac{|a^2|}{\epsilon}}dt
\end{aligned}$$

since $t, -\log\epsilon > 0$. Since $\epsilon \to 0$, $-\log\epsilon + \pi, \epsilon + \frac{|a^2|}{\epsilon} \to \infty$, by L'Hopital,

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{-\log \epsilon + \pi}{\epsilon+\frac{|a^2|}{\epsilon}} &= \lim_{\epsilon\to 0} \frac{-1/\epsilon}{1-\frac{|a^2|}{\epsilon^2}} \\
&= \lim_{\epsilon\to 0} \frac{-1}{\epsilon-\frac{|a^2|}{\epsilon}} \\
&= 0.
\end{aligned}$$

Thus, as $\epsilon \to 0$, the contribution along γ_4 also vanishes to 0. For the integral over γ_1, γ_3, we have

$$\int_{\gamma_1+\gamma_3} \frac{\log z}{z^2+a^2}dz = \int_{\epsilon}^{R} \frac{\log t}{t^2+a^2}dt + \int_{-R}^{-\epsilon} \frac{\log s}{s^2+a^2}ds.$$

Letting $s = -t$, we have

$$\begin{aligned}
\int_{\gamma_1+\gamma_3} \frac{\log z}{z^2+a^2}dz &= \int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \int_R^\epsilon \frac{\log -t}{(-t)^2+a^2}(-1)dt \\
&= \int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \int_\epsilon^R \frac{\log -t}{t^2+a^2}dt \\
&= \int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \int_\epsilon^R \frac{\log t + i\pi}{t^2+a^2}dt \\
&= 2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt + i\pi \int_\epsilon^R \frac{1}{t^2+a^2}dt \\
&= 2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \frac{i\pi}{a}\arctan\frac{t}{a}\bigg|_\epsilon^R
\end{aligned}$$

Thus we have, as $R \to \infty, \epsilon \to 0$ and as $\mathrm{res}_{ia} f = \frac{\log a}{2ia} + \frac{\pi}{2a}$, we have

$$\begin{aligned}
\lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \frac{i\pi}{a}\arctan\frac{t}{a}\bigg|_\epsilon^R\right) &= 2\pi i\left(\frac{\log a}{2ia} + \frac{\pi}{2a}\right) \\
\lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt\right) + \lim_{R\to\infty,\epsilon\to 0}\left(\frac{i\pi}{a}\arctan\frac{t}{a}\bigg|_\epsilon^R\right) &= \frac{\pi\log a}{a} + \frac{i\pi^2}{a} \\
\lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt\right) + \frac{i\pi^2}{a} &= \frac{\pi\log a}{a} + \frac{i\pi^2}{a} \\
\lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt\right) &= \frac{\pi\log a}{a} \\
\int_0^\infty \frac{\log t}{t^2+a^2}dt &= \frac{\pi\log a}{2a}
\end{aligned}$$

as desired. □

Additional Problems

1. *Find all poles of the function* $f(z) = 1/(1-z^2)^4$ *and find the residues at the poles.*

Let $g(x) = 1/f(z) = (1 - z^2)^4 = ((1+z)(1-z))^4$. We see that the zeros of g are ± 1, each with order 4. Hence, the residues are

$$\begin{aligned}
\text{res}_1(f) &= \lim_{z \to 1} \frac{1}{(4-1)!} \left(\frac{d}{dz} \right)^{4-1} (z-1)^4 \frac{1}{(1-z^2)^4} \\
&= \lim_{z \to 1} \frac{1}{6} \left(\frac{d}{dz} \right)^3 \frac{1}{(1+z)^4} \\
&= \lim_{z \to 1} \frac{1}{6} (-4)(-5)(-6) \frac{1}{(1+z)^7} \\
&= \lim_{z \to 1} \frac{-20}{(1+z)^7} \\
&= \frac{-20}{2^7} \\
&= \frac{-5}{32}
\end{aligned}$$

and

$$\begin{aligned}
\text{res}_{-1}(f) &= \lim_{z \to -1} \frac{1}{(4-1)!} \left(\frac{d}{dz} \right)^{4-1} (z+1)^4 \frac{1}{(1-z^2)^4} \\
&= \lim_{z \to -1} \frac{1}{6} \left(\frac{d}{dz} \right)^3 \frac{1}{(z-1)^4} \\
&= \lim_{z \to -1} \frac{1}{6} (4)(5)(6) \frac{-1}{(z-1)^7} \\
&= \lim_{z \to -1} \frac{-20}{(z-1)^7} \\
&= \frac{-20}{-2^7} \\
&= \frac{5}{32}
\end{aligned}$$

Thus we have found the desired residues. $\square$

We sketch an alternative proof. We have

$$\begin{aligned} f(z) &= \frac{1}{(z-1)^4}\frac{1}{(z+1)^4} \\ &= \frac{1}{(z-1)^4}\frac{1}{(z-1+2)^4} \\ &= \frac{1}{(z-1)^4}\frac{1}{2^4}\frac{1}{(1+\frac{z-1}{2})^4} \\ &= \frac{1}{(z-1)^4}\frac{1}{16}\left(1-\frac{z-1}{2}+\frac{(z-1)^2}{4}-\frac{(z-1)^3}{8}+\cdots\right)^4 \quad (5.10) \end{aligned}$$

The difficulty is we have to expand the factor to the fourth power well enough to identify the coefficient of $(z-1)^3$. A little algebra shows it is $-\frac{5}{2}(z-1)^3$, and thus (remembering the factor 1/16) the residue is just $-5/32$.

2. *Consider the map $f(z) = (z-i)/(z+i)$. Show that this is a one-to-one and onto map from the upper half plane (all $z = x+iy$ with $y > 0$) to the unit disk.*

2. First we'll show that the range of f is the unit disk. Writing $z = x + iy$ where $x, y \in \mathbb{R}$, $y > 0$, then we have

$$\begin{aligned} |f(x+iy)| &= \left|\frac{x+(y-1)i}{x+(y+1)i}\right| \\ &= \frac{\sqrt{x^2+(y-1)^2}}{\sqrt{x^2+(y+1)^2}} \end{aligned}$$

and since $y > 0$, $\sqrt{x^2+(y-1)^2} < \sqrt{x^2+(y+1)^2}$, $f(x+iy) < 1$, so the range of f is the unit disk.

Now we'll show that f is injective. Suppose for z_1, z_2 with imaginary part positive, $f(z_1) = f(z_2)$. Then

$$\begin{aligned} \frac{z_1-i}{z_1+i} &= \frac{z_2-i}{z_2+i} \\ (z_1-i)(z_2+i) &= (z_2-i)(z_1+i) \\ z_1Z_2+z_1i-Z_2i+1 &= z_1z_2-z_1i+z_2i+1 \\ 2i(z_1-z_2) &= 0 \\ z_1 &= z_2. \quad (5.11) \end{aligned}$$

Here's another, faster way to do the algebra. We add zero:

$$
\begin{aligned}
\frac{z_1 - i}{z_1 + i} &= \frac{z_2 - i}{z_2 + i} \\
\frac{z_1 + i - 2i}{z_1 + i} &= \frac{z_2 + i - 2i}{z_2 + i} \\
1 - \frac{2i}{z_1 + i} &= 1 - \frac{2i}{z_2 + 1};
\end{aligned}
\tag{5.12}
$$

it is clear that the only solution is when $z_1 = z_2$.

Now we'll show that f is surjective. Given any $w \in \mathbb{D}$, setting $z = (w + 1)i/(1 - w)$, we see that

$$
\begin{aligned}
f(z) &= \frac{\frac{(w+1)i}{(1-w)} - i}{\frac{(w+1)i}{(1-w)} + i} \\
&= \frac{(w+1)i - (1-w)i}{(w+1)i + (1-w)i} \\
&= w.
\end{aligned}
$$

Now we'll show that z has positive imaginary part. Writing $w = x + iy$ with $x, y \in \mathbb{R}, x^2 + y^2 < 1$, we have

$$
\begin{aligned}
z &= i\frac{(x+1) + iy}{(1-x) - iy} \\
&= \frac{-2y + i(1 - y^2 - x^2)}{(1-x)^2 + y^2}.
\end{aligned}
$$

So the imaginary part is $1 - (x^2 + y^2) > 0$, so z has positive imaginary part. $\square$

3. *Calculate the Weierstrass product for* $\cos(\pi z)$ *(this is also problem 10b in Chapter 5, and the answer is listed there) and for* $\tan(\pi z)$.

3. By the Euler formulas for sine and cosine, we see that

$$\begin{aligned}
\cos(\pi z) &= \frac{e^{i\pi z} + e^{-i\pi z}}{2} \\
&= \frac{e^{i\frac{\pi}{2}}(e^{i\pi z} + e^{-i\pi z})}{2i} \\
&= \frac{(e^{i\pi(z+\frac{1}{2})} + e^{-i\pi(z-\frac{1}{2})})}{2i} \\
&= \frac{e^{i\pi(\frac{1}{2}-z)} - e^{-i\pi(\frac{1}{2}-z)}}{2i} \\
&= \sin(\pi(\frac{1}{2} - z))
\end{aligned}$$

and since the zeros of $\sin \pi z$ occur only at the integers, the zeros of $\cos \pi z$ occur at $m + \frac{1}{2}$ for all $m \in \mathbb{Z}$. Thus, define the sequence $\{a_{2n-1} = n + \frac{1}{2}, a_{2n} = -(n + \frac{1}{2})\}_{n=1}^{\infty}$, which are precisely the zeros of $\cos \pi z$. Furthermore, since the zeros of sine are of order 1, the zeros of cosine are also of order one. Thus we have, for $h_k(z) = \sum_{j=1}^{k} \frac{z^j}{j}$, grouping together the pairs $2n$ and $2n-1$, the Weierstrauss product of $\cos \pi z$ is, up to a factor of $e^{h(z)}$ for some entire function h,

$$\begin{aligned}
\prod_{m=0}^{\infty}(1 - \frac{z}{a_m})e^{h_m(z)} &= \prod_{n=0}^{\infty}(1 - \frac{z}{n+\frac{1}{2}})(1 - \frac{z}{-(n+\frac{1}{2})}) \prod_{m=1}^{\infty} e^{h_m(z)} \\
&= \prod_{n=0}^{\infty}(1 - \frac{z^2}{(n+\frac{1}{2})^2})e^{\sum_{m=1}^{\infty} h_m(z)} \\
&= \prod_{n=0}^{\infty}(1 - \frac{4z^2}{(2n+1)^2})e^{\sum_{m=1}^{\infty} h_m(z)}.
\end{aligned}$$

Considering $\prod_{n=0}^{\infty}(1 - \frac{4z^2}{(2n+1)^2})$, we'll show this product converges. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}
\end{aligned}$$

so since the sum on the RHS is bounded $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent series, the series on the RHS converges as well, and as the convergence is absolute, the product converges. Thus (up to the exponential of an entire function) the Weierstrauss product of $\cos \pi z$ is $\prod_{n=0}^{\infty}(1 - \frac{4z^2}{(2n+1)^2})$.

Next, notice that $\tan(\pi z)$, has poles at odd integer multiples of $\frac{\pi}{2}$, and so by definition does not have a Weierstrass product.

6 Homework #6: Kung, Lin, Waters

HW: Due Friday, November 5: (1) Evaluate $\int_{-\infty}^{\infty} \cos(4x)dx/(x^4+1)$. (2) Let U be conformally equivalent to V and V conformally equivalent to W with functions f: U –> V and g: V –> U. Prove g $\circ$ f (g composed with f) is a bijection. (3) The Riemann mapping theorem asserts that if U and V are simply connected proper open subsets of the complex plane then they are conformally equivalent. Show that simply connected is essential. In other words, find a bounded open set U that is not simply connected and prove that it cannot be conformally equivalent to the unit disk. (4) Chapter 8, Page 248: #4. (5) Chapter 8: Page 248: #5. (6) Chapter 8: Page 251: #14.

1. Evaluate $\int_{-\infty}^{\infty} \cos(4x)dx/(x^4+1)$. *Evaluate* $\int_{-\infty}^{\infty} \cos(4x)dx/(x^4+1)$

First observe that $\cos(4x) = \frac{1}{2}(e^{4ix} + e^{-4ix})$, since $e^{4ix} = \cos(4x) + i\sin(4x)$ and $e^{-4ix} = \cos(4x) - i\sin(4x)$. We can rewrite this integral, then, as

$$\int_{-\infty}^{\infty} \frac{\cos(4x)}{x^4+1}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{4ix}+e^{-4ix}}{x^4+1} = \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{e^{4ix}}{x^4+1}dx + \int_{-\infty}^{\infty} \frac{e^{-4ix}}{x^4+1}dx\right)$$

and we can evaluate both halves separately.

For both halves, observe that the poles are located at $z = e^{\frac{1}{4}\pi i}, e^{\frac{3}{4}\pi i}, e^{\frac{5}{4}\pi i}, e^{\frac{7}{4}\pi i}$, since those are the solutions to $z^4 + 1 = 0$. We can now choose a contour over which to integrate and apply the residue theorem. Our choice of contour is motivated by the decay of the functions. We need to work in the upper half plane for $\exp(4iz)$ to decay, and in the lower half plane for $\exp(-4iz)$ to decay.

For $\int_{-\infty}^{\infty} \frac{e^{4ix}}{x^4+1} dx$, consider the contour γ_1 that traverses the semicircle of radius R in the upper half-plane and the real axis, with standard orientiation. This contour will enclose only the poles at $z = e^{\frac{1}{4}\pi i}, e^{\frac{3}{4}\pi i}$, so it suffices to find the residues at those two points in order to apply the residue theorem.

The simplest way to compute the residues is to note that we have simple poles and we may write $f(z) = g(z)/h(z)$ with $h(z)$ having simple zeros and g, h holomorphic. Then the residue of f at a pole z_0 is just $g(z_0)/h'(z_0)$. For us, $g(z_0) = \exp(4iz_0)$, while $h'(z_0) = 4z_0^3$.

At $e^{\frac{1}{4}\pi i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, the residue will be

$$\frac{\exp(4i\exp(\pi i/4))}{4\exp(i\pi/4)^3} = \frac{\exp(2i(\sqrt{2}+i\sqrt{2}))}{4\exp(3i\pi/4)} = \frac{\exp(-2\sqrt{2}+i2\sqrt{2}))}{-2\sqrt{2}+i2\sqrt{2}}.$$

We can compute this another way as well:

$$\lim_{z\to e^{\frac{1}{4}\pi i}} (z - e^{\frac{1}{4}\pi i}) \frac{e^{4iz}}{z^4+1} dz = \frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{(e^{\frac{1}{4}\pi i} - e^{\frac{3}{4}\pi i})(e^{\frac{1}{4}\pi i} - e^{\frac{5}{4}\pi i})(e^{\frac{1}{4}\pi i} - e^{\frac{7}{4}\pi i})} = \frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{2\sqrt{2}(-1+i)}.$$

At $e^{\frac{3}{4}\pi i}$, the residue will be

$$\lim_{z\to e^{\frac{3}{4}\pi i}} (z - e^{\frac{3}{4}\pi i}) \frac{e^{4iz}}{z^4+1} dz = \frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{(e^{\frac{3}{4}\pi i} - e^{\frac{1}{4}\pi i})(e^{\frac{3}{4}\pi i} - e^{\frac{5}{4}\pi i})(e^{\frac{3}{4}\pi i} - e^{\frac{7}{4}\pi i})} = \frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{2\sqrt{2}(1+i)}.$$

Thus, the countour integral over γ_1 is equal to

$$2\pi i \left(\frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{2\sqrt{2}(-1+i)} + \frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{2\sqrt{2}(1+i)} \right).$$

Now let the radius R tend to infinity, and observe that the portion of gamma that is not on the real axis (i.e. the semicircle of radius R) will make a zero contribution to the integral. In the upper half-plane, the integral is at most the maximum value of the integrand on our contour times the length of the contour. Since the length of the contour is πR, then, we have

$$\Big| \lim_{R\to\infty} \int_{\gamma_1,\text{semicircle}} \frac{e^{4iz}}{z^4+1} dz \Big| < \Big| \lim_{R\to\infty} \frac{\pi R e^{Ri}}{R^4-1} dx \Big| = 0$$

(note we need R^4-1 and note R^4+1 in the denominator, as the upper bound occurs when the denominator is as small as possible in absolute value; this happens when z^4 is negative, which occurs for $z = R\exp(i\pi/4)$).

Only the portion of the contour integral that lies on the real axis makes any non-zero contribution to the integral, then, so

$$\int_{-\infty}^{\infty} \frac{e^{4ix}}{x^4+1}dx = 2\pi i \left(\frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{2\sqrt{2}(-1+i)} + \frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{2\sqrt{2}(1+i)} \right).$$

As our denominator is non-zero and decays rapidly, and $\exp(4ix) = \cos(4x) + i\sin(4x)$, we see we may drop the integral from the sine term. The reason is that this is an odd, rapidly decaying function integrated over a symmetric region, and thus it gives zero. We therefore find

$$\int_{-\infty}^{\infty} \frac{\cos 4x}{x^4+1}dx = 2\pi i \left(\frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{2\sqrt{2}(-1+i)} + \frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{2\sqrt{2}(1+i)} \right).$$

WE MAY STOP HERE! There is no need to evaluate the other contour, as it will simply give us another calculation of our desired integral. For completeness, we include how the calculation would go in the lower half plane, but again, there is no need to do this!

For $\int_{-\infty}^{\infty} \frac{e^{-4ix}}{x^4+1}dx$, we can repeat the same process, but we must use a different contour. For this function, $\frac{e^{-4iz}}{z^4+1}$ won't vanish as $R \to \infty$ for z in the upper half-plane, since $-4iz$ will have a large positive real component, but it will vanish in the lower half-plane. Use the contour γ_2 consisting of the semicircle of radius R in the lower half-plane and the real axis; *it is very important to note that we are traversing the real axis in the opposite orientation, running from* ∞ *to* $-\infty$. Now, with z restricted to the lower half-plane, our integrand will again vanish, so we have

$$\lim_{R\to\infty} \int_{\gamma_2,\text{semi-circle}} \frac{e^{-4iz}}{z^4+1}dz = 0,$$

and we see that

$$\int_{\gamma_2} \frac{e^{-4iz}}{z^4+1}dz = \int_{\infty}^{-\infty} \frac{e^{-4ix}}{x^4+1}dx = -\int_{-\infty}^{\infty} \frac{e^{-4ix}}{x^4+1}dx.$$

Our contour γ_2 encloses the poles at $\exp(\frac{5}{4}\pi i)$ and $\exp(\frac{7}{4}\pi i)$, so we need to find the residues at those two points.

At $e^{\frac{5}{4}\pi i}$, the residue will be

$$\lim_{z\to e^{\frac{5}{4}\pi i}}(z-e^{\frac{5}{4}\pi i})\frac{e^{-4iz}}{z^4+1}dz = \frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{(e^{\frac{5}{4}\pi i}-e^{\frac{1}{4}\pi i})(e^{\frac{5}{4}\pi i}-e^{\frac{3}{4}\pi i})(e^{\frac{1}{4}\pi i}-e^{\frac{7}{4}\pi i})} = \frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{2\sqrt{2}(1-i)}.$$

At $e^{\frac{7}{4}\pi i}$, the residue will be

$$\lim_{z\to e^{\frac{7}{4}\pi i}}(z-e^{\frac{7}{4}\pi i})\frac{e^{-4iz}}{z^4+1}dz = \frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{(e^{\frac{7}{4}\pi i}-e^{\frac{1}{4}\pi i})(e^{\frac{7}{4}\pi i}-e^{\frac{3}{4}\pi i})(e^{\frac{7}{4}\pi i}-e^{\frac{5}{4}\pi i})} = \frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{2\sqrt{2}(-1-i)}.$$

The integral over γ_2, then, is equal to

$$2\pi i\left(\frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{2\sqrt{2}(1-i)}+\frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{2\sqrt{2}(-1-i)}\right).$$

As $R\to\infty$, this equals the integral over the real line; however, remember that we are proceeding with the opposite orientation, running from ∞ to $-\infty$ as we are using a semi-circle in the lower half plane, and thus we traverse the real line in the opposite orientation as usual. To fix this and restore the correct orientation requires a minus sign, and we find

$$\int_{-\infty}^{\infty}\frac{e^{-4ix}}{x^4+1}dx = -2\pi i\left(\frac{e^{-2\sqrt{2}+2\sqrt{2}i}}{2\sqrt{2}(1-i)}+\frac{e^{-2\sqrt{2}-2\sqrt{2}i}}{2\sqrt{2}(-1-i)}\right).$$

We then argue as before, namely that $\exp(-4ix)=\cos(4x)-i\sin(4x)$, and the sine integral does not contribute as it leads to an odd integral over a symmetric region. Arguing along these lines, we find the same answer as before.

2. Let U be conformally equivalent to V and V conformally equivalent to W with functions f: U –> V and g: V –> U. Prove g ∘ f (g composed with f) is a bijection.

To prove that g ∘ f is a bijection, we need to show that g ∘ f is one-to-one and onto.

One-to-one: Consider an arbitrary x_1, x_2 in U and assume that $g \circ f(x_1) = g \circ f(x_2)$. We need to show that $x_1 = x_2$. First observe that, since g is one-to-one, $g \circ f(x_1) = g \circ f(x_2)$ implies $f(x_1) = f(x_2)$. Since f is also one-to-one, we have that $x_1 = x_2$, and we are done.

Onto: Consider an arbitrary $x \in U$. Since g is onto, there is some $v \in V$ such that $g(v) = x$. Since f is also onto, there is some $u \in U$ such that $f(u) = v$. Therefore, $g \circ f(u) = x$, so $g \circ f$ is onto.

Then $g \circ f$ is one-to-one and onto, so it is a bijection. □

3. The Riemann mapping theorem asserts that if U and V are simply connected proper open subsets of the complex plane then they are conformally equivalent. Show that simply connected is essential. In other words, find a bounded open set U that is not simply connected and prove that it cannot be conformally equivalent to the unit disk.

Solution: Consider the punctured unit disc, $\mathbb{D} - \{0\}$, a bounded open set that is not simply connected. Consider function f(z)=1/z on a circle of radius 1/2. Then f(z) is holomorphic on the set, since the origin is not included.

If a conformal map g exists from $\mathbb{D}$ to the punctured disc, then the function $f(z)$ will map to a holomorphic function on $\mathbb{D}$, and the circle will be mapped to a closed curve in $\mathbb{D}$. (Technically we proved Cauchy's theorem, which we'll use in a moment, only for simple, non-intersecting curves. One can show that the image of our closed curve is also a simple, non-intersecting closed curve. If it intersected itself, that would violate the 1-1 property of our conformal map g between the two regions.)

We first compute $\frac{1}{2\pi i} \int_{|z|=1/2} f(z) dz$. As $f(z) = 1/z$, a brute-force computation (or use the Residue Theorem) tells us that this is just 1.

What if we look at the inverse image of the circle of radius $1/2$ in the unit disk? Let's call the inverse image γ, so $g(\gamma) = \{z : |z| = 1/2\}$. Then, using the change of variables formulas, if $z = g(w)$ (recall g is our assumed conformal map from $\mathbb{D}$ to the punctured disk), then $dz = g'(w)dw$ and

$$\frac{1}{2\pi i}\int_{|z|=1/2} f(z)dz \;=\; \frac{1}{2\pi i}\int_{\gamma} f(g(w))g'(w)dw.$$

As f and g are holomorphic, so too is $f(g(w))g'(w)$. As we are integrating a holomorphic function over a closed curve, it is just zero.

We've thus computed the integral two different ways, getting 1 as well as 0. As $1 \neq 0$, we have a contradiction and thus the unit disk and the punctured unit disk are not conformally equivalent.

4. Chapter 8, Page 248: #4. *Does there exist a holomorphic surjection from the unit disc to the complex plane* $\mathbb{C}$*?*

Solution: From 8.1.1 in the book, we know that there exists a conformal map from the disc to the upper half-plane:

$$f(z) = i\frac{1-z}{1+z} \tag{6.1}$$

Now we just map this image to the complex plane. We can do so by moving it down i units and then squaring it. The upper half-plane $\mathbb{H}$ represents complex numbers with positive imaginary part (z=x+iy, y>0); however, a better way to view this is to note that the upper half plane are all numbers of the form $r\exp(i\theta)$ with $r > 0$ and $0 < \theta < \pi$. If we were just to square this as is, we would get every angle we need but $\theta = 0$ and every radius we need but $r = 0$. The problem is that the upper half plane is an open set and does not include its boundary, the real axis. We may rectify this by mapping the image of the unit disk under f, namely the upper half plane, down i units. We now include the entire real line as well. While our resulting map won't be 1-1, it will be onto. *Now* our region includes all $r \geq 0$ and all $\theta \in [0, \pi]$. Squaring this gives all $r \geq 0$ and all $\theta \in [0, 2\pi]$, as desired. Thus our next maps are

$$g(z) = z - i \tag{6.2}$$

and

$$h(z) = z^2 \tag{6.3}$$

The functions f, g, and h are all holomorphic surjections on the complex plane, so $h(g(f(z)))$ is a holomorphic surjection that will map $\mathbb{D} \to \mathbb{H} \to \mathbb{C}$.

$$
\begin{aligned}
h(g(f(z))) &= h(g(i\frac{1-z}{1+z})) \\
&= h(i\frac{1-z}{1+z}-i) \\
&= h(i(\frac{1-z}{1+z}-1)) \\
&= h(-i\frac{2z}{1+z}) \\
&= -\frac{4z^2}{(1+z)^2}.
\end{aligned} \tag{6.4}
$$

5. Chapter 8: Page 248: #5. Prove $f(z) = -\frac{1}{2}(z + z^{-1})$ us a conformal map from the half-disk $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half plane.

First, we check that $f(z)$ is holomorphic. We have that $f^{-1}(z) = \frac{-1}{2}(1 - \frac{1}{z^2})$ and so it is as $z \neq 0$. We next check that this mapping will give us a value in the upper half plane. We take $z = x + iy$. Because z is in the upper half disk, $y > 0$. Thus,

$$
\begin{aligned}
f(z) &= \frac{-1}{2}(x + iy + \frac{1}{x+iy}) \\
&= -\frac{1}{2}\left(x + iy + \frac{x - iy}{x^2 + y^2}\right).
\end{aligned}
$$

Because $|z| < 1$, we have that $|x^2 + y^2| < 1$, and thus the imaginary part inside the parentheses above is negative, and thus becomes positive upon multiplication by $-1/2$. Thus $f(z)$ is in $\mathbb{H}$.

We now show that $f(z)$ is onto. That is, given w in the upper half plane, we must find a z in the upper half disk such that $f(z) = \frac{-1}{2}(z + \frac{1}{z}) = w$. Thus, we have to solve

$$
\begin{aligned}
z + \frac{1}{z} &= -2w \\
-2wz &= z^2 + 1 \\
z^2 + 2wz + 1 &= (z+w)^2 - (w^2 - 1) \ = \ 0 \\
(z+w)^2 &= w^2 - 1.
\end{aligned} \tag{6.5}
$$

Therefore,

$$z = \sqrt{w^2 - 1} - w$$

and so proving $f(z)$ is onto is equivalent to showing that $\sqrt{w^2 - 1} - w$ is in the upper half disk whenever w is in the upper half plane. **NEED MORE INFO HERE.**

We now show that $f(z)$ is one-to-one. To do this, we take $f(a) = f(b)$, with a, b in the upper half disk. Thus, we have

$$\begin{aligned} -\frac{1}{2}\left(a + \frac{1}{a}\right) &= -\frac{1}{2}\left(b + \frac{1}{b}\right) \\ a + \frac{1}{a} &= b + \frac{1}{b} \\ a^2 b + b &= ab^2 + a \\ a^2 b - ab^2 - a + b &= (a - b)(ab - 1) \;=\; 0. \end{aligned}$$

We then see that, if $(ab - 1) = 0$, then $ab = 1$, so $a = 1/b$. Because b is in the upper half disk, $|b| < 1$. This would cause $|1/b| = |a| > 1$. Because we know that a is in the upper half disk as well, this cannot be the case, and so $ab - 1 \neq 0$. This means then that $a - b = 0$, and so $a = b$. Therefore, $f(z)$ is one-to-one, and so $f(z)$ is a conformal map from the upper half disk to the upper half plane.

6. Chapter 8: Page 251: #14. Prove all conformal maps of the upper half plane to the unit disk are of the form $e^{i\theta}(z - \beta)/(z - \overline{\beta})$ for θ real and β in the upper half plane.

We first see that, given f and g two conformal maps from $\mathbb{H}$ to $\mathbb{D}$, we then have that $g^{-1} : \mathbb{D} \to \mathbb{H}$ and $f \circ g^{-1}$ is a conformal map from $\mathbb{D}$ to $\mathbb{D}$. That is, $f \circ g^{-1}$ is an automorphism of $\mathbb{D}$. From the book, we know that $f \circ g^{-1}$ is of the form $e^{i\theta}\frac{\alpha - z}{1 - \overline{\alpha} z}$ for some α in the unit disk. In order to then solve for a general form for f, we can use the inverse of any function $g : \mathbb{H} \to \mathbb{D}$. In other words, $f(z) = (f \circ g^{-1} \circ g)(z)$. We choose $g(z) = \frac{z-i}{z+i}$.

We find

$$\begin{aligned} f(z) &= (f \circ g^{-1} \circ g)(z) \\ &= e^{i\theta} \frac{\alpha - g(z)}{1 - \overline{\alpha} g(z)} \\ &= e^{i\theta} \frac{\alpha - \frac{z-i}{z+i}}{1 - \overline{\alpha}\frac{z-i}{z+i}} \\ &= e^{i\theta} \frac{\frac{\alpha z + \alpha i - z + i}{z+i}}{\frac{z + i - \overline{\alpha} z + \overline{\alpha} i}{z+i}} \\ &= e^{i\theta} \frac{z - i - \alpha z - \alpha i}{z + i - \overline{\alpha} z + \overline{\alpha} i} \\ &= e^{i\theta} \frac{(1-\alpha)z - i(1+\alpha)}{(1-\overline{\alpha}z) + i(1+\overline{\alpha})}. \end{aligned} \tag{6.6}$$

We have to be a bit careful in simplifying the above. Note the goal is to get a rotation times $z - \beta$ over $z - \overline{\beta}$. We thus need to have just a z plus or minus a constant in the numerator and denominator. We therefore pull out a $1 - \alpha$ from the numerator and a $1 - \overline{\alpha}$ from the denominator. Note these two quantities have the same norm, and thus their ratio is of size 1. We can thus write their ratio as $\exp(i\theta')$ for some θ', and hence $\exp(i\theta)\exp(i\theta') = \exp(i\theta'')$. We find

$$f(z) = \exp(i\theta'') \frac{z - i(1+\alpha)(1-\alpha)^{-1}}{z + i(1+\overline{\alpha})(1-\overline{\alpha})^{-1}}.$$

If we set

$$\beta = -i(1+\alpha)(1-\alpha)^{-1}$$

then clearly we do have

$$\overline{\beta} = i(1+\overline{\alpha})(1-\overline{\alpha})^{-1}.$$

We thus have

$$f(z) = \exp(i\theta'') \frac{z-\beta}{z-\overline{\beta}};$$

all that remains is to show that β is in the upper half plane. **NEED TO DO THIS.**

7 Homework #7: Thompson, Schrock, Tosteson

HW: Due Friday, November 12: DO ANY FIVE OUT OF THE FOLLOWING SIX: IF YOU DO MORE, THAT'S GOOD BUT ONLY THE FIRST FIVE WILL BE GRADED. (1) Consider the functions $f_n(x) = n/(1+nx^2)$ where n is a positive integer. Prove that each f_n is uniformly continuous on the real line. Is the family {f_n: n a positive integer} equicontinuous on compact sets? (2) Consider a 2x2 matrix M with integer entries and top row (a,b) and bottom row (c,d) such that ad-bc = 1; we denote the set of all such matrices by SL(2,Z). Consider the map $f_M(z) = (az+b)/(cz+d)$ with z in the upper half plane. Is the family {f_M: M in SL(2,Z)} uniformly bounded on compact sets of the upper half plane? Hint: I think each map is bounded on compact subsets of the upper half plane, but you can find a sequence of matrices such that no bound works simultaneously. (3) Let $f_n(x) = 1 - nx$ for 0 <= x <= 1/n and 0 otherwise, and let F = {f_n: n a positive integer}. Prove that $\lim f_n$ exists and determine it. (4) Consider the family from (3). Prove it is not normal (the problem is that the convergence is not uniform). Specifically, to be normal not only must it converge, but given any epsilon there is an N such that, for all n > N, $|f_n(x) - f(x)|$ < epsilon (or this must hold for a subsequence). (5) Evaluate the integral from -oo to oo of $x^2/(x^4+x^2+1)$. (6) Integrate from 0 to 2pi the function 1 / (a + b sin theta) where a and b are real numbers. What restrictions must we place on a and b in order for this to make sense?

(1) Consider the functions $f_n(x) = n/(1+nx^2)$ where n is a positive integer. Prove that each f_n is uniformly continuous on the real line. Is the family {f_n: n a positive integer} equicontinuous on compact sets?

We must show that, given any $\epsilon > 0$ that there exists a δ such that, for any $x, y \in \mathbb{R}$ and any f_n in our family that whenever $|x-y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$.

Suppose $|x-y| < \delta$. Then, by the Mean Value Theorem,

$$|f_n(x) - f_n(y)| \;=\; |f'(c)||x-y| \;<\; |f'(c)|\delta$$

So, all we need to show is that f' is bounded. Why? If $|f'(x)| \leq B$ for all x, then the above gives

$$|f_n(x) - f_n(y)| \;<\; B|x-y| \;<\; B\delta.$$

If we take $\delta < \epsilon/(B+1)$ then we see that, whenever $|x-y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$, as desired.

We now show f' is bounded. We easily find that

$$f'(x) = \frac{-2n^2 x}{(1+nx^2)^2}$$
$$f''(x) = \frac{-2n^2}{(1+nx^2)^2} - \frac{8n^2x^2}{(1+nx^2)^3}$$

Now, setting the second derivative to zero to get

$$x^2 = \frac{1}{3n}$$

so there are at most two local extrema. Notice that

$$\lim_{x\to\pm\infty} f'(x) = \lim_{x\to\pm\infty} \frac{-2n^2x}{(1+nx^2)^2} = \lim_{x\to\pm\infty} \frac{-2n^2x}{x^4(n+\frac{1}{x^2})^2} = 0,$$

which implies that the maximum of f' cannot occur as $x \to \pm\infty$. Thus the maximum value of f' occurs at both of $x = \pm 1/\sqrt{3n}$, and this is the desired bound.

Alternatively, we could argue as follows. We have

$$|f'(x)| \ = \ 2n^2 \cdot \frac{|x|}{(1+nx^2)^2}.$$

Once $x \geq 1/n$ the denominator exceeds the numerator; as $|x|/(1+nx^2)^2$ is continuous on $[-1/n, 1/n]$, it is bounded on this interval. Thus f' is bounded.

(2) Consider a 2x2 matrix M with integer entries and top row (a,b) and bottom row (c,d) such that ad-bc = 1; we denote the set of all such matrices by SL(2,Z). Consider the map $f_M(z) = (az+b)/(cz+d)$ with z in the upper half plane. Is the family {f_M: M in SL(2,Z)} uniformly bounded on compact sets of the upper half plane? Hint: I think each map is bounded on compact subsets of the upper half plane, but you can find a sequence of matrices such that no bound works simultaneously.

If we let K be an arbitrary compact subset of the upper half plane, we know each $z \in K$ has its imaginary part bounded above and below, and similarly for the real parts. To show that our family is not uniformly bounded, we must find a sequence of matrices and points such that the maps applied to these bounds become arbitrarily large in absolute value.

We're studying maps of the form

$$f_M(z) = \frac{az+b}{cz+d}$$

For problems like this, it is often useful to try and analyze special cases, where the algebra is simpler. Wouldn't it be nice if the denominator were just one? Well, to get that and satisfy the conditions, we would have to study matrices of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

which are in our family. These lead to $f_M(z) = z+n$. Clearly, as n increases, this is not bounded (as the real and imaginary parts of z are bounded, so by sending $n \to \infty$ we see it is unbounded.

(3) Let $f_n(x) = 1 - nx$ for 0 <= x <= 1/n and 0 otherwise, and let F = {f_n: n a positive integer}. Prove that $\lim f_n$ exists and determine it.

Let $x_0 \neq 0$ be a point on the positive real line. Then for all $n > N$, where $N > 1/|x_0|$, we have $f_n(x_0) = 0$. This is because

$$f_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

and

$$n > \frac{1}{|x_0|} \Rightarrow |x_0| > \frac{1}{n} \Rightarrow f(x_0) = 0.$$

So as $n \to \infty$, $f_n \to f$ where

$$f_n(x) = \begin{cases} 1 & \text{x=0} \\ 0 & \text{otherwise} \end{cases}$$

Of course, we haven't said anything about $\lim_n f_n(0)$; however, as each $f_n(0) = 1$, it is clear that the limit is 1 as well. Finally, what happens for x negative? Well, as $f_n(x) = 0$ for $x < 0$ by definition, then $\lim_n f_n(x) = 0$ for x negative.

(4) Consider the family from (3). Prove it is not normal (the problem is that the convergence is not uniform). Specifically, to be normal not only must it converge, but given any epsilon there is an N such that, for all n > N, $|f_n(x) - f(x)|$ **< epsilon (or this must hold for a subsequence).**

Take $x = \epsilon$, $y = 0$. Then obviously $|x - y| \leq \epsilon$. But for n such that $\frac{1}{n} < \epsilon$:

$$|f_n(x) - f_n(y)| = 1.$$

So, not normal.

(5) Evaluate the integral from -oo to oo of $x^2/(x^4 + x^2 + 1)$**.**

Using the quadratic formula we find that the equation $z^2 + z + 1 = 0$ has roots at $e^{2\pi i/3}$ and $e^{4\pi i/3}$. Therefore the function $p(z) = z^4 + z^2 + 1$ has roots at $e^{\pi i/3}$, $e^{2\pi i/3}$, $e^{4\pi i/3}$ and $e^{5\pi i/3}$. Thus we can rewrite our integral as

$$\int_{-\infty}^{\infty} \frac{x^2}{(x - e^{\pi i/3})(x - e^{2\pi i/3})(x - e^{4\pi i/3})(x - e^{5\pi i/3})} dx.$$

For our contour we will take a semicircle in the upper halfplane of radius R centered at the origin. In this region we have poles at $z = e^{\pi i/3}$ and $z = e^{2\pi i/3}$. To find what these residues are at the poles, we recall that if we can write a function $h(z)$ as a ratio of two entire functions $f(z)$ and $g(z)$, with $g(z)$ having a simple zero at the point z_0, then the residue of h at z_0 is simply $f(z_0)/g'(z_0)$. Using this we see the residue of $p(z)$ at $e^{\pi i/3}$ is:

$$\frac{e^{2\pi i/3}}{(e^{\pi i/3} - e^{2\pi i/3})(e^{\pi i/3} - e^{4\pi i/3})(e^{\pi i/3} - e^{5\pi i/3})} = \frac{1}{12}(3 - i\sqrt{3}).$$

Similarly, the residue of $p(z)$ at $e^{2\pi i/3}$ is:

$$\frac{e^{4\pi i/3}}{e^{2\pi i/3} - e^{\pi i/3})(e^{2\pi i/3} - e^{4\pi i/3})(e^{2\pi i/3} - e^{5\pi i/3})} = \frac{1}{12}(-3 - i\sqrt{3}).$$

The sum of the residuals is therefore $-\sqrt{3}i/6 = -i/(2\sqrt{3})$. We now show that the integral over the circular portion of the contour, call it γ_2, contributes nothing in the limit as $R \to \infty$. Since the length of γ_2 is πR, we have:

$$\left| \int_{\gamma_2} \frac{z^2}{z^4 + z^2 + 1} dz \right| \leq \pi R \frac{R^2}{R^4 - R^2 - 1} \to 0.$$

Therefore in the limit we have:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{x^2}{x^4+x^2+1}dx = -i/(2\sqrt{3}),$$

which gives

$$\int_{-\infty}^{\infty}\frac{x^2}{x^4+x^2+1}dx = \frac{\pi}{\sqrt{3}}.$$

(6) Integrate from 0 to 2pi the function 1 / (a + b sin theta) where a and b are real numbers. What restrictions must we place on a and b in order for this to make sense?

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}$$

$z = e^{i\theta},\ e^{-i\theta} = 1/z,\ dz = izd\theta,\ d\theta = -idz/z$

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \int_\gamma \frac{-idz}{z(a+b(z-1/z)/2i)} = \int_\gamma \frac{2dz}{2iaz+b(z^2-1)}$$

where γ is $\partial\mathbb{D}$ (the circle bounding the unit disk).

The following lines are the original write-up of the solution; these are based on the previous line having a factor of $2iaz$ instead of $2iaz^2$.

This has poles at

$$z_0 = \frac{i}{b}\left(-a \pm \sqrt{a^2-b^2}\right)$$

where the only one inside the unit circle is the plus root. This gives residue:

$$\frac{b}{i\sqrt{a^2-b^2}}$$

So

$$\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \frac{2\pi b}{\sqrt{a^2-b^2}}$$

as long as $a^2 > b^2$.

Unfortunately, the above *cannot* be correct, as a simple test shows. If we double a and b, then the original integral decreases by a factor of 2, while our answer here does not change. Thus there *must* be an algebra error. Below is the corrected argument.

Consider the integral

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}.$$

Making the change of variables $z = e^{i\theta},\ e^{-i\theta} = 1/z,\ dz = izd\theta,\ d\theta = -idz/z$, we find

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \int_\gamma \frac{-idz}{z(a + b(z - 1/z)/2i)} = \int_\gamma \frac{2dz}{b(2i(a/b)z + z^2 - 1)}$$

where γ is $\partial\mathbb{D}$ (ie, γ is the unit circle centered at the origin). From the quadratic formula, we see that the integrand has poles at

$$z_0 = i\left(-\frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)$$

where the only one inside the unit circle is the plus root. To compute the residue, we use the following fact: if $A(z) = B(z)/C(z)$ and $C(z)$ is a holomorphic function with a simple zero at z_0 and $B(z)$ is holomorphic, then the residue at z_0 is just $B(z_0)/C'(z_0)$. This gives a residue of

$$\frac{2}{ib\sqrt{(\frac{a}{b})^2 - 1}}$$

So

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

as long as $a^2 > b^2$ (remember that the residue formula requires the integral to be multiplied by $1/2\pi i$, thus in our case we must multiply the residue by $2\pi i$ as our integral was unadorned).

Alternatively, if we factor out a b from the denominator we have

$$\frac{1}{b}\int_0^{2\pi} \frac{d\theta}{(a/b) + \sin\theta}.$$

This is solved exactly like the problem on the midterm, except instead of having $a + \sin\theta$ we now have $(a/b) + \sin\theta$, with an extra factor of $1/b$ outside. Thus the answer is just

$$\frac{1}{b} \cdot \frac{2\pi}{\sqrt{(a/b)^2 - 1}} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

Notice this solution has all the desired properties. It doesn't make sense for $|a| < |b|$. For b fixed and $a \to \infty$ it converges to $2\pi/a$, et cetera. It is always good to do these quick consistency checks.

8 Homework #8: Xiong, Webster, Wilcox

HW: Due Tuesday, November 23rd: DO ANY FIVE OUT OF THE FOLLOWING SIX: IF YOU DO MORE, THAT'S GOOD BUT ONLY THE FIRST FIVE WILL BE GRADED. (1) Let Omega be the subset of the complex plane of all z = x+iy with |x| < |y|. Does there exist a logarithm on Omega? If yes, what does the image of Omega under the logarithm look like? (2) Let Omega be the region from (4); is Omega conformally equivalent to the unit disk? Prove your assertion. Hint: remember the full definition of what it means to be simply connected. (3) Let Omega be the subset of the complex plane of all z = x + iy with |x| < |y| + 1. Conformally map Omega onto an open subset of the disk – you must give an explicit form for the map. (Note: The Riemann Mapping Theorem asserts that you can get a map that is onto the disk; here you are just being asked to get a map that is holomorphic and 1-1). (4) Evaluate $\int_0^{2pi} \cos(x)^m dx$, where m is a positive integer. (5) Evaluate $\int_0^1 (1-x^2)^n dx$, where n is a positive integer. Hint: Let x = Sin[theta], dx = Cos[theta] d theta. (6) $\int_0^\infty \log(x)dx/(1+x^2)$.

Problem: Let $\Omega = \{z = x + iy \in \mathbb{C} \ : \ |x| < |y|\}$. Does there exist a logarithm on Ω. If yes, what does the image of Ω under the logarithm look like?
Solution: First note that Ω contains the imaginary axis save the origin. Now imagine the lines $y = \pm x$ in the plane. Ω looks like the open hourglass strictly between the lines and containing the imaginary axis. Noting that $\Omega \subseteq \mathbb{C}\backslash(-\infty, 0]$, we can take the principal branch of the logarithm and use the restriction of this logarithm to Ω (see Chapter 3, Theorem 6.1 in the book). To see the image of Ω under this logarithm, write $z = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi$, and then we have $\log z = \log r + i\theta$ where $\log r$ is the standard logarithm on the positive reals. Note that for example $-i$ would be written $1 \cdot e^{-\pi/2}$. Evaluating at some test points, we see

$$\begin{aligned}
\log(ei) &= \log(e^{1+i\pi/2}) = 1 + i\pi/2 \\
\log(e^{1+i\pi/4}) &= 1 + i\pi/4 \\
\log(-ei) &= 1 - i\pi/2,
\end{aligned}$$

etc. Evaluating points along the boundary lines shows us that the image of Ω under this logarithm is two horizontal strips, defined by

$$\log[\Omega] = \left\{ z = x + iy \in \mathbb{C} \;\middle|\; y \in \left(\frac{-3\pi}{4}, \frac{-\pi}{4} \right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4} \right) \right\}.$$

$\square$

Problem: Let Ω be the region from Problem 1. Is Ω conformally equivalent to the unit disk $\mathbb{D}$? Prove your assertion.

Solution: Ω is not conformally equivalent to $\mathbb{D}$ since Ω is not simply connected due to the hole at the origin. Citing Hw 6 $\#3$, being simply connected is necessary for Ω to be uniformly equivalent to $\mathbb{D}$.

A simple illustration: we may choose one point $a \in \Omega \cap \mathbb{H}$ (the upper-half complex plane) and another point b in the lower half. Suppose that Ω is conformally equivalent to $\mathbb{D}$, then $\exists$ a conformal map $f : \Omega \to \mathbb{D}$. If we draw a continuous curve γ connecting $f(a)$ and $f(b)$ in $\mathbb{D}$, we may use the reverse conformal map $f^{-1} : \mathbb{D} \to \Omega$ to map γ into Ω and still obtain a continuous curve. However, since Ω excludes the origin, it is impossible to draw a continuous curve in Ω connecting a and b. $\square$

Problem: Let Ω be the set from Problem 1. Conformally map Ω onto an open subset of the (unit) disk – you must give an explicit form for the map. (Note: The Riemann Mapping Theorem asserts that you can get a map that is onto the disk; here you are just being asked to get a map that is holomorphic and 1-1).

Solution: Define a map $f_1 : \Omega \to f_1(\Omega) \subset \mathbb{C}$ by

$$f_1(z) = z + 2010.$$

Clearly, f_1 is conformal, and $f_1(\Omega)$ lies outside the unit disk. Define a map $f_2 : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C}$ be defined

$$f_2(z) = \frac{1}{z},$$

which conformally maps regions outside the disk onto regions inside of the disk. Thus, $f_2(f_1(\Omega))$ lies inside the disk. Let

$$f(z) = f_2(f_1(z)) = \frac{1}{z + 2010}.$$

As desired, f maps Ω conformally onto $f_2(f_1(\Omega)) \subseteq \mathbb{D}$, which is open because Ω is open and f is continuous. $\square$

Problem: Evaluate

$$\int_0^{2\pi} (\cos(x))^m dx,$$

where $m \in \mathbb{N}_+$.

Solution: Begin by setting $x = \theta$ and applying Euler's formula for cosine.

$$\int_0^{2\pi} (\cos(x))^m dx = \int_0^{2\pi} \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^m d\theta$$

Let $z = e^{i\theta}$. Notice that as θ goes from 0 to 2π, z travels around the unit circle. We then have

$$\begin{aligned} dz &= ie^{i\theta} d\theta \\ d\theta &= \frac{dz}{ie^{i\theta}} \\ &= -iz^{-1} dz, \end{aligned}$$

and substituting yields

$$\begin{aligned} \int_0^{2\pi} \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^m d\theta &= \int_{|z|=1} \left(\frac{z + z^{-1}}{2}\right)^m dz \\ &= \frac{-i}{2^m} \int_{|z|=1} (z + z^{-1})^m z^{-1} dz \\ &= \frac{-i}{2^m} \int_{|z|=1} \sum_{k=0}^{m} \binom{m}{k} z^{2k-m-1} dz \\ &= 2\pi i \frac{-i}{2^m} a_m \\ &= \frac{\pi}{2^{m-1}} a_m, \end{aligned}$$

where a_m is the coefficient of z^{-1} in

$$\sum_{k=0}^{m} \binom{m}{k} z^{2k-m-1}.$$

If m is odd, then $a_m = 0$ and our integral is 0. If m is even, then

$$a_m = \binom{m}{m/2},$$

and our integral is

$$\frac{\pi}{2^{m-1}}\binom{m}{m/2}.$$

□

Problem: Evaluate

$$\int_0^1 (1-x^2)^n dx,$$

where $n \in \mathbb{N}_+$.

Solution: Note that we could expand using the binomial formula and power through some algebra. Instead, we translate to the unit circle and use complex analysis and recurrence relations to simplify the computation. So let $x = \sin\theta$ where $\theta \in [0, \frac{\pi}{2}]$, and then $dx = \cos\theta d\theta$ and note $1 - x^2 = \cos^2\theta$. Substituting yields

$$\int_0^1 (1-x^2)^n dx = \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1} d\theta.$$

Now write

$$I_n = \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1} d\theta$$

for $n \in \mathbb{N}$, and note that

$$\begin{aligned} I_0 &= \int_0^{\frac{\pi}{2}} \cos\theta d\theta \\ &= \sin\theta \Big|_0^{\frac{\pi}{2}} \\ &= 1. \end{aligned}$$

Then for $n \geq 1$, we have

$$(\cos\theta)^{2n+3} = (\cos\theta)^{2n+1}(1 - (\sin\theta)^2),$$

thus

$$\begin{aligned} I_{n+1} &= \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+3} d\theta \\ &= \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1} d\theta - \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1} (\sin\theta)^2 d\theta, \end{aligned}$$

and observe

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1} d\theta = I_n.$$

Now let $u = \sin\theta$ and $dv = (\cos\theta)^{2n+1}\sin\theta$, then $du = \cos\theta d\theta$ and

$$v = -\frac{(\cos\theta)^{2n+2}}{2n+2}.$$

Then using integration by parts, we have

$$\begin{aligned} \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1}(\sin\theta)^2 d\theta &= uv - \int v du \\ &= -\left[\sin\theta \frac{(\cos\theta)^{2n+2}}{2n+2}\right]_{\theta=0}^{\frac{\pi}{2}} + \frac{1}{2n+2}\int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+3} d\theta. \end{aligned}$$

Evaluating the left term we find

$$-\left[\sin\theta \frac{(\cos\theta)^{2n+2}}{2n+2}\right]_{\theta=0}^{\frac{\pi}{2}} = 0,$$

and we are left with

$$\frac{1}{2n+2}\int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+3} d\theta = \frac{I_{n+1}}{2n+2}.$$

In the end we have,

$$\begin{aligned} I_{n+1} &= I_n - \frac{I_{n+1}}{2n+2}, \\ \text{so} \quad I_{n+1} &= \frac{2n+2}{2n+3} I_n \\ \text{and finally,} \quad I_n &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot I_0 \\ &= \frac{(2n)!!}{(2n+1)!!}. \end{aligned}$$

While we could try to solve this problem by replacing the cosines with z and $1/z$, that unfortunately leads to a bit of algebra as the resulting expression won't just be integrals of z^k for k an integer, but rather for $2k$ (or $4k$, depending on how clever we are with our choice of variables) an integer.

□

Problem: Evaluate

$$\int_0^\infty \frac{\log x}{1+x^2}\, dx.$$

Solution: ' Rather than resort to the branch cut method, whose details have already been exposed[3]. We apply the Cauchy residue formula more directly, using the indented semicircle contour. Note that this requires that we take advantage of the symmetry of our integrand. In particular, it is easily integrable along the negative real axis. So, let

$$f(z) = \frac{\log z}{1+z^2},$$

(where we take the branch cut for the log to be along the negative imaginary axis) and let $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where γ_1 is the line segment along the positive real axis from ϵ to R, γ_2 is the upper half semicircle of radius R with counterclockwise orientation, γ_3 the segment from $-R$ to $-\epsilon$ and γ_4 the upper half semicircle of

[3] A very similar integral is done via the branch cut method on pages 10 and 11 of the wikipedia printout handed out in class if you're interested.

radius ϵ and with clockwise orientation. We will show that the integrals along γ_2 and γ_4 go to zero in the limit $\epsilon \to 0$ and $R \to \infty$. Then we may evaluate the integral by collecting terms and computing residues. We jump to the punchline first.

Note that f has a simple pole at $z = i$, and (applying Theorem 1.4 from Chapter 3) the residue there is

$$\begin{aligned}\lim_{z\to i}(z-i)f(z) &= \lim_{z\to i}\frac{\log(z)}{z+i}\\ &= \frac{\log(i)}{2i}\\ &= \frac{\log(1)+i\frac{\pi}{2}}{2i}\\ &= \frac{\pi}{4}.\end{aligned}$$

And when applying the residue theorem, we will need

$$2\pi i\sum_{\text{poles } z_0}\operatorname{res}_f(z_0) = 2\pi i\cdot\frac{\pi}{4} = \frac{i\pi^2}{2}.$$

Now consider the integral over γ_3, and note that for $x \in \mathbb{R}$, with $x < 0$, we have $\log x = \log|x| + i\pi$ where the $\log$ on the right hand side is taken to be the standard logarithm for the positive reals. Then

$$\begin{aligned}\int_{\gamma_3} f(z)\,dz &= \int_{-\infty}^{0}\frac{\log x}{1+x^2}\,dx\\ &= \int_0^\infty\frac{\log x + i\pi}{1+x^2}\,dx\\ &= \int_0^\infty\frac{\log x}{1+x^2}\,dx + i\pi\int_0^\infty\frac{1}{1+x^2}\,dx\\ &= \int_0^\infty\frac{\log x}{1+x^2}\,dx + \frac{i\pi^2}{2}.\end{aligned}$$

Then if we can show that the integrals over $\gamma_2 + \gamma_4$ vanish in the limit, we will have by Cauchy's residue theorem that

$$\frac{i\pi^2}{2} = \int_{\gamma_1+\gamma_3} f(z)\,dz = 2\int_0^\infty\frac{\log x}{1+x^2}\,dx + \frac{i\pi^2}{2},$$

at which point we may conclude

$$\int_0^\infty \frac{\log x}{1+x^2}\, dx = 0.$$

Now on to show that the required integrals vanish. We begin with γ_2. We have

$$\int_{\gamma_2} f(z)\, dz = \int_{\gamma_2} \frac{\log z}{1+z^2}\, dz,$$

and then letting $z = \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta} d\theta$ and $\theta \in [0, \pi]$

$$\int_{\gamma_2} f(z)\, dz = \int_0^\pi \frac{\log(\epsilon e^{i\theta})}{1+\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta}\, d\theta.$$

Then we note that for our branch of the logarithm, $\log(\epsilon e^{i\theta}) = \log \epsilon + i\theta$ where the log on the right is the standard, and next take absolute values to see

$$\begin{aligned}\left|\int_{\gamma_2} f(z)\, dz\right| &\leq \epsilon\pi \max_{0\leq\theta\leq\pi} \left|\frac{\log\epsilon + i\theta}{1+\epsilon^2 e^{2i\theta}}\right| \\ &\leq \epsilon\pi \frac{|\log \epsilon| + \pi}{1-\epsilon^2}.\end{aligned}$$

Now citing the fact that $\lim_{\epsilon\to 0} \epsilon \log \epsilon = 0$, we see that this integral vanishes in the limit. Moving more quickly through γ_4, we see

$$\int_{\gamma_4} f(z)\, dz = \int_0^\pi \frac{\log(Re^{i\theta})}{1+R^2 e^{2i\theta}} Rie^{i\theta}\, d\theta,$$

and so in absolute value we have

$$\left|\int_{\gamma_4} f(z)\, dz\right| \leq \pi R \frac{|\log R| + \pi}{R^2 - 1}.$$

Thus the integral grows like $(\log R)/R$ in the limit as $R \to \infty$, and by l'Hopital's rule, this goes to 0. So the integrals over γ_2 and γ_4 vanish as claimed, and we now have that the original integral is also 0.

□