Math 324: Linear Algebra

2.5: Applications of Matrix Operations

Mckenzie West

Last Updated: February 19, 2020

Last Time.

- Theorem 2.15
- Elementary Matrices
- Row Equivalence
- Writing a Matrix as a Product of Elementary Matrices
- Elementary Matrices and Inverses

Today.

- Stochastic Matrices
- Consumer Preference Models
- Cryptography-Encoding and Decoding using Matrices
- Leontief Input-Output Models
- Least Squares Approximation

Definition.

A `stochastic matrix` is an $n \times n$ matrix P where every entry is in the interval $[0, 1]$ and the sum of the entries in each column is 1.

Note.

Stochastic matrices are used to describe probabilities of going from one state to the next, also called `transition probabilities`.

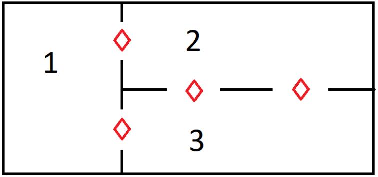
Exercise 1.

A population of 10,000 is grouped as follows: 5000 nonsmokers, 2500 smokers of one pack or less per day, and 2500 smokers of more than one pack per day. During any month, there is a 5% probability that a nonsmoker will begin smoking a pack or less per day, and a 2% probability that a nonsmoker will begin smoking more than one pack a day. For smokers who smoke a pack or less per day, there is a 10% probability of quitting and a 10% probability of increasing to more than a pack per day. For smokers who smoke more than a pack per day, there is a 5% probability of quitting and a 10% probability of dropping to a pack or less per day. How many will be in each group (a) in 1 month and (b) in 2 months? [a]

[a] Section 2.5 Exercise 7

Exercise 2.

An escape room designer places a person in a locked space with three compartments, as show in the figure below–the diamonds represent doors. The person has been instructed to select a door at random whenever a bell is rung and to move through it into the next compartment. Note that the person is equally likely to go through any of the doors in their current room.



(a) Write down a stochastic matrix where the (i, j) entry describes the probability that the person moves from room j to room i.

Exercise 2. (Continued)

(b) If the person starts in room 1, what is the probability that the person will be in room 2 after the bell has rung twice? three times?

(c) It turns out that "escape room designer" was never going to let the person leave but instead will just keep ringing the bell into eternity. In the long run, what proportion of time will the person spend in each room?

An $n \times n$ matrix can be used to encode a message by associating each letter with a number:

$$A = 1, B = 2, C = 3, \ldots, Z = 26$$

with 0 representing a space.

Example.

The message `HELLO WORLD` could be written as:

$$\begin{array}{ccccccccccc} H & E & L & L & O & & W & O & R & L & D \\ 8 & 5 & 12 & 12 & 15 & 0 & 23 & 15 & 18 & 12 & 4 \end{array}$$

If $n = 4$, we write the message using

$$\begin{bmatrix} 8 & 5 & 12 & 12 \end{bmatrix} \begin{bmatrix} 15 & 0 & 23 & 15 \end{bmatrix} \begin{bmatrix} 18 & 12 & 4 & 0 \end{bmatrix}$$

To encode the message, we pick a random 4×4 invertible matrix (that we must remember or we won't be able to decode), and multiply it on the left by each of the 1×4 row vectors.

Example.

Continuing, let

$$A = \begin{bmatrix} 3 & 4 & -3 & 2 \\ 4 & -2 & 5 & 4 \\ 5 & -4 & 3 & -3 \\ 1 & -1 & -2 & 1 \end{bmatrix}$$

Each encoded string is:

$$\begin{aligned} \begin{bmatrix} 8 & 5 & 12 & 12 \end{bmatrix} A &= \begin{bmatrix} 116 & -38 & 13 & 12 \end{bmatrix} \\ \begin{bmatrix} 15 & 0 & 23 & 15 \end{bmatrix} A &= \begin{bmatrix} 175 & -47 & -6 & -24 \end{bmatrix} \\ \begin{bmatrix} 18 & 12 & 4 & 0 \end{bmatrix} A &= \begin{bmatrix} 122 & 32 & 18 & 72 \end{bmatrix} \end{aligned}$$

What we would pass to our friend (who knows what A is as well):

$$116, -38, 13, 12, 175, -47, -6, -24, 122, 32, 18, 72$$

Exercise 3.

Suppose we're given $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and a message that was encoded using A:

$$11, 21, 64, 112, 25, 50, 29, 53, 23, 46, 40, 75, 55, 92$$

What was the original message?[a]

[a] Section 2.5 Exercise 15

Brain Break.

What was a “total first year move” you made when you began college?

Leontief Input-Output Models

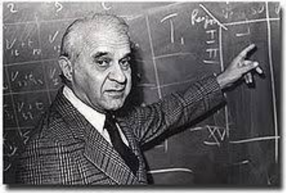


Image from: goodreads.com

Wassily Leontief, a Russian-American economist, won the 1973 Nobel prize in Economics for his work using the models he discovered.

Idea.

In an economic system, one company may use the outputs of another to produce its goods. Example 7 in the book explains how Electricity, Water, and Coal power may need to supply one another in order to provide power to the public.

Definition.

A `closed` system is one where all products are sold only to industries in the system.
An `open` system is one where some outputs are sold to non-producing industries outside the system.

Definition.

Let d_{ij} be the amount of output the jth industry needs from the ith industry to produce one unit of output per year. The matrix formed by these values is called the `input-output matrix`:

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}.$$

That is rows correspond to input, columns correspond to output.

Exercise 4.

A system composed of two industries, coal and steel, has the following input requirements:

- to produce \$1.00 worth of output, the coal industry requires \$0.10 of its own product and \$0.80 of steel;
- to produce \$1.00 worth of output, the steel industry requires \$0.10 of its own product and \$0.20 of coal.

Find D, the input-output matrix for this system. [a]

---

[a] Section 2.5, Exercise 23

Definition.

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, the output matrix, lists the total amounts produced by each industry, then

$$X = DX + E$$

where E, the external demand matrix, is the amount of product from each industry that is sold to the external industries.

Exercise 5.

If you're given D and E, how would you solve for X?

Exercise 4. (Continued)

For the coal and steel industries established in slide 13, assume the external demand matrix is $E = \begin{bmatrix} 10000 \\ 20000 \end{bmatrix}$.

(a) What do the 10,000 and 20,000 represent in the external demand matrix?

(b) Solve for the output matrix X in the equation $X = DX + E$. *Be careful here about factoring.

Note.

One of the most incredible mathematical discoveries related to this work is the fact that $I - D$ is *always* going to be invertible, given standard industry conditions.

Exercise 6.

[a] An industrial system with three industries–chocolate, milk, and cookies–has the following input-output matrix D and external demand matrix E:

$$D = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.0 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} \text{ and } E = \begin{bmatrix} 3000 \\ 3500 \\ 8500 \end{bmatrix}$$

(a) Interpret each of the entries in the first column in terms of each of the industries.

(b) Interpret each of the entries in the second row in terms of each of the industries.

(c) Solve for the output matrix X in the equation $X = DX + E$.

[a] Chapter 2 Review Exercise 72

Definition.

For a set of points $(x_1, y_1),\ (x_2, y_2), \ldots,\ (x_n, y_n)$, the `least squares regression line` is given by the linear function $f(x) = a_0 + a_1 x$ that minimizes the sum of squared error:

$$[y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_n - f(x_n)]^2$$

Finding the least squares regression line.

- For the least squares approximation, we want

$$a + bx_1 \approx y_1, a + bx_2 \approx y_2, \ldots, a + bx_n \approx y_n$$

- Let $X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, and $A = \begin{bmatrix} a \\ b \end{bmatrix}$. We then condense all of the equations from above via: $XA \approx Y$.
- In fact, $A = (X^T X)^{-1} X^T Y$.
- The individual errors for each Y value are given by $E = Y - XA$. The sum of square error is: $E^T E$.

Exercise 7.

Find the least squares regression line for the data:

$$(-5, 1), (1, 3), (2, 3), (2, 5).$$

What is the sum of squared error?

Exercise 8.

The table shows the average salaries y (in millions of dollars) or Major League Baseball players on opening day of baseball season from 2005 through 2010. (Source: Major League Baseball)[a]

Year	2005	2006	2007	2008	2009	2010
Salary, y	2.6	2.9	2.9	3.2	3.2	3.3

(a) Find the least squares regression line for the data. (I suggest translating the years like in Chapter 1).

(b) Use the linear model to create a table of estimated values for y during the years 2005 to 2010.

(c) What is the sum of squared errors?

(d) What is the estimated salary for 2019?

(e) When, approximately, will the average salary be 6 million?

[a] Chapter 2 Review Exercise 79