MATH 34032
Greens functions, integral equations and applications

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Syllabus

- **Section 1: Introduction and motivation.** What use are Green's functions and integral equations? Some example applications. (0.5 lecture)

- **Section 2: Green's functions in 1D.**
Ordinary differential equations review, influence function, Linear differential operators, Green's identity, adjoint and self-adjoint operators, Sturm-Liouville eigenvalue ODE problems, Fredholm Alternative, Green's functions as eigenfunction expansions, dirac delta function and generalized functions, direct approach for determining Green's functions via method of variation of parameters, the wave equation, adjoint Green's function, non Sturm-Liouville problems, modified Green's function and inhomogeneous boundary conditions. (9.5 lectures).

- **Section 3: Green's functions in 2 and 3D.**
Sturm-Liouville problems in 2 and 3D, Green's identity, Multidimensional eigenvalue problems associated with the Laplacian operator and eigenfunction expansions, basics of Bessel functions, Green's function for Laplace's equation in 2 and 3D (unbounded and simple bounded domains) and associated applications, Green's function for Helmholtz equation in 2D (unbounded and simple bounded domains) and associated wave scattering and cloaking problems. (7 lectures).

- **Section 4: Integral equations in 1D.**
Linear integral operators and integral equations in 1D, Volterra integral equations govern initial value problems, Fredholm integral equations govern boundary value problems, separable (degenerate) kernels, Neumann series solutions and iterated kernels, applications to scattering. (5 lectures)

Course lecturer

The course lecturer is Dr. William Parnell (william.parnell@manchester.ac.uk). My office is 2.238 in the School of Mathematics, Alan Turing building. If you have any questions please use either email or preferably ask me questions directly after the lectures. You will have plenty of time to discuss further aspects in the examples classes.

Course arrangements

There will be two lectures per week in weeks 1-11 and one examples class per week in weeks 2-12. 9 example sheets will be set, distributed appropriately between weeks 1-12. Sheet 1 is mainly revision material - ensure you know it! Examples classes will be held in weeks 2-12. There is no class in week 1. If you cannot do the material on Sheet 1, look back at your MT10121 and MT20401 notes but of course ask me if in the end you are still having problems. Students should work on the examples sheets ***before*** the Example class so that they can flag up any difficulties. Some hours in week 12 will be set aside for revision as should be expected.

Lectures are held on Mondays, 11.00-11.50 in the Schuster Moseley Lecture theatre and Fridays 13.00-13.50 in Alan Turing, G.107. The examples class follows the Friday class, 14.00-14.50 also in G.107. The purpose of this is for you to work through some of the problems on the examples sheet that you have already looked at and ask for help if you need it.

The end of semester 2 hour examination accounts for 80 % and a mid-term 50 minute test on the Friday of week 7 accounts for 20 %. The test will be on material from Section 2 only and the accompanying Example sheets 1-5 of the course. This test will help you with revision and it is good to get it out of the way before Easter.

A note about the notes

These notes are pretty comprehensive. You should not really need to look at any other books as a result of this. You may also have to look back at your notes from MT10121 and MT20401 from time to time. There are plenty of examples provided both in the notes and on the Examples sheets. In the lectures I will go through most of the notes but not always all of the details. The notes accompany the lectures and you should certainly still attend and listen carefully even though I provide these notes. I certainly will not necessarily *write* all of the text on the board although I will mention and describe all of the related mathematical ideas. It is up to you to read the notes carefully. In lectures I will mainly focus on the mathematics, the theory and model examples to aid understanding. Sometimes I will ask you to work through some of the examples in the notes in your own time. And remember you need to spend a great deal of your own time reading through the notes to understand them!

You will notice that at the end of each section I provide a revision check-list. This should help you to understand what you do and do not understand at the end of the section with the aid of the notes and the related examples sheets.

I urge you to look at the examples sheets *before* the examples class. Otherwise you will not make the most of the help available in the session and you may fall behind.

1 Introduction and motivation

In this course and these notes we will discuss the solution to a broad class of problems in applied mathematics. We will largely focus on solving ordinary differential equations (ODEs) and partial differential equations (PDEs). These will take the form

$$\mathcal{L}u(x) = f(x)$$

for ODEs and we are interested in boundary value problems where $x \in [a, b]$ for some real a and b with boundary conditions prescribed on a and b. In the end we want to solve for the field variable $u(x)$. We can also analyse initial value problems where initial conditions are specified at $x = 0$ but we only have 22 lectures! Here $\mathcal{L}$ is known as an *ordinary differential operator*, e.g. $\mathcal{L} = d^2/dx^2$. The function $f(x)$ is a "forcing" function. PDEs will take the form

$$\mathcal{L}u = Q(\mathbf{x}) \tag{1.1}$$

on some domain $\mathbf{x} \in D$ where $\mathbf{x} = (x, y, z)$ in three dimensional problems. Here $\mathcal{L}$ is a *partial differential operator*, e.g. $\mathcal{L} = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$, the Laplacian. Note that for reasons of clarity and time restrictions we do *not* consider problems with explicit time dependence or forcing, or rather we consider certain *types* of time dependent problems, e.g. $\exp(-i\omega t)$ for wave problems[1] which yield problems in the form (1.1) where (1.1) is the steady-state forcing. We also restrict attention to scalar problems so that u is a scalar field (temperature, pressure, etc.). As a result of the above, the PDEs that we consider in this course are all *elliptic*. This therefore includes the steady state heat equation (Laplace's equation $\nabla^2 u = Q(\mathbf{x})$) and the time harmonic wave equation (Helmholtz equation $\nabla^2 u + k^2 u = Q(\mathbf{x})$). "We've done all this before" you may say. Well you have done some of it, but we will be learning about a special technique to solve *inhomogeneous* PDEs, i.e. when the forcing terms $f(x)$ and $Q(\mathbf{x})$ above are non-zero. This technique is the method of *Green's functions*[2]. It transpires that the solution to the problem can (in general) be written as a weighted integral of the forcing over the domain, where the weighting is the Green's function. This is a topic that has been and is still of great interest as a research topic in applied mathematics. Green's functions have pervaded many areas of mathematics, science, engineering and computation, often in surprising ways. In particular, Green's functions can be used in order to re-write the differential equation forms of the problems in *integral equation* form. The subject of *boundary element methods*, an area of great interest for solving problems numerically, stems from this development.

In addition to the fact that they are of great use, they are also very interesting mathematically. We will be able to discuss various ideas and theoretical aspects pertaining to the theory of ordinary and partial differential equations.

As an example of the use of Green's functions, consider the simple ordinary differential equation of the form

$$\frac{d^2u}{dx^2} = f(x) \tag{1.2}$$

[1]In some of the Example Sheets we do consider a small subset of time dependent PDEs. The reason for doing this is to see the context in which our problems *without* time dependence reside.

[2]named after the brilliant applied mathematician and Nottingham Miller *George Green* (1793-1841) who developed them as a tool in the 1830s

where f is some forcing function, on a domain $x \in [0, L]$ with homogeneous boundary conditions e.g. $u(0) = 0, du/dx(L) = 0$. This corresponds to the steady state heat equation in one dimension with heat source term $f(x)$ and with fixed temperature at $x = 0$ and an insulated boundary at $x = L$ (no heat flux across the boundary). This problem is of course a *boundary value problem*, i.e. an ODE governing some function u (the temperature) with corresponding boundary conditions at the edge of the domain.

It transpires that a solution of the problem can be written in the form

$$u(x) = \int_0^L G(x, x_0) f(x_0)\ dx_0 \tag{1.3}$$

where $G(x, x_0)$ is the corresponding *Green's function* which satisfies an associated boundary value problem. We will not describe this here but will of course in detail in later chapters. Note that (1.3) is strictly an integral equation, although it does not have to be solved so it can be said to be an integral *expression* for the function $u(x)$.

In two and three dimensions, the corresponding solution can be written[3]

$$u(\mathbf{x}) = \int_V G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0)\ d\mathbf{x}_0 \tag{1.4}$$

where D is the two/three dimensional domain and G is the corresponding Green's function.

We will describe the theory behind the above analysis and describe in particular some applications in the context of heat conduction and wave propagation. In particular for problems involving inhomogeneous media (think of a solid body with an "inclusion" embedded inside it) we are able to write down *integral equations* which govern the scalar field $u(\mathbf{x})$. We shall describe methods to solve these interesting problems. Indeed in later chapters we will make links to some modern research topics. These include "acoustic scattering theory" i.e. how sound waves are scattered from obstacles, "acoustic cloaking theory" i.e. how we can try to make objects "invisible" to sound and the study of "composite materials", although we probably will not have the time to consider *all* of these applications. I will of course make it clear what *is* and *is not* examinable.

Here are some brief details of the application areas described above.

Acoustic scattering theory

Suppose that we have a uniform medium and within this domain we embed an "inclusion", it could have arbitrary shape. Imagine that sound (acoustic) waves are incident on the inclusion. This causes the waves to be *scattered*. How do we solve for this scattered field? One example is shown in figure 1. We shall describe how we do this for simple geometries in section 3 via Green's functions. In section 5 we describe a more general case and describe how the problem can be reformulated in terms of integral equations. We describe a technique that can be implemented in order to predict the scattered field.

[3] In harder problems this not the case - we will consider some of these in sections 4 and (**??**) when we discuss integral equations.

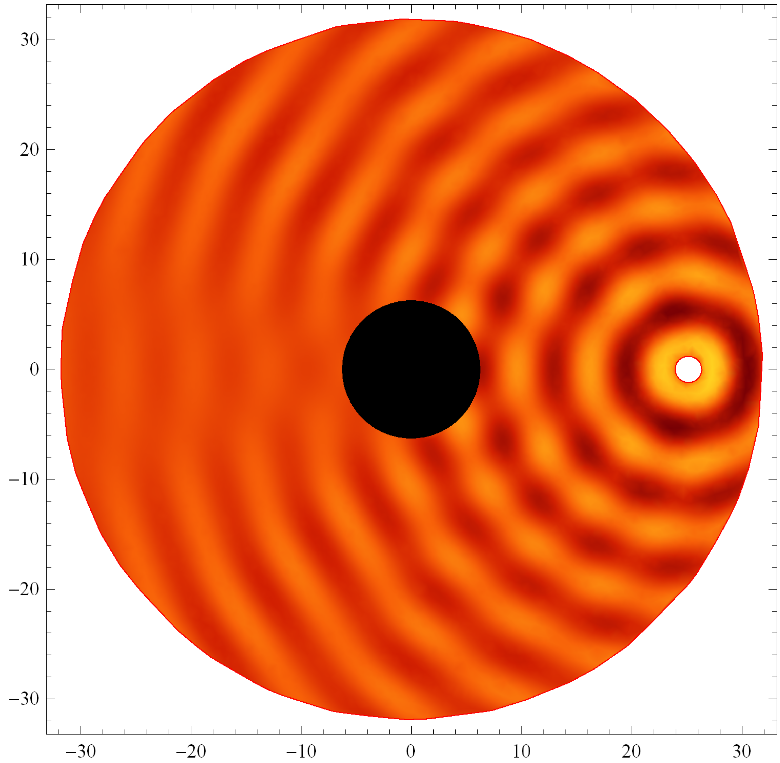

Figure 1: An acoustic (sound) field is generated by "forcing" at the point in the white circle. Outgoing circular waves are generated. These outgoing waves are subsequently scattered by the circular black region. Because in this instance the *wavelength* is commensurate with the size of the circular region, scattering is strong: we see a clear *shadow region* and *backscattered field*. The field is time harmonic so that we are showing the amplitude of the wave field at a single instant in time.

Acoustic cloaking theory

Suppose that we did not want the field to be scattered from the circular region above. How could we enable this to happen? The development of the two and three dimensional Green's function enables us to easily describe the concept of *acoustic cloaking*. This is a topic of great interest presently. The idea is to design an acoustic material which possesses properties in order to "guide" the acoustic waves around a region of interest. See figure 2. This is of interest in a number of applications mainly due to the fact that outside the cloak region, one cannot tell at all that there is a circular region or anything inside it. We will describe how this concept of cloaking can be achieved theoretically in section 3.

Composite materials

Suppose that we have a material which consists of lots of small inclusions embedded inside an otherwise uniform "host" medium (see figure 3). This type of so-called *composite* material is used in thousands of applications in engineering, medical science, the automotive and defence industries and aerospace sector amongst many others. If the inclusions and host medium have different thermal conductivities, how do we theoretically predict

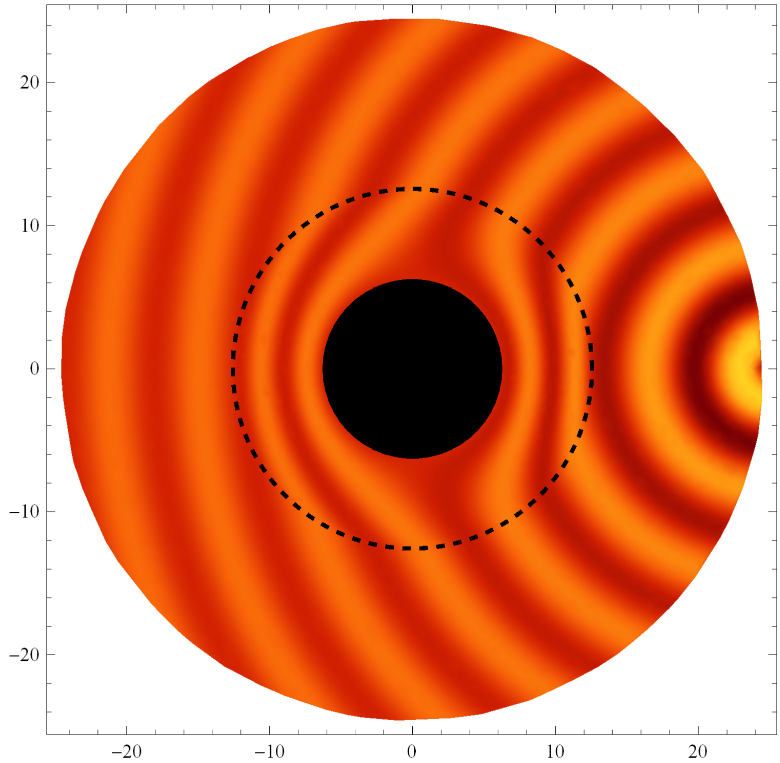

Figure 2: A material with special material properties is wrapped around the black circular region. These properties guide the incoming acoustic (sound) wave, generated at the "point" just to the right of the image, around the region. The region is therefore *cloaked* and anything inside will not be "seen" in the far-field. The field is time harmonic so that we are showing the amplitude of the wave field at a single instant in time.

what the so-called overall (or *effective*) thermal conductivity is and how it depends on the volume fraction (relative quantities of the different constituents), conductivities and shape of the constituents of the material in question? In section 5 we will use integral equations in order to motivate one approach to solving this problem. It transpires that we can introduce a small amount of the inclusion material in order to significantly influence (and improve) the overall (or *effective*) thermal conductivity of the material. This can assist in decreasing the cost, improving the effectiveness, etc. of the material.

Interesting mathematics underlies these applications!

The three applications above will be considered in this course but note that above all we will be interested in the interesting mathematics that sits underneath and describes these important phenomena. Understanding the mathematics is key to getting sensible predictions in these application areas. These research areas are of great current interest and many scientists are currently undertaking related mathematical research with associated applications in physics, materials science, chemistry, medical imaging and diagnostics, medical implants, non destructive evaluation of components in industry and many more.

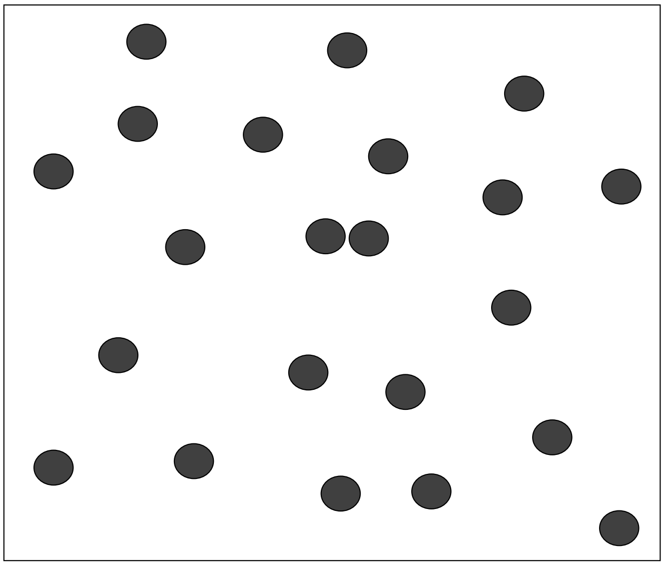

Figure 3: We show a composite material which consists of many small inclusions distributed throughout a uniform "host" material. The question is *how do we predict the overall material properties from knowledge of the constituent materials?*

2 Green's functions in 1D

We now come on to the introduction of the concept of a Green's function and we shall start in one dimension, i.e. with *ordinary differential equations* (ODEs). We will usually be interested in solutions of *second order* (highest derivative is two) ODEs. This includes many problems that are of interest in practice, for example the (steady state) heat equation and the wave equation at fixed frequency.

2.1 Ordinary Differential Equations: review

You have seen the material here before (MT10121). We will review it briefly but look back at your notes to ensure that you know it *thoroughly*!

Let us consider second order Ordinary Differential Equations (ODEs) of the form

$$p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x) \tag{2.1}$$

where $p(x), r(x), q(x)$ and $f(x)$ are real functions. Two type of problems can be considered: *Boundary Value Problems* (BVPs) and *Initial Value Problems*. For BVPs, x is a spatial variable e.g. $x \in [a, b]$ and we require associated *boundary conditions* (BCs) e.g. $\mathcal{B} = \{u(a) = 0, u(b) = 1\}$, etc. For IVPs, x is time so $x \in [0, \infty)$ and we require associated *initial conditions* (ICs) e.g. $\mathcal{I} = \{u(0) = 0, u'(0) = 1\}$. If the BCs or ICs have a zero right hand side they are known as *homogeneous*. Otherwise they are known as *inhomogeneous*. We will consider exclusively BVPs in this section. We will consider IVPs in section 4 (integral equations in 1D).

Note that often we can divide through by $p(x)$ in order to give a unit coefficient of $u''(x)$. However in general we have to be careful with this. Some singular problems (that are physical) do not allow us to do this.

The *general solution* of the ODE is in general written in the form

$$u(x) = u_c(x) + u_p(x) \tag{2.2}$$

where $u_c(x)$ is known as the *complementary function* and is the solution to the *homogeneous* ODE

$$p(x)u_c''(x) + r(x)u_c'(x) + q(x)u_c(x) = 0 \tag{2.3}$$

whereas $u_p(x)$ is known as the *particular solution* and is the solution to the *inhomogeneous* ODE

$$p(x)u_p''(x) + r(x)u_p'(x) + q(x)u_p(x) = f(x). \tag{2.4}$$

Once we have determined (2.2) it will have some undetermined constants (these are always in the complementary function) which are then determined by *imposing the BCs or ICs on the* ***general solution***.

How do we determine the complementary function and particular solution? Let us discuss this now. We note that in particular we are interested in two types of ODEs: Constant coefficient ODEs and those of Euler type since these may be solved analytically. ODEs that cannot be solved analytically can of course be treated by numerical methods but this is outside the scope of this course.

2.1.1 Homogeneous ODEs: The complementary function

For constant coefficient ODES, with $r, q \in \mathbb{C}$ we can write

$$u_c''(x) + ru_c'(x) + qu_c(x) = 0. \tag{2.5}$$

Here we really can take the coefficient of $u_c''(x)$ to be unity since we can divide through by the constant p. We know that since the ODE is second order there will be two fundamental solutions say $u_1(x)$ and $u_2(x)$ that contribute to the complementary function and it can be written as $u_c(x) = c_1u_1(x) + c_2u_2(x)$ for some real constants $c_1, c_2 \in \mathbb{R}$. To find u_1 and u_2, seek solutions of the form $\exp(mx)$ where $m \in \mathbb{R}$ and find the λ that ensure solutions from $m^2 + rm + q = 0$. There will either be two real, two complex conjugate or repeated roots. In the case of the latter one of these solutions must be multiplied by x in order to obtain the second linearly independent solution (see question 4 of Example sheet 1).

Example 2.1 *Find the solution of*

$$u_c''(x) + u_c'(x) - 2u_c(x) = 0. \tag{2.6}$$

Seeking solutions in the form $\exp(mx)$ *gives* $m^2 + m - 2 = (m + 2)(m - 1) = 0$ *so that* $m = -2, 1$. *The solution is therefore*

$$u_c(x) = c_1 \exp(-2x) + c_2 \exp(x) \tag{2.7}$$

for some constants c_1, c_2.

Example 2.2 *Find the solution of*

$$u_c''(x) + 2u_c'(x) + u_c(x) = 0. \tag{2.8}$$

Seeking solutions in the form $\exp(mx)$ *gives* $m^2 + 2m + 1 = (m + 1)^2 = 0$ *so that* $\lambda = -1$ *(repeated). The solution is therefore*

$$u_c(x) = c_1 \exp(-x) + c_2x \exp(-x) \tag{2.9}$$

for some constants c_1, c_2.

Euler equations are of the form

$$x^2u_c''(x) + rxu_c'(x) + qu_c(x) = 0 \tag{2.10}$$

for some $r, q \in \mathbb{R}$ and $x \neq 0$. Solutions are then sought in the form x^m.

Example 2.3 *Find the solution of the Euler ODE*

$$x^2u_c''(x) + 2xu_c'(x) - 6u_c(x) = 0. \tag{2.11}$$

Seeking solutions in the form x^m *gives* $m(m-1)+2m-6 = m^2+m-6 = (m+3)(m-2)$ *so that* $m = 2$ *and* $m = -3$. *The solution is therefore*

$$u_c(x) = c_1x^2 + \frac{c_2}{x^3} \tag{2.12}$$

for some constants c_1, c_2.

2.1.2 Inhomogeneous ODEs

Let us now consider how we find the particular solution $u_p(x)$. We can obtain this by two alternative techniques: the *method of undetermined coefficients* and the *method of variation of parameters.*

Inhomogeneous ODEs: Method of undetermined coefficients

Consider again the general second-order ODE of the form

$$p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x). \tag{2.13}$$

We must seek particular solutions $u_p(x)$ in order to take care of the inhomogeneous term $f(x)$ on the right hand side. A simple method is known as the method of undetermined coefficients. This is sometimes also called the *method of intelligent guessing*!

Example 2.4 *Find the particular solution for the ODE*

$$u''(x) + u'(x) - 2u(x) = 10\exp(3x) \tag{2.14}$$

We note that $\exp(3x)$ *is not one of the fundamental solutions (you can check this). Therefore pose a particular solution in the form* $u_p(x) = a\exp(3x)$ *for some* $a \in \mathbb{R}$ *to be determined. Substituting this into the ODE we find that*

$$a(9\exp(3x) + 3\exp(3x) - 2\exp(3x)) = 10\exp(3x) \tag{2.15}$$

and so for consistency we note that we require $a = 1$*.*

If the right hand side of the ODE is one of the fundamental solutions we multiply our choice by x (note the special case of an Euler ODE with fundmental solution $1/x$ with forcing term $1/x$ would have $u_p(x) = (a/x)\ln x$). Clearly this method can sometimes be difficult to apply because we are using our judgement as to what we should choose as a candidate solution. It would be preferable if we could derive a more algorithmic approach.

Inhomogeneous ODEs: Method of variation of parameters

We cannot always use the method of undetermined coefficients. Sometimes we just cannot "see" the particular solution. Consider again the general second-order ODE of the form

$$p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x). \tag{2.16}$$

We will now briefly describe the method of variation of parameters. In order to apply this method we need to know the complementary function. This is imperative (remember that this was not the case with the method of undetermined coefficients). We know from section 2.1.1 that the complementary function has the form

$$u_c(x) = c_1u_1(x) + c_2u_2(x). \tag{2.17}$$

We will pose a particular solution of the form

$$u_p(x) = v_1(x)u_1(x) + v_2(x)u_2(x) \tag{2.18}$$

and so we need to determine the two unknown functions $v_1(x)$ and $v_2(x)$.

Let us differentiate $u_p(x)$:

$$u_p'(x) = v_1'(x)u_1(x) + v_1(x)u_1'(x) + v_2'(x)u_2(x) + v_2(x)u_2'(x) \tag{2.19}$$

and make the assumption that

$$v_1'(x)u_1(x) + v_2'(x)u_2(x) = 0. \tag{2.20}$$

Differentiate $u_p''(x)$ again

$$u_p''(x) = v_1'(x)u_1'(x) + v_2'(x)u_2'(x) + v_1(x)u_1''(x) + v_2(x)u_2''(x). \tag{2.21}$$

Substituting $u_p(x)$ and its derivatives into the governing ODE and rearranging we find

$$\begin{aligned} p(x)[v_1'(x)u_1'(x) + v_2'(x)u_2'(x)] \\ + v_1(x)[p(x)u_1''(x) + r(x)u_1'(x) + q(x)u_1(x)] \\ + v_2(x)[p(x)u_2''(x) + r(x)u_2'(x) + q(x)u_2(x)] = f(x). \end{aligned} \tag{2.22}$$

Of course in the second and third terms on the left hand side, the terms in square brackets are zero. Therefore

$$p(x)(v_1'(x)u_1'(x) + v_2'(x)u_2'(x)) = f(x). \tag{2.23}$$

This together with the assumption (2.20) gives us two equations to solve for $v_1'(x)$ and $v_2'(x)$. We solve to find

$$v_1'(x) = \frac{-u_2(x)f(x)}{p(x)(u_1(x)u_2'(x) - u_2(x)u_1'(x))}, \quad v_2'(x) = \frac{u_1(x)f(x)}{p(x)(u_1(x)u_2'(x) - u_2(x)u_1'(x))}. \tag{2.24}$$

We note that since $u_1(x)$ and $u_2(x)$ are fundamental solutions the Wronskian is non-zero:

$$W(x) = u_1(x)u_2'(x) - u_2(x)u_1'(x) \neq 0. \tag{2.25}$$

So, we can integrate in each of (2.24) between a and x to find

$$v_1(x) = \int_a^x -\frac{u_2(x_0)f(x_0)}{p(x_0)W(x_0)}\, dx_0 + v_1(a), \quad v_2(x) = \int_a^x \frac{u_1(x_0)f(x_0)}{p(x_0)W(x_0)}\, dx_0 + v_2(a). \tag{2.26}$$

We can set $v_1(a) = v_2(a) = 0$ because from (2.18) these merely generate additional terms that are of the form of the complementary function. Therefore

$$v_1(x) = \int_a^x -\frac{u_2(x_0)f(x_0)}{p(x_0)W(x_0)}\, dx_0, \qquad v_2(x) = \int_a^x \frac{u_1(x_0)f(x_0)}{p(x_0)W(x_0)}\, dx_0. \tag{2.27}$$

Therefore we can assert that the general solution to the ODE is

$$u(x) = u_c(x) + u_p(x) \tag{2.28}$$

$$= (c_1 + v_1(x))u_1(x) + (c_2 + v_2(x))u_2(x) \tag{2.29}$$

2.2 General forcing and the influence (Green's) function

In order to give a full description of Green's functions, what they are and why they are useful we need a lot more ODE theory some (most?) of which you will not have come across before. We will come on to this in a moment but let us consider a simple problem here first in order to motivate the idea of a Green's function.

In particular we should ask if we can obtain a solution form for an ODE with an arbitrary forcing term $f(x)$ on the right hand side? In order to answer this question let us consider a canonical problem and one that has a very important application. Consider the simple equation

$$d^2u/dx^2 = u''(x) = f(x) \tag{2.30}$$

on the domain $x \in [0, L]$ subject to homogeneous boundary conditions $\mathcal{B} = \{u(0) = 0, u(L) = 0\}$. This problem is in fact the *steady state heat equation*. I.e. the heat equation without any time dependence[4]. Temperature is fixed to be zero on the boundaries.

In order to solve this problem, we note that the complementary function satisfies

$$u_c''(x) = 0 \tag{2.31}$$

and by direct integration, the fundamental solutions are 1 and x. However it turns out to be very convenient to have fundamental solutions one of which satisfies one of the homogeneous boundary conditions and one of which satisfies the other. Therefore we choose linear combinations, to obtain

$$u_1(x) = x, \qquad u_2(x) = L - x \tag{2.32}$$

satisfying the left and right boundary condition respectively.

Using (2.27), since $W = u_1u_2' - u_2u_1' = x(-1) - (L - x)(1) = -L$, we find that

$$v_1(x) = \frac{1}{L}\int_0^x f(x_0)(L - x_0)\ dx_0 \tag{2.33}$$

$$v_2(x) = -\frac{1}{L}\int_0^x f(x_0)x_0\ dx_0 \tag{2.34}$$

The full solution is therefore

$$u(x) = (c_1 + v_1(x))x + (c_2 + v_2(x))(L - x) \tag{2.35}$$

so finally let us apply the BCs. Setting $x = 0$ means that $c_2 = 0$ and for $x = L$ we find

$$0 = (c_1 + v_1(L))L \tag{2.36}$$

so that $c_1 = -v_1(L)$. We then note that

$$c_1 + v_1(x) = -v_1(L) + v_1(x) \tag{2.37}$$

$$= -\frac{1}{L}\int_0^L f(x_0)(L - x_0)\ dx_0 + \frac{1}{L}\int_0^x f(x_0)(L - x_0)\ dx_0 \tag{2.38}$$

$$= -\frac{1}{L}\int_x^L f(x_0)(L - x_0)\ dx_0. \tag{2.39}$$

[4] In reality all problems have to have some time dependence of course. What usually happens is that after some initial transients have decayed we are left with a steady state solution which may or may not be the trivial one $u = 0$.

We can therefore write

$$u(x) = \frac{x}{L}\int_x^L (x_0 - L)f(x_0)\ dx_0 + \frac{(x-L)}{L}\int_0^x x_0 f(x_0)\ dx_0$$

Finally this means we can write the solution in the form

$$u(x) = \int_0^L G(x, x_0)f(x_0)\ dx_0 \tag{2.40}$$

where

$$G(x, x_0) = \begin{cases} \frac{x_0}{L}(x-L), & 0 \le x_0 \le x, \\ \frac{x}{L}(x_0 - L), & x \le x_0 \le L. \end{cases} \tag{2.41}$$

The function $G(x, x_0)$ can be thought of as an "influence function". It is in fact the Green's function for this problem and we will say more about this later on. Note that $G(x, x_0) = \overline{G(x_0, x)} = G(x_0, x)$ here, i.e. it is symmetric (the overline or "bar" denotes the complex conjugate, recall $z = a + ib$, $\overline{z} = a - ib$). The Green's function does not always possess this full symmetry; it only occurs for special types of boundary value problems. In particular $G(x, x_0) = \overline{G(x_0, x)}$ always occurs for a special class of problems called *self-adjoint* operator problems (which we will consider shortly).

Note that we may write (2.41) in the form

$$G(x, x_0) = \frac{x}{L}(x_0 - L)H(x_0 - x) + \frac{x_0}{L}(x - L)H(x - x_0) \tag{2.42}$$

which also illustrates the symmetry, where

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}$$

is the so-called Heaviside step function.

When determining Green's function later, I would always encourage you to write them in this form. It helps a great deal, especially when integrating them!

Finally we note that by directly integrating twice we could in fact obtain the solution in the form (see question 5 on Example Sheet 1)

$$u(x) = \int_0^x \int_0^{x_0} f(x_1)\ dx_1 dx_0 + c_1 x + c_2. \tag{2.43}$$

You are asked to show that this is equivalent to (2.40) in question 5 on Example Sheet 1.

2.3 Linear differential operators

It turns out to be very useful to define the notation $\mathcal{L}$ to mean a linear operator, which means that

$$\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}u_1 + c_2\mathcal{L}u_2.$$

for (possibly complex) constants c_j. In this chapter it will be associated with a second order *ordinary* differential operator, e.g. $\mathcal{L} = d^2/dx^2$. In the next chapter it will be associated with partial differentiation. Remember that in general an operator will take a function and turn it into another function. The functions in general will belong to some function space which possess some specific properties, i.e. $L^2[a,b]$ which means that they are square integrable on $[a,b]$, (i.e. $f \in L^2[a,b]$ means $\int_a^b |f(x)|^2 \ dx < \infty$) etc.

We are interested in the linear BVP

$$\mathcal{L}u = p(x)\frac{d^2u}{dx^2} + r(x)\frac{du}{dx} + q(x)u = f(x) \tag{2.44}$$

where for now we do not make any restrictions on the functions $p(x), r(x), q(x)$ and $f(x)$ but they can be complex functions and we usually consider them as continuous. The (real) domain on which the ODE holds is $x \in [a,b]$ and it is of course subject to BCs on $x = a, b$ which we shall denote as $\mathcal{B}$. We will restrict attention to *homogeneous* BCs and for now these could be of any form, e.g.

$$\mathcal{B} = \{u(a) = 0, u(b) = 0\}, \quad \text{Dirichlet} \tag{2.45}$$
$$\mathcal{B} = \{u'(a) = 0, u(b) = 0\}, \quad \text{Dirichlet-Neumann}, \tag{2.46}$$
$$\mathcal{B} = \{u'(a) = 0, u'(b) = 0\}, \quad \text{Neumann}, \tag{2.47}$$
$$\mathcal{B} = \{u(a) + hu'(a) = 0, u(b) = 0\}, \quad \text{Robin-Dirichlet}, \tag{2.48}$$
$$\mathcal{B} = \{u(a) = u(b), u'(a) = u'(b)\}, \quad \text{Periodic}, \tag{2.49}$$
$$\mathcal{B} = \{u(a) + hu'(b) = 0, u(b) = 0\}, \quad \text{Mixed-Dirichlet}. \tag{2.50}$$

Extension to the case of inhomogeneous BCs is not too difficult - we shall discuss this in section 2.12.

The BVP therefore consists of the equation $\mathcal{L}u = f(x)$ and the BCs $\mathcal{B}$.

2.3.1 Inner products

The function spaces to which the functions that we are interested in belong, are endowed with an inner product. This means that they are "inner product spaces". This basically means that they possess nice properties such as Cauchy-Schwarz and the triangle inequality. We do not worry too much about this here, usually assuming that the functions we are interested in are in $L^2[a,b]$. The notion and notation of an inner product is useful. We define the usual inner product as

$$\langle f, g\rangle = \int_a^b \overline{f(x)}g(x) \ dx \tag{2.51}$$

where we note that $\overline{f(x)}$ denotes the complex conjugate of the function f, i.e. we have defined this inner product over the set of complex valued functions (this includes the set of real functions of course).

We have the important properties of inner product spaces that

$$\langle f, g\rangle = \overline{\langle g, f\rangle}, \tag{2.52}$$

$$\langle f, \alpha g_1 + \beta g_2\rangle = \alpha\langle f, g_1\rangle + \beta\langle f, g_2\rangle, \tag{2.53}$$

$$\langle f, f\rangle \geq 0 \text{ with equality if and only if } f = 0. \tag{2.54}$$

$$\langle \alpha g_1 + \beta g_2, f\rangle = \overline{\alpha}\langle g_1, f\rangle + \overline{\beta}\langle g_2, f\rangle. \tag{2.55}$$

2.3.2 The adjoint operator

It is useful to define a so-called *adjoint BVP* associated with the original BVP above. This adjoint problem consists of an *adjoint operator* $\mathcal{L}^*$ and associated *adjoint BCs* $\mathcal{B}^*$. These are defined by

$$\langle v, \mathcal{L}w\rangle = \langle \mathcal{L}^* v, w\rangle$$

noting that this prescribes *both* an operator and BCs and in general $\mathcal{L}^* \neq \mathcal{L}$ and $\mathcal{B}^* \neq \mathcal{B}$.

Example 2.5 *Assuming that $u, v \in L^2[a, b]$ (i.e. they are square integrable on $[a, b]$), find the adjoint operator and BCs for the following problems*

$$(i) \qquad \mathcal{L} = \frac{d^2}{dx^2}, \qquad \mathcal{B} = \{u(0) = 0, u(1) = 0\}, \tag{2.56}$$

$$(ii) \qquad \mathcal{L} = \frac{d^2}{dx^2} + \frac{d}{dx} + 1, \qquad \mathcal{B} = \{u(0) = 0, u(1) = 0\}, \tag{2.57}$$

$$(iii) \qquad \mathcal{L} = \frac{d^2}{dx^2} + 1, \qquad \mathcal{B} = \{u(0) = 0, u(1) = u'(0)\}, \tag{2.58}$$

$$(iv) \qquad \mathcal{L} = \frac{d^2}{dx^2} + 1, \qquad \mathcal{B} = \{u(0) = u(1), u'(0) = u'(1)\}, \tag{2.59}$$

$$(v) \qquad \mathcal{L} = \frac{d^2}{dx^2} + i\frac{d}{dx} + 1, \qquad \mathcal{B} = \{u(0) = 0, u(1) = 0\}, \tag{2.60}$$

$$(vi) \qquad \mathcal{L} = x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + 1, \qquad \mathcal{B} = \{u(1) = 0, u'(2) = 0\}, \tag{2.61}$$

$$(vii) \qquad \mathcal{L} = \frac{d^2}{dx^2} + k^2, \qquad \mathcal{B} = \{u'(x) \pm iku(x) \to 0 \text{ as } x \to \pm\infty\}, \tag{2.62}$$

The trick is to use integration by parts to interchange the order of integration onto the "other" function.

(i) Let us follow through the argument, using integration by parts:

$$\begin{aligned}
\langle v, \mathcal{L}u\rangle &= \int_0^1 \overline{v}\frac{d^2u}{dx^2}\ dx \\
&= \left[\overline{v}\frac{du}{dx}\right]_0^1 - \int_0^1 \frac{du}{dx}\frac{d\overline{v}}{dx}\ dx \\
&= \left[\overline{v}\frac{du}{dx}\right]_0^1 - \left[\left[u\frac{d\overline{v}}{dx}\right]_0^1 - \int_0^1 u\frac{d^2\overline{v}}{dx^2}\ dx\right] \\
&= \left[\overline{v}\frac{du}{dx} - u\frac{d\overline{v}}{dx}\right]_0^1 + \int_0^1 \overline{\frac{d^2v}{dx^2}}u\ dx \\
&= \left[\overline{v}\frac{du}{dx}\right]_0^1 + \langle \mathcal{L}^* u, v\rangle
\end{aligned} \tag{2.63}$$

where $\mathcal{L}^* = d^2/dx^2$ and in the last step we have imposed the BCs on u. In order to ensure that the term in brackets is zero we must choose $\mathcal{B}^* = \{\overline{v}(0) = \overline{v}(1) = 0\}$ but this is equivalent to having $v(0) = v(1) = 0$ (If v is a complex function then it being zero means both its real and imaginary parts must be zero and hence these conditions are equivalent). We see that $\mathcal{L}^* = \mathcal{L}$ and the adjoint BCs are the same as the original BCs, i.e. $\mathcal{B}^* = \mathcal{B}$.

(ii) The first term of the operator is identical with that in (i) so we can use that result.

$$\begin{aligned}
\langle v, \mathcal{L}u\rangle &= \int_0^1 \overline{v}\left(\frac{d^2u}{dx^2} + \frac{du}{dx} + u\right)\ dx \\
&= \left[\overline{v}\frac{du}{dx} - u\frac{d\overline{v}}{dx}\right]_0^1 + \int_0^1 \frac{d^2\overline{v}}{dx^2}u\ dx + [u\overline{v}]_0^1 - \int_0^1 u\frac{d\overline{v}}{dx}\ dx + \int_0^1 \overline{v}u\ dx \\
&= \left[\overline{v}\frac{du}{dx} - u\frac{d\overline{v}}{dx} + u\overline{v}\right]_0^1 + \int_0^1 \overline{\left(\frac{d^2v}{dx^2} - \frac{dv}{dx} + v\right)}u\ dx \\
&= \left[\overline{v}\frac{du}{dx}\right]_0^1 + \langle \mathcal{L}^*v, u\rangle
\end{aligned} \tag{2.64}$$

and we note here that $\mathcal{L}^* \neq \mathcal{L}$ due to the first derivative term. The adjoint BCs are unchanged however, $\mathcal{B}^* = \mathcal{B}$.

(iii) Using (i) above it is easily shown that

$$\begin{aligned}
\langle v, \mathcal{L}u\rangle &= \int_0^1 \overline{v}\left(\frac{d^2u}{dx^2} + u\right)\ dx \\
&= \left[\overline{v}\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1 + \int_0^1 \overline{\left(\frac{d^2v}{dx^2} + v\right)}u\ dx \\
&= \left[\overline{v}\frac{du}{dx} - u\frac{dv}{dx}\right]_0^1 + \langle \mathcal{L}^*v, u\rangle.
\end{aligned} \tag{2.65}$$

so that $\mathcal{L}^* = \mathcal{L}$. Let us now determine the adjoint BCs, $\mathcal{B}^*$. We need

$$\left[\overline{v}\frac{du}{dx} - u\frac{d\overline{v}}{dx}\right]_0^1 = 0 \tag{2.66}$$

and using the BCs $u(0) = 0$ and $u(1) = u'(0)$ we see that

$$\begin{aligned}
\left[\overline{v}\frac{du}{dx} - u\frac{d\overline{v}}{dx}\right]_0^1 &= (\overline{v(1)}u'(1) - u(1)\overline{v'(1)}) - (\overline{v(0)}u'(0) - u(0)\overline{v'(0)}), \\
&= \overline{v(1)}u'(1) - u(1)\overline{v'(1)} - \overline{v(0)}u'(0), \\
&= \overline{v(1)}u'(1) - \overline{(v'(1) + v(0))}u'(0)
\end{aligned} \tag{2.67}$$

which implies that we require the adjoint BCs to be

$$v(1) = 0, \qquad\qquad v'(1) = -v(0). \tag{2.68}$$

Note in particular that in this example, although $\mathcal{L}^* = \mathcal{L}$ the adjoint BCs are different from the original BCs, $\mathcal{B}^* \neq \mathcal{B}$.

(iv)-(vii) See question 5 on Example Sheet 2.

In question 6 of Example Sheet 2 you are asked to show that the adjoint operator associated with the general ODE (2.44) is

$$\mathcal{L}^* = \overline{p(x)}\frac{d^2}{dx^2} + \overline{\left(2\frac{dp}{dx} - r\right)}\frac{d}{dx} + \overline{\left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)}. \tag{2.69}$$

Lagrange's[5] identity

Lagrange derived a very useful identity. This is:

$$\overline{v}\mathcal{L}u - \overline{\mathcal{L}^* v}u = \frac{d}{dx}\left[p\left(\frac{du}{dx}\overline{v} - u\frac{\overline{dv}}{dx}\right) + \left(r - \frac{dp}{dx}\right)u\overline{v}\right] \tag{2.70}$$

You are asked to prove this in question 7 on Example Sheet 2.

Green's[6] second identity

We can integrate both sides of Lagrange's identity (2.70) between $x = a$ and $x = b$ to get

$$\int_a^b \overline{v}\mathcal{L}u - \overline{\mathcal{L}^* v}u \ dx = \left[p\left(\overline{v}\frac{du}{dx} - u\frac{\overline{dv}}{dx}\right) + \left(r - \frac{dp}{dx}\right)u\overline{v}\right]_a^b. \tag{2.71}$$

Note that this general identity is very useful in order to determine the adjoint BCs $\mathcal{B}^*$ required above.

With inner product notation we note that we can write (2.71) as

$$\langle v, \mathcal{L}u\rangle - \langle \mathcal{L}^* v, u\rangle = \left[p\left(\overline{v}\frac{du}{dx} - u\frac{\overline{dv}}{dx}\right) + \left(r - \frac{dp}{dx}\right)u\overline{v}\right]_a^b. \tag{2.72}$$

I had a sentence here which referred to "real function spaces"; please delete and ignore - it was very confusing and did not add anything! Apologies.

2.3.3 Self-adjoint operators

Self-adjoint (S-A) operators are special operators with the property that the adjoint problem is identical to the original problem, i.e. both the adjoint operator *and* the adjoint BCs are the same as the original physical BVP. I.e. $\mathcal{L}^* = \mathcal{L}$ *and* $\mathcal{B}^* = \mathcal{B}$. E.g. Example 2.5(i) above. It is sometimes the case that the differential operator is the same, i.e. $\mathcal{L}^* = \mathcal{L}$ but the boundary conditions are not, e.g. Example 2.5(iii) above. In this case the operator is said to be *formally self-adjoint.*

[5] Joseph-Louis Lagrange (1735-1813) was a brilliant Italian-born French mathematician and astronomer. He made significant contributions in many branches of science, in particular to analysis, number theory, and classical and celestial mechanics. Note that France has an incredible history in mathematics and engineering - if you are ever in Paris, go to the Eiffel Tower and look at the names engraved on each side of the lower part of the tower. You can also see this on wiki: `http://en.wikipedia.org/wiki/List_of_the_72_names_on_the_Eiffel_Tower` and note that Lagrange is present!

[6] We have already mentioned Green - he was the Nottingham miller!

Example 2.6 *Referring to Example 2.5 above, determine which of (i)-(vi) are self adjoint.*
(i) $\mathcal{L}^* = \mathcal{L}$ *and* $\mathcal{B}^* = \mathcal{B}$*. So self-adjoint.*
(ii) $\mathcal{L}^* \neq \mathcal{L}$ *so not self-adjoint.*
(iii) Although $\mathcal{L}^* = \mathcal{L}$, $\mathcal{B}^* \neq \mathcal{B}$ *so not self-adjoint. (This is called only formally self-adjoint)*
(iv)-(vii) See Question 5 on Example Sheet 2.

In general, mixed BCs do not lead to self-adjoint operators, although if $p(x)$ =constant, then periodic BCs (which *are* mixed) do yield self-adjoint operators.

For complex linear operators (i.e. where p, r and q are complex functions, the conditions for self-adjointness are complicated. They are in fact that $p(x)$ has to be a real function with $p' = \mathrm{Re}(r)$ and $2\mathrm{Im}(q) = (\mathrm{Im}(r))'$ (here Re and Im denote the real and imaginary parts of the function respectively. See Question 8 on Example Sheet 2.

For simplicity let us restrict attention from now on to real operators, so that p, q and r are real functions. Of course the functions u and v could still be complex. We see then from the form of the general adjoint operator in (2.69) that a *necessary* condition for a second order differential operator to be formally self-adjoint (i.e. $\mathcal{L}^* = \mathcal{L}$) is that $r(x) = p'(x)$. The operator can then be written as

$$\mathcal{L}u = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x). \tag{2.73}$$

In this case Green's second identity simplifies to

$$\langle v, \mathcal{L}u\rangle - \langle \mathcal{L}v, u\rangle = \Big[p(x)(\overline{v(x)}u'(x) - \overline{v'(x)}u(x))\Big]_a^b \tag{2.74}$$

and in order to derive the adjoint BCs $\mathcal{B}^*$ satisfied by v we choose them such that the right hand side of (2.74) is zero. For a given $\mathcal{B}$ satisfied by u this defines the conditions $\mathcal{B}^*$ satisfied by v. This also shows that even if $\mathcal{L}^* = \mathcal{L}$, we may not have $\mathcal{B}^* = \mathcal{B}$. *We therefore reiterate here that the property of self-adjointness requires properties of BCs, not just the operator itself.* In particular it could be that the operator is formally self-adjoint so that $\mathcal{L}^* = \mathcal{L}$ but the required adjoint BCs in order to ensure that (2.74) is satisfied are *not* the same as the original BCs.

2.3.4 Forcing formal self-adjointness

In fact we can use what we know about first order ODEs in order to write *all* second order ODEs in a formal self-adjoint form as we show in section 2.11. However, even though we can do this, we note that the BCs may not lead to a fully self-adjoint operator.

Let us now consider a very special type of BVP, the so-called *Sturm-Liouville problems.*

2.4 Sturm-Liouville (S-L) eigenvalue problems

The problems that we will be concerned with in this section are the so-called Sturm-Liouville[7] ODE BVPs which take the form of an operator in S-A form, i.e.

$$\mathcal{L}u = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) \tag{2.75}$$

with $x \in [a, b]$: this could also be the whole real line or the semi-infinite domain, e.g. $x \in [0, \infty)$. The functions p, q and μ are real and continuous. In general p is non-negative (and usually positive almost everywhere) and μ is positive. We will associate some homogeneous BCs with this ODE shortly.

If the operator if NOT in the form (2.75), the problem is NOT a S-L problem.

Naturally arising problems in the physical sciences often lead to the equation

$$\mathcal{L}\phi(x) + \lambda\mu(x)\phi(x) = 0 \tag{2.76}$$

where $\mu(x)$ arises via the physics in the derivation of the governing equations. This is accompanied by boundary conditions. Solutions to this problem exist only for particular values of λ say λ_k (the *eigenvalues*), for $k = 1, 2, 3, ...$, with associated solution $\phi_k(x)$ (the *eigenfunctions*). The eigenvalues and eigenfunctions are usually of great physical interest and significance.

Regular S-L problem

The *regular Sturm-Liouville eigenvalue problem* is defined by the ODE

$$\mathcal{L}\phi(x) + \lambda\mu(x)\phi(x) = 0 \tag{2.77}$$

with $\mathcal{L}$ as defined in (2.75) and homogeneous boundary conditions of the form

$$\mathcal{B} = \{\alpha_1\phi(a) + \alpha_2\frac{d\phi}{dx}(a) = 0, \beta_1\phi(b) + \beta_2\frac{d\phi}{dx}(b) = 0\}, \tag{2.78}$$

where α_n, β_n are real, $x \in [a, b]$ (a finite interval), the functions $p(x), q(x)$ and $\mu(x)$ are real and continuous, $p'(x)$ exists and is continuous, and $p(x), \mu(x)$ are positive.

We note that the BCs here are not mixed. This is important as we shall see later.

Also, note that the fact that α_n and β_n are real ensures the self-adjointness (i.e. full S-A not just formal) of the problem: **Regular S-L problems are fully self-adjoint!** (but note the many conditions required for regularity!)

Singular S-L problem

We sometimes want to relax the conditions above since physical problems are often not quite as constrained. We will not be too prescriptive here about the type of non-regular

[7]named after the French mathematicians Jacques Charles Francois Sturm (1803-1855) and Joseph Liouville (1809-1893) who studied these in the early 19th century. This work was very influential for the theory of ODEs.

S-L problem we consider but will occasionally refer to them as we proceed. What often happens in singular S-L problems is that e.g. $p(x)$ vanishes at one of the end points of the interval $[a, b]$ or e.g. the boundary conditions are not quite of the form in (2.78), e.g. periodic conditions with $p =$constant.

2.4.1 Theorems associated with Regular S-L problems for ODEs

For a *regular* S-L ODE problem we have the following important theorems:

1. All eigenvalues λ are real
2. There are an infinite number of eigenvalues
$$\lambda_1 < \lambda_2 < ... < \lambda_n < \lambda_{n+1} < .... \tag{2.79}$$
There is a smallest eigenvalue λ_1 but no largest eigenvalue: $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue λ_n there is an eigenfunction say $\phi_n(x)$ which is unique to within an arbitrary multiplicative constant. $\phi_n(x)$ has $n-1$ zeros for $x \in (a, b)$.
4. The eigenfunctions form a *complete set*. This means that any piecewise smooth function $g(x)$ can be represented in the form
$$g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \tag{2.80}$$
Importantly this series is convergent, converging to $(g(x+)+g(x-))/2$ where $x+$ and $x-$ denote approaching x from above and below respectively. Thus for continuous functions this series converges to $g(x)$.
5. Eigenfunctions associated with different eigenvalues are orthogonal relative to the weight function $\mu(x)$. I.e. if $\lambda_m \neq \lambda_n$ $(m \neq n)$
$$\int_a^b \mu(x)\overline{\phi_m(x)}\phi_n(x) = 0 \tag{2.81}$$

If the S-L problem is singular, these theorems may still hold, but not necessarily.

Since this is a course on Green's functions rather than ODEs, we do not go into the details of these theorems too much. Although let us discuss a simple example to illustrate their usefulness in a simple important case.

2.4.2 A model example to illustrate the theorems

Example 2.7 *We set $p = 1, q = 0$ in (2.75) and thus consider the associated eigenvalue problem for the Laplacian operator in one dimension, with the weighting $\mu(x) = 1$. These eigenfunctions are therefore appropriate for the heat equation and wave problems in one space dimension as you will have seen in MT20401. The eigenfunction equation is*

$$\phi''(x) + \lambda\phi(x) = 0 \tag{2.82}$$

for $x \in [0, L]$. Let us consider the case when $\mathcal{B} = \{\phi(0) = 0, \phi(L) = 0\}$. This is therefore a regular S-L problem.

The solutions of this problem take the form (see question 2 on Example Sheet 3)

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \tag{2.83}$$

with $n = 1, 2, ...$ and therefore the solution is of the form of a Fourier sine series:

$$u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x) \tag{2.84}$$

for some real coefficients a_n (i.e. $u(x)$ is a real function).

Real eigenvalues

In determining this result you usually assume real eigenvalues. Seeking complex ones can be hard! This theorem tells us that once we have found all of the real eigenvalues we can stop as there are no complex ones!

Eigenvalue ordering

We see that indeed we have an infinite number of eigenvalues $\lambda_n = (n\pi/L)^2$ and that indeed we have a smallest: $(\pi/L)^2$, but no largest.

Zeros of eigenfunctions

Eigenfunctions $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ should have $n-1$ zeros inside (a, b). This is clearly true.

Eigenfunction convergence

The eigenfunction expansion (2.84) is a Fourier Sine series and we know (from MT20401) via Fourier's convergence theorem that any piecewise smooth function can be represented as so. Remember that this helped in MT20401 as we could use separation of variables successfully in many cases.

Eigenfunction orthogonality

The weight function $\mu(x)$ here is simply unity. We can use the inner product notation and we know that if $m \neq n$

$$\langle \phi_n, \phi_m \rangle = \int_0^L \sin(n\pi x/L) \sin(m\pi x/L) = 0.$$

Orthogonality of the eigenfunctions enables the coefficients a_n to be determined in a straightforward manner as

$$a_n == \frac{\langle u(x), \phi_n \rangle}{\langle \phi_m, \phi_m \rangle} = \frac{\int_0^L u(x)\phi_n \; dx}{\int_0^L \phi_m^2(x) \; dx}.$$

2.4.3 Proofs of S-L Theorems 1. and 5.

Some of the Theorems 1-5 above relating to S-L problems are difficult to prove. Two of them are relatively simple however: Theorem 1 pertaining to real eigenvalues and Theorem 5 pertaining to orthogonal eigenfunctions. For reasons that will become clear shortly, we will prove Theorem 5 first.

Theorem 5 - A modified inner product and orthogonal eigenfunctions

Take two eigenfunctions, ϕ and ψ (associated with a regular S-L operator $\mathcal{L}$) corresponding to distinct eigenvalues λ and ν say, so that

$$\mathcal{L}\phi = -\lambda\mu(x)\phi(x), \qquad \mathcal{L}\psi = -\nu\mu(x)\psi(x). \tag{2.85}$$

Since the operator is regular S-L, it is S-A so that

$$\begin{aligned} 0 &= \langle \mathcal{L}\phi, \psi \rangle - \langle \phi, \mathcal{L}\psi \rangle, & (2.86)\\ &= -\overline{\lambda}\langle \mu\phi, \psi \rangle + \nu \langle \phi, \mu\psi \rangle, & (2.87)\\ &= -\lambda\langle \mu\phi, \psi \rangle + \nu \langle \phi, \mu\psi \rangle, & (2.88)\\ &= (\nu - \lambda) \int_a^b \mu(x)\overline{\phi}(x)\psi(x) \; dx & (2.89) \end{aligned}$$

and therefore since the eigenvalues are distinct, using standard inner product notation

$$\int_a^b \mu(x)\overline{\phi(x)}\psi(x) \; dx = \langle \phi, \mu\psi \rangle = 0.$$

I.e. the *weighted* eigenfunctions are orthogonal with respect to the usual inner product defined in (2.51).

Given the above however, it is convenient to define a modified inner product

$$\langle f, g \rangle = \int_a^b \mu(x)\overline{f(x)}g(x) \; dx. \tag{2.90}$$

Then the eigenfunctions themselves are orthogonal with respect to this newly defined inner product. **Unless otherwise stated, we assume that the weighting $\mu(x) = 1$.**

Theorem 1 - Real eigenvalues

Take the eigenvalue λ corresponding to the eigenfunction $\phi(x)$ associated with a regular S-L operator $\mathcal{L}$. We have, working with the modified inner product (2.90) above,

$$\langle \mathcal{L}\phi, \phi \rangle = -\langle \lambda\phi, \phi \rangle, \tag{2.91}$$
$$= -\overline{\lambda}\langle \phi, \phi \rangle. \tag{2.92}$$

Also we have

$$\langle \phi, \mathcal{L}\phi \rangle = \langle \phi, -\lambda\phi \rangle, \tag{2.93}$$
$$= -\lambda\langle \phi, \phi \rangle. \tag{2.94}$$

Therefore, since problem is S-A,

$$0 = \langle \mathcal{L}\phi, \phi \rangle - \langle \phi, \mathcal{L}\phi \rangle, \tag{2.95}$$
$$= (\lambda - \overline{\lambda})\langle \phi, \phi \rangle \tag{2.96}$$

so that

$$\lambda = \overline{\lambda}$$

and therefore the eigenvalues must be real.

It transpires that this result holds for regular S-L problems, singular S-L problems in the sense that $p(x) = 0$ at an end point, *and* also if the BCs are periodic.

2.5 Existence and uniqueness of BVPs for ODEs: The Fredholm Alternative

Recall the following theorem for *Initial Value Problems* associated with ODEs:

Theorem 2.1 *Given the ODE*

$$u''(t) + p(t)u'(t) + q(t)u(t) = f(t)$$

subject to ICs $u(t_0) = x_0, u'(t_0) = v_0$, *if* $p(t), q(t)$ *and* $f(t)$ *are continuous on the interval* $[a, b]$ *containing* t_0, *the solution of the IVP exists and is unique.*

Unfortunately the situation is not as simple for BVPs. It can be the case that BVPs have (i) no solution, (ii) a unique solution or (iii) infinitely many solutions! Let us first state the following theorem which guarantees the existence of two fundamental solutions to a homogeneous ODE:

Theorem 2.2 *Given the homogeneous ODE*

$$p(x)u''(x) + r(x)u'(x) + q(x)u(x) = 0,$$

with p, r *and* q *continuous and* p *never zero on the domain of interest, there always exist two fundamental solutions* $u_1(x)$ *and* $u_2(x)$ *which generate the general solution* $u(x) = c_1u_1(x) + c_2u_2(x)$.

Therefore whether a solution exists or not depends on the BCs. As a very simple example to illustrate that BVPs can have a unique solution, no solution or infinitely many solutions, let us consider the following problem.

Example 2.8 *Consider the homogeneous ODE*

$$u''(x) + u(x) = 0$$

subject to inhomogeneous BCs

$$\begin{array}{ll} (i) & u(0) = 1, \;\; u(\pi) = 1, \\ (ii) & u(0) = 1, \;\; u(\pi/2) = 1, \\ (iii) & u(0) = 1, \;\; u(2\pi) = 1. \end{array}$$

The fundamental solutions are $\cos x$ *and* $\sin x$ *so that* $u(x) = c_1 \cos x + c_2 \sin x$.

The BCs in (i) are inconsistent and therefore there is no solution.

The BCs in (ii) yield the unique solution $u(x) = \cos x + \sin x$.

The BCs in (iii) yield the infinite family of solutions $u(x) = \cos x + c_2 \sin x$ *where* c_2 *is arbitrary.*

Let us now consider the case of an inhomogeneous ODE subject to homogeneous BCs (recall that this is the main thrust of our enquiries in this course). We are able to state a rather general theorem regarding existence and uniqueness of solutions to this problem. We consider the additional effect of inhomogeneous BCs in section 2.12. We shall consider an example which illustrates the main issues that arise.

Example 2.9 *Find the solution to the ODE*

$$u''(x) + u(x) = f(x)$$

subject to $u(0) = u(L) = 0$.

Fundamental solutions of the homogeneous ODE are $\sin x$ *and* $\cos x$ *but remember that the general solution can be any linear combination of these and it is convenient to use* $u_1(x) = \sin x$ *and* $u_2(x) = \sin(x - L)$ *(since* $\sin(x - L) = \sin x \cos L - \cos L \sin x$*). This is convenient since they satisfy the left and right BCs respectively.*

Let us therefore write the solution to the homogeneous problem as $u(x) = c_1 \sin x + c_2 \sin(x - L)$. *We then know from* (2.29) *that the solution to the inhomogeneous problem can be written*

$$u(x) = (c_1 + v_1(x)) \sin x + (c_2 + v_2(x)) \sin(x - L) \tag{2.97}$$

where

$$v_1(x) = \int_a^x -\frac{u_2(x_0)f(x_0)}{p(x_0)W(x_0)}\ dx_0 = -\int_0^x \frac{\sin(x_0 - L)f(x_0)}{\sin L}\ dx_0, \tag{2.98}$$

$$v_2(x) = \int_a^x \frac{u_1(x_0)f(x_0)}{p(x_0)W(x_0)}\ dx_0 = \int_0^x \frac{\sin x_0 f(x_0)}{\sin L}\ dx_0. \tag{2.99}$$

noting that $W(x) = \sin x \cos(x - L) - \sin(x - L) \cos x = \sin(x - (x - L)) = \sin L$ *is a constant.*

Now impose boundary conditions, with $u(0) = 0$ *giving*

$$c_2 = 0, \tag{2.100}$$

whilst $u(L) = 0$ *gives*

$$c_1 \sin L = \int_0^L \sin(x_0 - L)f(x_0)\ dx_0. \tag{2.101}$$

This last equation giving c_1 *is perfectly valid, unless* $L = n\pi$ *which knocks out the left hand side! In that case there is then only a solution if*

$$\int_0^{n\pi} f(x_0) \sin(x_0 - n\pi)\ dx_0 = (-1)^n \int_0^{n\pi} f(x_0) \sin x_0\ dx_0 = 0.$$

Take, e.g. $L = \pi$. *Even if this condition is satisfied then there are infinitely many solutions because we can add on any multiple of* $\sin x$ *to the solution, i.e.*

$$u(x) = u_{PS}(x) + c \sin x$$

Often the only solution to the homogeneous BVP is the zero solution. When $L = \pi$ above we see that $\sin x$ is a *non trivial* solution to the *homogeneous* BVP. This corresponds to an existence of a so-called *zero eigenvalue*. Interestingly, this tells us something very special about the existence and uniqueness of the solution to the *inhomogeneous* problem as we now describe via a general theorem. We will return to the example above after we have stated the theorem to see how it aligns with the theorem.

The Fredholm Alternative for ODE BVPs

We can state the following theorem

Theorem 2.3 *We introduce the BVP consisting of the linear ODE*

$$\mathcal{L}u = p(x)u''(x) + r(x)u'(x) + q(x)u(x) = f(x)$$

subject to homogeneous BCs $\mathcal{B}$ *with* $p(x), r(x), q(x)$ *and* $f(x)$ *real and continuous on the interval* $[a, b]$, *with* $p(x) \neq 0$ *on* $[a, b]$. *Consider also the associated homogeneous adjoint problem*

$$\mathcal{L}^* v = 0$$

with associated homogeneous BCs $\mathcal{B}^*$.

Then ***EITHER***

1. *If the only solution to the homogeneous adjoint problem is the trivial solution* $v(x) = 0$ *then the solution to the inhomogeneous problem* $u(x)$ *exists and is unique*
   OR

2. *If there are non-trivial solutions to the homogeneous adjoint problem* $v(x) \neq 0$ *then* ***either***
   - *There are infinitely many solutions if* $\int_a^b \overline{v(x)} f(x) = 0$,
     or
   - *There is no solution if* $\int_a^b \overline{v(x)} f(x) \neq 0$.

See question 1 of Example Sheet 4 for some more details of this theorem.

Clearly if the problem is self-adjoint then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{B}^* = \mathcal{B}$ and so the adjoint homogeneous problem is simply the homogeneous version of the original BVP.

Example 2.10 *How is the Fredholm Alternative Theorem consistent with example 2.9?*

Firstly the BVP is S-A and so the adjoint problem is merely the homogeneous version of the original problem. It therefore has solution

$$v(x) = d_1 \sin x + d_2 \cos x.$$

Imposing $v(0) = 0$ *yields* $d_2 = 0$ *and* $v(L) = 0$ *gives*

$$d_1 \sin L = 0 \tag{2.102}$$

which means that if $L \neq n\pi$ *we need* $d_1 = 0$ *and therefore the only solution to this problem is the trivial one* $v(x) = 0$. *From the Fredholm Alternative Theorem, the solution* $u(x)$ *to the original problem is unique.*

If $L = n\pi$ *then* (2.102) *is trivially satisfied for any* d_1. *Therefore a non-trivial solution to the homogeneous adjoint problem is*

$$v(x) = \sin x$$

which from the Fredholm Theorem means that if

$$\int_0^L \sin x_0 f(x_0)\ dx_0 = 0$$

there are infinitely many solution to the original problem, whereas if

$$\int_0^L \sin x_0 f(x_0)\ dx_0 \neq 0$$

there are no solutions. This corresponds exactly to the Example above.

The existence of a non-trivial solution to the homogeneous problem corresponds to the existence of a zero eigenvalue. We will see later in section 2.13 that when this happens, the standard Green's function (as we will define shortly) does not exist and a modified form has to be considered.

One final point. This theorem allows us to say a great deal about the existence and uniqueness of solutions to inhomogeneous ODEs without actually having to solve the problems! We illustrate this with an example.

Example 2.11 *For the following ODE/BC pairings use the Fredholm Alternative to state if a solution exists and if so if it is unique (note that you do not solve the inhomogeneous BVP in order to show this!).*

$$u''(x) + \psi u(x) = \sin x$$

with

(a)	$\psi = 1,$	$\mathcal{B} = \{u(0) = 0, u(\pi) = 0\}$
(b)	$\psi = 1,$	$\mathcal{B} = \{u'(0) = 0, u'(\pi) = 0\}$
(c)	$\psi = -1,$	$\mathcal{B} = \{u(0) = 0, u(\pi) = 0\}$
(d)	$\psi = 2,$	$\mathcal{B} = \{u(0) = 0, u(\pi) = 0\}$

All problems are self-adjoint.
(a) A non-trivial solution to the homogeneous problem is $v(x) = \sin x$*. But we note that*

$$\int_0^\pi \sin^2 x\ dx \neq 0$$

so therefore a solution does not exist. (Verify this yourself by trying to solve the inhomogeneous problem).

Parts (b)-(d) are considered in question 2 on Example Sheet 4.

The Fredholm Alternative for Linear Systems

As perhaps should be expected, the Fredholm Alternative is far more general than just governing ODEs.

Theorem 2.4 *We introduce the linear system*

$$\mathbf{L}\mathbf{u} = \mathbf{f}$$

where $\mathbf{L}$ *is an* $m \times n$ *matrix and* $\mathbf{u}$ *and* $\mathbf{f}$ *are* $1 \times n$ *vectors where* $\mathbf{f}$ *is given and* $\mathbf{u}$ *is unknown. Consider the homogeneous adjoint (transpose) problem*

$$\mathbf{L}^T\mathbf{v} = 0$$

where superscript T *denotes the transpose of the matrix. Then* ***EITHER***

1. *If the only solution to the homogeneous adjoint problem is the trivial solution* $\mathbf{u} = \mathbf{0}$ *then the solution to the inhomogeneous problem* $\mathbf{u}$ *exists and is unique*
   OR

2. *If there are non-trivial solutions to the homogeneous adjoint problem* $\mathbf{v} \neq 0$ *then* ***either***

   - *There are infinitely many solutions if* $\mathbf{v} \cdot \mathbf{f} = 0$*,*
     or
   - *There is no solution if* $\mathbf{v} \cdot \mathbf{f} \neq 0$*.*

It transpires that this theorem is useful for linear integral equations in later sections.

2.6 What is a Green's function?

Having addressed many aspects of ODE theory, let us now focus on the main issue of this course - defining and using Green's functions. The method of Green's functions is simply a method in order to solve inhomogeneous BVPs. One of the interesting aspects of Green's functions is that they enable the solution to be written down in a very general form for a variety of forcing functions. The Green's function also often corresponds to something physically important. We have already seen one example where the Green's function enables the solution to be written in general form in section 2.2. At that time we did not think of it as a Green's function, it was considered merely as an "influence" function for the inhomogeneous forcing term $f(x)$.

2.7 Green's functions for Regular S-L problems via eigenfunction expansions

Consider again the regular S-L problem of the form

$$\mathcal{L}u = f(x) \tag{2.103}$$

with $\mathcal{L}$ given by (2.75), $x \in [a, b]$ and u is subject to two homogeneous BCs of the form (2.78). Also consider the related eigenvalue problem

$$\mathcal{L}u = -\lambda\mu(x)u \tag{2.104}$$

with some appropriately chosen $\mu(x)$. We can solve (2.103) by posing an eigenfunction expansion of the form (see Example Sheet 3)

$$u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x). \tag{2.105}$$

This can be differentiated term-by-term (see MT20401) so that, applying $\mathcal{L}$ we find

$$\mathcal{L}u(x) = -\sum_{n=1}^{\infty} a_n\lambda_n\mu(x)\phi_n(x) = f(x). \tag{2.106}$$

Let us multiply by $\phi_m(x)$ and integrate over the domain $x \in [a, b]$. The orthogonality of the eigenfunctions (with respect to the weight $\mu(x)$) allows us to then show that

$$-a_n\lambda_n = \frac{\int_a^b f(x)\phi_n(x)\ dx}{\int_a^b \phi_n^2(x)\mu(x)\ dx}. \tag{2.107}$$

Therefore

$$u(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \left(\frac{-\phi_n(x)\phi_n(x_0)}{\lambda_n \int_a^b \phi_n^2\mu(x_1)\ dx_1} \right)\ dx_0 \tag{2.108}$$

and so we recognize that we can write

$$u(x) = \int_a^b f(x_0)G(x, x_0)\ dx_0 \tag{2.109}$$

where

$$G(x,x_0) = \sum_{n=1}^{\infty}\left(\frac{-\phi_n(x)\phi_n(x_0)}{\lambda_n\int_a^b \phi_n^2(x_1)\mu(x_1)\ dx_1}\right) \tag{2.110}$$

which is therefore an eigenfunction expansion of the Green's function. Note that $G(x,x_0) = G(x_0,x)$ in this setting.

We note that the definition (2.110) would run into difficulty if one of the eigenvalues is zero (i.e. if there is a non-trivial solution to the homogeneous adjoint problem!). We return to this point later on in section 2.13.

Example 2.12 *Let us return to the familiar example*

$$\mathcal{L}u = \frac{d^2u}{dx^2} = f(x) \tag{2.111}$$

with $u(0) = u(L) = 0$ *and the related eigenvalue problem*

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \tag{2.112}$$

with $\phi(0) = \phi(L) = 0$. *We already know from example 2.7 that* $\lambda_n = (n\pi/L)^2$ *and* $\phi_n(x) = \sin(n\pi x/L)$ *with* $n = 1, 2, 3....$ *Therefore with reference to the theory above,* $u(x)$ *is given by*

$$u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x), \tag{2.113}$$

$$= \int_0^L f(x_0)G(x,x_0)\ dx_0 \tag{2.114}$$

where

$$G(x,x_0) = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin(n\pi x/L)\sin(n\pi x_0/L)}{(n\pi/L)^2}. \tag{2.115}$$

Finally we ask, how is this representation of the Green's function in terms of eigenfunctions related to the form derived in (2.41) or (2.42) above. They must be equivalent! We discuss this in question 4 on Example Sheet 4.

2.8 Green's functions for Regular S-L problems using a direct approach

For problems of regular (and some singular) S-L type we have shown above in equations (2.109)-(2.110) that the equation

$$\mathcal{L}u = f(x) \tag{2.116}$$

has the solution

$$u(x) = \int_a^b G(x, x_0) f(x_0) \, dx_0 \tag{2.117}$$

for some appropriately defined function $G(x, x_0)$ which we have termed the *Green's function*. We have an eigenfunction representation for the Green's function defined in (2.110). This approach shows that the Green's function exists provided that there is no "zero" eigenvalue, see section 2.13. We can obtain the Green's function for S-L using variation of parameters. We will describe this shortly but first we need some discussion of a few rather unusual "functions".

2.8.1 The Dirac delta "function"

The representation of the solution in the form (2.117) shows that the source term $f(x)$ represents a forcing at all of the points at which it is non-zero. We can isolate the effect of each point in the following manner. First we take a function $f(x)$ and consider splitting it up in order to take into account the separate contributions from intervals of width Δx_i such as we do when carrying out the process of Riemann integration, see figure 4. Consider decomposing the function $f(x)$ into a linear combination of unit pulses starting at the points x_i and being of width Δx_i, see figure 5.

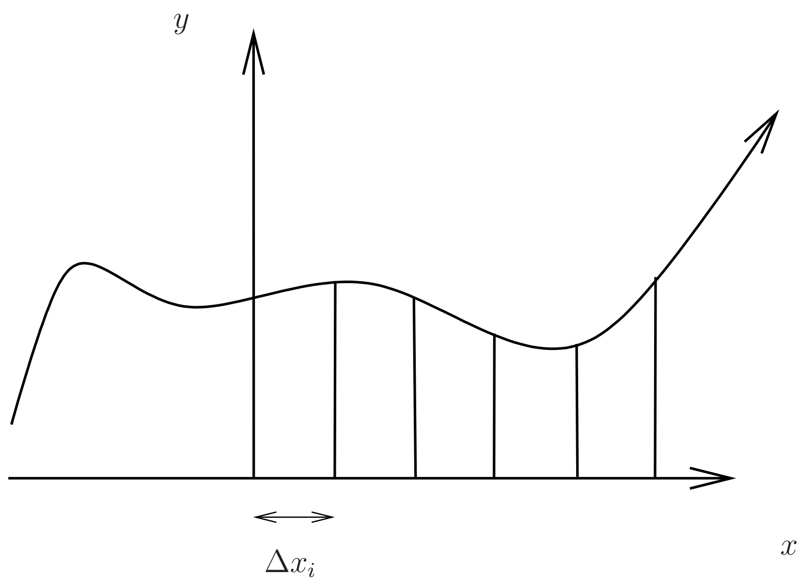

Figure 4: Figure depicting the partition of a function $f(x)$ into linear contributions of unit pulses, similarly to the process of Riemann integration.

So we would write

$$f(x) \simeq \sum_i f(x_i) \times (\text{unit pulse starting at } x = x_i). \tag{2.118}$$

and we know that this is only a good approximation if the intervals are small (infinitesimal in fact!)

Indeed this is very similar to something like an integral. Only the Δx_i is missing! Let us now introduce this and a limiting process in the following manner:

$$f(x) = \lim_{\Delta x_i \to 0} \sum_i f(x_i) \frac{(\text{unit pulse})}{\Delta x_i} \Delta x_i \tag{2.119}$$

$$= \lim_{\Delta x_i \to 0} \sum_i f(x_i)(\text{Dirac pulse}) \Delta x_i. \tag{2.120}$$

We now appear to have introduced a strange object - what we have termed here the *Dirac pulse*. It has height $1/\Delta x_i$ and width Δx_i. We picture this in figure 6. Note that this pulse has unit area. In the limit as $\Delta x_i \to 0$ this pulse represents a concentrated pulse of infinite amplitude located at a single point. It is not really a function but is often termed a *generalized function*. We will call this object the Dirac Delta function[8], which when located at the point $x = x_i$ is written as $\delta(x - x_i)$. It cannot be written down in the form $\delta(x - x_i) = ...$ We think of this object as a concentrated source or *impulsive force*, and according to (2.120), in the limit, we have the definition

$$f(x) = \int_{-\infty}^{\infty} f(x_i)\delta(x - x_i)\ dx_i. \tag{2.121}$$

The interval of integration here is all x_i. The property in (2.121) is known as the *sifting property* of the Dirac delta function. The dirac delta function can be thought of as the limit of the sequence of various different functions, not only the rectangular type depicted above.

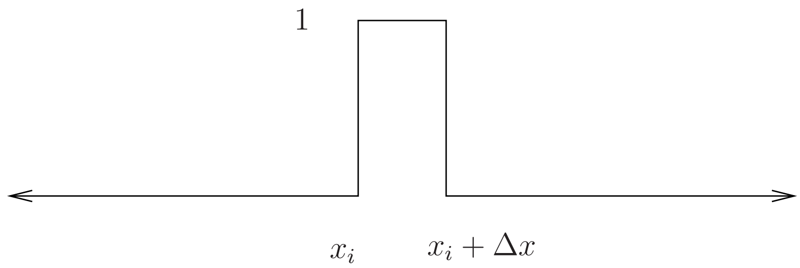

Figure 5: A unit pulse.

We now note some important properties of the function. Firstly, we note that with $f(x) = 1$

$$1 = \int_{-\infty}^{\infty} \delta(x - x_i)\ dx_i. \tag{2.122}$$

[8]Named after the brilliant twentieth century mathematical physicist Paul Dirac (1902-1984)

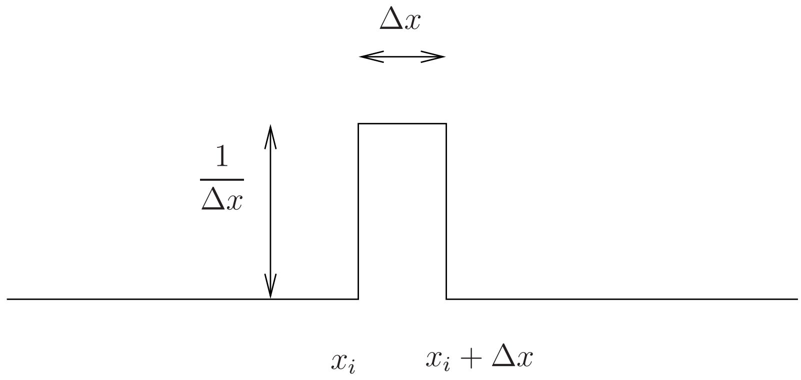

Figure 6: A Dirac pulse.

The function is even, $\delta(x - x_i) = \delta(x_i - x)$. Furthermore it is strongly linked with the Heaviside function $H(x - x_i)$ which we have already defined above, but repeat here for completeness, as

$$H(x - x_i) = \begin{cases} 1, & x > x_i, \\ 0, & x < x_i \end{cases} \tag{2.123}$$

via the expression

$$H(x - x_i) = \int_{-\infty}^{x} \delta(y - x_i)\ dy. \tag{2.124}$$

The Heaviside function is not defined at $x = x_i$ - we have freedom to choose its value there. Usually the most convenient is to choose its value as 1/2. This is the average of the limit from both sides of x_i of course.

Finally we note that

$$\delta[c(x - x_i)] = \frac{1}{|c|}\delta(x - x_i) \tag{2.125}$$

for some constant c. For proofs of the last two properties see question 6 on Example Sheet 4.

The introduction of this function now allows us to determine an equation governing the Green's function.

2.8.2 Relationship between the Dirac delta function and the Green's function

Given the solution (2.117), we note that the Green's function $G(x, x_0)$ is an "influence function" for the source $f(x)$. As an example, let us suppose that $f(x)$ is now a concentrated source at $x = s$, i.e. $f(x) = \delta(x - s)$ with $a < s < b$. This then gives

$$u(x) = \int_a^b \delta(x_0 - s) G(x, x_0)\ dx_0 = G(x, s) \tag{2.126}$$

by the sifting property (2.121). We therefore obtain the fundamental interpretation of the Green's function: it is the response at x due to a concentrated source at x_0:

$$\mathcal{L}_x G(x, x_0) = \delta(x - x_0) \tag{2.127}$$

where the subscript x on the operator ensures that we know that the derivatives are with respect to x. The source position x_0 is a parameter in the problem.

We can check that (2.117) satisfies (2.116) via the definition of the Green's function (2.127) by operating on each side of (2.117) with $\mathcal{L}_x$ to give

$$\mathcal{L}_x u = \int_a^b f(x_0)\mathcal{L}_x[G(x-x_0)]\ dx_0 = \int_a^b f(x_0)\delta(x-x_0)\ dx_0 = f(x) \tag{2.128}$$

via the sifting property (2.121).

2.8.3 Boundary conditions for the Green's function BVP

If we take (2.127) together with appropriate homogeneous boundary conditions as an independent definition of the Green's function (which we shall!) then we also want to derive the solution starting with this independent definition. Start with Green's identity in 1D for ODEs with operators in S-L form (2.75), which turns out to be

$$\langle v, \mathcal{L}u\rangle - \langle \mathcal{L}v, u\rangle = \left[p(x)\left(\overline{v}\frac{du}{dx} - u\overline{\frac{dv}{dx}}\right)\right]_a^b. \tag{2.129}$$

Let $v = G(x, x_0)$. The right hand side vanishes as long as we choose the homogeneous BCs for the Green's function to be the same as those for the original problem associated with u. Then

$$\langle G(x, x_0), \mathcal{L}u(x)\rangle - \langle \mathcal{L}G(x, x_0), u(x)\rangle = 0. \tag{2.130}$$

Using the definition of the Dirac delta function and interchanging variables we obtain

$$u(x) = \int_a^b f(x_0)\overline{G(x_0, x)}\ dy. \tag{2.131}$$

For *regular S-L operators* the Green's function is (Hermitian) symmetric ($\overline{G(x_0, x)} = G(x, x_0)$) as we shall show shortly, so that

$$u(x) = \int_a^b f(x_0)G(x, x_0)\ dx_0. \tag{2.132}$$

2.8.4 Reciprocity and symmetry of the Green's function for fully S-A problems.

Let us suppose that the BVP is *fully self-adjoint* (e.g. a regular S-L problem). Let us once again use (2.129) and let $u = G(x, x_1)$ and $v = G(x, x_2)$. Both satisfy homogeneous boundary conditions of the form (2.78). Furthermore since $\mathcal{L}_x u = \delta(x-x_1)$ we use Green's second identity to find

$$\int_a^b \left[\overline{G(x, x_2)}\delta(x-x_1) - \overline{\mathcal{L}G(x, x_2)}G(x, x_1)\right]\ dx = 0. \tag{2.133}$$

Therefore from the sifting property of the Dirac function,

$$\overline{G(x_1, x_2)} = \int_a^b \overline{\mathcal{L}G(x, x_2)} G(x, x_1)\ dx \tag{2.134}$$

$$= \overline{\int_a^b \mathcal{L}G(x, x_2) \overline{G(x, x_1)}\ dx} \tag{2.135}$$

$$= \overline{\int_a^b \delta(x - x_2) \overline{G(x, x_1)}\ dx} \tag{2.136}$$

$$= \overline{\overline{G(x_2, x_1)}} \tag{2.137}$$

$$= G(x_2, x_1) \tag{2.138}$$

Note that this is all reliant on the fact that the operator is fully self-adjoint. If it is not, much of the above theory has to be modified as we shall see in section 2.10.

Physically, the property (2.138) says that the response at x_1 due to a concentrated source at x_2 is the same as the response at x_2 due to a concentrated source at x_1. This is *not* immediately physically obvious!

2.8.5 Jump conditions at $x = x_0$

The Green's function can be determined from the governing equation (2.127). For $x < x_0$, $G(x, x_0)$ satisfies this equation with a homogeneous BC at $x = a$. Similarly for $x > x_0$ with a homogeneous BC at $x = b$. What happens at the point $x = x_0$? We need to consider the type of singularity that arises in (2.127) with reference to the property (2.124). Suppose firstly that $G(x, x_0)$ has a jump discontinuity at $x = x_0$ (a property shared by the Heaviside function $H(x - x_0)$). Then $dG(x, x_0)/dx$ would have a delta function singularity and so $d^2G(x, x_0)/dx^2$ would be more singular than the actual right hand side of (2.127). Therefore we conclude that $G(x, x_0)$ must be continuous at $x = x_0$ which we denote by

$$[G(x, x_0)]_{x=x_0^-}^{x=x_0^+} = 0 \tag{2.139}$$

where x_0^+ and x_0^- denote approaching $x = x_0$ from above and below respectively, e.g. $x_0^+ = \lim_{\epsilon \to 0 x_0 + \epsilon}$, $x_0^- = \lim_{\epsilon \to x_0 - \epsilon}$ with $\epsilon > 0$.

On the other hand $dG(x, x_0)/dx$ *does* have a jump discontinuity at $x = x_0$. In order to illustrate this for S-L problems, integrate

$$\mathcal{L}u = f(x)$$

where $\mathcal{L}$ is the S-L operator (2.75), between $x = x_0^-$ and $x = x_0^+$ to give (since q and G are continuous at $x = x_0$)

$$\left[p(x)\frac{dG}{dx}\right]_{x=x_0^-}^{x=x_0^+} = 1. \tag{2.140}$$

Since $p(x)$ is a continuous function this then gives

$$\left[\frac{dG}{dx}\right]_{x=x_0^-}^{x=x_0^+} = \frac{1}{p(x_0)}. \tag{2.141}$$

2.8.6 Summary: Green's function for regular S-L problems

Given a regular S-L problem of the form

$$\mathcal{L}_x u = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = f(x) \tag{2.142}$$

together with homogeneous boundary conditions $\mathcal{B}$ at $x = a, b$, the corresponding Green's function will be defined by

$$\mathcal{L}_x G(x, x_0) = \delta(x - x_0) \tag{2.143}$$

together with the same homogeneous boundary conditions $\mathcal{B}$ at $x = a, b$, and the following conditions at $x = x_0$:

$$[G(x, x_0)]_{x=x_0^-}^{x=x_0^+} = 0 \tag{2.144}$$

and

$$\left[\frac{dG}{dx}\right]_{x=x_0^-}^{x=x_0^+} = \frac{1}{p(x_0)}. \tag{2.145}$$

Let us use these steps to construct the Green's function for a simple example.

Example 2.13 *Consider again the steady state heat equation*

$$\frac{d^2u}{dx^2} = f(x) \tag{2.146}$$

with $u(0) = 0, u(L) = 0$. *We can write the solution to this problem in the form*

$$u(x) = \int_0^L f(x_0)G(x, x_0)\ dx_0 \tag{2.147}$$

where the Green's function $G(x, x_0)$ *satisfies*

$$\frac{d^2G(x, x_0)}{dx^2} = \delta(x - x_0) \tag{2.148}$$

with $G(0, x_0) = 0$ *and* $G(L, x_0) = 0$.

Now that we have a governing equation for the Green's function we can easily obtain its solution for $x \neq x_0$*:*

$$G(x, x_0) = \begin{cases} a + bx, & x < x_0, \\ c + dx, & x > x_0 \end{cases} \tag{2.149}$$

but we note that the "constants" could be different for different x_0 *- the source location.*

The BC at $x = 0$ *applies for* $x < x_0$ *and imposing this gives* $a = 0$. *Similarly* $G(L, x_0) = 0$ *gives* $c + dL = 0$. *Therefore we have*

$$G(x, x_0) = \begin{cases} bx, & x < x_0, \\ d(x - L), & x > x_0. \end{cases} \tag{2.150}$$

We also know from the discussion above that $G(x, x_0)$ *is continuous at* $x = x_0$. *This gives*

$$bx_0 = d(x_0 - L). \tag{2.151}$$

The jump condition on the derivative at $x = x_0$*, gives (since* $p = 1$*)*

$$d - b = 1. \tag{2.152}$$

Solve (2.151) *and* (2.152) *to obtain*

$$b = \frac{(x_0 - L)}{L}, \qquad d = \frac{x_0}{L} \tag{2.153}$$

noting in particular the dependence on x_0 *here. This gives*

$$G(x, x_0) = \begin{cases} \frac{x}{L}(x_0 - L), & 0 \leq x \leq x_0, \\ \frac{x_0}{L}(x - L), & x_0 \leq x \leq L, \end{cases} \tag{2.154}$$

which agrees with what we found in (2.41).

In fact, for regular (and some singular) S-L problems we can derive the Green's function explicitly via the method of variation of parameters as we describe in these steps:

2.8.7 Explicit solution for the Green's function for regular S-L problems

1. Find the two independent solutions of the homogeneous equation (2.142) (i.e. the complementary function u_c) say $u_1(x)$ and $u_2(x)$.

2. Take linear combinations of these solutions in order to find a solution which satisfies the *left* (at $x = a$) and *right* (at $x = b$) homogeneous boundary conditions. Call these $u_L(x)$ and $u_R(x)$ respectively.

3. Write the Green's function as
$$G(x, x_0) = \begin{cases} c_L(x_0)u_L(x), & a \le x \le x_0, \\ c_R(x_0)u_R(x), & x_0 \le x \le b \end{cases} \tag{2.155}$$
where we have noted the explicit dependence of c_L and c_R on the source location x_0. Note that we are able to put $\le$ here because the Green's function is continuous at $x = x_0$.

4. Enforce the condition on $G(x, x_0)$ at $x = x_0$:
$$c_L(x_0)u_L(x_0) = c_R(x_0)u_R(x_0) \tag{2.156}$$

5. Enforce the condition on dG/dx at $x = x_0$:
$$c_R(x_0)\frac{du_R}{dx}(x_0) - c_L(x_0)\frac{du_L}{dx}(x_0) = \frac{1}{p(x_0)}. \tag{2.157}$$

6. Finally we can solve (2.156) and (2.157) for $c_L(x_0)$ and $c_R(x_0)$ to get
$$c_L(x_0) = \frac{u_R(x_0)}{p(x_0)W(x_0)}, \qquad c_R(x_0) = \frac{u_L(x_0)}{p(x_0)W(x_0)}, \tag{2.158}$$
where we have defined the associated Wronskian
$$W(x_0) = \begin{vmatrix} u_L(x_0) & u_R(x_0) \\ u_L'(x_0) & u_R'(x_0) \end{vmatrix}. \tag{2.159}$$
And therefore the Green's function is known immediately once u_L and u_R have been determined.

Although such an explicit form is also available when the problem is *not* of Sturm-Liouville type it needs a little more justification, see section 2.11. Let us first consider some examples which *do* conform to S-L type.

Example 2.14 *Let us reconsider the steady state heat equation from Example 2.13 where we determined the Green's function by using a direct method, instead of the variation of parameters procedure above. We note that* $p(x) = 1$ *and we have the two homogeneous solutions* $u_L(x) = x$ *and* $u_R(x) = x - L$*. Therefore since from* (2.159) *we have* $W(x_0) = x_0 - (x_0 - L) = L$*, from* (2.155) *and* (2.158) *we see that*
$$G(x, x_0) = \begin{cases} \frac{1}{L}x(x_0 - L), & 0 \le x \le x_0, \\ \frac{1}{L}x_0(x - L), & x_0 \le x \le L \end{cases} \tag{2.160}$$
which agrees with (2.154) *which we note in turn agreed with the alternative derivation of* (2.41)*. Note how quickly we could derive the Green's function with the procedure above!*

Example 2.15 *Let us consider once again the Green's function associated with the steady state heat equation*

$$\frac{d^2u}{dx^2} = f(x) \tag{2.161}$$

but now with boundary conditions $u(0) = 0, u'(L) = 0$. *The Green's function* $G(x, x_0)$ *satisfies*

$$\frac{d^2G(x,x_0)}{dx^2} = \delta(x - x_0) \tag{2.162}$$

with $G(0, x_0) = 0$ *and* $dG/dx(L, x_0) = 0$.

By the argument above, all we have to do is to find the homogeneous solutions which satisfy separately the left and right boundary conditions and we can immediately write down the Green's function. These are respectively $u_L(x) = x$ *and* $u_R(x) = 1$. *Therefore* $W(x_0) = -1$ *and*

$$G(x,x_0) = \begin{cases} -x, & 0 \le x \le x_0, \\ -x_0, & x_0 \le x \le L \end{cases}$$
$$= -xH(x_0 - x) - x_0H(x - x_0) \tag{2.163}$$

The next example is an example of a *singular* S-L problem showing that the method also works in that case.

This is the example that was inconsistent in the lectures. Note that now I have changed the forcing $f(x)$ so that the problem IS consistent . Note that the actual construction of the Green's function has not changed.

Example 2.16 *Construct the Green's function associated with the BVP consisting of the ODE*

$$x^2u''(x) + 2xu'(x) - 2u(x) = f(x) \tag{2.164}$$

together with the BCs $u(0) = 0$ *and* $u(1) = 0$. *Use the Green's function to find the solution when* $f(x) = x^2$ *and confirm this by finding the solution via standard techniques.*

First we note that the ODE is of S-L type since it may be written

$$\frac{d}{dx}\left(x^2\frac{du}{dx}\right) - 2u(x) = f(x). \tag{2.165}$$

Therefore we can appeal to all of the theory above.

Since the ODE is of Euler type we seek solutions in the form

$$u(x) = x^m \tag{2.166}$$

and therefore we require $m^2 + m - 2 = 0$ *so that* $(m+2)(m-1) = 0$ *and therefore* $m = -2$ *or* $m = 1$ *and therefore we have* $u_1(x) = 1/x^2$ *and* $u_2(x) = x$. *these can be combined to yield the solutions*

$$u_L(x) = x, \qquad u_R(x) = x - \frac{1}{x^2}. \tag{2.167}$$

We find that $p(x_0)W(x_0) = 3$ *and therefore*

$$c_L(x_0) = \frac{1}{3}u_R(x_0) = \frac{(x_0^3 - 1)}{3x_0^2}, \tag{2.168}$$

$$c_R(x_0) = \frac{1}{3}u_L(x_0) = \frac{x_0}{3} \tag{2.169}$$

so that

$$G(x, x_0) = \begin{cases} \dfrac{x}{3}\left(\dfrac{x_0^3 - 1}{x_0^2}\right), & 0 \leq x \leq x_0, \\ \dfrac{x_0}{3}\left(\dfrac{x^3 - 1}{x^2}\right), & x_0 \leq x \leq 1, \end{cases}$$

$$= \frac{x}{3}\left(\frac{x_0^3 - 1}{x_0^2}\right) H(x_0 - x) + \frac{x_0}{3}\left(\frac{x^3 - 1}{x^2}\right) H(x - x_0) \tag{2.170}$$

The solution is therefore

$$u(x) = \int_0^1 G(x, x_0) f(x_0) dx_0$$

and for us since $f(x_0) = x_0^2$*, using the (Heaviside form of the) Green's function above,*

$$\begin{aligned} u(x) &= \frac{x}{3}\int_x^1 \left(x_0^3 - 1\right)\ dx_0 + \frac{(x^3 - 1)}{3x^2}\int_0^x x_0^3\ dx_0 \\ &= \frac{x}{3}\left[\frac{1}{4}x_0^4 - x_0\right]_x^1 + \frac{1}{12}\left(x - \frac{1}{x^2}\right)x^4 \\ &= \frac{x}{3}\left(-\frac{3}{4} - \frac{1}{4}x^4 + x\right) + \frac{x^5}{12} - \frac{x^2}{12} \\ &= \frac{1}{4}x(x - 1). \end{aligned} \tag{2.171}$$

This can be confirmed using the standard solution form $u(x) = u_c(x) + u_p(x)$ *where*

$$u_c(x) = c_1 x + \frac{c_2}{x^2}.$$

It is easily verified that $u_p = x^2/4$*. BCs require* $c_2 = 0$ *and* $c_1 = -1/4$ *which recovers (2.171).*

2.9 Green's functions for the wave equation with time harmonic forcing

Consider the wave equation in one spatial dimension,

$$\frac{\partial^2 U}{\partial \xi^2} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} + F(\xi, t)$$

where c is the wavespeed and $F(\xi, t)$ is some forcing term. This applies to a number of physical systems, e.g. transverse waves on a string. Suppose that the forcing is time harmonic $F(\xi, t) = g(\xi)e^{-i\omega t}$ so that the waves are also time harmonic, i.e.

$$U(\xi, t) = u_0(\xi)e^{-i\omega t}$$

and then we have

$$u_0''(\xi) + k^2 u_0(\xi) = g(\xi)$$

where $k = \omega/c$ is the wavenumber (dimensions $1/L$)and ω is the frequency (dimensions $1/T$). Note that using a complex exponential is for convenience. It is much neater mathematically than using a sine or cosine function. After all analysis the idea is to simply take the *real part* as clearly in reality we need a real solution.

We can non-dimensionalize (scale) ξ on the wavenumber, introducing $x = k\xi$ and $u(x) = ku_0$ so that

$$u''(x) + u(x) = f(x) \tag{2.172}$$

where $f(x) = \frac{1}{k}g(x/k)$.

We have already determined the existence and uniqueness properties of this problem for the BCs $u(0) = u(L) = 0$ in Example 2.9. This corresponds to the situation when the ends of the string are fixed - a common scenario!

Let us assume, for now, that $L \neq n\pi$ so that there are no existence and uniqueness issues. We then know the solution to the problem, it is (combining and simplifying all of the terms in Example 2.9)

$$u(x) = \int_x^L \frac{\sin(x_0 - L)\sin x}{\sin L} f(x_0)\ dx_0 + \int_0^x \frac{\sin(x - L)\sin x_0}{\sin L} f(x_0)\ dx_0 \tag{2.173}$$

and we note that we can therefore write this as

$$u(x) = \int_0^L G(x, x_0) f(x_0)\ dx_0 \tag{2.174}$$

where

$$G(x, x_0) = \frac{1}{\sin L}\begin{cases} \sin(x_0 - L)\sin x, & 0 \leq x \leq x_0, \\ \sin x_0 \sin(x - L), & x_0 \leq x \leq L \end{cases} \tag{2.175}$$

and we note that $G(x, x_0) = G(x_0, x)$. In particular it is helpful to write

$$G(x, x_0) = \frac{1}{\sin L}\left(\sin(x_0 - L)\sin x H(x_0 - x) + \sin x_0 \sin(x - L) H(x - x_0)\right).$$

In question 1 on Example Sheet 5 we show that this Green's function can be derived directly by following the steps in the "explicit solution" rather than the method of variation of parameters above.

Let us consider a slightly different problem now: an infinite domain.

Example 2.17 *Consider a string of infinite length. What this means in reality is that it is so long that the end conditions are never important in the problem. What is the associated Green's function for forcing at the origin? Since we are interested in physical problems, let us work in the physical domain so let us solve*

$$G''(x,0) + k^2 G(x,0) = \delta(x)$$

where $k = \omega/c$ is the wavenumber defined above. This is subject to the usual continuity conditions at $x = x_0 = 0$. Since the string is of infinite extent we do not have any BCs! So what do we use as additional conditions?! Well we know that if we are forcing the string harmonically at the point $x = 0$ then waves must be moving away from that point to infinity. They clearly cannot be coming in from infinity. So this applies for $x \to \pm\infty$ which gives us another two conditions.

It is convenient in this case to write our fundamental solutions as $u_1(x) = \exp(ix)$ and $u_2(x) = \exp(-ix)$. In order for the solution to be outgoing as $x \pm \infty$ it is clear that we need to take

$$G(x,0) = \begin{cases} c_L(0)\exp(-ikx), & x \leq 0, \\ c_R(0)\exp(ikx), & x \geq 0. \end{cases} \tag{2.176}$$

We have retained the general form with argument $x_0 = 0$ here since we will generalize to arbitrary x_0 later on. Why did we choose this form? Well we have time dependence of $\exp(-i\omega t)$ so that for $x > 0$, we have a solution of the form $\exp(i(kx - \omega t))$. That this corresponds to a wave moving in the positive x direction is clear since when we increase time, if we want to stay at the same point on the wave we have to increase x. Similarly $\exp(-i(kx+\omega t))$ corresponds to a left propagating wave and so is valid for $x < 0$. Another way of saying this is to impose that

$$G'(x,0) - ikG(x,0) \to 0 \qquad \text{as } x \to \infty, \tag{2.177}$$

$$G'(x,0) + ikG(x,0) \to 0 \qquad \text{as } x \to -\infty. \tag{2.178}$$

Continuity at $x = 0$ gives $c_L(0) = c_R(0)$. And the jump condition on the derivative gives

$$\left[\frac{dG}{dx}\right]_{x=0^-}^{x=0^+} = ikc_R(0) + ikc_L(0) = 2ikc_L(0) = 1 \tag{2.179}$$

so that $c_L(0) = 1/(2ik)$ and

$$G(x,0) = \frac{1}{2ik}\exp(ik|x|). \tag{2.180}$$

It is not difficult to show that (see question 2 of Example Sheet 4) for a general point of forcing $x = x_0$*,*

$$G(x, x_0) = \frac{1}{2ik} \exp(ik|x - x_0|). \tag{2.181}$$

Note that $G(x, x_0) = G(x_0, x)$*. This is interesting because this is not Hermitian symmetry and therefore this problem cannot be self-adjoint. But this is strange. Why not?! It looks like it should be! Well, this is subtle: note that the coefficients in the "boundary (radiation) conditions"* (2.177)*,* (2.178) *are complex and this means that we are not guaranteed self-adjointness. See part (ii) of question 2 on Example Sheet 4. In part (iii) of that question we also consider the problem of waves on a semi-infinite string forced harmonically.*

Why work with *complex* solutions $\exp(\pm ikx)$ rather trigonometric functions $\sin kx$ and $\cos kx$? We do this for waves problems mainly because it is convenient in terms of algebra: when we take products of exponentials we can combine terms additively in the exponent. A good example is the time harmonic dependence:

$$\exp(ikx)\exp(-i\omega t) = \exp(i(kx - \omega t)).$$

Of course at the end we want REAL solutions so we have to take the real part of whatever solution we obtain in practice.

2.10 The adjoint Green's function

We derived the result (2.132) only for S-A problems and in these cases we note that the Green's function is also symmetric. What if the problem is *not* S-A? Can we still define a Green's function and if so, what does the solution form for $u(x)$ look like? We will deal with this here, although we note that physically many of the equations we deal with are of the S-A form.

First we introduce the governing equation

$$\mathcal{L}_x u = f(x) \tag{2.182}$$

where now $\mathcal{L}_x$ is *not* S-A. Homogeneous BCs $\mathcal{B}$ accompany (2.182). We note that we can introduce the corresponding Green's function in the usual manner

$$\mathcal{L}_x G(x, x_0) = \delta(x - x_0) \tag{2.183}$$

with equivalent homogeneous BCs to the BVP.

Theorem 2.5 *For non S-A problems with homogeneous BCs, the solution can still be written*

$$u(x) = \int_a^b G(x, x_0) f(x_0)\ dx_0 \tag{2.184}$$

The proof of this is easy. We can perform $\mathcal{L}_x$ on both sides of (2.184) as we did in (2.128).

However, how is this consistent with what was derived above using Green's identity, because there we had to use the symmetry properties of the Green's function? In fact what we can do is use the adjoint operator and define the so-called *adjoint Green's function* G^* which satisfies

$$\mathcal{L}_x^* G^* = \delta(x - x_0) \tag{2.185}$$

where $\mathcal{L}^*$ is the adjoint operator together with the necessary adjoint BCs $\mathcal{B}^*$.

We use the definition of the adjoint operator,

$$\langle v, \mathcal{L}u \rangle = \langle \mathcal{L}^* v, u \rangle \tag{2.186}$$

Let us choose $u = G(x, x_1)$ and $v = G^*(x, x_2)$, noting that G satisfies the same BCs as the original problem whereas G^* satisfies the adjoint BCs (note also here that we have seen a few examples of cases where $\mathcal{L}^* = \mathcal{L}$ but the adjoint BCs are different and therefore this would clearly give $G^*(x, y) \neq G(x, y)$). This all gives

$$\langle G^*(x, x_2), \mathcal{L}G(x, x_1) \rangle = \langle \mathcal{L}^* G^*(x, x_2), G(x, x_1) \rangle. \tag{2.187}$$

Using the definitions of these functions we find that

$$\int_a^b \overline{G^*(x, x_2)} \delta(x - x_1)\ dx = \int_a^b \overline{\mathcal{L}^* G^*(x, x_2)} G(x, x_1)\ dx \tag{2.188}$$

and the sifting property of the delta function gives

$$\begin{aligned}\overline{G^*(x_1, x_2)} &= \overline{\int_a^b \mathcal{L}^* G^*(x, x_2) \overline{G(x, x_1)}\ dx} \\ &= \overline{\int_a^b \delta(x - x_2) \overline{G(x, x_1)}\ dx} \\ &= G(x_2, x_1) \end{aligned} \tag{2.189}$$

So we have proved the following result:

Theorem 2.6 *For operators that are not fully S-A, the Green's function is not Hermitian symmetric. But there is a symmetry relation relating the Green's function and its adjoint:*

$$G(x, x_0) = \overline{G^*(x_0, x)}. \tag{2.190}$$

Next, choose u to satisfy the original inhomogeneous problem and $v = G^*$ in (2.186) so that we obtain

$$\langle G^*, \mathcal{L}_x u \rangle - \langle u, \mathcal{L}^* G^* \rangle = 0. \tag{2.191}$$

Once again, using the sifting property of the delta function and interchanging variables we obtain

$$u(x) = \int_a^b \overline{G^*(x_0, x)} f(x_0)\ dx_0. \tag{2.192}$$

Finally, we use Theorem 2.6 to get

$$u(x) = \int_a^b G(x, x_0) f(x_0)\ dx_0. \tag{2.193}$$

What the theory just presented regarding the adjoint Green's function tells us is that we do not have to ever worry about constructing the adjoint Green's function! We can always just construct the Green's function (if it exists), satisfying the same BCs as the original problem, even when it is not S-A and still write the solution in the form

$$u(x) = \int_a^b G(x, x_0) f(x_0)\ dx_0.$$

This is convenient!

In the next section we will construct a Green's function for a non S-A operator.

2.11 Green's functions for non S-A BVPs

The theory in this section (in blue below) is non-examinable.

What you should take from this section is that for Non-SA problems, you can use the same explicit method to derive the Green's function as we derived for regular S-L problems BUT ONLY WHEN BCs are NOT MIXED! I WILL NOT SET ANY PROBLEMS ON THE EXAM INVOLVING FINDING THE GREENS FUNCTION WITH MIXED BCs!

Above we showed explicitly how to determine the Green's function when the problem is of regular S-L type. What if it is not of this type? Three different cases arise. These are:

(i) When the ODE is in S-A form, with $\mathcal{B}^* \neq \mathcal{B}$ (but BCs are *not* mixed)

(ii) When the ODE is *not* in S-A form with non-mixed BCs.

(iii) When any BC is of mixed type.

In case (i) we can follow the explicit approach described above for regular S-L problems with no problem.

In case (ii) we now show that we can transform the ODE into S-A form and hence there is no issue. One can approach the Green's function construction in exactly the manner described for regular S-L problems above. We can *force* the ODE into S-A form as now explain.

Take the usual ODE

$$p(x)\frac{d^2G}{dx^2} + r(x)\frac{dG}{dx} + q(x)G = \delta(x - x_0). \tag{2.194}$$

Divide by $p(x)$, generate the integrating factor

$$I(x) = \exp\left(\int r(x)/p(x)\ dx\right)$$

and then multiply through by this to get

$$I(x)\frac{d^2G}{dx^2} + \frac{r(x)}{p(x)}I(x)\frac{dG}{dx} + q(x)I(x)G = \frac{I(x)}{p(x)}\delta(x - x_0).$$

and now re-write the left hand side in the form

$$\frac{d}{dx}\left(I(x)\frac{dG}{dx}\right) + q(x)I(x)G = \frac{I(x)}{p(x)}\delta(x - x_0). \tag{2.195}$$

It may not be beneficial to do this in order to solve the problem. Indeed solving (2.194) may be easier (but they must give the same solution!). But we can now integrate (2.195) between $x = x_0^-$ and $x = x_0^+$ to get the jump condition as before

$$\left[\frac{dG}{dx}\right]_{x_0^-}^{x_0^+} = \frac{1}{p(x_0)}. \tag{2.196}$$

Hence we can in fact follow exactly the same procedure as for regular S-L problems in order to generate the Green's function. The only difference is that the Green's function will not be Hermitian symmetric. Instead we have the alternative symmetry relation $G(x_0, x) = \overline{G^*(x, x_0)}$ as proven above.

Example 2.18 *Solve the BVP consisting of the ODE*

$$x^2u'' + 4xu' + 2u = f(x)$$

and the BCs $u(1) = u(2) = 0$ by finding the Green's function and write down the explicit solution in the case when $f(x) = x$. Confirm that this is what one would expect by solving via direct methods for this specific $f(x)$.

Fundamental solutions to the homogeneous problem are given by solving

$$x^2u'' + 4xu + 2u = 0$$

which is an Euler equation so we seek solutions of the form $u(x) = x^m$ which gives

$$m^2 + 3m + 2 = (m + 2)(m + 1) = 0$$

so that $u(x) = x^{-2}$ and $u(x) = x^{-1}$.

Combination of these solutions satisfying the left and right BCs are

$$u_L(x) = \frac{1}{x} - \frac{1}{x^2}, \qquad u_R(x) = \frac{1}{x} - \frac{2}{x^2} \tag{2.197}$$

The associated Wronskian is (check this!)

$$\begin{aligned} W(x_0) &= u_L(x_0)u_R'(x_0) - u_R(x_0)u_L'(x_0) \\ &= \frac{1}{x_0^4} \end{aligned}$$

so that $p(x_0)W(x_0) = 1/x_0^2$. Therefore

$$c_L(x_0) = \frac{u_R(x_0)}{p(x_0)W(x_0)} = x_0 - 2, \qquad c_R(x_0) = \frac{u_L(x_0)}{p(x_0)W(x_0)} = x_0 - 1. \tag{2.198}$$

and the Green's function is

$$\begin{aligned} G(x, x_0) &= \begin{cases} c_L(x_0)u_L(x), & 1 \le x \le x_0, \\ c_R(x_0)u_R(x), & x_0 \le x \le 2. \end{cases} \\ &= \begin{cases} \frac{1}{x^2}(x_0 - 2)(x - 1), & 1 \le x \le x_0, \\ \frac{1}{x^2}(x_0 - 1)(x - 2), & x_0 \le x \le 2. \end{cases} \\ &= \frac{1}{x^2}(x_0 - 2)(x - 1)H(x_0 - x) + \frac{1}{x^2}(x_0 - 1)(x - 2)H(x - x_0). \end{aligned}$$

The solution of the problem is

$$u(x) = \int_1^2 f(x_0)G(x, x_0)\ dx_0.$$

With $f(x) = x$, carrying out the integrations we obtain (check this!)

$$u(x) = \frac{1}{x^2} - \frac{7x}{6} + \frac{x}{6}$$

In question 4 on Example Sheet 5 you are asked to show that the same result would have been obtained if you had used the adjoint Green's function.

THE REST OF SECTION 2.11 IS NON-EXAMINABLE.

If BCs are of mixed type (i.e. (iii) above), then strictly there is no "left" or "right" BC to satisfy. What this means is that we cannot solve the problem in the same manner as the explicit method above. In particular it means that the x and x_0 dependence in the Green's function is NOT separable.

Once we have determined fundamental solutions u_1 and u_2 what we must do is to pose a solution in the form

$$G(x, x_0) = \begin{cases} c_L(x_0)u_1(x) + d_L(x_0)u_2(x), & a \leq x \leq x_0, \\ c_R(x_0)u_1(x) + d_R(x_0)u_2(x), & x_0 \leq x \leq b \end{cases} \tag{2.199}$$

and then determine c_L, d_L, c_R, d_R from the two boundary conditions and continuity conditions imposed at $x = x_0$.

Let us illustrate with an example.

Example 2.19 *Determine the Greens function for the ODE*

$$u''(x) + 3u'(x) + 2u(x) = 0$$

subject to the mixed BCs $u(0) = u'(1)$ and $u(1) = 2u(0) + u'(0)$.

Both BCs are mixed so there is no "left" or "right" BC. However we can easily solve the homogeneous ODE with

$$u_1(x) = e^{-x}, \qquad\qquad u_2(x) = e^{-2x}$$

so let us take the Green's function in the form above:

$$G(x, x_0) = \begin{cases} c_L(x_0)e^{-x} + d_L(x_0)e^{-2x}, & 0 \leq x \leq x_0, \\ c_R(x_0)e^{-x} + d_R(x_0)e^{-2x}, & x_0 \leq x \leq 1 \end{cases} \tag{2.200}$$

Imposing the BCs leads to the conditions

$$c_L + d_L = -c_R\frac{1}{e} - 2\frac{d_R}{e^2},$$
$$\frac{c_R}{e} + \frac{d_R}{e^2} = c_L$$

together with the standard continuity conditions at $x = x_0$ (which give an addition two conditions) we can then determine c_L, d_L, c_R, d_R. (see question 8 on Example Sheet 5).

2.12 Inhomogeneous boundary conditions

In general we would like to solve problems which are not restricted to boundary conditions that are homogeneous. There are two approaches to solving such problems.

Consider the following problem

$$\mathcal{L}U = f(x) \tag{2.201}$$

subject to $U(a) = \alpha, U(b) = \beta$. Since the problem is linear we can decompose the solution in the form

$$U(x) = u(x) + v(x) \tag{2.202}$$

where u and v satisfy

$$\mathcal{L}u = f(x), \qquad \mathcal{L}v = 0 \tag{2.203}$$

subject to $u(a) = 0, u(b) = 0$ and $v(a) = \alpha, v(b) = \beta$. The problem for v is simply to find a linear combination of the fundamental solutions such that the boundary conditions are met. *However note that from Example* (2.8) *we are not always guaranteed that such a solution will exist!*

The problem for u is equivalent to the homogeneous boundary condition problems above with associated Green's function that also satisfies homogeneous BCs. Thus if both $v(x)$ and the Green's function exists, the solution will be

$$U(x) = \int_a^b G(x, x_0)f(x_0)\ dx_0 + v(x) \tag{2.204}$$

Alternatively we can derive the solution directly via the Green's function and application of Lagrange's identity. For conciseness let us consider the fully self-adjoint case. Then

$$\begin{aligned}\int_a^b G\mathcal{L}U - U\mathcal{L}G\ dx &= \left[G\frac{dU}{dx} - U\frac{dG}{dx}\right]_a^b \\ &= U(a)\frac{dG}{dx}(a, x_0) - U(b)\frac{dG}{dx}(b, x_0) \\ &= \alpha\frac{dG}{dx}(a, x_0) - \beta\frac{dG}{dx}(b, x_0)\end{aligned} \tag{2.205}$$

Therefore simplifying the left hand side, the solution is

$$U(x) = \int_a^b G(x, x_0)f(x_0)\ dx_0 + \beta\frac{dG}{dx}(b, x) - \alpha\frac{dG}{dx}(a, x). \tag{2.206}$$

The two solutions (2.204) and (2.206) are equivalent of course.

An example associated with non-homogeneous boundary conditions can be found in question 9 on Example Sheet 5.

THIS LAST SECTION IS NOT EXAMINABLE. BUT:
You should ensure you know what goes wrong when there is a zero eigenvalue, i.e. the normal green's function cannot be defined and therefore you need a "modified" Green's function. But you won't be asked to construct any or know the theory of them, etc.

2.13 Existence of a zero eigenvalue - modified Green's functions

As is clear from the eigenfunction expansion of the Green's function in (2.110), there is clearly a problem when there exists a zero eigenvalue. In this instance the Green's function does not exist! As we know, a zero eigenvalue corresponds to a non-trivial solution of the homogeneous BVP, and this exists for the fixed ends string problem above, when $L = n\pi$ so let us consider that problem. Try to determine the Green's function using the method above - it will not work!

In this case as we know from the Fredholm Alternative, there will only exist solutions to the inhomogeneous BVP if

$$\int_0^L \sin x_0 f(x_0)\ dx_0 = 0,$$

so we assume that this holds. If n is even, any even function $f(x_0)$ will ensure this.

The reason that the Green's function does not exist is because, once again by the Fredholm Alternative, for its existence (it is itself defined by an inhomogeneous BVP) we require

$$\int_0^L \sin x_0 \delta(x - x_0)\ dx_0 = \sin x = 0$$

for all $x \in [0, L]$ but this clearly does not hold.

Therefore instead of using that Green's function as defined classically, what we should do is introduce a different comparison problem. We define the so-called *modified Green's function* $\mathcal{G}_m(x, x_0)$ via the governing equation

$$\mathcal{L}\mathcal{G}_m(x, x_0) = \delta(x - x_0) + c\phi(x)$$

where $c \in \mathbb{R}$ and $\phi(x)$ is the non-trivial eigenfunction corresponding to the zero eigenvalue (here it is $\phi(x) = \sin x$). All other conditions on the Green's function remain the same.

In the general case then, we must *choose* c so that the right hand side is orthogonal to the eigenfunction corresponding to the zero eigenvalue. I.e. here we choose

$$\int_a^b (\delta(z - x_0) + c\phi(z))\phi(z)\ dz = 0$$

so that

$$c = -\frac{\phi(x_0)}{\int_a^b \phi^2(z)\ dz}$$

where x_0 is the location of the source. Therefore the modified Green's function $\mathcal{G}_m(x, x_0)$ is defined by the equation

$$\mathcal{L}\mathcal{G}_m(x, x_0) = \delta(x - x_0) - \frac{\phi(x)\phi(x_0)}{\int_a^b \phi^2(z)\ dz}.$$

subject to the usual homogeneous BCs. Unfortunately since the right hand side is orthogonal to $\phi(x)$, by the Fredholm Alternative there are an infinite number of solutions so that the modified Green's function is not uniquely defined.

It transpires that the *particular* solution of the BVP can be written as usual in terms of the modified Green's function $\mathcal{G}_m(x, x_0)$ (which can be chosen to be symmetric) in the form

$$u(x) = \int_a^b \mathcal{G}_m(x, x_0) f(x_0)\ dx_0.$$

Example 2.20 *Derive the modified Green's function for the problem*

$$u''(x) = f(x)$$

subject to $u'(0) = 0, u'(L) = 0$.

As we have discussed above, a constant c *is a homogeneous solution (an eigenfunction corresponding to a zero eigenvalue). We note that by the Fredholm Alternative, for a solution to exist therefore we must have*

$$\int_0^L f(x)\ dx = 0.$$

Let us assume that we have such an $f(x)$*. In that case the modified Green's function satisfies*

$$\frac{d^2\mathcal{G}_m}{dx^2} = \delta(x - x_0) + c.$$

subject to $\mathcal{G}_m'(0, x_0) = 0, \mathcal{G}_m'(L, x_0) = 0$*. For a modified Green's function to exist, again by the Fredholm Alternative the right hand side has to be orthogonal to a constant, i.e.*

$$\int_0^L \delta(x - x_0) + c\ dx = 0$$

so that $c = -1/L$.

Thus for $x \neq y$*, we must have*

$$\frac{d^2\mathcal{G}_m}{dx^2} = -\frac{1}{L}.$$

By direct integration,

$$\frac{d\mathcal{G}_m}{dx} = \begin{cases} -\dfrac{x}{L} + c_1, & 0 \leq x \leq x_0, \\ -\dfrac{x}{L} + c_2, & x_0 \leq x \leq L. \end{cases}$$

If we choose c_1 *and* c_2 *to satisfy the BCs at* $x = 0, L$ *we find* $c_1 = 0, c_2 = 1$*. The jump condition on* $d\mathcal{G}_m/dx$ *is already satisfied. Integrating once again we find*

$$\mathcal{G}_m(x, x_0) = \begin{cases} -\dfrac{x^2}{2L} + d_1, & 0 \leq x \leq x_0, \\ -\dfrac{x^2}{2L} + x + d_2, & x_0 \leq x \leq L. \end{cases}$$

Imposing continuity of the Green's function at $x = x_0$ *we get* $d_1 = x_0 + d_2$. *So we find*

$$\mathcal{G}_m(x, x_0) = \begin{cases} -\dfrac{x^2}{2L} + x_0 + d_2, & 0 \le x \le x_0, \\ -\dfrac{x^2}{2L} + x + d_2, & x_0 \le x \le L. \end{cases}$$

which illustrates that it is not unique. Imposing symmetry of the Green's function, i.e. we find that (see question **??** *on Example Sheet 5)*

$$d_2 = -\frac{1}{L}\frac{x_0^2}{2} + \beta$$

where β *is a constant, independent of* x_0.

Therefore we have,

$$\mathcal{G}_m(x, x_0) = \begin{cases} -\dfrac{x^2}{2L} + x_0 + \beta, & 0 \le x \le x_0, \\ -\dfrac{x^2}{2L} + x + \beta, & x_0 \le x \le L. \end{cases}$$

and a solution of the BVP is

$$u(x) = \int_0^L \mathcal{G}_m(x, x_0) f(x_0)\ dx_0.$$

since $f(x_0)$ *is orthogonal to a constant, we can take* $\beta = 0$ *without loss of generality.*

2.14 Revision checklist

The following is a guide to what you should know. Read each point and ask yourself if you understand what it means! Also, remember that associated theory from the relevant sections is examinable.

- For constant coefficient and Euler ODEs:
  - be able to find the complementary function u_c and
  - be able to find the particular solution u_p by method of undetermined coefficients and variation of parameters.
- Use integration by parts with inner products to derive the adjoint operator and BCs.
- Know Lagrange's and Green's identities and be able to use them.
- Be able to identify self adjoint operators - this requires both $\mathcal{L}^* = \mathcal{L}$ *and* $\mathcal{B}^* = \mathcal{B}$.
- Be able to identify a Sturm-Liouville problem and know the difference between regular and singular Sturm-Liouville problems.
- Know the 5 theorems associated with regular S-L problems and be able to see how they relate to model problems.
- Understand the Fredholm Alternative theorem and be able to apply it to determine if solutions to inhomogeneous ODEs (with homogeneous BCs) exist and are unique.
- Be able to derive the Green's function via eigenfunction expansions
- Understand some basic properties of the Dirac delta function and its relationship to the Green's function
- Understand the conditions that define the Green's function (governing BVP, conditions at $x = x_0$)
- Be able to derive the Green's function for regular S-L problems by variation of parameters (i.e. applying v.o.p. to the original ODE), and the direct approach (explicit solution - section 2.8.7)
- Be able to derive Green's functions for the wave equation at fixed frequency (time harmonic)
- Understand the problem that arises if a zero eigenvalue exists, understand its relationship with the Fredholm Alternative and the fact that a Green's function does not exist in this case
- Be able to define and derive the adjoint Green's function and its relationship to the Green's function
- Be able to derive Green's functions for non S-L problems (WHEN BCs are NOT MIXED).
- Understand linear superposition in order to derive solutions to inhomogeneous ODEs with inhomogeneous BCs

3 Green's functions in 2 and 3D

Unlike the one dimensional case where Green's functions can be found explicitly for a number of different problems, the two and three dimensional case is somewhat more difficult. Explicit Green's functions are rather difficult to determine especially for bounded domains. However some special cases turn out to be extremely important and we can do a great deal with them.

We shall denote a two *or* three dimensional domain by D and its boundary by ∂D. It will be clear from the problem whether we are in 2 or 3 dimensions or alternatively, the benefit of this unified notation is that it allows us to write down equations which apply to both cases. For example, if D denotes all space (two *or* three dimensional space) then we can prove some rather general theorems associated with the scalar partial differential equation

$$\nabla \cdot (p(\mathbf{x})\nabla\phi(\mathbf{x})) + q(\mathbf{x})\phi(\mathbf{x}) + \lambda\mu(\mathbf{x})\phi(\mathbf{x}) = 0 \tag{3.1}$$

on the two or three dimensional domain D subject to some homogeneous BCs on ∂D. Note that this equation has great similarity to that discussed in the Sturm-Liouville (one dimensional) context in section (2.4). We do not have the time to consider this in detail, although some aspects are discussed.

Instead let us consider the eigenvalue problem associated with the Laplacian operator $\mathcal{L} = \nabla^2$ in two and three dimensions which amounts to the consideration of the problem

$$\nabla^2\phi + \lambda\phi = 0 \tag{3.2}$$

subject to the homogeneous BC

$$\alpha(\mathbf{x})\phi + \beta(\mathbf{x})\nabla\phi \cdot \mathbf{n} = 0, \qquad \mathbf{x} \in \partial D. \tag{3.3}$$

Equation (3.2) is known as *Helmholtz equation*[9]. Note that this eigenvalue problem is important in (at least) two physical contexts. The first is the heat equation governing the temperature field $u(\mathbf{x})$ which is

$$\nabla \cdot (k(\mathbf{x})\nabla U) = \rho(\mathbf{x})c(\mathbf{x})\frac{\partial U}{\partial t} + \mathcal{Q}(\mathbf{x}, t) \tag{3.4}$$

where $k(\mathbf{x})$ is the conductivity, $\rho(\mathbf{x})$ the mass density and $c(\mathbf{x})$ the specific heat. The term $+\mathcal{Q}(\mathbf{x}, t)$ is a heat source. The second is the wave equation

$$\nabla \cdot (E(\mathbf{x})\nabla U) = \rho(\mathbf{x})\frac{\partial^2 U}{\partial t^2} + \mathcal{Q}(\mathbf{x}, t) \tag{3.5}$$

where $E(\mathbf{x})$ is some physical property to be specified and $\rho(\mathbf{x})$ is again, the mass density. The term $\mathcal{Q}(\mathbf{x}, t)$ is any applied forcing. The physical interpretation of $u(\mathbf{x})$ depends on the context.

[9]Hermann Ludwig Ferdinand von Helmholtz (1821 1894) was a German physician and physicist who made significant contributions to several widely varied areas of modern science. He is most known in the waves community for Helmholtz' equation, often also known as the reduced wave equation having assumed time harmonic behaviour

When the medium is *homogeneous* (material parameters do not depend on $\mathbf{x}$) these equations reduce to

$$k\nabla^2 U = \rho c \frac{\partial U}{\partial t} + \mathcal{Q}(\mathbf{x}, t) \tag{3.6}$$

and

$$\nabla^2 U = \frac{1}{c_p^2}\frac{\partial^2 U}{\partial t^2} + \mathcal{Q}(\mathbf{x}, t) \tag{3.7}$$

where c_p is the wave speed.

If you remember back to MT20401 you will have determined eigenfunction expansions for two and three dimensional problems for the Laplacian operator. It transpires that you can find such eigenfunction expansions explicitly in a number of cases, but the separability depends on the geometry (coordinate systems). Things work out nice and cleanly in a number of systems but in particular in rectangles and circles as we shall recall. Partial differential operators in cylindrical and spherical coordinate systems (again see MT20401) are reviewed in Appendix A.

We consider problems for (3.8) when $\mathcal{Q}$ is independent of time and the resulting solution is also independent of time, $U(\mathbf{x}, t) = u(\mathbf{x})$ so that

$$\nabla^2 u = \frac{1}{k}\mathcal{Q}(\mathbf{x}) \tag{3.8}$$

and (3.9) when $\mathcal{Q}$ is time-harmonic, $\mathcal{Q}(\mathbf{x}, t) = Q(\mathbf{x})e^{-i\omega t}$ so that $U(\mathbf{x}, t) = u(\mathbf{x})e^{-i\omega t}$ and so

$$\nabla^2 u + K^2 u = Q(\mathbf{x}) \tag{3.9}$$

where $K = \omega/c_p$ is known as the *wavenumber*.

Both of these are very important problems in practice.

3.1 Self-adjointness

Assuming that $\mathcal{L}$ is now some *partial* differential operator, we can define the adjoint operator in exactly the same manner as we did in one dimension. Taking an inner product

$$\langle v, u\rangle = \int_D \overline{v} u \; d\mathbf{x}$$

which holds in two or three dimensions, we can define the adjoint operator $\mathcal{L}^*$ via

$$\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}^* v, u\rangle$$

but actually determining $\mathcal{L}^*$ is often difficult! See however question 2 on Example sheet 6. In this section we will consider only the Laplacian and Helmholtz operators. As we will show shortly in this case $\mathcal{L} = \mathcal{L}^*$.

3.1.1 Lagrange's identity and Green's second identity

First, note that (question 1 on Example sheet 6) for two functions $f(\mathbf{x}), g(\mathbf{x})$, show that

$$\nabla \cdot (f \nabla g) = \nabla f \cdot \nabla g + f \nabla^2 g. \tag{3.10}$$

In the case when $\mathcal{L} = \nabla^2 + K^2$ (the Helmholtz operator) with $K \in \mathbb{R}$ (a very useful case!), we can write down a multidimensional Lagrange's identity as introduced in one dimension in section 2.3:

$$\overline{v}(\nabla^2 u + K^2 u) - u\overline{(\nabla^2 v + K^2 v)} = \nabla \cdot (u \nabla \overline{v} - \overline{v} \nabla u). \tag{3.11}$$

And it follows then that Green's second identity also has analogues in two and three dimensions:

$$\int_D \left[\overline{v}(\nabla^2 u + K^2 u) - u\overline{(\nabla^2 v + K^2 v)} \right] = \int_{\partial D} (u \nabla \overline{v} - \overline{v} \nabla u) \cdot \mathbf{n} ds \tag{3.12}$$

where s denotes a parametrization of the boundary ∂D of D. The notation $\mathbf{n}$ denotes an outward pointing normal to ∂D and ∂D is traversed in a counter-clockwise manner.

With the homogeneous boundary condition $\mathcal{B}$:

$$\alpha(\mathbf{x}) u + \beta(\mathbf{x}) \nabla u \cdot \mathbf{n} = 0$$

with α, β real functions, choosing $\mathcal{B}^* = \mathcal{B}$ (from the right hand side of (3.12)) we have $\mathcal{L} = \mathcal{L}^*$ (left hand side of (3.12)). That is

$$\int_D \left[\overline{v} \mathcal{L} u - \overline{\mathcal{L} v} u \right] = 0. \tag{3.13}$$

3.2 An eigenvalue problem on a rectangular domain

Example 3.1 *Consider the two dimensional eigenvalue problem associated with either heat distribution or a vibrating rectangular membrane on a rectangular domain. The full problems are considered in question 3 on Example sheet 6. In that question it is shown that the relevant two dimensional eigenvalue problem is*

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \lambda \phi = 0 \tag{3.14}$$

subject to the BCs which for simplicity here we shall take as zero temperature (heat) or pinned (membrane), i.e.

$$\phi(x, 0) = 0, \qquad \phi(x, H) = 0, \tag{3.15}$$

$$\phi(L, y) = 0, \qquad \phi(L, y) = 0 \tag{3.16}$$

for $x \in [0, L]$ *and* $y \in [0, H]$.

In that question you also show, by separating variables, that the two dimensional eigenfunctions are

$$\phi_{mn}(x, y) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \tag{3.17}$$

for $m, n = 1, 2, 3, ...,$ *with associated eigenvalues*

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2. \tag{3.18}$$

Circular domains lead to eigenvalue expansions in terms of *Bessel functions* which you may have seen before. These can be rather messy but we deal with one of the simplest Bessel functions later when we discuss the free space Green's function for Helmholtz equation.

3.3 Eigenvalue problem for the Laplacian operator

As in the one dimensional case, we can make some rather general statements about certain eigenvalue problems. In particular let us consider the eigenvalue problem associated with the Laplacian operator:

$$\nabla^2\phi + \lambda\phi = 0 \tag{3.19}$$

subject with homogeneous BC

$$\alpha(\mathbf{x})\phi + \beta(\mathbf{x})\nabla\phi\cdot\mathbf{n} = 0 \tag{3.20}$$

where α, β are real functions. As we see in question 2 on Example sheet 6, this is a self-adjoint BVP.

Associated with the above eigenvalue problem we have the following theorems:

1. All eigenvalues are real
2. There are an infinite number of eigenvalues. There is a smallest eigenvalue but no largest.
3. Corresponding to any eigenvalue there *may* be many eigenfunctions (note that this was *not* the case for the 1D Sturm-Liouville problems)
4. The eigenfunctions form a complete set, i.e. any piecewise smooth function can be represented by a generalized Fourier Series in terms of the eigenfunctions.
5. Eigenfunctions associated with *different* eigenvalues are orthogonal, relative to a weight $\mu(\mathbf{x})$ with $\mu(\mathbf{x}) = 1$ here. [Furthermore, even different eigenfunctions that correspond to the *same* eigenvalue can be made orthogonal via Gram-Schmidt orthogonalization (but we will not consider this here)].

3.4 Multidimensional Dirac Delta Function

The Dirac delta function extends naturally to higher dimensions. For Cartesian coordinates and two and three dimensions we have

$$\delta(\mathbf{x}-\mathbf{x}_0) = \delta(x-x_0)\delta(y-y_0), \tag{3.21}$$
$$\delta(\mathbf{x}-\mathbf{x}_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \tag{3.22}$$

respectively. There are also versions in cylindrical an spherical coordinates but we do not need them here. We use the notation $\mathbf{x} = (x, y, z)$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$.

The filtering property of the Dirac delta function in higher dimensions is written as

$$\int_D f(\mathbf{x}_0)\delta(\mathbf{x}-\mathbf{x}_0)\ d\mathbf{x}_0 = \begin{cases} f(\mathbf{x}) & \mathbf{x}\in D, \\ 0, & \mathbf{x}\notin D. \end{cases} \tag{3.23}$$

3.5 Green's functions for the Laplace and Poisson equation

Consider the equation

$$\nabla^2 u = Q(\mathbf{x}) \tag{3.24}$$

on a domain D in either two or three dimensions. It will also be subject to some homogeneous BCs usually of the form $u = 0$ or $\nabla u \cdot \mathbf{n} = 0$ or $\alpha u + \beta \nabla u \cdot \mathbf{n} = 0$ for some real functions α, β. We note that with $Q = 0$ corresponds to the case of Laplace's equation.

For simplicity we will assume that all functions and parameters, including the Green's function associated with the Laplacian problem are real. This means we don't have to worry about conjugates, etc. in this subsection.

Motivated by the one dimensional case, the associated Green's function is therefore defined by the equation

$$\nabla^2 G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) \tag{3.25}$$

with equivalent homogeneous BCs, i.e.

$$\alpha G(\mathbf{x}, \mathbf{x}_0) + \beta \nabla G(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n} = 0, \qquad \mathbf{x} \in \partial D.$$

This is often known as Green's function for Laplace's or Poisson's equation. It is perhaps more accurate to describe it as Green's function associated with the Laplacian operator.

We note that equation (3.24) is the forced steady state (independent of time) heat equation (3.8), i.e. it is that temperature field that persists after all transients have decayed.

3.5.1 Symmetry for the Laplacian operator

Use Green's second identity with $\mathcal{L} = \nabla^2$ in (3.13) (i.e. take $K = 0$) and with $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_1)$ and $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_2)$ so that (remembering everything is real!)

$$\int_D \left(G(\mathbf{x}, \mathbf{x}_1)\nabla^2 G(\mathbf{x}, \mathbf{x}_2) - \nabla^2 G(\mathbf{x}, \mathbf{x}_1) G(\mathbf{x}, \mathbf{x}_2)\right)\, d\mathbf{x} = 0, \tag{3.26}$$

which can be written

$$\begin{aligned}
\int_D G(\mathbf{x}, \mathbf{x}_1)\delta(\mathbf{x} - \mathbf{x}_2)\, d\mathbf{x} &= \int_D \nabla^2 G(\mathbf{x}, \mathbf{x}_1) G(\mathbf{x}, \mathbf{x}_2))\, d\mathbf{x} && (3.27)\\
&= \int_D \nabla^2 G(\mathbf{x}, \mathbf{x}_1) G(\mathbf{x}, \mathbf{x}_2)\, d\mathbf{x} && (3.28)\\
&= \int_D \delta(\mathbf{x} - \mathbf{x}_1) G(\mathbf{x}, \mathbf{x}_2)\, d\mathbf{x} && (3.29)
\end{aligned}$$

and therefore using the filtering property of the Dirac Delta function (3.23), when $\mathbf{x}_1, \mathbf{x}_2 \in D$,

$$G(\mathbf{x}_2, \mathbf{x}_1) = G(\mathbf{x}_1, \mathbf{x}_2) \tag{3.30}$$

and as with the one dimensional case, this in fact holds for *all* self-adjoint operators.

3.5.2 Solution representation for the Laplacian operator

Use Green's second identity with $u(\mathbf{x})$ satisfying the boundary value problem above and $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$ to get (again remembering we assume everything is real)

$$\int_D G(\mathbf{x}, \mathbf{x}_0)\nabla^2 u(\mathbf{x})\ d\mathbf{x} = \int_D \nabla^2 G(\mathbf{x}, \mathbf{x}_0) u(\mathbf{x})\ d\mathbf{x}, \tag{3.31}$$

so that

$$\int_D G(\mathbf{x}, \mathbf{x}_0) Q(\mathbf{x})\ d\mathbf{x} = \int_D \nabla^2 G(\mathbf{x}, \mathbf{x}_0) u(\mathbf{x})\ d\mathbf{x} \tag{3.32}$$

$$= \int_D \delta(\mathbf{x} - \mathbf{x}_0) u(\mathbf{x})\ d\mathbf{x} \tag{3.33}$$

$$= u(\mathbf{x}_0). \tag{3.34}$$

Interchange $\mathbf{x}$ and $\mathbf{x}_0$ and use (3.30) so that we get

$$u(\mathbf{x}) = \int_D G(\mathbf{x}, \mathbf{x}_0) Q(\mathbf{x}_0)\ d\mathbf{x}_0. \tag{3.35}$$

So, the solution representation in higher dimensions is entirely analogous to the one dimensional case. Now let us consider finding some Green's functions associated with the Laplacian operator.

3.5.3 Multidimensional eigenfunction expansions

Let us use eigenfunctions to construct the Green's function for Laplace's equation. We note here that this approach may not be ideal - it may be very slowly convergent! Write

$$G(\mathbf{x}, \mathbf{x}_0) = \sum_\lambda a_\lambda \phi_\lambda(\mathbf{x})$$

where ϕ_λ is the eigenfunction corresponding to the eigenvalue λ. Then apply the Laplacian operator so that

$$\nabla^2 G(\mathbf{x}, \mathbf{x}_0) = \sum_\lambda a_\lambda \nabla^2 \phi_\lambda(\mathbf{x}) = -\sum_\lambda a_\lambda \lambda \phi_\lambda(\mathbf{x})$$

Since the left hand side is $\delta(\mathbf{x} - \mathbf{x}_0)$, by orthogonality and the filtering property of the Dirac delta function we have

$$-a_\lambda \lambda = \frac{\int_D \phi_\lambda(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0)\ d\mathbf{x}}{\int_D \phi_\lambda^2(\mathbf{x})\ d\mathbf{x}} = \frac{\phi_\lambda(\mathbf{x}_0)}{\int_D \phi_\lambda^2(\mathbf{x})\ d\mathbf{x}}.$$

Example 3.2 *Use the two dimensional eigenfunction from example 3.1 to construct the associated Green's function, solving*

$$\nabla^2 G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) \tag{3.36}$$

subject to homogeneous BCs on $x = 0, L$ and $y = 0, H$.

From Example 3.1 we know that the corresponding eigenvalues are

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

with $m = 1, 2, 3, ...$ *and* $n = 1, 2, 3, ...$ *The corresponding eigenfunctions are*

$$\phi_{mn}(x, y) = \sin(n\pi x/L)\sin(m\pi y/H)$$

and in this case

$$\int_D \phi_{mn}^2(x, y)\ d\mathbf{x} = (L/2)(H/2).$$

Therefore the eigenfunction expansion of the Green's function is

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{-4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(n\pi x/L)\sin(m\pi y/H)\sin(n\pi x_0/L)\sin(m\pi y_0/H)}{(n\pi/L)^2 + (m\pi/H)^2}.$$

3.5.4 Fredholm Alternative

As in the one dimensional context, if there is a zero eigenvalue ($\lambda = 0$) then there are problems in terms of defining a Green's function. We do not worry too much about this here as we do not have time! Let us assume from now on that λ will never be zero and therefore the Green's function always exists.

3.5.5 Nonhomogeneous boundary conditions

We can also cater for nonhomogeneous boundary conditions in a fairly simple way as in the one dimensional case. We use Green's identity. For example if we have $u(\mathbf{x}) = h(\mathbf{x})$ on the boundary for the Laplacian problem, we still retain the homogeneity of the Greens function BCs which enables us to write, using (3.12) with $v = G$ (and remembering everything is real!)

$$\int_D G(\mathbf{x}, \mathbf{x}_0)\nabla^2 u - \nabla^2 G(\mathbf{x}, \mathbf{x}_0)u\ d\mathbf{x} = \int_{\partial D}(G\nabla u - u\nabla G)\cdot\mathbf{n}\ ds \tag{3.37}$$

$$= -\int_{\partial D} h(\mathbf{x})\mathbf{n}\cdot\nabla_{\mathbf{x}}G(\mathbf{x}, \mathbf{x}_0)\ ds \tag{3.38}$$

using the boundary conditions $G = 0$ and $u = h$ on ∂D.

Therefore, given G we know the right hand side of this equation. Exploiting the forms of the left hand side we find

$$u(\mathbf{x}_0) = \int_D Q(\mathbf{x})G(\mathbf{x}, \mathbf{x}_0)\ d\mathbf{x} + \int_{\partial D} h(\mathbf{x})\mathbf{n}\cdot\nabla_{\mathbf{x}}G(\mathbf{x}, \mathbf{x}_0)\ ds.$$

Finally, interchanging $\mathbf{x}_0$ and $\mathbf{x}$ and using (3.30) (**in the first integral only!**) we find

$$u(\mathbf{x}) = \int_D G(\mathbf{x}, \mathbf{x}_0)Q(\mathbf{x}_0)\ d\mathbf{x}_0 + \int_{\partial D} h(\mathbf{x}_0)\mathbf{n}\cdot\nabla_{\mathbf{x}_0}G(\mathbf{x}_0, \mathbf{x})\ ds. \tag{3.39}$$

Note that we have to be very careful with the last surface integral term. **Please consider this carefully and look at the relevant Questions on Example Sheet 7 and also the examples that we will consider later in Section 3.6 to help you with your understanding!**.

3.5.6 Conditions satisfied by the Green's function

Thus far, by analogy with the one dimensional case we have said that the Green's function satisfies

$$\nabla^2 G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) \tag{3.40}$$

subject to the boundary condition

$$\alpha G(\mathbf{x}, \mathbf{x}_0) + \beta \mathbf{n} \cdot \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) = 0 \tag{3.41}$$

for $\mathbf{x} \in \partial D$ and α, β can be real functions of $\mathbf{x}$ in general.

In order to derive the GF directly as in the one dimensional case we need to understand what happens near the source point $\mathbf{x} = \mathbf{x}_0$. In one dimension, remember that the Green's function was continuous but its derivative was discontinuous. To determine what happens in two and three dimensions, let us first define a local polar coordinate system in two and three dimensions (with reference to figure 7):

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{r} \tag{3.42}$$

where

$$\mathbf{r} = \begin{cases} r(\cos\theta, \sin\theta) & \text{in two dimensions,} \\ r(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi) & \text{in three dimensions} \end{cases} \tag{3.43}$$

noting that $r = |\mathbf{x} - \mathbf{x}_0| \geq 0$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$. In both cases we note that $r = |\mathbf{x} - \mathbf{x}_0|$. This is just the system of cylindrical and spherical polar coordinates remember.

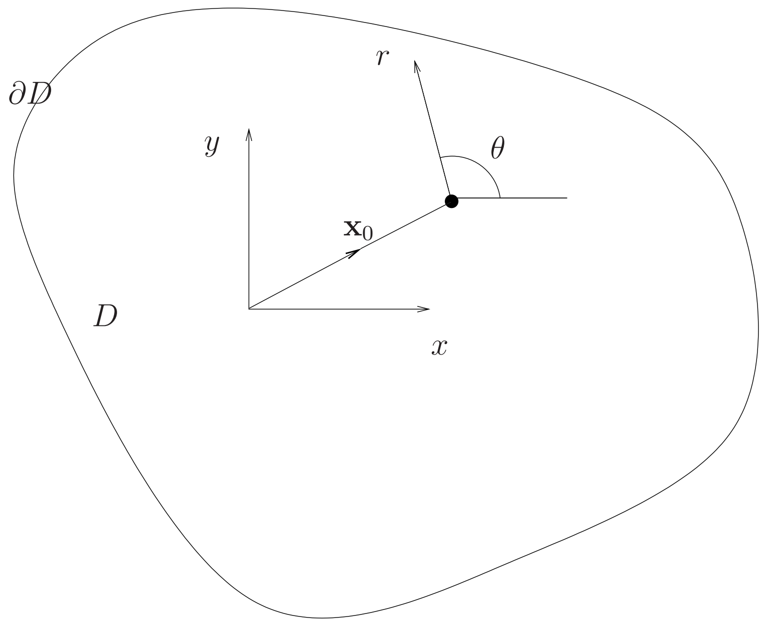

Figure 7: Figure illustrating the local coordinate system introduce around the source at $\mathbf{x} = \mathbf{x}_0$ (in two dimensions). The three dimensional case is analogous but harder to draw!

Let us now introduce these coordinates, and integrate (3.40) over a small circle(sphere) $S_\epsilon = \mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \leq \epsilon$ in two(three) dimensions, of radius ϵ with its centre on the source point, with the intention of letting $\epsilon \to 0$. Note that we could also define S_ϵ with respect to the local coordinate system $\mathbf{r}$ via $S_\epsilon = \mathbf{r} : |\mathbf{r}| = r \leq \epsilon$. Therefore

$$\int_{S_\epsilon} \nabla^2 G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x} = \int_{S_\epsilon} \delta(\mathbf{x} - \mathbf{x}_0)\ d\mathbf{x} \tag{3.44}$$

$$= \int_{S_\epsilon} \delta(\mathbf{r})\ d\mathbf{r} \tag{3.45}$$

$$= 1 \tag{3.46}$$

where we have used the filtering property of the delta function on the right hand side. On the left hand side we use Green's theorem so that

$$\int_{\partial S_\epsilon} \nabla G(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n} ds = 1. \tag{3.47}$$

This dictates the type of singularity that the Green's function must possess. We note in particular that this implies that the Green's function is *not* continuous as it is in the one dimensional context. And remember we are interested in the limiting case when $\epsilon \to 0$ to understand the singular nature of the Green's function at the source location.

We can easily see the type of singularity required in two and three dimensions as we now show. In two dimensions S_ϵ is a circle, $\mathbf{n} = \mathbf{e}_r$, $ds = r d\theta$ so that (3.47) says that we need

$$\lim_{\epsilon \to 0} \int_0^{2\pi} \left(\frac{\partial G}{\partial r}\ r \right) \Big|_{r=\epsilon} d\theta = 1$$

and therefore we need the leading order behaviour of the Green's function to be

$$\frac{\partial G}{\partial r} \sim \frac{1}{2\pi r}$$

so that upon integration for $\mathbf{x}$ close to $\mathbf{x}_0$, i.e. $|\mathbf{r}| \ll 1$,

$$G \sim \frac{1}{2\pi} \ln r = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0|.$$

Importantly, note that this is the local behaviour of the Green's function near the source point $\mathbf{x} = \mathbf{x}_0$. It is not the actual Green's function globally.

In three dimensions S_ϵ is a sphere, $\mathbf{n} = \mathbf{e}_r$, $ds = r^2 \sin\phi d\theta$ so we need

$$\lim_{\epsilon \to 0} \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \left(\frac{\partial G}{\partial r}\ r^2 \right) \Big|_{r=\epsilon} \sin\phi d\phi d\theta = 1$$

and so if

$$G \sim -\frac{1}{4\pi r} = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}_0|}$$

we get the correct behaviour.

The behaviour exhibited by the Green's functions in the cases above close to the source point means that the Green's function is clearly *not* continuous (it is singular) but it possesses a type of singularity that is integrable, i.e.

$$\int_D G(\mathbf{x}, \mathbf{x}_0)\ d\mathbf{x}_0$$

exists for all $\mathbf{x}$.

We stress again, as we stated above that this is just the local behaviour close to the source point. The Green's function could behave quite differently away from the source point, depending upon the geometry and boundary conditions of the problem. However there is a special (very important) case where the above allows us to determine the Green's function immediately as we now describe.

3.5.7 "Free-space" Green's function

What happens if we do not have any boundaries? I.e. we seek Green's functions for infinite space, or as it is often known, the *free-space* Green's function. Well what do we require for the Green's function for the Laplacian? It must solve Laplace's equation and also have the correct singular behaviour near the source. So immediately we are able to say that the free space Green's function for Laplace's equation in two and three dimensions are respectively

$$G_{2\infty}(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}_0|, \qquad G_{3\infty}(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} \tag{3.48}$$

where the subscripts $n\infty$ refers to n dimensions and free space. Why is this? Well, clearly both possess the correct singular behaviour - they are exactly the singular functions that we required above! But importantly they also satisfy Laplace's equation - you are asked to check this in question 4 on Example Sheet 6.

We must be careful however when we use the free space Green's functions. In order to derive the solution representation for physical problems, we usually apply Green's identity but this requires a condition on the boundary. Here the boundary is "at infinity". In order to understand what happens we must apply Green's identity to a finite domain and then let the outer boundary tend to infinity, ensuring that the contribution to the solution from that boundary term does not diverge in the limit. Note that this is not mysterious - it just substitutes for a boundary condition that would be present in finite problems.

Let us focus on the solution of $\nabla^2 u = Q(\mathbf{x})$ for free space problems. We can apply Green's identity from (3.12) with $K = 0$ on a finite domain D to obtain (remembering again that everything is real!)

$$\int_D (G(\mathbf{x}, \mathbf{x}_0)\nabla^2 u(\mathbf{x}) - \nabla^2 G(\mathbf{x}, \mathbf{x}_0)u(\mathbf{x})) \ d\mathbf{x} = \int_{\partial D} (G\nabla u - u\nabla G) \ ds. \tag{3.49}$$

Let us take D to be a circle(sphere) of radius R, centred on $\mathbf{x} = \mathbf{x}_0$, the source location.

Carrying out the usual steps on the left hand side we obtain

$$u(\mathbf{x}_0) = \int_D Q(\mathbf{x})G(\mathbf{x}, \mathbf{x}_0) \ d\mathbf{x} + \int_{\partial D} (G(\mathbf{x}, \mathbf{x}_0)\nabla_{\mathbf{x}} u(\mathbf{x}) - u(\mathbf{x})\nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0)) \cdot \mathbf{n} \ ds. \tag{3.50}$$

Interchanging $\mathbf{x}$ and $\mathbf{x}_0$ and using the symmetry of the Green's function (except in the very last term where we cannot!) we find

$$u(\mathbf{x}) = \int_D Q(\mathbf{x}_0)G(\mathbf{x}, \mathbf{x}_0) \ d\mathbf{x}_0 + \int_{\partial D} (G(\mathbf{x}, \mathbf{x}_0)\nabla_{\mathbf{x}_0} u(\mathbf{x}_0) - u(\mathbf{x}_0)\nabla_{\mathbf{x}_0} G(\mathbf{x}_0, \mathbf{x})) \cdot \mathbf{n} \ ds. \tag{3.51}$$

We could require the integral "at infinity" (second term on the right hand side) to do a number of things. It could be chosen to give certain behaviour to the solution at infinity (e.g. it could be a constant), or alternatively it could be chosen so that it decays (is zero in the limit). In the latter case, using the forms of the Green's functions in two and three dimensions, we require

$$\lim_{R\to\infty}\left(u - r\ln r\frac{\partial u}{\partial r}\right)\Big|_{r=R} = 0, \tag{3.52}$$

$$\lim_{R\to\infty}\left(u + r\frac{\partial u}{\partial r}\right)\Big|_{r=R} = 0. \tag{3.53}$$

You are asked to confirm this in question 5 on Example Sheet 6.

3.5.8 Bounded domains

In fact, it turns out that we may use the free space solution to solve for Green's functions on domains that *do* have boundaries. In particular the nice thing is that the free space solution contains the correct singular behaviour and so we can write the solution for a new problem as

$$G(\mathbf{x},\mathbf{x}_0) = G_{n\infty}(\mathbf{x},\mathbf{x}_0) + V(\mathbf{x},\mathbf{x}_1) \tag{3.54}$$

where $V(\mathbf{x},\mathbf{x}_1)$ is chosen to satisfy the correct boundary conditions on the boundaries. We use the so-called *method of images* which you may have come across in fluid dynamics for potential flow problems.

As usual let us restrict attention to homogeneous boundary conditions here.

Example 3.3 *Determine the two dimensional Green's function for Laplace's equation on the semi-infinite domain* $D = \{-\infty < x < \infty, 0 \le y < \infty\}$, *(upper half-space) subject to* $G = 0$ *on the boundary* $y = 0$.

Let us seek the solution in the form (3.54), *with*

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}_0| + V(\mathbf{x},\mathbf{x}_1) \tag{3.55}$$

so that the first term takes care of the singularity at the source. We need the function V *to ensure that the boundary condition is satisfied. Let us choose*

$$V(\mathbf{x},\mathbf{x}_1) = -\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}_1| \tag{3.56}$$

where $\mathbf{x}_1 = (x_0, -y_0)$. *By symmetry we see that the response should be zero at* $y = 0$ *and we will verify this shortly. We think of* $\mathbf{x}_1$ *as an "image source". Thus the Green's function is*

$$\begin{aligned} G(\mathbf{x},\mathbf{x}_0) &= \frac{1}{2\pi}\left(\ln|\mathbf{x}-\mathbf{x}_0| - \ln|\mathbf{x}-\mathbf{x}_1|\right) \\ &= \frac{1}{4\pi}\ln\left(\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right). \end{aligned} \tag{3.57}$$

Evaluating this on $y = 0$ we find

$$G(\mathbf{x}, \mathbf{x}_0)\Big|_{y=0} = \frac{1}{4\pi} \ln\left(\frac{(x-x_0)^2 + y_0^2}{(x-x_0)^2 + y_0^2}\right) = \frac{1}{4\pi} \ln 1 = 0 \tag{3.58}$$

as required.

3.6 Applications of Poisson's equation

The beauty of Green's functions is that they are useful! Let us consider an example here where we determine the steady state heat distribution on a two dimensional domain. In question 5 of Example Sheet 6 you are asked to consider a three dimensional problem relating to the distribution of electrical charge. This was the problem for which Green originally devised Green's functions! How great is that?!

Example 3.4 *Suppose that we have a very large two dimensional domain and suppose that it is so large that boundary effects can be ignored, i.e. the boundary is so far away that any heat source generates a heat distribution which has decayed to zero along time and distance before the boundary. We use a heat source to heat a circular region C of radius a uniformly with magnitude q and this source is maintained for all times. What is the steady state temperature field that results? You may use without proof the result*

$$\int_0^{2\pi} \ln(\beta^2 + 1 - 2\beta\cos\psi)\ d\psi = \begin{cases} 0, & \beta^2 < 1, \\ 2\pi\beta^2, & \beta^2 > 1. \end{cases} \tag{3.59}$$

Since we can ignore boundary effects we can assume that the circular domain C has its centre at the origin of our coordinate system, i.e. at $\mathbf{x} = \mathbf{0}$. The problem is two dimensional and the field that results (after transients have decayed) is simply

$$u(\mathbf{x}) = \int_C G(\mathbf{x}, \mathbf{x}_0) q\ d\mathbf{x}_0$$

and we approximate the Green's function as the free-space one since the question says we can ignore boundary effects. So we have

$$u(\mathbf{x}) = \frac{q}{2\pi} \int_C \ln(|\mathbf{x} - \mathbf{x}_0|)\ d\mathbf{x}_0.$$

Introduce the two polar coordinate systems

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{3.60}$$

$$x_0 = r_0\cos\theta_0, \qquad y_0 = r_0\sin\theta_0, \tag{3.61}$$

and thus, upon simplifying the argument we find

$$\begin{aligned} u(r,\theta) &= \frac{q}{4\pi}\int_0^{2\pi}\int_0^a \ln(r^2 + r_0^2 - 2rr_0\cos(\theta_0 - \theta))\ r_0 dr_0 d\theta_0 \\ &= \frac{q}{4\pi}\int_0^{2\pi}\int_0^a \ln r^2\left(1 + \frac{r_0^2}{r^2} - 2\frac{r_0}{r}\cos(\theta_0 - \theta)\right)\ r_0 dr_0 d\theta_0 \\ &= \frac{q}{4\pi}\int_0^{2\pi}\int_0^a \left[\ln r^2 + \ln\left(1 + \frac{r_0^2}{r^2} - 2\frac{r_0}{r}\cos(\theta_0 - \theta)\right)\right]\ r_0 dr_0 d\theta_0 \qquad (3.62) \\ &= \frac{qa^2}{2}\ln r + \frac{q}{4\pi}\int_0^{2\pi}\int_0^a \ln\left(1 + \frac{r_0^2}{r^2} - 2\frac{r_0}{r}\cos(\theta_0 - \theta)\right)\ r_0 dr_0 d\theta_0 \qquad (3.63) \end{aligned}$$

Let $\beta = r_0/r$ and $\psi = \theta_0 - \theta$ so that ($dr_0 = rd\beta$ and $d\psi = d\theta_0$)

$$u(r,\theta) = \frac{qa^2}{2}\ln r + \frac{qr^2}{4\pi}\int_0^{a/r}\beta\int_0^{2\pi}\ln[\beta^2 + 1 - 2\beta\cos\psi]\ d\beta d\psi \tag{3.64}$$

where we note that the limits of the θ_0 *integral simplified because we could write*

$$\int_{-\theta}^{2\pi-\theta} = \int_{-\theta}^{0} + \int_{0}^{2\pi} - \int_{2\pi-\theta}^{2\pi}$$

and since the integrand is periodic with period 2π*, the first and last terms on the right hand side cancel.*

We now use the result given in the question, so that if $r > a$*, the* β *integral runs from* 0 *to* $a/r < 1$ *and therefore is zero. Therefore for* $r > a$*,* $u(r,\theta) = (qa^2/2)\ln r$*. If* $r < a$*, the* β *integral runs from* 0 *to* $a/r > 1$ *so we write it as*

$$\frac{qr^2}{4\pi}\left(\int_0^1 + \int_1^{a/r}\right)\beta \int_0^{2\pi} \ln[\beta^2 + 1 - 2\beta\cos\psi]\ d\psi d\beta.$$

The first term (integral between 0 and 1) is zero (using the result (3.59)*), the second, again using the result* (3.59) *again*

$$\begin{aligned} qr^2\int_1^{a/r}\beta\ln\beta\ d\beta &= qr^2\left[-\frac{\beta^2}{4} + \frac{1}{2}\beta^2\ln\beta\right]_1^{a/r} \\ &= -\frac{qa^2}{4} + \frac{qa^2}{2}[\ln a - \ln r] + \frac{qr^2}{4} \end{aligned} \tag{3.65}$$

where we have evaluated this using integration by parts. Finally when combined with the first term in (3.64)*, this reduces to*

$$\frac{q}{4}(r^2 - a^2) + \frac{qa^2}{2}\ln a$$

and therefore we have

$$u(r,\theta) = \begin{cases} \dfrac{q}{4}(r^2 - a^2) + \dfrac{qa^2}{2}\ln a, & r \le a, \\ \dfrac{qa^2}{2}\ln r, & r \ge a. \end{cases} \tag{3.66}$$

which we note is continuous at $r = a$ *on the boundary of the forcing disc* C*.*

Finally we just have to check that the integral "at infinity" from Green's identity does not diverge as described above. If it diverges then the solution (3.4) *is not the correct one and our assumptions from the outset are wrong. So we must see if* (3.52) *holds. Well it clearly does since with* $u = (qa^2)/2\ln r$*,*

$$\lim_{R\to\infty}\left(u - r\ln r\frac{\partial u}{\partial r}\right)\Big|_{r=R} = \lim_{R\to\infty}\frac{qa^2}{2}\left(\ln r - r\ln r\frac{1}{r}\right)\Big|_{r=R} \tag{3.67}$$

$$= 0. \tag{3.68}$$

so the boundary term is not just zero in the limit, it is identically zero.

We plot the solution in figure 8

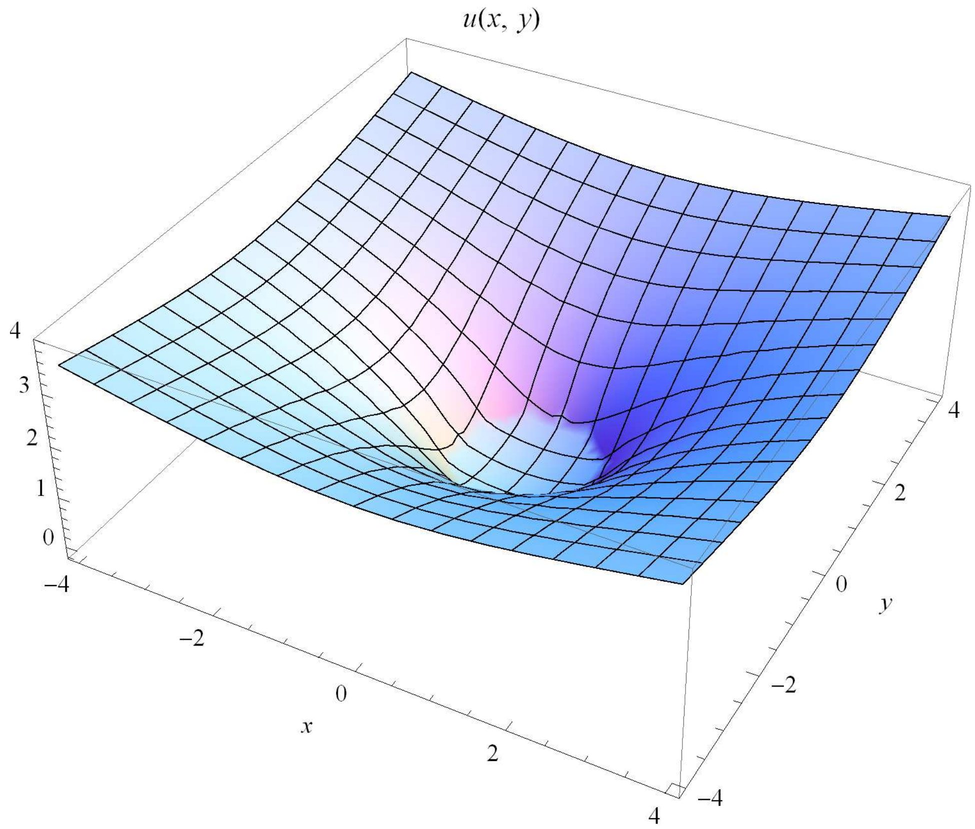

Figure 8: Plot of the steady state temperature distribution $u(x, y)$ in a domain where boundaries may be neglected with the heat source inside the circular domain $x^2 + y^2 < 1$. Here we have $a = q = 1$.

Example 3.5 *Consider the steady state heat distribution governed by*

$$\nabla^2 u = 0$$

on the semi-infinite domain $D = \{-\infty < x < \infty, y \geq 0\}$ *subject to the inhomogeneous boundary condition*

$$u(x, 0) = h(x).$$

Use the semi-infinite Green's function to determine the solution to the problem in integral form and determine the solution explicitly when

$$h(x) = \begin{cases} qx, & -a \leq x \leq a, \\ 0, & \text{otherwise} \end{cases} \tag{3.69}$$

for $a, q \in \mathbb{R}$*, with* $a > 0$*.*

Since we have a semi-infinite domain let us employ the semi-infinite Green's function that we determined in (3.57) *above. We can use the expression* (3.39) *derived in section 3.5.5 with* $Q(\mathbf{x}) = 0$*, i.e.*

$$u(\mathbf{x}) = \int_{\partial D} h(\mathbf{x}_0) \mathbf{n} \cdot \nabla_{\mathbf{x}_0} G(\mathbf{x}_0, \mathbf{x}) \, ds. \tag{3.70}$$

Firstly, we note that the dummy variable of integration is $\mathbf{x}_0$*. This is important! The boundary should be partitioned as* $\partial D = \partial D_0 \cup \partial D_\infty$*, i.e. one along the boundary* $y = 0$*,* ∂D_0 *and one from "infinity",* ∂D_∞*. See figure 9. We assume that the boundary integral along* ∂D_∞*, i.e. the boundary contribution from "infinity" is zero. But we will check this later! The orientation of the integral is anti-clockwise with the normal pointing outwards.*

Therefore, since also $h(\mathbf{x}_0) = 0$ *outside* $-a \leq x_0 \leq a$*, and it is* qx_0 *inside that interval, we have*

$$u(\mathbf{x}) = \int_{-a}^{a} qx_0 \mathbf{n} \cdot \nabla_{\mathbf{x}_0} G(\mathbf{x}_0, \mathbf{x})\Big|_{y_0=0} \, dx_0. \tag{3.71}$$

where we note that $s = x_0$ *on* D_0 *and this integral is along* $y_0 = 0$ *(figure 9.*

Next, we note that $\mathbf{n} = (0, -1)$ *so that*

$$\nabla_{\mathbf{x}_0} G(\mathbf{x}_0, \mathbf{x}) = -\frac{\partial}{\partial y_0} G(\mathbf{x}_0, \mathbf{x})$$

Next we use the Green's function that we determined in Example 3.3, in the form

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \left(\ln |\mathbf{x} - \mathbf{x}_0| - \ln |\mathbf{x} - \mathbf{x}_1| \right)$$

where $\mathbf{x}_0 = (x_0, y_0)$ *with* $y_0 > 0$*, and* $\mathbf{x}_1 = (x_0, -y_0)$*. We have* $G(\mathbf{x}_0, \mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$*. Therefore*

$$-\frac{\partial}{\partial y_0} G(\mathbf{x}_0, \mathbf{x}) = -\frac{1}{2\pi} \left(-\frac{y - y_0}{(x - x_0)^2 + (y + y_0)^2} - \frac{y + y_0}{(x - x_0)^2 + (y - y_0)^2} \right)$$

and we have to evaluate this on $y_0 = 0$*.*

$$\begin{aligned} -\frac{\partial}{\partial y_0} G(\mathbf{x}_0, \mathbf{x})\Big|_{y_0=0} &= -\frac{1}{2\pi} \left(-\frac{y}{(x - x_0)^2 + y^2} - \frac{y}{(x - x_0)^2 + y^2} \right) \\ &= \frac{y}{\pi((x - x_0)^2 + y^2)} \end{aligned}$$

Finally then

$$u(\mathbf{x}) = \frac{qy}{\pi} \int_{-a}^{a} \frac{x_0}{((x - x_0)^2 + y^2)} \, ds. \tag{3.72}$$

where we have taken qy *outside the integral. You will hopefully remember how to evaluate integrals of this form from the first year! What we have to do is to re-write it by using the standard trick of "adding zero", i.e.*

$$u(\mathbf{x}) = -\frac{qy}{\pi} \int_{-a}^{a} \frac{-x_0}{((x - x_0)^2 + y^2)} \, dx_0. \tag{3.73}$$

$$= -\frac{qy}{\pi} \int_{-a}^{a} \frac{x - x_0 - x}{((x - x_0)^2 + y^2)} \, dx_0. \tag{3.74}$$

$$= -\frac{qy}{\pi} \int_{-a}^{a} \frac{x - x_0}{((x - x_0)^2 + y^2)} \, dx_0 + \frac{qxy}{\pi} \int_{-a}^{a} \frac{1}{((x - x_0)^2 + y^2)} \, dx_0. \tag{3.75}$$

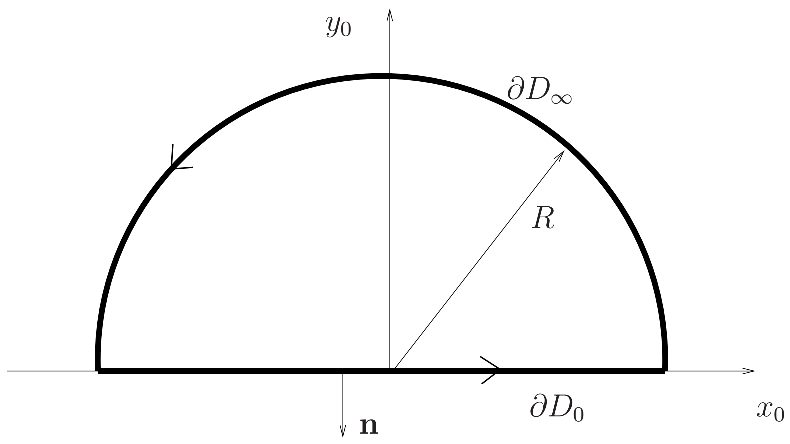

Figure 9: Figure illustrating the decomposition of the boundary ∂D into a contribution from ∂D_∞, i.e. from "infinity" (as $R \to \infty$) and one on $y = 0$, i.e. ∂D_0.

The first integral is

$$-\frac{qy}{\pi}\int_{-a}^{a}\frac{x-x_0}{((x-x_0)^2+y^2)}\;dx_0 = \frac{qy}{2\pi}\left[\ln((x-x_0)^2+y^2)\right]_{-a}^{a}, \tag{3.76}$$

$$= \frac{qy}{2\pi}\left[\ln((x-a)^2+y^2)-\ln((x+a)^2+y^2)\right], \tag{3.77}$$

$$= \frac{qy}{2\pi}\ln\left(\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right). \tag{3.78}$$

The second integral is

$$\frac{qxy}{\pi}\int_{-a}^{a}\frac{1}{((x-x_0)^2+y^2)}\;dx_0 = \frac{qx}{\pi y}\int_{-a}^{a}\frac{1}{(1+(x_0-x)^2/y^2)}dx_0$$

and let

$$p = \frac{(x_0-x)}{y}$$

so that $dx_0 = y\;dp$ *and the lower and upper limits become* $(-a-x)/y$ *and* $(a-x)/y$ *respectively:*

$$\frac{qxy}{\pi}\int_{-a}^{a}\frac{1}{((x-x_0)^2+y^2)}\;dx_0 = \frac{qx}{\pi}\int_{(-a-x)/y}^{(a-x)/y}\frac{1}{(1+p^2)}dp \tag{3.79}$$

$$= \frac{qx}{\pi}\left[\arctan(p)\right]_{(-a-x)/y}^{(a-x)/y} \tag{3.80}$$

$$= \frac{qx}{\pi}\left[\arctan\left(\frac{(a-x)}{y}\right)-\arctan\left(\frac{-a-x}{y}\right)\right] \tag{3.81}$$

$$= \frac{qx}{\pi}\left[\arctan\left(\frac{(x+a)}{y}\right)-\arctan\left(\frac{x-a}{y}\right)\right] \tag{3.82}$$

where the last step used the fact that arctan *is an odd function.*

So combing (3.78) *and* (3.82), *the solution is*

$$u(\mathbf{x}) = \frac{qy}{2\pi}\ln\left(\frac{((x-a)^2+y^2)}{((x+a)^2+y^2)}\right)+\frac{qx}{\pi}\left[\arctan\left(\frac{(x+a)}{y}\right)-\arctan\left(\frac{x-a}{y}\right)\right] \tag{3.83}$$

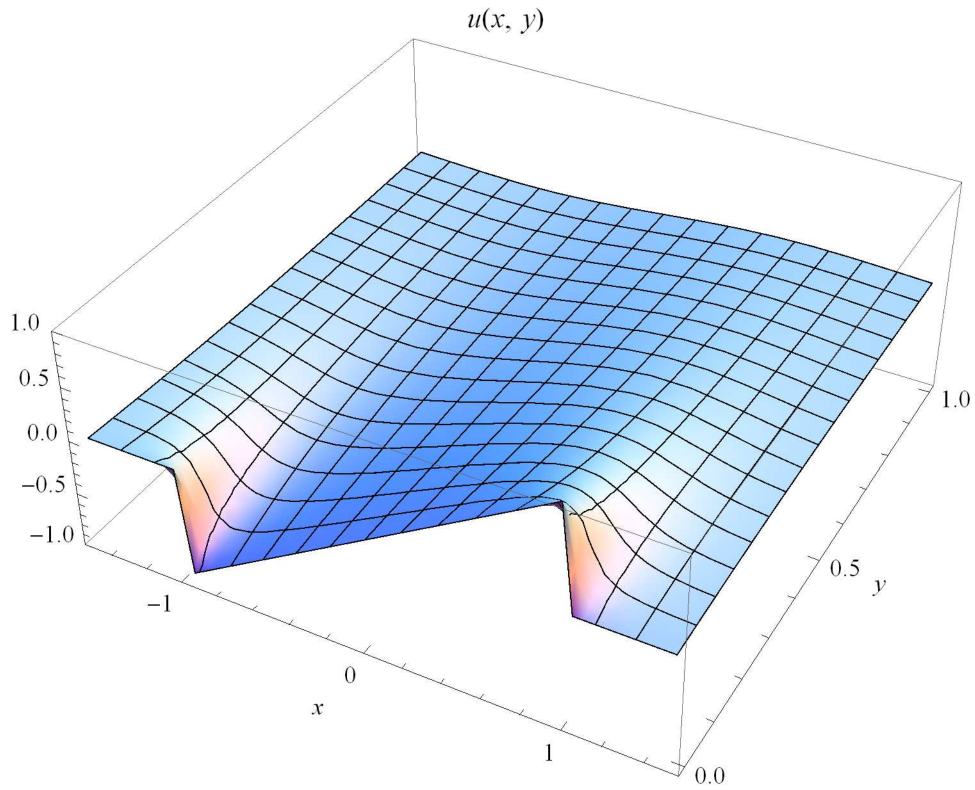

Figure 10: Plot of the steady state temperature distribution $u(x, y)$ in the semi-infinite domain $y \geq 0$ with boundary distribution $h(x)$ given by (3.69).

Let us check that this recovers the right behaviour as $y \to 0$. The first term tends to zero so we eliminate that straight away from our investigations. The behaviour of the second term is a little more subtle. Let us fix $q = a = 1$ as in Figure 10. We have to consider three different domains for x. We study

$$F(x, y) = \frac{x}{\pi}\left[\arctan\left(\frac{(x+1)}{y}\right) - \arctan\left(\frac{x-1}{y}\right)\right]$$

in the limit as $y \to 0^+$ (i.e. from above). We should find that

$$\lim_{y \to 0^+} F(x, y) = h(x) \tag{3.84}$$

First consider $x < -1$. In this case the arguments of arctan *become*

$$\lim_{y \to 0^+} \frac{(x+1)}{y} \to -\infty, \tag{3.85}$$

$$\lim_{y \to 0^+} \frac{(x-1)}{y} \to -\infty, \tag{3.86}$$

so both arctans yield $-\pi/2$ and the respective contribution cancels, as is required for $h(x)$.

Second consider $-1 < x < 1$*. In this case the arguments of* arctan *become*

$$\lim_{y\to 0^+} \frac{(x+1)}{y} \to \infty, \tag{3.87}$$

$$\lim_{y\to 0^+} \frac{(x-1)}{y} \to -\infty, \tag{3.88}$$

Then

$$\lim_{y\to 0^+} \arctan(\frac{(x+1)}{y}) = \pi/2, \tag{3.89}$$

$$\lim_{y\to 0^+} \arctan(\frac{(x-1)}{y}) = -\pi/2 \tag{3.90}$$

and so their respective contributions add to get, for $-1 < x < 1$*,*

$$\lim_{y\to 0^+} F(x,y) = \frac{x}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = x \tag{3.91}$$

as is required for $h(x)$*.*

Finally consider $x > 1$*. In this case the arguments of* arctan *become*

$$\lim_{y\to 0^+} \frac{(x+1)}{y} \to \infty, \tag{3.92}$$

$$\lim_{y\to 0^+} \frac{(x-1)}{y} \to \infty, \tag{3.93}$$

so both arctans yield $\pi/2$ *and the respective contribution cancels, as is required for* $h(x)$*.*

So we have indeed proved that

$$\lim_{y\to 0^+} F(x,y) = \begin{cases} x, & -1 < x < 1, \\ 0, & \text{otherwise} \end{cases} \tag{3.94}$$

$$= h(x) \tag{3.95}$$

as required.

Finally we should show that the contribution from the integral at infinity is zero. This is rather difficult! But just involves taking limits of the solution and its derivative, in order to show that as $r \to \infty$

$$u(r,\theta) \sim \frac{1}{r^2}, \qquad \frac{\partial u}{\partial r}(r,\theta) \sim \frac{1}{r^3}. \tag{3.96}$$

and therefore the contribution from this integral is zero.

Some other slightly more simple examples involving boundary forcing are considered on Sheet 7. We did this example here so that you could see all of the separate elements that can occur.

The rest of this section is non-examinable (but hopefully interesting!). I have not provided too many details. If you are interested in further reading let me know.....

3.7 Helmholtz' equation in two spatial dimensions

Consider the equation known as Helmholtz' equation as introduced at the start of this section

$$\nabla^2 u(\mathbf{x}) + K^2 u(\mathbf{x}) = Q(\mathbf{x}) \tag{3.97}$$

on a domain D in two dimensions with $K \in \mathbb{R}$. It will also be subject to some homogeneous BCs usually of the form $u = 0$ or $\nabla u \cdot \mathbf{n} = 0$ or $\alpha u + \beta \nabla u \cdot \mathbf{n} = 0$ for some real constants α, β. The associated Green's function is therefore defined by the equation

$$\nabla^2 G(\mathbf{x}, \mathbf{x}_0) + K^2 G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) \tag{3.98}$$

with equivalent homogeneous BCs. This is often known as Green's function for Helmholtz equation.

3.7.1 Symmetry

In wave problems we often use the complex exponential (as we did in the one dimensional case) and therefore we have to consider all functions as complex.

Use Green's second identity (3.13) with $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_1)$ and $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_2)$ so that

$$\int_D \left(\overline{G(\mathbf{x}, \mathbf{x}_1)}(\nabla^2 + K^2)G(\mathbf{x}, \mathbf{x}_2) - G(\mathbf{x}, \mathbf{x}_2)\overline{(\nabla^2 + K^2)G(\mathbf{x}, \mathbf{x}_1)}\right)\ d\mathbf{x} = 0. \tag{3.99}$$

Use our usual tricks on the right hand side, and then since $(\nabla^2 + K^2)G(\mathbf{x}, \mathbf{x}_j) = \delta(\mathbf{x} - \mathbf{x}_j), j = 1, 2$, we exploit the property of the Dirac delta function to find

$$\overline{G(\mathbf{x}_2, \mathbf{x}_1)} = G(\mathbf{x}_1, \mathbf{x}_2) \tag{3.100}$$

in both two and three dimensions. So, for the Helmholtz operator, the Green's function is Hermitian symmetric.

3.7.2 Solution representation

We can obtain the solution representation exactly as in the Laplacian case, using Green's identity (3.13) with $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$ and u the solution to the physical problem in (3.97) so that

$$\int_D \left(\overline{G(\mathbf{x}, \mathbf{x}_0)}(\nabla^2 + K^2)u - u(\mathbf{x})\overline{(\nabla^2 + K^2)G(\mathbf{x}, \mathbf{x}_0)}\right)\ d\mathbf{x} = 0 \tag{3.101}$$

and therefore following our usual steps

$$u(\mathbf{x}) = \int_D Q(\mathbf{x}_0)G(\mathbf{x}, \mathbf{x}_0)\ d\mathbf{x}_0.$$

3.7.3 Conditions satisfied by the Green's function

We require that the Green's function satisfy (3.98) subject to some homogeneous BC. But what about its singular nature? It transpires that the Green's function has to have the same singular nature as that associated with the Laplacian. Since we are only interested in two dimensional problem for Helmholtz therefore, its local behaviour near the source is

$$\phi \sim \ln |\mathbf{x} - \mathbf{x}_0|.$$

3.7.4 "Free-space" Green's function

Let us construct the Green's function for Helmholtz on an unbounded domain: the so-called "free-space" Green's function. Let us work with a cylindrical coordinate system that is centred on the source position, i.e.

$$\mathbf{x} = \mathbf{y} + r(\cos\theta, \sin\theta). \tag{3.102}$$

Because there are no boundaries and the source is a circularly symmetric, we must seek a solution that is independent of θ. This means we seek solutions of the following equation for $r \neq 0$:

$$(\nabla^2 + k^2)\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + K^2\phi = 0$$

which is amounts to

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + K^2\phi = \delta(\mathbf{r}).$$

This equation is special - it is a special case of the more general *Bessel's equation of order* m which takes the form

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + \left(K^2 - \frac{m^2}{r^2}\right)\phi = 0$$

and whose two linearly independent solutions are written as $J_m(kr), Y_m(kr)$ which are known as the Bessel functions of order m. You can think of these as *special functions* (just like the circular functions sin and cos) which have many special properties which are also tabulated. Since we have $m = 0$ we are only interested in the Bessel functions of order zero. In particular we need to know how these functions behave close to the source and far away.

It turns out that as $x \to 0$ we have

$$J_0(x) \sim 1 + O(x^2), \qquad Y_0(x) \sim \frac{2}{\pi}\left(\gamma - \ln(2) + \ln(x)\right) \tag{3.103}$$

where $\gamma = 0.5772...$ is the Euler-Mascheroni constant. Furthermore as $x \to \infty$ we have

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x - \pi/4), \qquad Y_0(x) \sim \sqrt{\frac{2}{\pi x}}\sin(x - \pi/4). \tag{3.104}$$

It also turns out to be useful to define a slightly different Bessel function:

$$H_0^{(1)}(x) = J_0(x) + iY_0(x). \tag{3.105}$$

Its asymptotic behaviour is

$$H_0^{(1)}(x) \sim 1 + \frac{2i}{\pi}\left(\gamma - \ln(2) + \ln(x)\right) \qquad x \to 0, \tag{3.106}$$

$$H_0^{(1)}(x) \sim \sqrt{\frac{1}{\pi x}}(1-i)e^{ix} \qquad x \to \infty. \tag{3.107}$$

It would appear therefore that the function $H_0^{(1)}(Kr)$ possesses everything that we require in order for it to be the free-space Green's function associated with Helmholtz equation. Firstly, it is independent of θ, secondly it possesses a logarithmic singularity as $r \to 0$ and finally it behaves as an outgoing wave as $r \to \infty$.

All this means that the free space Green's function for Helmholtz in two dimensions is the Hankel function of zero order:

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4i} H_0^{(1)}(K|\mathbf{x} - \mathbf{y}|). \tag{3.108}$$

where the constant $1/4i$ ensures the $1/(2\pi)$ constant in front of the logarithm that is required, as in the Laplacian case.

3.8 Where next?

We have not had time to fully appreciate the Green's function for Helmholtz equation. The fact that we have found it (and that Bessel functions are easily to evaluate on modern computers) means that we can do a great deal with it. We can extend to the case of Green's functions for Helmholtz equation with boundaries as in the case of the Laplacian via the method of images, and these cases have great application. They give rise to the solution of *scattering problems*, e.g. we could solve the half-space problem with a source in the upper half-space this gives scattering of the incident wave source from the boundary. Via some transformations we can also find in a straightforward manner the Green's function for a boundary consisting of unbounded space but with a cylinder inside it. The associated Green's function is then the solution for scattering of an incident source from a cylinder. We can also solve for domains with periodic arrays of scatterers which have a huge variety of applications such as acoustic filters, ensuring noise reduction for example. In wave problems, if the obstacles have "corners", i.e. the boundary of the obstacle is not smooth, as well as standard "scattering", interesting "diffraction" effects occur creating very nice effects that can be investigated via ray theory and geometric diffraction.

Additionally, a not-so-complicated extension of the above, also shows that you can construct approximate "acoustic cloaks", layers of materials around obstacles which guide waves around the obstacle undisturbed (approximately). This is a huge area of current research. I hope you have an opportunity to study about these kinds of things in the future.........

3.9 Revision checklist

The following is a guide to what you should know from this section. Read each point and ask yourself if you understand what it means! Also, remember that associated theory from the relevant sections is examinable

- Know the Laplacian and Helmholtz operators.
- Be able to use vector identities to show that the Laplacian and Helmholtz are self-adjoint (sec. 3.1.1)
- Be able to "simple" multi-dimensional eigenvalue problems in Cartesian coordinates, such as Example 3.1
- Know properties 1-5 of eigenvalue problems from sec. 3.3
- Understand the filtering property of the multdimensional Dirac delta function
- Understand a great deal about Green's functions for Poisson's equation, in particular:
  - Be able to prove that it is symmetric (i.e. (3.30))
  - Understand the solution form (3.35) and be able to derive it
  - Be able to derive the solution forms for non-homogeneous BCs, e.g. (3.39) (IMPORTANT!)
  - Understand the conditions satisfied by the Green's function
  - Know the free-space Green's functions and be able to show that they solve Laplace's equation
  - Be able to derive Green's functions for bounded domains using the method of images
  - Use the solution forms for non-homogeneous BCs, together with the appropriate Green's function to derive the explicit solution for a variety of problems (Examples 3.4, 3.5 and exercises on Example Sheet 7)

4 Theory of integral equations and some examples in 1D

4.1 Linear integral operators

In the past two chapters we have discussed a linear *differential* operator $\mathcal{L}$. Here we shall move on to discuss linear *integral* operators. We shall denote such an operator by $\mathcal{K}$. A good example is one that we have already seen for BVPs associated with the Green's function, e.g.

$$u(x) = \int_a^b G(x,y)f(y)\ dy = \mathcal{K}f \tag{4.1}$$

More generally we shall denote a linear integral operator as

$$\mathcal{K}u = \int_a^b K(x,y)u(y)\ dy \tag{4.2}$$

where $K(x,y)$ is known as the *kernel* function.

An integral operator takes a function (belonging to some function space) and maps it to a function (in possibly a different function space). We are interested in *linear* integral operators, so analogously to the linear differential operator we have the property that

$$\mathcal{K}(c_1u_1 + c_2u_2) = c_1\mathcal{K}u_1 + c_2\mathcal{K}u_2.$$

We can define an adjoint integral operator $\mathcal{K}^*$, in the same way as for differential operators, via the inner product notation, i.e.

$$\langle v, \mathcal{K}u\rangle = \langle \mathcal{K}^*v, u\rangle$$

and the operator is self-adjoint if $\mathcal{K}^* = \mathcal{K}$.

4.2 What is an integral equation?

Remember that a *differential* equation is an equation involving *derivatives* of a function $u(x)$. We have to solve this equation for $u(x)$. So it should be clear then that an *integral* equation is an equation involving *integrals* of $u(x)$, possibly (usually) with some weighting function involved.

Example 4.1 *A very simple example of an integral equation is to find $u(x)$ such that*

$$\int_0^1 u(y)\ dy = 1.$$

Of course the solution is not unique! We could have e.g. $u(x) = (n+1)x^n$ for any $n > 0$ and other functions also satisfy this.

Note that the above example is of the form

$$\mathcal{K}u = \int_a^b K(x,y)u(y)\ dy = f(x) \tag{4.3}$$

which is known as a *Fredholm integral equation of the first kind.* A slightly more general form, and one that often arises in applications is an integral equation of the form

$$u - \lambda\mathcal{K}u = u(y) - \lambda\int_a^b K(x,y)u(y)\ dy = f(x) \tag{4.4}$$

which is known as a *Fredholm integral equation of the second kind.*

If $f(x) = 0$ the integral equation is called homogeneous and then we have the form

$$\lambda\mathcal{K}u = u$$

which is of similar form to eigenvalue problems for ODEs. Note the slight difference in ordering/positioning here. This is the usual form for eigenvalue problems associated with Fredholm integral equations.

Finally suppose that

$$K(x,y) = \begin{cases} k(x,y), & a \le y \le x, \\ 0, & x \le y \le b \end{cases}$$

so that (4.3) and (4.4) become

$$\mathcal{K}u = \int_a^x k(x,y)u(y)\ dy = f(x), \tag{4.5}$$

$$u - \lambda\mathcal{K}u = u(y) - \lambda\int_a^x k(x,y)u(y)\ dy = f(x) \tag{4.6}$$

and these integral equation types are known as *Volterra integral equations of the first and second kind* respectively.

4.3 Volterra integral equations govern IVPs

It transpires that Volterra integral equations are equivalent to IVPs. Due to time restrictions we will not consider these type of integral equations here however.

4.4 Fredholm integral equations govern BVPs

Consider now the eigenvalue BVP

$$\mathcal{L}u = \lambda\mu(x)u + g(x)$$

for $x \in [a,b]$ subject to some homogeneous BCs $\mathcal{B}$ on $x = a$ and $x = b$.

Given that we are able to construct a Green's function $G(x,y)$ associated with the operator $\mathcal{L}$ we can treat the right hand side as the forcing term and immediately write

$$u(x) = \lambda\int_a^b K(x,y)u(y)\ dy + f(x)$$

where the kernel $K(x, y) = \mu(y)G(x, y)$ and

$$f(x) = \int_a^b g(y)G(x, y)\ dy.$$

BCs are satisfied by construction of the Green's function.

Example 4.2 *Write down the corresponding Fredholm integral equation for the eigenvalue BVP*

$$u''(x) = \lambda u(x) + 1$$

subject to $u(0) = 0$ *and* $u(1) = 0$.

We have determined the Green's function for the Laplacian operator $\mathcal{L} = d^2/dx^2$ *many times in Section 2. It is*

$$\begin{aligned} G(x, y) &= \begin{cases} x(y-1), & 0 \le x \le y, \\ y(x-1), & y \le x \le 1 \end{cases}, \\ &= x(y-1)H(y-x) + y(x-1)H(x-y) \end{aligned}$$

so that the corresponding Fredholm integral equation is

$$u(x) = \lambda \int_0^1 G(x, y)u(y)\ dy + f(x)$$

where

$$\begin{aligned} f(x) &= \int_0^1 G(x, y)\ dy = x\int_x^1 (y-1)\ dy + (x-1)\int_0^x y\ dy, && (4.7) \\ &= \frac{1}{2}x(x-1) && (4.8) \end{aligned}$$

4.5 Separable (degenerate) kernels

We have seen that a Fredholm integral equation of the second kind is defined as

$$u(x) = \lambda \int_a^b K(x,y)\, u(y)\, dy + f(x). \tag{4.9}$$

Definition 4.1 *The kernel $K(x,y)$ is said to be* ***degenerate (separable)*** *if it can be written as a sum of terms, each being a product of a function of x and a function of y. Thus,*

$$K(x,y) = \sum_{j=1}^{n} w_j(x)\, v_j(y) = \mathbf{w}(x) \cdot \mathbf{v}(y) \tag{4.10}$$

where $\mathbf{w}$ and $\mathbf{v}$ are vectors of order n.

Equation (4.9) may be solved by reduction to a set of simultaneous linear algebraic equations as we shall now show. Substituting (4.10) into (4.9) gives

$$u(x) = \lambda \int_a^b \left[\sum_{j=1}^{n} w_j(x)\, v_j(y) \right] f(y)\, dy + f(x)$$

$$= \lambda \sum_{j=1}^{n} \left[w_j(x) \int_a^b v_i(y)\, u(y)\, dy \right] + f(x)$$

and letting

$$c_j = \int_a^b v_j(y)\, u(y)\, dy = \langle v_j, f \rangle, \tag{4.11}$$

then

$$u(x) = \lambda \sum_{j=1}^{n} c_j u_j(x) + f(x). \tag{4.12}$$

For this class of kernel, it is sufficient to find the c_j in order to obtain the solution to the integral equation. Eliminating u between equations (4.11) and (4.12) (i.e. take inner product of both sides with v_i) gives

$$c_i = \int_a^b v_i(y) \left[\lambda \sum_{j=1}^{n} c_j w_j(y) + f(y) \right] dy,$$

or interchanging the summation and integration,

$$c_i = \lambda \sum_{j=1}^{n} c_j \int_a^b v_i(y)\, w_j(y)\, dy + \int_a^b v_i(y)\, f(y)\, dy. \tag{4.13}$$

Writing

$$a_{ij} = \int_a^b v_i(y)\, w_j(y)\, dy = \langle v_i, w_j \rangle, \tag{4.14}$$

and

$$f_i = \int_a^b v_i(y)\, f(y)\, dy = \langle v_i, f \rangle, \tag{4.15}$$

then (4.13) becomes

$$c_i = \lambda \sum_{j=1}^{n} a_{ij} c_j + f_i. \tag{4.16}$$

By defining the matrices

$$\mathbf{A} = (a_{ij}), \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

this equation may be written in matrix notation as

$$\mathbf{c} = \lambda \mathbf{A}\mathbf{c} + \mathbf{f}$$

i.e.

$$(I - \lambda \mathbf{A})\,\mathbf{c} = \mathbf{f} \tag{4.17}$$

where I is the identity matrix. **This is just a simple linear system of equations for c**. We therefore need to understand how we solve the canonical system $\mathbf{L}\mathbf{u} = \mathbf{f}$ where $\mathbf{L}$ is a given matrix, $\mathbf{f}$ is the given forcing vector and $\mathbf{u}$ is the vector to be determined. Let us remind ourselves of the Fredholm Alternative from Section 2, but adapted to linear algebraic systems of equations rather then ODEs.

The Fredholm Alternative for Linear Systems

Theorem 4.2 *We introduce the linear system*

$$\mathbf{L}\mathbf{u} = \mathbf{f}$$

where $\mathbf{L}$ *is an* $m \times n$ *matrix and* $\mathbf{u}$ *and* $\mathbf{f}$ *are* $1 \times n$ *vectors where* $\mathbf{f}$ *is given and* $\mathbf{u}$ *is unknown. Consider the homogeneous adjoint (transpose) problem*

$$\mathbf{L}^T \mathbf{v} = 0$$

where superscript T *denotes the transpose of the matrix. Then* ***EITHER***

1. *(When Det*$\mathbf{L} \neq 0$*). If the only solution to the homogeneous adjoint problem is the trivial solution* $\mathbf{u} = \mathbf{0}$ *then the solution to the inhomogeneous problem* $\mathbf{u}$ *exists and is unique*
   OR

2. *(When Det*$\mathbf{L} = 0$*). If there are non-trivial solutions to the homogeneous adjoint problem* $\mathbf{v} \neq 0$ *then* ***either***

   - *There are infinitely many solutions if* $\mathbf{v} \cdot \mathbf{f} = 0$*,*
     or
   - *There is no solution if* $\mathbf{v} \cdot \mathbf{f} \neq 0$*.*

(N.B. Of course the comments regarding Det$\mathbf{L}$ *only apply if the matrix ix square, which it will be for our applications of this theorem to integral equations).*

[Reminder, rank($\mathbf{L}$) is the number of linearly independent rows (or columns) of the matrix $\mathbf{L}$.]

In the first case of 2. then there are infinitely many solutions because the theorem states that we can find a particular solution $\mathbf{u}_{PS}$ and furthermore, the homogeneous system

$$\mathbf{L}\mathbf{u} = \mathbf{0} \tag{4.18}$$

has $p = n -$ rank $(A) > 0$ non-trivial linearly independent solutions

$$\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p.$$

so that there are infinitely many solutions because we can write

$$\mathbf{u} = \mathbf{u}_{PS} + \sum_{j=1}^{p} \alpha_j \hat{\mathbf{u}}_j$$

where α_j are **arbitrary** constants (and hence there are infinitely many solutions).

To illustrate this theorem consider the following simple 2×2 matrix example:

Example 4.3 *Determine the solution structure of the linear system* $\mathbf{L}\mathbf{u} = \mathbf{f}$ *when*

$$\text{(I)} \quad \mathbf{L} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad\qquad \text{(II)} \quad \mathbf{L} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \tag{4.19}$$

and in the case of (II) when

$$\mathbf{f} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad\qquad \mathbf{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.20}$$

(I) Since Det($\mathbf{L}$*)* $= 1 \neq 0$ *the (unique) solution exists for any* $\mathbf{f}$*, given by* $\mathbf{u} = \mathbf{L}^{-1}\mathbf{f}$*.*

(II) Here Det($\mathbf{L}$*)* $= 0$ *so we have to consider solutions to the adjoint homogeneous system, i.e.*

$$\mathbf{L}^T \hat{\mathbf{v}} = 0 \tag{4.21}$$

i.e.

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \hat{\mathbf{v}} = 0. \tag{4.22}$$

This has the 1 non-trivial linearly independent solution $\hat{\mathbf{v}}_1 = (2 \quad -1)^T$*. It is clear that there should be 1 such solution, because* $p = n -$*rank(*$\mathcal{A}$*)* $= 2 - 1 = 1$*.*

The Fredholm Alternative then says that there are infinitely many solutions if

$$\mathbf{v} \cdot \mathbf{f} = 0$$

and no solution if this does not hold. So in the first case of $\mathbf{f}$ *above there are infinitely many solutions whereas in the second there is no solution.*

We can see that this is the case by studying the original homogeneous system

$$\mathbf{L}\hat{\mathbf{u}} = 0 \tag{4.23}$$

i.e.

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \hat{\mathbf{x}} = 0 \tag{4.24}$$

one can see that this has the 1 non-trivial linearly independent solution $\mathbf{u}_1 = (1 \quad -1)^T$. *Therefore if we can find a particular solution there will be an infinite number of solutions of the form* $\mathbf{u} = \mathbf{u}_{PS} + \alpha_1 \mathbf{u}_1$ *for some constant* α_1. *The Fredholm alternative (as applied above) then tells us when we can and cannot find particular solutions. So when*

$$\mathbf{f} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{4.25}$$

a solution to the problem $\mathbf{L}\mathbf{u} = \mathbf{f}$ *will exist since* $\hat{\mathbf{v}} \cdot \mathbf{f} = 0$. *Indeed it does. Note that* $\mathbf{u}_{PS} = (1/2 \ \ 1/2)^T$.

On the other hand when

$$\mathbf{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.26}$$

a solution to the problem $\mathbf{L}\mathbf{u} = \mathbf{f}$ *will NOT exist since* $\hat{\mathbf{v}} \cdot \mathbf{f} \neq 0$. *The equations in the linear system are incompatible.*

Let us now use the above to find the solution or otherwise of a degenerate integral equation.

Example 4.4 *Consider the integral equation*

$$u(x) = \lambda \int_0^\pi \sin(x-y)\, u(y)\, dy + f(x). \tag{4.27}$$

Find
(i) the values of λ for which it has a unique solution,
(ii) the solution in this case,
(iii) the resolvent kernel.

For those values of λ for which the solution is not unique, find
(iv) a condition which $g(x)$ must satisfy in order for a solution to exist,
(v) the general solution in this case.

The solution proceeds as follows. Expand the kernel:

$$\begin{aligned} u(x) &= \lambda \int_0^\pi \sin(x-y)\, u(y)\, dy + f(x) \\ &= \lambda \int_0^\pi (\sin x \cos y - \cos x \sin y)\, u(y)\, dy + f(x) \end{aligned}$$

and hence it is clear that

$$K(x,y) = \sin x \cos y - \cos x \sin y$$

is separable. Thus

$$u(x) = \lambda \left[\sin x \int_0^\pi u(y) \cos y dy - \cos x \int_0^\pi u(y) \sin y dy \right] + f(x),$$

and so write

$$c_1 = \int_0^\pi u(y) \cos y dy, \tag{4.28}$$

$$c_2 = \int_0^\pi u(y) \sin y dy, \tag{4.29}$$

which gives

$$u(x) = \lambda [c_1 \sin x - c_2 \cos x] + f(x). \tag{4.30}$$

Substituting this value of $u(x)$ into (4.28) gives

$$c_1 = \int_0^\pi \{\lambda [c_1 \sin y - c_2 \cos y] + f(y)\} \cos y dy$$

$$= \lambda c_1 \int_0^\pi \sin y \cos y dy - \lambda c_2 \int_0^\pi \cos^2 y dy + \int_0^\pi f(y) \cos y dy.$$

Defining

$$f_1 = \int_0^\pi f(y) \cos y dy,$$

and noting the values of the integrals

$$\int_0^\pi \sin y \cos y dy = \frac{1}{2}\int_0^\pi \sin 2y dy = \left[-\frac{1}{4}\cos 2y\right]_0^\pi = 0,$$

$$\int_0^\pi \cos^2 y dy = \frac{1}{2}\int_0^\pi (1+\cos 2y)\, dy = \frac{1}{2}\left[y + \frac{1}{2}\sin 2y\right]_0^\pi = \frac{1}{2}\pi,$$

yields

$$c_1 = -\frac{1}{2}\pi\lambda c_2 + f_1.$$

Repeating this procedure, putting $u(x)$ from (4.30) into (4.29), gives

$$c_2 = \int_0^\pi \{\lambda\left[c_1 \sin y - c_2 \cos y\right] + f(y)\} \sin y dy$$

$$= \lambda c_1 \int_0^\pi \sin^2 y dy - \lambda c_2 \int_0^\pi \cos y \sin y dy + \int_0^\pi f(y) \sin y dy.$$

Observing that

$$\int_0^\pi \sin^2 y dy = \frac{1}{2}\int_0^\pi (1-\cos 2y)\, dy = \frac{1}{2}\left[y - \frac{1}{2}\sin 2y\right]_0^\pi = \frac{1}{2}\pi,$$

and writing

$$f_2 = \int_0^\pi f(y) \sin y dy,$$

then we obtain

$$c_2 = \frac{1}{2}\pi\lambda c_1 + f_2.$$

Thus, there is a pair of simultaneous equations for c_1, c_2:

$$\begin{aligned} c_1 + \tfrac{1}{2}\pi\lambda c_2 &= f_1, \\ -\tfrac{1}{2}\pi\lambda c_1 + c_2 &= f_2, \end{aligned}$$

or in matrix notation

$$\begin{bmatrix} 1 & \frac{1}{2}\pi\lambda \\ -\frac{1}{2}\pi\lambda & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{4.31}$$

Case (i): *These equations have a unique solution provided*

$$\det \begin{bmatrix} 1 & \frac{1}{2}\pi\lambda \\ -\frac{1}{2}\pi\lambda & 1 \end{bmatrix} \neq 0,$$

i.e.

$$1 + \frac{1}{4}\pi^2\lambda^2 \neq 0 \quad \text{or} \quad \lambda \neq \pm\frac{2i}{\pi}.$$

In this case

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}\pi\lambda \\ -\frac{1}{2}\pi\lambda & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

or

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{1+\frac{1}{4}\pi^2\lambda^2} \begin{bmatrix} 1 & -\frac{1}{2}\pi\lambda \\ \frac{1}{2}\pi\lambda & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{1}{1+\frac{1}{4}\pi^2\lambda^2} \begin{bmatrix} f_1 - \frac{1}{2}\pi\lambda f_2 \\ \frac{1}{2}\pi\lambda f_1 + f_2 \end{bmatrix}$$

$$= \frac{1}{1+\frac{1}{4}\pi^2\lambda^2}\begin{bmatrix} \int_0^\pi f(y)\cos y dy - \frac{1}{2}\pi\lambda\int_0^\pi f(y)\sin y dy \\ \frac{1}{2}\pi\lambda\int_0^\pi f(y)\cos y dy + \int_0^\pi f(y)\sin y dy \end{bmatrix}$$

$$= \frac{1}{1+\frac{1}{4}\pi^2\lambda^2}\int_0^\pi \begin{bmatrix} \cos y - \frac{1}{2}\pi\lambda\sin y \\ \frac{1}{2}\pi\lambda\cos y + \sin y \end{bmatrix} f(y)\, dy.$$

Hence,

$$\begin{aligned} u(x) &= \lambda\left[c_1\sin x - c_2\cos x\right] + f(x) \\ &= \frac{\lambda}{1+\frac{1}{4}\pi^2\lambda^2} \\ &\quad \times \left[\int_0^\pi \left[\sin x\left(\cos y - \frac{1}{2}\pi\lambda\sin y\right) - \cos x\left(\frac{1}{2}\pi\lambda\cos y + \sin y\right)\right] f(y)\, dy\right] \\ &\quad + f(x) \end{aligned}$$

or

$$u(x) = \frac{\lambda}{1+\frac{1}{4}\pi^2\lambda^2}\int_0^\pi \left[\sin(x-y) - \frac{1}{2}\pi\lambda\cos(x-y)\right] f(y)\, dy + f(x).$$

This is the required solution. Often such solutions are written in the form

$$u(x) = \lambda\int_0^\pi R(x,y,\lambda) f(y) dy + f(x)$$

where $R(x,y,\lambda)$ is known as the ***resolvent kernel****:*

$$R(x,y,\lambda) = \frac{\sin(x-y) - \frac{1}{2}\pi\lambda\cos(x-y)}{1+\frac{1}{4}\pi^2\lambda^2}.$$

Case (ii): *If*

$$\det\begin{bmatrix} 1 & \frac{1}{2}\pi\lambda \\ -\frac{1}{2}\pi\lambda & 1 \end{bmatrix} = 0$$

i.e.

$$\lambda = +\frac{2i}{\pi} \text{ or } -\frac{2i}{\pi}$$

then there is either no solution or infinitely many solutions. With this, (4.31) becomes

$$\begin{bmatrix} 1 & \pm i \\ \mp i & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Solving these equations using row operations, $R2 \mapsto R2 \pm iR1$, gives

$$\begin{bmatrix} 1 & \pm i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \pm i f_1 \end{bmatrix}$$

or

$$c_1 \pm i c_2 = f_1,$$
$$0 = f_2 \pm i f_1.$$

The second equation places a restriction on $f(x)$, which by definition of the f_i, is

$$\int_0^\pi (\sin y \pm i\cos y)\, g(y) dy = 0. \tag{4.32}$$

This is the condition that $f(x)$ *must satisfy for the integral equation to be soluble, i.e. if* $f(x)$ *does not satisfy this, then (4.27) does not have a solution. Suppose this condition holds then we can set* c_2 *to take any arbitrary constant value,* $c_2 = \alpha$*, say. Thus,*

$$c_1 = \mp i\alpha + f_1,$$

and hence from (4.30), the solution of (4.27) is, when $\lambda = \pm 2i/\pi$*,*

$$u(x) = \pm\frac{2i}{\pi}\left[(\mp i\alpha + f_1)\sin x - \alpha\cos x\right] + f(x)$$

$$= \frac{2\alpha}{\pi}(\sin x \mp i\cos x) \pm \frac{2i}{\pi} f_1 \sin x + f(x)$$

for arbitrary α*, with constraint* $f_2 = \mp i f_1$ *or equivalently (4.32).*

We arrived at the above conclusions via simple row operations. Fredholm's theorem would have also told us the same information regarding constraints on f_1 *and* f_2*.*

4.6 Neumann series solution

As we have seen, a Fredholm integral equation of the second kind may be written as

$$u(x) = \lambda \int_a^b K(x,y)\, u(y)\, dy + f(x). \tag{4.33}$$

We also saw that if the Kernel is separable, then we may take steps to solve the integral equation analytically, by reducing it to a system of linear equations. However, what if the Kernel is *not* separable? In general these problems are much harder. One approach exploits the fact that for sufficiently small values of λ it looks like we can neglect the integral term to obtain the approximate solution

$$u(x) \simeq f(x) = u_0(x). \tag{4.34}$$

We can then improve on this approximation by substituting (4.34) into the right hand side of (4.33), i.e.

$$\begin{aligned} u(x) &\simeq u_0(x) + \lambda \int_a^b K(x,y)\, u_0(y)\, dy \\ &= u_0(x) + \lambda u_1(x). \end{aligned} \tag{4.35}$$

Substituting (4.35) into the right hand side of (4.33) then yields an even better approximation

$$\begin{aligned} u(x) &\simeq u_0(x) + \lambda \int_a^b K(x,y)\left[u_0(y) + \lambda u_1(y)\right] dy \\ &= u_0(x) + \lambda \int_a^b K(x,y)\, u_0(y)\, dy + \lambda^2 \int_a^b K(x,y)\, u_1(y)\, dy \\ &= u_0(x) + \lambda u_1(x) + \lambda^2 \int_a^b K(x,y)\, u_1(y)\, dy \\ &= u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x), \end{aligned}$$

say. This suggests letting

$$u(x) = \sum_{n=0}^{\infty} \lambda^n u_n(x), \tag{4.36}$$

which may be substituted into (4.33),

$$\sum_{n=0}^{\infty} \lambda^n u_n(x) = \lambda \int_a^b K(x,y) \left[\sum_{n=0}^{\infty} \lambda^n u_n(y)\right] dy + f(x)$$

$$= \sum_{n=0}^{\infty} \lambda^{n+1} \int_a^b K(x,y)\, u_n(y)\, dy + f(x).$$

Equating coefficients of like powers in λ gives

$$\begin{array}{rr} \lambda^0: & u_0(x) = f(x), \\ \lambda^n, \quad n>0: & u_n(x) = \int_a^b K(x,y)\, u_{n-1}(y)\, dy. \end{array} \tag{4.37}$$

Definition 4.3 *Formula (4.36), with (4.37), is called the* ***Neumann series*** *for the integral equation (4.33).*

Example 4.5 *Find the first 3 terms of the Neumann series for the integral equation*

$$u(x) = 1 + \epsilon \int_0^{1/2} (x-y)u(y)\,dy.$$

where $\epsilon \in \mathbb{R}$.

We note that $\lambda = \epsilon$ *and* $K(x,y) = (x-y)$ *so that*

$$u(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \ldots.$$

and it remains for us to determine each term. Clearly neglecting the integral term gives $u_0(x) = 1$, *and then*

$$u_1(x) = \int_0^{1/2} (x-y)u_0(y)dy \tag{4.38}$$

$$= \left[xy - \frac{y^2}{2}\right]_0^{1/2} \tag{4.39}$$

$$= \left(\frac{x}{2} - \frac{1}{8}\right) \tag{4.40}$$

and then

$$u_2(x) = \int_0^{1/2} (x-y)u_1(y)dy \tag{4.41}$$

$$= \int_0^{1/2} (x-y)\left(\frac{y}{2} - \frac{1}{8}\right) dy \tag{4.42}$$

$$= \left[x\frac{y^2}{4} - x\frac{y}{8} - \frac{y^3}{6} + \frac{y^2}{16}\right]_0^{1/2} \tag{4.43}$$

$$= -\frac{1}{192} \tag{4.44}$$

so that the Neumann series takes the form

$$u(x) = 1 + \epsilon\left(\frac{x}{2} - \frac{1}{8}\right) - \frac{\epsilon^2}{192} + \ldots. \tag{4.45}$$

Note that this equation is also separable! We revisit this problem in Exercise 6 on Example Sheet 9.

But this approach is clearly only useful if the resulting series converges! Here is a simple theorem (without proof) which assess the convergence of the Neumann Series.

Theorem 4.4 *The Neumann series for the Fredholm equation*

$$u(x) = \lambda \int_a^b K(x,y)\, u(y)\, dy + f(x),$$

where $K(x,y)$ and $g(x)$ are bounded and absolutely integrable, converges absolutely for all values of λ such that

$$|\lambda| < \frac{1}{M(b-a)},$$

where

$$M = \sup_{\substack{a \le x \le b \\ a \le y \le b}} |K(x,y)|.$$

Example 4.6 *Determine for which values of ϵ the Neumann series converges in Example 4.5 above.*

Using Theorem 4.6, we see that $b = 1/2, a = 0$ and

$$\begin{aligned} M &= \sup_{\substack{a \le x \le b \\ a \le y \le b}} |K(x,y)| \\ &= \sup_{\substack{a \le x \le b \\ a \le y \le b}} |x - y| \\ &= 1/2 \end{aligned}$$

so that the series converges for

$$|\epsilon| < \frac{1}{M(b-a)} = 4.$$

Let us now consider an application of integral equations.

4.7 Wave propagation in heterogeneous media

4.7.1 Problem set-up in terms of integral equations

Frequently, we are interested in media where the material properties are functions of space (they are called *heterogeneous* or *inhomogeneous* media). The wave propagation properties of these media can be very complicated indeed. Let us take the simplest possible example. We shall consider wave propagation in one dimension, e.g. the transverse motion of a string, as considered in section 2.9. In that case we considered the string to be homogeneous and therefore, time harmonic waves satisfy

$$\frac{d^2u}{dx^2} + k_0^2 u = 0 \tag{4.46}$$

where $k_0 \in \mathbb{R}$ is the so-called wavenumber and we note that it is a constant. This means that the wave speed is fixed and the waves can propagate without reflection. Suppose now however that the density of the string changes continuously or with only a finite number of discontinuities in density so that instead of (4.47), transverse displacements satisfy

$$\frac{d^2u}{dx^2} + k^2(x)u = 0. \tag{4.47}$$

where $k(x)$ has at most a finite number of discontinuities. We note that we will always consider problems where the string and its slope are continuous.

Let us consider an infinite string, and take $k(x) \to k_0$ as $|x| \to \pm\infty$. In particular we are interested in what happens when an *incident wave* travelling to the right (since we always assume time dependence of $\exp(-i\omega t)$) of the form

$$u_i(x) = e^{ik_0x}$$

is scattered (reflected and transmitted) by some inhomogeneous medium. We usually write the *total field* $u(x)$ as

$$u(x) = u_i(x) + u_s(x)$$

where $u_s(x)$ is the "scattered" field (the unknown bit). See figure 11. Exact solutions to problems involving (4.47) are not easy to find. It transpires that we can use the Green's function that we derived in Section 2.9 associated with an infinite string to put (4.47) into a convenient form in order to find solutions.

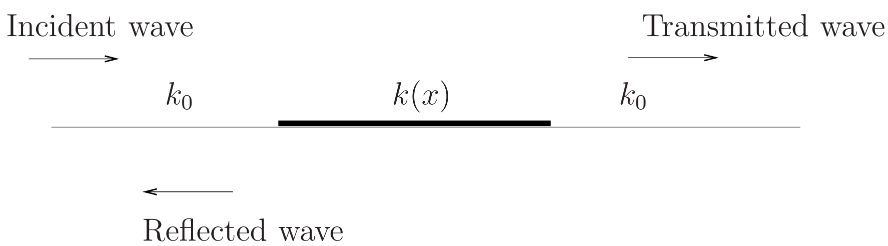

Figure 11: Figure depicting reflection and transmission of transverse waves on a stretched string. The region $k(x) \neq k_0$ and this inhomogeneity causes the wave scattering.

Firstly note that u_i satisfies

$$\frac{d^2u_i}{dx^2} + k_0^2 u_i = 0. \tag{4.48}$$

The idea is to re-write (4.47) so that it is an equation for the scattered field with constant coefficients but with a forcing term on the right hand side. So we write $u = u_i + u_s$ so that

$$u_i'' + u_s'' + k^2(x)(u_i + u_s) = 0$$

and then we "add zero", $k^2(x) = k^2(x) - k_0^2 + k_0^2$ so that

$$u_i'' + u_s'' + (k^2(x) - k_0^2 + k_0^2)(u_i + u_s) = 0.$$

and regroup the terms. Therefore

$$(u_i'' + k_0^2 u_i) + (u_s'' + k_0^2 u_s) + (k^2(x) - k_0^2)(u_i + u_s) = 0.$$

The first term is zero and therefore we have

$$u_s''(x) + k_0^2 u_s(x) + (k^2(x) - k_0^2)u(x) = 0.$$

or rather

$$u_s''(x) + k_0^2 u_s(x) = (k_0^2 - k^2(x))u(x) = f(x).$$

Note that the right hand side is unknown as it involves $u(x)$. However, since we know a Green's function for the operator (as determined in section 2.9) we can simply write

$$u_s(x) = \int_{-\infty}^{\infty} f(x_0)G(x, x_0)\ dx_0 \tag{4.49}$$

$$= \int_{-\infty}^{\infty} (k_0^2 - k^2(x_0))u(x_0)G(x, x_0)\ dx_0 \tag{4.50}$$

and therefore

$$u(x) = u_i(x) + \int_{-\infty}^{\infty} (k_0^2 - k^2(x_0))u(x_0)G(x, x_0)\ dx_0. \tag{4.51}$$

We can see that this is a Fredholm integral equation for the wave field $u(x)$. Furthermore in general the Kernel is not separable.

Note at this point that this derivation is possible because the function $u(x)$ and its derivative $u'(x)$ are continuous. In more general problems (even in one dimension) and in acoustics in higher dimensions the dependent variable does not have the property that its derivative is continuous which makes the analogous integral equations more complicated. But you can still write them down in terms of a Green's function!.

4.7.2 A Neumann series solution and its convergence

Let us now assume that $k^2(x) = k_0$ for $x \notin [a, b]$. We want to investigate under what circumstances we can look for Neumann series solutions of the problem above. We write

$$u(x) = u_i(x) + \int_a^b (k_0^2 - k^2(x_0))u(x_0)G(x, x_0)\ dx_0, \tag{4.52}$$

$$= u_i(x) + k_0 \int_a^b (k_0^2 - k^2(x_0))u(x_0)\frac{1}{k_0}G(x, x_0)\ dx_0. \tag{4.53}$$

And we apply (4.4) with $\lambda = k_0$ and

$$K(x, x_0) = \frac{k_0^2 - k^2(x)}{2ik_0^2} \exp\left(ik_0|x - x_0|\right).$$

It is easy to show that

$$|K(x, x_0)| = \left|\frac{1 - \kappa^2(x)}{2}\right|$$

where $\kappa(x) = k(x)/k_0$, so that we have

$$M = \sup_{a \le x \le b} \left|\frac{1 - \kappa^2(x)}{2}\right|$$

and so

$$\frac{1}{M} = \inf_{a \le x \le b} \frac{2}{|1 - \kappa^2(x)|}.$$

Therefore the Neumann series converges when

$$k_0 < \inf_{a \le x \le b} \left[\frac{2}{(b - a)(\kappa^2(x) - 1)}\right]$$

or rather

$$k_0(b - a) < \inf_{a \le x \le b} \left[\frac{2}{(\kappa^2(x) - 1)}\right] \tag{4.54}$$

where $x \in [a, b]$. What this tells us is that for a given thickness of inhomogeneous region (which is $b - a$), and for a given type of inhomogeneity $\kappa(x)$, the solution will only be valid for waves with wavenumber k_0 below a certain value. Given that the wavenumber is inversely proportional to the wavelength $k_0 = 2\pi/\Lambda$, this means that the solution only convergese if the wavelength is large enough as compared with the size of the inhomogeneous layer. In the other limit, when we have high frequency wave propagation a different approximation (the so-called WKBJ approximation which you may have seen in courses on asymptotics) can be used.

Example 4.7 *Suppose that the inhomogeneous region is a layer of thickness 1 metre (m) ($b - a = 1$) and with wavenumber $k_1 = 2k_0$. For what wavenumbers would a Neumann series solution of (4.51) be valid?*

Well we just use these values in (4.54) to get

$$k_0 < \frac{2}{3} m^{-1}$$

where we note that we have added units. Since wavelengths Λ are related to wavenumber via $k_0 = 2\pi/\Lambda$ this means

$$\Lambda > 3\pi m.$$

So if the incident wavelength is less than $3\pi m$ the Neumann series solution would not converge.

4.7.3 The Born approximation

The Born approximation is very often used in scattering problems. The Born approximation is essentially the correction to the incident field that arises due to the first correction term in the Neumann series. Let us develop this idea via an example.

Example 4.8 *Suppose that $a = 0, b = L$ and suppose $k(x) = k_1$ which is a constant. Of course in this case the problem is easy! We can solve using other methods since we always know the solution of the problem for constant coefficients, but it gives us a simple problem to look at initially. Harder problems are on the Examples sheet. In this case however we get*

$$u(x) = u_i(x) + (k_0^2 - k_1^2) \int_0^L u(x_0)G(x, x_0)\ dx_0. \tag{4.55}$$

The Born approximation is then to put $u(x_0) = u_i(x_0)$ under the integral on the right hand side and take as the solution whatever comes out. We get

$$u(x) = u_i(x) + (k_0^2 - k_1^2) \int_0^L u_i(x_0)G(x, x_0)\ dx_0. \tag{4.56}$$

We see that $u_i(x)$ is the first term in the Neumann series. To only take this term would not make sense as it would mean we were ignoring all scattering in the medium. The integral term is the first approximation (the Born approximation) to the scattered field.

Now, consider the integral term on its own. We have to be careful!

$$\int_0^L u_i(x_0)G(x, x_0)\ dx_0 = \frac{1}{2ik_0} \int_0^L e^{ik_0x_0}e^{ik_0|x-x_0|}dx_0. \tag{4.57}$$

In particular it matters where we are, i.e. whether $x < x_0$ or $x > x_0$.

Case 1, $x < 0$.

In this case since x_0 is the integration variable we always have $x_0 > x$ so that

$$|x - x_0| = x_0 - x$$

and so

$$e^{-ik_0x} \int_0^L e^{2ik_0x_0}dx_0 = \frac{(e^{2ik_0L} - 1)}{2ik_0}e^{-ik_0x}$$

and so

$$u_s(x) = \frac{(k_1^2 - k_0^2)(e^{2ik_0L} - 1)}{4k_0^2}e^{-ik_0x}.$$

This is the Born approximation to the reflected field. The first approximation to the reflection coefficient is

$$R = \frac{(k_1^2 - k_0^2)(e^{2ik_0L} - 1)}{4k_0^2}.$$

Case 2, $x > 1$**.**

In this case since x_0 is the integration variable we always have $x_0 < x$ so that

$$|x - x_0| = x - x_0$$

and so

$$\int_0^L e^{ik_0 x} dx_0 = Le^{ik_0 x}$$

and so

$$u_s(x) = \frac{(k_0^2 - k_1^2)L}{2ik_0} e^{ik_0 x}$$

Since $u(x) = u_i(x) + u_s(x)$, the Born approximation to the transmitted field is thus

$$u(x) = \left(1 + \frac{(k_0^2 - k_1^2)L}{2ik_0}\right) e^{ik_0 x}.$$

One can also determine the field inside the layer, but we will not do this here.

Note that we used complex notation in the above. In the end we would either take the real or imaginary parts corresponding to an input of $\cos(k_0 x)$ or $\sin(k_0 x)$ respectively.

4.7.4 Non destructive testing

The formulation above says that given k_1 and k_0 we can determine the scattered field (this is called the *forward problem*). In many problems however we want to use the wave field to *determine* k_1 (this is called the *inverse problem*). So for example suppose we know the layer thickness (which is unity as above) and we know k_0. If we measure the scattered field $u_s(x)$ at a given location then the expression (4.8) allows us to determine k_1! What is generally measured is the *reflection coefficient* which here is $R = |(k_1^2 - k_0^2)(e^{2ik_0} - 1)/(4k_0^2)|$. And given that k_0 is known we could therefore determine k_1.

Such "non-destructive testing" or "imaging" (although much more advanced versions!) is the key to a huge variety of important problems, for example ultrasonic imaging of babies in the womb and scans for detecting tumours, but also the key to finding oil and also radar detection mechanisms. There are a huge number of interesting problems still to solve in these and related areas and mathematics has a huge part to play.

4.8 Revision checklist

The following is a guide to what you should know from this section. Read each point and ask yourself if you understand what it means! Also, remember that associated theory from the relevant sections is examinable

- Understand what a linear integral operator is and know what "adjoint" and "fsel-adjoint" means (I would not ask you to find the adjoint integral operator).
- Be able to identify different types of integral operator.

- Know that Volterra integral equations govern IVPs whereas Fredholm integral equations govern BVPs
- Be able to take an eigenvalue BVP and use the associated Green's function to write the BVP in the form of an integral equation (with the Green's function as its kernel).
- Know what a separable/degenerate kernel is
- Be able to solve Fredholm equations when the kernel is separable which includes:
  - being able to understand and able to apply the Fredholm Alternative for linear algebraic systems
- Know what a Neumann series solution is and know the theorem regarding their convergence
- Understand the application to wave propagation in heterogeneous media, including the derivation of the integral equation (4.51).
- Know the difference between the Neumann series and the Born approximation
- Be able to apply Neumann series and/or the Born approximation to find the reflected and/or scattered field from a heterogeneous medium

A Some helpful stuff (which you should know!)

Cartesian coordinates

With $\mathbf{x} = (x, y, z)$, with $x, y, z \in (-\infty, \infty)$ we have

$$\nabla u = \frac{\partial u}{\partial x}\mathbf{e}_x + \frac{\partial u}{\partial y}\mathbf{e}_y + \frac{\partial u}{\partial z}\mathbf{e}_z \tag{A.1}$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are unit vectors pointing in the x, y, z directions. Finally

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{A.2}$$

Volume and surface elements are

$$dV = dxdydz, \tag{A.3}$$
$$dS = dxdy \tag{A.4}$$

for surfaces in the xy plane for example (constant z).

Cylindrical Polar Coordinates

With $\mathbf{x} = (x, y, z)$, cylindrical polar coordinates r, θ, z are defined via

$$x = r\cos\theta, \tag{A.5}$$
$$y = r\sin\theta, \tag{A.6}$$
$$z = z. \tag{A.7}$$

with $r \geq 0, \theta \in [0, 2\pi), z \in (-\infty, \infty)$. We have

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta + \frac{\partial u}{\partial z}\mathbf{e}_z \tag{A.8}$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are unit vectors pointing in the r, θ, z directions.

We also have

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}. \tag{A.9}$$

Volume and surface elements are

$$dV = rdrd\theta dz, \tag{A.10}$$
$$dS = rd\theta dz \tag{A.11}$$

for surfaces parallel to the z-axis (constant r).

Spherical Polar Coordinates

With $\mathbf{x} = (x, y, z)$, cylindrical polar coordinates r, θ, ϕ are defined via

$$x = r\cos\theta\sin\phi, \tag{A.12}$$
$$y = r\sin\theta\sin\phi, \tag{A.13}$$
$$z = r\cos\phi. \tag{A.14}$$

with $r \geq 0, \theta \in [0, 2\pi), \phi \in [0, \pi)$. We have

$$\nabla u = \frac{\partial u}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\phi}\frac{\partial u}{\partial \phi}\mathbf{e}_\phi \tag{A.15}$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ are unit vectors pointing in the r, θ, ϕ directions.

We also have

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \phi^2}. \tag{A.16}$$

Volume and surface elements are

$$dV = r^2\sin\phi dr d\theta d\phi, \tag{A.17}$$
$$dS = r^2\sin\phi d\theta d\phi \tag{A.18}$$

for surfaces with constant r.

Divergence Theorem

$$\int_D \nabla\cdot\mathbf{f}\, dV = \int_{\partial D}\mathbf{f}\cdot\mathbf{n} dA \tag{A.19}$$

where V and A denote the volume and surface of the domain D of integration.

B Example sheets

There are 9 examples sheets to work distributed throughout weeks 1-12. Students should ensure that they have looked at, and studied the examples sheet together with the accompanying notes, *before* the examples class to get the most out of the class.

Example Sheet 1

Topics covered: solution of ODEs by method of undetermined coefficients and variation of parameters, the "influence function", separation of variables

All of this sheet is material that should be revision! Questions 1-4 relate to ODE solution methods which you studied in MT10121. Questions 5-7 are asking for solutions of ODEs in certain forms which are convenient for this course. Questions 8 and 9 discuss separation of variables which you have done in MT20401. Ensure you know this work thoroughly! If you don't understand it, go and read some of your first (MT10121) and second year (MT20401) notes!

1. Determine the general solution of the following ODEs:

$$\begin{array}{llllll} \text{(i)} & u''-u'-6u=0, & \text{(ii)} & u''+u=0, & \text{(iii)} & u''-u=0 \\ \text{(iv)} & u''-2u'+u=0, & \text{(v)} & x^2u''+7xu'+5u=0, & \text{(vi)} & x^2u''-xu'-8u=0 \end{array}$$

2. Using what you determined in Q1. determine the particular solution of the following ODEs

$$\begin{array}{llll} \text{(i)} & u''-u'-6u=a(x), & \text{(ii)} & u''+u=b(x), \\ \text{(iii)} & u''-u=c(x), & \text{(iv)} & x^2u''+7xu'+5u=d(x) \end{array}$$

   using the method of undetermined coefficients, where

$$\begin{array}{llll} \text{(a)} & a(x)=\exp(3x), & \text{(b)} & a(x)=\exp(-3x), \\ \text{(c)} & b(x)=\sin(x), & \text{(d)} & b(x)=\exp(\mathrm{i}x), \\ \text{(e)} & c(x)=\exp(x), & \text{(f)} & c(x)=\exp(-x), \\ \text{(g)} & d(x)=x, & \text{(h)} & d(x)=1/x. \end{array}$$

3. Confirm the results in Q2 by using the method of variation of parameters (do enough questions to understand the method!)

4. Consider the ODE
$$u''+2u'+u=0.$$
   One solution is $u_1(x)=\exp(-x)$. Determine the second solution by the method of reduction of order, i.e. pose the solution $u_2(x)=v(x)u_1(x)$.

5. In section 2.2 we showed that the solution to the BVP
$$u''(x)=f(x)$$
   subject to the BCs $u(0)=0, u(L)=0$ could be written in the form
$$u(x)=\int_0^L G(x,x_0)f(x_0)dx_0$$
   where $G(x,x_0)=\frac{x}{L}(x_0-L)H(x_0-x)+\frac{x_0}{L}(x-L)H(x-x_0)$.

(i) If we modify the problem so that the BCs are inhomogeneous, i.e. $u(0) = u_0, u(L) = u_L$ show that the solution is

$$u(x) = \int_0^L G(x, x_0) f(x_0) dx_0 + \frac{x}{L}(u_L - u_0) + u_0.$$

(ii) By integrating the ODE twice, show that the solution can also be written in the form

$$u(x) = \int_0^x \int_0^{x_0} f(x_1) \; dx_1 dx_0 + c_1 x + c_2. \tag{B.1}$$

(iii) Impose the BCs and integrate by parts in (B.1) to show that these solution forms are equivalent.

6. Using the method of variation of parameters, determine the solution to the BVP

$$u''(x) = f(x)$$

subject to the BCs $u(0) = 0, u'(L) = 0$ in the form

$$u(x) = \int_0^L G(x, x_0) f(x_0) dx_0$$

determining the form of the influence (or Green's) function $G(x, x_0)$.

7. (i) Using the method of variation of parameters try to determine a solution to the BVP

$$u''(x) = f(x)$$

subject to the BCs $u'(0) = 0, u'(L) = 0$. What constraint is required on $f(x)$ and why does this make sense physically, interpreting this problem in the sense of the steady state heat equation with insulated BCs?

(ii) With $L = 2\pi$ does a solution exist if (a) $f(x) = e^x$ and (b) $f(x) = \sin x$? If so, is the solution unique?

8. Using separation of variables $u(x, t) = X(x)T(t)$, determine the solution of the following boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

given that $-\lambda$ is the "separation constant" and λ *must* be real and positive **in this case** (see later for why, or look back at your notes from MT20401) so that

$$X''(x) + \lambda X(x) = 0, \qquad T'(t) + \lambda T(t) = 0.$$

We note that $x \in [0, 1]$ and $t \geq 0$ with *initial conditions*

$$u(x, 0) = u_0(x)$$

and *boundary conditions*

$$u(0, t) = 0, \qquad u(1, t) = 0.$$

Note that this problem corresponds to the way that heat diffuses inside a bar of unit length with ends fixed at a temperature of zero degrees and with initial heat distribution $u_0(x)$.

9. Using separation of variables $u(x,t) = X(x)T(t)$, determine the solution of the following boundary value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

given that $-\lambda$ is the "separation constant" and λ *must* be real and positive **in this case** (see later for why, or look back at your notes from MT20401) so that

$$X''(x) + \lambda X(x) = 0, \qquad T''(t) + \lambda T(t) = 0.$$

We note that $x \in [0,1]$ and $t \geq 0$ with *initial conditions*

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = v_0(x)$$

and *boundary conditions*

$$u(0,t) = 0, \qquad u(1,t) = 0.$$

Note that this problem corresponds to plucking a string which is pinned at its ends, having initial plucked shape $u_0(x)$ and initial velocity $v_0(x)$.

Discuss the problems satisfied individually by $X(x)$ and $T(t)$. Why are they different even though they are governed by the same equation?

10. In questions 8 and 9 above, is there a "steady" solution, i.e. does the solution tend to any limit as $t \to \infty$? In these problems what is "forcing" the solution to act in the way that it does?

Example Sheet 2

Topics covered: Linear differential operators and their properties, inner products, the adjoint operator, Lagrange and Green's identities, Self-adjoint operators

Unless otherwise stated below the inner product on $[a, b]$ is the usual one with weighting $\mu(x) = 1$, i.e.

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) \ dx.$$

1. (i) Using inner product notation, the definition of the adjoint operator, and $\langle f, g \rangle = \overline{\langle g, f \rangle}$, prove that $(\mathcal{L}^*)^* = \mathcal{L}$.
   (ii) An operator $\mathcal{L}$ is called *skew adjoint* if $\mathcal{L}^* = -\mathcal{L}$. Show that
   (a) $(\mathcal{L} + \mathcal{L}^*)$ is self adjoint.
   (b) $(\mathcal{L} - \mathcal{L}^*)$ is skew adjoint.
   (iii) Show that any linear operator can be expressed as the sum of a self adjoint and skew adjoint operator

2. Consider the linear operator

$$\mathcal{L} = \frac{d^2}{dx^2} - \frac{d}{dx} + 2$$

   with homogeneous BCs $u(0) = u(1) = 0$.

   (i) Use integration by parts to determine the adjoint operator $\mathcal{L}^*$ and BCs $\mathcal{B}$.
   (ii) Partition $\mathcal{L}$ into the sum of a self adjoint and skew adjoint operator.

3. (i) Show that a necessary and sufficient condition for a linear operator $\mathcal{L}$ to be self-adjoint is that the inner product $\langle f, \mathcal{L}f \rangle$ is real valued for all f. *Hint: for sufficiency consider $\langle f + mg, f + mg \rangle$ for some constant m and functions f, g.*
   (ii) An operator is known as *positive* if $\langle f, \mathcal{L}f \rangle$ is real and positive for all f. Show that a positive operator is self-adjoint.

4. With respect to the usual inner product with weighting $\mu(x) = 1$ on $[a, b]$, show that the following operator is positive:

$$\mathcal{L}u = -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) - q(x)u$$

   where $p(x) > 0$ and $q(x) \leq 0$, with associated BCs $u(a) = u(b) = 0$.

5. Referring to Example 2.5 in the notes, determine the adjoint operator for

$$\begin{aligned}
&(iv) && \mathcal{L} = \frac{d^2}{dx^2} + 1, && \mathcal{B} = \{u(0) = u(1), u'(0) = u'(1)\},\\
&(v) && \mathcal{L} = \frac{d^2}{dx^2} + i\frac{d}{dx} + 1, && \mathcal{B} = \{u(0) = 0, u(1) = 0\},\\
&(vi) && \mathcal{L} = x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + 1, && \mathcal{B} = \{u(1) = 0, u'(2) = 0\},\\
&(vii) && \mathcal{L} = \frac{d^2}{dx^2} + k^2, && \mathcal{B} = \{u'(x) \mp iku(x) \to 0 \text{ as } x \to \pm\infty\},
\end{aligned}$$

with $k \in \mathbb{R}$ in (vii) and state whether the operator is self-adjoint, formally self-adjoint or neither.

6. Show that the adjoint operator associated with the general ODE

$$\mathcal{L} = p(x)\frac{d^2}{dx^2} + r(x)\frac{d}{dx} + q(x) \tag{B.2}$$

is

$$\mathcal{L}^* = \overline{p(x)}\frac{d^2}{dx^2} + \overline{\left(2\frac{dp}{dx} - r\right)}\frac{d}{dx} + \overline{\left(\frac{d^2p}{dx^2} - \frac{dr}{dx} + q\right)}.$$

7. With $\mathcal{L}$ as defined in (B.2), prove Lagrange's identity

$$\overline{v}\mathcal{L}u - \overline{\mathcal{L}^* v}u = \frac{d}{dx}\left[p(\frac{du}{dx}\overline{v} - u\frac{d\overline{v}}{dx}) + \left(r - \frac{dp}{dx}\right)u\overline{v}\right]$$

8. Suppose that $p(x), r(x)$ and $q(x)$ are complex functions.
(i) Show that the conditions for $\mathcal{L}$ defined in (B.2) to be (formally) self-adjoint are that $p(x)$ has to be a real function with $p' = \text{Re}(r)$ and $2\text{Im}(q) = (\text{Im}(r))'$ (where Re and Im denote the real and imaginary parts of the function respectively).
(ii) Using the result in (i), confirm the result that is the operator is real it is self-adjoint if $p' = r$.

9. Consider the fourth order differential operator

$$\mathcal{L} = \frac{d^4}{dx^2}$$

and take real functions u and v.

(i) Show that $v\mathcal{L}u - u\mathcal{L}v$ is an exact differential
(ii) Evaluate

$$\int_0^1 v\mathcal{L}u - u\mathcal{L}v \ dx$$

in terms of boundary data on v and u.
(iii) Show that

$$\int_0^1 v\mathcal{L}u - u\mathcal{L}v \ dx = 0$$

if u and v are any two functions satisfying

$$\phi(0) = 0, \qquad \phi(1) = 0,$$
$$\phi'(0) = 0, \qquad \phi''(1) = 0$$

where ϕ here is either of u or v. (iv) Give another example of BCs for which

$$\int_0^1 v\mathcal{L}u - u\mathcal{L}v = 0$$

Example Sheet 3

Topics covered: Sturm-Liouville operators and theorems, eigenvalues and eigenfunctions, modified inner product and orthogonal eigenfunctions

1. State whether the following are regular or singular Sturm-Liouville problems, or neither

(i)

$$\mathcal{L}u = \frac{d^2u}{dx^2} + u = 0, \qquad \mathcal{B} = \{u(0) = 0, u'(1) = 0\}$$

(ii)

$$\mathcal{L}u = \frac{d^2u}{dx^2} + u = 0, \qquad \mathcal{B} = \{u(0) = u(1), u'(0) = u'(1)\}$$

(iii)

$$x^2\frac{d^2u}{dx^2} + 3x\frac{du}{dx} + u = 0, \qquad \mathcal{B} = \{u(0) = 0, u(1) = 0\}$$

(iv)

$$x^2\frac{d^2u}{dx^2} + 2x\frac{du}{dx} + u = 0, \qquad \mathcal{B} = \{u(1) = 0, u(2) = 0\}$$

2. Confirm that the eigenfunctions and eigenvalues for the following BVP (considered in Example 2.7)

$$\mathcal{L} = \frac{d^2}{dx^2}$$

for $x \in [0, L]$ with $\mathcal{B} = \{\phi(0) = 0, \phi(L) = 0\}$ are

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

3. Confirm theorems 2-5 (you can assume Theorem 1, that eigenvalues are real) for the S-L eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0$$

for $x \in [0, L]$ with $\mathcal{B} = \{\phi(0) = 0, \phi'(L) = 0\}$.

4. Prove that the eigenvalues of a skew-adjoint operator ($\mathcal{L}^* = -\mathcal{L}$ as defined on Example Sheet 2) are purely imaginary, starting by considering $\langle f, \mathcal{L}f\rangle$.

5. Consider the eigenvalue problem with weighting function $\mu(x) = 1$.

$$\phi''(x) + \lambda\phi(x) = 0.$$

such that $\phi'(0) = 0$ and $\phi'(L) = 0$. Determine all eigenfunction and eigenvalues, justifying the choice of these.

6. Consider the fourth order operator and eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$$

subject to the same BCs as consider in question 9 on Example Sheet 2, i.e.

$$\phi(0) = 0, \qquad \phi(1) = 0,$$
$$\phi'(0) = 0, \qquad \phi''(1) = 0$$

Using the results from question 9 on Example Sheet 2 *and without trying to solve the eigenvalue problem*, show that eigenfunctions corresponding to distinct eigenvalues are orthogonal. What is the weighting function?

7. Consider the eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0$$

subject to the BCs $\mathcal{B} = \{\phi(0) = 0, \phi(\pi) = a\phi'(0)\}$ where $a \in \mathbb{R}$ and $a > 0$.
(i) Show that if $0 < a < \pi$ there exist a finite number of real eigenvalues given by the real roots of the equation $\sin\sqrt{\lambda}\pi = a\sqrt{\lambda}$, i.e. $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding eigenfunctions $\sin\sqrt{\lambda_k}x$.
(ii) What happens when $a = \pi$?
(iii) Show that if $a > \pi$ there are no real roots.
(iv) Theorem 2 associated with regular S-L problems states that there are an infinite number of eigenvalues for S-L problems. Why does this example not contradict this theorem?

8. The case of periodic BCs is not a regular S-L problem. Some of theorems 1-5 associated with S-L problems do not apply. In particular eigenfunctions are *not* unique (theorem 3). Confirm that this is true for the following problem with periodic BCs:

$$\phi''(x) + \lambda\phi(x) = 0$$

with periodic BCs: $\mathcal{B} = \{\phi(-L) = \phi(L), \phi'(-L) = \phi'(L)\}$.

Example Sheet 4

Topics covered: Fredholm Alternative, Eigenfunction expansion representation of the Green's function, Properties of the Dirac Delta function, Determination of Green's function by direct methods for S-L operators

1. (i) Prove the statement in the Fredholm Alternative that any solution of the homogeneous adjoint problem $v(x)$ will always be orthogonal to the forcing $f(x)$, i.e. in general $\langle v, f\rangle = 0$.
(ii) For regular S-L problems, if the associated BCs are modified to be inhomogeneous, say $u(0) = u_0$ and $u(L) = u_L$ but we retain the homogeneity of the adjoint BCs, use Green's identity to determine a modified Fredholm condition relating the solution to the adjoint problem and the forcing.

2. (i) Complete (b)-(d) of Example 2.11 (and i've added another example for good measure!), i.e. for the following ODE/BC pairings use the Fredholm Alternative to state if a solution exists and if so if it is unique (note that you do not solve the inhomogeneous BVP in order to show this!).
$$u''(x) + \psi u(x) = \sin x$$
with
$$\begin{array}{lll} \text{(b)} & \psi = 1, & \mathcal{B} = \{u'(0) = 0, u'(\pi) = 0\} \\ \text{(c)} & \psi = -1, & \mathcal{B} = \{u(0) = 0, u(\pi) = 0\} \\ \text{(d)} & \psi = 2, & \mathcal{B} = \{u(0) = 0, u(\pi) = 0\} \\ \text{(e)} & \psi = 1, & \mathcal{B} = \{u(0) = 0, u'(\pi/2) = 0\} \end{array}$$
(ii) Confirm the results in (i) by trying to solve the BVPs directly.

3. Consider the BVP
$$\mathcal{L}u = u'' - 3u' + 2u = f(x)$$
subject to the BCs $\mathcal{B} = \{u(1) = 0, u'(1) = (e^2/(1-e))u(0)\}$.
(i) Construct the homogeneous adjoint problem $\mathcal{L}^*v = 0$ subject to $\mathcal{B}^*$.
(ii) Use the Fredholm Alternative to determine the existence and uniqueness conditions on solutions to the original BVP.

4. Show that the eigenfunction expansion of the Green's function determined in (2.115) of Example 2.12, i.e.
$$G(x, x_0) = -\frac{2}{L}\sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)\sin(n\pi x_0/L)}{(n\pi/L)^2}.$$
is equivalent to the polynomial solution obtained in equation (2.41), i.e.
$$G(x, x_0) = \begin{cases} \dfrac{x}{L}(x_0 - L), & 0 \le x \le x_0, \\ \dfrac{x_0}{L}(x - L), & x_0 \le x \le L. \end{cases}$$
by posing a Fourier Series expansion. [Reminder - Fourier series are defined on intervals $x \in [-L, L]$, so you must define the "odd extension" of the Green's function above (see MATH20401)]

5. Determine the eigenfunction expansion for the Green's function associated with the following BVP:
$$u''(x) = f(x)$$
subject to the BCs $u(0) = 0, u'(L) = 0$.

   Show that this expansion is equivalent to the form for $G(x, y)$ determined in Q6 on Sheet 1. [Note the comment above regarding Fourier series again]

6. Prove the expressions (2.124) and (2.125) in the notes, i.e.
$$H(x - x_i) = \int_{-\infty}^{x} \delta(x_0 - x_i)\ dx_0,$$
$$\delta[c(x - x_i)] = \frac{1}{|c|}\delta(x - x_i)$$

7. Determine the Green's function associated with each of the following self-adjoint BVPs

   (a) $u''(x) = f(x), \qquad \mathcal{B} = \{u'(0) = 0, u(1) = 0\},$

   (b) $u''(x) = f(x), \qquad \mathcal{B} = \{u'(0) - u(0) = 0, u(1) = 0\},$

   (c) $u''(x) + u(x) = f(x), \qquad \mathcal{B} = \{u'(0) = 0, u(1) = 0\}.$

8. (i) Determine the Green's function associated with the BVP
$$x^2u''(x) + 2xu'(x) - 2u(x) = f(x)$$
with $u(1) = 0$ and $u'(2) = 0$.

   (ii) Thus solve the problem if $f(x) = x^2$.

   (iii) Check that the answer you obtain in (ii) is the same solution that you would obtain with standard direct methods.

9. Consider the BVP
$$\mathcal{L}u = u''(x) = f(x)$$
subject to the boundary conditions $u'(0) + u(0) = 0$ and $u(1) = 0$.

   (i) Does the operator have a zero eigenvalue?

   (ii) Given what you know about eigenfunction expansions of Green's function, will the Green's function corresponding to the BVP above exist?

   (iii) Using (i) and the Fredholm Alternative write down a solvability condition for the BVP in terms of $f(x)$.

Example Sheet 5

Topics covered: Green's function for the wave equation, adjoint Green's function, Green's functions for non S-A problems, Inhomogeneous BCs, separation of variables again

In all cases below time dependence is assumed of the form $\exp(-i\omega t)$.

1. In section 2.9 we derived the Green's function (by using earlier information) for the problem associated with time-harmonic wave propagation on a finite string with fixed ends and length $L \neq n\pi, n \in \mathbb{Z}$. Use the direct (explicit) approach to derive this here.

2. (i) In Example 2.17 we showed that the Green's function associated with an infinite string that is time harmonically forced a the point $x = 0$ is

$$G(x, 0) = \frac{1}{2ik} \exp(ik|x|). \tag{B.3}$$

Show that if we force the problem at $x = x_0$ then the Green's function becomes

$$G(x, x_0) = \frac{1}{2ik} \exp(ik|x - x_0|). \tag{B.4}$$

(ii) **By using the relationship between the adjoint Green's function and the Green's function** find the adjoint Green's function. Note that $G^*(x, x_0) \neq G(x, x_0)$. Why is this? In order to understand this derive the full adjoint problem referring to question (5)(vii) on Example Sheet 2 and making reference to the radiation conditions (2.177), (2.178) where appropriate.

(iii) Derive the Green's function associated with the semi-infinite problem

$$u''(x) + k^2 u(x) = f(x)$$

for $x \in [0, \infty)$, $u(0) = 0$ and u having "outgoing" form as $x \to \infty$.

3. Time-harmonic, flexural waves on a beam are governed by

$$\frac{d^4 u}{dx^4} - k^2 u = 0.$$

for some constant $k > 0$. Determine the associated Green's function for the infinite beam problem with $x_0 = 0$, given boundary conditions $G(0, x_0) = G(L, x_0) = 0$ and the following continuity conditions at $x = 0$:

$$[G(x, x_0)]_{x_0^-}^{x_0^+} = 0, \qquad \left[\frac{dG}{dx}(x, x_0)\right]_{x_0^-}^{x_0^+} = 0, \tag{B.5}$$

$$\left[\frac{d^2 G}{dx^2}(x, x_0)\right]_{x_0^-}^{x_0^+} = 0, \qquad \left[\frac{d^3 G}{dx^3}(x, x_0)\right]_{x_0^-}^{x_0^+} = 1. \tag{B.6}$$

4. With reference to Example 2.18 from the notes, show that you would obtain the same solution to the BVP if you used the adjoint Green's function.

5. (i) Consider the BVP

$$\mathcal{L}u = x^2u''(x) - xu'(x) - 3u(x) = f(x)$$

with $u(1) = 0$ and $u(2) = 0$. Is this problem S-A?

(ii) Determine the adjoint Green's function $G^*(x, x_0)$ for this problem.

(iii) Determine the Green's function $G(x, x_0)$ for this problem and show that $G(x_0, x) = G^*(x, x_0)$.

(iv) Solve the problem if $f(x) = x - 3$.

(v) Check that the answer you obtain in (iv) is the same solution of the BVP that you would obtain with standard direct methods.

6. Determine the Green's function associated with the BVP

$$u''(x) + 2u'(x) + u(x) = f(x)$$

with $u(0) = 0$ and $u(1) = 0$ and write the solution in integral form.

7. A differential operator is defined such that

$$\mathcal{L}u = \frac{d^2u}{dx^2} + \frac{x}{1+x}\frac{du}{dx} - \frac{1}{1+x}u$$

for $x > 0$.

(i) Verify that $u = Ax + Be^{-x}$ is the general solution to the differential equation.
(ii) Construct the Green's function associated with the BVP

$$\mathcal{L}u = f(x), \qquad \mathcal{B} = \{u(0) = 0, u \to 0 \text{ as } x \to 0\} \qquad \text{(B.7)}$$

(iii) Use the Green's function to solve the BVP

$$\mathcal{L}u = (1 + x)e^{-x}$$

subject to $u(0) = 0$ and $u \to 0$ as $x \to \infty$.

8. Determine the Green's function for Example 2.19 in the notes, noting that this is an example where the BCs are mixed (and hence is not of S-L type). **This is not examinable since the BCs are mixed**.

9. Solve the (fully inhomogeneous) problem

$$u''(x) = f(x), \qquad \mathcal{B} = \{u(0) = \alpha, u(1) = \beta\}$$

by *both* approaches described in section 2.12 of the notes.

Example Sheet 6

This sheet covers material covered in weeks 6, 7 and 8.

Topics covered: Eigenvalues of the Laplacian, Green's identity in two/three dimensions, Green's functions for the Laplacian: eigenfunction expansions, free-space formulation and image method

1. For Cartesian coordinates in two dimensions, for two functions $f(\mathbf{x}), g(\mathbf{x})$, show that

$$\nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g.$$

2. (i) Using 1, show that

$$\int_D \overline{v}\nabla \cdot (p\nabla u) - u\overline{\nabla \cdot (\overline{p}\nabla v)}\ d\mathbf{x} = \int_{\partial D} p(\overline{v}\nabla u - u\nabla\overline{v}) \cdot \mathbf{n} ds$$

(ii) Using (i), determine the adjoint operator $\mathcal{L}^*$ and $\mathcal{B}^*$ given that

$$\mathcal{L} = \nabla \cdot (p\nabla) + q$$

for two (possibly complex) functions $p(\mathbf{x}), q(\mathbf{x})$ and

$$\alpha(\mathbf{x})u + \beta(\mathbf{x})\nabla u \cdot \mathbf{n} = 0, \qquad \mathbf{x} \in \partial D$$

for two (possibly complex) functions $al(\mathbf{x}), \beta(\mathbf{x})$.

(iii) Write down conditions on the functions p, q, α, β to ensure that $\mathcal{L}$ as defined in (ii) above is self-adjoint.

3. Consider the full rectangular domain eigenvalue problem as considered briefly in Example 3.1 in the notes. Since the domain is rectangular $D = \{x \in [0, L], y \in [0, H]\}$ we use Cartesian coordinates.

Assume governing equations of the form

$$\nabla^2 u = \frac{\partial u}{\partial t}$$

and

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$$

subject to boundary conditions

$$u(x, 0, t) = 0, \qquad u(x, H, t) = 0, \tag{B.8}$$

$$u(L, y, t) = 0, \qquad u(L, y, t) = 0 \tag{B.9}$$

and some prescribed initial conditions. Use separation of variables to show that these problems yield the same eigenvalue boundary value problem and show that the two dimensional eigenfunctions are

$$\phi_{mn}(x, y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right) \tag{B.10}$$

for $m, n = 1, 2, 3, ...$, with associated eigenvalues

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2. \tag{B.11}$$

4. Confirm that the free-space Green's functions $G_{2\infty}$ and $G_{3\infty}$ defined in (3.48) satisfy laplace's equation $\nabla_{\mathbf{x}}^2 G = 0$ for $\mathbf{x} \neq \mathbf{x}_0$.

5. Show that if we wish the solution to decay at infinity, we must have the conditions (3.52) and (3.53), i.e.

$$\lim_{R\to\infty} \left(u - r \ln r \frac{\partial u}{\partial r} \right) \Bigg|_{r=R} = 0, \tag{B.12}$$

$$\lim_{R\to\infty} \left(u + r \frac{\partial u}{\partial r} \right) \Bigg|_{r=R} = 0. \tag{B.13}$$

   [Hint: to do this use the fact that the domain D is centred on the source so that the boundary is located on $r = R$.]

6. Using the method of images, construct the Green's function for the Laplacian operator in two dimensions for a semi-infinite domain $x \geq 0$ with $G = 0$ on $x = 0$.

7. Using the method of images, construct the Green's function for the Laplacian operator in two dimensions for a semi-infinite domain $y \geq 0$ with $\partial G/\partial y = 0$ on $y = 0$.

8. Using the method of images, construct the Green's function for the Laplacian operator in two dimensions for the quarter-plane domain $y \geq 0$ and $x \geq 0$ with $G = 0$ on $x = 0, y \geq 0$ and $y = 0, x \geq 0$.

Example Sheet 7

Topics covered: Solution forms for 2/3D problems and applications of Poisson's equation

1. Consider a general domain D on which u satisfies Poisson's equation with forcing $Q(\mathbf{x})$ and suppose that the boundary condition is, for $\mathbf{x} \in \partial D$

$$\mathbf{n} \cdot \nabla u(\mathbf{x}) = j(\mathbf{x}).$$

   where $\mathbf{n}$ is an outward facing normal to D. Introduce an appropriate Green's function and use Green's identity to show that

$$u(\mathbf{x}) = \int_D G(\mathbf{x}, \mathbf{x}_0) u(\mathbf{x}_0) \; d\mathbf{x}_0 - \int_{\partial D} G(\mathbf{x}, \mathbf{x}_0) j(\mathbf{x}) \; ds$$

   where s parametrizes the boundary ∂D.

2. Consider the boundary value problem

$$\nabla^2 u = 0$$

   in two dimensions, on the upper half-plane $D = \{-\infty < x < \infty, y \geq 0\}$, with the boundary condition $u = h(x)$ on $y = 0$. Determine the associated Greens's function and use (3.39) to derive the integral form of solution. Thus determine the solution when

$$h(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

   Analyze the solution carefully in the limit as $y \to 0^+$ to ensure that it yields the correct limit.

   Are you convinced that the "integral from infinity" is zero?

3. Consider the boundary value problem

$$\nabla^2 u = 0$$

   in two dimensions, on the quarter-plane $D = \{x \geq 0, y \geq 0\}$, with the boundary condition $u = h(x)$ on $y = 0$ and $u = j(y)$ on $x = 0$. Use the Greens's function you determined on Example Sheet 6, question 8 to determine the form of solution in terms of integrals over the boundaries.

   Evaluate these integrals when $h(x) = 0$ and

$$j(y) = \begin{cases} 1, & 1 \leq y \leq 2, \\ 0, & \text{otherwise} \end{cases}$$

   Analyze the solution carefully in the limit as $x \to 0^+$ to ensure that it yields the correct limit.

   [HINT: You may assume that the integral from infinity is zero]

4. Consider the boundary value problem

$$\nabla^2 u(r,\theta) = 0$$

on a wedge shaped region $D = \{r \geq 0, 0 \leq \theta \leq \pi/3\}$ with boundary conditions

$$u(r,\pi/3) = 0, \qquad\qquad u(r,0) = h(r).$$

(i) Determine the appropriate Green's function $G(\mathbf{x},\mathbf{x}_0)$ using the method of images.

[HINT: You will need 5 image source]

(ii) Show that (assuming the integral from infinity is zero)

$$u(r,\theta) = -\int_0^\infty h(r)\left(\frac{\partial}{\partial\theta_0}G(\mathbf{x},\mathbf{x}_0)\right)\Big|_{\theta_0=0}\; dr_0$$

5. Consider a constant source of electric charge q over a spherical region of radius a in three dimensions where we can ignore boundaries. The electric field $u(\mathbf{x})$ is thus

$$u(\mathbf{x}) = q\int_D G(\mathbf{x},\mathbf{x}_0)\; d\mathbf{x}_0$$

where D is the sphere is of radius a.

Since we consider "free space" we can position the centre of the sphere at the origin of our coordinate systems $\mathbf{x}$ and $\mathbf{x}_0$ (which coincide). Therefore the surface of the sphere is defined by

$$x_0^2 + y_0^2 + z_0^2 = a^2 \tag{B.14}$$

Use the free space Green's function $G_{3\infty}(\mathbf{x},\mathbf{x}_0)$ to show that for $|\mathbf{x}| < a$, i.e. for points inside the sphere.

$$u(\mathbf{x}) = \frac{q}{8\pi}\int_0^{2\pi}\int_0^{\pi} R^2(\theta,\phi)\sin\phi\;\, d\phi d\theta.$$

where $r = R(\theta,\phi)$ defines the surface of the sphere.

[HINT: Introduce shifted coordinates

$$\mathbf{x}_0 = \mathbf{x} + \mathbf{r}$$

where

$$\mathbf{r} = r(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi). \tag{B.15}$$

and define the surface of the sphere at $r = R(\theta,\phi)$ in terms of a quadratic equation (which you should determine by using (B.14)).]

Example Sheet 8

Topics covered: Linear Integral Operators, Integral equation types, Fredholm equations - BVP equivalence, Fredholm Alternative for algebraic systems

1. Show that the linear integral operator $\mathcal{K}$ defined by

$$\mathcal{K}f = \int_a^b K(x,y)f(y)\ dy$$

is self adjoint if $K(x,y) = \overline{K(y,x)}$.

2. Identify the types of the following integral equations

(i)

$$x + x^2 = \int_0^1 u(y)\ dy$$

(ii)

$$x + u(x) = \lambda \int_0^1 \sin(x-y)u(y)\ dy$$

(iii)

$$u(x) + \int_0^x [1 + (x-y)]u(y)\ dy = x + 1$$

3. Use a Green's function to determine the integral equation associated with the following BVPs and where possible evaluate any integrals you can.

(i)

$$u''(x) = \lambda u(x) + x$$

subject to $u'(0) = 0$, $u(1) = 0$.

(ii)

$$u''(x) + u(x) = \lambda u(x) + 2$$

subject to $u(0) = 0$, $u(1) = 0$

4. Use the Fredholm Alternative to understand the existence and uniqueness properties of the linear system $\mathbf{Lu} = \mathbf{f}$ where

$$\mathbf{L} = \begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} \tag{B.16}$$

with

$$\text{(i)} \qquad \mathbf{f} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad\qquad \text{(ii)} \qquad \mathbf{f} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{B.17}$$

and find the explicit solution(s) when you can.

5. Determine values of λ such that the linear system $\mathbf{Lu} = \mathbf{f}$ has a unique solution where

$$\mathbf{L} = \begin{pmatrix} 1 & 3-\lambda \\ 2-\lambda & 2 \end{pmatrix} \tag{B.18}$$

When λ does *not* satisfy the above criterion, can the system still possess a solution?

Example Sheet 9

Topics covered: Degenerate kernels, Neumann series solutions, Waves in heterogeneous media

1. Solve the integral equation

$$u(x) = e^{-x} + \lambda \int_0^1 \cos(2\pi x)\sin(2\pi y)u(y)\,dy$$

2. Consider the integral equation

$$u(x) = x + \lambda \int_0^\pi \cos(x-y)u(y)\,dy.$$

For which values of λ does this equation have a unique solution? Find the solution in this case. When λ does not take any of these values, can a non-unique solution still exist?

3. (a) Show that the eigenvalues of the integral equation

$$u(x) = \lambda \int_0^{2\pi} \sin(x+y)u(y)\,dy$$

are

$$\pm\frac{1}{\pi}$$

with corresponding eigenfunctions

$$\sin x \pm \cos x.$$

(b) For what values of λ does the integral equation

$$u(x) = \lambda \int_0^{2\pi} \sin(x+y)u(y)\,dy + x$$

have a unique solution? Find the solution in those cases.

4. Consider the integral equation

$$u(x) = 2 + \lambda \int_0^\pi \sin(x+y)u(y)\,dy$$

Find the values of λ for which this integral equation has a unique solution and thus determine the solution in those cases.

When λ does not take these values, can a solution still exist?

5. Show that the integral equation

$$u(x) = 1 + \lambda \int_0^1 \ln(xy)u(y)\,dy$$

has a unique solution provided that

$$(\lambda-1)^2 + \frac{1}{2}\lambda^2(e-1)^3 \neq 0.$$

6. (a) We solved the integral equation

$$u(x) = 1 + \lambda \int_0^{1/2} (x - y)u(y)\, dy.$$

in Example 4.5 by a Neumann Series approach. However, note that the Kernel is in fact degenerate. Therefore solve the integral equation exactly.

(b) Expand your solution for $\lambda = \epsilon \ll 1$ to show that the Neumann series solution derived in Example 4.5 is correct.

7. Consider the integral equation

$$u(x) = \lambda \int_0^1 xyu(y)\, dy + f(x).$$

(a) When $\lambda = 1$, determine c_1 in terms of an integral of the function $f(x)$.

(i) Suppose that $f(x) = 1/x^p, p \in \mathbb{R}$. For what range of p does the solution exist?

(ii) Find the solution when (i) $p = 1$, (ii) $p = 0$.

(b) Suppose now that $|\lambda| < 3$ and $f(x) = 1$. Find the Neumann series solution and show that it can be formally summed in order to find an explicit solution of the form

$$u(x) = 1 + \frac{x}{2}\left(\frac{3\lambda}{3 - \lambda}\right).$$

Does this generate the correct solution for (a)(ii) above?

8. What is the difference between a Neumann series solution and the Born approximation?

9. (i) Consider wave reflection and transmission as in section 4.7 from a layer with $a = 0, b = 1$ and layer wavenumber

$$k(x) = k_0\sqrt{2}e^x.$$

Determine the Born approximation to the reflected and transmitted field from the layer.

(ii) Determine the range of wavenumbers for which the associated Neumann series converges