Math 365 – Monday 2/11/19
Section 2.5: Cardinality of Sets

Exercise 15. Show that each of the following sets are countably infinite by giving a bijective function between that set and the positive integers.

(a) the integers greater than 10.
(b) the odd negative integers
(c) the set $A \times \mathbb{Z}^+$, where $A = \{2, 3\}$
(d) the integers that are multiples of 10

Exercise 16.

(a) Determine whether each of these sets is finite, countable, or uncountable. For those that are countably infinite, exhibit a bijective correspondence between the set of positive integers and that set.
  (i) The integers that are multiples of 10.
  (ii) Integers not divisible by 3.
  (iii) The real numbers with decimal representations consisting of all 1s.
  (iv) The real numbers with decimal representations of all 1s or 9s.
  (v) The integers with absolute value less than $1,000$.
  (vi) The real numbers between 0 and 2
(b) Give an example of two uncountable sets A and B such that $A - B$ is
  (i) finite;
  (ii) countably infinite;
  (iii) uncountable.
(c) Explain why the power set of $\mathbb{Z}_{\geq 1}$ is not countable as follows:
  (i) First, for each subset $A \subset \mathbb{Z}_{\geq 1}$, represent A as an infinite bit string (a sequence of 1's and 0's with no end to the right) with ith bit 1 if i belongs to the subset and 0 otherwise. For example, we represent

$$\begin{aligned} \{3\} &\quad \text{as } 001000000000\ldots, \\ \{1,3,4\} &\quad \text{as } 101100000000\ldots, \text{ and} \\ \{2x \mid x \in \mathbb{Z}_{\geq 1}\} &\quad \text{as } 010101010101\ldots. \end{aligned}$$

  Give the bit-string expansions for the sets $\{2, 4, 6, 7\}$ and $\{3x \mid x \in \mathbb{Z}_{\geq 1}\}$ (i.e. the positive multiples of 3); and give the set corresponding to the bitstring expansions 0000000000000. . . and 111111111. . . . Finally, explain why this coding of sets as bit strings is actually a bijection between $\{\text{infinite bit strings}\}$ and $\mathcal{P}(\mathbb{Z}_{\geq 1})$.
  (ii) Suppose that you can list these infinite strings in a list labeled by the positive integers (as we saw, this is the same as saying that there is some bijective map $f : \{\text{infinite bit strings}\} \to \mathbb{Z}_{\geq 1}$). Construct a new bit string one bit at a time, so that it doesn't match the ith string in the ith bit. Conclude that your new string can't be in the list, so that the list wasn't actually complete.
  (iii) Finally, explain how to use (i) and (ii) together to show that the sets in $\mathcal{P}(\mathbb{Z}_{\geq 1})$ aren't listable (and therefore aren't countable).
(d) Show that if A and B are sets and $A \subset B$, then $|A| \leq |B|$.
  [Hint: Start with thinking about the definition of what it means for $|A| \leq |B|$.]
(e) Show that a subset of a countable set is also countable.
  [Hint: Start with "Suppose A is a countable set, and that $B \subseteq A$. Since A is countable, there is a bijective function. . . ".]
(f) Use the Schröder-Bernstein theorem to show that $(0, 1)$ and $[0, 1]$ have the same cardinality.

Getting to know

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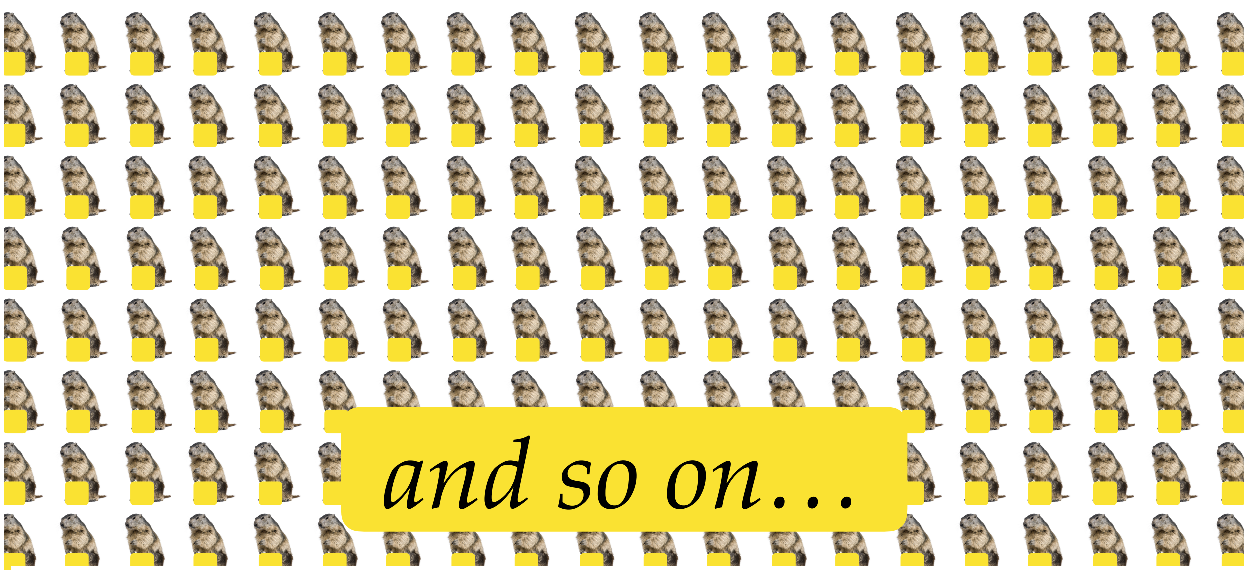
(Section 2.5)

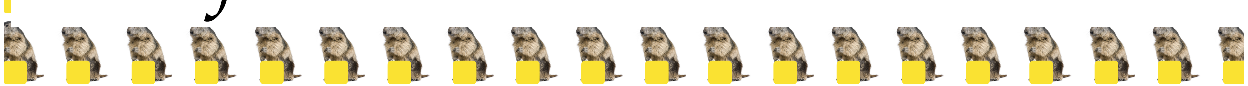
How do we count?

Match numbers to groundhogs.
Every groundhog gets **exactly** one number;
every number gets **exactly** one groundhog.

Defn. Two sets A and B have the same **cardinality** if there is a bijection $f\colon A \to B$.



Counting numbers never run out, because there are an *infinite* number of them.



Countably infinite *a.*

Capable of being matched bijectively with the natural numbers.

Example 1:
The counting numbers are countably infinite.

Example 2:
The integers from -2 and above are countably infinite.

Example 3:
The integers from **any** fixed number (like -11) and above are countably infinite.

Being precise

Recall: A bijection is a function that is both injective and surjctive. We also showed that a function is bijective if and only if it is invertible.

Example 1: The sets $\mathbb{Z}_{\geqslant 1}$ and $\mathbb{Z}_{\geqslant 1}$ have the same cardinality since

$$\begin{array}{rcl} f : \mathbb{Z}_{\geqslant 1} & \to & \mathbb{Z}_{\geqslant 1} \\ x & \mapsto & x \end{array}$$

is a bijective map.

Example 2: The sets $\mathbb{Z}_{\geqslant -2}$ and $\mathbb{Z}_{\geqslant 1}$ have the same cardinality since

$$\begin{array}{rcl} f : \mathbb{Z}_{\geqslant -2} & \to & \mathbb{Z}_{\geqslant 1} \\ x & \mapsto & x+3 \end{array}$$

is a bijective map.

Example 3: The sets $\mathbb{Z}_{\geqslant -11}$ and $\mathbb{Z}_{\geqslant 1}$ have the same cardinality since

$$\begin{array}{rcl} f : \mathbb{Z}_{\geqslant -11} & \to & \mathbb{Z}_{\geqslant 1} \\ x & \mapsto & x+12 \end{array}$$

is a bijective map.

Example 4: The sets $\mathbb{Z}$ and $\mathbb{Z}_{\geqslant 1}$ have the same cardinality since...

$$\begin{array}{cccccccccccc} \mathbb{Z}_{\geqslant 0}: & \cdots & 9 & 7 & 5 & 3 & 1 & 2 & 4 & 6 & 8 & \cdots \\ & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ \mathbb{Z}: & \cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \end{array}$$

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z}_{\geqslant 1} \\ x &\mapsto \begin{cases} 2x & x > 0 \\ -2x + 1 & x \leqslant 0 \end{cases} \end{aligned}$$

is a bijective map.

Alternatively, turning that bijective function around: the sets $\mathbb{Z}$ and $\mathbb{Z}_{\geqslant 1}$ have the same cardinality since

$$\begin{array}{cccccccccccc} \mathbb{Z}_{\geqslant 0}: & \cdots & 9 & 7 & 5 & 3 & 1 & 2 & 4 & 6 & 8 & \cdots \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathbb{Z}: & \cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \end{array}$$

$$\begin{aligned} f : \mathbb{Z}_{\geqslant 1} &\to \mathbb{Z} \\ x &\mapsto \begin{cases} x/2 & x > 0 \\ -(x-1)/2 & x \leqslant 0 \end{cases} \end{aligned}$$

is a bijective map.

Countably infinite *a.*

Capable of being matched bijectively with the natural numbers.

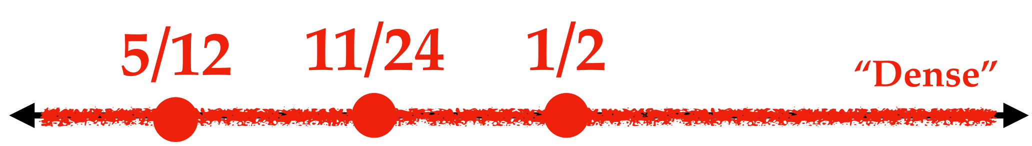
Examples:

The counting numbers and the integers are both countably infinite sets.

Rational numbers *n.*

Can't list in order of size!

Those numbers that can be expressed as integer fractions, i.e. 2/5, -17/3, 0/1, …



	1	2	3	4	5	6	7	8	9	· · ·
1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	
2	**1/2**	**2/2**	**3/2**	**4/2**	**5/2**	**6/2**	**7/2**	**8/2**	**9/2**	
3	**1/3**	**2/3**	**3/3**	**4/3**	**5/3**	**6/3**	**7/3**	**8/3**	**9/3**	
4	**1/4**	**2/4**	**3/4**	**4/4**	**5/4**	**6/4**	**7/4**	**8/4**	**9/4**	
5	**1/5**	**2/5**	**3/5**	**4/5**	**5/5**	**6/5**	**7/5**	**8/5**	**9/5**	
6	**1/6**	**2/6**	**3/6**	**4/6**	**5/6**	**6/6**	**7/6**	**8/6**	**9/6**	
7	**1/7**	**2/7**	**3/7**	**4/7**	**5/7**	**6/7**	**7/7**	**8/7**	**9/7**	
8	**1/8**	**2/8**	**3/8**	**4/8**	**5/8**	**6/8**	**7/8**	**8/8**	**9/8**	
9	**1/9**	**2/9**	**3/9**	**4/9**	**5/9**	**6/9**	**7/9**	**8/9**	**9/9**	
⋮										⋱

	1	2	3	4	5	6	7	8	9	· · ·
1	1	2	5	6	**5**	**6**	**7**	**8**	**9**	
2	3		7		**5/2**		**7/2**		**9/2**	
3	4	**2/3**		**4/3**	**5/3**		**7/3**	**8/3**		
4	**1/4**		**3/4**		**5/4**		**7/4**		**9/4**	
5	**1/5**	**2/5**	**3/5**	**4/5**		**6/5**	**7/5**	**8/5**	**9/5**	
6	**1/6**				**5/6**		**7/6**			
7	**1/7**	**2/7**	**3/7**	**4/7**	**5/7**	**6/7**		**8/7**	**9/7**	
8	**1/8**		**3/8**		**5/8**		**7/8**		**9/8**	
9	**1/9**	**2/9**		**4/9**	**5/9**		**7/9**	**8/9**		
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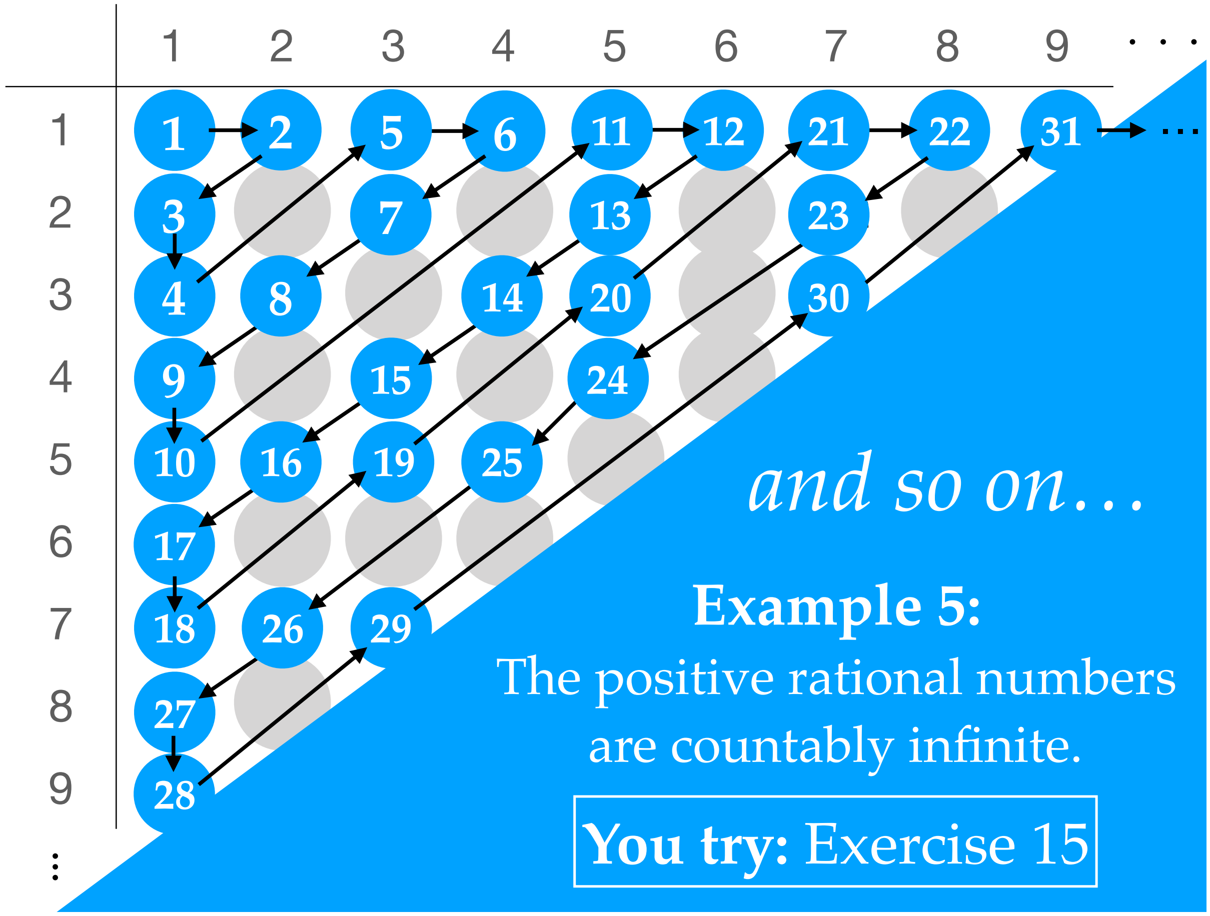


Countably infinite *a.*

Capable of being matched bijectively with the natural numbers.

Is there anything that isn't countable?

(Spoiler: yes!)

Real numbers *n.*

Numbers obtained by appending an infinite decimal expansion to an integer.

Examples:

1.000000000…

-2.12121212121…

3.1415926535…

Includes:

Integers, rational numbers.

Excludes: $\sqrt{-1}$

Starting with *any* list of real numbers…

we can build a number that's *not* on that list!

1. 12.32156019…
2. 0.12121212…
3. −5.00000000…
4. 95.33333333…
5. 1.12300000…
6. 0.00000000…
7. 3.14159265…

0.0
0
1
0
1
1
0

Starting with *any* list of real numbers…

we can build a number that's *not* on that list!

1. 0.00101101…
2. 12.32156019…
3. 0.12121212…
4. −5.00000000…
5. 95.33333333…
6. 1.12300000…
7. 0.00000000…

0.00101101…

Any list of real numbers is incomplete.
So the real numbers are **not countable!**

Cantor's Diagonal Argument

Finite

Countably infinite $\aleph_0$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 2

Uncountably infinite

Real numbers $\aleph_1$

Sets of real numbers $\aleph_2$

Sets of sets of real numbers $\aleph_3$

Uncountably infinite

Real numbers $\aleph_1$

Sets of real numbers $\aleph_2$

Sets of sets of real numbers $\aleph_3$

Sets of sets of sets of real numbers $\aleph_4$

Sets of sets of sets of sets of real numbers $\aleph_5$

Sets of sets of sets of sets of sets of real numbers $\aleph_6$

Sets of sets of sets of sets of sets of sets of real numbers $\aleph_7$

Sets of sets of sets of sets of sets of sets of sets of real numbers $\aleph_8$

Infinite sets

A set is countable if it is either finite or the same cardinality as the natural numbers ($\mathbb{N} = \mathbb{Z}_{\geqslant 0}$). If a set A is not finite but is countable, we say A is "countably infinite" and write $|A| = \aleph_0$ (pronounced "aleph naught" or "aleph null"). To show that $|A| = \aleph_0$: show A is not finite, and give a bijection $f : \mathbb{Z}_{\geqslant 1} \to A$.

A set A is not countable if there is **no** bijection between $\mathbb{Z}_{\geqslant 0}$ and A. To show that, start with "Suppose $f : \mathbb{Z}_{\geqslant 1} \to A$ is a bijection" and find a universal problem—that means that no such bijection could possibly exist!

Other cardinalities:

- The real numbers $\mathbb{R}$ have size $\aleph_1$;
- the power set of the real numbers $\mathcal{P}(\mathbb{R})$ has size $\aleph_2$;
- the power set of the power set of the real numbers $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ has size $\aleph_3$;

and so on...

Finite

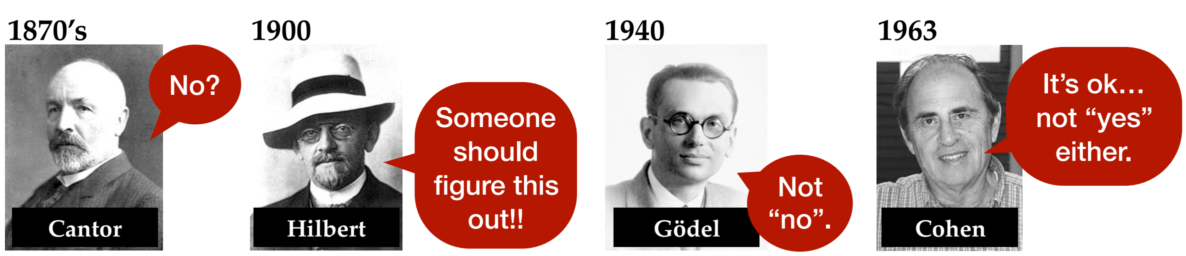
Countably infinite

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 2

Uncountably infinite

Fact:
There are *at least* a countably infinite number of "sizes" of infinite sets.

Question:
Are there more?



Independence of the continuum hypothesis:
You can't prove or disprove that there are or aren't others.

More facts

Unions

If A and B are countable, then so is $A \cup B$.
Therefore, if $A_1, A_2, \ldots, A_n$ are all countable, then so is $\bigcup_{i=1}^n A_i$.
In fact, if $\{A_x \mid x \in C\}$ is a countable collection of countable sets (i.e. C is a countable set), then so is $\bigcup_{x \in C} A_x$.
Example: Let $A_x = \{y/x \mid y \in \mathbb{Z}\}$ for each $x \in \mathbb{Z}_{>0}$. Then

$$\bigcup_{x \in \mathbb{Z}_{>0}} A_x = \mathbb{Q}, \quad \text{which is countable.}$$

However, if $\{A_x \mid x \in U\}$ is an uncountable collection of countable sets, then $\bigcup_{x \in U} A_x$ could be countable or uncountable (we can't tell).
Countable: $\bigcup_{x \in [0,1)} A_x$ where $A_x = \mathbb{Q}$.
Uncountable: $\bigcup_{x \in [0,1)} A_x$ where $A_x = \{x\}$.

Containment

If A is not countable and $A \subseteq B$, then B is not countable.

Ex: Any subset of $\mathbb{Z}$ is countable.

If B is countable and $A \subseteq B$, then A is countable (prove on HW).

Ex. Since $\mathbb{R}$ is not countable, then neither is $\mathbb{C}$.

Comparing sizes

If there is an injective function $f : A \to B$ then we write $|A| \leqslant |B|$.
(Makes sense for finite sets; take as a definition for infinite sets.)

Theorem (Schröder-Bernstein Theorem)

If $|A| \leqslant |B|$ *and* $|B| \leqslant |A|$, *then* $|A| = |B|$.

Example

We can show $|(0,1)| = |(0,1]|$ by using the fact that

$$f : (0,1) \to (0,1] \qquad x \mapsto x$$

and

$$g : (0,1] \to (0,1) \qquad x \mapsto x/2$$

are both injective, even though they are not surjective nor are they inverses of each other.

You try: Exercise 16