# Math 372: Solutions to Homework

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**Abstract**

Below are detailed solutions to the homework problems from Math 372 Complex Analysis (Williams College, Fall 2013, Professor Steven J. Miller, sjm1@williams.edu). The course homepage is

`http://www.williams.edu/Mathematics/sjmiller/public_html/372`

and the textbook is *Complex Analysis* by Stein and Shakarchi (ISBN13: 978-0-691-11385-2). Note to students: it's nice to include the statement of the problems, but I leave that up to you. **I am only skimming the solutions. I will occasionally add some comments or mention alternate solutions. If you find an error in these notes, let me know for extra credit.**

## Contents

# 1 Math 372: Homework #1: Yuzhong (Jeff) Meng and Liyang Zhang (2010)

**Due by 11am Friday, September 13: Chapter 1: Page 24: #1abcd, #3, #13.**

***Problem:*** Chapter 1: #1: Describe geometrically the sets of points $z$ in the complex plane defined by the following relations: (a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$; (b) $1/z = \overline{z}$; (c) $\mathrm{Re}(z) = 3$; (d) $\mathrm{Re}(z) > c$ (resp., $\geq c$) where $c \in \mathbb{R}$.

***Solution:*** (a) When $z_1 \neq z_2$, this is the line that perpendicularly bisects the line segment from $z_1$ to $z_2$. When $z_1 = z_2$, this is the entire complex plane.
(b)

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}. \tag{1.1}$$

So

$$\frac{1}{z} = \overline{z} \Leftrightarrow \frac{\overline{z}}{|z|^2} = \overline{z} \Leftrightarrow |z| = 1. \tag{1.2}$$

This is the unit circle in $\mathbb{C}$.
(c) This is the vertical line $x = 3$.
(d) This is the open half-plane to the right of the vertical line $x = c$ (or the closed half-plane if it is $\geq$).

***Problem:*** Chapter 1: #3: With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where $n$ is a natural number. How many solutions are there?

***Solution:*** Notice that

$$\omega = se^{i\varphi} = se^{i(\varphi + 2\pi m)}, m \in \mathbb{Z}. \tag{1.3}$$

It's worth spending a moment or two thinking what is the best choice for our generic integer. Clearly $n$ is a bad choice as it is already used in the problem; as we often use $t$ for the imaginary part, that is out too. The most natural is to use $m$ (though $k$ would be another fine choice); at all costs do not use $i$!

Based on this relationship, we have

$$z^n \ = \ se^{i(\varphi + 2\pi m)}. \tag{1.4}$$

So,

$$z \ = \ s^{1/n} e^{\frac{i(\varphi + 2\pi m)}{n}}. \tag{1.5}$$

Thus, we will have $n$ unique solutions since each choice of $m \in \{0, 1, \dots, n-1\}$ yields a different solution so long as $s \neq 0$. Note that $m = n$ yields the same solution as $m = 0$; in general, if two choices of $m$ differ by $n$ then they yield the same solution, and thus it suffices to look at the $n$ specified values of $m$. If $s = 0$, then we have only $1$ solution.

***Problem:*** Chapter 1: #13: Suppose that $f$ is holomorphic in an open set $\Omega$. Prove that in any one of the following cases $f$ must be constant:
(a) Re($f$) is constant;

(b) Im($f$) is constant;
(c) $|f|$ is constant.

***Solution:*** Let $f(z) = f(x, y) = u(x, y) + iv(x, y)$, where $z = x + iy$.
(a) Since Re$(f) = constant$,

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0. \tag{1.6}$$

By the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0. \tag{1.7}$$

Thus, in $\Omega$,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0. \tag{1.8}$$

Thus $f(z)$ is constant.

(b) Since Im$(f)$ = $constant$,

$$\frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0. \tag{1.9}$$

By the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0. \tag{1.10}$$

Thus in $\Omega$,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + 0 = 0. \tag{1.11}$$

Thus $f$ is constant.

(c) We first give a mostly correct argument; the reader should pay attention to find the difficulty. Since $|f| = \sqrt{u^2 + v^2}$ is constant,

$$\begin{cases} 0 = \frac{\partial(u^2+v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}. \\ 0 = \frac{\partial(u^2+v^2)}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \end{cases} \tag{1.12}$$

Plug in the Cauchy-Riemann equations and we get

$$u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0. \tag{1.13}$$

$$-u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0. \tag{1.14}$$

$$(1.14) \Rightarrow \frac{\partial v}{\partial x} = \frac{v}{u}\frac{\partial v}{\partial y}. \tag{1.15}$$

Plug (1.15) into (1.13) and we get

$$\frac{u^2 + v^2}{u}\frac{\partial v}{\partial y} = 0. \tag{1.16}$$

So $u^2 + v^2 = 0$ or $\frac{\partial v}{\partial y} = 0$.

If $u^2 + v^2 = 0$, then, since $u$, $v$ are real, $u = v = 0$, and thus $f = 0$ which is constant.

Thus we may assume $u^2 + v^2$ equals a non-zero constant, and we may divide by it. We multiply both sides by $u$ and find $\frac{\partial v}{\partial y} = 0$, then by (1.15), $\frac{\partial v}{\partial x} = 0$, and by Cauchy-Riemann, $\frac{\partial u}{\partial x} = 0$.

$$f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0. \tag{1.17}$$

Thus $f$ is constant.

Why is the above only mostly a proof? The problem is we have a division by $u$, and need to make sure everything is well-defined. Specifically, we need to know that $u$ is never zero. We do have $f' = 0$ except at points where $u = 0$, but we would need to investigate that a bit more.

Let's return to

$$\begin{cases} 0 = \frac{\partial(u^2+v^2)}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}. \\ 0 = \frac{\partial(u^2+v^2)}{\partial y} = 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}. \end{cases} \tag{1.18}$$

Plug in the Cauchy-Riemann equations and we get

$$\begin{aligned} u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} &= 0 \\ -u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{1.19}$$

We multiply the first equation $u$ and the second by $v$, and obtain

$$\begin{aligned} u^2\frac{\partial v}{\partial y} + uv\frac{\partial v}{\partial x} &= 0 \\ -uv\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{1.20}$$

Adding the two yields

$$u^2\frac{\partial v}{\partial y} + v^2\frac{\partial v}{\partial y} = 0, \tag{1.21}$$

or equivalently

$$(u^2 + v^2)\frac{\partial v}{\partial y} = 0. \tag{1.22}$$

We now argue in a similar manner as before, except now we don't have the annoying $u$ in the denominator. If $u^2 + v^2 = 0$ then $u = v = 0$, else we can divide by $u^2 + v^2$ and find $\partial v/\partial y = 0$. Arguing along these lines finishes the proof. $\square$

One additional remark: we can trivially pass from results on partials with respect to $v$ to those with respect to $u$ by noting that if $f = u + iv$ has constant magnitude, so too does $g = if = -v + iu$, which essentially switches the roles of $u$ and $v$. Though this isn't needed for this problem, arguments such as this can be very useful.

*The following is from Steven Miller.* Let's consider another proof. If $|f| = 0$ the problem is trivial as then $f = 0$, so we assume $|f|$ equals a non-zero constant. As $|f|$ is constant, $|f|^2 = f\overline{f}$ is constant. By the quotient rule, the ratio of two holomorphic functions is holomorphic, assuming the denominator is non-zero. We thus find $\overline{f} = |f|^2/f$ is holomorphic. Thus $f$ and $\overline{f}$ are holomorphic, and satisfy the Cauchy-Riemann equations. Applying these to $f = u + iv$ yields

$$\frac{\partial u}{\partial x} \ = \ \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} \ = \ -\frac{\partial v}{\partial x},$$

while applying to $\overline{f} = u + i(-v)$ gives

$$\frac{\partial u}{\partial x} \ = \ \frac{\partial(-v)}{\partial y}, \quad \frac{\partial u}{\partial y} \ = \ -\frac{\partial(-v)}{\partial x}.$$

Adding these equations together yields

$$2\frac{\partial u}{\partial x} \ = \ 0, \quad 2\frac{\partial u}{\partial y} \ = \ 0.$$

Thus $u$ is constant, and by part (a) this implies that $f$ is constant. If we didn't want to use part (a) we could subtract rather than add, and similarly find that $v$ is constant.

*The following is from Craig Corsi, 2013 TA.* The problem also follows from the polar form of the Cauchy-Riemann equations.

It's worth mentioning that (a) and (b) follow immediately from (c). For example, assume we know the real part of $f$ is constant. Consider

$$g(z) \ = \ \exp(f(z)) \ = \ \exp(u(x,y))\exp(iv(x,y)).$$

As $|g(z)| = \exp(u(x,y))$, we see that the real part of $f$ being constant implies the function $g$ has constant magnitude. By part (c) this implies that $g$ is constant, which then implies that $f$ is constant.

# 2 Math 372: Homework #2: Solutions by Nick Arnosti and Thomas Crawford (2010)

**Due at the start of class by 11am Friday, September 20: Chapter 1: Page 24: #16abc, #24, #25ab. Chapter 2: (#1) We proved Goursat's theorem for triangles. Assume instead we know it holds for any rectangle; prove it holds for any triangle. (#2) Let $\gamma$ be the closed curve that is the unit circle centered at the origin, oriented counter-clockwise. Compute $\oint_\gamma f(z)dz$ where $f(z)$ is complex conjugation (so $f(x+iy) = x - iy$). Repeat the problem for $\oint_\gamma f(z)^n dz$ for any integer $n$ (positive or negative), and compare this answer to the results for $\oint_\gamma z^n dz$; is your answer surprising? (#3) Prove that the four triangles in the subdivision in the proof of Goursat's theorem are all similar to the original triangle. (#4) In the proof of Goursat's theorem we assumed that f was complex differentiable (ie, holomorphic). Would the result still hold if we only assumed f was continuous? If not, where does our proof break down?**

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**Due at the start of class by 11am Friday, September 20: Chapter 1: Page 24: #16abc, #24, #25ab. Chapter 2: (#1) We proved Goursat's theorem for triangles. Assume instead we know it holds for any rectangle; prove it holds for any triangle. (#2) Let $\gamma$ be the closed curve that is the unit circle centered at the origin, oriented counter-clockwise. Compute $\oint_\gamma f(z)dz$ where $f(z)$ is complex conjugation (so $f(x+iy) = x - iy$). Repeat the problem for $\oint_\gamma f(z)^n dz$ for any integer $n$ (positive or negative), and compare this answer to the results for $\oint_\gamma z^n dz$; is your answer surprising? (#3) Prove that the four triangles in the subdivision in the proof of Goursat's theorem are all similar to the original triangle. (#4) In the proof of Goursat's theorem we assumed that f was complex differentiable (ie, holomorphic). Would the result still hold if we only assumed f was continuous? If not, where does our proof break down?**

**Problem:** If $\gamma$ is a curve in $\mathbb{C}$, show that $\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$.

Parameterize $\gamma$ by $z = g(t)$ for $t$ in $[a, b]$, and define $w(t) = g(a + b - t)$. Then $w(t)$ is a parameterization of $-\gamma$ on the interval $[a, b]$ (note that $w(a) = g(b), w(b) = g(a)$). Additionally, $w'(t) = -g'(a + b - t)$. It follows that

$$\int_{-\gamma} f(z)dz = \int_a^b f(w(t))w'(t)dt = -\int_a^b f(g(a+b-t))g'(a+b-t)dt.$$

Making the substitution $u = a + b - t$, we get that

$$\begin{aligned} -\int_{t=a}^{b} f(g(a+b-t))g'(a+b-t)dt &= \int_{u=b}^{a} f(g(u))g'(u)du \\ &= -\int_{u=a}^{b} f(g(u))g'(u)du. \end{aligned} \tag{3.1}$$

But

$$-\int_{u=a}^{b} f(g(u))g'(u)du = -\int_\gamma f(z)dz,$$

which proves the claim.

**Problem:** If $\gamma$ is a circle centered at the origin, find $\int_\gamma z^n dz$.

We start by parameterizing $\gamma$ by $z = re^{i\theta}, 0 \le \theta < 2\pi$, so $dz = ire^{i\theta}d\theta$. Then

$$\int_\gamma z^n dz = \int_0^{2\pi} r^n e^{in\theta}(ire^{i\theta})d\theta = ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta.$$

If $n=-1$, this is $ir^0\int_0^{2\pi} d\theta = 2\pi i$. Otherwise, we get

$$ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta = \frac{r^{n+1}}{n+1}e^{i(n+1)}\theta\Big|_0^{2\pi} = 0.$$

**Problem:** If $\gamma$ is a circle not containing the origin, find $\int_\gamma z^n dz$.

If $n\neq -1$, the function $f(z)=z^n$ has a primitive (namely $\frac{z^{n+1}}{n+1}$), so by Theorem 3.3 in Chapter 1 of our book, $\int_\gamma f(z)dz = 0$.

If $n=-1$, we parameterize $\gamma$ by $z=z_0+re^{i\theta}, 0\leq\theta<2\pi$, so $dz = ire^{i\theta}d\theta$.Then

$$\int_\gamma \frac{1}{z}dz = \int_0^{2\pi}\frac{ire^{i\theta}}{z_0+re^{i\theta}}d\theta = \frac{ir}{z_0}\int_0^{2\pi}\frac{e^{i\theta}}{1+\frac{r}{z_0}e^{i\theta}}d\theta.$$

Note that because our circle does not contain the origin, $|z_0|>r$, so $|\frac{r}{z_0}e^{i\theta}|<1$. Thus, we can write this expression as a geometric series:

$$\frac{ir}{z_0}\int_0^{2\pi}\frac{e^{i\theta}}{1+\frac{r}{z_0}e^{i\theta}}d\theta = \frac{ir}{z_0}\int_0^{2\pi} e^{i\theta}\sum_{k=0}^{\infty}(\frac{-r}{z_0}e^{i\theta})^k d\theta.$$

Interchanging the sum and the integral, we see that this is just

$$-i\sum_{k=0}^{\infty}(\frac{-r}{z_0})^{k+1}\int_0^{2\pi} e^{i(k+1)\theta}d\theta = -\sum_{k=0}^{\infty}(\frac{-r}{z_0})^{k+1}\frac{e^{i(k+1)\theta}}{k+1}\Big|_0^{2\pi} d\theta = 0.$$

Why may we interchange? We can justify the interchange due to the fact that the sum of the absolute values converges.

**Problem:** If $\gamma$ is the unit circle centered at the origin, find $\int_\gamma \bar{z}^n dz$.

We start by parameterizing $\gamma$ by $z=e^{i\theta}, 0\leq\theta<2\pi$, so $\bar{z}=e^{-i\theta}$ and $dz = ie^{i\theta}d\theta$. Then

$$\int_\gamma \bar{z}^n dz = \int_0^{2\pi} e^{-in\theta}(ie^{i\theta})d\theta = i\int_0^{2\pi} e^{-i(n-1)\theta}d\theta.$$

If $n=1$, this is $i\int_0^{2\pi} d\theta = 2\pi i$. Otherwise, we get

$$i\int_0^{2\pi} e^{-i(n-1)\theta}d\theta = \frac{e^{i(1-n)\theta}}{1-n}\Big|_0^{2\pi} = 0.$$

Note that instead of doing the algebra, we could have observed that on the unit circle $\bar{z} = z^{-1}$, so $\int_\gamma \bar{z}^n dz = \int_\gamma z^{-n} dz$. Applying our work from Problem 3, we get the answer above.

**Problem:** Where in the proof of Goursat's theorem do we use the fact that the function $f$ is holomorphic? Is it sufficient to know that $f$ is continuous?

Start by recapping the main ideas behind the proof. We began by continually splitting our triangle $T$ into smaller triangles. These triangles converge to a point in the limit, and we called this point $z_0$. We then established the bound

$$\left| \int_T f(z)dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)dz \right|.$$

Our goal was to show that this quantity tends to zero as $z \to z_0$.

To do this, we Taylor expanded $f(z)$ around the point $z_0 : f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$. Note that $(z - z_0)$ divides $\psi(z)$, so $\psi(z) \to 0$ as $z \to z_0$.

$$\left| \int_{T^{(n)}} f(z)dz \right| \leq \left| \int_{T^{(n)}} f(z_0) + f'(z_0)(z - z_0)dz \right| + \int_{T^{(n)}} |\psi(z)(z - z_0)|dz$$

The first integrand in this sum has a primitive, so the value of this integral is zero. Let $M_n = \max_{z \text{ on } T^{(n)}} |\psi(z)|$. Then $|\psi(z)| \leq M_n$, and $z - z_0 \leq \text{diam}(T^{(n)})$. Hence, the value of the second integral is at most $\text{perim}(T^{(n)}) \cdot \text{diam}(T^{(n)}) \cdot M_n$.

Since the perimeter and diameter of $T^{(n)}$ both decay at a rate of $2^{-n}$, we establish the bound that $\left| \int_{T^{(n)}} f(z)dz \right| \leq 4^{-n} C M_n$ for some constant $C$. Hence, $CM_n$ is an upper-bound for $\left| \int_T f(z)dz \right|$, and since $\psi(z) \to 0$ as $z \to z_0$, $M_n \to 0$ as desired.

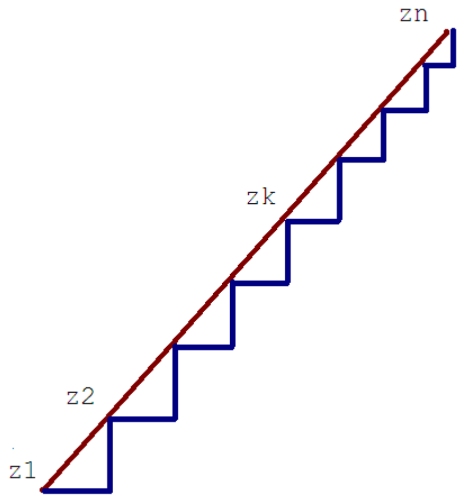
Now let us see what happens if we don't know that $f$ is differentiable. Using only continuity, we can approximate $f(z)$ by $f(z_0) + \psi(z)(z - z_0)$. Defining $M_n$ as before, we can still bound our integral by $CM_n$. We want to say that $M_n$ tends to 0, but $\lim_{z \to z_0} \psi(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$, which may not exist if $f$ is not differentiable (and certainly may not tend to zero). Thus, this approach fails.

We could also try the expression $f(z) = f(z_0) + \psi(z)$, and then $\psi(z) \to 0$ as $z \to z_0$. Unfortunately, without the factor of $(z - z_0)$, our bound on $| \int_{T^{(n)}} f(z)dz|$ will simply be $\text{perim}(T^{(n)}) \cdot M_n = 2^{-n} C M_n$. Thus, our bound for $| \int_{T^{(n)}} f(z)dz|$ is $4^n 2^{-n} C M_n = 2^n C M_n$. Even though $M_n$ tends to 0, the factor of $2^n$ may overwhelm it, so this approach fails. From these attempts, it seems that knowing that $f$ was differentiable was a fairly important step in the proof.

**Problem:** Prove Goursat's theorem for triangles using only the fact that it holds for rectangles.

Note that it suffices to prove that the integral along any right triangle is zero, since any triangle can be divided into two right triangles by dropping an altitude.

Given a right triangle ABC, by drawing a series of rectangles inside the triangle, we can reduce the desired integral to the integral along a series of $n$ congruent triangles similar to ABC, each of which border the original hypotenuse (as shown in the figure).




Since $f$ is continuous on the original triangle ABC (a compact set) we know that $f$ is uniformly continuous on the region of interest.

Thus, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any two points $x, y$ in ABC with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$. If $h$ is the length of the hypotenuse of ABC, choose $n$ large enough so that the diameter of each small triangle, $h/n$, is less than $\delta$. Then for any triangle $T_k$ and any point $z_k$ on that triangle write $f(z) = f(z_k) + \psi(z)$, so that

$$\int_{T_k} f(z)dz = \int_{T_k} f(z_k) + \psi(z)dz = \int_{T_k} f(z_k)dz + \int_{T_k} \psi(z)dz$$

Since $f(z_k)$ is a constant, it has a primitive, so the first integral is zero. Meanwhile, since any point on triangle $T_k$ is within $h/n$ of $z_k$, and we chose $n$ to be such that $h/n < \delta$, we know that $|\psi(z)| = |f(z) - f(z_k)| < \varepsilon$. Thus, $|\int_{T_k} \psi(z)dz| <$ perim$(T_k) \cdot \varepsilon$. But perim$(T_k) < 3h/n$, so the integral of $f(z)$ along triangle $T_k$ is at most $3h\varepsilon/n$. Summing over all $n$ triangles, we see that the integral of $f(z)$ along the entire curve is at most $3h\varepsilon$. Since this technique works for arbitrarily small $\varepsilon$, this implies that the integral of $f$ along any right triangle is zero, proving the claim.

## 4 Math 372: Homework #3: Carlos Dominguez, Carson Eisenach, David Gold

**HW: Due at the start of class by 11am Friday, September 27: Chapter 2, Page 64: #1, #8. Also do: Chapter 2: (Problems from me): (#1) In the proof of Liouville's theorem we assumed $f$ was bounded. Is it possible to remove that assumption? In other words, is it enough to assume that $|f(z)| < g(z)$ for some real-valued, non-decreasing function $g$? If yes, how fast can we let $f$ grow? (#2) a) Find all $z$ where the function $f(z) = 1/(1 + z^4)$ is not holomorphic; b) Let $a, b, c$, and $d$ be integers such that $ad - bc = 1$. Find all $z$ where the function $g(z) = (az + b)/(cz + d)$ is not holomorphic. (#3) Compute the power series expansion of $f(z) = 1/(1 - z)$ about the point $z = 1/2$ (it might help to do the next problem first, or to write $1 - z$ as $1/2 - (z - 1/2)$). (#4) Do Chapter 1, Page 29, #18.**

**Math 372: Complex Analysis**

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1. Let $\gamma_1$ denote the straight line along the real line from 0 to $R$, $\gamma_2$ denote the eighth of a circle from $R$ to $Re^{i\frac{\pi}{4}}$, and $\gamma_3$ denote the line from $Re^{i\frac{\pi}{4}}$ to 0. Then by Cauchy's theorem,

$$\int_{\gamma_1+\gamma_2+\gamma_3} e^{-z^2}\,dz = 0.$$

We can calculate

$$\begin{aligned}
-\int_{\gamma_3} e^{-z^2}\,dz &= \int_0^R e^{-(e^{i\pi/4}t)^2} e^{i\pi/4}\,dt \\
&= e^{i\pi/4}\int_0^R e^{-it^2}\,dt \\
&= e^{i\pi/4}\int_0^R \cos(-t^2)dt + i\sin(-t^2)\,dt \\
&= e^{i\pi/4}\int_0^R \cos(t^2)dt - i\sin(t^2)\,dt
\end{aligned}$$

So we can calculate the Fresnel integrals by calculating $\int_{\gamma_1+\gamma_2} e^{-z^2}\,dz$, taking $R \to \infty$, dividing by $e^{i\pi/4}$, and looking at the real and negative imaginary parts. First we show the integral over $\gamma_2$ goes to zero:

$$\begin{aligned}
\left|\int_{\gamma_2} e^{-z^2}\,dz\right| &= \left|\int_0^{\pi/4} e^{-R^2e^{2i\theta}} iRe^{i\theta}\,d\theta\right| \\
&\le R\int_0^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta \\
&= R\int_0^{\pi/4-1/R\log R} e^{-R^2\cos 2\theta}\,d\theta + R\int_{\pi/4-1/R\log R}^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta \\
&\le R\left(\frac{\pi}{4} - \frac{1}{R\log R}\right) e^{-R^2\cos\left(\frac{\pi}{2}-\frac{2}{R\log R}\right)} + R\cdot\frac{1}{R\log R} \\
&\le \frac{\pi}{4}Re^{-R^2\sin\left(\frac{2}{R\log R}\right)} + \frac{1}{\log R}
\end{aligned}$$

The $\frac{1}{\log R}$ term goes to zero as $R$ goes to infinity. So we need to show that the first term goes to zero. Note that $\sin x \geq x/2$ for positive $x$ sufficiently close to 0, since $\sin 0 = 0$ and $\frac{d}{dx} \sin x \geq 1/2$ for sufficiently small $x$. So for sufficiently large $R$ the first term is less than or equal to

$$\frac{\pi}{4} R e^{-R^2 \cdot \frac{1}{R \log R}} = \frac{\pi}{4} e^{\log R - \frac{R}{\log R}},$$

which goes to zero as $R$ goes to infinity. So, as $R \to \infty$, the contribution from $\gamma_2$ goes to zero. And we know that as $R \to \infty$, $\int_0^R e^{-x^2} \, dx = \sqrt{\pi}/2$. So, finally,

$$\begin{aligned}
\int_0^\infty \cos(t^2) dt - i \sin(t^2) \, dt &= \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{2}/2 + i\sqrt{2}/2} \\
&= \frac{\sqrt{2\pi}}{4} - \frac{\sqrt{2\pi}}{4} i
\end{aligned}$$

as desired.

8. Since $x \in \mathbb{R}$, $f$ is holomorphic in an open circle of radius $\epsilon$ centered at $x$, $0 < \epsilon < 1$. And by Cauchy's inequality,

$$|f^{(n)}(x)| \leq \frac{n! ||f||_C}{R^n}$$

Case 1: $\eta \geq 0$. For some $0 < \epsilon < 1$,

$$|z| \leq |x + \epsilon|$$

thus,

$$|f(z)| \leq A(1 + |x + \epsilon|)^\eta \leq A(1 + \epsilon + |x|)^\eta$$

by both the given and the triangle inequality. And in Cauchy's inequality R is just $\epsilon$. So by combining results from above

$$\begin{aligned}
|f^{(n)}(x)| &\leq \frac{n! ||f||_C}{\epsilon^n} \\
&\leq \frac{An!}{\epsilon^n} (1 + \epsilon + |x|)^\eta \\
&\leq \frac{An!}{\epsilon^n} (1 + \epsilon + |x| + \epsilon |x|)^\eta \\
&\leq \frac{An!}{\epsilon^n} (1 + \epsilon)^\eta (1 + |x|)^\eta. \qquad (4.1)
\end{aligned}$$

Now let

$$A_n = \frac{A(n!)}{\epsilon^n} (1 + \epsilon)^\eta$$

thus,

$$|f^{(n)}(x)| \leq A_n (1 + |x|)^\eta.$$

Case 2: $\eta < 0$. For some $0 < \epsilon < 1$,

$$\epsilon \geq |x - z| \geq |x| - |z|$$

by the reverse triangle inequality. When we rearrange the inequality we see that

$$|z| \geq |x| - |\epsilon| = |x| + \epsilon$$

Since $\eta$ is negative, our goal is to minimize (1+|z|) in order to get an upper bound. Now, by combining our result above with the Cauchy inequality we get that:

$$\begin{aligned}
|f^{(n)}(x)| &\leq \frac{n!||f||_C}{\epsilon^n} \leq \frac{An!}{\epsilon^n}(1 - \epsilon + |x|)^\eta \\
&\leq \frac{An!}{\epsilon^n}(1 - \epsilon + |x| - \epsilon|x|)^\eta \\
&\leq \frac{An!}{\epsilon^n}(1 - \epsilon)^\eta(1 + |x|)^\eta. \qquad (4.2)
\end{aligned}$$

Now let

$$A_n = \frac{A(n!)}{\epsilon^n}(1 - \epsilon)^\eta$$

thus,

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^\eta.$$

q.e.d.

1. In the proof of Liouville's theorem, we had that

$$|f'(z_0)| \leq \frac{B}{R}$$

where $B$ was an upper bound for $f$. It only matters that $B$ is an upper bound for $f$ in a disc of radius $R$ about $z_0$, however. Let $B_R$ be the smallest upper bound for $f$ in a disc of radius $R$ about $z_0$. Liouville's theorem still holds if $B_R \to \infty$ as long as $B_R/R \to 0$ for every choice of $z_0$. Alternatively, we just need $f$ to grow slower than linear; say $|f(z)|$ is less than $C|z|^{1-\epsilon}$ or $C|z|/\log|z|$ or anything like this (for those who have seen little-oh notation, $f(z) = o(z)$ suffices).

2. (a) $f$ is holomorphic wherever its derivative exists:

$$f'(z) = -\frac{4z^3}{1 + z^4}$$

That is, whenever $z^4 \neq -1$. This gives $z = e^{i\pi/4}$, $e^{3i\pi/4}$, $e^{5i\pi/4}$, and $e^{7i\pi/4}$, or $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, and $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

(b) The $ad - bc = 1$ condition prevents $g$ from being a mostly-constant function with an undefined value at $z = -d/c$. (That is, if $ad - bc = 0$, then $a/c = b/d$, and so the function would simply collapse to the value of $a/c$.) So

$$g'(z) = \frac{(cz + d)a - (az + b)c}{(cz + d)^2} = \frac{1}{(cz + d)^2}$$

The function is then not holomorphic at $z = -d/c$.

3. Just use the geometric series formula:

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{1/2-(z-1/2)}\\ &= \frac{2}{1-2(z-1/2)}\\ &= \sum_{n=0}^{\infty} 2^{n+1}(z-1/2)^n.\end{aligned}$$

4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$\begin{aligned}f(z) &= \sum_{n=0}^{\infty} a_n (z_0 + (z-z_0))^n\\ &= \sum_{n=0}^{\infty} a_n \left[\sum_{m=0}^{n} \binom{n}{m} (z-z_0)^m z_0^{n-m}\right]\\ &= \sum_{m=0}^{\infty} (z-z_0)^m \left(\sum_{n=m}^{\infty} a_n \binom{n}{m} z_0^{n-m}\right).\end{aligned}$$

The inner sum converges by the root test:

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{a_n \binom{n}{m}} &= \frac{1}{R} \lim_{n\to\infty} \sqrt[n]{\binom{n}{m}}\\ &= \frac{1}{R}\end{aligned}$$

where R is the radius of convergence of the original power series for $f$ and second limit is evaluated by noting $1 \leq \sqrt[n]{\binom{n}{m}} \leq n^{m/n}$ and $\lim_{n\to\infty} n^{m/n} = 1$. Since the inner sum has the same radius of convergence as the original sum, $z_0$ still lies in the disc of convergence in the inner sum; hence all the coefficients of $z - z_0$ converge, and $f$ has a power series expansion about $z_0$.

**Homework 4: Due at the start of class by 11am Friday, October 11 (even if this is Mountain Day): Chapter 3, Page 103: #1, #2, #5 (this is related to the Fourier transform of the Cauchy density), #15d, #17a (hard). Additional: Let $f(z) = \sum_{n=-5}^{\infty} a_n z^n$ and $g(z) = \sum_{m=-2}^{\infty} b_m z^m$ be the Laurent expansions for two functions holomorphic everywhere except possibly at $z = 0$. a) Find the residues of $f(z)$ and $g(z)$ at $z = 0$; b) Find the residue of $f(z) + g(z)$ at $z = 0$; c) Find the residue of $f(z)g(z)$ at $z = 0$; d) Find the residue of $f(z)/g(z)$ at $z = 0$.**

## 5 Math 372: Homework #4: Due Friday, October 4, 2013: Pham, Jensen, Koloğlu

**HW: Due at the start of class by 11am Friday, October 11 (even if this is Mountain Day): Chapter 3, Page 103: #1, #2, #5 (this is related to the Fourier transform of the Cauchy density), #15d, #17a (hard). Additional: Let $f(z) = \sum_{n=-5}^{\infty} a_n z^n$ and $g(z) = \sum_{m=-2}^{\infty} b_m z^m$ be the Laurent expansions for two functions holomorphic everywhere except possibly at $z = 0$. a) Find the residues of $f(z)$ and $g(z)$ at $z = 0$; b) Find the residue of $f(z) + g(z)$ at $z = 0$; c) Find the residue of $f(z)g(z)$ at $z = 0$; d) Find the residue of $f(z)/g(z)$ at $z = 0$.**

### 5.1 Chapter 3, Exercise 1

**Exercise 5.1.** *Using Euler's formula $\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$, show that the complex zeros of $\sin \pi z$ are exactly the integers, and that they are each of order 1. Calulate the residue of $\frac{1}{\sin \pi z}$ at $z = n \in \mathbb{Z}$.*

***Solution:*** To show that the complex zeros of $\sin \pi z$ are exactly the integers, we will show that $\frac{e^{i\pi z_0} - e^{-i\pi z_0}}{2i} = 0$ if and only if $z_0 \in \mathbb{Z}$.

First prove the forward direction. We see that $\frac{e^{i\pi z_0} - e^{-i\pi z_0}}{2i} = 0$ gives

$$e^{i\pi z_0} = e^{-i\pi z_0}. \tag{5.1}$$

Since $z_0 = x + iy$ with $x, y \in \mathbb{R}$,

$$e^{i\pi x} e^{-\pi y} = e^{-i\pi x} e^{\pi y}. \tag{5.2}$$

For complex numbers to be equivalent, their magnitudes must be the same. Thus,

$$e^{-\pi y} = e^{\pi y}. \tag{5.3}$$

This implies that $-\pi y = \pi y$, so $y = 0$. The angles corresponding to Equation 5.2 must be congruent modulo $2\pi$ as well. Thus,

$$\pi x \equiv -\pi x \bmod 2\pi, \tag{5.4}$$

which means $\pi x \equiv 0$ or $\pi$. So we have

$$2\pi x \mod 2\pi \equiv 0, \tag{5.5}$$

which implies that $x$ is an integer. Thus $x \in \mathbb{Z}$. Since $y = 0$, we have $z_0 = x$, implying $z_0 \in \mathbb{Z}$.

To prove the backward direction, consider $z_0 \in \mathbb{Z}$ for $z_0$ even,

$$\begin{aligned} \sin \pi z_0 &= \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \\ &= \frac{1-1}{2i} = 0. \end{aligned} \tag{5.6}$$

Similarly for $z_0$ odd,

$$\begin{aligned} \sin \pi z_0 &= \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \\ &= \frac{-1+1}{2i} = 0. \end{aligned} \tag{5.7}$$

Thus $\sin \pi z_0 = 0$ if and only if $z_0 \in \mathbb{Z}$. So the zeros of $\sin \pi z$ are exactly the integers.

Next we must show that each zero has order 1. We refer to Theorem 1.1 in Stein and Shakarchi.

**Theorem 5.2.** *Suppose that f is holomorphic in a connected open set $\Omega$, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in $\Omega$. Then there exists a neighborhood $U \subset \Omega$ of $z_0$, a non-vanishing holomorphic function g on U, and a unique positive integer n such that $f(z) = (z - z_0)^n g(z)$ for all $z \in U$.*

Since $\sin \pi z$ is analytic, take its Taylor series about $z_0$. We add zero to write $z$ as $z - z_0 + z_0$. Using properties of the sine function, we claim

$$\sin \pi z \;=\; \sin \pi(z + z_0 - z_0) \;=\; \sin \pi(z - z_0) \cos \pi z_0 + \cos \pi(z - z_0) \sin \pi z_0. \tag{5.8}$$

Note this statement does require proof, but will follow from standard properties of the exponential function (or from analytic continuation). The reason some work needs to be done is that $z - z_0$ need not be real, but the relation above does hold when $z$ is real. What we are trying to do is understand the behavior of the function near $z_0$ from knowledge near $0$ (as $z - z_0$ is close to zero). This is a common trick, but of course what makes this tractable is that we have the angle addition formula for sine.

When $z_0$ is an integer, we always have $\sin \pi z_0 = 0$. If $z_0$ is odd then $\cos \pi z_0$ is -1 while if $z_0$ is even it is 1. Thus for odd $z_0$,

$$\sin \pi z = -\frac{\pi}{1!}(z - z_0)^1 + \frac{\pi^3}{3!}(z - z_0)^3 - \frac{\pi^5}{5!}(z - z_0)^5 + \cdots \tag{5.9}$$

and for even $z_0$,

$$\sin \pi z = \frac{\pi}{1!}(z - z_0)^1 - \frac{\pi^3}{3!}(z - z_0)^3 + \frac{\pi^5}{5!}(z - z_0)^5 - \cdots . \tag{5.10}$$

We thus see that all zeros are simple.

We now turn to finding the residue at $z = n$ for $1/\sin \pi z$. From our Taylor expansion above, we have

$$\frac{1}{\sin \pi z} \;=\; \frac{1}{\sin \pi(z - n) \cos \pi n} \;=\; \frac{1}{\cos \pi n} \frac{1}{\sin \pi(z - n)}. \tag{5.11}$$

The problem is now solved by using the Taylor expansion of sine and the geometric series. We have $\cos \pi n = (-1)^n$, so

$$\begin{aligned} \frac{1}{\sin \pi z} &= (-1)^n \frac{1}{\pi(z - z_0) - \frac{1}{3!}\pi^3 (z - z_0)^3 + \cdots} \\ &= \frac{(-1)^n}{\pi(z - z_0)} \frac{1}{1 - \left(\frac{1}{3!}\pi^2 (z - z_0)^2 + \cdots\right)} \\ &= \frac{(-1)^n}{\pi(z - z_0)} \left(1 + \left(\frac{1}{3!}\pi^2 (z - z_0)^2 + \cdots\right) + \left(\frac{1}{3!}\pi^2 (z - z_0)^2 + \cdots\right)^2 + \cdots\right). \end{aligned} \tag{5.12}$$

Note that each term in parentheses in the last line is divisible by $(z - z_0)^2$, and thus *none* of these will contribute to the residue, which is simply $(-1)^n/\pi$.

## 5.2 Chapter 3, Exercise 2

**Exercise 5.3.** *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

***Solution:*** Consider the function $f(z) = \frac{1}{1+z^4}$. This function has poles at

$$\begin{aligned} 1/f(z) &= 0 \\ 1 + z^4 &= 0 \\ z &= e^{i\left(\frac{\pi}{4}+n\frac{\pi}{2}\right)}. \end{aligned} \tag{5.13}$$

Consider the contour of the semicircle in the upper half plane of radius $R$, denoted $\gamma$. Denote the part of the contour along the real line $\gamma_1$ and the part along the arc $\gamma_2$. Note that two of the poles of $f(z)$ lie inside this contour. Thus by Cauchy's residue theorem,

$$\frac{1}{2\pi i}\oint_\gamma f dz = \mathrm{Res}_f(e^{i\pi/4}) + \mathrm{Res}_f(e^{i3\pi/4}). \tag{5.14}$$

To find the residues, write

$$f(z) = \frac{1}{1+z^4} = \left(\frac{1}{z - e^{i\frac{\pi}{4}}}\right)\left(\frac{1}{z - e^{i\frac{3\pi}{4}}}\right)\left(\frac{1}{z - e^{i\frac{5\pi}{4}}}\right)\left(\frac{1}{z - e^{i\frac{7\pi}{4}}}\right).$$

Thus

$$\begin{aligned} \mathrm{Res}_f(e^{i\pi/4}) &= \left(\frac{1}{e^{i\frac{\pi}{4}} - e^{i\frac{3\pi}{4}}}\right)\left(\frac{1}{e^{i\frac{\pi}{4}} - e^{i\frac{5\pi}{4}}}\right)\left(\frac{1}{e^{i\frac{\pi}{4}} - e^{i\frac{7\pi}{4}}}\right) \\ &= e^{-i\frac{3\pi}{4}}\left(\frac{1}{1-i}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1+i}\right) \\ &= -\frac{1+i}{4\sqrt{2}} \end{aligned} \tag{5.15}$$

and similarly

$$\begin{aligned} \mathrm{Res}_f(e^{i\frac{3\pi}{4}}) &= e^{-i\frac{9\pi}{4}}\left(\frac{1}{1+i}\right)\left(\frac{1}{1-i}\right)\left(\frac{1}{2}\right) \\ &= \frac{1-i}{4\sqrt{2}} \end{aligned} \tag{5.16}$$

Thus we have

$$\begin{aligned} \frac{1}{2\pi i}\oint_\gamma f dz &= -\frac{1+i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}} \\ &= -\frac{i}{2\sqrt{2}} \\ \oint_\gamma f dz &= \frac{\pi}{\sqrt{2}}. \end{aligned} \tag{5.17}$$

Now, note that

$$\oint_\gamma f dz = \oint_{\gamma_1+\gamma_2} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz. \tag{5.18}$$

Observe that

$$\int_{\gamma_1} f dz = \int_{-R}^{R} \frac{1}{1+x^4} dx$$

and that

$$\begin{aligned}
\int_{\gamma_2} f dz &= \int_{-R}^{R} \frac{1}{1+z^4} dx \\
\left|\int_{\gamma_2} f dz\right| &= \left|\int_{-R}^{R} \frac{1}{1+z^4} dx\right| \\
&\leq \max_{z\in\gamma_2} \left|\frac{1}{1+z^4}\right| \pi R \\
&= \frac{1}{R^4-1}\pi R. \qquad (5.19)
\end{aligned}$$

Thus

$$\lim_{R\to\infty} \left|\int_{\gamma_2} f dz\right| \leq \lim_{R\to\infty} \frac{\pi R}{R^4-1} = 0. \qquad (5.20)$$

Hence, as $R \to \infty$, $\int_{\gamma_2} f dz \to 0$. Therefore as $R \to \infty$ we get our final result;

$$\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4} dx + \lim_{R\to\infty} \int_{\gamma_2} f dz &= \frac{\pi}{\sqrt{2}} \\
\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}. \qquad (5.21)
\end{aligned}$$

## 5.3 Chapter 3, Exercise 5

**Exercise 5.4.** *Use contour integration to show that* $\int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\xi}}{(1+x^2)^2} dx = \frac{\pi}{2}(1+2\pi|\xi|)e^{-2\pi|\xi|}$ *for all* $\xi$ *real.*

***Solution:*** Let $f(z) = \frac{e^{-2\pi i z\xi}}{(1+z^2)^2} = \frac{e^{-2\pi i z\xi}}{(z+i)^2(z-i)^2}$. We see that $f(z)$ has poles of order 2 at $z = \pm i$. Thus

$$\text{res}_{z_0} f(z) = \lim_{z\to z_0} \frac{d}{dz}(z-z_0)^2 f(z). \qquad (5.22)$$

Alternatively, we could write our function as

$$f(z) = \frac{g(z)}{(z-z_0)^2}, \qquad (5.23)$$

and then we need only compute the coefficient of the $z - z_0$ term of $g$.

Now consider the residue at $z_0 = i$:

$$\begin{aligned}
\text{res}_{z_0=i} f(z) &= \lim_{z\to i} \frac{d}{dz}(e^{-2\pi i z\xi}(z+i)^{-2}) \\
&= \lim_{z\to i}(-2\pi i\xi e^{-2\pi i z\xi}(z+i)^{-2} - 2e^{-2\pi i z\xi}(z+i)^{-3}) \\
&= \frac{1}{2}\pi i\xi e^{2\pi\xi} - \frac{1}{4} i e^{2\pi\xi}. \qquad (5.24)
\end{aligned}$$

For $z_0 = -i$, we have:

$$\begin{aligned}
\mathrm{res}_{z_0=-i} f(z) &= \lim_{z\to i} \frac{d}{dz}(e^{-2\pi i z\xi}(z-i)^{-2}) \\
&= \lim_{z\to -i}(-2\pi i\xi e^{-2\pi i z\xi}(z-i)^{-2} - 2e^{-2\pi i z\xi}(z-i)^{-3}) \\
&= \frac{1}{2}\pi i\xi e^{-2\pi\xi} + \frac{1}{4} i e^{-2\pi\xi}. \qquad (5.25)
\end{aligned}$$

Now let us first consider the case when $\xi < 0$. We will use the contour $\gamma$ of a semicircle oriented counterclockwise in the upper half-plane with radius $R$. Call the portion of $\gamma$ along the real line $\gamma_1$ and the arc portion $\gamma_2$. Note that there is a pole inside $\gamma$ at $z_0 = i$. By the residue formula, we have that

$$\int_\gamma f(z)dz = 2\pi i\left(\frac{1}{2}\pi i\xi e^{2\pi\xi} - \frac{1}{4} i e^{2\pi\xi}\right) = -\pi^2\xi e^{2\pi\xi} + \frac{1}{2}\pi e^{2\pi\xi}. \qquad (5.26)$$

We also know that

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{R\to\infty}\int_{\gamma_1} f(z)dz. \qquad (5.27)$$

Along $\gamma_2$, $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\theta$, where $z = R\cos\theta + iR\sin\theta$. Thus

$$\int_{\gamma_2} f(z)dz = \int_0^\pi \frac{e^{-2\pi i\xi Re^{i\theta}} iRe^{i\theta}}{(1-R^2e^{i2\theta})^2}d\theta. \qquad (5.28)$$

Then it follows that

$$\begin{aligned}
\left|\int_{\gamma_2} f(z)dz\right| &\leq \int_0^\pi \left|\frac{e^{-2\pi i\xi R\cos\theta}e^{2\pi\xi R\sin\theta} iRe^{i\theta}}{(1-R^2e^{i2\theta})^2}\right| d\theta \\
&\leq \int_0^\pi \left|\frac{Re^{-2\pi|\xi|R\sin\theta}}{(1-R^2)^2}\right| d\theta \\
&\leq \int_0^\pi \frac{R}{(1-R^2)^2}d\theta = \frac{\pi R}{(R^2-1)^2}. \qquad (5.29)
\end{aligned}$$

Taking the limit as $R$ goes to infinity, we have

$$\lim_{R\to\infty}\left|\int_{\gamma_2} f(z)dz\right| \leq \lim_{R\to\infty}\frac{\pi R}{(R^2-1)^2} = 0. \qquad (5.30)$$

Thus

$$\lim_{R\to\infty}\int_{\gamma_2} f(z)dz = 0. \qquad (5.31)$$

So $\lim_{R\to\infty}\int_\gamma f(z) = \lim_{R\to\infty}\int_{\gamma_1} f(z)$. It thus follows from Equation 5.26 that

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{e^{-2\pi i x\xi}}{(1+x^2)^2}dx &= -\pi^2\xi e^{2\pi\xi} + \frac{1}{2}\pi e^{2\pi\xi} \\
&= \frac{\pi}{2}(1+2\pi|\xi|)\, e^{-2\pi|\xi|} \qquad (5.32)
\end{aligned}$$

Now consider $\xi \geq 0$. We will use the contour $\gamma$ of a semicircle oriented counterclockwise in the lower half-plane with radius $R$. Call the portion of $\gamma$ along the real line $\gamma_1$ and the arc portion $\gamma_2$. Note that there is a pole inside $\gamma$ at $z_0 = -i$. By the residue formula, we have that

$$\int_\gamma f(z)dz = 2\pi i \left( \frac{1}{2}\pi i \xi e^{-2\pi\xi} + \frac{1}{4} i e^{-2\pi\xi} \right) = -\pi^2 \xi e^{-2\pi\xi} - \frac{1}{2}\pi e^{-2\pi\xi}. \tag{5.33}$$

Also note that,

$$\int_{-\infty}^{+\infty} f(x)dx = -\lim_{R\to\infty} \int_{\gamma_1} f(z)dz. \tag{5.34}$$

Along $\gamma_2$, $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\theta$, where $z = R\cos\theta + iR\sin\theta$. Thus,

$$\int_{\gamma_2} f(z)dz = \int_{-\pi}^{0} \frac{e^{-2\pi i \xi R\cos\theta} e^{2\pi\xi R\sin\theta} iRe^{i\theta}}{(1 - R^2 e^{i2\theta})^2} d\theta. \tag{5.35}$$

Accordingly,

$$\begin{aligned} \left| \int_{\gamma_2} f(z)dz \right| &\leq \int_{\gamma_2} |f(z)|dz \\ &\leq \int_{-\pi}^{0} \left| \frac{Re^{2\pi|\xi|R\sin\theta}}{(1 - R^2 e^{i2\theta})^2} \right| d\theta \\ &\leq \int_{-\pi}^{0} \left| \frac{R}{(1 - R^2)^2} \right| d\theta \\ &= \frac{\pi R}{(1 - R^2)^2} \end{aligned} \tag{5.36}$$

Taking the limit as $R$ goes to infinity, we have

$$\lim_{R\to\infty} \left| \int_{\gamma_2} f(z)dz \right| \leq \lim_{R\to\infty} \frac{\pi R}{(R^2 - 1)^2} = 0. \tag{5.37}$$

And thus,

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)dz = 0. \tag{5.38}$$

So $\lim_{R\to\infty} \int_\gamma f(z) = \lim_{R\to\infty} \int_{\gamma_1} f(z)$. It thus follows from Equation 5.33 that

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\xi}}{(1 + x^2)^2} dx &= -\left( -\pi^2 \xi e^{-2\pi\xi} - \frac{1}{2}\pi e^{-2\pi\xi} \right) \\ &= \frac{\pi}{2} \left(1 + 2\pi|\xi|\right) e^{-2\pi|\xi|} \end{aligned} \tag{5.39}$$

Thus for all $\xi$ real,

$$\int_{-\infty}^{+\infty} \frac{e^{-2\pi i x\xi}}{(1 + x^2)^2} dx = \frac{\pi}{2} \left(1 + 2\pi|\xi|\right) e^{-2\pi|\xi|} \tag{5.40}$$

### 5.4 Chapter 3 Exercise 15d

[1] For any entire function $f$, let's consider the function $e^{f(x)}$. It is an entire function and furthermore we have the real part of $f$ is bounded so:

$$|e^f| = |e^{u+iv}| = |e^u| \leq \infty$$

Hence $e^f$ is bounded and therefore, by Louisville's Theorem, $e^f$ is constant. It then follows that $f$ is constant .

Alternatively, we could argue as follows. We are told the real part of $f$ is bounded. Let's assume that the real part is always at most $B-1$ in absolute value. Then if we consider $g(z) = 1/(B - f(z))$ we have $|g(z)| \leq 1$. To see this, note the real part of $B - f(z)$ is at least 1. We again have constructed a bounded, entire function, and again by Liouville's theorem we can conclude $g$ (and hence $f$) is constant.

### 5.5 Chapter 3 Exercise 17a

**Exercise 5.5.** *Let $f$ be non-constant and holomorphic in an open set containing the closed unit disc. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then the image of $f$ contains the unit disc.*

***Solution:*** Suppose $f(z)$ does not have a zero in the unit disc, $\mathbb{D}$. Then $1/f(z)$ is holomorphic in $\mathbb{D}$. Note that since $|f(z)| = 1$ whenever $|z| = 1$, $|1/f(z)| = 1/|f(z)| = 1$ whenever $|z| = 1$ as well. But $f(z)$ is holomorphic in $\mathbb{D}$, implying $|f(z)| \leq 1$ in $\mathbb{D}$ by the maximum modulus principle since $|f(z)| = 1$ on the boundary of $\mathbb{D}$. We find $1 \leq |f(z)| \leq 1$ in the unit disk, which implies that our function is constant as its modulus is constant (we would like to use Exercise 15d, but that requires our function to be entire; fortunately we can obtain constancy by the Open Mapping Theorem), contradicting the assumption that $f$ is not constant!

Let $w_0 \in \mathbb{D}$. Consider the constant function $g(z) = -w_0$. On the unit circle, $|f(z)| = 1 > |w_0| = |g(z)|$ for all $|z| = 1$. Thus by Rouché's theorem, $f(z)$ and $f(z) + g(z)$ have the same number of zeroes inside the unit circle (ie, in $\mathbb{D}$). But we have shown that $f(z)$ has at least one zero, thus for some $z_w$, $0 = f(z_w) + g(z_w) = f(z_w) - w_0$. Thus for all $w_0 \in \mathbb{D}$, there exists $z_w$ such that $f(z_w) = w_0$. Thus the image of $f(z)$ contains the unit disc. □

### 5.6 Additional Problem 1

[2] Let:

$$f(z) = \sum_{n=-5}^{\infty} a_n \, z^n \qquad g(z) = \sum_{m=-2}^{\infty} b_m \, z^m$$

1. We have:

$$\mathrm{res}_0 f = a_{-1} \qquad \mathrm{res}_0 g = b_{-1}$$

2. We have

$$f(z) + g(z) = \sum_{n=-5}^{-3} a_n z^n \; + \; \sum_{n=-2}^{\infty} (a_n + b_n) z^n$$

So $\mathrm{res}_0(f+g) = a_{-1} + b_{-1}$.

[1]Hint from Professor Miller
[2]Hint from Professor Miller

3. We have $-1 = -5+4 = -4+3 = -3+2 = -2+1 = -1+0 = 0-1 = 1-2$ so:

$$\text{res}_0(f\,g) = a_{-5}\,b_4 + a_{-4}\,b_3 + a_{-3}\,b_2 + a_{-2}\,b_1 + a_{-1}\,b_0 + a_0\,b_{-1} + a_1\,b_{-2}$$

4. We have (assuming $b_2 \neq 0$):

$$\begin{aligned} \frac{f(z)}{g(z)} &= \frac{\sum_{n=-5}^{\infty} a_n\,z^n}{\sum_{m=-2}^{\infty} b_m\,z^m} \\ &= \frac{1}{z^3}\frac{\sum_{n=-2}^{\infty} a_{n-3}\,z^n}{\sum_{m=-2}^{\infty} b_m\,z^m} \\ &= \frac{1}{b_{-2}z}\;\frac{\sum_{n=-2}^{\infty} a_{n-3}\,z^n}{1-(-\frac{1}{b_{-2}}\sum_{m=1}^{\infty} b_{m-2}\,z^m)}. \end{aligned} \tag{5.41}$$

As $z \to 0$ the final quantity in parentheses tends to zero, and thus we can expand using the geometric series formula. We only care about the constant term of this fraction, as it is multiplied by $1/b_{-2}z$ and thus only the constant term contributes to the pole. This is a very useful observation. It means that, when we expand with the geometric series, we can drop many terms, as we only need to keep terms that contribute to the constant term. Remember, we are not trying to find the Taylor expansion of this function, but rather just one particular term. We can thus write:

$$\begin{aligned} \frac{f(z)}{g(z)} &= \frac{1}{b_{-2}z}\,(\sum_{n=-2}^{\infty} a_{n-3}\,z^n)\sum_{k=0}^{\infty}\left(-\frac{1}{b_{-2}}\sum_{m=1}^{\infty} b_{m-2}\,z^m)\right)^k \\ &= \frac{1}{b_{-2}z}\Bigg[(a_{-5}z^{-2})\left(\frac{-1}{b_{-2}}\,(b_0z^2+\cdots)+\frac{1}{b_{-2}^2}\,(b_{-1}^2z^2+\cdots)+\cdots\right) \\ &\quad +(a_{-4}z^{-1})\left(\frac{-1}{b_{-2}}\,(b_{-1}z^1+\cdots)+\cdots\right)+(a_{-3}z^0)\,(1+\cdots)+\cdots.\Bigg] \end{aligned} \tag{5.42}$$

So:

$$\text{res}_0(\frac{f}{g}) = \frac{1}{b_{-2}}\left[a_{-5}(-\frac{b_0}{b_{-2}}+\frac{b_{-1}}{b_{-2}^2})+a_{-4}(-\frac{b_{-1}}{b_{-2}})+a_{-3}\right]$$

.

Homework due Friday October 18 (though you are strongly encouraged to hand it in on Friday, October 18, you may hand it in by 10am on Monday October 21, but 10am does not mean 10:05am!!!): The Midterm!

**HW: Due at the start of class by 11am Friday, October 25: Chapter 5: Page 155: #6, #7, #9 (extra credit: what is the combinatorial significance of this problem?). Chapter 3: Page 104: #10. Additional Problems: (1) Find all poles of the function $f(z) = 1/(1-z^2)^4$ and find the residues at the poles. (2) Consider the map $f(z) = (z-i)/(z+i)$. Show that this is a 1-to-1 and onto map from the upper half plane (all $z = x+iy$ with $y > 0$) to the unit disk. (3) Calculate the Weierstrass product for $\cos(\pi z)$ (this is also problem #10b in Chapter 5, and the answer is listed there), and for $\tan(\pi z)$.**

## 6 Math 372: Homework #5: Due Friday October 25: Pegado, Vu

**HW: Due at the start of class by 11am Friday, October 25: Chapter 5: Page 155: #6, #7, #9 (extra credit: what is the combinatorial significance of this problem?). Chapter 3: Page 104: #10. Additional Problems: (1) Find all poles of the function $f(z) = 1/(1-z^2)^4$ and find the residues at the poles. (2) Consider the map $f(z) = (z-i)/(z+i)$. Show that this is a 1-to-1 and onto map from the upper half plane (all $z = x + iy$ with $y > 0$) to the unit disk. (3) Calculate the Weierstrass product for $\cos(\pi z)$ (this is also problem #10b in Chapter 5, and the answer is listed there), and for $\tan(\pi z)$.**

**6.** *Prove Wallis's product formula*

$$\frac{\pi}{2} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdots \frac{2m \cdot 2m}{(2m-1) \cdot (2m+1)} \cdots.$$

*[Hint: Use the product formula for $\sin z$ at $z = \pi/2$.]*

**6.** We know (from p. 142) the product formula for the sine function is

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Let $z = 1/2$. Then,

$$\frac{\sin(\pi/2)}{\pi} = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 - \frac{(1/2)^2}{n^2}\right).$$

Using $\sin(\pi/2) = 1$, we simplify this equation:

$$\begin{aligned} \frac{1}{\pi} &= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{(2n)^2}\right) \\ \frac{2}{\pi} &= \prod_{n=1}^{\infty} \left(\frac{(2n)^2 - 1}{(2n)^2}\right) = \prod_{n=1}^{\infty} \left(\frac{(2n+1)(2n-1)}{(2n)^2}\right). \end{aligned} \tag{6.1}$$

But this implies that

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{(2n)^2}{(2n+1)(2n-1)}\right),$$

proving the identity.

**7.** *Establish the following properties of infinite products.*

(a) *Show that if $\Sigma |a_n|^2$ converges, and $a_n \neq -1$, then the product $\prod(1 + a_n)$ converges to a non-zero limit if and only if $\Sigma a_n$ converges.*

(b) *Find an example of a sequence of complex numbers $\{a_n\}$ such that $\Sigma a_n$ converges but $\prod(1 + a_n)$ diverges.*

(c) *Also find an example such that $\prod(1 + a_n)$ converges and $\Sigma a_n$ diverges.*

7. a) Let $\sum |a_n|^2$ converge with $a_1 \neq -1$.

($\Leftarrow$) First assume $\sum a_n$ converges to a nonzero limit. Without loss of generality we may assume that each $a_n$ satisfies $|a_n| \leq 1/2$; this is clearly true in the limit (as the sum converges, the summands must tend to zero). We assume this to facilitate expanding with logarithms. Consider the product $\prod(1+a_n)$. Taking logs, we see $\log\left(\prod(1+a_n)\right) = \sum \log(1+a_n)$. Setting $x = -a_n$ and using the Taylor expansion

$$\log(1+x) \;=\; -x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots\,,$$

we see that

$$\log\left(\prod(1+a_n)\right) = \sum\left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right).$$

In general, notice that

$$\sum_{k=2}^{\infty} -|x|^k \leq \left|-\frac{x^2}{2} + \frac{x^3}{3} - \cdots\right| \leq \sum_{k=2}^{\infty} |x|^k,$$

or

$$-|x|^2(1+|x|+|x|^2+\ldots) \leq \left|-\frac{x^2}{2} + \frac{x^3}{3} - \cdots\right| \leq |x|^2(1+|x|+|x|^2+\ldots).$$

If a sum $\sum x$ converges to a nonzero limit, we know that $|x|$ converges to zero; thus we may assume (without changing convergence) that $|x| \leq \frac{1}{2}$. Thus using the geometric expansion, we see that $1+|x|+|x|^2+\cdots = \frac{1}{1-|x|}$. Because $|x| \leq \frac{1}{2}$, we have that $\frac{1}{1-|x|} \leq 2$. Hence we have that

$$-2|x|^2 \leq \left|-\frac{x^2}{2} + \frac{x^3}{3} - \cdots\right| \leq 2|x|^2.$$

Recall that we were looking at $\log\left(\prod(1+a_n)\right) = \sum\left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right)$. Since $\sum a_n$ converges, we know eventually we must have $|a_n| < 1/2$, so we can assume $|a_n| < 1/2$ without changing convergence, and thus use the simplification involving the geometric series expansion developed in the previous paragraph. Thus we write

$$\begin{aligned} \log\left(\prod(1+a_n)\right) &= \sum\left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right) \\ &\leq \sum\left(a_n + 2|a_n|^2\right) \;=\; \sum a_n + 2\sum|a_n|^2. \end{aligned} \tag{6.2}$$

**A QUICK WORD OF WARNING. THE ABOVE EQUATION, AND THE ONES BELOW, ARE A LITTLE ODD. REMEMBER THAT OUR SEQUENCE NEED NOT BE JUST REAL NUMBERS. AS SUCH, WE MUST BE CAREFUL WITH THE DEFINITION OF ABSOLUTE VALUE. WE ABUSE NOTATION A BIT – WHEN WE WRITE $a \leq b + c$, THIS MEANS THE DESIRED RELATION IS TRUE UP TO A LINEAR RESCALING. REALLY WHAT WE MEAN IS $a = b$ UP TO AN ERROR AT MOST $|c|$. WE REALLY SHOULD WRITE THINGS LIKE $|a - b| \leq c$, BUT IN A HOPEFULLY OBVIOUS ABUSE OF NOTATION....**

Since by assumption both $\sum a_n$ and $\sum |a_n|^2$ converge, we must have that $\sum a_n + 2\sum|a_n|^2$ is finite, call it $L$. Thus $\log\left(\prod(1+a_n)\right) \leq L$, so $\prod(1+a_n) \leq e^L$, which is again finite. Thus the product converges.

($\Rightarrow$) Next assume $\prod(1+a_n)$ converges to a nonzero limit. Since $\prod(1+a_n)$ is converging to a nonzero limit, the terms in the product must be converging to 1, so we must have $|a_n|$ approaching zero and we can assume $|a_n| < 1/2$ without affecting convergence. We now write:

$$\begin{aligned}\log\left(\prod(1+a_n)\right) &= \sum\left(a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} - \cdots\right) \\ &\geq \sum\left(a_n - \frac{|a_n|^2}{2} - \frac{|a_n|^3}{3} - \cdots\right) \geq \sum\left(a_n - |a_n|^2 - |a_n|^3 - \cdots\right).\end{aligned} \tag{6.3}$$

As before, we substitute in using the geometric series expansion:

$$\begin{aligned}\log\left(\prod(1+a_n)\right) &\geq \sum\left(a_n - |a_n|^2 - |a_n|^3 - \cdots\right) \\ &= \sum\left(a_n - |a_n|^2(1+|a_n|+|a_n|^2+\cdots)\right) \\ &\geq \sum\left(a_n - 2|a_n|^2\right) = \sum a_n - 2\sum|a_n|^2.\end{aligned} \tag{6.4}$$

Thus we see that $\log\left(\prod(1+a_n)\right) + 2\sum|a_n|^2 \geq \sum a_n$. Since $\prod(1+a_n)$ and $\sum|a_n|^2$ converge, we must have that $\log\left(\prod(1+a_n)\right) + 2\sum|a_n|^2$ are both finite. Thus our sum $\sum a_n$ is bounded by finite terms, and so the sum must also be finite itself. Hence the sum $\sum a_n$ must converge to a finite limit.

b) Let $\{a_n\} = \{\frac{i}{\sqrt{1}}, \frac{-1}{\sqrt{1}}, \frac{i}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, \ldots\}$. The sum $\sum a_n$ converges by the alternating series test, since the absolute value of the terms approaches zero (one can show this by showing that first the odd terms tend to zero in absolute value and then that the even terms do as well).

Consider now the product $\prod\left(1+a_n\right)$. For an arbitrary integer $N$, look at the $2N$-th partial product:

$$\begin{aligned}\prod_{n=1}^{2N}\left(1+a_n\right) &= \left(1+\frac{i}{\sqrt{1}}\right)\left(1-\frac{i}{\sqrt{1}}\right)\cdots\left(1+\frac{i}{\sqrt{2N}}\right)\left(1-\frac{i}{\sqrt{2N}}\right) \\ &= \left(1-\frac{i^2}{\sqrt{1^2}}\right)\cdots\left(1-\frac{i^2}{\sqrt{(2N)^2}}\right) \\ &= \left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{2N}\right) = \left(\frac{2}{1}\right)\cdots\left(\frac{2N+1}{2N}\right) = 2N+1.\end{aligned} \tag{6.5}$$

Thus when we evaluate at an even term $2N$, we see that

$$\lim_{2N\to\infty}\prod_{n=1}^{2N}(1+a_n) = \lim_{2N\to\infty}(2N+1) = \infty,$$

so the product diverges. Hence the product diverges at even terms and thus cannot converge in general.

c) For a trivial example, let $\{a_n\} = \{1, -1, 1, -1, \dots\}$. The sum $\sum a_n$ does not converge because the limit of the $N$th partial sum as $N$ tends to infinity does not converge; it alternates between $0$ and $1$. However, the product will clearly converge:

$$\prod a_n = (1+1)(1-1)(1+1)(1-1)\cdots = (1)(0)(1)(0)\cdots = 0.$$

For an example in which the sum diverges but the product converges to a nonzero limit, consider the sequence $\{a_n | a_{2n-1} = 1/\sqrt{n}, a_{2n} = -1/(1+\sqrt{n})\}_{n=1}^{\infty}$. Grouping the pairs $2n$ and $2n-1$ together, we see that

$$\sum_{m=1}^{\infty} a_m \;=\; \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{1+\sqrt{n}}\right) \;=\; \sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}}.$$

We'll show that this series diverges. Notice that for every n,

$$\sum_{n=1}^{\infty}\frac{1}{n+\sqrt{n}} \geq \sum_{n=1}^{\infty}\frac{1}{2n}$$

and since the series on the RHS diverges, by comparison test, so does the series on the LHS. So $\sum a_n$ diverges. However, grouping again the even and odd pair terms, for even $N$, we have

$$\begin{aligned}
\prod_{m=1}^{N}(1+a_m) &= \prod_{n=1}^{N/2}(1+\frac{1}{\sqrt{n}})(1-\frac{1}{\sqrt{n}+1}) \\
&= \prod_{n=1}^{N/2}(1+\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n}+1} - \frac{1}{\sqrt{n}+n}) \\
&= \prod_{n=1}^{N/2}(1-\frac{-\sqrt{n}+\sqrt{n}+1-1}{\sqrt{n}+n}) \;=\; \prod_{n=1}^{N/2} 1 = 1
\end{aligned}$$

and for odd N,

$$\prod_{m=1}^{N}(1+a_m) = (1+\frac{1}{\sqrt{N}})$$

which converges to 1 as $N \to \infty$. Thus,

$$\prod_{n=1}^{\infty}(1+a_n) = 1.$$

Hence $\{a_n\}$ is the desired sequence. □

**9.** *Prove that if* $|z| < 1$, *then*

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \prod_{k=0}^{\infty}\left(1+z^{2^k}\right) = \frac{1}{1-z}.$$

9. Consider the product $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots$. Suppose we tried to multiply this product out: to get one term, we would need to choose either the $1$ or the power of $z$ in each term to multiply by. For example,

one term we could get out is simply $z$, where we would choose the $z$ in the first term and the $1$ in every succeeding term; another way to say this is to write $z = z \times 1 \times 1 \times \cdots$. To write out the entire product, we would have to make sure we evaluated every possible choice of ones and powers of $z$.

But this isn't so bad if we think of choosing terms as counting in binary. In binary counting, a number is written entirely in terms of 0s and 1s. For any given number, each digit represented a choice between the digit 0 and the digit 1. If we think of selecting the power of $z$ in a term as picking $1$ for a given digit in binary counting, and selecting the $1$ in a term as picking $0$ for a given digit in binary, we can identify a bijective correspondence between integers written in binary and products from our term (with the exception that $000000000\cdots = 1$ in our product). For example, the binary number $101 = \cdots 000101 = 2^2 \times 1 + 2^1 \times 0 + 2^0 \times 1 = 5$, and if choose the terms $(z)(1)(z^4)(1)(1)\cdots$, we see that we get the product $z^5$.

To evaluate our product we must sum over all such possible choices. Since all possible binary numbers together yield precisely the nonnegative integers, this bijective correspondence importantly tells us that the sum over all such products will be the sum over all nonnegative powers of z, or $1 + z + z^2 + z^3 + \ldots$. Thus we have $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = 1 + z + z^2 + z^3 + \ldots$. Since $|z| < 1$, we can use the geometric expansion of $z$ to write $(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \frac{1}{1-z}$, as desired.

Significance for combinatorics: notice the way in which our solution invokes combinatorics (such as seeing how many ways we can choose our terms to make a product).

Alternatively, we can truncate the product and multiply by $1 - z$. Note that $(1-z)(1+z) = (1-z^2)$, then $(1-z^2)(1+z^2) = (1-z^4)$, and so

$$(1-z)(1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^k}) \;=\; 1 - z^{2^{k+1}};$$

as $|z| < 1$ the latter tends to 1, and thus

$$(1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^k}) \;=\; \frac{1}{1-z} - \frac{z^{2^{k+1}}}{z-1} \;\to\; \frac{1}{1-z}.$$

**Chapter 3**

**10.** *Show that if $a > 0$, then*

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

*[Hint: Use the contour in Figure 10.]*

10. We will first find the residue at $ia$ and then integrate over the given contour. Let $f(z) = \frac{\log z}{z^2+a^2}$, where we take the branch cut of the logarithm along $-ib$ for all $b \in [0, \infty)$. Furthermore, $ia$ is a zero of order $1$. Finding the residue at $ia$, we have

$$\mathrm{res}_{ia} f = \lim_{z \to ia} (z - ia) \frac{\log z}{z^2 + a^2} \;=\; \lim_{z \to ia} (\frac{\log z}{z + ia}) \;=\; \frac{\log ia}{2ia} \;=\; \frac{\log a}{2ia} + \frac{\pi}{2a}$$

Label the contours from the portion on the positive real axis $\gamma_1$, the larger arc $\gamma_2$, the portion on the negative real axis $\gamma_3$, and the smaller arc $\gamma_4$. Choose $\epsilon < \min\{a, 1\}a$, $R > \max\{a, 1\}$. Parametrize $\gamma_1$ with $z(t) = t$ from $\epsilon$

to $R$, $\gamma_2$ with $z(t) = Re^{it}$ from $0$ to $\pi$, $\gamma_3$ with $z(t) = t$ from $-R$ to $-\epsilon$, and $\gamma_4$ with $z(t) = \epsilon e^{it}$ from $\pi$ to $0$. Integrating over the $\gamma_2$ and taking absolute values, we have

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{\log z}{z^2+a^2} dz\right| &= \left|\int_0^\pi \frac{\log Re^{it}}{(Re^{it})^2+a^2} Rie^{it} dt\right| \\
&\leq \int_0^\pi \left|\frac{\log Re^{it}}{R^2e^{2it}+a^2} Rie^{it}\right| dt \\
&= \int_0^\pi \left|\frac{\log Re^{it}}{R^2e^{2it}+a^2}\right| Rdt \\
&= \int_0^\pi \left|\frac{\log R+it}{R^2e^{2it}+a^2}\right| Rdt \\
&\leq \int_0^\pi \frac{\log R+|it|}{|R^2e^{2it}|+|a^2|} Rdt \\
&= \int_0^\pi \frac{\log R+t}{|Re^{2it}|+\frac{|a^2|}{R}} dt \;\leq\; \int_0^\pi \frac{\log R+t}{R+\frac{|a^2|}{R}} dt \;\leq\; \pi\frac{\log R+\pi}{R+\frac{|a^2|}{R}}
\end{aligned}$$

since $t, \log R > 0$. Since $R \to \infty$, $\log R + \pi, R + \frac{|a^2|}{R} \to \infty$, by L'Hopital,

$$\lim_{R\to\infty} \frac{\log R+\pi}{R+\frac{|a^2|}{R}} = \lim_{R\to\infty} \frac{1/R}{1-\frac{|a^2|}{R^2}} = \lim_{R\to\infty} \frac{1}{R-\frac{|a^2|}{R}} = 0.$$

Thus, as $R \to \infty$, the contribution along $\gamma_2$ vanishes to $0$. Similarly, for $\gamma_4$, we have

$$\begin{aligned}
\left|\int_{\gamma_4} \frac{\log z}{z^2+a^2} dz\right| &= \left|\int_\pi^0 \frac{\log \epsilon e^{it}}{(\epsilon e^{it})^2+a^2} \epsilon e^{it} dt\right| \\
&\leq \int_\pi^0 \left|\frac{\log \epsilon e^{it}}{\epsilon^2e^{2it}+a^2} \epsilon e^{it}\right| dt \\
&= \int_\pi^0 \left|\frac{\log \epsilon e^{it}}{\epsilon^2e^{2it}+a^2}\right| \epsilon dt \\
&= \int_\pi^0 \left|\frac{-\log \epsilon+it}{\epsilon^2e^{2it}+a^2}\right| \epsilon dt \\
&\leq \int_\pi^0 \frac{-\log \epsilon+|it|}{|\epsilon^2e^{2it}|+|a^2|} \epsilon dt \\
&\leq \int_\pi^0 \frac{-\log \epsilon+t}{|\epsilon e^{2it}|+\frac{|a^2|}{\epsilon}} dt \;\leq\; \int_\pi^0 \frac{-\log \epsilon+t}{\epsilon+\frac{|a^2|}{\epsilon}} dt \;\leq\; \pi\frac{-\log \epsilon+\pi}{\epsilon+\frac{|a^2|}{\epsilon}} dt
\end{aligned}$$

since $t, -\log \epsilon > 0$. Since $\epsilon \to 0$, $-\log \epsilon + \pi, \epsilon + \frac{|a^2|}{\epsilon} \to \infty$, by L'Hopital,

$$\lim_{\epsilon\to 0} \frac{-\log \epsilon+\pi}{\epsilon+\frac{|a^2|}{\epsilon}} = \lim_{\epsilon\to 0} \frac{-1/\epsilon}{1-\frac{|a^2|}{\epsilon^2}} = \lim_{\epsilon\to 0} \frac{-1}{\epsilon-\frac{|a^2|}{\epsilon}} = 0.$$

Thus, as $\epsilon \to 0$, the contribution along $\gamma_4$ also vanishes to $0$. For the integral over $\gamma_1$, $\gamma_3$, we have

$$\int_{\gamma_1+\gamma_3} \frac{\log z}{z^2+a^2}dz = \int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \int_{-R}^{-\epsilon} \frac{\log s}{s^2+a^2}ds.$$

Letting $s = -t$, we have

$$\begin{aligned}\int_{\gamma_1+\gamma_3} \frac{\log z}{z^2+a^2}dz &= \int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \int_R^\epsilon \frac{\log -t}{(-t)^2+a^2}(-1)dt \\ &= \int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \int_\epsilon^R \frac{\log -t}{t^2+a^2}dt \\ &= \int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \int_\epsilon^R \frac{\log t + i\pi}{t^2+a^2}dt \\ &= 2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt + i\pi \int_\epsilon^R \frac{1}{t^2+a^2}dt \ = \ 2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \frac{i\pi}{a}\arctan\frac{t}{a}\bigg|_\epsilon^R\end{aligned}$$

Thus we have, as $R \to \infty, \epsilon \to 0$ and as $\text{res}_{ia} f = \frac{\log a}{2ia} + \frac{\pi}{2a}$, we have

$$\begin{aligned}\lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt + \frac{i\pi}{a}\arctan\frac{t}{a}\bigg|_\epsilon^R\right) &= 2\pi i\left(\frac{\log a}{2ia} + \frac{\pi}{2a}\right) \\ \lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt\right) + \lim_{R\to\infty,\epsilon\to 0}\left(\frac{i\pi}{a}\arctan\frac{t}{a}\bigg|_\epsilon^R\right) &= \frac{\pi\log a}{a} + \frac{i\pi^2}{a} \\ \lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt\right) + \frac{i\pi^2}{a} &= \frac{\pi\log a}{a} + \frac{i\pi^2}{a} \\ \lim_{R\to\infty,\epsilon\to 0}\left(2\int_\epsilon^R \frac{\log t}{t^2+a^2}dt\right) &= \frac{\pi\log a}{a} \\ \int_0^\infty \frac{\log t}{t^2+a^2}dt &= \frac{\pi\log a}{2a}\end{aligned}$$

as desired. $\square$

**Additional Problems**

**1.** *Find all poles of the function $f(z) = 1/(1-z^2)^4$ and find the residues at the poles.*

Let $g(x) = 1/f(z) = (1-z^2)^4 = ((1+z)(1-z))^4$. We see that the zeros of $g$ are $\pm 1$, each with order $4$.

Hence, the residues are

$$\begin{aligned}\mathrm{res}_1(f) &= \lim_{z\to 1}\frac{1}{(4-1)!}\left(\frac{d}{dz}\right)^{4-1}(z-1)^4\frac{1}{(1-z^2)^4}\\ &= \lim_{z\to 1}\frac{1}{6}\left(\frac{d}{dz}\right)^{3}\frac{1}{(1+z)^4}\\ &= \lim_{z\to 1}\frac{1}{6}(-4)(-5)(-6)\frac{1}{(1+z)^7}\\ &= \lim_{z\to 1}\frac{-20}{(1+z)^7} \;=\; \frac{-20}{2^7} \;=\; \frac{-5}{32}\end{aligned}$$

and

$$\begin{aligned}\mathrm{res}_{-1}(f) &= \lim_{z\to -1}\frac{1}{(4-1)!}\left(\frac{d}{dz}\right)^{4-1}(z+1)^4\frac{1}{(1-z^2)^4}\\ &= \lim_{z\to -1}\frac{1}{6}\left(\frac{d}{dz}\right)^{3}\frac{1}{(z-1)^4}\\ &= \lim_{z\to -1}\frac{1}{6}(4)(5)(6)\frac{-1}{(z-1)^7}\\ &= \lim_{z\to -1}\frac{-20}{(z-1)^7} \;=\; \frac{-20}{-2^7} \;=\; \frac{5}{32}\end{aligned}$$

Thus we have found the desired residues. □

We sketch an alternative proof. We have

$$\begin{aligned}f(z) \quad &= \quad \frac{1}{(z-1)^4}\frac{1}{(z+1)^4}\\ &= \quad \frac{1}{(z-1)^4}\frac{1}{(z-1+2)^4}\\ &= \quad \frac{1}{(z-1)^4}\frac{1}{2^4}\frac{1}{(1+\frac{z-1}{2})^4}\\ &= \quad \frac{1}{(z-1)^4}\frac{1}{16}\left(1-\frac{z-1}{2}+\frac{(z-1)^2}{4}-\frac{(z-1)^3}{8}+\cdots\right)^4. \end{aligned} \tag{6.6}$$

The difficulty is we have to expand the factor to the fourth power well enough to identify the coefficient of $(z-1)^3$. A little algebra shows it is $-\frac{5}{2}(z-1)^3$, and thus (remembering the factor 1/16) the residue is just $-5/32$.

**2.** *Consider the map $f(z) = (z-i)/(z+i)$. Show that this is a one-to-one and onto map from the upper half plane (all $z = x+iy$ with $y > 0$) to the unit disk.*

2. First we'll show that the range of $f$ is the unit disk. Writing $z = x+iy$ where $x, y \in \mathbb{R}$, $y > 0$, then we have

$$|f(x+iy)| = \left|\frac{x+(y-1)i}{x+(y+1)i}\right| \;=\; \frac{\sqrt{x^2+(y-1)^2}}{\sqrt{x^2+(y+1)^2}}$$

and since $y > 0$, $\sqrt{x^2 + (y-1)^2} < \sqrt{x^2 + (y+1)^2}$, $f(x + iy) < 1$, so the range of $f$ is the unit disk.

Now we'll show that f is injective. Suppose for $z_1, z_2$ with imaginary part positive, $f(z_1) = f(z_2)$. Then

$$\begin{aligned} \frac{z_1 - i}{z_1 + i} &= \frac{z_2 - i}{z_2 + i} \\ (z_1 - i)(z_2 + i) &= (z_2 - i)(z_1 + i) \\ z_1 Z_2 + z_1 i - Z_2 i + 1 &= z_1 z_2 - z_1 i + z_2 i + 1 \\ 2i(z_1 - z_2) &= 0 \\ z_1 &= z_2. \end{aligned} \tag{6.7}$$

Here's another, faster way to do the algebra. We add zero:

$$\begin{aligned} \frac{z_1 - i}{z_1 + i} &= \frac{z_2 - i}{z_2 + i} \\ \frac{z_1 + i - 2i}{z_1 + i} &= \frac{z_2 + i - 2i}{z_2 + i} \\ 1 - \frac{2i}{z_1 + i} &= 1 - \frac{2i}{z_2 + 1}; \end{aligned} \tag{6.8}$$

it is clear that the only solution is when $z_1 = z_2$.

Now we'll show that $f$ is surjective. Given any $w \in \mathbb{D}$, setting $z = (w+1)i/(1-w)$, we see that

$$f(z) = \frac{\frac{(w+1)i}{(1-w)} - i}{\frac{(w+1)i}{(1-w)} + i} = \frac{(w+1)i - (1-w)i}{(w+1)i + (1-w)i} = w.$$

Now we'll show that $z$ has positive imaginary part. Writing $w = x + iy$ with $x, y \in \mathbb{R}, x^2 + y^2 < 1$, we have

$$z = i\frac{(x+1) + iy}{(1-x) - iy} = \frac{-2y + i(1 - y^2 - x^2)}{(1-x)^2 + y^2}.$$

So the imaginary part is $1 - (x^2 + y^2) > 0$, so $z$ has positive imaginary part. $\square$

**3.** *Calculate the Weierstrass product for* $\cos(\pi z)$ *(this is also problem 10b in Chapter 5, and the answer is listed there) and for* $\tan(\pi z)$.

3. By the Euler formulas for sine and cosine, we see that

$$\begin{aligned} \cos(\pi z) &= \frac{e^{i\pi z} + e^{-i\pi z}}{2} \\ &= \frac{e^{i\frac{\pi}{2}}(e^{i\pi z} + e^{-i\pi z})}{2i} \\ &= \frac{(e^{i\pi(z+\frac{1}{2})} + e^{-i\pi(z-\frac{1}{2})})}{2i} \\ &= \frac{e^{i\pi(\frac{1}{2}-z)} - e^{-i\pi(\frac{1}{2}-z)}}{2i} = \sin(\pi(\frac{1}{2} - z)) \end{aligned}$$

and since the zeros of $\sin \pi z$ occur only at the integers, the zeros of $\cos \pi z$ occur at $m + \frac{1}{2}$ for all $m \in \mathbb{Z}$. Thus, define the sequence $\{a_{2n-1} = n + \frac{1}{2}, a_{2n} = -(n + \frac{1}{2})\}_{n=1}^{\infty}$, which are precisely the zeros of $\cos \pi z$. Furthermore, since the zeros of sine are of order 1, the zeros of cosine are also of order one. Thus we have, for $h_k(z) = \sum_{j=1}^{k} \frac{z^j}{j}$, grouping together the pairs $2n$ and $2n-1$, the Weierstrauss product of $\cos \pi z$ is, up to a factor of $e^{h(z)}$ for some entire function $h$,

$$\begin{aligned}
\prod_{m=0}^{\infty}(1 - \frac{z}{a_m})e^{h_m(z)} &= \prod_{n=0}^{\infty}(1 - \frac{z}{n+\frac{1}{2}})(1 - \frac{z}{-(n+\frac{1}{2})}) \prod_{m=1}^{\infty} e^{h_m(z)} \\
&= \prod_{n=0}^{\infty}(1 - \frac{z^2}{(n+\frac{1}{2})^2})e^{\sum_{m=1}^{\infty} h_m(z)} \\
&= \prod_{n=0}^{\infty}(1 - \frac{4z^2}{(2n+1)^2})e^{\sum_{m=1}^{\infty} h_m(z)}.
\end{aligned}$$

Considering $\prod_{n=0}^{\infty}(1 - \frac{4z^2}{(2n+1)^2})$, we'll show this product converges. Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
\frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}
\end{aligned}$$

so since the sum on the RHS is bounded $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a convergent series, the series on the RHS converges as well, and as the convergence is absolute, the product converges. Thus (up to the exponential of an entire function) the Weierstrauss product of $\cos \pi z$ is $\prod_{n=0}^{\infty}(1 - \frac{4z^2}{(2n+1)^2})$.

Next, notice that $\tan(\pi z)$, has poles at odd integer multiples of $\frac{\pi}{2}$, and so by definition does not have a Weierstrass product.

**HW: Due Friday, November 1: (1) Evaluate $\int_{-\infty}^{\infty} \cos(4x)dx/(x^4+1)$. (2) Let $U$ be conformally equivalent to $V$ and $V$ conformally equivalent to $W$ with functions $f : U \to V$ and $g : V \to U$. Prove $g \circ f$ ($g$ composed with $f$) is a bijection. (3) The Riemann mapping theorem asserts that if $U$ and $V$ are simply connected proper open subsets of the complex plane then they are conformally equivalent. Show that simply connected is essential: find a bounded open set $U$ that is not simply connected and prove that it cannot be conformally equivalent to the unit disk. (4) Chapter 8, Page 248: #4. (5) Chapter 8: Page 248: #5. (6) Chapter 8: Page 251: #14.**