Math 3A - Calculus with Applications 1

Brent Albrecht

University of California, Santa Barbara

Spring 2012

Course Instructor

Brent Albrecht
Office: SH 6431M
Office Hours: MWH 10:00 - 11:00 AM or By Appointment
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Course Information

Course Description: Differential calculus including analytic geometry, functions and limits, derivatives, techniques and applications of differentiation, logarithmic and trigonometric functions.

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Text: Single Variable Calculus - Early Transcendentals, 7th Ed. by James Stewart or a Similar Text.

Course Information

Discussion Sections: Each student must enroll in one of the associated discussion sections. Attendance at discussion sections is mandatory.

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Homework: Homework assignments are to be completed online. Homework assignments are due at 11:50 AM on Fridays (when assigned on Mondays or on Fridays) and at 11:50 AM on Wednesdays (when assigned on Wednesdays).

Course Information

Calculators: Students may use calculators when completing homework assignments. Graphing calculators are prohibited during quizzes and exams.

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- Midterm I: 12:00 - 12:50 PM on Wednesday, 25 April 2012

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- Midterm I: 12:00 - 12:50 PM on Wednesday, 25 April 2012
- Midterm II: 12:00 - 12:50 PM on Monday, 14 May 2012

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- Midterm I: 12:00 - 12:50 PM on Wednesday, 25 April 2012
- Midterm II: 12:00 - 12:50 PM on Monday, 14 May 2012
- Final: 12:00 - 3:00 PM on Tuesday, 12 June 2012

Course Information

Grading:

Course Information

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- Quizzes 10%

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- Homework 10%

Course Information

Grading:

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- Homework 10%
- Midterm I 20%

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- Midterm II 20%
- Final 40%

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- Homework 10%
- Midterm I 20%
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Academic Integrity: Students are expected to uphold the highest standards of academic integrity.

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Additional Resources:

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- The Math Lab in South Hall 1607

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Accommodations: Any student requiring accommodations or services due to a disability must contact the Disabled Students Program. I encourage any such students to contact me as well.

Mathematics and Philosophy

Galileo Galilei
1564-1642

Functions

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A *function* f is a rule that assigns to each element x in a set D exactly one corresponding element, called $f(x)$, in a set E. The set D is called the *domain* of the function f. The *range* of the function f is the set of all possible values of $f(x)$ as x varies throughout the domain D.

Four Ways to Represent Functions

From now on, we will assume that D and E are sets of real numbers.

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Proposition (The Vertical Line Test)

A curve in the xy-plane is the graph of a function f of x if and only if no vertical line intersects the curve more than once.

Properties of Functions

A function $f : D \to E$ is said to be *injective* if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

for all $x_1, x_2 \in D$. (This means that no two distinct elements of the set D are sent by f to the same element of the set E, that is, the function f never takes on the same value twice.)

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Proposition (The Horizontal Line Test)

A function is injective if and only if no horizontal line intersects its graph more than once.

Properties of Functions

A function $f : D \to E$ is said to be *surjective* if for every element y of E there exists an element x of D such that $f(x) = y$. (This means that the set E is equal to the range of the function f, that is, the function f succeeds [at some point or another of the set D] in taking on the value of each and every element of the set E.)

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A function $f : D \to E$ that is both injective and surjective is said to be *bijective*.

Properties of Functions

A function $f : D \to E$ is said to be *increasing (nondecreasing)* on an interval I belonging to its domain D if

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad (f(x_1) \leq f(x_2))$$

for all $x_1, x_2 \in I$.

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A function $f : D \to E$ is said to be *decreasing (nonincreasing)* on an interval I belonging to its domain D if

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \;\; (f(x_1) \geq f(x_2))$$

for all $x_1, x_2 \in I$.

New Functions from Old Functions

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  - $y = f(x - c)$, shift the graph of $y = f(x)$ a distance of c units to the right.

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  - $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.

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- (Compositions) Given two functions f and g, we define the *composite function* $(f \circ g)$ by $(f \circ g)(x) = f(g(x))$ with the appropriate domain restrictions.

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- $f\left(\frac{3}{2}\right) = 2^{\frac{3}{2}}$ is one of the solutions to $x^2 = 2^3$ or $x^2 = 8$.

But what is the value of f when it is applied to an irrational number like π?

Exponential Functions

We want to define such values of the function f in such a way so as to keep f an increasing function on all of $\mathbb{R}$. In particular, given values x and y such that $x < \pi < y$, we must require that $2^x < 2^\pi < 2^y$.

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It turns out that there is exactly one number that is greater than all of the numbers $2^3, 2^{3.1}, 2^{3.14}, \ldots$ and less than all of the numbers $2^4, 2^{3.2}, 2^{3.15}, \ldots$. We define 2^π to be this number. It is equal to $8.8249778\ldots$.

Laws of Exponents

Proposition (Laws of Exponents)

If a and b are positive numbers and x and y are any real numbers, then

$$a^{x+y} = a^x a^y, \quad a^{x-y} = \frac{a^x}{a^y}, \quad (a^x)^y = a^{xy},$$

$$\text{and} \quad (ab)^x = a^x b^x.$$

The Number e

Of all of the possible bases for an exponential function, there is one that turns out to be the most convenient for the purposes of calculus. Indeed, some of the formulas of calculus will be greatly simplified if we can choose the base a in such a way that the slope of the tangent line to the graph of the exponential function $f(x) = a^x$ at the point $(0, 1)$ is exactly 1.

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Of all of the possible bases for an exponential function, there is one that turns out to be the most convenient for the purposes of calculus. Indeed, some of the formulas of calculus will be greatly simplified if we can choose the base a in such a way that the slope of the tangent line to the graph of the exponential function $f(x) = a^x$ at the point $(0, 1)$ is exactly 1.

There is a base value with this property, and we denote it by the letter e. It is equal to $2.7182818284\ldots$, and we refer to the exponential function $f(x) = e^x$ as the *natural exponential function*.

Inverse Functions

Suppose that we are interested in studying the growth rate of a certain type of bacteria. Working very carefully, we could introduce a single such bacterium into a closed and nutrient-rich environment. Having done this, we would find that the number of bacteria N in the closed environment is a function of the amount of time t that has elapsed since the beginning, that is, $N = f(t)$.

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We would know how many bacteria there are at any given time t_0. (We could just feed the given time t_0 into the function f to get the answer.) But what if we turned this question around? What if we asked instead how long it would take for the population level to rise to some prescribed value N_0 of N?

Inverse Functions

We would then need to understand the time t that has elapsed since the beginning as a function of the number of bacteria N. This function is the one that we would call the *inverse function of f* and denote by f^{-1}. Supposing the existence of this function, we would write that $t = f^{-1}(N)$ and find the value of $f^{-1}(N_0)$ to answer our question.

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Not all functions possess inverse functions. Injective functions are important mainly because they are precisely the ones that do possess inverse functions. Indeed, given an injective function f with domain A and range B, we are enabled to formulate the precise defininition of the *inverse function of* f:

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for all $y \in B$. (Notice that the function f^{-1} so defined has domain B and range A.)

Inverse Functions

The letter x is traditionally used as the independent variable. As a result, when we are more interested in f^{-1} than in f, we will usually reverse the roles of x and y and write the defining statement as

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Proposition (Cancellation Equations)

$$f^{-1}(f(x)) = x \quad \textit{for all} \quad x \in A$$

and

$$f(f^{-1}(x)) = x \quad \textit{for all} \quad x \in B.$$

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- Interchange x and y to obtain the equation $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Logarithmic Functions

If a is a positive value different from 1, then the exponential function $f(x) = a^x$ passes the Horizontal Line Test and is therefore injective. In this situation, it follows that the exponential function $f(x) = a^x$ has an inverse function. We call this inverse function the *logarithmic function with base a* and denote it by $f^{-1}(x) = \log_a(x)$.

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Looking back at the defining statement of the inverse function of f, we quickly obtain that

$$\log_a(x) = y \Leftrightarrow a^y = x$$

and hence that $\log_a(x)$ is the value to which the base a must be raised in order to obtain x.

Logarithmic Functions

The Cancellation Equations, when applied to the functions $f(x) = a^x$ and $f^{-1}(x) = \log_a(x)$, become the following:

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Proposition (Cancellation Equations for Logarithmic Functions)

$$\log_a(a^x) = x \quad \textit{for all} \quad x \in \mathbb{R}$$

and

$$a^{\log_a(x)} = x \quad \textit{for all} \quad x \in (0, \infty).$$

Logarithmic Functions

The following properties of logarithmic functions follow from the corresponding properties of exponential functions:

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Proposition (Laws of Logarithms)

If x and y are positive numbers and r is any real number, then

$$\log_a(xy) = \log_a(x) + \log_a(y), \quad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y),$$

$$\text{and} \quad \log_a(x^r) = r\log_a(x).$$

Natural Logarithms

The logarithmic function with base e is called the *natural logarithmic function* and is denoted by $\log_e(x) = \ln(x)$.

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Proposition (Change of Base Formula)

For any positive number a different from 1, we have that

$$\log_a(x) = \frac{\ln x}{\ln a}.$$

Inverse Trigonometic Functions

The trigonometric functions are not injective. As a result, they do not possess inverse functions. That said, if we restrict the domains of the trigonometric functions so that they become injective, it is possible to define the so-called *inverse trigonometric functions*:

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- $$\tan^{-1} x = y \Leftrightarrow \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Limits

We begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus - the derivative.

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We start with an important conceptual definition:

Suppose that the function $f(x)$ is defined when x is near the number a, i.e., suppose that $f(x)$ is defined on some open interval that contains a, except possibly at a itself. If we can make the values of $f(x)$ arbitrarily close to some number L by taking x to be sufficiently close to a (on either side of a) but not equal to a, then we say that *the limit of $f(x)$, as x approaches a, equals L* and write

$$\lim_{x \to a} f(x) = L.$$

Limits

Notice that this definition contains the phrase "but not equal to a". This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. Indeed, the function $f(x)$ need not even be defined at the point $x = a$ in order for its limit as x approaches a to be discussed. The only thing that matters is how the function $f(x)$ is defined near $x = a$.

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Let's look at an example. Even though the function

$$f(x) = \frac{\sin x}{x}$$

is not defined at the point $x = 0$, we can perform increasingly precise calculations that suggest that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 :$$

Limits

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

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± 1.0	0.84147098
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± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

This is, in fact, the true value of

$$\lim_{x \to 0} \frac{\sin x}{x}$$

as we shall later see.

The Tangent Problem

Now that we have some conceptual idea of the definition of a limit, let's see how limits arise in attempting to find the tangent to a curve or the velocity of an object.

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The word *tangent* is derived from the Latin word *tangens*, which means "touching". Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. But how can we make this idea precise?

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The word *tangent* is derived from the Latin word *tangens*, which means "touching". Thus a tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact. But how can we make this idea precise?

In the case of a circle, we could follow the Greek mathematician Euclid (the Father of Geometry) and say that a tangent line is a line that intersects the circle at precisely one point. But this will not work for more complicated curves. We need some other means of precisely defining our concept of a tangent.

The Tangent Problem

The word *secant* is derived from the Latin word *secans*, which means "cutting". Thus a secant line is a line that cuts or intersects a curve more than once. We can use secant lines to define our notion of a tangent line.

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Suppose that we wish to find the tangent line to the graph of the parabola $f(x) = x^2$ at the point $P = (1, 1)$. All we really need to find is the slope of the tangent line to the graph of the parabola at that point. We could choose a nearby point $Q = (x, x^2)$, $x \neq 1$ belonging to the graph of the parabola and compute the slope of the corresponding secant line:

$$m_{PQ} = \frac{x^2 - 1}{x - 1}.$$

The Tangent Problem

The closer that the point Q is to the point P, the better this computed slope value approximates the slope that we are seeking. Indeed, the slope of the tangent line to the graph of the parabola $f(x) = x^2$ at the point $P = (1, 1)$ is given by

$$\lim_{Q \to P} m_{PQ} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

and increasingly precise calculations suggest that

$$\lim_{Q \to P} m_{PQ} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 :$$

The Tangent Problem

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001
0.999	1.999
0.99	1.99
0.9	1.9
0.5	1.5
0	1

The Tangent Problem

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0.5	1.5
0	1

We will always define the slope of the tangent line in this way, that is, we will always define the slope of the tangent line to be the limit of the slopes of the secant lines.

The Velocity Problem

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We know that the velocity of a car driving in city traffic is not constant. (Just take a look at the speedometer.) We assume from looking at the speedometer that the car has a definite velocity at each moment, but how is this *instantaneous velocity* defined?

The Velocity Problem

Remember our friend Galileo? He discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. If the time t is measured in seconds and the distance s is measured in meters, then Galileo's discovery is expressed by the equation

$$s(t) = 4.9t^2.$$

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Suppose that Galileo drops a cannon ball from the top of the Leaning Tower of Pisa (a height of roughly 56 meters above the ground on the low side). What would be the velocity of the cannon ball after 1 second?

The Velocity Problem

The difficulty in finding the velocity after 1 second is that we are dealing with a single instant of time ($t = 1$), so no time interval is involved. Fortunately for us, we can approximate the desired velocity by computing the average velocity over the brief time interval of a tenth of a second from $t = 1$ to $t = 1.1$:

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$$= 10.29\ m/s.$$

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The following table shows the results of similar calculations of the average velocity v_{avg} over successively smaller time periods:

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Time Interval	v_{avg} (m/s)
$1 \le t \le 2$	14.7
$1 \le t \le 1.1$	10.29
$1 \le t \le 1.01$	9.849
$1 \le t \le 1.001$	9.8049
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It appears that as we shorten the time period, the average velocity is becoming closer to 9.8 m/s. The instantaneous velocity at $t = 1$ second is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = 1$ second. Thus the instantaneous velocity of the cannon ball after 1 second is 9.8 m/s.

The Velocity Problem

We will always define the instantaneous velocity of an object at a time moment in this way, that is, we will always define the instantaneous velocity of an object at a time moment to be the limiting value of the average velocities over shorter and shorter time periods that start at that time moment.

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There is a close connection between the tangent problem and the velocity problem:

- The slope of the secant line to the graph of the distance function of a moving object over a time interval is equal to the average velocity of that object over that time interval.
- The slope of the tangent line to the graph of the distance function of a moving object at a time moment is equal to the instantaneous velocity of that object at that time moment.

Calculating Limits

Unfortunately, our method of performing increasingly precise calculations isn't always going to produce correct limiting values. To see this, let's try to use our method to find the limit of the function

$$f(x) = \sin \frac{\pi}{x}$$

as x approaches 0:

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as x approaches 0:

x	$\sin\frac{\pi}{x}$
± 1	0
$\pm\frac{1}{2}$	0
$\pm\frac{1}{10}$	0
$\pm\frac{1}{100}$	0
$\pm\frac{1}{1000}$	0

Calculating Limits

Based upon our computations, we are tempted to say that

$$\lim_{x \to 0} \sin \frac{\pi}{x} = 0.$$

But we know that $f(x) = 1$ for infinitely many values of x that approach 0! (Just take as x values the numbers of the form $\frac{2}{4n+1}$ where n is an integer.)

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But we know that $f(x) = 1$ for infinitely many values of x that approach 0! (Just take as x values the numbers of the form $\frac{2}{4n+1}$ where n is an integer.)

Since the values of the function

$$f(x) = \sin \frac{\pi}{x}$$

do not approach a fixed number as x approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x}$$

does not exist.

Calculating Limits

Proposition (Limit Laws)

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \textit{and} \quad \lim_{x\to a} g(x)$$

exist. Then

- $$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$
- $$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

Calculating Limits

Proposition (Limit Laws (Continued))

- $$\lim_{x \to a}(cf(x)) = c \lim_{x \to a} f(x).$$

- $$\lim_{x \to a}(f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x).$$

- $$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \textit{if} \quad \lim_{x \to a} g(x) \neq 0.$$

Calculating Limits

Proposition (Limit Laws (Continued))

- $$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x) \right)^n \quad \textit{where } n \textit{ is a positive integer.}$$

- $$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad \textit{where } n \textit{ is a positive integer.}$$

Calculating Limits

Proposition (Direct Substitution Property)

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

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Proposition

If $f(x) = g(x)$ when $x \neq a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x),$$

provided the limits exist.

Calculating Limits

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Calculating Limits

Theorem (The Squeeze Theorem)

If $f(x) \leq g(x) \leq h(x)$ *when* x *is near* a *(except possibly at* a*) and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$

Calculating Limits

The Squeeze Theorem is a powerful tool for calculating limits. To see this, let's attempt to find the value of the following limit:

$$\lim_{x \to 0} x^2 \sin \frac{1}{x}.$$

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Notice that we cannot write

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}.$$

(Why?)

Calculating Limits

We know that

$$-1 \leq \sin\frac{1}{x} \leq 1$$

for all $x \in \mathbb{R}$, $x \neq 0$.

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Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, it follows that

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

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Since $x^2 \geq 0$ for all $x \in \mathbb{R}$, it follows that

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

for all $x \in \mathbb{R}$, $x \neq 0$.

Since

$$\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0,$$

this implies by the Squeeze Theorem that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

as well.

One-Sided Limits

We will need to understand one-sided limits in order to fully appreciate the statement of our next theorem.

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Let's consider the so-called Heaviside function H given by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

(This function is named for the English electrical engineer Oliver Heaviside and can be used to model an electric current that is switched on at time $t = 0$.)

One-Sided Limits

As t approaches 0 from the left (through small negative values of t), the Heaviside function approaches 0. As t approaches 0 from the right (through small positive values of t), the Heaviside function approaches 1. There is no single number that $H(t)$ approaches as t approaches 0, so

$$\lim_{t \to 0} H(t)$$

does not exist.

One-Sided Limits

As t approaches 0 from the left (through small negative values of t), the Heaviside function approaches 0. As t approaches 0 from the right (through small positive values of t), the Heaviside function approaches 1. There is no single number that $H(t)$ approaches as t approaches 0, so

$$\lim_{t \to 0} H(t)$$

does not exist.

But this function seems pretty nice. (It doesn't behave as strangely at $t = 0$ as the function

$$\sin \frac{\pi}{x}$$

at $x = 0$.) Isn't there something that we can say about this situation?

One-Sided Limits

Of course there is something that we can say. We can say what we have just said. We write

$$\lim_{t \to 0^-} H(t) = 0 \quad \text{and} \quad \lim_{t \to 0^+} H(t) = 1$$

to indicate that the Heaviside function approaches 0 as t approaches 0 from the left (through small negative values of t) and that the Heaviside function approaches 1 as t approaches 0 from the right (through small positive values of t), respectively.

One-Sided Limits

In general, we write

$$\lim_{x \to a^-} f(x) = L \quad \left(\lim_{x \to a^+} f(x) = L \right)$$

and say that the *left-hand limit of $f(x)$ as x approaches a* (*right-hand limit of $f(x)$ as x approaches a*) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than (greater than) a.

One-Sided Limits

In general, we write

$$\lim_{x \to a^-} f(x) = L \quad \left(\lim_{x \to a^+} f(x) = L \right)$$

and say that the *left-hand limit of* $f(x)$ *as* x *approaches* a (*right-hand limit of* $f(x)$ *as* x *approaches* a) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than (greater than) a.

Theorem

$$\lim_{x \to a} f(x) = L \quad \textit{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Infinite Limits

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Let's consider the function g given by

$$g(x) = \frac{1}{x^2}.$$

As x becomes close to 0, x^2 also becomes close to 0, and

$$g(x) = \frac{1}{x^2}$$

becomes very, very large:

Infinite Limits

x	$\frac{1}{x^2}$
± 1.0	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	$10,000$
± 0.001	$1,000,000$

Infinite Limits

The values of $g(x)$ do not approach a single number, so

$$\lim_{x \to 0} \frac{1}{x^2}$$

does not exist. But isn't there something that we can say about the particular way in which this limit does not exist? More particularly, doesn't it appear that the values of $g(x)$ can be made arbitrarily large by taking x sufficiently close to 0, but not equal to 0?

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It certainly does, and we write

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

to indicate this type of behavior.

Infinite Limits

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty \ \left(\lim_{x \to a} f(x) = -\infty \right)$$

means that the values of $f(x)$ can be made arbitrarily large (negative) by taking x sufficiently close to a, but not equal to a.

Infinite Limits

Let f be a function defined on both sides of a, except possibly at a itself. Then

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means that the values of $f(x)$ can be made arbitrarily large (negative) by taking x sufficiently close to a, but not equal to a.

The vertical line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \quad \lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \quad \lim_{x\to a^-} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty.$$

Limits (Again)

It is now time for us to learn the precise definition of a limit. In order to motivate this definition, let's consider the function f given by

$$f(x) = \begin{cases} 5x - 1 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases},$$

and let's try to answer the following question: How close to 1 does x have to be so that $f(x)$ differs from 4 by less than 0.1?

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$$f(x) = \begin{cases} 5x - 1 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases},$$

and let's try to answer the following question: How close to 1 does x have to be so that $f(x)$ differs from 4 by less than 0.1?

The distance from x to 1 is written $|x - 1|$. Similarly, the distance from $f(x)$ to 4 is written $|f(x) - 4|$. Thus our problem is to find a number δ such that

$$|f(x) - 4| < 0.1 \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

Limits (Again)

Notice that if

$$0 < |x - 1| < \frac{0.1}{5} = 0.02,$$

then

$$|f(x) - 4| = |(5x - 1) - 4| = |5x - 5| = 5|x - 1| < 5(0.02) = 0.1,$$

that is,

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$$|f(x) - 4| < 0.1 \quad \text{whenever} \quad 0 < |x - 1| < 0.02.$$

Thus an answer to our problem is to take $\delta = 0.02$, that is, if x is within a distance of 0.02 from 1, then $f(x)$ will be within a distance of 0.1 from 4. (We could also choose δ to be any positive value smaller than 0.02 as well.)

Limits (Again)

If we replace the *error tolerance* 0.1 in our question with an even smaller positive number, say 0.01 (0.001), then by using the same method as before we obtain that $f(x)$ differs from 4 by less than 0.01 (0.001) provided that x differs from 1 by less than

$$\frac{0.01}{5} = 0.002 \quad \left(\frac{0.001}{5} = 0.0002\right).$$

In general, given a small error tolerance $\varepsilon > 0$, we find that

$$|f(x) - 4| < \varepsilon \text{ whenever } 0 < |x - 1| < \delta = \frac{\varepsilon}{5}.$$

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In general, given a small error tolerance $\varepsilon > 0$, we find that

$$|f(x) - 4| < \varepsilon \text{ whenever } 0 < |x - 1| < \delta = \frac{\varepsilon}{5}.$$

This is a very precise way of saying that $f(x)$ is close to 4 when x is close to 1. It says that we can make the values of $f(x)$ within an arbitrary distance ε from 4 by taking the values of x within a certain related distance $\delta = \delta(\varepsilon)$ from 1 (but $x \neq 1$).

Limits (Again)

Here then is the precise definition of a limit:

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Here then is the precise definition of a limit:

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the *limit of $f(x)$ as x approaches a is L*, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$

Limits (Again)

Notice that the condition $|x - a| < \delta$ is equivalent to

$$-\delta < x - a < \delta,$$

which in turn can be written as

$$a - \delta < x < a + \delta.$$

Similarly, $|f(x) - L| < \varepsilon$ is equlivalent to

$$L - \varepsilon < f(x) < L + \varepsilon.$$

In particular, the precise definition of a limit has a very nice geometric interpretation.

One-Sided Limits (Again)

We need also the precise definitions of the other kinds of limits that we are studying:

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$$\lim_{x \to a^-} f(x) = L \ \left(\lim_{x \to a^+} f(x) = L \right)$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if

$$a - \delta < x < a \ (a < x < a + \delta)$$

then

$$|f(x) - L| < \varepsilon.$$

Infinite Limits (Again)

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty \ \left(\lim_{x \to a} f(x) = -\infty \right)$$

means that for every positive (negative) number M there is a positive number δ such that if

$$0 < |x - a| < \delta$$

then

$$f(x) > M \ (f(x) < M).$$

Continuity

A function f is said to be *continuous at a number a* if

$$\lim_{x \to a} f(x) = f(a).$$

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Classification of Discontinuities

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- A function f is said to have a *removable discontinuity at a number a* provided that we could remove the discontinuity by redefining f at just the single number a.
- A function f is said to have a *jump discontinuity at a number a* provided that the function jumps from one value to another at the number a.
- A function f is said to have an *essential discontinuity at a number a* provided that at least one of the one-sided limits

$$\lim_{x \to a^-} f(x) \quad \text{and} \quad \lim_{x \to a^+} f(x)$$

does not exist or is infinite.

One-Sided Continuity

A function f is said to be *continuous from the left (right) at a number* a if

$$\lim_{x\to a^-} f(x) = f(a) \ \left(\lim_{x\to a^+} f(x) = f(a)\right).$$

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At each integer n, the greatest integer function $f(x) = [[x]]$ is continuous from the right but discontinuous from the left. This is because

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [[x]] = n = f(n)$$

but

$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [[x]] = n - 1 \neq f(n).$$

Continuous Functions

A function f is said to be *continuous on an interval* if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand "continuous" at that endpoint to mean whichever one of "continuous from the left" or "continuous from the right" is appropriate.)

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Theorem

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Theorem

Any rational function is continuous wherever it is defined, that is, any rational function is continuous on its domain.

Continuous Functions

Theorem

The following types of functions are continuous at every number in their domains:

- *Polynomials*
- *Rational Functions*
- *Root Functions*
- *Trigonometric Functions*
- *Inverse Trigonometric Functions*
- *Exponential Functions*
- *Logarithmic Functions*

New Continuous Functions from Old Continuous Functions

Theorem

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

- $f + g$
- $f - g$
- cf
- fg
- $$\frac{f}{g} \quad \text{if} \quad g(a) \neq 0.$$

New Continuous Functions from Old Continuous Functions

Theorem

If f is continuous at b and

$$\lim_{x \to a} g(x) = b,$$

then

$$\lim_{x \to a} f(g(x)) = f(b).$$

In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

New Continuous Functions from Old Continuous Functions

Theorem

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

The Intermediate Value Theorem

Theorem (The Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

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If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms, it says that the graph of f can't jump over any horizontal line $y = N$ lying betweeen $y = f(a)$ and $y = f(b)$.

The Intermediate Value Theorem

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The Intermediate Value Theorem is used to locate the roots of polynomials:

Consider the polynomial $f(x) = 4x^3 - 6x^2 + 3x - 2$. We calculate that

$$f(1) = -1 < 0$$

and that

$$f(2) = 12 > 0.$$

Since the function f is continuous, we have by the Intermediate Value Theorem that the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has a solution between $x = 1$ and $x = 2$.

Limits at Infinity

We will need to understand limits at infinity in order to fully appreciate the definition of a horizontal asymptote. Here is the conceptual definition:

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We will need to understand limits at infinity in order to fully appreciate the definition of a horizontal asymptote. Here is the conceptual definition:

Let f be a function defined on some interval (a, ∞) $((-\infty, a))$. Then

$$\lim_{x \to \infty} f(x) = L \ \left(\lim_{x \to -\infty} f(x) = L \right)$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large (negative).

Limits at Infinity

The horizontal line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

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Consider the function g given by

$$g(x) = \frac{x^2 - 1}{x^2 + 1}.$$

We have that

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = \lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1,$$

so the horizontal line $y = 1$ is the only horizontal asymptote of the curve $y = g(x)$.

Limits at Infinity

The curve $y = \tan^{-1} x$ has two horizontal asymptotes since

$$\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

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Theorem

If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Derivatives

The problem of finding the tangent line to a curve and the problem of finding the instantaneous velocity of an object both involve finding the same type of limit. This special type of limit is known as a *derivative*.

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The *tangent line* to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists.

Derivatives

There is another useful expression for the slope of a tangent line. Indeed, setting $h = x - a$, we have that $x = a + h$ and hence that the limit of interest can be rewritten as follows:

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The *instantaneous velocity* of an object with equation of motion $s = f(t)$ at the time moment $t = a$ is given by

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Derivatives

The *derivative of a function f at a number a*, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided that this limit exists. Equivalently, $f'(a)$ is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists.

Derivatives

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f^{'}(a)$, the derivative of f at a.

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The derivative $f^{'}(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Derivatives as Functions

We have seen that the derivative of a function f at a fixed number a is given by

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Given any number x for which this limit exists, we assign to x the number $f'(x)$. In this manner, we obtain a new function f' called the *derivative of* f.

Derivatives as Functions

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x).$$

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The symbols D and $\frac{d}{dx}$ are called *differentiation operators* because they indicate the operation of *differentiation*, the process of calculating a derivative.

Differentiable Functions

A function f is *differentiable at a* if $f'(a)$ exists. It is *differentiable on an open interval* if it is differentiable at every number in the interval.

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Theorem

If f is differentiable at a, then f is continuous at a.

Differentiation Rules

It would be tedious if we always had to use the limit definition of a derivative in order to calculate the derivatives of functions.

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It would be tedious if we always had to use the limit definition of a derivative in order to calculate the derivatives of functions.

Fortunately, there exist several differentiation rules which enable us to calculate with relative ease the derivatives of polynomials, rational functions, algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions.

Derivatives of Polynomials

The derivative of the constant function f given by $f(x) = c$ is the zero function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

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Let n be a positive integer, and consider the function f given by $f(x) = x^n$. We can compute the derivative of this function as follows:

$$f^{'}(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

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$$= \lim_{t\to x} \frac{t^n - x^n}{t - x}$$

$$= \lim_{t\to x} \frac{(t^{n-1} + t^{n-2}x + \ldots + tx^{n-2} + x^{n-1})(t - x)}{t - x}$$

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This is true for every real number n:

Proposition (The Power Rule)

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

New Derivatives from Old Derivatives

Proposition

Suppose that c is a constant and the functions f and g are differentiable. Then

- $$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x).$$
- $$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$
- $$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x)-\frac{d}{dx}g(x).$$

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$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

Derivatives of Exponential Functions

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$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \lim_{h \to 0} \frac{a^h - 1}{h}. \end{aligned}$$

Derivatives of Exponential Functions

But this is precisely $a^x f'(0)$! This equation says that the rate of change of any exponential function is proportional to the function itself. (The slope is proportional to the height.)

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This motivates the definition of the number e that we gave previously, that is, the statement that the number e is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

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Proposition

$$\frac{d}{dx}(e^x) = e^x.$$

The Product Rule

Although the derivative of a sum (difference) is the sum (difference) of the derivatives, it is not true that the derivative of a product (quotient) is the product (quotient) of the derivatives.

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Proposition (The Product Rule)

If f and g are both differentiable, then

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x)).$$

The Quotient Rule

Proposition (The Quotient Rule)

If f and g are both differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}.$$

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Derivatives of Trigonometric Functions

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We previously performed increasingly precise calculations that suggest that

$$\lim_{h\to 0}\frac{\sin h}{h} = 1.$$

The fact that

$$\lim_{h\to 0}\frac{\sin h}{h} = 1$$

can be seen by using the Squeeze Theorem:

$$\cos h < \frac{\sin h}{h} < 1.$$

Derivatives of Trigonometric Functions

In order to calculate the value of the remaining limit, we write

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Altogether then we have that

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

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Derivatives of Trigonometric Functions

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The derivatives of the remaining trigonometric functions can also be found by applying the Quotient Rule:

$$\frac{d}{dx}(\csc x) = -\csc x \cot x, \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\text{and} \quad \frac{d}{dx}(\cot x) = -\csc^2 x.$$

The Chain Rule

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Proposition (The Chain Rule)

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The Chain Rule

Let's make explicit the special case of the Chain Rule where the outer function f is a power function. If $y = (g(x))^n$, then we can write $y = f(u) = u^n$ where $u = g(x)$. By using the Chain Rule and then the Power Rule, we obtain the following statement:

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Proposition (The Power Rule Combined with the Chain Rule)

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Alternatively,

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x).$$

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Suppose that $y = f(u)$, $u = g(x)$, and $x = h(t)$, where f, g, and h are differentiable functions. Then, to compute the derivative of y with respect to t, we simply use the Chain Rule twice:

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$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}.$$

Implicit Differentiation

The functions that we have encountered so far can be described by expressing one variable explicitly in terms of another variable, that is, the functions that we have encountered so far can be written in the form $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y.

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The functions that we have encountered so far can be described by expressing one variable explicitly in terms of another variable, that is, the functions that we have encountered so far can be written in the form $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y.

Think of the circle $x^2 + y^2 = 36$ of radius 6 centered at the origin. This is a perfectly good function (of the radius r in terms of the angle θ from the positive x-axis), but it cannot be written as a single function of y in terms of x.

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$$g(x) = -\sqrt{36 - x^2}$$

(the lower semicircle of radius 6 centered at the origin).

Implicit Differentiation

Fortunately, we don't have to solve an equation for y in terms of x in order to find the derivative of y with respect to x. We can think of y as a function of x and use the method of implicit differentiation as follows:

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Implicit Differentiation

Let's use the method of implicit differentiation to find the derivatives of the inverse trigonometric functions. Recall that

$$\sin^{-1} x = y \Leftrightarrow \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

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$$\text{and } \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}.$$

Derivatives of Logarithmic Functions

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- Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
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If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined. In such a situation, we can write $|y| = |f(x)|$ and use the fact that

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

together with the Chain Rule.

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$$= n\ln|x|, \quad x \neq 0.$$

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and therefore that

$$\frac{dy}{dx} = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

as desired.

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Related Rates

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In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

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- Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.
- Use the Chain Rule to differentiate both sides of the equation with respect to t.
- Substitute the given information into the resulting equation and solve for the unknown rate.

Exponential Growth and Decay

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This equation is called the *law of natural growth (decay)* if $k > 0$ $(k < 0)$. It is our first example of a *differential equation*, an equation that involves an unknown function and its derivative(s).

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In turns out also that *any* function that satisfies our differential equation must be of the form $y(t) = Ce^{kt}$:

Theorem

The only solutions of the differential equation

$$\frac{dy}{dt} = ky$$

are the exponential functions $y(t) = y(0)e^{kt}$.

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- In finance, the value of a savings account with continuously compounded interest increases at a rate proportional to that value.

Maximum and Minimum Values

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- At what angle should blood vessels branch so as to minimize the energy expended by the heart in pumping blood?

Maximum and Minimum Values

These problems can be reduced to finding the maximum or minimum values of a function.

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Let c be a number in the domain D of a function f. We call $f(c)$ the *absolute maximum (absolute minimum)* value of f on D if $f(c) \geq f(x)$ $(f(c) \leq f(x))$ for all x in D. This value is sometimes called the *global maximum (global minimum)* value.

Maximum and Minimum Values

The number $f(c)$ is called a *local maximum (local minimum)* value of f if $f(c) \geq f(x)$ $(f(c) \leq f(x))$ when x is near c, that is, if $f(c) \geq f(x)$ $(f(c) \leq f(x))$ for all values of x belonging to some open interval containing c.

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Theorem (Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Maximum and Minimum Values

Theorem (Fermat's Theorem)

If f has a local maximum or minimum at c, and if $f^{'}(c)$ exists, then $f^{'}(c) = 0$.

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If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

A *critical number* of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The Closed Interval Method

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- Find the values of f at the critical numbers of f in (a, b).
- Find the values of f at the endpoints of the interval.
- The largest (smallest) of the values found in the previous steps is the absolute maximum (absolute minimum) value.

The Mean Value Theorem

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following three hypotheses:

- *f is continuous on the closed interval $[a, b]$.*
- *f is differentiable on the open interval (a, b).*
- *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f^{'}(c) = 0$.

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- *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

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Corollary

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Derivatives and Graphs

Many of the applications of calculus depend on our ability to deduce facts about a function f from information concerning its derivatives.

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Proposition (Increasing/Decreasing Test)

Let f be a function.

- *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*
- *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Derivatives and Graphs

Proposition (The First Derivative Test)

Suppose that c is a critical number of a continuous function f.

- *If f' changes from positive to negative at c, then f has a local maximum at c.*
- *If f' changes from negative to positive at c, then f has a local minimum at c.*
- *If f' does not change sign at c, then f has no local maximum or minimum at c.*

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If the graph of f lies above all of its tangents on an interval I, then it is called *concave upward* on I. If the graph of f lies below all of its tangents on I, it is called *concave downward* on I.

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Proposition (Concavity Test)

Let f be a function.

- *If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.*
- *If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.*

Derivatives and Graphs

A point P on a curve $y = f(x)$ is called an *inflection point* if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

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Proposition (The Second Derivative Test)

Suppose f'' is continuous near c.

- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Indeterminate Forms and L'Hospital's Rule

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an *indeterminate form of type* $\frac{0}{0}$.

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Similarly, there are *indeterminate forms of type* $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, and 1^∞.

We now introduce a systematic method, known as L'Hospital's Rule, for the evaluation of indeterminate forms:

Indeterminate Forms and L'Hospital's Rule

Proposition (L'Hospital's Rule)

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty.$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).