Math 4C Course Overview; Inequalities (Friday, September 23, 11-1150am, CSB 002)

Math 4C is the preparatory course for Math 20A (Calculus for Physical Sciences & Engineering). Math 20A is more rigorous than Math 10A, covers more material than 10A, and has stronger math students than 10A (after all they are mostly Physical Sciences and/or Engineering majors). Math 4C covers much of precalculus including trigonometry but not all (not enough time) and develops (or strengthens for those who've taken precalculus or calculus in high school) precalculus problem-solving skills.

For good tips about surviving and enjoying college, see Professor David Jordan's webpage at www.anthro.ucsd.edu/~dkjordan/resources/studyclass.html. I have one tip for this class: work as many problems as you can. You learn how to solve math problems by solving math problems! Work all of the assigned homework problems on your own first (with a friend later) and not with the solution manual open. Rework all lecture examples so you really understand what your professor was saying in class. If you can't do a homework problem, ask for help (your TA, the Calculus Lab, OASIS study tables, your study partners, your Professor, a homework forum) to understand your mistake, then rework the problem (understanding your error isn't enough). Rework all errors made on quizzes and midterms. Work all text examples on your own – don't just read the text. To study for midterms and the final exam, work new problems (chapter reviews are good sources).

Inequalities on the Number Line

Definition of inequality. *Let a and b be any two real numbers represented by points on a horizontal number line. Then* $a > b$ *if and only if the point representing a lies to the right of the point representing b.*

Clearly, $3 > 2$. But is $-3 > -2$?

Alternate definition of inequality. *Let a and b be any two real numbers. Then* $a > b$ *if and only if* $a - b > 0$.

Asking if $-3 > -2$ is equivalent to asking if $-3 - (-2) > 0$. But $-3 - (-2) = -3 + 2 = -1$. So the answer is no.

Axiom. *If a and b are real numbers, then one and only one of the following relationship holds:*

$a = b,\qquad a > b,\qquad b > a$

For example, if $b = 0$, then one of the following cases holds:

$a = 0,\qquad a > 0,\qquad 0 > a$

Additional inequality relationships.

1. Instead of $b > a$, we can equally write $a < b$.
2. "$>$" and "$<$" are called strict inequalities.
3. "$\geq$" and "$\leq$" are called mixed inequalities. $a \leq b$ means that either $a < b$ or $a = b$.
4. Transitivity. If $a > b$ and $b > c$, then $a > c$.
5. Addition. If $a > b$ and $c > d$, then $a + c > b + d$.
6. Multiplication. If $a > b$ and $c > 0$, then $ac > bc$. If $a > b$ and $c < 0$, then $ac < bc$. (See first example)
7. Subtraction. If $a > b$ and c is any real number, then $a - c > b - c$.

Which of the following statements are true?

1. $-2 \geq -3$ 2. $0 < -1$ 3. $1/2 < 1/3$ 4. $-1/2 < -1/3$

5. $2 > -1$ 6. $1 - (-4) \geq -(-4)$ 7. $7/3 < 14/5$ 8. $2/9 < 4/15$

9. Since $-3 < -2$ and $-4 < -2$, then $-3 < -4$.

10. If $0 < a < b$, then $a/b < (a+1)/(b+1)$.

Linear Inequalities

Solve $2x + 1 > 3$. Such inequalities are called *linear inequalities*, since the variable is linear. The solution to a linear equality is a statement of the form $x > a$ or $x < a$ for some real number a. Graph both possibilities (separately). To solve a linear inequality like $2x + 1 > 3$ we apply the inequality relationships we saw earlier.

$2x + 1 > 3$ is equivalent to

$2x + 1 - 1 > 3 - 1$ or $2x > 2$

$2x > 2$ is equivalent to $2x/2 > 2/2$ or $x > 1$ [graph solution]

We can also write our solution using interval notation, i.e., $x > 1$ can be written as $(1,\infty)$.

Definition. *An interval is a set of real numbers that contains all numbers between any two numbers in the set.*

Four cases are possible, depending on whether one or both endpoints are included in the set. Assume $a < b$.

1. (a, b) or $\{x: a < x < b\}$
2. $[a, b]$ or $\{x: a \leq x \leq b\}$
3. $[a, b)$ or $\{x: a \leq x < b\}$
4. $(a, b]$ or $\{x: a < x \leq b\}$

Solve $1 - 2x \geq x + 4$. Recall our goal is to use inequality relationships to end up with a statement of the form $x \geq a$ or $x \leq a$. We have two basic options here: move the variables to the left or move them to the right. Let's move them to the left first.

$1 - 2x \geq x + 4$ is equivalent to

$-3x \geq 3$ that in turn is equivalent to

$x \leq -1$ or $(-\infty, -1]$ [graph solution]

Note we switched the direction of the inequality when we divided both sides by -3. We could have avoided this by solving using the second option. Here we move the variable to the right side *so we don't have a negative coefficient for our variable*.

$1 - 2x \geq x + 4$ is equivalent to

$-3 \geq 3x$ that in turn is equivalent to

$-1 \geq x$ or $x \leq -1$

Polynomial Inequalities

Solve $x^2 + 2x \geq 8$. From our axiom, we know that this inequality is either true or false for every real number. Our goal is to find those real numbers (if any) where it is true. Let's begin by trying to solve the *equality part* of the inequality.

$x^2 + 2x = 8$ is equivalent to
$x^2 + 2x - 8 = 0$ which factors to
$(x+4)(x-2) = 0$ and by the Zero Product Property
$x = -4$ and $x = 2$

Now we know where the equality part of the inequality is satisfied (i.e., where the inequality is "true"). For all other real numbers, the inequality is either true or false. We have two ways to proceed. First, we use logic to determine when $x^2 + 2x > 8$.

$x^2 + 2x > 8$ is equivalent to
$x^2 + 2x - 8 > 0$ that in turn is equivalent to
$(x+4)(x-2) > 0$

The product of these two terms, $x + 4$ and $x - 2$, is positive. When is the product of two numbers positive? When both numbers are positive or when both numbers are negative. So we have two cases to consider.

Case 1 (both factors are positive). This means that ***both*** x + 4 > 0 and x – 2 > 0. This occurs when x > 2 or $(2,\infty)$. So the inequality $x^2 + 2x \geq 8$ is now true for x = -4, 2, and $(2,\infty)$ or for x = -4 and $[2,\infty)$.

Case 2 (both factors are negative). This means that ***both*** x + 4 < 0 and x – 2 < 0. This occurs when x < -4 or $(-\infty,-4)$. So the inequality $x^2 + 2x \geq 8$ is now true for x = -4 and $[2,\infty)$ and $(-\infty,-4)$ or for $(-\infty,-4]$ and $[2,\infty)$. We can write this solution as the union of both sets or $(-\infty,-4] \cup [2,\infty)$. [Graph solution]

I mentioned a second solution method. This method is quite useful and uses test values or "points". We know the inequality, $x^2 + 2x \geq 8$, is true for x = -4 and x = 2 and is either true or false for every other real numbers. Graph the numbers -4 and 2 on a number line. We see that the number line is now divided into three parts, x < - 4 and -4 < x < 2 and x > 2. We can evaluate a single test point in each interval to determine whether the inequality is true or false for that number. We will then state that the inequality has the same "truth value" for all other points in that interval. Why is this reasonable?

For x < -4. Let's choose any number less than -4, e.g., -5, and evaluate the inequality for x = -5.

$x^2 + 2x \geq 8$

$(-5)^2 + 2(-5) \geq 8$ is equivalent to

$25 - 10 \geq 8$ or $15 \geq 8$ which is true

So the inequality is true for all x < -4 or $(-\infty,-4)$.

For -4 < x < 2. Let us choose x = 0 and evaluate the inequality.

$(0)^2 + 2(0) \geq 8$ is equivalent to

$0 + 0 \geq 8$ or $0 \geq 8$ which is false

So the inequality is false for all x between -4 and 2.

For x > 2. Let us choose x = 3 and evaluate the inequality.

$(3)^2 + 2(3) \geq 8$ is equivalent to

$9 + 6 \geq 8$ or $15 \geq 8$ which is true

So the inequality is true for all x > 2 or $(2,\infty)$.

Finally, the inequality is true when x = -4, x = 2, x < -4, and x > 2 or for or $(-\infty,-4] \cup [2,\infty)$.

Absolute Value Inequalities

Definition of opposite. *The opposite of real number* a *is the number* –a *such that* (a) + (-a) = 0.

Absolute value of a number. *The absolute value* |a| *of the real number* a *(* $a \neq 0$*) is the positive member of the set* {a, -a}.

Solve |x| = 1. $\{-1,1\}$ Solve |x| < 1. $(-1,1)$ Solve |x| ≥ 1. $(-\infty,-1]\cup[1,\infty)$

Solve |x – 1| ≤ 2. $[-1,3]$

Solve |2x + 5| > 1. $(-\infty,-3)\cup(-1,\infty)$

Solve |x + 5| - |x – 1| < 0. $(-\infty,-2)$

Inequalities in the Plane

Graphs of Linear Inequalities

To graph a linear inequality, we first graph the line. To graph the linear inequality $-x+3y<12$, we first graph the line $-x+3y=12$ or $\frac{x}{-12}+\frac{y}{4}=1$ (This form of a linear equation is called the double-intercept form, because it shows both intercepts) or $y=\frac{1}{3}x+4$. Sketch graph. Any non-vertical line divides the plane into three areas: points above the line, points on the line, and points below the line. Here, these three planar regions correspond to $-x+3y>12$, $-x+3y=12$, and $-x+3y<12$, respectively. To match the correct inequality with each half-plane, choose a *test point* in the half-plane. E.g., the origin lies in the half-plane below the line $-x+3y=12$ and thus satisfies the inequality $-x+3y<12$. So the half-plane below the line corresponds to the inequality $-x+3y<12$. The half-plane above the line corresponds to the inequality $-x+3y>12$

Graphs of Polynomial Inequalities

Solve the polynomial inequality $x^2+2x\geq 8$ graphically. Earlier we solved this inequality algebraically. Now we'll solve it graphically.

1. We begin by graphing the inequality in the plane. Graph $y_1=x^2+2x$ equivalent to $y_1=x(x+2)$ and $y_2=8$. Label both graphs. Where is $y_1\geq y_2$?
2. Recall that the inequality $x^2+2x\geq 8$ is equivalent to $x^2+2x-8\geq 0$ which in turn is equivalent to $(x+4)(x-2)\geq 0$. Graph $y=(x+4)(x-2)$. Where is $y\geq 0$? Sketch solution. This is a quick and efficient method of solving polynomial inequalities *if we can sketch their graphs easily*.

Sketch the solution to $3(x+6)(x+1)(x-3)<0$.