Math 501 - Differential Geometry
Professor Gluck
January 26, 2012

2. SURFACES

Definition. A subset $S \subset R^3$ is a ***regular surface*** if, for each point $p \in S$, there is an open neighborhood V of p in R^3, an open set $U \subset R^2$ and a map

$$X: U \to V \cap S ,$$

such that

(1) X is *smooth*, meaning that if we write

$$X(u, v) = (x(u, v), y(u, v), z(u, v)) ,$$

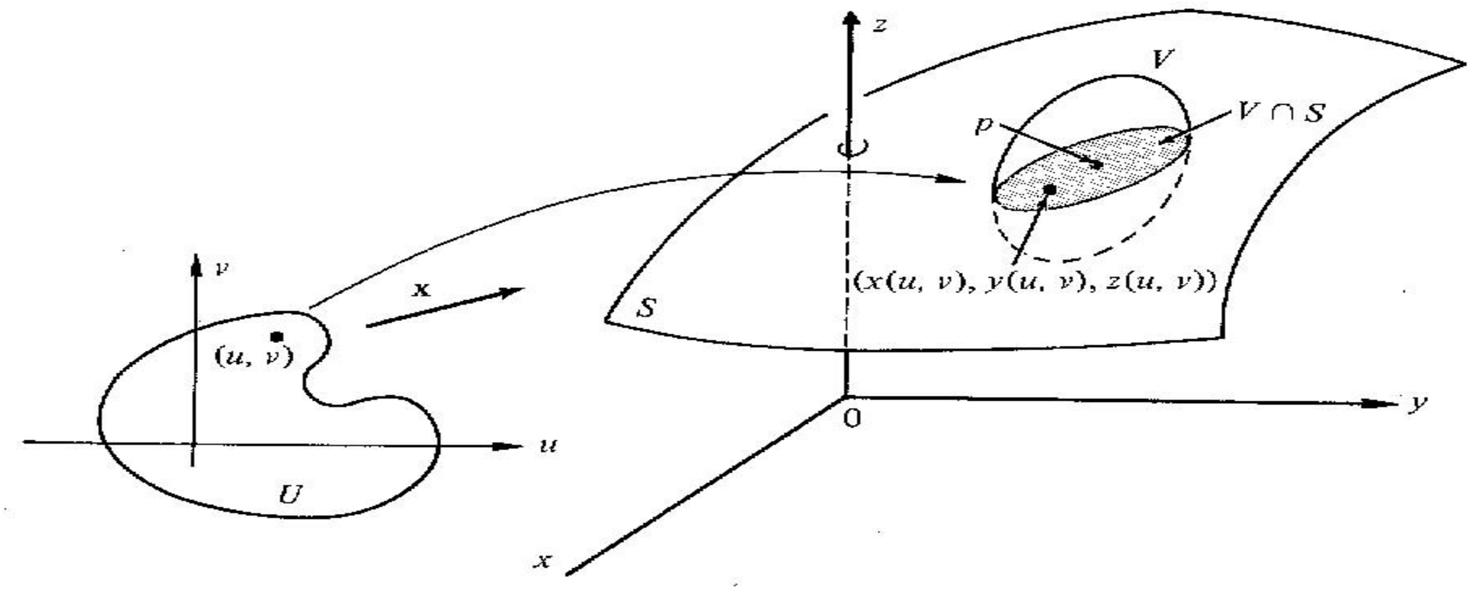
then the real-valued functions $x(u, v)$, $y(u, v)$ and $z(u, v)$ have continuous partial derivatives of all orders in U .

(2) X is a homeomorphism, meaning that it is a one-to-one correspondence between the points of U and $V \cap S$ which is continuous in both directions.

(3) For each point $q \in U$, the linear map

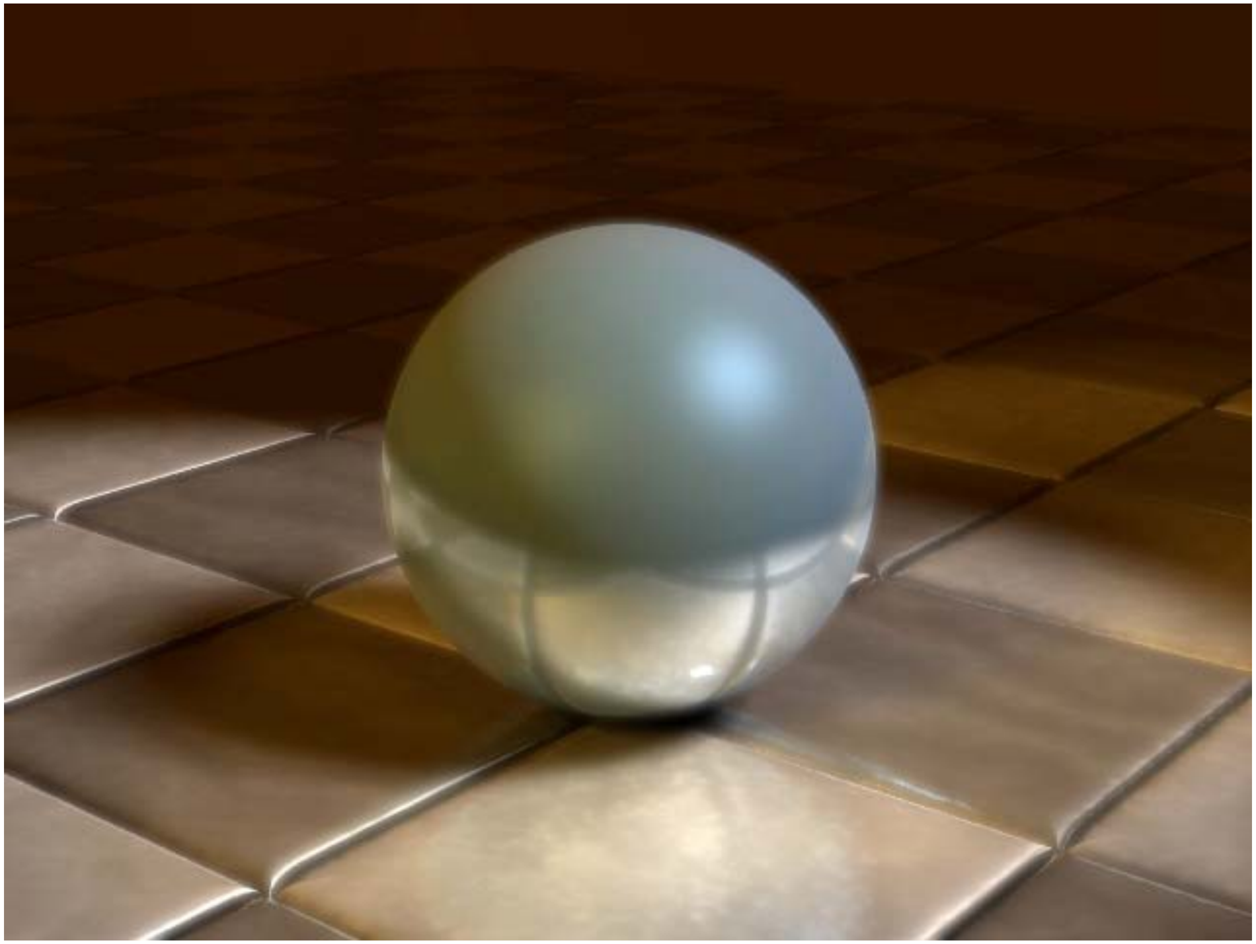
$$dX_q : R^2 \rightarrow R^3 ,$$

called the *differential of* X *at* q, is one-to-one.

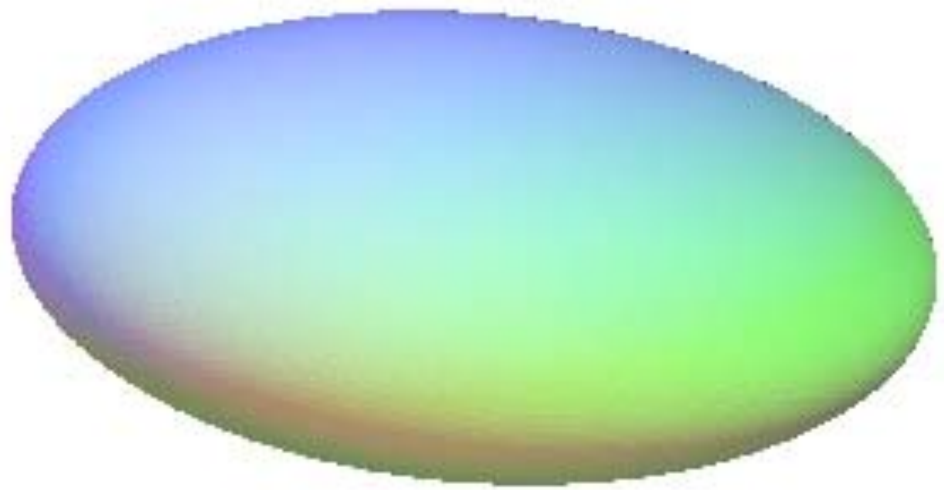


The mapping $X: U \rightarrow V \cap S$ is called a ***parametrization*** or a ***system of local coordinates*** for the surface S in the ***coordinate neighborhood*** $V \cap S$ of p .

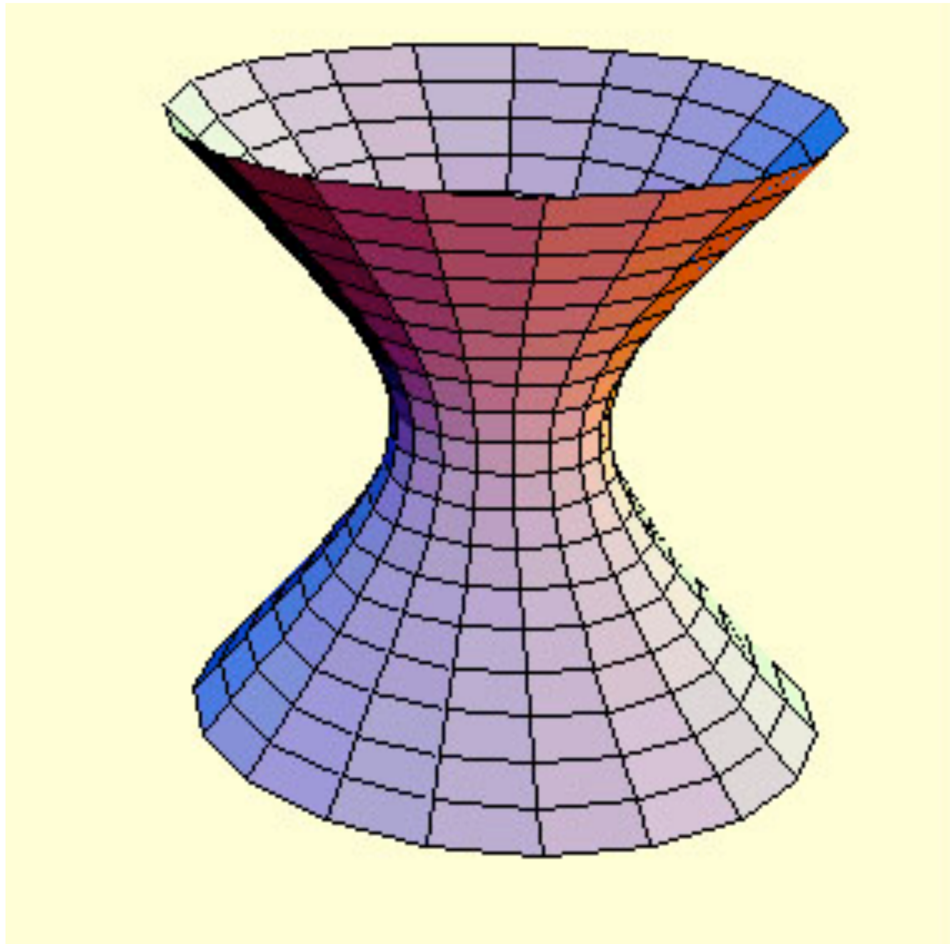
For simplicity of notation, we will henceforth use V , rather than $V \cap S$, to denote an open set on the surface S .



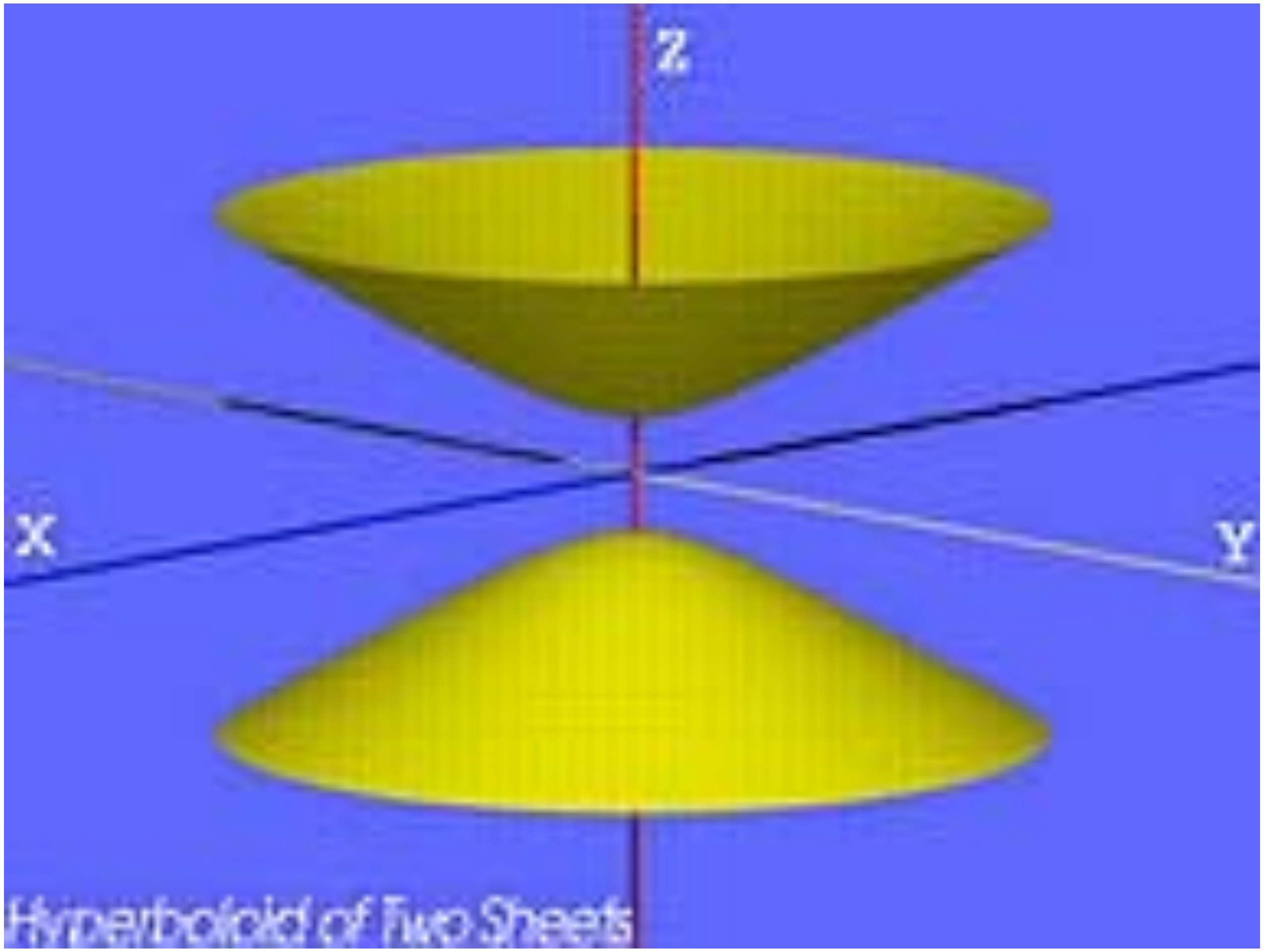
Sphere $x^2 + y^2 + z^2 = r^2$



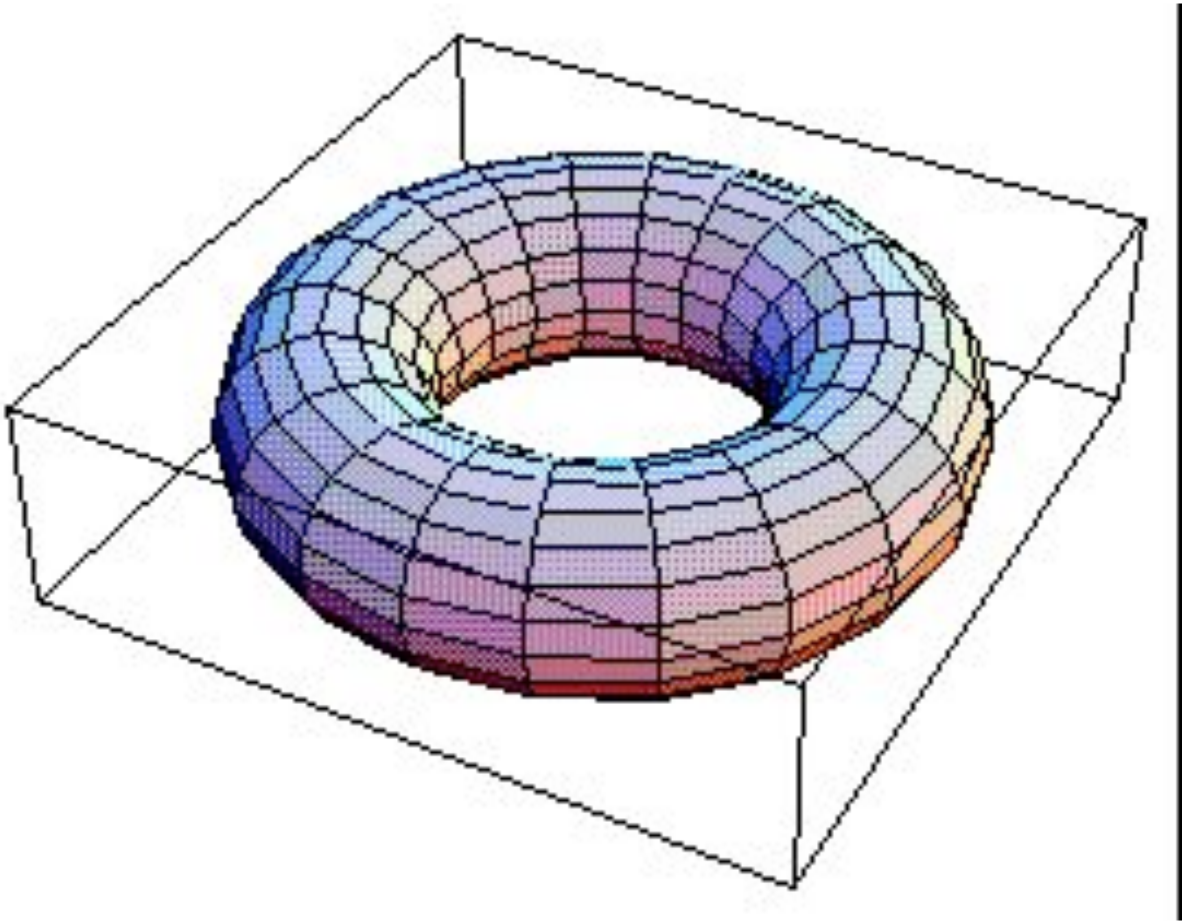
Ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$



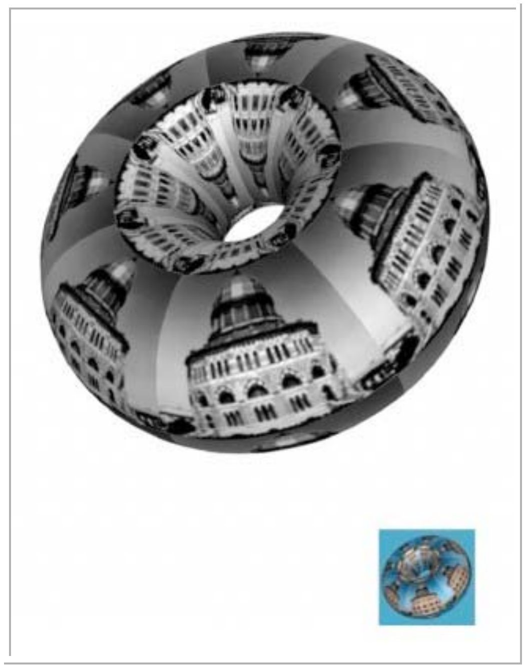
One-sheeted hyperboloid $x^2 + y^2 = 1 + z^2$



Two-sheeted hyperboloid $z^2 = 1 + x^2 + y^2$



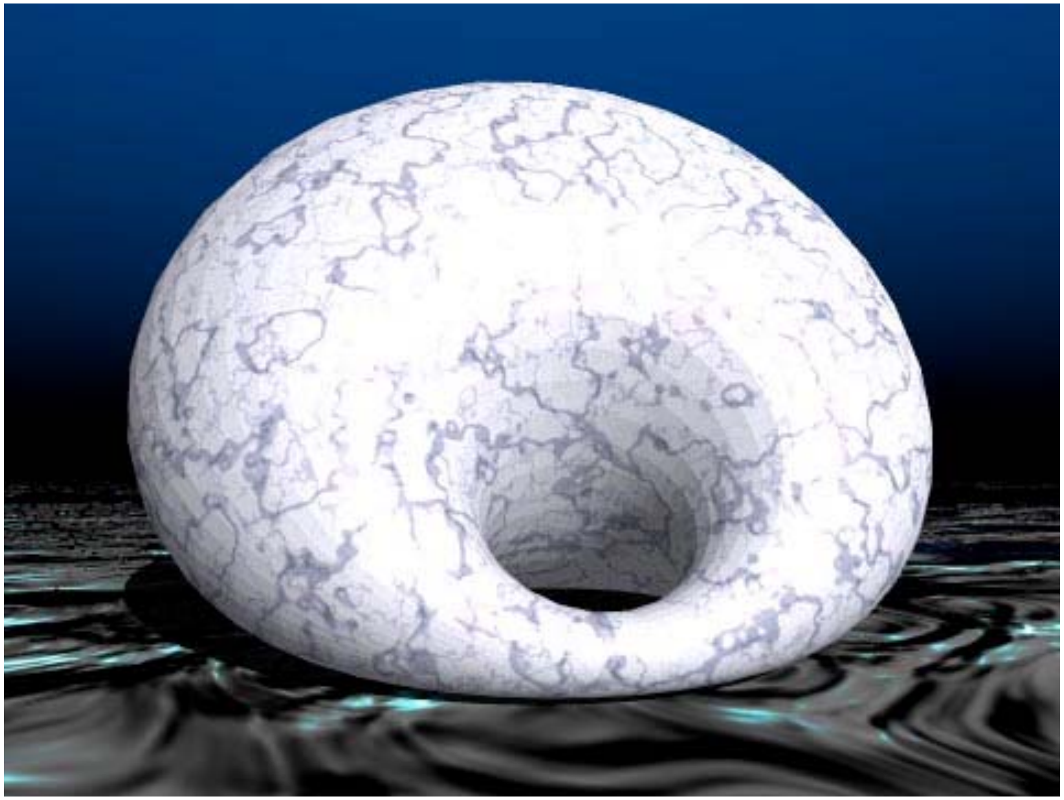
Torus $(r - 3)^2 + z^2 = 1$
in cylindrical (r, φ, z)**-coordinates**

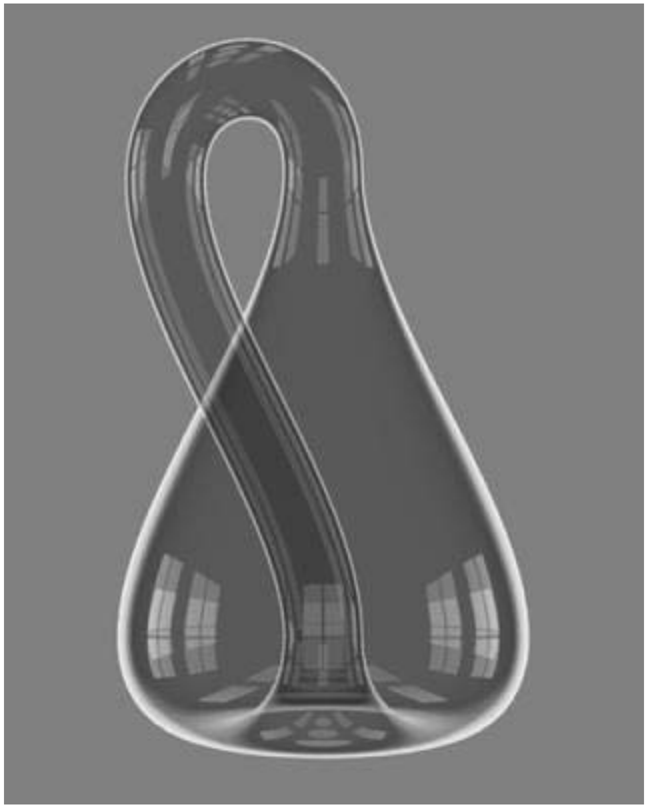


Nott Torus

This image was developed for the Hudson River Undergraduate Mathematics Conference when it was held at Union College in the spring of 1998. Bill Zwicker, the coordinator of the conference here at Union, requested the design. He used it in the advertising for the conference, and on all the signage during the meeting. It also appeared on the T-shirts that were sold to commemorate the event. For these uses, it was printed either smaller or in lower resolution, so the pixelation that is visible in this large, high-resolution image was not apparent.
The small color version is one of the icons that can appear at the top of the Math Department web site home page ▸; a random image is selected each time the page is loaded. Again, the rough appearance of this version is due to the low resolution necessitated by its use on the web.

Miscellaneae





Klein bottle

Picture by John M. Sullivan
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Minimal Surfaces

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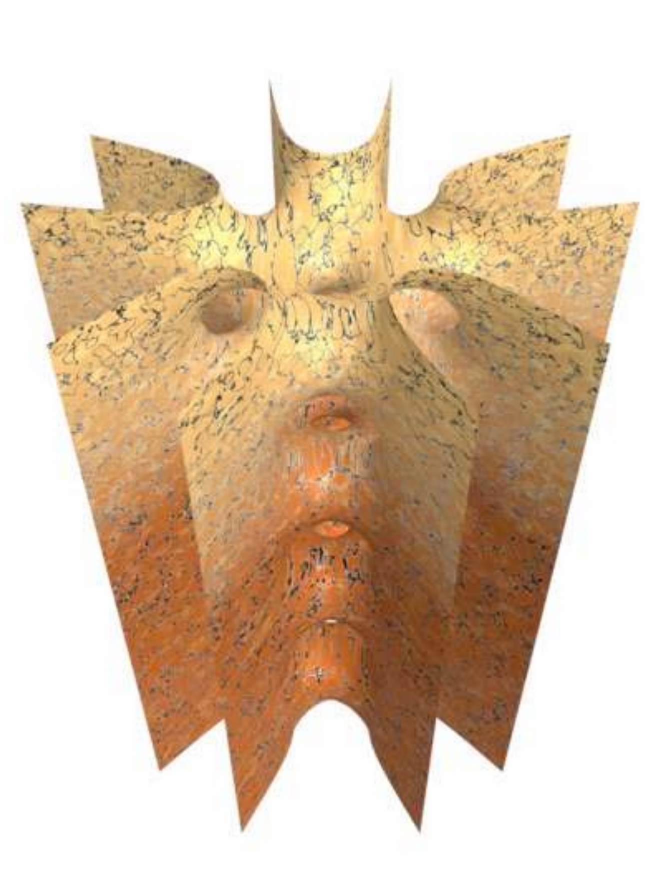
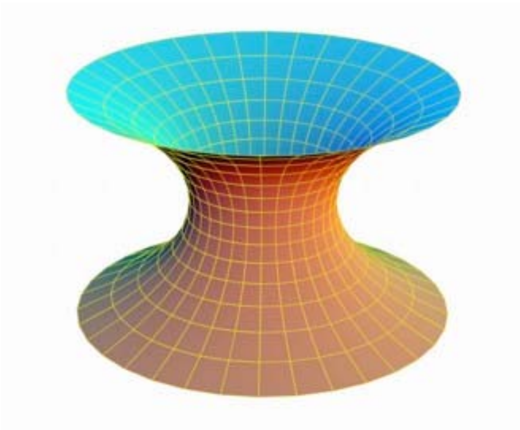
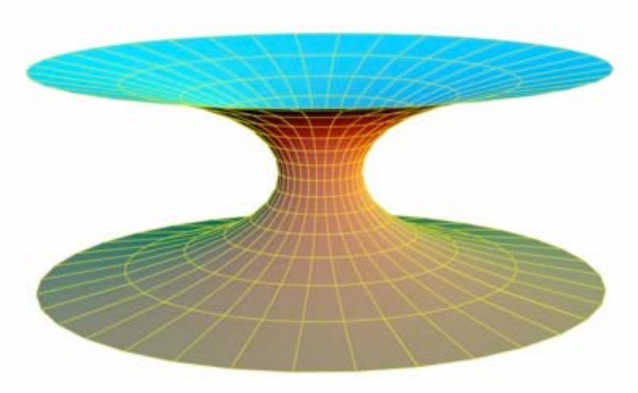
Bloomington's Virtual Minimal Surface Museum

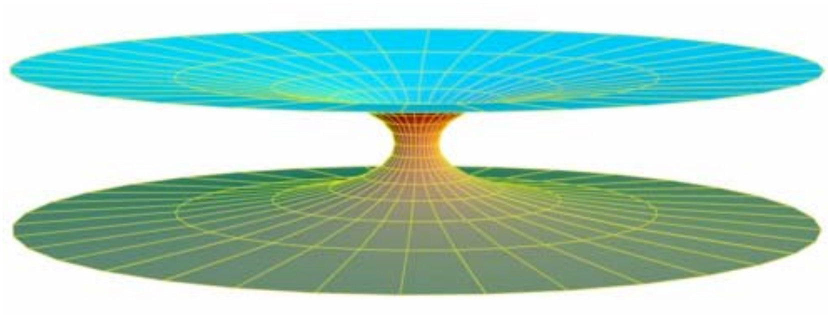


The Catenoid

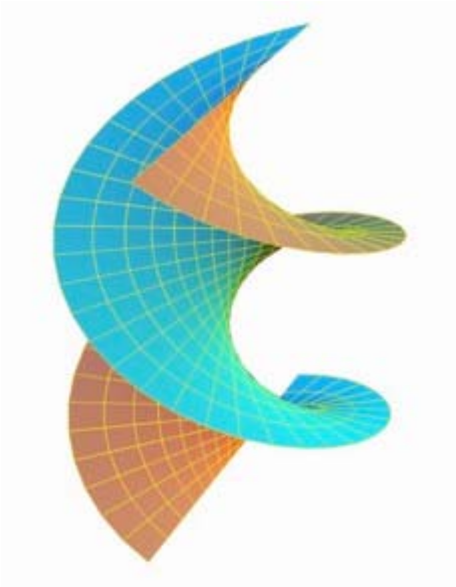




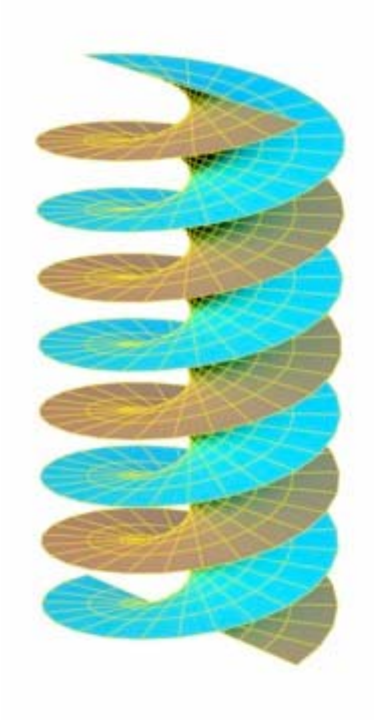
The Catenoid is the only minimal surface of revolution.



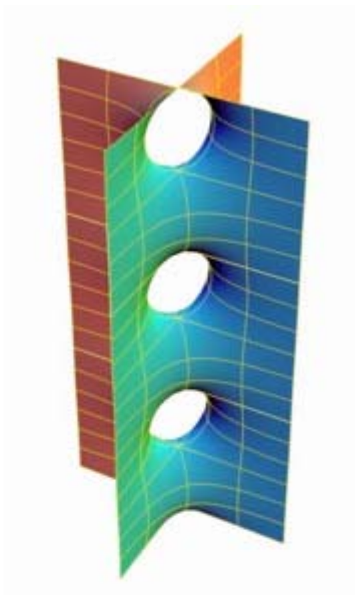
The Helicoid



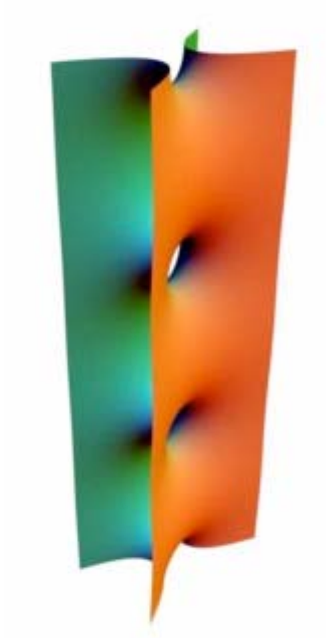
The Helicoid is the only ruled minimal surface.



The singly periodic Scherk surface

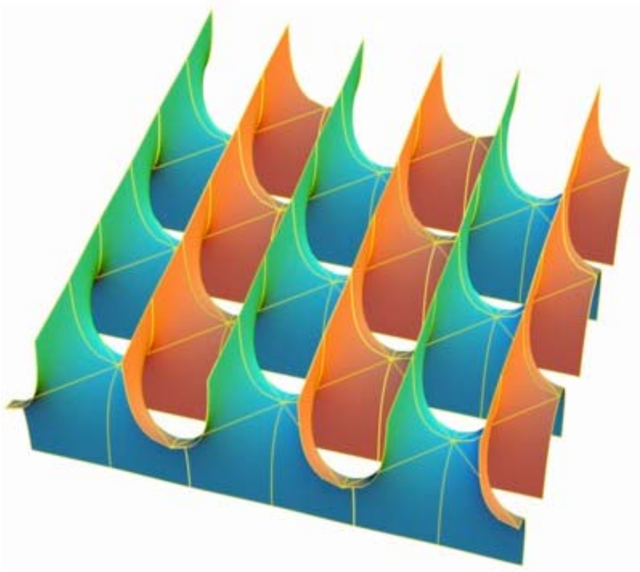


The singly periodic Scherk surface approaches two orthogonal planes.

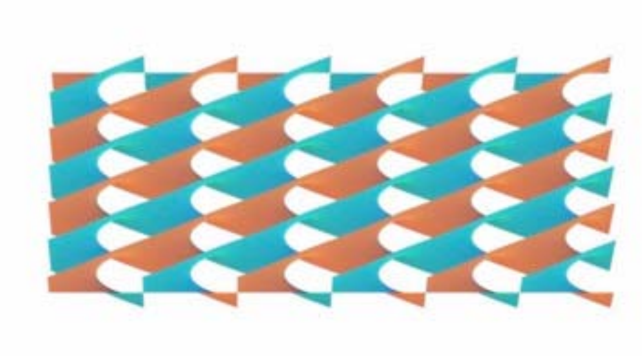


Here is a variation where the two planes are not orthogonal.

The doubly periodic Scherk surface

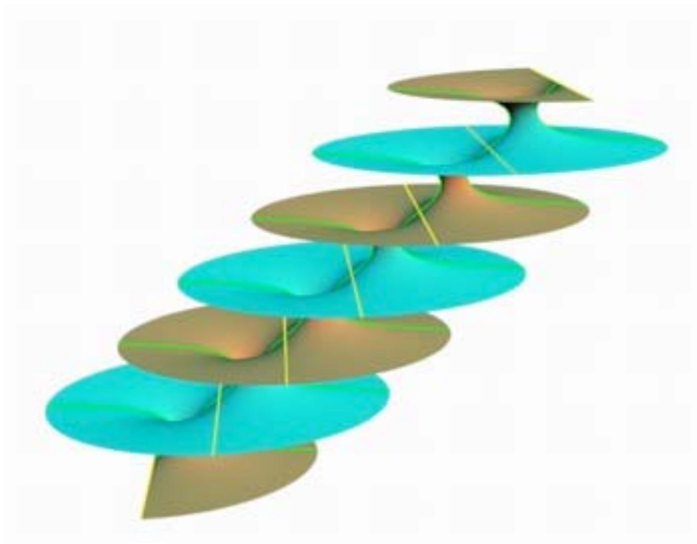


The doubly periodic Scherk surface approaches two families of orthogonal planes.



*Here is a variation where the two families are not orthogonal. One can see **helicoids** forming in the limit!*

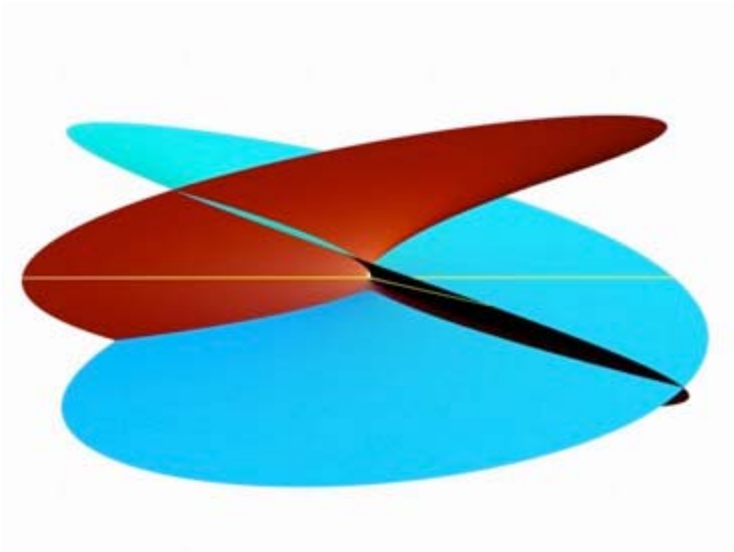
The Riemann minimal surface



Riemann found a family of singly periodic minimal surface whose intersections with horizontal planes are circles.

To the right is a pretty degenerate example where one can see two helicoids developing.

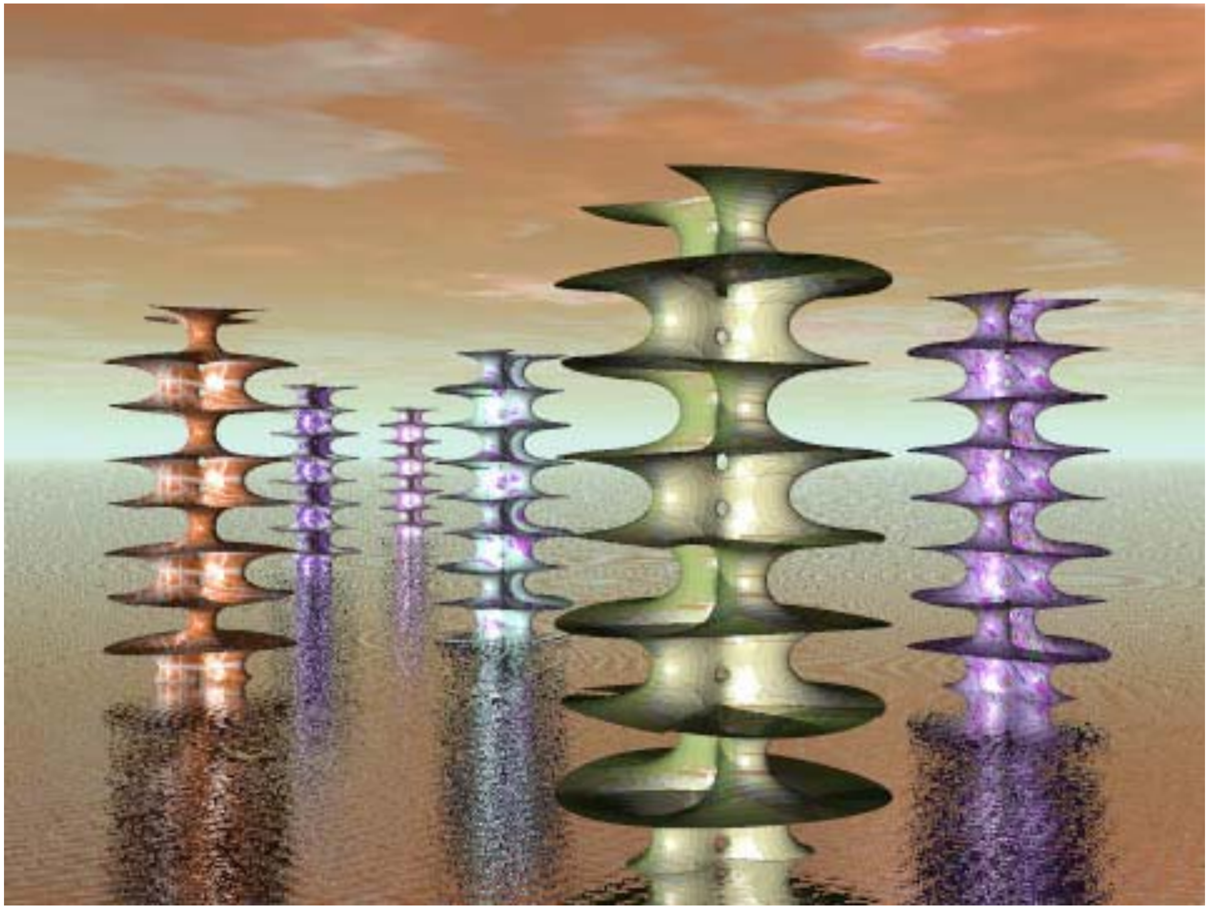
The Enneper surface



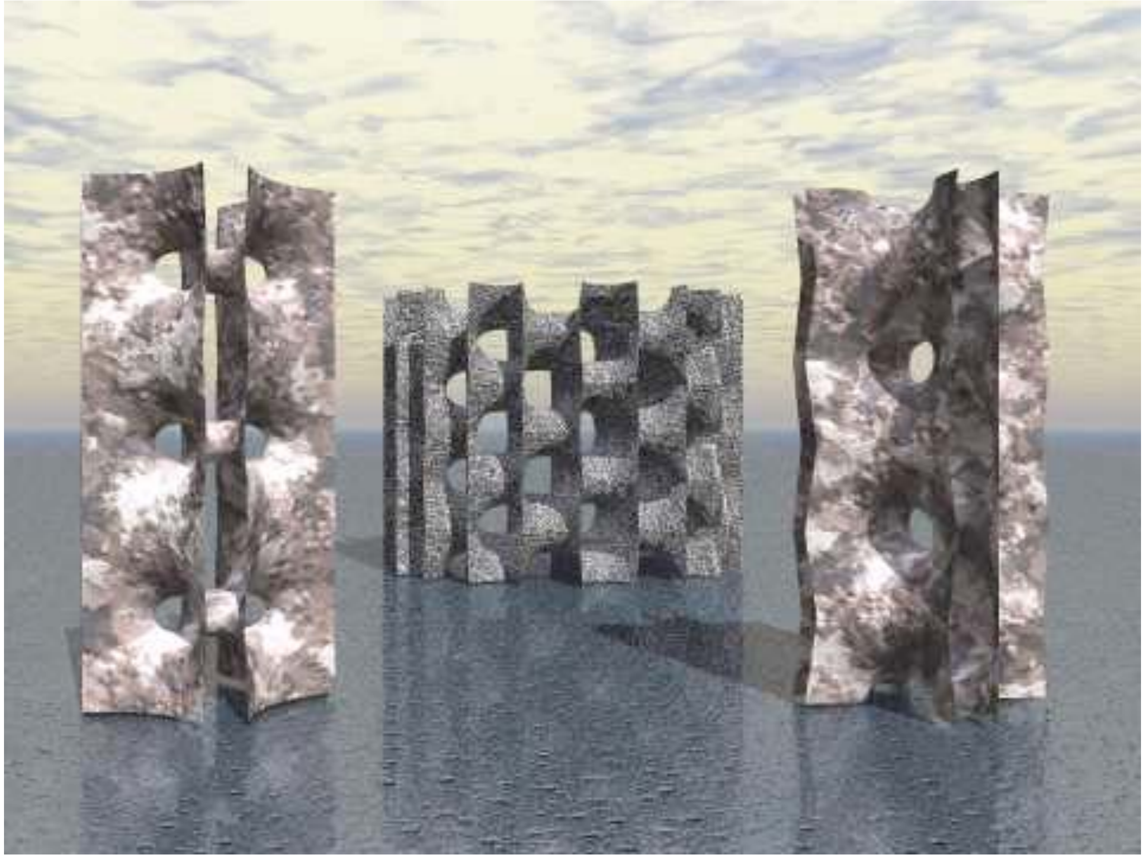
The Enneper surface is a complete minimal surface with two straight lines on it. It is not embedded:

From far away, it looks like a plane covering itself three times.

Minimal Surface Gallery

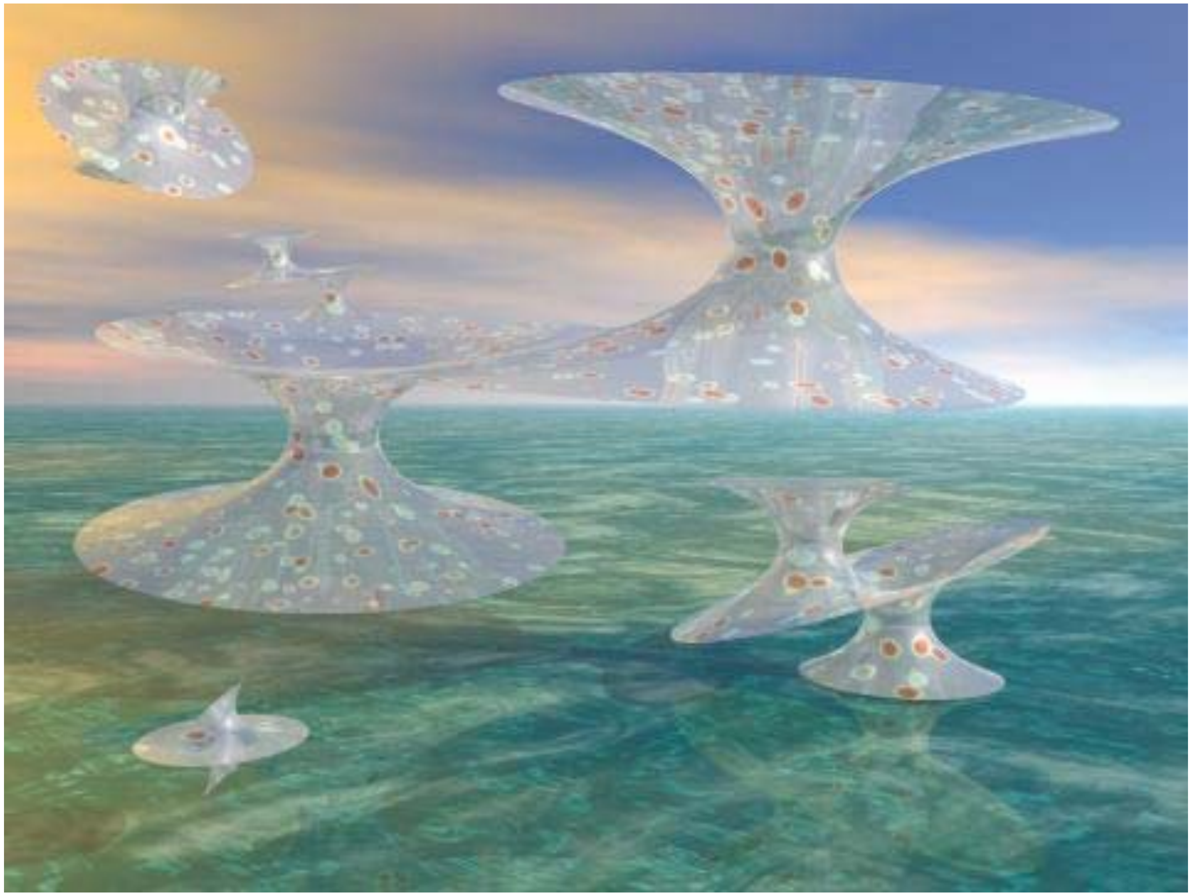


Minimal Surface Gallery

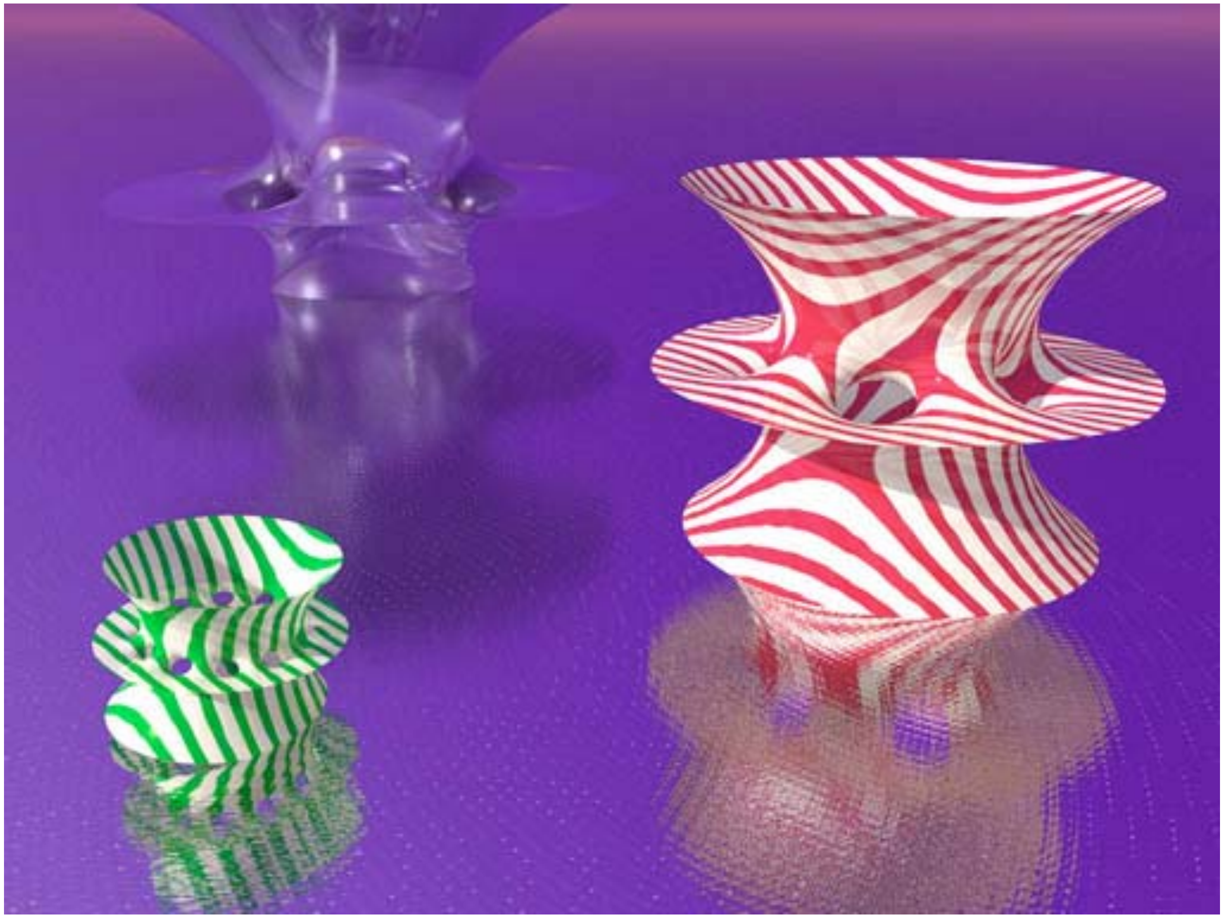


Scherk's surface

Minimal Surface Gallery

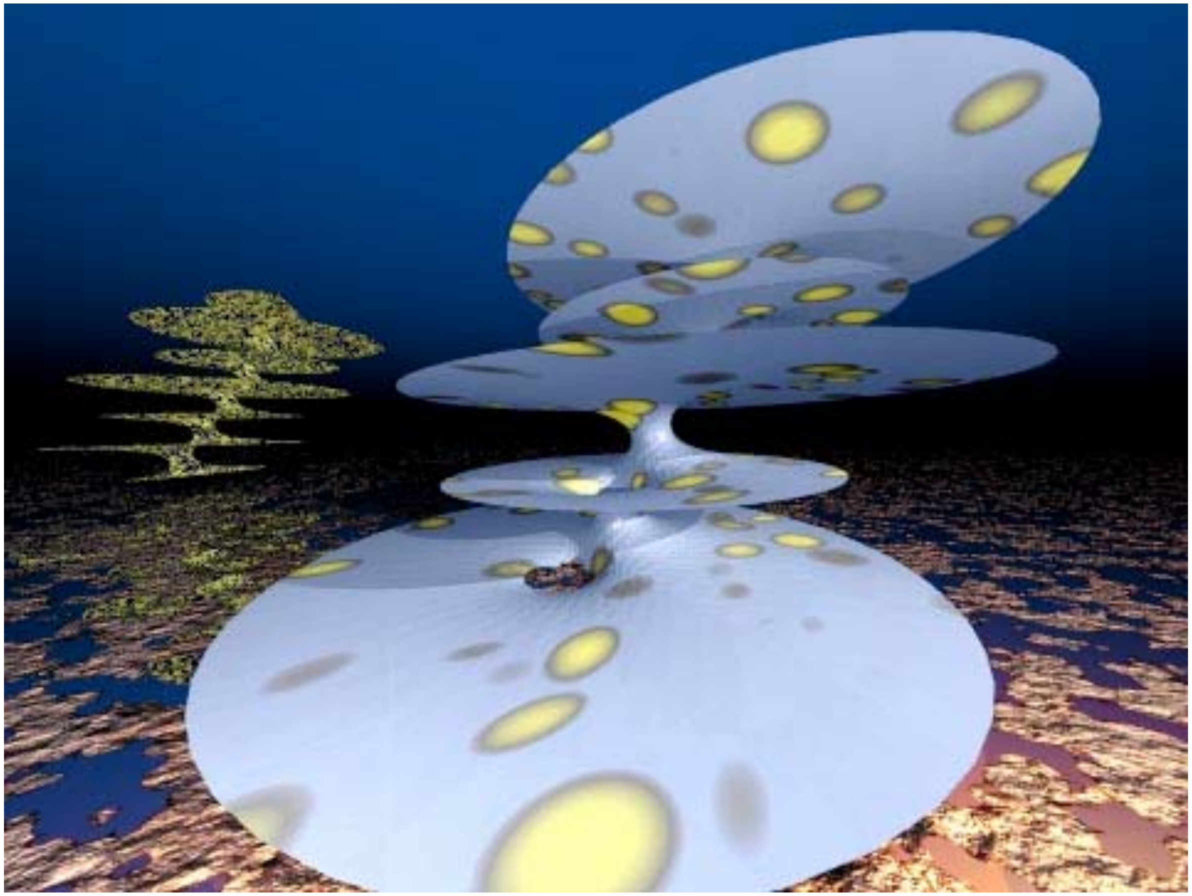


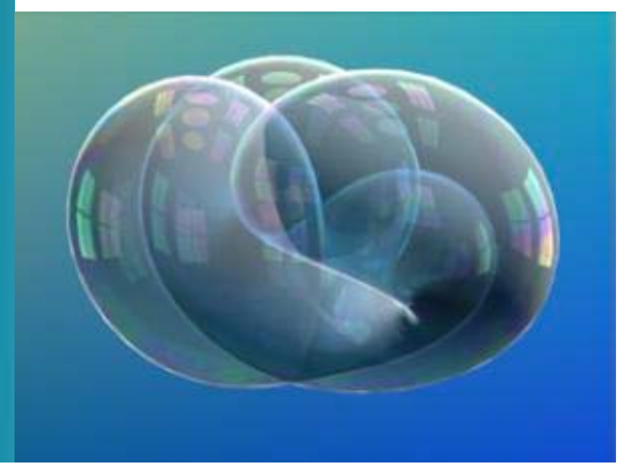
Minimal Surface Gallery



Minimal Surface Gallery

Minimal Surface Gallery



The Willmore bending energy of a surface is the integral (over that surface) of squared mean curvature.

A ***Willmore surface*** is a minimum (or any critical point) for this energy.

One way to get a Willmore surface in R^3 is to stereographically project a minimal surface in S^3.

The two surfaces pictured here arise in this way from a pair of conjugate minimal surfaces in S^3.

Picture and text by John M. Sullivan
torus.math.uiuc.edu/jms/images

Problem 1. What surface is described by the equations

$$x = (a \cos\theta + b) \cos\varphi$$
$$y = (a \cos\theta + b) \sin\varphi$$
$$z = a \sin\theta \ ,$$

where $0 < a < b$ are positive constants, and θ and φ are angular variables?

Draw this surface, and indicate on the drawing what the constants a and b measure.

Problem 2. Let $f: U \rightarrow R$ be a smooth real-valued function defined on the open set $U \subset R^2$.

The *graph* of f is the subset of R^3 given by

$$\{(x, y, f(x, y) : (x, y) \in U\} .$$

Show that the graph of f is a regular surface.

Definition. Let U be an open set of R^m and $F: U \rightarrow R^n$ a smooth map. A point $p \in U$ is called a ***critical point*** of F is the differential $dF_p: R^m \rightarrow R^n$ is *not* onto. The image $F(p)$ of a critical point is called a ***critical value*** of F. A point of R^n which is not a critical value of F is called a ***regular value*** of F.

Note that any point of R^n which is not in the image $F(U)$ is, by default, a regular value of F.

Problem 3. Let U be an open subset of R^3 and $f: U \rightarrow R$ a smooth function. If a is a regular value of f, show that $f^{-1}(a)$ is a regular surface in R^3.

Hint. Use the inverse function theorem.

Problem 4. Show that 1 is a regular value of the function

$$f(x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2 ,$$

and conclude that the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

is a regular surface.

Problem 5. Let S be a regular surface in R^3 and $p \in S$. Show that there is a neighborhood V of p in S which is the graph of a differentiable function having one of the following three forms:

$$z = f(x, y)\ , \quad y = g(x, z)\ , \quad x = h(y, z)\ .$$

Problem 6 (Change of parameters). Let

$$X_1 : U_1 \to V_1 \quad \text{and} \quad X_2 : U_2 \to V_2$$

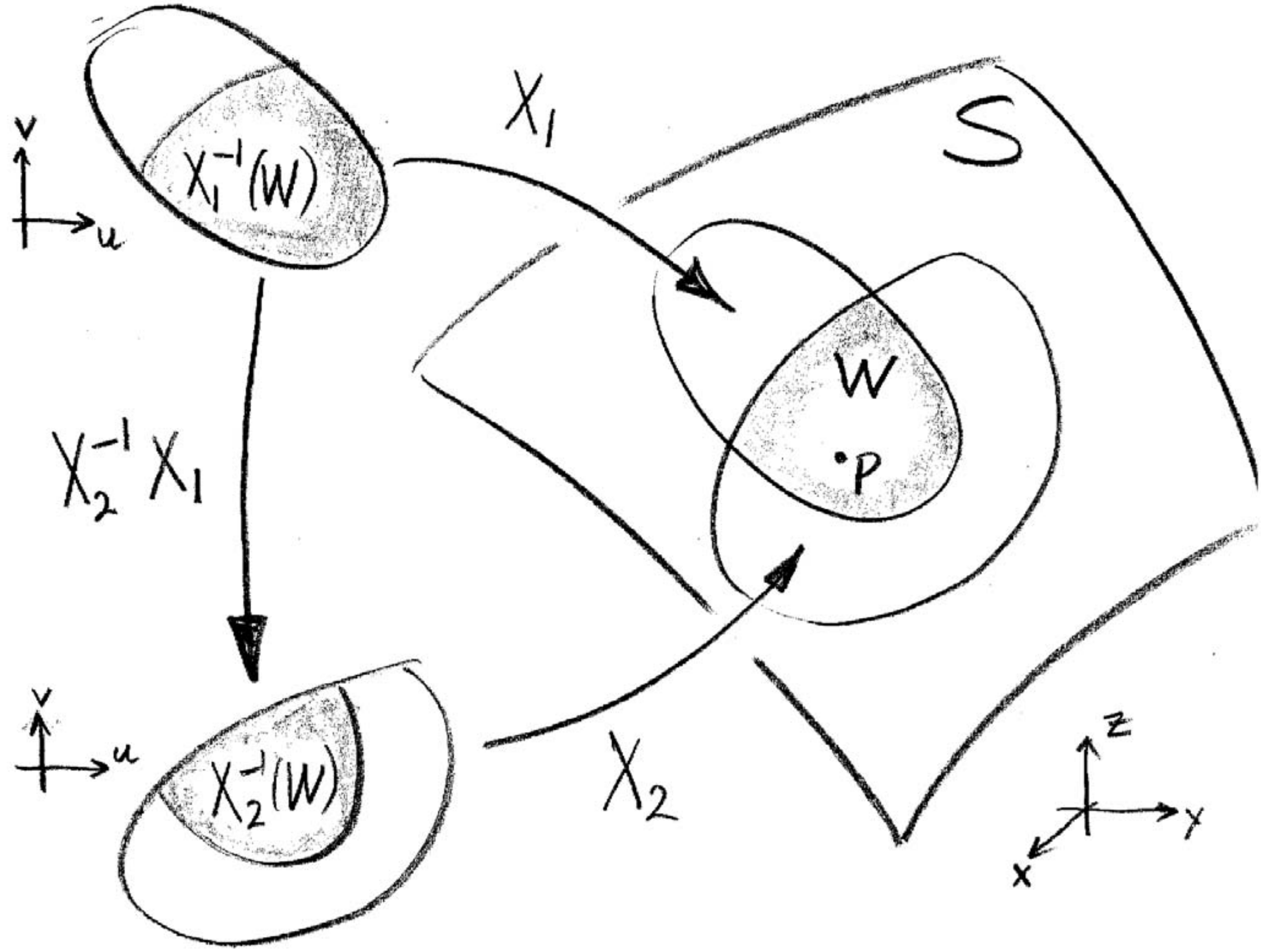
be two parametrizations of the regular surface S, and suppose that the point p of S lies in the image of both:

$$p \in W = X_1(U_1) \cap X_2(U_2) = V_1 \cap V_2 .$$

Show that the map

$$X_2^{-1} X_1 : X_1^{-1}(W) \to X_2^{-1}(W)$$

is a diffeomorphism (that is, a one-to-one correspondence which is smooth in both directions).



v
u
$X_1^{-1}(W)$
X_1
S
W
P
$X_2^{-1}X_1$
v
u
$X_2^{-1}(W)$
X_2
z
y
x

Definition. Let $f : S \to R$ be a real-valued function defined on the regular surface S in R^3. We will say that f is ***smooth*** if for every parametrization $X: U \to V$ of an open set V on S, the composite map $f \circ X : U \to R$ is smooth.

Problem 7. Show that to check that a given real-valued function $f : S \to R$ is smooth, you don't really have to check that the compositions $f \circ X$ are smooth for *all* parametrizations X of S. It's enough to do it for any family of parametrizations whose images cover S.

Problem 8. Let S be a regular surface in R^3, let V be an open subset of R^3 which contains S, and let $f : V \to R$ be a smooth function. Show that the restriction of f to S is also a smooth function.

Problem 9. Given two surfaces S_1 and S_2 in R^3 and a map $f : S_1 \to S_2$.

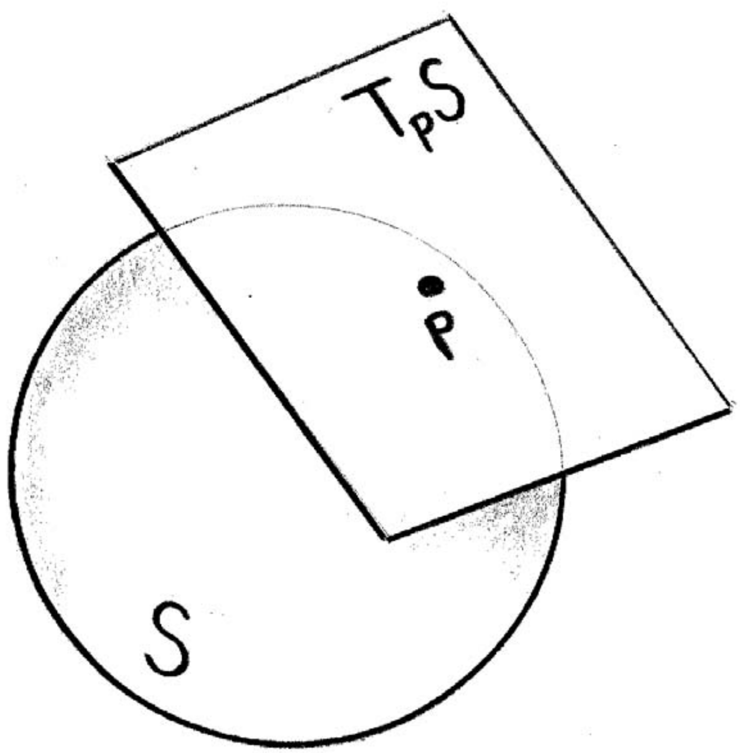
Figure out how to use parametrizations of S_1 and S_2 to define smoothness of f.

The tangent plane to a regular surface at a point.

Definition. Let S be a regular surface in R^3 and p a point of S. Pick any parametrization of S, $X: U \rightarrow V \cap S$, with p lying in the open set $V \cap S$. Let q be the unique point of U such that $X(q) = p$. The linear map $dX_q: R^2 \rightarrow R^3$, that is, the differential of X at q, is one-to-one, and hence its image, $dX_q(R^2)$, is a 2-dimensional subspace of R^3. We call this the ***tangent space to S at p***, and denote it by T_pS.

Problem 10. Show that this definition of T_pS is independent of the choice of parametrization X.

The common convention is to draw the tangent space to S at p so that it goes through p rather than through the origin, and simply remember that it is a vector space.



Problem 11. Let S be a regular surface in R^3 .
Show that the tangent space to S at p consists of the tangent vectors at p to all the regular curves which lie on S and go through p .

Problem 12. If S_1 and S_2 are regular surfaces in R^3 and $f: S_1 \to S_2$ is a smooth map, show how to define its differential

$$df_p : T_pS_1 \to T_{f(p)}S_2 .$$

Then prove the chain rule for the differentials of smooth maps between regular surfaces in R^3 .

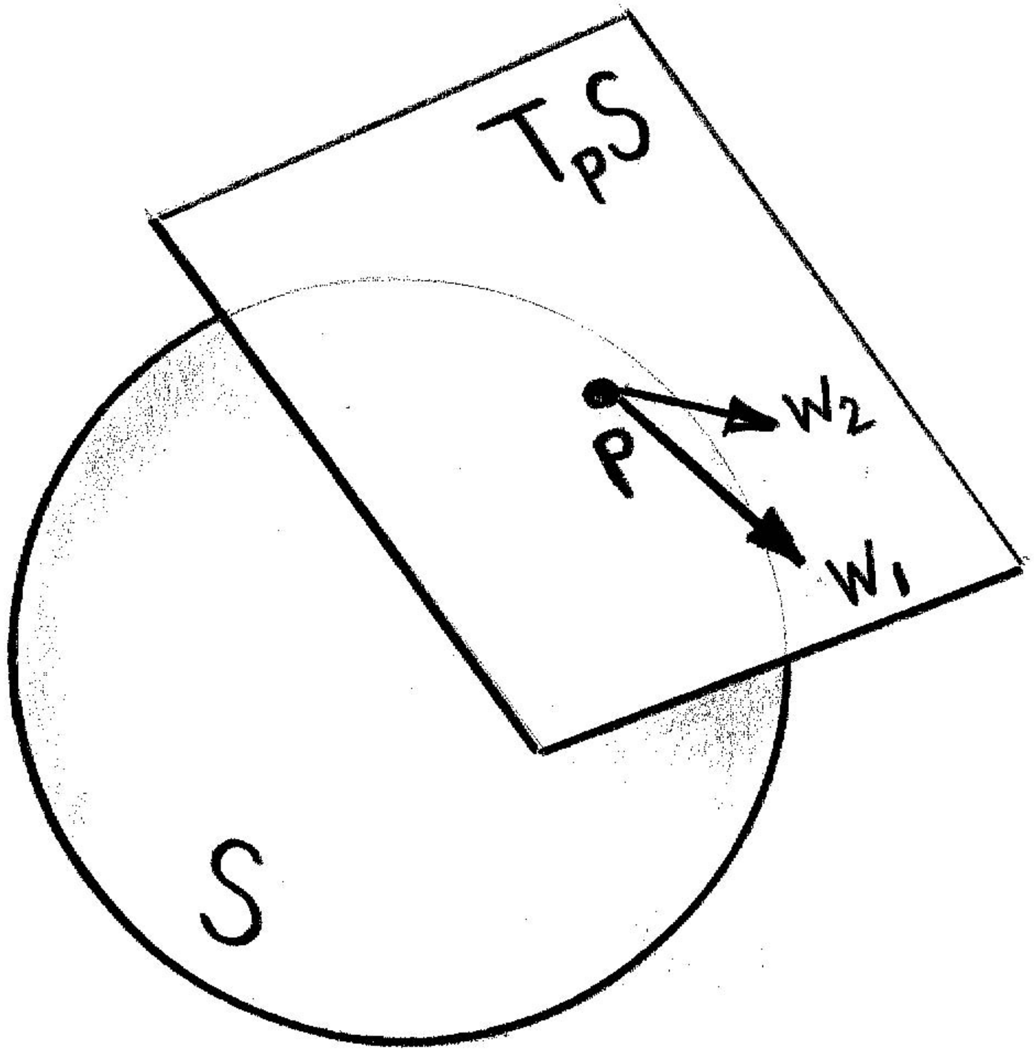
Problem 13. Suppose $f: S_1 \rightarrow S_2$ is a smooth map between regular surfaces in R^3. Suppose that at the point $p \in S_1$, the differential $df_p : T_pS_1 \rightarrow T_{f(p)}S_2$ is an isomorphism.

Prove that f is a diffeomorphism from some open neighborhood of p on S_1 to some open neighborhood of $f(p)$ on S_2.

The first fundamental form.

Let S be a regular surface in R^3 and $p \in S$. Then the tangent plane T_pS to S at p is a 2-dimensional subspace of R^3, meaning that it is a 2-plane passing through the origin, even though when drawing it, we usually move it parallel to itself so that it passes through the point p.

Thus T_pS inherits an inner product $< , >_p$ from R^3, and if W_1 and W_2 are two tangent vectors to S at p, their inner product is written as $< W_1 , W_2 >_p$.



T_pS
w_2
p
w_1
S

The inner product $< W_1 , W_2 >_p$ is a ***symmetric bilinear form*** on the tangent space T_pS , meaning that it is a map

$$< , >_p : T_pS \times T_pS \rightarrow R$$

which is linear in each of the arguments W_1 and W_2 when the other is held fixed, and that

$$< W_1, W_2 >_p = < W_2 , W_1 >_p .$$

The associated ***quadratic form***

$$I_p : T_pS \rightarrow R$$

is defined by

$$I_p(W) = < W , W >_p .$$

Note that the original bilinear form $< , >_p$ can be recovered from the associated quadratic form because

$$I_p(W_1 + W_2) = < W_1 + W_2 , W_1 + W_2 >$$

$$= <W_1 , W_1> + <W_1 , W_2> + <W_2 , W_1> + <W_2 , W_2>$$

$$= I_p(W_1) + 2 <W_1 , W_2> + I_p(W_2) ,$$ and hence

$$<W_1 , W_2> = \tfrac{1}{2} (I_p(W_1 + W_2) - I_p(W_1) - I_p(W_2)) .$$

Thus there is no loss of information in focusing on the associated quadratic form, and on the plus side we gain in notational symplicity because we only have to evaluate it on one vector W instead of on two, W_1 and W_2 .

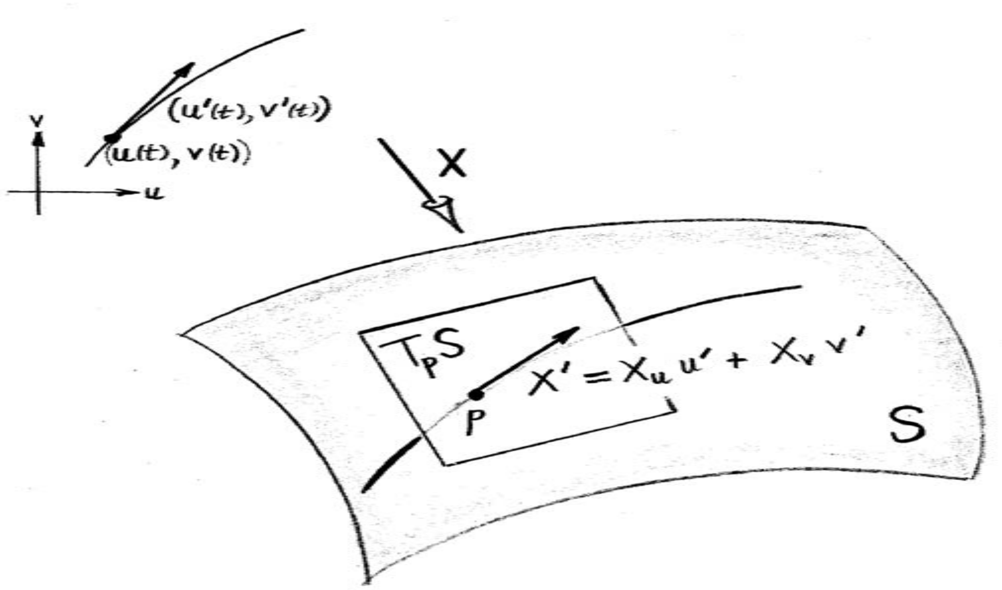
Definition. The quadratic form $I_p(W) = \langle W, W \rangle_p$ is called the ***first fundamental form*** of the regular surface S at the point p .

The first fundamental form simply encodes how the surface S inherits the natural inner product of R^3 .

We will see shortly that the first fundamental form allows us to make geometric measurements on the surface, such as lengths of curves, angles between tangent vectors, and areas of regions on the surface, without referring to the ambient space R^3 where the surface lies.

We will see later that the first fundamental form also encodes some, but not all, of the information about the "curvature" of the surface in R^3 .

Notation for tangent vectors to S .



Let S be a surface in R^3 and p a point of S .

Let $X : U \rightarrow V$ be a parametrization of a neighborhood V of p on S , and let (u, v) be Euclidean coordinates on U .

If (u(t), v(t)) describes a curve in U , then X(u(t), v(t)) describes its image on S .

If $(u'(t), v'(t))$ is the velocity vector to the curve in U, then

$$X'(t) = X_u\, u'(t) + X_v\, v'(t)$$

is the velocity vector to the image curve on S, and hence a tangent vector to S at the point $p = X(u(t), v(t))$.

The vector partial derivatives X_u and X_v provide a basis for the tangent space T_pS.

Notation for the first fundamental form.

Let $W = X_u u' + X_v v'$ be a tangent vector to S at p, as just explained.

Then evaluating the first fundamental form I_p on W, we get

$$I_p(W) = \langle W, W \rangle_p$$

$$= \langle X_u u' + X_v v', X_u u' + X_v v' \rangle_p$$

$$= \langle X_u, X_u \rangle_p (u')^2 + 2 \langle X_u, X_v \rangle u' v' + \langle X_v, X_v \rangle (v')^2$$

$$= E(u, v) (u')^2 + 2 F(u, v) u' v' + G(u, v) (v')^2 .$$

The three real-valued functions

$$E(u, v) = \langle X_u , X_u \rangle,$$

$$F(u, v) = \langle X_u , X_v \rangle,$$

$$G(u,v) = \langle X_v, X_v \rangle$$

encode complete information about the first fundamental form throughout the given coordinate neighborhood on S .

Notice that we have dropped the subscript p from the notation for the inner product, since it is clear from context.

Examples.

• Consider the ***xy-coordinate plane*** in R^3 as a surface S parametrized by itself:

$$X(u, v) = (u, v, 0) .$$

Then $X_u = (1, 0, 0)$ and $X_v = (0, 1, 0)$, hence

$$E(u, v) = \langle X_u , X_u \rangle = 1$$
$$F(u, v) = \langle X_u , X_v \rangle = 0$$
$$G(u, v) = \langle X_v , X_v \rangle = 1 .$$

• Consider the ***torus of revolution*** in R^3 as a surface S parametrized by

$$X(\theta, \varphi) = ((a \cos \theta + b)\cos \varphi, (a \cos \theta + b)\sin \varphi, a \sin \theta) .$$

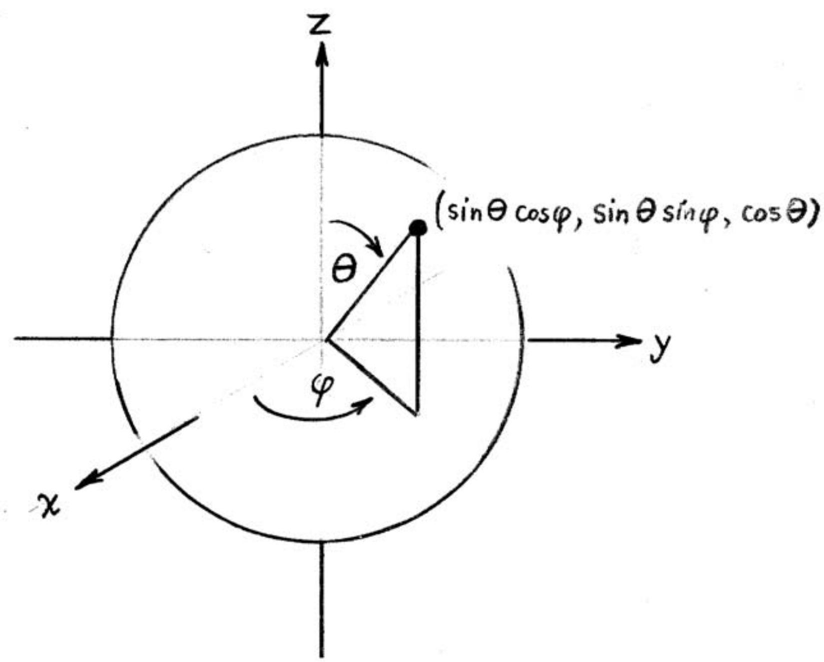
Then

$$X_\theta = (-a \sin \theta \cos \varphi, -a \sin \theta \sin \varphi , a \cos \theta)$$
$$X_\varphi = (-(a \cos \theta + b)\sin \varphi, (a \cos \theta + b)\cos \varphi, 0) ,$$

$$E(\theta, \varphi) = \langle X_\theta , X_\theta \rangle = a^2$$
$$F(\theta, \varphi) = \langle X_\theta , X_\varphi \rangle = 0$$
$$G(\theta, \varphi) = \langle X_\varphi , X_\varphi \rangle = (a \cos \theta + b)^2 .$$

- Consider the ***unit 2-sphere*** S^2 in R^3 parametrized by

$$X(\theta, \varphi) = (\sin\theta \cos\varphi , \sin\theta \sin\varphi , \cos\theta) .$$



The parametrization is singular when $\theta = 0$ (north pole) and when $\theta = \pi$ (south pole), so we restrict $0 < \theta < \pi$.

To make the parametrization one-to-one rather than many-to-one, we can require that $0 < \varphi < 2\pi$.

Then

$$X_\theta = (\cos\theta\cos\varphi , \cos\theta\sin\varphi , -\sin\theta)$$

$$X_\varphi = (-\sin\theta\sin\varphi , \sin\theta\cos\varphi , 0)$$

$$E(\theta, \varphi) = \langle X_\theta , X_\theta \rangle = 1$$

$$F(\theta, \varphi) = \langle X_\theta , X_\varphi \rangle = 0$$

$$G(\theta, \varphi) = \langle X_\varphi , X_\varphi \rangle = \sin^2\theta .$$

Arc length of curves on a surface.

Let $X: U \to V \subset S$ be a parametrization of a portion of the regular surface S in R^3.

Give a curve on S, the portion of it which runs within the open set V can be expressed as $X(u(t), v(t))$, where $(u(t), v(t))$ is a curve in U.

The arc length s of this curve is then given by

$$s = \int |dX/dt|\, dt = \int < X'(t) , X'(t) >^{1/2} \, dt$$

$$= \int I(X'(t))^{1/2} \, dt$$

$$= \int [E(u, v) (u')^2 + 2 F(u,v) u' v' + G(u, v) (v')^2]^{1/2} \, dt .$$

In view of this, it is customary to write

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2 ,$$

as a short hand for the formula

$$(ds/dt)^2 = E\,(du/dt)^2 + 2\,F\,(du/dt)\,(dv/dt) + G\,(dv/dt)^2 .$$

Problem 14. Let $X: U \to V \subset S$ be a parametrization of a portion of the regular surface S in R^3 . The image under X of the curves $u = \text{constant}$ and $v = \text{constant}$ are called the ***coordinate curves*** on V . Show that the angle $\theta(u, v)$ between these curves is given by

$$\cos \theta(u, v) = F(u, v) / \sqrt{(E(u,v)\, G(u,v))} = F / \sqrt{(EG)} .$$

Problem 15. Let $X: U \to V \subset S$ be a parametrization of a portion of the regular surface S in R^3. Let U_o be a subdomain of U and $V_o = X(U_o)$ the corresponding subdomain of V in S. Justify the formula

$$\text{area}(V_o) = \int_{U_o} |X_u \times X_v| \; du\, dv ,$$

and use it to compute the total surface area of the unit 2-sphere $S^2 \subset R^3$.

Problem 16. Use this same formula to compute the total surface area of the torus of revolution

$$X(\theta, \varphi) = ((a \cos \theta + b)\cos \varphi, (a \cos \theta + b)\sin \varphi, a \sin \theta) ,$$

where $0 < a < b$.