Math 501 - Differential Geometry
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4. INTRINSIC GEOMETRY OF SURFACES

Let S and S' be regular surfaces in 3-space.

Definition. A diffeomorphism $\varphi: S \rightarrow S'$ is an ***isometry*** if for all points $p \in S$ and tangent vectors $W_1, W_2 \in T_pS$ we have

$$< W_1 , W_2 >_p \; = \; < d\varphi_p(W_1) , d\varphi_p(W_2) >_{\varphi(p)} .$$

The surfaces S and S' are then said to be ***isometric***.

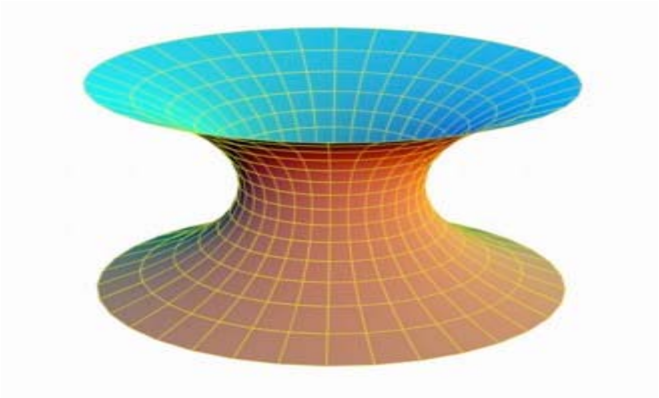
Problem 1. Show that an isometry between surfaces preserves lengths of tangent vectors, angles between tangent vectors, the first fundamental form, lengths of curves, angles between curves and areas of domains.

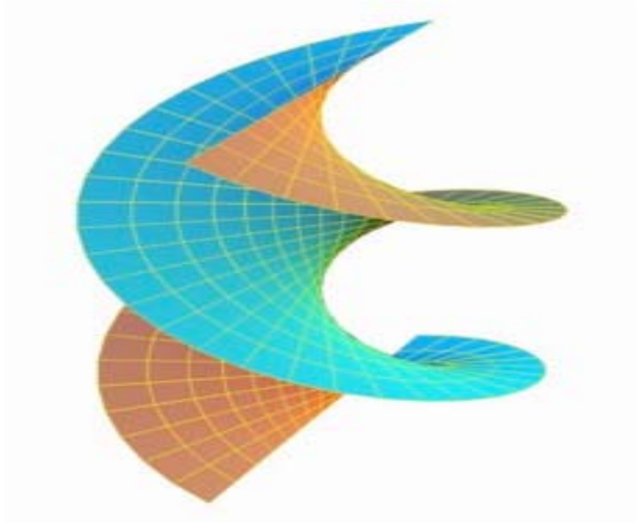
Problem 2. Show that a diffeomorphism between regular surfaces which preserves the first fundamental form is an isometry.

Definition. Two regular surfaces S and S' are said to be ***locally isometric*** if each point on each surface has an open neighborhood isometric to an open set on the other surface.

Example. A cylinder is locally isometric to a plane.
But the two surfaces are not isometric.

Problem 3. Show that a catenoid and helicoid are locally isometric. See do Carmo, Problem 14 on page 213, and also Example 2 on pages 221-222.





Definition. A diffeomorphism $\varphi: S \to S'$ between regular surfaces is called a ***conformal map*** if for all points $p \in S$ and tangent vectors $W_1, W_2 \in T_pS$ we have

$$< d\varphi_p(W_1) , d\varphi_p(W_2) >_{\varphi(p)} = \lambda^2(p) < W_1 , W_2 >_p ,$$

where λ^2 is a strictly positive smooth function on S. The surfaces S and S' are then said to be ***conformally equivalent.***

Problem 4. Show that a conformal map preserves angles between tangent vectors, but not necessarily the lengths of tangent vectors, and that likewise it preserves angles between curves, but not necessarily lengths of curves.

Definition. Two regular surfaces S and S' are said to be ***locally conformal*** if each point on each surface has an open neighborhood conformally equivalent to an open set on the other surface.

DEEP THEOREM. Any two regular surfaces are locally conformal.

Remark of do Carmo. The proof is based on the possibility of parametrizing a neighborhood of any point of a regular surface in such a way that the coefficients of the first fundamental form are

$$E = \lambda^2(u, v) > 0 , \quad F = 0 , \quad G = \lambda^2(u, v) .$$

Problem 5. Let $\varphi: R^3 \rightarrow R^3$ be a map such that

$$|\varphi(p) - \varphi(q)| = |p - q| .$$

Show there is a linear isometry $L: R^3 \rightarrow R^3$ such that

$$\varphi(p) = L(p) + \varphi(0) .$$

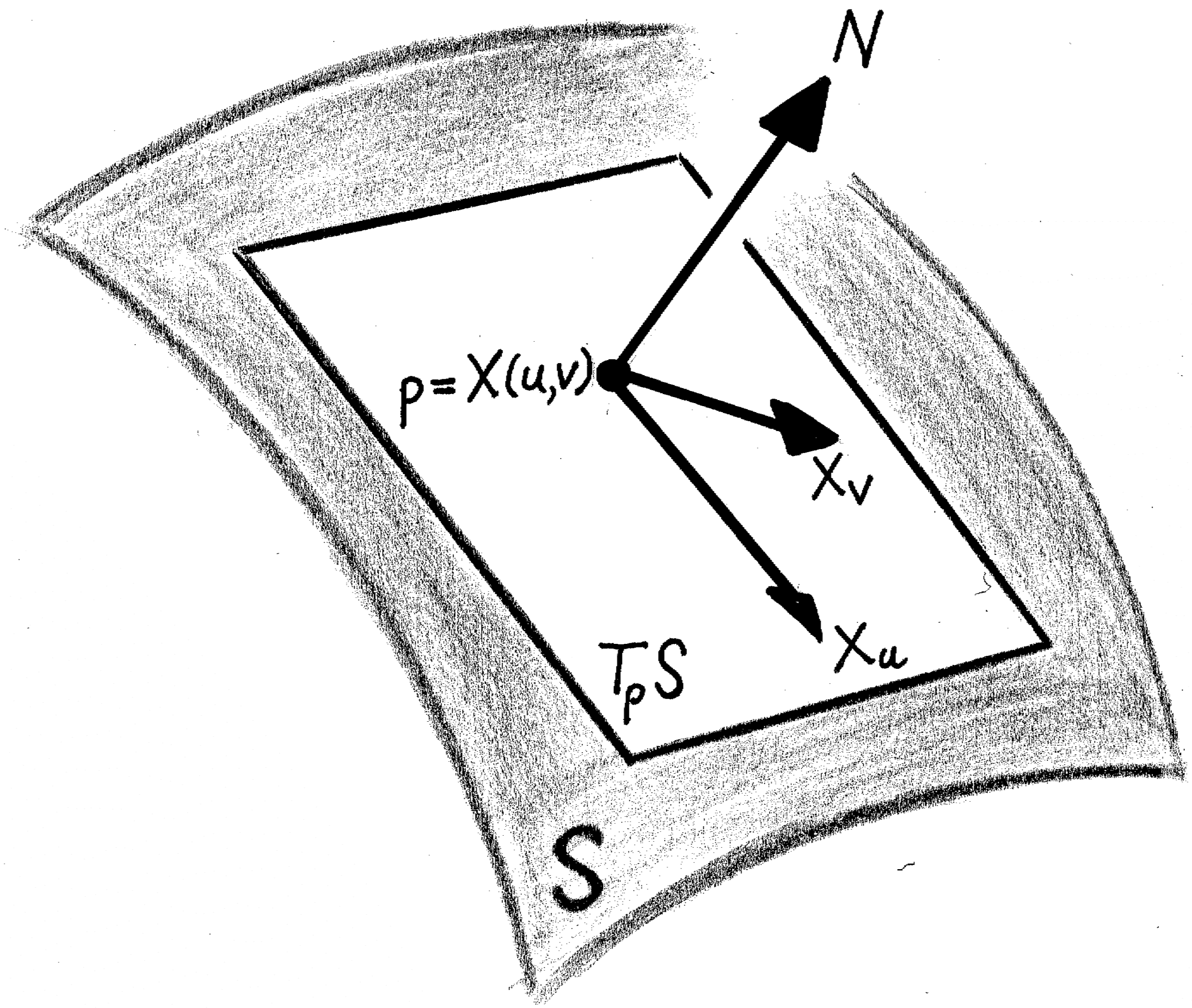
Problem 6. Suppose that the diffeomorphism $\varphi: S \rightarrow S'$ between two regular surfaces is conformal and preserves areas of regions. Show that φ is an isometry.

Set-up.

Let S be a regular, oriented surface in R^3, let U be an open set in R^2 and $X: U \to V \subset S$ a smooth parametrization of the open set V in S.

For each point $p = X(u, v)$ of V, the tangent vectors X_u and X_v form a basis for the tangent space T_pS.

Let $N(u, v)$ denote the unit normal vector to S at the point $p = X(u, v)$. Then the vectors X_u, X_v, N form a basis for R^3.


N
p=X(u,v)
Xv
Xu
TpS
S

Plan.

When we studied curves in 3-space, the Frenet frame T , N , B was defined as long as the curvature of the curve was nonzero.

The rates of change of T , N and B along the curve gave rise to the Frenet equations, involving both the curvature and torsion, and these were fundamental in studying the geometry of the curve.

Now we want to do the analogous thing for the surface S , that is, we want to study the rates of change of the frame X_u , X_v , N along the surface.

The Christoffel symbols.

$$
\begin{array}{l}
X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e\,N \\
X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + f\,N \\
\;\;\| \qquad\quad \| \qquad\qquad\quad \| \\
X_{vu} = \Gamma^1_{21} X_u + \Gamma^2_{21} X_v + f\,N \\
X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + g\,N
\end{array}
$$

$$
\begin{array}{l}
N_u = a_{11} X_u + a_{21} X_v \\
N_v = a_{12} X_u + a_{22} X_v
\end{array}
$$

The coefficients Γ^k_{ij} are called the ***Christoffel symbols*** of the surface S in the parametrization $X: U \to S$.

They are smooth functions of (u, v) which are symmetric relative to the two lower indices i, j .

Recall that the coefficients e , f , g of the second fundamental form of S were defined by

$$e = -\langle N_u , X_u \rangle = \langle N , X_{uu} \rangle$$

$$f = -\langle N_u , X_v \rangle = \langle N , X_{uv} \rangle = \langle N , X_{vu} \rangle = -\langle N_v , X_u \rangle$$

$$g = -\langle N_v , X_v \rangle = \langle N , X_{vv} \rangle$$

and were introduced and discussed in the previous chapter.

Likewise, the functions a_{11} , a_{12} , a_{21} , a_{22} are the entries in the 2×2 matrix expressing the differential $dN_p : T_pS \rightarrow T_pS$ of the Gauss map $N: S \rightarrow S^2$ with respect to the basis X_u and X_v of T_pS .

Example 1. Define $X: R^2 \rightarrow R^2 \subset R^3$ by $X(u, v) = (u, v, 0)$, the simplest parametrization of the xy-plane in R^3. Then

$$X_u \equiv (1, 0, 0)\,, \quad X_v \equiv (0, 1, 0)\,, \quad N = (0, 0, 1)\,.$$

Hence all Christoffel symbols $\Gamma^k_{ij} \equiv 0$.

Likewise for the coefficients of the second fundamental form:

$$e = f = g \equiv 0\,.$$

Likewise for the coefficients a_{ij} of the matrix representing the differential dN_p of the Gauss map $N: S \rightarrow S^2$.

Example 2. Define $X: R^2 \rightarrow R^2 \subset R^3$ by

$X(u, v) = (au + bv , cu + dv , 0)$, with $ad - bc \neq 0$,

a linear parametrization of the xy-plane in R^3 .

Then $X_u = (a , c , 0)$ and $X_v = (b , d , 0)$.

Hence $X_{uu} = X_{uv} = X_{vv} \equiv 0$,
so again all Christoffel symbols $\Gamma^k_{ij} \equiv 0$.

LEMMA. *The Christoffel symbols* Γ^k_{ij} *can be computed in terms of the coefficients* E , F *and* G *of the first fundamental form, and of their derivatives with respect to* u *and* v . *Thus all concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries of the surface.*

Proof. Consider the equations that define the Christoffel symbols.

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e\,N$$
$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + f\,N$$
$$X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + g\,N$$

We omit the equation for X_{vu} since it duplicates the one for X_{uv} .

Problem 7. Show that

$$\langle X_{uu}, X_u \rangle = \tfrac{1}{2} E_u, \qquad \langle X_{uu}, X_v \rangle = F_u - \tfrac{1}{2} E_v$$
$$\langle X_{uv}, X_u \rangle = \tfrac{1}{2} E_v, \qquad \langle X_{uv}, X_v \rangle = \tfrac{1}{2} G_u$$
$$\langle X_{vv}, X_u \rangle = F_v - \tfrac{1}{2} G_u, \qquad \langle X_{vv}, X_v \rangle = \tfrac{1}{2} G_v .$$

Now take the inner product of each of the following equations, first with X_u and then X_v .

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e\,N$$
$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + f\,N$$
$$X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + g\,N$$

Using the result of the preceding problem, we get six equations, grouped into three pairs:

$$
\begin{aligned}
\tfrac{1}{2} E_u &= \Gamma^1_{11} E + \Gamma^2_{11} F \\
F_u - \tfrac{1}{2} E_v &= \Gamma^1_{11} F + \Gamma^2_{11} G
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{1}{2} E_v &= \Gamma^1_{12} E + \Gamma^2_{12} F \\
\tfrac{1}{2} G_u &= \Gamma^1_{12} F + \Gamma^2_{12} G
\end{aligned}
$$

$$
\begin{aligned}
F_v - \tfrac{1}{2} G_u &= \Gamma^1_{22} E + \Gamma^2_{22} F \\
\tfrac{1}{2} G_v &= \Gamma^1_{22} F + \Gamma^2_{22} G
\end{aligned}
$$

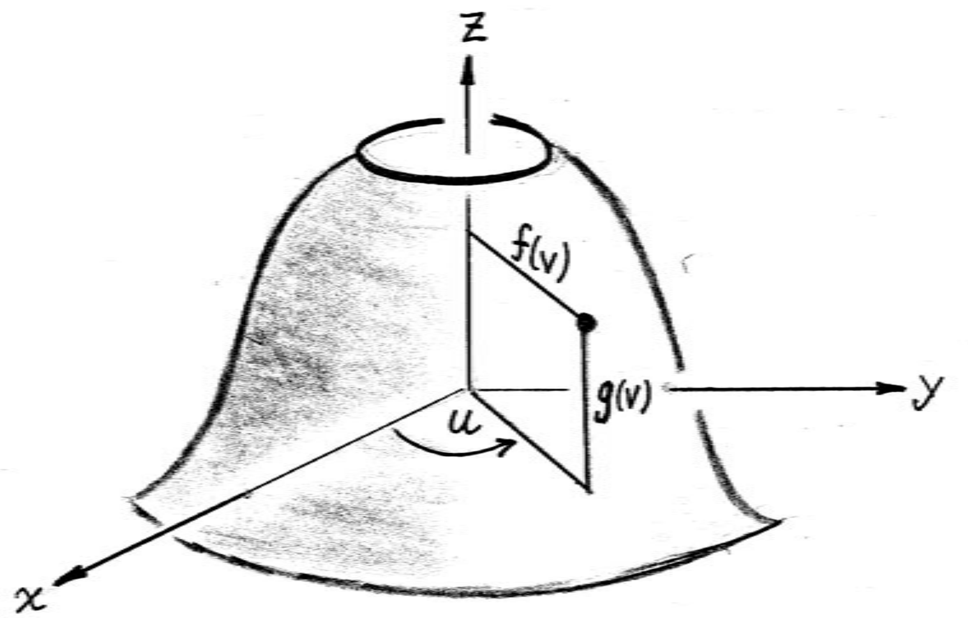
Each pair of equations has $EG - F^2 \neq 0$ as the determinant of its coefficient matrix, and is hence solvable for the unknown Christoffel symbols in terms of E, F and G and their first partials w.r.t. u and v.

This completes the proof of the Lemma.

Example 3. We will compute the Christoffel symbols Γ^k_{ij} for a surface of revolution parametrized by

$$X(u, v) = (f(v) \cos u , f(v) \sin u , g(v))$$

with $f(v) \neq 0$.



$$X(u, v) = (f(v) \cos u ,\ f(v) \sin u ,\ g(v))$$

$$X_u = (- f \sin u ,\ f \cos u ,\ 0)$$

$$X_v = (f' \cos u ,\ f' \sin u ,\ g')$$

$$E = \langle X_u , X_u \rangle = f^2$$

$$F = \langle X_u , X_v \rangle = 0$$

$$G = \langle X_v , X_v \rangle = f'^2 + g'^2$$

$$E = f^2 \qquad F = 0 \qquad G = f'^2 + g'^2$$

$$E_u = 0 \qquad F_u = 0 \qquad G_u = 0$$

$$E_v = 2\, f\, f' \qquad F_v = 0 \qquad G_v = 2\, f'\, f'' + 2\, g'\, g''$$

$$\tfrac{1}{2} E_u = \Gamma^1_{11} E + \Gamma^2_{11} F$$

$$F_u - \tfrac{1}{2} E_v = \Gamma^1_{11} F + \Gamma^2_{11} G$$

$$0 = \Gamma^1_{11} E \qquad \Rightarrow \Gamma^1_{11} = 0$$

$$-f\, f' = \Gamma^2_{11} (f'^2 + g'^2)$$

$$\Rightarrow \Gamma^2_{11} = -f\, f' / (f'^2 + g'^2)$$

$$E = f^2 \qquad F = 0 \qquad G = f'^2 + g'^2$$

$$E_u = 0 \qquad F_u = 0 \qquad G_u = 0$$

$$E_v = 2 f f' \qquad F_v = 0 \qquad G_v = 2 f' f'' + 2 g' g''$$

$$\tfrac{1}{2} E_v = \Gamma^1{}_{12} E + \Gamma^2{}_{12} F$$

$$\tfrac{1}{2} G_u = \Gamma^1{}_{12} F + \Gamma^2{}_{12} G$$

$$f f' = \Gamma^1{}_{12} f^2 \qquad \Rightarrow \Gamma^1{}_{12} = f' / f$$

$$0 = \Gamma^2{}_{12} G \qquad \Rightarrow \Gamma^2{}_{12} = 0$$

$$E = f^2 \qquad F = 0 \qquad G = f'^2 + g'^2$$

$$E_u = 0 \qquad F_u = 0 \qquad G_u = 0$$

$$E_v = 2 f f' \qquad F_v = 0 \qquad G_v = 2 f' f'' + 2 g' g''$$

$$F_v - \tfrac{1}{2} G_u = \Gamma^1_{22} E + \Gamma^2_{22} F$$

$$\tfrac{1}{2} G_v = \Gamma^1_{22} F + \Gamma^2_{22} G$$

$$0 = \Gamma^1_{22} E \qquad \Rightarrow \Gamma^1_{22} = 0$$

$$f' f'' + g' g'' = \Gamma^2_{22} (f'^2 + g'^2)$$

$$\Rightarrow \Gamma^2_{22} = (f' f'' + g' g'') / (f'^2 + g'^2)$$

Summary so far.

Given a parametrization $X : U \to S \subset R^3$ of an open subset of a regular surface S in R^3, we have the following associated functions of u and v.

- The coefficients E, F and G of the first fundamental form, which express the "intrinsic geometry" of the surface.

- The Christoffel symbols Γ^k_{ij}, as defined on page 11.

Earlier, we wrote six equations which show how the Christoffel symbols may be computed from knowledge of the coefficients E, F and G of the first fundamental form, and of their first partial derivatives w.r.t. u and v.

• The coefficients e , f and g of the second fundamental form, which express how the surface is bent and curved in 3-space.

• The entries a_{11} , a_{12} , a_{21} , a_{22} of the matrix of the linear map $dN_p: T_pS \rightarrow T_pS$ which is the differential of the Gauss map $N: S \rightarrow S^2$.

The Weingarten equations from the previous chapter express the a_{ij} in terms of E , F , G , e , f , g :

$$a_{11} = (f F - e G) / (EG - F^2)$$
$$a_{12} = (g F - f G) / (EG - F^2)$$
$$a_{21} = (e F - f E) / (EG - F^2)$$
$$a_{22} = (f F - g E) / (EG - F^2) .$$

Thus the coefficients E, F, G, e, f, g of the first and second fundamental forms seem to be the basic pieces of information, in terms of which we can compute the Christoffel symbols Γ^k_{ij} and the entries a_{ij} in the matrix expressing the differential of the Gauss map.

The coefficients E, F, G remind us of the speed along a curve, while the coefficients e, f, g remind us of the curvature and torsion of a curve.

Question. *To what extent are the six functions*

E(u, v) , F(u, v) , G(u,v) , e(u, v) , f(u, v) , g(u, v)

independent, and to what extent are they constrained by some inter-relations?

Recall the formulas

$$
\begin{aligned}
(1) \qquad X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e\,N \\
X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + f\,N \\
X_{vv} &= \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + g\,N \\
N_u &= a_{11} X_u + a_{21} X_v \\
N_v &= a_{12} X_u + a_{22} X_v \,.
\end{aligned}
$$

To obtain relations among the above coefficients, consider the expressions

(2)
$$(X_{uu})_v - (X_{uv})_u = 0$$
$$(X_{vv})_u - (X_{uv})_v = 0$$
$$(N_u)_v - (N_v)_u = 0 .$$

Using the equations (1) , the relations (2) can be written as

(2a)
$$A_1 X_u + B_1 X_v + C_1 N = 0$$
$$A_2 X_u + B_2 X_v + C_2 N = 0$$
$$A_3 X_u + B_3 X_v + C_3 N = 0 .$$

The coefficients A_i , B_i and C_i , $i = 1, 2, 3$, can be written initially in terms of the coefficients of the first and second fundamental forms, the Christoffel symbols Γ^k_{ij} and the entries a_{ij} of dN_p , and their first derivatives with respect to u and v . But ultimately, they are just functions of E , F , G , e , f , g and their first derivatives with respect to u and v .

Since the vectors X_u , X_v and N are linearly independent, the three vector equations in (2a) give us nine scalar equations,

$$A_i = 0 , \quad B_i = 0 , \quad C_i = 0 \qquad \text{for } i = 1, 2, 3.$$

As an important example, we will consider in detail the equation $B_2 = 0$, which comes from the equation $(X_{uu})_v - (X_{uv})_u = 0$ by setting the coefficient of X_v equal to 0.

We start with the equation

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e N ,$$

and differentiate it with respect to v, getting

$$\begin{aligned}(X_{uu})_v = \Gamma^1_{11,v} X_u &+ \Gamma^1_{11} X_{uv} \\ &+ \Gamma^2_{11,v} X_v + \Gamma^2_{11} X_{vv} \\ &\qquad + e_v N + e N_v .\end{aligned}$$

We are using the notation $\Gamma^1_{11,v}$ for $(\Gamma^1_{11})_v$.

If we insert the values for X_{uv} , X_{vv} and N_v from equations (1) , we get

$$
\begin{aligned}
(X_{uu})_v = \; & (\Gamma^1_{11,v} + \Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + e\, a_{12})\, X_u \\
& + (\Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11,v} + \Gamma^2_{11}\Gamma^2_{22} + e\, a_{22})\, X_v \\
& + (\Gamma^1_{11}\, f + \Gamma^2_{11}\, g + e_v)\, N .
\end{aligned}
$$

Likewise we get

$$
\begin{aligned}
(X_{uv})_u = \; & (\Gamma^1_{12,u} + \Gamma^1_{12}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^1_{12} + f\, a_{11})\, X_u \\
& + (\Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12,u} + \Gamma^2_{12}\Gamma^2_{12} + f\, a_{21})\, X_v \\
& + (\Gamma^1_{12}\, e + \Gamma^2_{12}\, f + f_u)\, N .
\end{aligned}
$$

Now equate the coefficients of X_v in these two expressions for $(X_{uu})_v$ and $(X_{uv})_u$, getting

$$(3) \quad e\, a_{22} - f\, a_{21} = \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12,u} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11,v} - \Gamma^2_{11}\Gamma^2_{22} .$$

The right hand side of (3) depends only on the coefficients of the first fundamental form and their first partials with respect to u and v.

Consider the left hand side:

$$e\,a_{22} - f\,a_{21} = e\,(f\,F - g\,E) / (EG - F^2)$$
$$- f\,(e\,F - f\,E) / (EG - F^2)$$
$$= (-e\,g\,E + f^2\,E) / (EG - F^2)$$
$$= -E\,(e\,g - f^2) / (EG - F^2)$$
$$= -E\,K\,,$$

where $K = (e\,g - f^2) / (EG - F^2)$ is the Gaussian curvature. Thus

$$(4) \quad -E\,K = \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12,u} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11,v} - \Gamma^2_{11}\Gamma^2_{22}\,.$$

Dividing through by $-E$ gives the ***Gauss formula*** for the Gaussian curvature K .

THEOREMA EGREGIUM (Gauss) . ***The Gaussian curvature*** **K** ***of a surface in 3-space depends only on the first fundamental form, and hence only on the intrinsic geometry of the surface.***

In other words, two surfaces which are locally isometric, such as the catenoid and helicoid, have the same Gaussian curvature at corresponding points.

Problem 8. (a) Consider the equation $A_1 = 0$, which comes from the equation $(X_{uu})_v - (X_{uv})_u = 0$ by setting the coefficient of X_u equal to 0. Show that this yields

$$\Gamma^1_{12,u} - \Gamma^1_{11,v} + \Gamma^2_{12}\Gamma^1_{12} - \Gamma^2_{11}\Gamma^1_{22} = F\,K .$$

If $F \neq 0$, this also shows that the Gaussian curvature depends only on the first fundamental form.

(b) Consider the equation $C_1 = 0$, which comes from the equation $(X_{uu})_v - (X_{uv})_u = 0$ by setting the coefficient of N equal to 0. Show that this yields

$$(5) \qquad e_v - f_u = e\,\Gamma^1_{12} + f\,(\Gamma^2_{12} - \Gamma^1_{11}) - g\,\Gamma^2_{11} .$$

This is one of the two ***Mainardi-Codazzi equations***.

(c) Consider the equations $A_2 = 0$ and $B_2 = 0$, which come from the equation $(X_{vv})_u - (X_{uv})_v = 0$ by setting the coefficients of X_u and X_v, respectively, equal to 0. Show that both of these equations again give the Gauss Formula for the Gaussian curvature K.

(d) Consider the equation $C_2 = 0$, which comes from the equation $(X_{vv})_u - (X_{uv})_v = 0$ by setting the coef-ficient of N equal to 0. Show that this yields

$$(6) \qquad f_v - g_u = e\,\Gamma^1_{22} + f\,(\Gamma^2_{22} - \Gamma^1_{12}) - g\,\Gamma^2_{12}\,.$$

This is the second of the two Mainardi-Codazzi equations.

(e) Do the same for the coefficients A_3, B_3 and C_3 of the third equation $(N_u)_v = (N_v)_u$. Show that the equations $A_3 = 0$ and $B_3 = 0$ yield again the two Mainardi-Codazzi equations, and that the equation $C_3 = 0$ is an identity.

The Gauss equation and the two Mainardi-Codazzi equations are known as the ***compatibility equations*** of the theory of surfaces.

Problem 9. Is there a regular surface $X: U \to R^3$ with $E \equiv 1,\ F \equiv 0,\ G \equiv 1,\ e \equiv 1,\ f \equiv 0,\ g \equiv -1$?

(a) Show that such a surface would have Gaussian curvature $\equiv -1$.

(b) Show that such a surface would have all Christoffel symbols $\Gamma^k_{ij} \equiv 0$.

(c) Show that such a surface would violate the Gauss formula.

Conclude that no such surface exists.

Problem 10. Is there a regular surface $X: U \rightarrow R^3$ with $E \equiv 1$, $F \equiv 0$, $G = \cos^2 u$, $e = \cos^2 u$, $f \equiv 0$, $g \equiv 1$?

(a) Show that such a surface would have Gaussian curvature $K \equiv 1$.

(b) Compute the Christoffel symbols for such a surface.

(c) Show that such a surface would satisfy the Gauss formula.

(d) Show that such a surface would satisfy the first Mainard-Codazzi equation, but violate the second one.

Conclude that no such surface exists.

Problem 11. In this problem, we will see how the Mainardi-Codazzi equations simplify when the coordinate neighborhood contains no umbilical points and the coordinate curves are lines of curvature ($F = 0$ and $f = 0$) .

(a) Show that in such a case, the Mainardi-Codazzi equations may be written as

$$e_v = e\,\Gamma^1_{12} - g\,\Gamma^2_{11} \quad \text{and} \quad g_u = g\,\Gamma^2_{12} - e\,\Gamma^1_{22}\,.$$

(b) Show also that

$$\Gamma^2_{11} = -E_v / 2G \qquad \Gamma^1_{12} = E_v / 2E$$
$$\Gamma^1_{22} = -G_u / 2E \qquad \Gamma^2_{12} = G_u / 2G\,.$$

(c) Conclude that the Mainardi-Codazzi equations take the following form:

$$e_v = \tfrac{1}{2} E_v\,(e/E + g/G) \quad \text{and} \quad g_u = \tfrac{1}{2} G_u\,(e/E + g/G)\,.$$

The fundamental theorem of the theory of surfaces.

It is natural to ask if there exist any further relations of compatibility between the first and second fundamental forms of a regular surface in 3-space, besides the Gauss formula (4) and the Mainardi-Codazzi equations (5, 6) .

THEOREM (Bonnet). *Let* $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ *be smooth functions defined on an open set* $\mathbf{U} \subset \mathbf{R}^2$, *with* $\mathbf{E} > 0$, $\mathbf{F} > 0$ *and* $\mathbf{EG} - \mathbf{F}^2 > 0$.

Suppose in addition that these functions satisfy the Gauss formula **(4)** *and the Mainardi-Codazzi equations* **(5, 6)** .

Then each point of $\mathbf{U}$ *has an open neighborhood* $\mathbf{U_0} \subset \mathbf{U}$ *for which there is a regular surface* $\mathbf{X}: \mathbf{U_0} \to \mathbf{R}^3$ *having the given functions* $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ *as coefficients of the first and second fundamental forms.*

Furthermore, if $\mathbf{U}$ *is a connected open set in* $\mathbf{R}^2$ *and* $\mathbf{X}$ *and* $\mathbf{Y}: \mathbf{U} \to \mathbf{R}^3$ *are regular surfaces having the same first and second fundamental forms, then there is a rigid motion (translation plus rotation) of* $\mathbf{R}^3$ *taking one surface to the other.*

Proof of Bonnet's Theorem.

Step 1. We are given the six smooth real-valued functions

$$E(u,v)\ ,\ F(u,v)\ ,\ G(u,v)\ ,\ e(u,v)\ ,\ f(u,v)\ ,\ g(u,v)\ ,$$

defined on the open set $U \subset R^2$, such that

$$E > 0\ ,\ G > 0\ ,\ EG - F^2 > 0\ ,$$

and satisfying the one Gauss equation (4) and the two Mainardi-Codazzi equations (5, 6) .

Using the functions E , F and G , we calculate the Christoffel symbols Γ^k_{ij} from the six equations on p. 11, just as we did on a surface. Usng E , F , G , e , f , g , we calculate the functions a_{11} , a_{12} , a_{21} and a_{22} from the Weingarten formulas (see p. 58 of section 6):

$$a_{11} = (f F - e G) / (EG - F^2)$$
$$a_{12} = (g F - f G) / (EG - F^2)$$
$$a_{21} = (e F - f E) / (EG - F^2)$$
$$a_{22} = (f F - g E) / (EG - F^2) .$$

In this step we will find the three vectors in R^3 ,

$$X_u(u, v) , \ X_v(u, v) \text{ and } N(u, v) .$$

To do this, we regard the equations

$$
(7) \qquad
\begin{aligned}
X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e\,N \\
X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + f\,N \\
\| \quad & \quad \| \qquad\qquad \| \\
X_{vu} &= \Gamma^1_{21} X_u + \Gamma^2_{21} X_v + f\,N \\
X_{vv} &= \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + g\,N
\end{aligned}
$$

$$
\begin{aligned}
N_u &= a_{11} X_u + a_{21} X_v \\
N_v &= a_{12} X_u + a_{22} X_v
\end{aligned}
$$

as telling us the partial derivatives of the desired vectors X_u , X_v and N with respect to u and v in terms of the preassigned functions Γ^k_{ij} , e , f , g and a_{ij} , and in terms of the vectors X_u , X_v and N themselves.

Standard PDE theory says that we can solve such a system *locally*, meaning in some unspecified open set about any point (u_0, v_0) in U, provided we have equality of mixed partials:

$$(X_u)_{uv} = (X_u)_{vu}\,, \quad (X_v)_{uv} = (X_v)_{vu} \quad \text{and} \quad N_{uv} = N_{vu}\,.$$

And these equations, we saw earlier, are precisely the Gauss and Mainardi-Codazzi equations.

So the conclusion of Step 1 is that, in some unspecified open set about any point $(u_0\,, v_0)$ in U, there are vector functions $X_u(u, v)$, $X_v(u, v)$ and $N(u, v)$ which satisfy the equations in (7) above.

It is important to emphasize at this point that the notation X_u , X_v and N for these three vector functions can be a bit misleading: we don't yet know that X_u and X_v are partials w.r.t. u and v of some vector function X , we don't yet know that N is a unit vector, and we don't yet know that N is orthogonal to X_u and X_v .

For this reason, it will be convenient to temporarily replace the names X_u , X_v and N by A , B and C .

So at this point, we have three vector functions

$$A(u, v)\ ,\ \ B(u, v)\ \text{ and }\ C(u, v)\ ,$$

defined in an open set about a chosen point $(u_0, v_0) \in U$, such that

$$
\begin{aligned}
(7') \qquad A_u &= \Gamma^1_{11} A + \Gamma^2_{11} B + e\, C \\
A_v &= \Gamma^1_{12} A + \Gamma^2_{12} B + f\, C \\
B_u &= \Gamma^1_{21} A + \Gamma^2_{21} B + f\, C \\
B_v &= \Gamma^1_{22} A + \Gamma^2_{22} B + g\, C \\
C_u &= a_{11}\, A + a_{21}\, B \\
C_v &= a_{12}\, A + a_{22}\, B
\end{aligned}
$$

This concludes Step 1.

Step 2. Here we settle on the initial values

$$A(u_0, v_0)\ ,\ \ B(u_0, v_0)\ \text{ and }\ C(u_0, v_0)$$

of the vectors A , B and C . We choose these so that

$$\begin{aligned} < A(u_0, v_0)\ , A(u_0, v_0) > &= E(u_0, v_0)\ , \\ < A(u_0, v_0)\ , B(u_0, v_0) > &= F(u_0, v_0)\ , \\ < B(u_0, v_0)\ , B(u_0, v_0) > &= G(u_0, v_0)\ . \end{aligned}$$

The possibility of doing this is guaranteed by the conditions

$$E > 0\ ,\ \ G > 0 \ \text{ and }\ EG - F^2 > 0\ .$$

Then we choose the vector $C(u_0, v_0)$ so that

$$< C(u_0, v_0) , C(u_0, v_0) > \ = \ 1 ,$$

$$< C(u_0, v_0) , A(u_0, v_0) > \ = \ 0 ,$$

$$< C(u_0, v_0) , B(u_0, v_0) > \ = \ 0 .$$

Step 3. Next we find the surface $X: U_0 \rightarrow R^3$, where U_0 is some open set about (u_0, v_0) in U .

Consider the system of PDEs,

$$X_u = A \quad \text{and} \quad X_v = B .$$

Note that

$$(X_u)_v = A_v = B_u = (X_v)_u ,$$

by the form of equations (7') , since $\Gamma^k_{12} = \Gamma^k_{21}$.

So again by the basic theory of first order PDEs, there is a solution of this system in some open set U_0 about (u_0, v_0) .

The initial value $X(u_0, v_0)$ is irrelevant, and we can choose it to be, for example, the origin in R^3 .

Now we have our proposed surface $X\colon U_0 \rightarrow R^3$.

Step 4. There's a lot that we still do not know.

- We do not yet know that E , F and G are the coefficients of the first fundamental form of our surface.

- We do not yet know that N is the unit normal vector field to our surface.

- We do not yet know that e , f and g are the coefficients of the second fundamental form of our surface.

For the first two items above, we already know that

$$< A, A > \ = \ E , \quad < A , B > \ = \ F , \quad < B, B > \ = \ G$$

$$< C, C > \ = \ 1 , \quad < C, A > \ = \ 0 , \quad < C, B > \ = \ 0$$

at the single point (u_0, v_0) , and must show that these six equations hold throughout the domain U_0 of our surface.

We will show this is true by invoking a uniqueness theorem for solutions of PDEs, as follows.

We already have the vector functions A , B and C defined throughout our open set U_0 , and therefore also the six real-valued functions

$\langle A, A\rangle$, $\langle A, B\rangle$, $\langle B, B\rangle$, $\langle C, C\rangle$, $\langle C, A\rangle$, $\langle C, B\rangle$.

Let us find the twelve first order PDEs satisfied by these six functions. For example,

$$
\begin{aligned}
\langle A, A\rangle_u &= 2\langle A, A_u\rangle \\
&= 2\langle A, \Gamma^1{}_{11} A + \Gamma^2{}_{11} B + e\, C\rangle \\
&= 2\,\Gamma^1{}_{11}\langle A, A\rangle + 2\,\Gamma^2{}_{11}\langle A, B\rangle + 2\,e\,\langle A, C\rangle ,
\end{aligned}
$$

and eleven more first order PDEs like this.

But we already know that

$$E_u = 2\,\Gamma^1_{11}\,E + 2\,\Gamma^2_{11}\,F + 2\,e\,0\,,$$

because this was one of the equations used to define the Christoffel symbols in terms of the coefficients of the first fundamental form and their partials w.r.t. u and v .

Continuing in this way, we see that the twelve first order PDEs satisfied by the six functions

<A, A> , <A, B> , <B, B> , <C, C> , <C, A> , <C, B>

are also satisfied by the six functions

$$E\,,\ F\,,\ G\,,\ 1\,,\ 0\,,\ 0\,.$$

Since the first six functions agree with the second six functions at the point (u_0, v_0) and satisfy the same PDEs, they must be equal throughout the open set U_0 .

Thus our surface $X: U_0 \rightarrow R^3$ has the given functions E , F and G as coefficients of its first fundamental form, and the vector field C as its unit normal N throughout the parameter domain U_0

Then the equations

$$X_{uu} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + e\,N$$
$$X_{uv} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + f\,N$$
$$X_{vv} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + g\,N$$

tell us that e , f and g are the coefficients of the second fundamental form of our surface throughout U_0 .

This completes the proof of the local existence of a surface in 3-space with prescribed first and second fundamental forms, subject to the compatibility conditions of Gauss and Mainardi-Codazzi.

Step 5. To complete the proof of Bonnet's Theorem, we need to show that if the domain U of our surface is connected, then, up to translations and rotations in R^3, there is only one surface with prescribed first and second fundamental forms.

By translation and rotation, we can move either surface so that at a given point $(u_0, v_0) \in U$ we have the same location $X(u_0, v_0)$ and the same first partials $X_u(u_0, v_0)$ and $X_v(u_0, v_0)$ and the same unit normal $N(u_0, v_0)$.

Then by the uniqueness theorems already quoted, the frames X_u , X_v and N will agree for both surfaces in a neighborhood of (u_0, v_0) . Then by integration, the positions X for both surfaces will also agree throughout this neighborhood, since they agreed at (u_0, v_0) .

By this argument, it follows that the set of points at which the two surfaces agree is an open subset of U .

It is, by continuity, also a closed subset of U .

Since U is connected, it is all of U , and hence the two surfaces coincide, as claimed, completing the proof of Bonnet's Theorem.