Math 542 Complex Variables I

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The following notes are for the course **Math 542 Complex Variables I**, taught during the Fall 2019 semester by Mehmet Erdogan. If you find any errors in these notes, feel free to contact me at `altizio2@illinois.edu`.

Contents

1 August 26

Today basically serves as review of material that should be familiar to everyone.

1.1 An Introduction to the Complex Numbers

We first begin with a definition.

Definition 1.1. Let $\mathbb{R}$ denote the set of real numbers.

1. We denote the set of *complex numbers* $\mathbb{C}$ as
$$\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\},$$
where here $i^2 = -1$.
2. Given a complex number $z = x + yi$, x is its *real part*, denoted $x = \Re(z)$, while y is its *imaginary part*, denoted $y = \Im(z)$.

Observe that we can add two complex numbers via
$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$
We can also multiply two complex numbers via the string of manipulations
$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + ix_1y_2 + ixy_1x_2 + i^2y_1y_2\\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).\end{aligned}$$

We now give a few simple definitions pertaining to complex numbers.

Definition 1.2. Let $z = x + iy$ be a complex number.

1. We define the *conjugate* of z, as $\bar{z} := x - iy$.
2. We define the *absolute value* of z as
$$|z| := \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

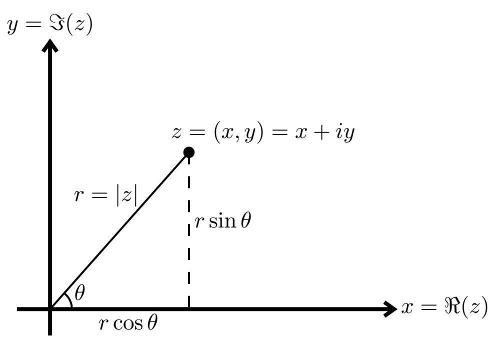
One important property of the absolute value is the *triangle inequality*, which states that for any complex numbers z_1 and z_2 we have
$$|z_1 + z_2| \leq |z_1| + |z_2|.$$
The proof is left as an exercise.

With these definitions, one can check that $\mathbb{C}$ is a field! This is basically an exercise, but let's check inverses: indeed, if $z \neq 0$, we have
$$z \cdot \frac{\bar{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1.$$

1.2 Polar Coordinates

It is useful to identify $\mathbb{C}$ with $\mathbb{R}^2$ by associating each complex number $z = x + yi$ with the ordered pair $(x, y) \in \mathbb{R}^2$. Then we can add and multiply two vectors by
$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad \text{and} \qquad (x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$
In this way, complex number addition becomes vector addition. However, complex number multiplication is somewhat less natural to describe.

In this regard, it is useful to think about complex numbers in *polar form*. In the diagram to the right, it is easy to see that $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$ and θ is the angle the vector (x, y) makes with the x-axis. Thus, we can rewrite z as
$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) =: re^{i\theta}.$$
This representation is unique assuming $r \neq 0$ and modulo 2π in θ.



While we have an appropriate definition for r, we do not have such a definition for θ. This leads into the next definition.

Definition 1.3. We set θ in the above diagram to be the *argument* of z. In particular, given a complex number z, we set $\operatorname{Arg}(z) = \theta \in (-\pi, \pi]$. This is the *principal branch* of the argument function.

This approach to exploring complex numbers is quite versatile. For example, one can check (using the addition formulas for sine and cosine) that $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ for any real numbers θ_1 and θ_2. Therefore any two complex numbers z_1 and z_2 satisfy

$$z_1 z_2 = |z_1| \cdot |z_2| \, e^{i(\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2))}.$$

This tells us that $|z_1 z_2| = |z_1| \cdot |z_2|$. More generally, though, multiplication by z consists of two parts: a dilation by $|z|$ and a rotation by $\operatorname{Arg}(z)$.

One thing worth pointing out is that we can think of multiplication by z as a *map* acting on $\mathbb{C}$. In this way, the map $w \mapsto zw$ preserves shapes of sets via dilation and rotation.

1.3 Matrix Definition of Complex Numbers

There is a third definition of complex numbers that is worth mentioning. We can identify $\mathbb{C}$ as

$$\mathbb{C} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

The crucial aspect to this definition is that addition and multiplication in $\mathbb{C}$ *naturally* turns into matrix addition and multiplication. Furthermore, taking the magnitude of a complex number corresponds to taking the determinant of the associated matrix, while complex conjugation amounts to transposition of the original matrix. In turn, we can write

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \sqrt{x^2+y^2} \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{pmatrix},$$

which decomposes the matrix into its dilation and rotation components.

Before we move on, here is a quick remark.

Remark. Although multiplication of complex numbers corresponds to addition of the arguments, it is *not* true that $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ - this is because the right hand side of the previous equality is not necessarily a number in the interval $(-\pi, \pi]$. It *is* true, however, that the two sides are congruent modulo 2π.

1.4 nth Roots of Complex Numbers

Recall that we define

$$z^n := \underbrace{z \cdot z \cdot \ldots \cdot z}_{n \text{ times}} = |z|^n \, e^{ni \operatorname{Arg}(z)}.$$

It is natural to ask the following question: given $w \neq 0$ and $n \in \mathbb{N}$, can we find some (all) z such that $z^n = w$?

Here is how we can do it. Write $w = re^{i\theta}$. Then $r = |z|^n$ and $n \operatorname{Arg}(z) \equiv \theta \pmod{2\pi}$. This means that

$$\begin{aligned} \{n^{\text{th}} \text{ roots of } w\} &= \left\{ r^{1/n} e^{i\phi} : n\phi \equiv \theta \pmod{2\pi} \right\} \\ &= \left\{ r^{1/n} e^{i(\theta+2\pi k)/n} : 0 \leq k \leq n-1 \right\}. \end{aligned}$$

These n roots plot to a regular n-gon on the complex plane.

1.5 $\mathbb{C}$ as a metric space

Because of the Triangle Inequality, the function $d(\cdot,\cdot)$ defined by $d(z_1, z_2) = |z_1 - z_2|$ turns $\mathbb{C}$ into a metric space. This metric agrees with the usual Euclidean notion of distance, i.e. $\mathbb{C}$ is isometric to $\mathbb{R}^2$ in the obvious way. This allows us to port over lots of concepts from real analysis into complex analysis, which will be very useful.

First up is the notion of a ball.

Definition 1.4. Let z_0 be a complex number and $r > 0$.

1. The *open ball* centered at z_0 with radius r is defined as
$$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$
2. The *closed ball* centered at z_0 with radius r is defined as
$$\overline{B(z_0, r)} := \{z \in \mathbb{C} : |z - z_0| \le r\}.$$
3. The *punctured open ball* centered at z_0 with radius r is defined as
$$B^*(z_0, r) := \{z \in \mathbb{C} : 0 < |z - z_0| < r\}.$$

With the notion of open and closed balls comes open and closed sets.

Definition 1.5. As usual let $\mathbb{C}$ denote the complex numbers.

1. A set $O \subset \mathbb{C}$ is *open* if for all $z_0 \in O$ there exists some $r > 0$ such that $B(z_0, r) \subset O$.
2. A set $F \subset \mathbb{C}$ is *closed* if its complement F^c is open.

We also have some definitions relating to convergent sequences.

Definition 1.6. Let $\bar{z} = \{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers.

1. We say that $\bar{z}$ is *convergent*, converging to some $z_0 \in \mathbb{C}$ (written $z_n \to z_0$ or $\lim_{n\to\infty} z_n = z_0$) if $\lim_{n\to\infty} |z_n - z_0| = 0$. Equivalently, for all $\varepsilon > 0$ there exists $N > 0$ such that
$$|z_n - z_0| < \varepsilon \text{ for every } n \ge N.$$
2. We say $\bar{z}$ is *Cauchy* if for all $\varepsilon > 0$ there exists $N > 0$ such that
$$|z_m - z_n| < \varepsilon \text{ for all } n, m > N.$$

Recall that since $\mathbb{R}^2$ is so-called *complete*, $\mathbb{C}$ is complete as well. This means that Cauchyness and convergence are the exact same thing.

We also have the notion of *compact sets*.

Definition 1.7. A set $K \subset \mathbb{C}$ is *compact* if every open cover has a finite subcover.

Equivalently (in the case of $\mathbb{C}$, that is), every sequence $\{z_n\}_{n=1}^{\infty} \subseteq K$ has a subsequence which converges to some $z \in K$.

Equivalently yet again (in the case of $\mathbb{C}$ that is), K is closed and bounded.

Finally, we can extend these notions of sequences to *functions*.

Definition 1.8. Fix some set $A \subset \mathbb{C}$, and let $f : A \to \mathbb{C}$ be a function. Further let z_0 be a limit point of A.

1. We say that $\lim_{z\to z_0} f(z) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that
$$f(z) \in B(L, \varepsilon) \text{ whenever } z \in B^*(z_0, \delta) \cap A.$$
Equivalently, whenever $\{z_n\}_{n=1}^{\infty} \subseteq A \setminus \{z_0\}$ is a sequence of complex numbers converging to z_0, the sequence $\{f(z_n)\}_{n=1}^{\infty}$ converges to L.
2. We say that f is *continuous* at $z_0 \in A$ if $\lim_{z\to z_0} f(z) = f(z_0)$. (Intuitively, f agrees with its limit at z_0.)

Remark. It is often useful to write $f(z) = u(x, y) + iv(x, y)$ as the "sum" of two real-valued functions u and v defined on the real and imaginary parts of z; we write $u = \Re(f)$ and $v = \Im(f)$ to denote this. Then a theorem from Real Analysis says that f is continuous at z_0 if and only if u and v are continuous at z_0.

2 August 28

We continue review of previous analysis material before transitioning to results in complex analysis.

2.1 Connected Sets

The notion of connectivity also carries over to $\mathbb{C}$.

Definition 2.1. Let $A \subset \mathbb{C}$.

1. We say that A is *connected* if the following holds: whenever U and V are disjoint open sets such that $A \subset U \cup V$, either $A \subset U$ or $A \subset V$.
2. We say that A is *path-connected* if for all w and z in A there exists some continuous function $\gamma : [0, 1] \to A$ such that $\gamma(0) = w$ and $\gamma(1) = z$.

In $\mathbb{R}$, both connectedness and path-connectedness are equivalent to A being an interval. In $\mathbb{C}$, this is not generally true, and connectedness is not necessarily equal to path-connectedness. However, in complex analysis we'll often be working with *domains*, and in this case we have a different story.

Definition 2.2. A set $A \subset \mathbb{C}$ is a *domain* if it is nonempty, open, and connected.

It turns out that connected always implies path-connected. While path-connected does not always imply connected, it does in the case that A is a domain.

Theorem 2.3. *If A is a domain in $\mathbb{C}$, then A is path-connected.*

Proof. For a domain $A \subset \mathbb{C}$, define an *equivalence relation* as follows: we say two points $w, z \in A$ are equivalent if they can be connected by a path. It's not hard to check that this is actually an equivalence relation.

Observe that equivalence classes are clearly disjoint. Also, each equivalence class is *open*. Indeed, for each $z \in A$, there exists $r > 0$ such that $B(z, r) \subset A$, and then every $w \in B(z, r)$ is equivalent to z. So there's only one equivalence class since A is connected. □

2.2 Extended Complex Plane and Stereographic Projection

For a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$, we can define $z_n \to \infty$ if $|z_n| \to \infty$ as a real-valued sequence. With this, every sequence in $\mathbb{C}$ has a convergent subsequence, either to ∞ or to some $z_0 \in \mathbb{C}$.

Thus we can let $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the *extended complex plane* (also called the 1-point compactification of $\mathbb{C}$), and in this case the topology consisting of all open sets of $\mathbb{C}$ and those sets of the form $\{|z| > r\} \cup \infty$ turns $\hat{\mathbb{C}}$ into a compact topological space.

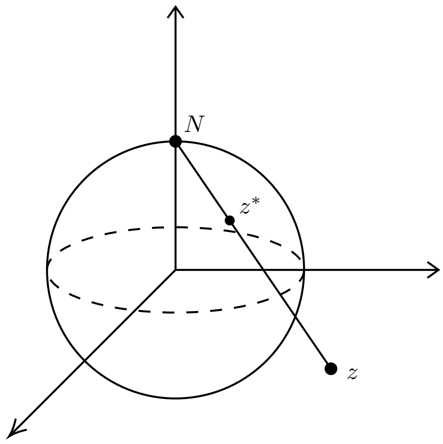
We can define arithmetic on $\hat{\mathbb{C}}$, and it carries over for the most part. If $z \neq \infty$, then

$$\infty \pm z = z \pm \infty = \infty \quad \text{and} \quad \frac{z}{\infty} = 0;$$

analogously, if $z \neq 0$ then

$$z \cdot \infty = \infty \cdot z = \infty \quad \text{and} \quad \frac{z}{0} = \infty.$$

We now seek to turn $\hat{\mathbb{C}}$ into a metric space; to do this, we will need the concept of *stereographic projection*. Identify $\mathbb{C} = \{(x, y, 0) : x, y \in \mathbb{R}\} \subset \mathbb{R}^3$. For $z \in \mathbb{C}$, consider the line starting at the point $N = (0, 0, 1)$ and passing through z. Then we define z^* to be the second intersection point of the unit sphere with this ray.



The upside of this transformation is that $\infty^* = N$. This means that $\hat{\mathbb{C}}$ is identified with $\mathbb{S}^2$.

Actually, we can say something stronger: for $z = (x, y, 0) \in \mathbb{R}^3$, we have

$$z^* = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right) = \frac{(2\Re(z), 2\Im(z), |z|^2-1)}{|z|^2+1}.$$

Hence we may define a metric on $\hat{\mathbb{C}}$ via e.g. coordinate distance in $\mathbb{S}^2$. This metric may seem contrived, but it actually has good properties.

Proposition 2.4. *Circles and lines in $\hat{\mathbb{C}}$ correspond to circles in $\mathbb{S}^2$.*

Proof. What follows is a sketch. Let $\mathcal{P} \subset \mathbb{R}^3$ be a plane, so that

$$\mathcal{P} = \{x \in \mathbb{R}^3 : Ax_1 + Bx_2 + Cx_3 = D\}$$

for some constants A, B, C, and D. So $z^* \in \mathcal{P}$ if and only if

$$2xA + 2yB + C(x^2+y^2-1) = D(x^2+y^2+1),$$

or

$$(C-D)(x^2+y^2) + 2Ax + 2By = D + C.$$

This is a line if $C = D$ and a circle otherwise. $\square$

Corollary 2.5. *The topology in $\hat{\mathbb{C}}$ is induced by the topology in $\mathbb{S}^2$.*

2.3 Analytic Functions

We now turn to the topic of differentiability in $\mathbb{C}$. We begin with a definition.

Definition 2.6. Let f be a complex function defined in the neighborhood of some $z_0 \in \mathbb{C}$. We say that f is *differentiable* at z_0 if the limit

$$L \coloneqq \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

exists and is finite. This limit is called the *derivative* of f at z_0 and is denoted by $f'(z_0)$.

Equivalently, f is differentiable at z_0 if there exists some complex number $f'(z_0) \in \mathbb{C}$ such that

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + E_{z_0}(z),$$

where $E_{z_0}(z)$ is an "error term" satisfying $\lim_{z\to z_0} \frac{E_{z_0}(z)}{z-z_0} = 0$. It's convenient here to use the notation $E_{z_0}(z) = o(|z-z_0|)$, and in some cases we will use this.

What does the above definition *mean*? Well, assuming that $f'(z_0) \neq 0$, it means that the map

$$z \mapsto L(z) \coloneqq f(z_0) + (z-z_0)f'(z_0)$$

"approximates" $f(z)$ near z_0. As a map, L rotates around z_0 by $\operatorname{Arg}(f'(z_0))$, dilates around z_0 by $|f'(z_0)|$, and translates by $f(z_0)$. (If $f'(z_0) = 0$, however, the error term $E_{z_0}(z)$ becomes the leading term, so stranger things can happen.)

There are several standard theorems and rules regarding differentiation, but we will not learn them and instead refer to the textbook.

Remark. Contrast the content of the previous discussion with that of the $\mathbb{R}^2$ case. Differentiation in $\mathbb{R}^2$ can be *any matrix*, and so derivatives can stretch balls in $\mathbb{R}^2$. In contrast, under the assumption $f'(z_0) \neq 0$, L takes open balls to *open balls* and not other convex shapes.

2.4 Cauchy-Riemann Equations

Recall that we may write $f(x, y) = u(x, y) + iv(x, y)$, where $u = \Re(f)$ and $v = \Im(f)$. Suppose f' exists. Then this limit exists regardless of which direction we take the limit in, which means we can restrict ourselves to any direction we like.

For instance, if h is restricted to be a real number, then

$$f'(z) = \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon, y) - u(x, y)}{\varepsilon} + i\frac{v(x+\varepsilon, y) - v(x, y)}{\varepsilon} = u_x(x, y) + iv_x(x, y).$$

However, if h is restricted to be purely imaginary, then we get

$$\begin{aligned} f'(z) = \lim_{\delta \to 0} \frac{u(x, y+\delta) - u(x, y)}{i\delta} &+ i\frac{v(x, y+\delta) - v(x, y)}{i\delta} \\ &= \frac{u_y(x, y)}{i} + \frac{iv_y(x, y)}{i} = v_y(x, y) - iu_y(x, y). \end{aligned}$$

But recall that the derivative must be unique! This means we may equate the real and imaginary parts of $f'(z)$ to get the following theorem.

Theorem 2.7 (Cauchy-Riemann Conditions)**.** *Suppose $f = u + iv$ is differentiable on some domain D. Then within D the partials of u and v exist and satisfy the two equations*

$$u_x = v_y \qquad \textit{and} \qquad v_x = -u_y. \tag{2.1}$$

3 August 30

3.1 Differentiability and Partial Derivatives

We first seek to prove the following result.

Theorem 3.1. *Let f be a complex-valued function. Then $f = u + iv$ is differentiable at the point $z = x + iy$ iff both u and v and differentiable at (x, y) in the $\mathbb{R}^2$ sense and the Cauchy-Riemann conditions* (2.1) *hold at (x, y). In this case we have e.g. $f'(z) = u_x(x, y) + iv_x(x, y)$.*

As a reminder, "differentiable in the $\mathbb{R}^2$ sense" means that

$$u((x, y) + (h_1, h_2)) = u(x, y) + (\nabla u) \cdot (h_1, h_2) + o(|h|), \tag{3.1}$$

for instance.

Proof. Last time, we showed that the Cauchy-Riemann conditions hold under the assumption that f is differentiable at z, i.e. that

$$f(z + h) = f(z) + f'(z)h + o(|h|).$$

Taking the real part of both sides of this equation[1] yields

$$\begin{aligned} u((x, y) + (h_1, h_2)) &= u(x, y) + \Re((u + iv)(h_1 + ih_2)) + o(|(h_1, h_2)|) \\ &= u(x, y) + u_x h_1 - v_x h_2 + o(|h|) \\ &= u(x, y) + u_x h_1 + u_y h_2 + o(|h|). \end{aligned}$$

Hence u satisfies (3.1), so u is differentiable. Similar computations show that v is also differentiable.

On the other hand, suppose u and v are differentiable and satisfy the Cauchy-Riemann equations. Taking the equalities (3.1) for u and v and adding them together yields

$$\begin{aligned} f(z + h) = u(z + h) + iv(z + h) &= (u + iv)(z) + (\nabla u) \cdot (h_1, h_2) + i(\nabla v) \cdot (h_1, h_2) + o(|h|) \\ &= u + iv + (u_x h_1 + u_y h_2) + i(v_x h_1 + v_y h_2) + o(|h|) \\ &= u + iv + (u_x h_1 - v_x h_2) + i(v_x h_1 + u_x h_2) + o(|h|) \\ &= u + iv + u_x(h_1 + ih_2) + iv_x(h_1 + ih_2) + o(|h|). \end{aligned}$$

So f is differentiable as a complex-valued function. □

3.2 Analytic functions

We are now ready to present an important definition in complex analysis.

Definition 3.2. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$.

1. We say that f is *analytic* on U if f is differentiable everywhere on U.[2]
2. We say that f is analytic at a point $z \in \mathbb{C}$ if it is analytic in some open ball around z.
3. We say that f is *entire* if it is analytic on all of $\mathbb{C}$.

Part of the allure of complex analysis lies in the fact that analytic functions possess some surprising properties. Here is one (classical!) example.

Proposition 3.3. *Let $U \subset \mathbb{C}$ be a domain, and let $f : U \to \mathbb{C}$ be analytic. If $|f|$ is constant on U, then so is f.*

Proof. Assume $|f| = c \neq 0$, or else there's nothing to prove. Then $|f|^2 = u^2 + v^2$ is constant, so

$$0 = (|f|^2)_x = 2uu_x + 2vv_x$$

[1] note that $(u + iv)(h_1 + ih_2)$ is shorthand for $u(h_1 + ih_2) + iv(h_1 + ih_2)$ and is *not* multiplication of two complex numbers

[2] Some people may recognize this as the definition of a *holomorphic* function, but the textbook calls these analytic, and in the end both definitions are identical anyway.

and
$$0 = (|f|^2)_y = 2uu_y + 2vv_y \overset{(*)}{=} 2vu_x - 2uv_x,$$
where in $(*)$ we use the fact that f is analytic and thus satisfies (2.1). Solving this system of equations yields $u_x = v_x = 0$, so $u_y = v_y = 0$ as well. Hence $\nabla u = \nabla v = 0$ on some connected open set, meaning that u and v are constant. □

Remark. We highlight here one more example with fewer details fleshed out. Suppose f is analytic at z_0 and $f'(z_0) \neq 0$. This condition implies (for example) that $u_x(x_0, y_0) \neq 0$, and by (2.1) we see $v_y(x_0, y_0) \neq 0$ as well. Thus, by the *implicit function theorem* (which in the $\mathbb{R}^2 \to \mathbb{R}$ case only requires individual entries of the gradients ∇u and ∇v to be nonzero), the level curves
$$\mathcal{C}_u = \{(x, y) : u(x, y) = u(x_0, y_0)\} \quad \text{and} \quad \mathcal{C}_v = \{(x, y) : v(x, y) = v(x_0, y_0)\}$$
exist in a neighborhood around z_0.

Now recall that the gradient vector ∇u is orthogonal to $\mathcal{C}_u$ at (x_0, y_0); for a proof of this, see here. Thus, since
$$(\nabla u)(x_0, y_0) \cdot (\nabla v)(x_0, y_0) = u_x v_x + u_y v_y = u_x v_x + (-v_x)(u_x) = 0,$$
the curves $\mathcal{C}_u$ and $\mathcal{C}_v$ are perpendicular at (x_0, y_0) as well.

3.3 Examples of Elementary Analytic and Multivalued Functions

Much of our previous discussion about analytic functions has been theoretical. Let's switch gears and discuss a few examples.

3.3.1 Polynomials

Recall that we can define the function
$$f(z) = z^n := \underbrace{z \cdot z \cdot \ldots \cdot z}_{n \text{ times}},$$
where $n \geq 1$ is an integer. Standard differentiation rules imply $f'(z) = nz^{n-1}$, so f is entire. In turn, polynomials are entire.

3.3.2 Exponential Function

For a complex number $z = x + iy$, define
$$e^z := e^x e^{iy} = e^x(\cos y + i \sin y).$$
A simple computation reveals that $u_x = v_y = e^x \cos y$ and $v_y = -u_y = e^x \sin y$. Hence by Theorem 3.1 we deduce that e^z is entire.

This function e^z satisfies a few interesting properties.

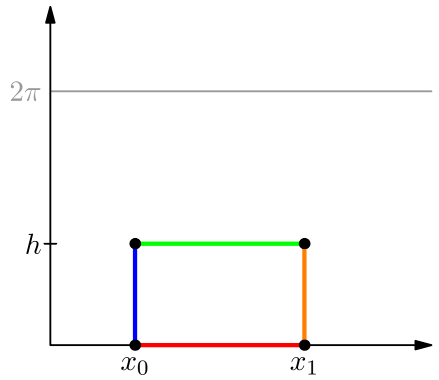
- First (and arguably foremost), its derivative by Theorem 3.1 is
$$(e^z)' = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z.$$
- For any complex numbers z_1 and z_2, we have
$$e^{z_1} e^{z_2} = e^{x_1} e^{iy_1} e^{x_2} e^{iy_2} = e^{x_1 + x_2} e^{i(y_1 + y_2)} = e^{z_1 + z_2}.$$
- We have $|e^z| = \left|e^x e^{iy}\right| = |e^x| = e^x$ and $\operatorname{Arg}(e^z) = y \bmod 2\pi$.
- e^z is $2\pi i$-periodic, meaning that $e^{z+2\pi i} = e^z$ for any $z \in \mathbb{C}$.

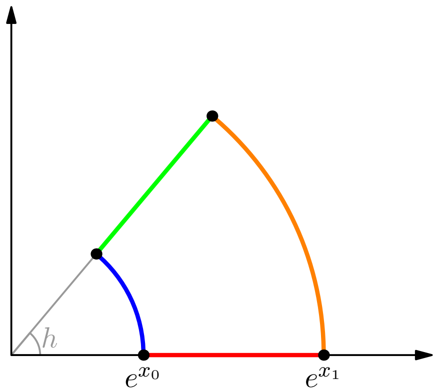
With these properties in tow, we may now answer the following question: what does the image of e^z look like? Well, observe the following.

- Suppose x is fixed but y varies. Then $|z|$ is fixed while $\operatorname{Arg} z$ is periodic modulo 2π. Hence vertical lines are taken to circles under the map $z \mapsto e^z$.

- Suppose y is fixed but x varies. Then $\operatorname{Arg} z$ is fixed while $|z|$ varies. Hence horizontal lines are taken to radial lines (i.e. lines that pass through the origin) under the map $z \mapsto e^z$.

Hence we deduce that, for example, rectangles are taken to annulus-shaped regions under the map $z \mapsto e^z$.





3.3.3 Trig Functions

With the exponential function in tow, we may now define the complex-valued functions $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ via

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Since e^z is entire, we deduce that sin and cos are entire as well.

3.4 Multivalued Functions and Branches

In order to discuss our last function, we need to discuss what it means for a function to be multi-valued.

Definition 3.4. Fix a set $A \subset \mathbb{C}$. Then any function $f : A \to 2^{\mathbb{C}}$ is called a *multivalued function.*

As an example, recall that we restricted the argument of a complex number to be in the interval $(-\pi, \pi]$. If we don't want to deal with this restriction, we may define

$$\arg(z) \coloneqq \{\operatorname{Arg}(z) + 2\pi k : k \in \mathbb{Z}\}$$

on the set $\mathbb{C} \setminus \{0\}$.

Of course, sometimes we want to go the other way around: given a multivalued function, can we transform it into a single-valued function by picking appropriate outputs for each input? This leads into a definition.

Definition 3.5. Let $f : A \to 2^{\mathbb{C}}$ be a multivalued function. Then a function $g : A \to \mathbb{C}$ is a *branch of f on A* if

1. for all $z \in A$, $g(z) \in f(z)$;
2. g is continuous on A.

In the context of the definition above, $\operatorname{Arg}(z)$ is a branch of $\arg(z)$ on the set $\mathbb{C} \setminus \{x \le 0\}$. The reason we need to exclude the negative x-axis is because otherwise Arg fails to be continuous: it approaches π as x approaches said axis from the second quadrant but it approaches $-\pi$ as x approaches the axis from the third quadrant.

In fact, we can say something stronger.

Proposition 3.6. *The multivalued function* arg *has no branch on* $\mathbb{C} \setminus \{0\}$.

Proof. Suppose g were such a branch. On the unit circle $\mathbb{S}^1$, we have

$$g(e^{i\theta}) = \theta \bmod 2\pi = \theta + 2\pi k_\theta$$

where k_θ is some integer that depends on θ. Since g is a branch, it is continuous on $\mathbb{S}^1$, so $2\pi k_\theta$ must be continuous as well. Hence in fact $2\pi k_\theta$ is constant on $\mathbb{S}^1$, but then continuity at $\theta = 2\pi$ breaks. $\square$

We are now ready to touch on our last function.

3.4.1 Complex Logarithm

Fix $z \in \mathbb{C} \setminus \{0\}$. Is it possible to solve the equation $e^w = z$ in complex numbers w? Well, write $z = |z|\, e^{i\,\mathrm{Arg}(z)}$. Then $e^{|w|} = |z|$ and $\mathrm{Arg}(w) \equiv \mathrm{Arg}(z) \pmod{2\pi}$. Hence

$$w = \ln(|z|) + i\,\mathrm{Arg}(z) + 2\pi i k \quad \text{for some } k \in \mathbb{Z}.$$

Hence the answer is "yes".

This means we can define

$$\log(z) \coloneqq \ln(|z|) + i\arg(z)$$

as a multivalued function, and furthermore we can set

$$\mathrm{Log}(z) \coloneqq \ln(|z|) + i\,\mathrm{Arg}(z)$$

as the so-called *principal branch* of $\log(z)$ on $\mathbb{C} \setminus \{0\}$. Observe that from $e^{w_1}e^{w_2} = e^{w_1+w_2}$ we obtain $\log(z_1 z_2) = \log(z_1) + \log(z_2)$; here the operation on the right hand side is *set addition*. (But that distinction doesn't matter that much here.)

Next time we'll show that $\mathrm{Log}(z)$ is analytic and that $(\mathrm{Log}(z))' = \frac{1}{z}$.

4 September 4

It's easy to see that $\operatorname{Log}(z)$ is analytic since it can be written as the sum $\ln|z| + i\operatorname{Arg}(z)$ of two analytic functions. To prove that its derivative is $\frac{1}{z}$, we will actually show something more general.

Theorem 4.1. *Suppose $f : U \to \mathbb{C}$ is analytic. Observe that we can write f^{-1} as a multivalued function. Let g be some branch of f^{-1} on a domain D, assuming such a branch exists. Given $z_0 \in D$, if $f'(g(z_0)) \neq 0$, then g is differentiable at z_0 and*

$$g'(z_0) = \frac{1}{f'(g(z_0))}.$$

Proof. Since g is continuous and bijective on D, we may write

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{g(z) - g(z_0)}{f(g(z)) - f(g(z_0))} = \left(\frac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)}\right)^{-1},$$

and sending $z \to z_0$ yields the desired. $\square$

As a corollary, we obtain the equality

$$(\operatorname{Log}(z))' = \frac{1}{(e^w)'\,|_{w=\operatorname{Log}(z)}} = \frac{1}{e^{\operatorname{Log}(z)}} = \frac{1}{z}.$$

4.1 General Power Functions

Given $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{0\}$, we may define $z^\alpha := e^{\alpha \log(z)}$ as a multivalued functions. The branches of log give the branches of $\cdot^\alpha$, and so the branches of $\cdot^\alpha$ are analytic. Also, we have the (multivalued) function equalities $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$ and $z_1 \alpha z_2^\alpha = (z_1 z_2)^\alpha$. We now compute (on some branch of z^α) that

$$(z^\alpha)' = (e^{\alpha \log(z)})' = \alpha \cdot \frac{1}{z} e^{\alpha \log z} = \alpha z^{\alpha - 1}.$$

For $n \in \mathbb{N}$ this agrees with the usual sense of differentiability, and in this case it's entire. For $\alpha = \frac{p}{q} \in \mathbb{Q}$, we may write

$$z^{p/q} = e^{p/q \cdot \log(z)} = e^{p/q \cdot \ln|z| + ip/q \cdot \arg(z)} = |z|^{p/q}\, e^{ip/q \cdot \arg(z)}.$$

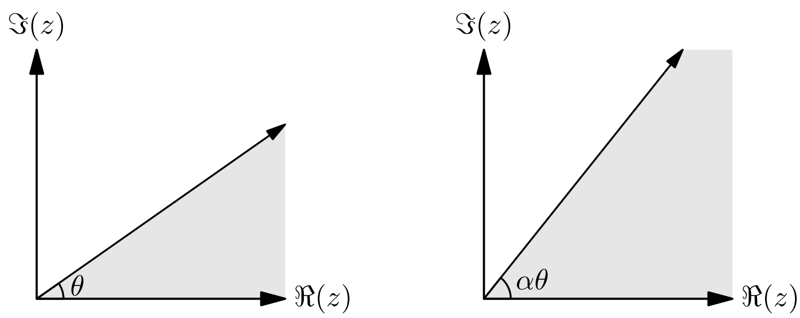
These are the q^{th} roots of p.

For another classical example, we may write

$$i^i = e^{i \log i} = \left\{ e^{i(\ln|i| + i\pi/2 + 2\pi k i)} : k \in \mathbb{Z} \right\} = \left\{ e^{-\pi/2 - 2\pi k} : k \in \mathbb{Z} \right\}.$$

So this tells us that i^i is real and multivalued! Strange.

Remark. What does the map $z \mapsto z^\alpha$ do to $\mathbb{C}$ on a principal branch? Let's assume that $\alpha > 0$ for simplicity. Then the ray with angle φ centered at the origin gets sent to the ray with angle $\alpha\varphi$ centered at the origin, so "sectors" are mapped to "sectors", as shown below.



4.2 Complex Integration

We begin with a definition.

Definition 4.2. Let $a < b$ be real numbers.

1. A *path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$.
2. A path is said to be *simple* if it has no self-intersections other than endpoints.
3. A path γ is *piecewise smooth* (also called a *contour*) if there exists some partition

$$t_0 = a < t_1 < \cdots < t_n = b$$

such that γ is continuously differentiable on each interval $[t_i, t_{i+1}]$ for $i = 0, \dots, n-1$. (At the endpoints, we must take one-sided limits.)

A few remarks are in order.

Remark. It's a quirk of terminology that both the function and its image are referred to as "paths". Fortunately, we do have notation for the image of a function γ: it's denoted $|\gamma|$.

Remark. Observe that we may write $\gamma(t) = x(t) + iy(t)$, and in this case we have

$$\gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h) - \gamma(t)}{h} = x'(t) + iy'(t).$$

One theorem we'll be using a lot is the following.

Theorem 4.3 (Jordan Curve Theorem)**.** *Any simple closed path γ divides $\mathbb{C}$ into two open disjoint and connected sets, namely* inside(γ) *and* outside(γ).

This is a classical example of something that's really easy to state but really hard to prove.

Now let $\gamma : [a, b] \to \mathbb{C}$ be a contour, and let $f : U \to \mathbb{C}$ be a continuous function, where U is a domain satisfying $U \supseteq |\gamma|$. Then we define

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

We attempt to make sense of this integral. Write $f = u + iv$ and $\gamma = x + iy$. Then $\gamma' = x' + iy'$ by the previous remark, and so a bit of expansion yields

$$f(\gamma(t))\gamma'(t) = u(\gamma(t))x'(t) - v(\gamma(t))y'(t) + i\left[u(\gamma(t))y'(t) + v(\gamma(t))x'(t)\right].$$

Hence the integral equals

$$\int_a^b u(\gamma(t))x'(t) - v(\gamma(t))y'(t)\, dt + i \int_a^b u(\gamma(t))y'(t) + v(\gamma(t))x'(t)\, dt,$$

which simplifies to

$$\int_\gamma f\, dz = \int_\gamma u\, dx - v\, dy + i \int_\gamma v\, dx + u\, dy.$$

These last integrals are essentially line integrals.

Example 4.4. Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$, and set $f(z) = (z - z_0)^k$ for some $k \in \mathbb{Z}$. Then

$$f(\gamma(t)) = (re^{it})^k = r^k e^{itk} \quad \text{and} \quad \gamma'(t) = ire^{it}.$$

So

$$\begin{aligned}\int_\gamma f(z)\, dz &= i \int_0^{2\pi} r^{k+1} e^{it(k+1)}\, dt \\ &= ir^{k+1} \int_0^{2\pi} e^{it(k+1)}\, dt = \begin{cases} 0 & \text{if } k \neq -1, \\ 2\pi i & \text{if } k = -1. \end{cases}\end{aligned}$$

This forms the basis for Cauchy's integral formula, as we shall see later.

Remark. Suppose f is analytic; then the Cauchy-Riemann conditions read $v_y = u_x$ and $v_x = -u_y$, so the vector fields $(u, -v)$ and (v, u) are conservative. Thus assuming f' is continuous[3] on an open set containing $|\gamma|$ and the interior of γ, where γ is a simple closed counterclockwise contour, then Green's Theorem tells us

$$\oint_\gamma u\,dx - v\,dy = \iint_{\text{inside}(\gamma)} \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_{\text{inside}(\gamma)} 0 = 0.$$

Similarly $\oint_\gamma v\,dx + u\,dy = 0$. Hence we deduce that $\int_\gamma f = 0$ for all such contours! Surprising.[4]

So in the case of $f(z) = (z - z_0)^k$, where $k \geq 1$, we have $f'(z) = k(z - z_0)^k$, which is continuous on $\mathbb{C}$. So $\int_\gamma f(z)\,dz = 0$.

[3] which always happens to be the case, but we haven't proven that yet

[4] In fact we can generalize this to all contours by splitting each contour into simple closed parts, but we will not prove this here.

5 September 6

5.1 Integration Properties

The complex integral we discussed last time has some nice properties, many of which carry over from real-valued integration.

- The integral is *linear*: $\int_\gamma (af + bg) = a \int_\gamma f + b \int_\gamma g$.
- If γ can be written as the concatenation of two paths $\gamma_1 \cup \gamma_2$, then $\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$.
- Given a path $\gamma : [a, b] \to \mathbb{C}$, define $-\gamma : [a, b] \to \mathbb{C}$ via $(-\gamma)(t) = \gamma(a + b - t)$. Then $\int_\gamma f = -\int_{-\gamma} f$.
- The integral satisfies the *change of parameter* property: let $\gamma : [a, b] \to \mathbb{C}$ be a contour, and let $\alpha : [c, d] \to [a, b]$ be a piecewise smooth, increasing, and onto bijection. Then $\int_\gamma f = \int_{\gamma \circ \alpha} f$. This says that it doesn't *really* matter what γ is, and so we may (without loss of generality) write $\int_\gamma f = \int_{|\gamma|} f$.
- Observe that

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\, dt \right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, dt$$
$$\leq \sup_{z \in |\gamma|} |f(z)| \int_a^b |\gamma'(t)|\, dt = \sup_{z \in |\gamma|} |f(z)| \cdot \mathrm{length}(\gamma).$$

  This will be very useful.
- If $f_n \to f$ uniformly on $|\gamma|$, then $\int_\gamma f_n(z)\, dz \to \int_\gamma f(z)\, dz$. Indeed,

$$\left| \int_\gamma f_n(z)\, dz - \int_\gamma f(z)\, dz \right| \leq \sup_{z \in |\gamma|} |f_n(z) - f(z)| \, \mathrm{length}(\gamma) \to 0$$

  as $n \to \infty$.

5.2 Primitives

We begin with a definition.

Definition 5.1. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. A function $F : U \to \mathbb{C}$ is called a *primitive* of f if F is analytic on U and its derivative is f.

As one might expect, there is a form of the Fundamental Theorem of Calculus that holds in the complex case.

Theorem 5.2. *Define U, f, and F as above, and let $\gamma : [a, b] \to U$. Then*

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Assume first that γ is smooth. Then

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b \frac{d}{dt} [F(\gamma(t))]\, dt = F(\gamma(b)) - F(\gamma(a)).$$

In the case where γ is piecewise smooth, apply the previous argument to each smooth component of γ and telescope. □

This theorem nas a very important corollary.

Corollary 5.3. *Define U, γ, f, and F as above, and assume γ is closed. Then $\int_\gamma f = 0$.*

Example 5.4. Recall that $(z - z_0)^k = \frac{d}{dz}\left[\frac{(z-z_0)^{k+1}}{k+1}\right]$ whenever $k \neq -1$. This primitive is analytic on $\mathbb{C} \setminus \{z_0\}$, (and in fact is analytic on $\mathbb{C}$ when $k \geq 0$), so $\int_\gamma (z - z_0)^k\, dz = 0$ for any $|\gamma| \subset \mathbb{C} \setminus \{z_0\}$ and $k \in \mathbb{Z} \setminus \{-1\}$.

Now observe the $k=-1$ case. Then $(z-z_0)^{-1} = \frac{d}{dz}g(z)$, where $g(z)$ is *any* branch of the function $z \mapsto \log(z-z_0)$. This means that $\int_\gamma (z-z_0)^{-1}$ changes depending on y!

There are two cases: either $z_0 \in \text{inside}(\gamma)$ or $z_0 \in \text{outside}(\gamma)$. In the former case, we may construct a (possibly curved) infinite ray ℓ with vertex at z_0; then a branch of $\log(z-z_0)$ exists on $\mathbb{C}\setminus\ell$, and we conclude that $\int_\gamma (z-z_0)^{-1}\,dz = 0$.

If instead we have e.g. $0 \in \text{inside}(\gamma)$, then we can still use the same techniques but more work is needed. Consider the case in which γ intersects the negative real axis exactly once, namely at w. We may re-parametrize $|\gamma|$ so that $\gamma(0)=\gamma(1)=w$ and γ is oriented counterclockwise. Then

$$\begin{aligned}\int_\gamma \frac{1}{z}\,dz &= \lim_{\varepsilon\to 0}\int_\varepsilon^{1-\varepsilon} f(\gamma(t))\gamma'(t)\,dt \\ &= \lim_{\varepsilon\to 0}\left[\operatorname{Log}(\gamma(1-\varepsilon)) - \operatorname{Log}(\gamma(\varepsilon))\right] \\ &= (\ln|w|+\pi i) - (\ln|w|-\pi i) = 2\pi i.\end{aligned}$$

This agrees with our earlier work when we computed this integral for γ a circle.

5.3 The Local Cauchy Theorem

We will now prove the following theorem.

Theorem 5.5 (Cauchy for Triangles). *Let $U\subset\mathbb{C}$ be open, and let $f:U\to\mathbb{C}$ be continuous on U and analytic on U except possibly at some point $p\in U$. Then for any triangle $\Delta\subset U$, we have*

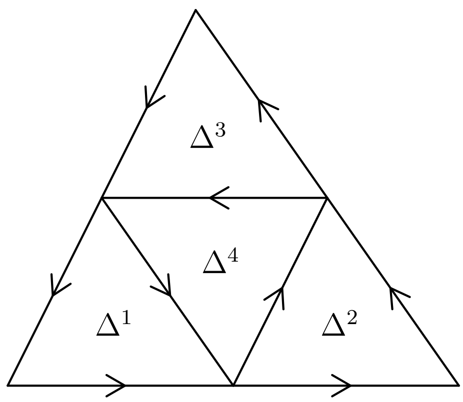
$$\int_{\partial\Delta} f(z)\,dz = 0.$$

The key observation is as follows: suppose Δ is a triangle with diameter ε. If ε is sufficiently small, then we may approximate f by its linear approximation at $z_0\in\Delta$ fairly well. That is,

$$\begin{aligned}\int_{\partial\Delta} f(z)\,dz &= \int_{\partial\Gamma}\left[f(z_0)+f'(z_0)(z-z_0)+o(|z-z_0|)\right]\,dz \\ &= f(z_0)\int_{\partial\Delta}1\,dz + f'(z_0)\int_{\partial\Delta}(z-z_0) + \int_{\partial\Delta} o(|z-z_0|)\,dz \\ &= f(z_0)\cdot 0 + f'(z_0)\cdot 0 + o(\varepsilon^2) = o(\varepsilon^2).\end{aligned}$$

We proceed with the proof.

Proof. Let's first assume that $p\notin\Delta$ and that $\partial\Delta$ is a counterclockwise contour. Divide Δ into four congruent triangles as shown. Observe that integrating f along Δ^1, Δ^2, and Δ^3 gives us an integral



along the original triangle Δ plus the *negative* of the integral along Δ^4. In particular, this means

$$\oint_{\partial\Delta} f(z)\,dz = \int_{\partial\Delta^1} f + \int_{\partial\Delta^2} f + \int_{\partial\Delta^3} f + \int_{\partial\Delta^4} f =: I.$$

Then there must exist some $1\le i\le 4$ such that $\left|\oint_{\partial\Delta^i} f(z)\,dz\right| \ge \frac{I}{4}$. Fix such a Δ^i and call it Δ_1.

By relating this process we obtain a nested sequence of triangles

$$\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \supset \cdots$$

satisfying $\left|\int_{\partial\Delta^n} f(z)\,dz\right| \geq \frac{|I|}{4^n}$. Since $\operatorname{diam}(\partial\Delta_n) = \frac{1}{2^n}\operatorname{diam}(\partial\Delta) \to 0$ as $n \to \infty$, the nested compact sets theorem tells us that

$$\bigcap_{n=1}^{\infty} \Delta_n = \{z_0\} \in \Delta$$

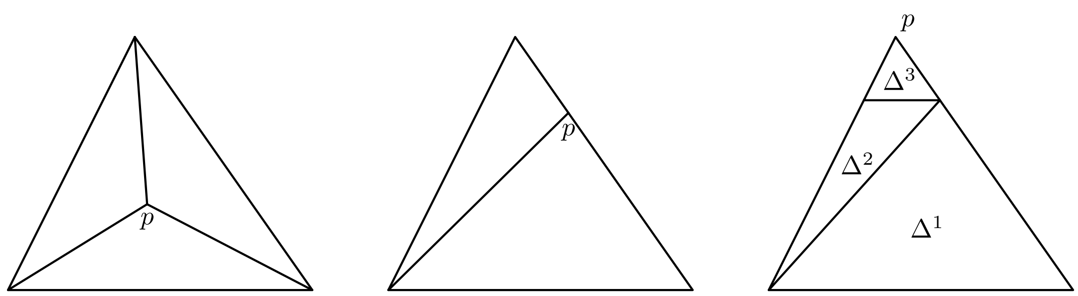
for some z_0.

As a result, we may write

$$\begin{aligned}\left|\int_{\partial\Delta_n} f(z)\,dz\right| &= \left|\int_{\partial\Delta_n} f(z) = f(z_0) + f'(z_0)(z-z_0) + o(|z-z_0|)\,dz\right| \\ &= \left|\int_{\partial\Delta} o(|z-z_0|)\,dz\right| \leq \max_{z\in\partial\Delta_n} o(|z-z_0|)\operatorname{length}(\partial\Delta_n) \\ &\qquad\qquad = o(2^{-n})\cdot o(2^{-n}) = o(4^{-n}).\end{aligned}$$

But we already established that $\left|\int_{\partial\Delta_n} f(z)\,dz\right| \geq \frac{|I|}{4^n}$; thus $|I| \leq o(4^{-n})4^n \to 0$ as $n \to \infty$.

Now we deal with the case where $p \in \Delta$. In this case, it suffices to show the result when p is a corner of Δ; this is because we may subdivide Δ into triangles and reduce to the corner case as shown below in the left two pictures.



Now assuming p is a corner, divide Δ into three triangles Δ^1, Δ^2, and Δ^3 as shown in the rightmost diagram above. Then

$$\left|\int_{\partial\Delta} f\right| = \left|\int_{\partial\Delta^3} f\right| \leq \sup_{\partial\Delta^3}|f|\cdot\operatorname{length}(\partial\Delta^3),$$

which goes to zero as we make Δ^3 smaller and smaller. $\square$

6 September 9

We proceed with a generalization of Theorem 5.5.

Theorem 6.1 (Cauchy's Theorem for Convex Sets)**.** *Let $U \subseteq \mathbb{C}$ be open, and suppose $f : U \to \mathbb{C}$ is continuous on U and analytic on $U \setminus \{p\}$. Then f has a primitive F on U. In particular, $\int_\gamma f = 0$ for any closed contour γ in U.*

Remark. The assumption that f is analytic can be replaced with the assumption that $\int_{\partial\Delta} f = 0$ for any triangle $\Delta \subset U$; in particular, this is the only consequence of f being analytic we need.

Proof. Fix a point $a \in U$. Let $[a, z]$ be the straight line from a to z, and overload notation by setting

$$[a, z](t) := a + (z - a)t \quad \text{for } t \in [0, 1].$$

Define $F(t) = \int_{[a,z]} f(\xi)\, d\xi$; our goal is to show $F' = f$. Indeed, write

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} &= \frac{\int_{[a,z]} f(\xi)\, d\xi - \int_{[a,z_0]} f(\xi)\, d\xi}{z - z_0} \\ &= \frac{\int_{[a,z]} f(\xi)\, d\xi + \int_{[z_0,a]} f(\xi)\, d\xi}{z - z_0} = \frac{\int_{[z_0,z]} f(\xi)\, d\xi}{z - z_0}. \end{aligned}$$

Observe that in the last step we use the fact that $\int_{\partial\Delta} f = 0$ for the triangle with vertices z, z_0, and a (which we know is completely contained in U by convexity). Thus

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{\int_{[z_0,z]} f(\xi)\, d\xi}{z - z_0} - \frac{\int_{[z_0,z]} f(z_0)\, d\xi}{z - z_0} = \frac{\int_{z_0}^{z} (f(\xi) - f(z_0))\, d\xi}{z - z_0}$$

and hence

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \le \frac{|z - z_0| \max_{\xi \in [z_0,z]} |f(\xi) - f(z_0)|}{|z - z_0|} = \max_{\xi \in [z_0,z]} |f(\xi) - f(z_0)|\,.$$

This last quantity tends to zero as $z \to z_0$, and so we obtain the desired conclusion. ☐

While the constraint that U is convex is somewhat restrictive, in reality this result is farther reaching than expected.

Remark. Thee previous result still is useful when U is not necessarily convex. In particular, we can break γ into pieces that are contained in convex subdomains of U.

6.1 Differentiation under the Integral Sign

We now take a quick tangent that will become immediately useful.

Theorem 6.2 (Differentiation under the Integral Sign)**.** *Let $U \subset \mathbb{C}$ and let γ be a contour in $\mathbb{C}$. Assume that $F : |\gamma| \times U \to \mathbb{C}$ is a function satisfying the following properties:*

- *F is continuous on $|\gamma| \times U$;*
- *for any $\xi \in |\gamma|$, $F(\xi, \cdot)$ is analytic in U;*
- *$\frac{\partial}{\partial z} F(\xi, z)$ is continuous on $|\gamma| \times U$.*

Then $G(z) := \int_\gamma F(\xi, z)\, d\xi$ is analytic in U, and

$$G'(z) = \int_\gamma \frac{\partial}{\partial z} F(\xi, z)\, d\xi.$$

Proof. Fix $z > 0$ and $\delta > 0$ such that $\overline{B(z, \delta)} \subset U$. For $|h| < \delta$ we may compute

$$\begin{aligned} \left|\frac{G(z + h) - G(z)}{h} - \int_\gamma \frac{\partial}{\partial z} F(\xi, z)\, d\xi\right| &= \left|\int_\gamma \left(\frac{F(\xi, z + h) - F(\xi, z)}{h} - \frac{\partial}{\partial z} F(\xi, z)\right) d\xi\right| \\ &= \left|\int_\gamma \frac{\int_{[z,z+h]} \partial_z F(\xi, \eta)\, d\eta - \int_{[z,z+h]} \partial_z F(\xi, z)\, d\eta}{h}\, d\xi\right| \\ &\le \text{length}(\gamma) \cdot \frac{|h|}{|h|} \cdot \max_{\substack{\eta \in [z,z+h] \\ \xi \in |\eta|}} |\partial_z F(\xi, \eta) - \partial_z F(\xi, z)|\,. \end{aligned}$$

Now recall that $|\gamma| \times \overline{B(z,\delta)}$ is compact, and so uniform continuity implies that the right hand side goes to zero as $h \to 0$. This completes the proof. $\square$

6.2 Winding Numbers

In a previous class, we showed that $\int_{B(z_0,r)} \frac{1}{z-z_0}\, dz = 2\pi i$ for any $r > 0$. Our goal now is to generalize this.

For any closed contour γ, recall that $\mathbb{C} \setminus |\gamma|$ has one unbounded connected component and multiple bounded connected components. This contour γ can "wind" around points in these bounded components multiple times. For example, in the diagram below observe that γ rotates -1 times around z_1, 1 time around z_2, 2 times around z_3, and so on.

It turns out we can make this notion of "turning" completely rigorous.

Definition 6.3. Given a contour γ, define the function $n(\gamma, \cdot) : \mathbb{C} \setminus |\gamma| \to \mathbb{C}$ via

$$n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z}\, d\xi.$$

It's not obvious at all that this function relates to "turning", but it does.

Lemma 6.4. *Let γ be a closed contour and $U = \mathbb{C} \setminus |\gamma|$. The following hold.*

- *$n(\gamma, z)$ is constant on each connected component of $\mathbb{C} \setminus |\gamma|$.*
- *$n(\gamma, z) \in \mathbb{Z}$ for all $z \in U$.*
- *$n(\gamma, z) = 0$ for all z in the unboundent component of U.*
- *If γ is simple, closed, and counterclockwise, then $n(\gamma, z) = 1$ for all $z \in \operatorname{inside}(\gamma)$.*

Proof. We will prove the first part now and leave parts 2 and 3 for tomorrow. The proof of 4 is surprisingly technical and we leave it to Palka.

By Theorem 6.2, we can write

$$\frac{\partial}{\partial z} n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial z} \frac{1}{\xi - z}\, d\xi = \frac{1}{2\pi i} \int_\gamma \frac{1}{(\xi - z)^2}\, d\xi = \frac{1}{2\pi i} \int_\gamma \frac{\partial}{\partial \xi} \left[\frac{1}{\xi - z}\right] d\xi.$$

Now observe that $\frac{\partial}{\partial \xi}\left[\frac{1}{\xi - z}\right]$ is a continuous function on U with primitive $\frac{1}{\xi - z}$; thus by Corollary 5.3 this integral is zero! This means that $n(\gamma, z)$ is constant on each connected component of U, and we're done. $\square$

7 September 11

As suggested, we will prove parts 2 and 3 of the previous lemma.

Proof. Assume $\gamma : [a,b] \to U$ is smooth; the piecewise smooth case follows by applying the smooth case to each part.

For part 2, recall that

$$n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\xi - z}\, d\xi = \frac{1}{2\pi i \int_a^b \frac{\gamma'(s)}{\gamma}(s) - z}\, ds.$$

Thus we may define $g : [a,b] \to \mathbb{C}$ via

$$g(t) := \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds\right).$$

Observe that $g(a) = 1$ and $g(b) = e^{2\pi i n(\gamma,z)}$. Our goal now is to show that $g(a) = g(b)$; thus $e^{2\pi i n(\gamma,z)} = 1$, and this can only happen when $n(\gamma, z) \in \mathbb{Z}$.

It is tempting to show that $g'(t) = 0$ for all $t \in (a,b)$, but this isn't actually the case. Instead, if you stare at this for a while, you will see that from $g'(t) = \frac{\gamma'(t)}{\gamma(t)-z} \cdot g(t)$ we have

$$\frac{d}{dt}\left[\frac{g(t)}{\gamma(t) - z}\right] = \frac{g'(t)}{\gamma(t) - z} - \frac{\gamma'(t)g(t)}{(\gamma(t) - z)^2} = 0.$$

Thus $\frac{g(a)}{\gamma(a)-z} = \frac{g(b)}{\gamma(b)-z}$, and from $\gamma(a) = \gamma(b) \neq z$ we obtain the desired conclusion.

Finally, we proceed with part 3. Suppose z lies in the unbounded component. Fix some disk D containing $|\gamma|$, and note that by part 1 we may write $n(\gamma, z) = n(\gamma, z_0)$, where z_0 is some point lying outside of D. But in this case, the function $\xi \mapsto \frac{1}{\xi - z_0}$ is analytic inside D, and hence the integral is zero. □

7.1 Cauchy's Integral Formula

We now prove (a special case of) one of the most important theorems in complex analysis.

Theorem 7.1 (Cauchy's Integral Formula in a convex set)**.** *Let $U \subset \mathbb{C}$ be convex and open, γ be a closed contour in U, and $f : U \to \mathbb{C}$ be analytic. Then for any $z \in U \setminus |\gamma|$,*

$$f(z)n(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi.$$

Proof. Fix $z \in U \setminus |\gamma|$, and let

$$g(\xi) = \begin{cases} \frac{f(\xi)-f(z)}{\xi - z} & \text{if } \xi \neq z, \\ f'(z) & \text{if } \xi = z. \end{cases}$$

Note that g is defined and actually continuous on U. Furthermore, it is differentiable on $U \setminus \{z\}$. So Cauchy's Theorem applies, and hence

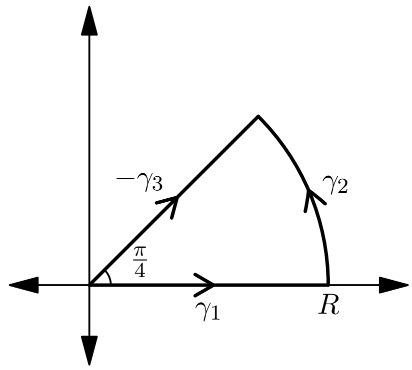
$$\begin{aligned} 0 = \int_\gamma g(\xi)\, d\xi = \int_\gamma \frac{f(\xi) - f(z)}{\xi - z}\, d\xi &= \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi - \int_\gamma \frac{f(z)}{\xi - z}\, d\xi \\ &= \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi - f(z) \cdot 2\pi i \cdot n(\gamma, z). \end{aligned}$$

This gives the desired equality. □

We illustrate the power of Cauchy's Integral Formula with an example.

Example 7.2. Our goal is to compute the value of the *Fresnel integral*

$$\int_0^\infty \cos(t^2)\, dt = \lim_{R\to\infty}\int_0^R \cos(t^2)\, dt.$$



Write this last expression as $\lim_{R\to\infty} \Re(I_R)$, where $I_R := \int_0^R e^{it^2}\,dt$.

Now consider the contour shown below, which consists of three segments γ_1, γ_2, and γ_3 oriented counterclockwise.

Let us analyze the integral of e^{iz^2} on each component separately.

- The contour γ_1 is parametrized by the function $\varphi_1 : [0, R] \to \mathbb{C}$ given by $\varphi_1(t) = t$. Thus,
$$\int_{\gamma_1} e^{iz^2}\,dz = \int_0^R e^{it^2}\,dt = I_R.$$
- The integral along $-\gamma_3$ is also easy to wrangle with, since $-\gamma_3$ is parametrized by the function $\varphi_3 : [0, R] \to \mathbb{C}$ given by $\varphi_3(t) = te^{i\pi/4}$. Thus
$$\int_{-\gamma_3} e^{iz^2}\,dz = \int_0^R e^{i(te^{i\pi/4})^2} \cdot e^{i\pi/4}\,dt = e^{i\pi/4}\int_0^R e^{-t^2}\,dt.$$
While this integral does not have a closed form, its limit as $R \to \infty$ does and equals $e^{i\pi/4}\frac{\sqrt{\pi}}{2}$.
- The hardest part of this question involves the integral along γ_2. This contour is parametrized by the function $\varphi_2 : [0, \pi/4] \to \mathbb{C}$ given by $\varphi_2(t) = Re^{it}$. Thus
$$\int_{\gamma_2} e^{iz^2}\,dz = \int_0^{\pi/4} e^{iR^2e^{2it}} Re^{it}\,dt.$$
We now claim this integral goes to zero as $R \to \infty$. To prove this, write
$$\left|\int_0^{\pi/4} e^{iR^2e^{2it}}\,dt\right| \le \frac{\pi}{4}\int_0^{\pi/4}\left|e^{iR^2e^{2it}}\right|\,dt = R\int_0^{\pi/4} e^{-R^2\sin(2t)}\,dt.$$
Now recall the bound $\frac{2}{\pi}x \le \sin x \le x$ for all $x \in [0, \frac{\pi}{2}]$. Thus
$$R\int_0^{\pi/4} e^{-R^2\sin(2t)}\,dt \le R\int_0^{\pi/4} e^{-2tR^2}\,dt \le R\cdot\frac{c}{R^2},$$
which goes to zero as $R \to \infty$.

As a result, we may write
$$0 = \int_{\gamma=\gamma_1\cup\gamma_2\cup\gamma_3} e^{iz^2}\,dz = I_R + \int_{\gamma_2} e^{iz^2}\,dz - e^{i\pi/4}\int_0^R e^{-t^2}\,dt$$
and send $R \to \infty$ to obtain
$$\lim_{R\to\infty} I_R = \lim_{R\to\infty} e^{i\pi/4}\int_0^R e^{-t^2}\,dt - \lim_{R\to\infty}\int_{\gamma_2} e^{iz^2}\,dz = e^{i\pi/4}\frac{\sqrt{\pi}}{2}.$$
Taking the real part of this last limit gives us our answer of $\frac{\sqrt{2\pi}}{4}$.

8 September 13

Let's see some applications of the Cauchy Integral Formula. Our first application is, in some sense, an extension.

Theorem 8.1. *Suppose f and U are defined as in Theorem 7.1. Then f is infinitely differentiable on the set $\{z : n(\gamma, z) \neq 0\}$ and*

$$n(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\xi)\, d\xi}{(\xi - z)^{k+1}}.$$

Proof. The key idea is to differentiate under the integral sign, observing that $n(\gamma, z)$ is constant on each connected component of $U \setminus |\gamma|$. This yields

$$f^{(k)}(z) n(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \left(\frac{\partial^k}{\partial z^k} \frac{f(\xi)}{\xi - z} \right) d\xi = \frac{k!}{2\pi i} \int_\gamma \frac{f(\xi)\, d\xi}{(\xi - z)^{k+1}}.$$

□

This theorem admits a very useful corollary.

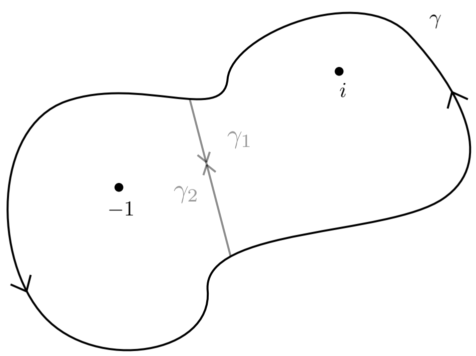
Corollary 8.2. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be analytic. Then f is infinitely differentiable and every derivative is analytic.*

Proof. Let $z \in U$; then there exists $r > 0$ such that $B(z, r) \subset U$. Now apply Theorem 8.1 to $\gamma = \partial B(z, r)$; this tells us that f is infinitely differentiable at z, and since z was arbitrary we deduce differentiability everywhere in U. □

Example 8.3. Consider the contour γ shown below, which is counterclockwise and contains the points i and -1. We will compute

$$\int_\gamma \frac{dw}{(w+1)^3 (w-i)^2}.$$

At the moment we are unable to apply Theorem 8.1: such an application requires us to write $\frac{1}{(w+1)^3(w-i)^2}$ in the form $\frac{f(w)}{\xi - r}$ for some analytic function f and some $r \in \text{inside}(\gamma)$, but this is not possible because of the existence of multiple singularities.



Instead, the idea is to *split* the contour. Divide the contour into two pieces γ_1 and γ_2 as shown above in gray. Then we may use Theorem 8.1 *twice* to obtain

$$\begin{aligned}
\int_\gamma \frac{dw}{(w+1)^3(w-i)^2} &= \int_{\gamma_1} \frac{1/(w+1)^3}{(w-i)^2}\, dw + \int_{\gamma_2} \frac{1/(w-i)^2}{(w+1)^3} \\
&= \frac{2\pi i}{1!} \left(\frac{1}{(w+1)^3} \right)' \Bigg|_{w=i} + \frac{2\pi i}{2!} \left(\frac{1}{(w-i)^2} \right)'' \Bigg|_{w=-1} \\
&= 2\pi i \cdot \frac{3}{4} + \pi i \cdot -\frac{3}{2} = 0.
\end{aligned}$$

Our next examples combine multiple results we have done previously.

Corollary 8.4 (Morera's Theorem)**.** *Let f be continuous on some open set $U \subset \mathbb{C}$ such that $\int_\Delta f(z)\,dz = 0$ for every triangle $\Delta \subset U$. Then f is analytic on U.*

Proof. Use Theorem 6.1 and the remark following it. In particular, the triangle condition tells us that f has a primitive F on U. So F is infinitely differentiable on U, and hence f is (infinitely) differentiable. □

Corollary 8.5. *Suppose $U \subset \mathbb{C}$ is open and $p \in U$. Assume f is analytic on $U \setminus \{p\}$ and continuous on U. Then f is analytic on U.*

Proof. Use Morera's Theorem. Alternatively, use the fact that f has a primitive on $B(p, r)$, where r is chosen so that $B(p, r) \subset U$. □

Our next proposition discusses taking logarithms of analytic functions.

Proposition 8.6. *Let U be open and convex, and let $f : U \to \mathbb{C}$ be analytic and nonvanishing. Then $\log(f)$ has a branch in U and is hence analytic.*

Proof. Recall that $(\log f)' = \frac{f'}{f}$. The goal is to find a primitive for $\frac{f'}{f}$ and prove that it is a branch of f.

The existence of a primitive F follows from the fact that f is nonvanishing, since $\frac{1}{f}$ and f' are both analytic.

Now observe that

$$(fe^{-F})' = f'e^{-F} - F'e^{-F}f = f'e^{-F} - \frac{f'}{f}e^{-F}f = 0.$$

Since U is connected we deduce that $fe^{-F} = c$ for some constant c.

Finally, observe that c is nonzero, which means we can write $fe^{-F} = e^{c_0}$ for some constant c_0. Then $e^{F+c_0} = f$ on U, and so $F + c_0$ is the desired branch. □

As a corollary, we see that for any $\alpha \in \mathbb{C}$, $f^\alpha = e^{\alpha \log f}$ has a branch in U.

Example 8.7. Let

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Note that $z \mapsto \frac{\sin z}{z}$ is analytic on $\mathbb{C} \setminus \{0\}$.

We now claim that $\lim_{z\to 0} \frac{\sin z}{z} = 1$; this matches what occurs in the real case, but the proof when z is complex is a bit harder. To prove this, recall the equality

$$\sin z = \int_{[0,z]} \cos\xi\,d\xi \quad \text{for all } z \in \mathbb{C}.$$

Thus

$$\left|1 - \frac{\sin z}{z}\right| = \left|\int_{[0,z]} \frac{1-\cos\xi}{z}\,d\xi\right| \le |z| \cdot \max_{[0,z]} \left|\frac{1-\cos\xi}{z}\right| = \max_{[0,z]} |1-\cos\xi|\,.$$

Now observe that $1 - \cos\xi$ is continuous and hence uniformly continuous on $\overline{B(0, r)}$ for some r. Hence as $z \to 0$, the quantity $\max_{[0,z]} |1 - \cos\xi|$ tends to zero, which is what we wanted.

Thus f is continuous on $\mathbb{C}$, and so $\frac{\sin z}{z}$ is analytic on $\mathbb{C}$.

We end today with one final theorem.

Theorem 8.8 (Derivative Estimates)**.** *Assume f is analytic on $B(z_0, r)$ and that there exists some constant $m > 0$ such that $|f(z)| \le m$ for $z \in B(z_0, r)$. Then for every $k \in \mathbb{N}$ and for all $z \in B(z_0, r)$ we have*

$$\left|f^{(k)}(z)\right| \le \frac{k!m}{(r - |z - z_0|)^k}.$$

In particular, $f^{(k)}(z_0) \le k!mr^{-k}$.

Proof. It suffices to prove the result when $z = z_0$; then to obtain the general result we can apply this specific case, noting that

$$B(z, r - |z - z_0|) \subseteq B(z_0, r).$$

Now for each $0 < s < r$ let $\gamma_s : [0, 2\pi] \to \mathbb{C}$ be defined via $\gamma_s(t) = z_0 + se^{it}$. Since $n(\gamma, z_0) = 1$,

$$\left| f^{(k)}(z_0) \right| = \frac{k!}{2\pi} \left| \int_\gamma \frac{f(\xi)\, d\xi}{(z - z_0)^{k+1}} \right| \le \frac{k!}{2\pi} \cdot 2\pi s \cdot \frac{m}{s^{k+1}} = \frac{k! m}{s^k}.$$

This holds for any such s, so letting $s \to r$ yields the desired inequality. ☐

9 September 16

Our estimates from Theorem 8.8 allow us to derive a very useful corollary.

Corollary 9.1 (Liouville)**.** *Let f be a bounded, entire function. Then f is constant.*

Proof. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then Theorem 8.8 tells us that $|f'(z)| \leq \frac{M}{r}$. Sending $r \to \infty$ yields $|f'(z)| \leq 0$. So $f'(z) = 0$, and since $\mathbb{C}$ is connected we deduce that f is constant on $\mathbb{C}$. $\square$

In turn, Liouville gives us a result that forms the basis of high school algebra.

Corollary 9.2 (Fundamental Theorem of Arithmetic)**.** *Any polynomial in z of degree at least* 1 *has a root in $\mathbb{C}$. As a result, polynomials of degree n have exactly n roots up to multiplicity.*

Proof. Suppose for the sake of contradiction that $P \in \mathbb{C}[z]$ has no real roots. Then the function $f = P^{-1}$ is entire. Furthermore, since P is continuous and nonzero, f is bounded on any compact subset of $\mathbb{C}$.

Now write

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$

Then for all sufficiently large values of $|z|$ (say when $|z| > R$ for some $R > 0$),

$$\begin{aligned} |P(z)| &\geq |z_n| \cdot |z|^n - |a_{n-1}| \cdot |z|^{n-1} - \cdots - |a_0| \\ &= |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \cdots - \frac{|a_0|}{|z|^n} \right) \geq |z|^n \cdot \frac{|a_n|}{2}. \end{aligned}$$

Hence f is bounded on both $\{z : |z| \leq R\}$ and $\{z : |z| > R\}$, so f is bounded on all of $\mathbb{C}$. Hence Liouville tells us that f is constant, which is a contradiction. $\square$

Another application of complex integration lies in something called the *Maximum Modulus Principle.*

Theorem 9.3 (Maximum Modulus Principle)**.** *Let D be a domain, and let $f : D \to \mathbb{C}$ be analytic. Suppose f attains its maximum somewhere in D, so that there exists $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Then f is constant.*

Remark. This can be generalized somewhat. Suppose a function $f : D \to \mathbb{R}$ is called *subharmonic* if f is continuous and, for all $z_0 \in \mathbb{C}$, the integral

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})$$

holds for sufficiently small r. It turns out that the Maximum Modulus Principle also holds for subharmonic functions. This will be clear from the proof.

Proof. Let $z_0 \in D$ be a maximum. Take $r_0 > 0$ such that $\overline{B(z_0, r_0)} \subset D$. Then for $0 < r < r_0$ we may write

$$f(z_0) = \frac{1}{2\pi i} \oint_{\partial B(z_0, r)} \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} \cdot rie^{rt}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

(Intuitively, $f(z_0)$ is the average of the values f takes on any circle centered at z_0.) Now taking the absolute value of both sides yields

$$|f(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \right| dt \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{it}) \right| dt,$$

and so

$$\int_0^{2\pi} \left| f(z_0 + re^{it}) \right| - |f(z_0)|\, dt \geq 0.$$

However, observe that $f(z_0)$ is a maximum for f in D, meaning that $\left| f(z_0 + re^{it}) \right| \leq |f(z_0)|$ for all t. The only way these contradicting facts can hold simultaneously is if $|f(z_0)| = \left| f(z_0 + re^{it}) \right|$. By varying r we deduce that $|f(t)|$ is constant on $B(z_0, r_0)$.

Now set

$$U = \{z \in D : |f(z)| = |f(z_0)|\} \quad \text{and} \quad V = \{z \in D : |f(z)| < |f(z_0)|\}.$$

We just proved that U is open in D, and furthermore V is open in D by continuity of $|f|$. Thus, since D is connected and $z_0 \in U$ we deduce that $V = \varnothing$. Hence $|f|$ is constant on D, and by Proposition 3.3 f is constant on D as well. □

The previous theorem admits an easy corollary.

Corollary 9.4. *Let D be a bounded domain, and let $f : \bar{D} \to \mathbb{C}$ be continuous. Assume f is analytic in D. Then f attains its maximum on ∂D. Moreover, if f is nonvanishing, it attains its minimum in ∂D as well.*

Proof. Since $\bar{D}$ is compact and f is continuous, f obtains its maximum at some point $z_0 \in \bar{D}$. If $z_0 \in \partial D$, we're done. Otherwise, $z_0 \in D$, and so the Maximum Modulus Principle implies f is constant in D. By continuity, f is constant on $\bar{D}$ as well, and we're done.

For the second part, apply the previous result to $\frac{1}{f}$. □

We finish with a final named result.

Theorem 9.5 (Schwarz's Lemma). *Let $f : B(0,1) \to \mathbb{C}$ be analytic. Assume that $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in B(0,1)$. Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all $z \in B(0,1)$. Moreover, these inequalities are strict unless $f(z) = ze^{i\theta}$ for some $\theta \in \mathbb{R}$.*

Proof. Define $g : B(0,1) \to \mathbb{C}$ via

$$g(z) = \begin{cases} f(z)/z & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0. \end{cases}$$

Observe that f is analytic on $B^*(0,1)$, and the $f(0) = 0$ condition ensures continuity at 0 as well. So 0 is a removable singularity of g and g is analytic on $B(0,1)$.

Fix $0 < r < 1$. On $\partial B(0,r)$, $|g(z)| = \frac{|f(z)|}{|z|} \le \frac{1}{r}$. Thus the Maximum Modulus Principle implies $|g(z)| \le \frac{1}{r}$ on $B(0,r)$. Sending $r \to 1$ yields $|g(z)| \le 1$ on $B(0,1)$, so $|g(0)| = |f'(0)| \le 1$ and $|f(z)| \le |z|$ for all $z \in B(0,1)$.

Furthermore, if either equality holds at some $z_0 \in B(0,1)$, $|g|$ attains its maximum at z_0. Hence g is constant with $|g| = 1$. This proves the claim. □

10 September 18

10.1 Phragmén–Lindelöf

Let $D \subset \mathbb{C}$ be unbounded, and let $f : \bar{D} \to \mathbb{C}$ be a function that is analytic on D and continuous on $\bar{D}$. If $|f(z)| \leq M$ on ∂D, it does *not* imply $|f(z)| \leq M$ on D. However, this estimate *can* hold if we assume f doesn't grow *too* fast as $|z| \to \infty$.

Theorem 10.1. *Fix some $\alpha \in [0, 2\pi]$, and let*

$$D \coloneqq \{z : |\mathrm{Arg}(z)| < \tfrac{1}{2}\alpha\}.$$

Let $f : \bar{D} \to \mathbb{C}$ be analytic on D and continuous on $\bar{D}$. Also assume that $|f(z)| \leq M$ for $z \in \partial D$ ad that

$$|f(z)| \leq Ce^{|z|^\rho} \text{ for all } z \in D,$$

where ρ is a real number in the interval $[0, \frac{\pi}{\alpha})$. Then $|f(z)| \leq M$ for all $z \in D$.

Remark. The strict inequality in the definition of D is necessary. Suppose

$$D = \{z : |\mathrm{Arg}(z)| \leq \pi\} = \{z : \Re(z) \geq 0\},$$

and take $f(z) = e^z$. Then f is bounded in ∂D (the imaginary axis), but f as a whole is not bounded in D.

Proof. Choose arbitrary ρ_1 with $\rho < \rho_1 < \frac{\pi}{\alpha}$. Given $\varepsilon > 0$, let $f_\varepsilon(z) = f(z)e^{-\varepsilon z^{\rho_1}}$, where recall that

$$z^{\rho_1} = e^{\rho_1 \mathrm{Log}(z)} = |z|^{\rho_1} e^{i\rho_1 \mathrm{Arg}(z)}.$$

Observe that f_ε is analytic on D and continuous on $\bar{D}$ as well. Now note the bound

$$|f_\varepsilon|(z) = |f(z)| e^{-\varepsilon \Re(z^{\rho_1})} = |f(z)| e^{-\varepsilon |z|^{\rho_1} \cos(\rho_1 \mathrm{Arg}(z))}.$$

But $|\mathrm{Arg}(z)| \leq \frac{\pi}{\alpha}$, so $|\rho_1 \mathrm{Arg}(z)| \leq \frac{\alpha}{2}\rho_1 < c < \frac{\pi}{2}$ for some constant c. Hence

$$\cos(\rho_1 \mathrm{Arg}(z)) \geq C > 0$$

for some constant C, implying

$$|f_\varepsilon(z)| \leq |f(z)| e^{-C\varepsilon|z|^{\rho_1}} \leq Me^{-C\varepsilon|z|^{\rho_1}} \leq M$$

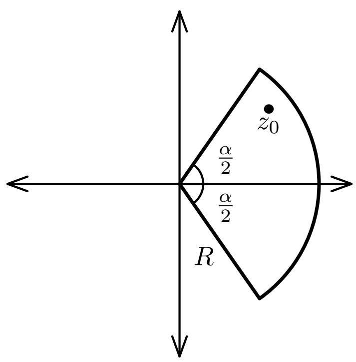
for all $z \in \partial D$.

Furthermore, for $z \in D$, we may write

$$|f_\varepsilon(z)| \leq Ce^{|z|^\rho - c|z|^{\rho_1}\varepsilon} = Ce^{-|z|^\rho(\varepsilon c|z|^{\rho_1-\rho} - 1)},$$

which is less than or equal to M if $|z| \geq R$ for some large R depending on ε, c, ρ_1, and ρ.

Now fix $z_0 \in D$, and consider the region D_{z_0} shown below, which consists of a circular sector of radius R. Observe that $|f(z)| \leq M$ for all $z \in \partial D_{z_0}$, so the Maximum Modulus Principle implies



that $|f_\varepsilon(z_0)| \leq M$. Since z_0 was arbitrary, we deduce that $|f_\varepsilon(z)| \leq M$ for all $z \in D$.

To obtain the desired result, send $\varepsilon \to 0$. $\square$

Before moving on, we briefly touch on one other result that is worth mentioning. The proof can be found in Palka.

Theorem 10.2 (Hadamard's Three Lines Lemma). *Let $D = (0,1) \times D$ be a domain and $f : D \to \mathbb{C}$ be analytic on D and continuous on $\bar{D}$. Further assume that f is bounded on D, with*

$$|f(iy)| \leq M_0 \quad \text{and} \quad |f(1+iy)| \leq M_1 \quad \text{for all } y \in \mathbb{R}.$$

Then

$$|f(x+iy)| \leq M_0^{1-x} M_1^x \quad \text{for all } x, y \in \mathbb{R}.$$

10.2 Sequences and Series and Functions

We now turn our attention to the analysis of multiple functions - that is, of series of functions.

Definition 10.3. Let D be a domain, and let $f, f_n : D \to \mathbb{C}$ be functions.

1. We say that $f_n \to f$ *pointwise* if

$$\lim_{n\to\infty} |f_n(z) - f(z)| = 0 \quad \text{for all } z \in D.$$

2. We say that $f_n \to f$ *uniformly* if

$$\lim_{n\to\infty} \sup_{z\in D} |f_n(z) - f(z)| = 0.$$

3. We say that $f_n \to f$ *normally* if $f_n \to f$ uniformly on every compact set $K \subset D$.

Observe that uniform convergence implies normal convergence implies pointwise convergence, but the other implications may fail.

Example 10.4. Let $D = B(0,1)$, and define $f_n : D \to \mathbb{C}$ via $f_n(z) = z^n$. Then $f_n \to 0$ both pointwise and normally but *not* uniformly.

We know from real analysis that the uniform limit of continuous functions is continuous. It turns out that normal limits of continuous functions are also continuous; this is not hard to show.

We also have a useful criterion for uniform convergence. It is analogous to the equivalence of Cauchy sequences and convergent sequences in $\mathbb{R}$.

Theorem 10.5 (Cauchy Criterion for Uniform Convergence). *Let $(f_n)_{n\geq 1}$ be a sequence of functions defined in a domain D. Then f_n converges uniformly on D if and only if for all $\varepsilon > 0$ there exists $N \geq 0$ such that*

$$\sup_{z\in D} |f_n(z) - f_m(z)| < \varepsilon \quad \text{for all } m, n \geq N.$$

Note that Cauchy's criterion also holds for series, since we define $\sum_{n=1}^\infty f_n(z) = \lim_{N\to\infty} \sum_{n=1}^N f_n(z)$. Cauchy's Criterion then says that the series converges uniformly on D if and only if for all $\varepsilon > 0$ there exists $N \geq 0$ such that

$$\sup_{z\in D} \left| \sum_{k=m}^n f_k(z) \right| < \varepsilon. \tag{10.1}$$

Before moving on to the main result, we state the *Weierstrass M-test*: if $(f_n)_{n\geq 1}$ is a sequence of functions satisfying $|f_n(z)| \leq M_n$, where $\sum_{n=1}^\infty M_n < \infty$, then the sequence f_n converges uniformly.

Theorem 10.6 (Weierstrass). *Let $(f_n)_{n\geq 1}$ be a sequence of analytic functions on an open set D. Assume that $f_n \to f$ normally on D. Then f is analytic, and furthermore $f_k^{(n)} \to f^{(n)}$ normally on D for all $n \geq 0$.*

Proof. We know that f is continuous since $f_n \to f$ normally. Now let $\Delta \subset D$ be ab arbitrary triangle, and observe that

$$\int_{\partial\Delta} f = \int_{\partial\Delta} \lim_{n\to\infty} f_n \, dz \overset{(*)}{=} \lim_{n\to\infty} \int_{\partial\Delta} f_n = 0.$$

Here, the step $(*)$ holds since $\partial\Delta$ is compact; hence $f_n \to f$ uniformly on D and we can swap the limit with the integral. Thus, by Morera's Theorem (Corollary 8.4), f is analytic. In particular, all its derivatives exist.

To prove the second part, recall by our derivative estimates (Theorem 8.8) that

$$\left|f_n^{(k)}(z_0) - f^{(k)}(z_0)\right| \leq \frac{\sup_{|z-z_0|<r} |f_n(z) - f(z)|}{r^k} \cdot k!. \tag{10.2}$$

Now let $K \subset D$ be compact. Since D^c is closed, the distance $\operatorname{dist}(K, D^c)$ is nonzero. Thus we may take $r_0 > 0$ such that F_{r_0}, defined as the closure of an r_0-neighborhood of K, is contained in D. Then applying (10.2) to each z in D yields

$$\sup_{z\in K}\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \frac{k!}{r^k} \sup_{z\in K_{r_0}} |f_n(z) - f(z)| \to 0,$$

as $n \to \infty$. Since K was arbitrary, we deduce that $f_n^{(k)} \to f^{(k)}$ normally in D. $\square$

11 September 20

We start with some remarks about Theorem 10.6.

Remark. The theorem can be applied to series as well, since series are nothing more than sequences of partial sums. In this case, let $(f_n)_{n\geq 1}$ be a sequence of functions. Then if (10.1) holds on every compact subset $K \subset D$, then the series $\sum_{n=1}^{\infty} f_n$ converges to an analytic function f, and

$$f'(z) = \frac{d}{dz}\left[\lim_{n\to\infty}\sum_{k=1}^{n} f_k(z)\right] = \lim_{n\to\infty}\sum_{k=1}^{n} f_k'(z) = \sum_{k=1}^{\infty} f_k'(z).$$

More succinctly, normally convergent series can be differentiated termwise.

Remark. We should compare this with the Weierstrass Approximation Theorem in real analysis, which says that polynomials are dense in $C([a,b];\mathbb{R})$. In particular, any continuous function f can be uniformly approximated by polynomials. Such a result is false in $\mathbb{C}$, since the limit of a sequence of polynomials must be analytic.

Example 11.1. Let D be a bounded domain, and let $(f_n)_{n\geq 1}$ be a sequence of analytic functions on D which are continuous on $\bar{D}$. Assume that $f_n \to f$ on ∂D. We will show that f_n converges uniformly to an analytic function on D.

Recall that since ∂D is compact, $(f_n)_{n\geq 1}$ is uniformly convergent and hence uniformly Cauchy. As a result, for all $\varepsilon > 0$ there exists N such that

$$\sup_{z\in\partial D} |f_n(z) - f_m(z)| < \varepsilon \quad \text{for all } m, n \geq N.$$

Now the Maximum Modulus Principle implies this bound carries over to the interior of D, i.e. for all $\varepsilon > 0$ there exists N such that

$$\sup_{z\in D} |f_n(z) - f_m(z)| < \varepsilon \quad \text{for all } m, n \geq N.$$

Hence f_n converges uniformly on D, and by Theorem 10.6 the limit is analytic.

11.1 Power Series

We focus now on series of functions. A few definitions are in order.

Definition 11.2. In what follows, let $(a_i)_{i\geq 1} \subset \mathbb{C}$.

1. A *power series centered at* $z_0 \in \mathbb{C}$ is a series of the form

$$\sum_{n\geq 0} a_n (z - z_0)^n.$$

   The sum is analytic on the open sets where the series converges normally.

2. Given such a series, we define

$$R := \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} \in [0, \infty]$$

   to be the *radius of convergence of the series*, with the understanding that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

3. The disc $B(z_0, R)$ is defined to be the *disc of convergence*.

These definitions have such names on purpose, as the next theorem shows.

Theorem 11.3. *A power series converges normally and absolutely in its disc of convergence $B(z_0, R)$ and diverges at each point outside $\overline{B(z_0, R)}$. In particular, the sum is analytic in $B(z_0, R)$, and*

$$\left[\sum_{n=0}^{\infty} a_n (z - z_0)^n\right] = \sum_{n=m}^{\infty} a_n \frac{d^m}{dz^m}\left[(z - z_0)^n\right].$$

Proof. Assume $R > 0$, or else there's nothing to prove. It suffices to show that for any $\rho < R$, the series converges uniformly on $B(z_0, \rho)$. Pick $R_1 \in (\rho, R)$¿ Observe that $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$ implies there exists N such that $|a_n|^{1/n} \le R_1^{-1}$ for all $n \ge N$. Then for any $m, n \ge N$,

$$\sum_{k=n}^{m} \left|a_k(z-z_0)^k\right| \le \sum_{k=n}^{m} \frac{1}{R^k}\rho^k \le \sum_{k=n}^{\infty} \left(\frac{\rho}{R}\right)^k = \frac{(\rho/R)^n}{1-\rho/R}.$$

This vanishes as $n \to \infty$, so the series is uniformly Cauchy and hence uniformly convergent.

For the second part, we may assume $R > \infty$ or else there's nothing to prove. Fix z satisfying $|z - z_0| > R$. Observe that since

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} > \frac{1}{|z-z_0|},$$

there eist infinitely many k satisfying $|a_k|^{1/k} \ge |z-z_0|^{-1}$. In turn, $|a_k|^{1/k}|z-z_0| \ge 1$ for infinitely many k, and so the series diverges.

The last part follows from Weierstrass and, in particular, the remark that follows it. □

We finish today with an instructive example.

Example 11.4. The radius of convergence of the series $\sum_{k\ge1} \frac{z^k}{k}$ is

$$\frac{1}{\limsup_{k\to\infty}(\frac{1}{k})^{1/k}} = \liminf_{k\to\infty} k^{1/k} = 1.$$

Hence the series converges to an analytic function in $B(0,1)$.

On the boundary $\partial B(0,1)$, $|z| = 1$, so we may write $z = e^{i\theta}$. If $\theta \equiv 0 \pmod{2\pi}$, then this series is the Harmonic series, which does not converge. However, miraculously the series converges for all other θ! This is a consequence of the following test.

Theorem 11.5 (Dirichlet Test)**.** *Suppose $(a_n)_{n\ge1} \subseteq \mathbb{R}$ and $(b_n)_{n\ge1} \subseteq \mathbb{C}$ are sequences of real and complex numbers, respectively, such that*

$$a_n \textit{ converges monotonically to } 0 \textit{ and } \sup_{N\ge1}\left|\sum_{k=0}^{N} b_k\right| < \infty.$$

Then the series $\sum_{n\ge1} a_n b_n$ converges.

The proof of this relies on Summation by Parts; for more information, see here.

In the $|z| < 1$ case, we may differentiate term by term to get

$$\frac{d}{dz}\sum_{k\ge1}\frac{z^k}{k} = \sum_{k\ge1}\frac{d}{dz}\left[\frac{z^k}{k}\right] = \sum_{k\ge1} z^{k-1} = \frac{1}{1-z}.$$

Thus the series $\sum_{k\ge1}\frac{z^k}{k}$ and $-\operatorname{Log}(1-z)$ are both primitives on $\mathbb{C}\setminus\{z : 1-z \le 0\}$, which contains the unit ball $B(0,1)$, meaning they are equal to each other modulo a constant. Plugging in $z = 0$ tells us this constant is actually zero, and so we obtain the equality

$$\sum_{k\ge1}\frac{z^k}{k} = -\operatorname{Log}(1-z) \text{ on } B(0,1).$$

Notice that the series in the previous example happens to represent $-\operatorname{Log}(1-z)$ on the largest disc centered at 0 for which $-\operatorname{Log}(1-z)$ is analytic. This is no coincidence, and we will prove an analogous result for general power series next time.

12 September 23

No class.

13 September 25

We start with one more example.

Example 13.1. Consider the power series $\sum_{k=0}^{\infty} 2^k (z-5)^{2k}$. Computing the coefficients of this power series is a bit tricky; it turns out they are

$$a_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2^{k/2} & \text{if } k \text{ is even.} \end{cases}$$

Thus $|a_k|^{1/k} \in \{0, \sqrt{2}\}$, so the radius of convergence is $R = 1/\sqrt{2}$ and the disc of convergence is $B(5, 1/\sqrt{2})$. Now observe that on the boundary of this disc, we may write $z = 5 + e^{i\theta}/\sqrt{2}$ for some θ, so

$$\sum_{k=0}^{\infty} 2^k (z-5)^k = \sum_{k=0}^{\infty} 2^k \left(\frac{e^{i\theta}}{\sqrt{2}}\right)^{2k} = \sum_{k=0}^{\infty} e^{2\theta ik}.$$

This series diverges regardless of what θ is.

Furthermore, on $B(5, 1/\sqrt{2})$, this sum equals $(1 - 2(z-5)^2)^{-1}$; again, the disci s the largest one in which the function $(1 - 2(z-5)^2)^{-1}$ is analytic.

13.1 Taylor Series

We are now ready to generalize the above examples.

Theorem 13.2 (Taylor Series)**.** *Let f be analytic on $B(z_0, R)$. Then f has a unique power series representation on $B(z_0, R)$, i.e. there exist complex numbers $c_0, c_1, \ldots$ such that*

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

Here it turns out that $c_k = \frac{f^{(k)}(z_0)}{k!}$.

Proof. Fix $R_1 < R$ and consider $z \in B(z_0, R_1)$. Then Cauchy's Integral Formula yields

$$f(z) = \frac{1}{2\pi i} \oint_{\partial B(z_0, R_1)} \frac{f(\xi)}{\xi - z}\, d\xi.$$

Now write

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \frac{1}{\xi - z_0} \cdot \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\xi - z_0)^k}.$$

This series converges uniformly on the contour $\partial B(z_0, R_1)$, and so

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B(z_0, R_1)} \sum_{k=0}^{\infty} \frac{f(\xi)(z - z_0)^k}{(\xi - z_0)^{k+1}}\, d\xi \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} (z - z_0)^k \int_{\partial B(z_0, R_1)} \frac{f(\xi)}{(\xi - z_0)^{k+1}}\, d\xi = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^{k+1}. \end{aligned}$$

This proves the series converges.

To show uniqueness of this power series expansion, write $f(z) = \sum_k d_k (z - z_0)^k$ for some sequence of complex numbers $(d_k)_{k \geq 0}$. Now differentiating m times and evaluating at z_0 yields $d_m = \frac{f^{(m)}(z_0)}{m!}$. $\square$

Some examples of Taylor Series in action are in order.

Example 13.3. Since $(e^z)^{(k)} = e^z$, we get $e^z = \sum_{k\geq 0}\frac{1}{k!}z^k$. This converges on all of $\mathbb{C}$ since e^z is entire.

Example 13.4. Recall Example 11.4, i.e. $f(z) = \operatorname{Log}(1-z)$. Then we can compute $f^{(k)}(0) = -(k-1)!$, and so $c_k = -\frac{1}{k}$.

We now state a useful corollary.

Corollary 13.5 (Cauchy's Inequality for Coefficients)**.** *Let f be analytic on $B(z_0, R)$, so that f has a power series expansion*

$$f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$$

within that ball. Denote

$$M_{z_0}(r) := \sup_{|z-z_0|=r} |f(z)|$$

for all $r \in (0, R)$. Then

$$|c_k| \leq \frac{M_{z_0}(r)}{r^k} \quad \text{for all } r \in (0, R).$$

Proof. Recall that $c_k = \frac{f^{(k)}(z_0)}{k!}$. Now use the derivative estimates (Theorem 8.8). □

Example 13.6. Let us suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function with $|f(z)| \leq |z|^2$ for all $z \in \mathbb{C}$. What possible functions f are there?

To solve this question, write $f(z) = \sum_{k=0}^{\infty} c_k z^k$ as a power series centered at 0. Now recall by Corollary 13.5 that

$$|c_k| \leq \frac{M_0(r)}{r^k} = \frac{\max_{|z|=r}|f(z)|}{r^k} \leq \frac{r^2}{r^k} = r^{2-k}$$

for *every* $r \in (0, \infty)$. Sending $r \to 0$ yields $c_0 = c_1 = 0$, while sending $r \to \infty$ yields $c_k = 0$ for all $k \geq 3$. Finally, observe that the $k = 2$ case yields $|c_2| \leq 1$. Thus $f(z) = cz^2$ for some complex number c with $|c| \leq 1$.

13.2 Zeroes of Analytic Functions

We now explore yet another instance in which complex functions behave more nicely than real-valued ones. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Observe that f is a C^∞ function. However, $f^{(k)}(0) = 0$ for every $k \geq 0$. This means that there cannot possibly exist a Taylor series for f centered at $x = 0$.

Fortunately, this phenomenon does not occur in the complex case.

Theorem 13.7. *Let f be analytic in a domain $D \subset \mathbb{C}$. Suppose there exists $z_0 \in D$ such that $f^{(k)}(z_0) = 0$ for all $k \geq 0$. Then f is identically zero in D.*

Proof. Define the set

$$A := \{z \in D : f^{(k)}(z) = 0 \text{for all } k \geq 0\}.$$

We claim that A is open in D. To prove this, fix $w \in A$. Since D is open there exists $R > 0$ such that $B(w, R) \subset D$. On this ball, f has a power series expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(w)}{k!}(z-w)^k = \sum_{k=0}^{\infty} \frac{0}{k!}(z-w)^k = 0.$$

Hence $B(w, R) \subset A$ as well, and so A is open.

We also claim that $D \setminus A$ is open in D. This proof is much simpler: if $f^{(j)}(w) \neq 0$ for some j, then by continuity of $f^{(j)}$ we may find an open ball B containing w such that $f^{(j)}(z) \neq 0$ for all $z \in B$.

Hence connectivity of D implies that one of A or $D \setminus A$ must be empty. But A is nonempty by assumption, so $A = D$. □

We end today with a similar-looking statement. We will not prove it today; rather, we will state the result and a definition that follows.

Theorem 13.8. *Suppose f is analytic on a domain D and not identically zero on D. Assume there exists $z_0 \in D$ such that $f(z_0) = 0$. Then there exists a unique $m \in \mathbb{N}$ and a unique $g : D \to \mathbb{C}$ analytic such that*

$$f(z) = (z - z_0)^m g(z) \qquad \textit{and} \qquad g(z_0) \neq 0.$$

Definition 13.9. The value of m is called the *multiplicity of zero* at z_0.

14 September 27

Did not attend class due to a sore throat. What follows is a transcription of the class lecture notes.

14.1 Multiplicity of Zeroes

We prove the result mentioned on Wednesday.

Proof. Uniqueness is easy and left to the reader.

Now we prove existence. Since f is not identically zero, by Theorem 13.7 there exists some $n \in \mathbb{N}$ for which $f^{(n)}(z_0) \neq 0$. Let m be the smallest such n, and set

$$g(z) = \frac{f(z)}{(z-z_0)^m} \quad \text{for all } z \in D \setminus \{z_0\}.$$

Obviously g is analytic on $D \setminus \{0\}$. Hence there exists some $R > 0$ such that for all $z \in B(z_0, R)$ we have the power series expansion

$$f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k = \sum_{k=m}^{\infty} c_k(z-z_0)^k = (z-z_0)^m \sum_{k=0}^{\infty} c_{k+m}(z-z_0)^k.$$

Hence

$$\lim_{z \to z_0} g(z) = c_m = \frac{f^{(m)}(z_0)}{m!} \neq 0.$$

Now we may define $g(z_0) = \frac{f^{(m)}(z_0)}{m!}$ so that g is continuous on D. Thus g is analytic on D. □

Corollary 14.1. *The zero set of an analytic function f (which is not identically zero) on a domain D is an isolated (or discrete) set; that is, the set $\{z : f(z) = 0\}$ can not have any accumulation points in D.*

Proof. If $f(z_0) = 0$, then $f(z) = (z-z_0)^m g(z)$ for some analytic function g with $g(z_0) \neq 0$. By continuity of g, we may find a small neighborhood $U \ni z_0$ with $g(z) \neq 0$ on U. Hence f is nonzero in some neighborhood of z_0, and it follows that the zero-set is discrete. □

This leads into a very important idea in complex analysis.

Corollary 14.2 (Principle of Analytic Continuation)**.** *If f and g are analytic on a domain D and $f(z) = g(z)$ on some set E which has an accumulation point in D, then $f \equiv g$ on D.*

Proof. The function $h := f - g$ is equal to zero on a set E having an accumulation point in D, and hence $f - g \equiv 0$ on D. □

Here are a few examples.

Example 14.3. Suppose f is an entire function with $f(\frac{1}{k}) = \frac{1}{k}$ for all $k \in \mathbb{N}$. What is f?

Well, $f(z) = z$ on the set $E = \{\frac{1}{k} : k \in \mathbb{N}\}$, and this set has an accumulation point - namely $z_0 = 0 \in \mathbb{C}$. Hence $f(z) = z$ on $\mathbb{C}$.

Example 14.4. Suppose f is an entire function with $f(\frac{1}{k}) = \frac{(-1)^k}{k}$ for all $k \in \mathbb{N}$. What is f?

It turns out that such an f cannot exist. Indeed, we see that $f(\frac{1}{k}) = \frac{1}{k}$ when k is even and $f(\frac{1}{k}) = -\frac{1}{k}$ when k is odd. The sets

$$E_1 = \{\tfrac{1}{k} : k \in \mathbb{N} \text{ even}\} \quad \text{and} \quad E_2 = \{\tfrac{1}{k} : k \in \mathbb{N} \text{ odd}\}$$

both have accumulation points in $\mathbb{C}$. But then $f(z) = z$ and $f(z) = -z$ simultaneously, which is not possible.

However, there are many cases of such functions f which are analytic on $\mathbb{C} \setminus \{0\}$; one example is $f(z) = z\sin(\frac{\pi}{2} + \frac{\pi}{z})$.

Corollary 14.2 naturally leads into a definition.

Definition 14.5. Let $S \subset \mathbb{C}$ be a set and let D be a domain containing S. Let $f : S \to \mathbb{C}$ be a function. We say $g : D \to \mathbb{C}$ is an *analytic continuation* of f to D if g is analytic on D and $g|_S = f$. By Corollary 14.2, g is unique if S has an accumulation point in D.

We may now phrase the previous two examples in the language of analytic continuations. The first example yields a unique analytic continuation to $\mathbb{C}$ - namely the function $g(z) = z$. In the second example, there is no analytic continuation from f to $\mathbb{C}$, but there are many to $\mathbb{C} \setminus \{0\}$.

Here is a much stranger example.

Example 14.6. Consider the series

$$f(z) = \sum_{k=0}^{\infty} z^{k!}.$$

Note that the radius of convergence of this series is 1, and so f is analytic on $B(0,1)$. This raises a natural question: is there an analytic continuation of f to any domain $D \supsetneq B(0,1)$? (Note that this is different from saying the radius of convergence is greater than 1.)

The answer is 'NO'. Notice that D would contain a boundary point z_0, and hence it would containing $B(z_0, \rho)$ for some $z_0 \in \partial B(0,1)$ and $\rho > 0$.

Now take $z = re^{2\pi ip/q}$, where p and q are relatively prime and $1-\rho < r < 1$ (so that $z \in B(z_0, \rho)$). Note that

$$f(z) = \sum_{k=0}^{q-1} r^k e^{2\pi ipk/q} + \sum_{k=q}^{\infty} r^k,$$

and hence $|f(z)| \to \infty$ as $r \to 1^-$. Therefore f cannot be continued analytically to any larger domain D.

A few more examples are in order.

Example 14.7. Let $S = [0,1]$, and let $f : S \to \mathbb{R}$ be defined via $f(x) = \frac{1}{1+x^2}$. Let g be an analytic continuation to a domain $D \supset S \cup \{-1\}$. What is $g(-1)$?

In this case, the answer is easy: note that $h(z) = \frac{1}{1+z^2}$ is an analytic continuation to $\mathbb{C} \setminus \{\pm i\}$ since $h = g$ on S. Therefore $g(-1)$ is forced to equal $h(-1) = \frac{1}{2}$.

Observe that since $|h| \to \infty$ as $z \to \pm i$, g cannot be analytic at $\pm i$, i.e. D cannot contain those points.

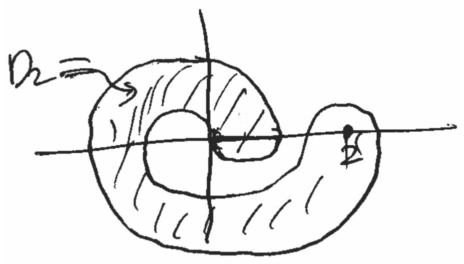
Example 14.8. Let $S = (0,1)$, and let $f : S \to \mathbb{R}^+$ be given by $f(x) = \sqrt{x}$. Let g be an analytic continuation to a domain $D \supset S \cup \{2\}$. What is $g(2)$?

Here the answer is not so clear; we consider two possible domains.

- First consider the domain $D_1 = \mathbb{C} \setminus (-\infty, 0]$. Then $g_1(z) = \sqrt{z} = e^{\operatorname{Log}(z)/2}$ is an analytic continuation of f onto D_1, and so $g_1(2) = \sqrt{2}$.
- Now consider the domain D_2 given pictorially below. In this case, the function

$$g_2(z) = \exp\left(\tfrac{1}{2}(\ln|z| + i\Theta(z))\right)$$

is an analytic continuation of f onto D_2, where Θ is any branch of arg on D_2 with $\Theta(\frac{1}{2}) = 0$. Then $\Theta(2) = 2\pi$, and hence $g_2(2) = -\sqrt{2}$.



Note that in the first example, g is unique, while in the second case g depends on D.

15 September 30

15.1 Regular and Singular Points on the Boundary of the D.O.C

Let $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$, where $0 < R < \infty$ is the radius of convergence of the power series. Recall that f is analytic on $B(z_0, R)$, but behavior on $\partial B(z_0, R)$ is unknown. This warrants a few definitions.

Definition 15.1. We say $\xi \in \partial B(z_0, R)$ is a *regular point* of the boundary if there exists $\delta > 0$ such that f has an analytic continuation to $B(z_0, R) \cup B(\xi, \delta)$. If no such δ exists, we say ξ is a *singular point.*

We follow with a few examples.

Example 15.2. Consider the power series $\sum_{k=0}^{\infty} z^k$, where $z \in B(0,1)$. We know that this series converges to $\frac{1}{1-z}$, so we have found an analytic continuation of the series to $\mathbb{C} \setminus \{1\}$. However, no extension containing $\{1\}$ exists since the series tends to ∞ as $z \to 1$. So 1 is a singular point on the boundary while all other points are regular.

Example 15.3. As per Example 14.6, all boundary points of the series $\sum_{k=0}^{\infty} z^{k!}$ are singular.

Remark. It is important to note that the idea of regularity has nothing to do with whether the series converges or diverges at that point. For instance, in the first example the series diverges everywhere on the boundary, and yet an analytic continuation still exists![5]

In all the examples we have done thus far, we have found the existence of at least one singular point. It turns out this is always the case, as was hinted a few lectures ago.

Theorem 15.4. *Any power series with a finite radius of convergence has at least one singular point on the boundary of its disc of convergence.*

As an example, $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ has radius of convergence 1 and even converges absolutely on $\partial B(0,1)$, and yet some singular point must exist! (It so happens that $z = 1$ works, as we shall see.)

Proof. Let $f(z) = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ be a power series with radius of convergence R. Suppose for the sake of contradiction that f has no singular points. Around each point $z \in \partial B(z_0, R)$ we may find an open ball B_z centered at z such that f has an analytic continuation to $B(z_0, R) \cup B_z$. Now the set of balls $(B_z)_{z \in \partial B(z_0,R)}$ is an open cover of the boundary, and so by compactness there exists a finite subcollection $B_1, B_2, \dots, B_N$ of these balls which also covers $B(z_0, R)$.

Now let

$$U := B(z_0, R) \cup \Big(\bigcup_{j=1}^{N} B_j \Big).$$

Observe that $\partial B(z_0, R)$ is compact and U^c is closed, so their distance is some nonzero δ. Hence there exists $\delta > 0$ such that $B(z_0, R+\delta) \supset U$.

Finally, let

$$g(z) := \begin{cases} f(z) & \text{if } z \in B(z_0, R), \\ f_j(z) & \text{if } z \in B_j \ (j = 1, \dots, N). \end{cases}$$

Observe that g is well-defined and analytic by uniqueness of analytic continuation. So we have extended f to be analytic on $B(z_0, R+\delta)$, and so the radius of convergence of the power series is strictly larger than R. This is a contradiction, and so a singular point must exist. □

We now state without proof a few tests to determine where singular points are located.

Proposition 15.5 (Test for Singular Points)**.** *A point ξ is a regular point if and only if the following holds: for some $b \in (z_0, \xi)$ (the open line segment connecting z_0 and ξ), the power series of $f(z)$ centered at b has radius of convergence strictly greater than $|\xi - b|$.*

Proposition 15.6 (Pringsheim's Theorem)**.** *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be a power series with radius of convergence $R \in (0, \infty)$. Assume that every coefficient c_k is positive. Then $z = R$ is a singular point of the boundary.*[6]

[5] More technically, this is because while the series still converge as one approaches the boundary, this convergence is not uniform.

[6] As a hint to this exercise, show that if R is regular, then *every* boundary point is regular.

15.2 Isolated Singularities of Analytic Functions

Unfortunately, singularities are less tame than zeroes. In order to get these ideas across, a few definitions are needed.

Definition 15.7. Let $U \subset \mathbb{C}$ be open and $z_0 \in U$. Suppose $f : U \setminus \{z_0\} \to \mathbb{C}$ is analytic. Then z_0 is said to be an *isolated singularity* of f.

As an example, note that $z \mapsto z^{-1}$ has an isolated singularity at zero, while $z \mapsto \operatorname{Log}(z)$ has non-isolated singularities on $(-\infty, 0]$.

Definition 15.8. Let z_0 be an isolated singularity of a complex-valued function f.

- We say that z_0 is a *removable singularity* if f can be assigned a value at z_0 such that the resulting function is analytic at z_0. (By previous discussions, it suffices to make f continuous at z_0.)
- We say that z_0 is a *pole* if $\lim_{z\to z_0} |f(z)| = \infty$.
- We say that z_0 is an *essential singularity* if it is neither removable nor a pole.

It turns out that we have a complete characterization of isolated singularities; these requirements are summarized in the theorem below.

Theorem 15.9. *Let z_0 be an isolated singularity of $f : U \setminus \{z_0\} \to \mathbb{C}$. Then the following hold.*

- *The singularity z_0 is removable if and only if f is bounded in a punctured neighborhood of z_0, which in turn holds if and only if $\lim_{z\to z_0} f(z)(z - z_0) = 0$.*
- *The singularity z_0 is a pole if and only if there exists some $n \in \mathbb{N}$ and some $h : U \to \mathbb{C}$ analytic with $h(z_0) \neq 0$ and*
$$f(z) = \frac{h(z)}{(z - z_0)^n} \quad \text{for all } z \in U \setminus \{z_0\}.$$
This value of n is called the order of the pole.
- *[Casorati-Weierstrass] The singularity z_0 is essential if and only if the following holds: for all $\varepsilon > 0$, the set*
$$f(B^*(z_0, \varepsilon) \cap U)$$
is dense in $\mathbb{C}$.

We start proving the theorem now and leave the rest to Wednesday.

Proof. The only nontrivial implication is the last one: $\lim_{z\to z_0} f(z)(z - z_0) = 0$ implies z_0 is removable.

In this case, define $g : U \to \mathbb{C}$ via

$$g(z) := \begin{cases} f(z)(z - z_0) & \text{if } z \neq z_0, \\ 0 & \text{if } z = z_0. \end{cases}$$

Then g is analytic on $U \setminus \{z_0\}$ and continuous on U, so g itself is analytic on U. This means there exists some $m \in \mathbb{N}$ such that $g(z) = (z - z_0)^m h(z)$ in a neighborhood around z_0. But $m - 1 \geq 0$, so $(z - z_0)^{m-1} h(z)$ (which equals $f(z)$ everywhere except z_0) is analytic. Hence z_0 is removable. $\square$

16 October 2

16.1 Isolated Singularities of Analytic Functions (cont.)

We continue with the proof of our theorem.

Proof. Next we prove the second part. The "$\Leftarrow$" direction is obvious. For the "$\Rightarrow$" direction, consider the function g defined by $g(z) = \frac{1}{f(z)}$. Observe that g is analytic on $B^*(z_0, \delta)$ for some $\delta > 0$. Furthermore, $g(z) \to 0$ as $z \to \infty$, so defining $g(0) = 0$ yields that g has a removable singularity at zero. Furthermore, it's actually an isolated zero. So there exists a unique $n \in \mathbb{N}$ and some function $\tilde{g}$ such that $\tilde{g}(z_0) \neq 0$ and $g(z) = (z - z_0)^n g(z_0)$.

Now let $h(z) = f(z)(z - z_0)^n$. Then h is analytic on U, and $h(z) = \frac{1}{\tilde{g}(z)}$ on $B^*(z_0, \delta)$. Since $\tilde{g}(z_0) \neq 0$, h has a removable singularity at z_0, and so h is analytic on U. This completes the proof of the second part.

Finally, we tackle the third part. Assume that z_0 is not a pole and that the condition in Casorati-Weierstrass does *not* hold; that is, there exists $\delta > 0$ and $w \in \mathbb{C}$ such that $f(B^*(z_0, \varepsilon) \cap U) \cap B(w, \delta) = \varnothing$. We will show that z_0 is actually a removable singularity, which proves the result.

With this in mind, let

$$g(z) = \frac{1}{f(z) - w},$$

noting that $|g(z)| \leq \frac{1}{\delta}$ on $B^*(z_0, \varepsilon) \cap U$. So g is bounded on a neighborhood containing z_0, implying that g has a removable singularity at z_0.

Can $g(z_0) = 0$? No, because $\lim_{z \to z_0} |f(z)| \neq \infty$. Thus $g(z_0) \neq 0$, which implies that

$$f(z) = \frac{1}{g(z)} + w \quad \text{is analytic at } z_0.$$

So our singularity z_0 is actually removable, which is what we were after. $\square$

We now go through some examples. Before doing so, however, we note the following connection between zeroes and poles.

Remark. Suppose g is analytic in a neighborhood containing z_0, and that

$$g(z_0) = g'(z_0) = g''(z_0) = \cdots = g^{(m-1)}(z_0) = 0 \quad \text{but} \quad g^{(m)}(z_0) \neq 0.$$

This means the order of zero is equal to m. As a consequence, $\frac{1}{g(z)}$ has a pole at m.

This gives us a good way to test the orders of poles of certain complex-valued functions.

Example 16.1. Let $f(z) = \frac{1}{z - z^4}$. Note that f is analytic on $\mathbb{C} \setminus \{0, 1, \omega, \omega^2\}$, where ω is a (primitive) third root of unity. Furthermore, we have the factorization

$$f(z) = \frac{1}{z(1 - z^3)} = \frac{1}{z(1 - z)(\omega - z)(\omega^2 - z)}.$$

Hence f has poles of order one (so-called *simple poles*) at each of these four points.

Example 16.2. Let $g(z) = \frac{1}{z^2(e^z + 1)}$. Note that g is analytic on $\mathbb{C} \setminus (\{0\} \cup \log(-1))$, where $\log(-1)$ is a set since log is a multi-valued function. Within $B^*(0, \delta)$ for sufficiently small δ, we may write

$$f(z) = \frac{1/(e^z + 1)}{z^2};$$

as $1/(e^z + 1)$ is analytic in this punctured disc, $z = 0$ is a pole of order 2. Furthermore,

$$(e^z + 1)' = e^z \neq 0 \text{ for all } z \in \mathbb{C},$$

so all poles in $\log 1$ are simple.

Example 16.3. Let $h(z) = \cos(z - \frac{1}{z})$. Note that h is analytic on $\mathbb{C} \setminus \{0\}$, which means $z = 0$ is an isolated singularity. However, this time this singularity is *essential*; indeed, since

$$\lim_{x \to \infty} \cos(x - \tfrac{1}{x}) \neq \infty \quad \text{while} \quad \lim_{y \to 0} \cos(iy - \tfrac{1}{iy}) = \infty,$$

this singularity is neither removable nor a pole.

16.2 Singularities at Infinity

We now turn our attention to a different type of singularity.

Definition 16.4. Let f be a complex-valued function.

1. We say f has an *isolated singularity* at ∞ if f is analytic on some set of the form $\{z : |z| > R\}$, where $R > 0$.
2. Let $g(z) = f(1/z)$. We say the singularity at ∞ is (removable/a pole of order m/essential) if and only if g has a (removable/pole of order m/essential) singularity at $z = 0$.

As usual, a few examples are in order.

Example 16.5. The function $f(z) = e^z$ has an isolated singularity at ∞ (but observe that its inverse function $z \mapsto \operatorname{Log}(z)$ does not). Since $g(z) = e^{1/z}$ has an essential singularity at 0, we see that f has an essential singularity at ∞.

Example 16.6. The function $f(z) = z^5 + 2z + 5$ has an isolated singularity at ∞. Since

$$g(z) = \frac{1}{z^5} + \frac{2}{z} + 5 = \frac{1 + 2z^4 + 5z^5}{z^5}$$

has a pole of order 5 at $z = 0$, we see that f has a pole of order 5 at ∞.

Finally, before moving on, we state some simple results regarding poles at ∞.

Proposition 16.7. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and has a removable singularity at ∞. Then f is constant.*

Proof. Since f has a removable singularity at ∞, there exists some $R > 0$ such that f is bounded on $\{z : |z| > R\}$. Furthermore, continuity of f implies that f is also bounded on the set $\{z : |z| \le R\}$. Therefore f is bounded on all of $\mathbb{C}$, and we are done by Liouville. $\square$

Proposition 16.8. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and has a pole of degree m at ∞. Then f is a polynomial of degree m.*

Proof. Since f is entire, we may express it as a power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

centered at zero. By the definition of isolated singularity at ∞, there exists some $R > 0$ such that f is analytic on the set $\{z : |z| > R\}$. In turn, $g(z) := f(\frac{1}{z})$ is analytic on $B^*(0, \frac{1}{R})$. Now g has a pole of order m at 0, which means we may write $g(z) = h(z)/z^m$ for some analytic function h with $h(0) \neq 0$. Therefore $f(z) = z^m h(\frac{1}{z})$.

Now let $r > R$, and fix $k > m$. Since h has a removable singularity at 0, there exists some constant M such that $\left|h(\frac{1}{z})\right| \le M$ for all z with magnitude greater than R. Now observe by the Cauchy estimates that

$$|c_k| \le \frac{\max_{|z|=r} |f(z)|}{r^k} \le \frac{Mr^m}{r^k} = \frac{M}{r^{k-m}}.$$

Sending $r \to \infty$ yields $|c_k| = 0$. Therefore all coefficients of the power series expansion past $k = m$ are equal to zero, proving the claim. $\square$

16.3 Meromorphic Functions

The above discussion about isolated singularities makes it convenient for us to discuss functions which are not necessarily analytic everywhere in a given domain. This warrants a definition.

Definition 16.9. We say f is *meromorphic* on a domain D if f is analytic on D modulo isolated poles.

This definition is somewhat abstract, but in practice many functions are meromorphic.

Proposition 16.10. *Suppose f is meromorphic on $\mathbb{C}$ and has an isolated singularity at ∞ which is not essential (i.e. f is "meromorphic on $\hat{\mathbb{C}}$"). Then $f(z) = \frac{P(z)}{Q(z)}$ for some polynomials P and Q.*

Proof. Since f has an non-essential isolated singularity at ∞, f grows at most polynomially on some set of the form $\{z : |z| > R\}$. Hence all poles of f are contained within the ball $B(0, R)$. But each pole of f is isolated, and so there can only exist finitely many poles $z_1, \ldots, z_n$ of f.

Denote by $m_1, \ldots, m_n$ the orders of these poles. By applying part two of Theorem 15.9 repeatedly, we may write

$$f(z) = \frac{h_1(z)}{(z - z_1)^{m_1}} = \cdots = \frac{h(z)}{\prod_{j=1}^{n}(z - z_j)^{m_j}},$$

where h is an entire function. Now observe that the singularity of h at ∞ is either a pole or removable, and so by the previous proposition we deduce that h is a polynomial. $\square$

17 October 4

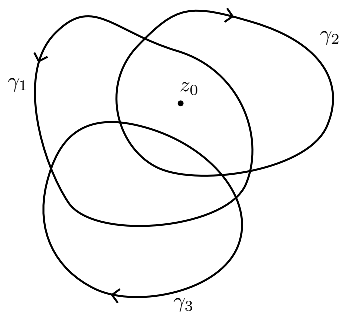
17.1 Global Cauchy Theorem

We now seek to generalize Cauchy's Theorem to non-convex sets. To accomplish this, we need a few definitions.

Definition 17.1. Let $U \subset \mathbb{C}$ be a domain.

1. A *cycle* (in U) is a finite collection of closed contours. We write $\sigma = (\gamma_1, \dots, \gamma_p)$, where γ_1 through γ_p are closed contours in U.
2. We define
$$\int_\sigma f(z)\, dz := \sum_{k=1}^{p} \int_{\gamma_k} f(z)\, dz.$$
3. Given $z \notin |\sigma| := \bigcup_{j=1}^p |\gamma_j|$, we may define/write
$$n(\sigma, z) := \int_\sigma \frac{2\pi i}{\xi - z}\, d\xi = \sum_{j=1}^{k} \int_{\gamma_k} \frac{2\pi i}{\xi - z}\, d\xi = n(\gamma_1, z) + \cdots + n(\gamma_p, z).$$
4. We say a cycle σ is 0-*homologous in U* if $n(\sigma, z) = 0$ for every $z \notin U$.

Here is an example.



Example 17.2. Fix $z_0 \in \mathbb{C}$. Let $U = \mathbb{C} \setminus \{z_0\}$, and consider the cycle $\sigma = (\gamma_1, \gamma_2, \gamma_3)$ shown to the right. Then
$$\begin{aligned} n(\sigma, z_0) &= n(\gamma_1, z_0) + n(\gamma_2, z_0) + n(\gamma_3, z_0) \\ &= 1 + (-1) + 0 = 0, \end{aligned}$$
and so σ is 0-homologous in U. We can run through the same logic to deduce that (γ_3) and (γ_1, γ_2) are also 0-homologous in U, but (γ_1), (γ_2), (γ_1, γ_3), and (γ_2, γ_3) are not.

With this, we are ready to state the Global version of the Cauchy Integral Formula.

Theorem 17.3 (Global CIF). *Let $U \subset \mathbb{C}$ be open. Suppose $f : U \to \mathbb{C}$ is analytic and that σ is a cycle that is 0-homologous in U. Then for every $z \in U \setminus |\sigma|$ we have*
$$n(\sigma, z) f(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(\xi)}{\xi - z}\, d\xi.$$

Proof. Define the function $G : U \times U \to \mathbb{C}$ via
$$G(z, \xi) = \begin{cases} \frac{f(z) - f(\xi)}{z - \xi} & \text{if } \xi \neq z, \\ f'(\xi) & \text{if } \xi = z. \end{cases}$$
Observe that G is analytic in z for fixed ξ and vice versa. Further, G is continuous on $U \times U$: for $\xi \neq z$ continuity at (ξ, z) is obvious, while for $\xi = z$ we may use the equality
$$f(z) - f(\xi) = \int_{[\xi, z]} f'(\eta)\, d\eta.$$

Now le $U' := \{z \in \mathbb{C} \setminus |\sigma| : n(\sigma, z) = 0\}$. Note that $U' \subset U^c$ by assumption, which implies $U' \cup U = \mathbb{C}$. Furthermore, U' is open, since it is a union of open components of $\mathbb{C} \setminus |\sigma|$.

Now define $g : \mathbb{C} \to \mathbb{C}$ via
$$g(z) := \begin{cases} \int_\sigma G(z, \xi)\, d\xi & \text{if } z \in U, \\ \int_\sigma \frac{f(\xi)}{z - \xi}\, d\xi & \text{if } x \notin U. \end{cases}$$
(Most of) the rest of this proof consists of a series of claims about g.

Claim 1: g is well-defined. To prove this, note that $z \in U \cap U'$ implies $n(\sigma, z) = 0$ and that $z \in U$. Thus

$$\int_\sigma G(z,\xi)\, d\xi = \int_\sigma \frac{f(z) - f(\xi)}{z - \xi}\, d\xi = \int_\sigma \frac{f(z)}{z-\xi}\, d\xi + \int_\sigma \frac{f(\xi)}{\xi - z}\, d\xi = \int_\sigma \frac{f(\xi)}{\xi - z}\, d\xi.$$

In the last step, we use the fact that the former integral is zero since $n(\sigma, z) = 0$.

Claim 2: g is entire. To prove this, we split into cases. If $z \in U'$ - that is, $g(z) = \int_\sigma \frac{f(\xi)}{z-\xi}\, d\xi$ - then we obtain continuity of g via integration by parts. If instead $z \in U$ - that is, $g(z) = \int_\sigma G(z,\xi)\, d\xi$ - then we need a different approach. Instead, observe that, for any triangle $\Delta \subset U$, Fubini's Theorem for Riemann integrals yields

$$\int_{\partial\Delta} \int_\sigma G(z,\xi)\, d\xi\, dz = \int_\sigma \int_{\partial\Delta} G(z,\xi)\, dz\, d\xi = \int_\sigma 0\, d\xi = 0.$$

Hence the integral of g over any triangle $\Delta \subset U$ is zero, and so Morera's Theorem yields the desired conclusion.

Claim 3: $\lim_{|z|\to\infty} g(z) = 0$. To prove this, take R sufficiently large so that $B(0,r) \supset |\sigma|$. If $z \notin B(0,R)$, then z lies in the unbounded component of $\mathbb{C} \setminus |\sigma|$. This means $n(\sigma, z) = 0$, so $z \in U'$. Now

$$|g(z)| = \left| \int_\sigma \frac{f(\xi)}{\xi - z}\, d\xi \right| \le \frac{\max_{\xi \in |\sigma|} |f(\xi)| \cdot \operatorname{length}(\sigma)}{\operatorname{dist}(z, |\sigma|)},$$

which tends to zero as $z \to 0$.

Thus g is a bounded entire function, which implies by Liouville that $g(z)$ is constant; since $g(z) \to 0$ as $|z| \to \infty$, this constant must be zero. Thus

$$0 = \int_\sigma G(z,\xi)\, d\xi = -2\pi i \cdot n(\sigma, z) f(z) + \int_\sigma \frac{f(\xi)}{\xi - z}\, d\xi$$

for every $z \in U$, and we may conclude. □

The previous theorem admits a corollary in much the same way the Local Cauchy Integral Formula did.

Corollary 17.4. *Define f, U, and σ as above. Then $\int_\sigma f(\xi)\, d\xi = 0$.*

Proof. Fix $z_0 \in U \setminus |\sigma|$, and apply the previous theorem to the function $\xi \mapsto f(\xi)(\xi - z_0)$ at $\xi = z_0$. □

17.2 Simply Connected Domains

It turns out there is another way to extend the Local Cauchy Integral Formula, namely to so-called *simply connected domains* (rather than just convex domains).

Definition 17.5. Let $U \subset \mathbb{C}$ be a domain. We say that U is *simply connected* if every cycle in U has zero modulus.

We summarize the main results of simply-connected domains (as they pertain to complex analysis) in the propositions below.

Proposition 17.6. *Suppose U is a simply connected domain. Then the following hold.*

1. *The Cauchy Integral Formula and Cauchy's Theorem both hold for any analytic function $f : U \to \mathbb{C}$ and any cycle in U.*
2. *If f is analytic in U, then f has a primitive in U and $\log f$ has a branch in U.*

Proposition 17.7. *Let $U \subset \mathbb{C}$ be a domain. Then the following are equivalent.*

1. *The domain U is simply connected.*
2. *Every closed contour in U is contractible.*
3. *The set $\hat{\mathbb{C}} \setminus U$ is a connected set in $\hat{\mathbb{C}}$.*

The second half of the Proposition 17.7 warrants further discussion. More specifically, what does it mean to be "contractible"?

Definition 17.8. Suppose $\gamma_0 : [0,1] \to \mathbb{C}$ and $\gamma_1 : [0,1] \to \mathbb{C}$ are closed contours in a domain U.

1. We say that γ_0 and γ_1 are *homotopic* if there exists some continuous map $H : [0,1] \times [0,1] \to U$ such that $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$, and $H(s,0) = H(s,1)$ for all $0 \le s \le 1$. (That is, for each $0 \le s \le 1$, the curve $H(s,t)$ is a closed contour in U.)
2. We say that a closed contour γ is *contractible* if it is homotopic to a constant path.

The punchline is that if γ_0 and γ_1 are homotopic, then $n(\gamma_0, z) = n(\gamma_1, z)$ for $z \notin U$, which (after a bit of work) implies nice results about integration over γ_0 and γ_1. The details are technical, though, and so we refer to Palka for them.

18 October 7

18.1 Laurent Series

We now extend our notion of power series to meromorphic functions.

Definition 18.1. Let $z_0 \in \mathbb{C}$.

1. A *Laurent series* centered at $z_0 \in \mathbb{C}$ is a series of the form
$$\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}.$$
2. Define
$$R_O := \frac{1}{\limsup_{k\geq 0} |a_k|^{1/k}} \qquad \text{and} \qquad R_I := \limsup_{k\to\infty} |a_k|^{1/k}$$
to be the *outer* and *inner* radii of convergence of the Laurent series, respectively.
3. The annulus $\{z : R_I < |z-z_0| < R_O\}$ is the *annulus of convergence* of the Laurent series.

These terms are so-named for the obvious reasons.

Theorem 18.2. *Consider the Laurent series $f(z) := \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ along with its inner and outer radii of convergence R_I and R_O. Then the following hold.*

1. *The sum $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely and normally on $B(z_0, R_O)$, while it diverges on $\overline{B(z_0,R_O)}^c$.*
2. *The sum $\sum_{k=0}^{\infty} a_{-k}(z-z_0)^{-k}$ converges absolutely and normally on $\overline{B(z_0,R_I)}^c$, while it diverges on $B(z_0,R_I)$.*
3. *The series for f converges absolutely and normally on its annulus of convergence, while it diverges outside of said annulus. Furthermore, the sum is analytic inside the annulus.*
4. *For each $k \in \mathbb{Z}$, we have the equality*
$$a_k = \frac{1}{2\pi i}\oint_{\partial B(z_0,r)} \frac{f(z)}{(z-z_0)^{k+1}} \quad \text{for all } R_I < r < R_O.$$

Proof. The proofs of the first three parts follow those from the power series unit. For the fourth part, use normal convergence to swap the sum with the limit. □

We now prove that Laurent series are unique.

Theorem 18.3. *Let f be analytic on the annulus $D = \{z : a < |z-z_0| < b\}$. Then f has a unique Lorent series representation $f(z) = \sum_{k\in\mathbb{Z}} a_k(z-z_0)^k$ on D, where the sequence $(a_k)_{k\in\mathbb{Z}}$ is defined as above.*

Proof. Fix $z \in D$. Choose R_1 and R_2 so that
$$a < R_1 < |z-z_0| < R_2 < b.$$
Then the cycle $\sigma = (\partial B(z_0,R_2), -\partial B(z_0,R_1))$ is zero-homologous, and furthermore $n(\sigma,z) = 1$¿ As a result,
$$f(z) = \frac{1}{2\pi i}\oint_{\partial B(z_0,R_2)} \frac{f(\xi)}{\xi - z}\,d\xi - \frac{1}{2\pi i}\oint_{\partial B(z_0,R_1)} \frac{f(\xi)}{\xi - z}\,d\xi.$$
Now we proceed as in the power series case. For the first integral, we may write
$$\frac{1}{\xi - z} = \frac{1}{1-\frac{z-z_0}{\xi-z_0}}\cdot\frac{1}{\xi-z_0} = \frac{1}{\xi-z_0}\sum_{k\geq 0}\left(\frac{z-z_0}{\xi-z_0}\right)^k,$$
while for the second, we may write
$$\frac{1}{\xi - z} = -\frac{1}{1-\frac{\xi-z_0}{z-z_0}}\cdot\frac{1}{z-z_0} = \frac{1}{z-z_0}\sum_{k\geq 0}\left(\frac{\xi-z_0}{z-z_0}\right)^k.$$
Now swap the summation with the integral to obtain the desired Laurent series expansion.

For uniqueness, fix $j \in \mathbb{Z}$. Multiply the sum by $(z-z_0)^{-j-1}$ and integrate over $\partial B(z_0,r)$ for some $r \in (a,b)$. □

This technique of building geometric series in multiple ways is very useful in determining Laurent series in different annuli. We observe this with an example.

Example 18.4. Let $f(z) = \frac{1}{(z-1)(z+2i)}$; we wish to determine representatives for f centered at $z_0 = 0$.

Notice that z is entire on the set

$$D = \{z : |z| < 1\} \cup \{z : 1 < |z| < 2\} \cup \{z : 2 < |z|\}.$$

Thus there are three possible annuli to consider.

Notice that we may write

$$\frac{1}{(z-1)(z+2i)} = \frac{(1+2i)^{-1}}{z-1} - \frac{(1+2i)^{-1}}{z+2i}$$

as the partial fraction decomposition of f. Now observe that

$$\frac{1}{z-1} = \begin{cases} -\sum_{k\geq 0} z^k & \text{if } |z| < 1, \\ \frac{1}{z}\sum_{k\geq 0}(\frac{1}{z})^k & \text{if } |z| > 1 \end{cases} \qquad \text{and} \qquad \frac{1}{z+2i} = \begin{cases} \frac{1}{2i}\sum_{k\geq 0}(\frac{-z}{2i})^k & \text{if } |z| < 2, \\ \frac{1}{z}\sum_{k\geq 0}(\frac{-2i}{z})^k & \text{if } |z| > 2. \end{cases}$$

Thus we may generate representatives in each annulus by picking the correct power series representation. For example, on the annulus $\{1 < |z| < 2\}$ we have

$$f(z) = \frac{1}{(1+2i)z}\sum_{k\geq 0}\left(\frac{1}{z}\right)^k - \frac{1}{2i(1+2i)}\sum_{k\geq 0}\left(\frac{-z}{2i}\right)^k.$$

Notice that, when the inner radius of the annulus is zero, we obtain a useful corollary.

Corollary 18.5. *Let f be analytic on the punctured disc $B^*(z_0, R)$. Then there exists a Laurent series representation*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k.$$

Moreover, we can classify the singularity at z_0 based on the coefficients of this series expansion:

- *The singularity z_0 is removable if and only if $a_k = 0$ for all $k \leq 0$;*
- *The singularity z_0 is a pole of order m if and only if $a_k = 0$ for all $k < -m$ while $a_{-m} \neq 0$;*
- *The singularity z_0 is essential if and only if $a_k \neq 0$ for infinitely many $k < 0$.*

Proof. The first part is a direct consequence of Theorem 18.3. For the second part, use the fact that

$$a_k = \frac{1}{2\pi i}\oint_{\partial B(z_0, r/2)} \frac{f(z)}{(z - z_0)^{k+1}}\, dz.$$

□

18.2 Residues

We now give, arguably, the most important definitions in Complex Analysis.

Definition 18.6. Fix $z_0 \in \mathbb{C}$.

1. Suppose f has an isolated singularity at z_0, so that there exists a Laurent series expansion

$$f(z) = \sum_{k\in\mathbb{Z}} a_k(z - z_0)^k$$

converging normally and absolutely in a punctured neighborhood $D \ni z_0$. The quantity

$$a_{-1} = \frac{1}{2\pi i}\oint_{\partial B(z_0, r)} f(z)\, dz,$$

where $B(z_0, r) \subset D$, is called the *residue* of f at z_0, and is denoted by $\operatorname{Res}(f; z_0)$.

2. Suppose f has an isolated singularity at ∞, so that there exists a Laurent series expansion

$$f(z) = \sum_{k\in\mathbb{Z}} a_k z^k$$

converging normally and absolutely on some unbounded annulus $D := \{z : |z| > R\}$. The quantity

$$a_{-1} = \frac{1}{2\pi i}\oint_{\partial B(0,r)} f(z)\,dz,$$

where $r > R$, is denoted by $-\operatorname{Res}(f;\infty)$.

Remark. The negative sign makes more sense when viewed through the lens of the one-point compactification of $\mathbb{C}$. On the Riemann sphere, the points 0 and ∞ are opposite each other, so from the perspective of ∞, the orientation of the contour $\partial B(0,R)$ is reversed.

Some examples of residues are in order. We will explore more involved applications of residues in the coming days.

Example 18.7. Consider the function $f(z) = e^{(1+z)/z}$. Then f has a Laurent series expansion

$$e^{(1+z)/z} = e\cdot e^{1/z} = e\sum_{k\geq 1}\frac{1}{z^k k!}.$$

around $z_0 = 0$. Hence $\operatorname{Res}(f;0) = e$.

Example 18.8. Consider the function $f(z) = \frac{1}{z^4-1}$. Observe that f has singularities precisely at the fourth roots of unity. Notice further that these singularities are simple poles, i.e. poles of order 1.

In general, suppose $f(z) = \frac{h(z)}{g(z)}$, where $g(z_0) = 0$ is a simple zero and $h(z_0) \neq 0$. Write $g(z) = (z-z_0)\tilde{g}(z_0)$. Then

$$\frac{h(z)}{g(z)} = \frac{h(z)}{(z-z_0)\tilde{g}(z_0)} = \frac{h(z)/\tilde{g}(z_0)}{z-z_0}$$

is a Laurent series expansion of f at $z = z_0$, so

$$\operatorname{Res}(f;z_0) = \frac{h(z_0)}{\tilde{g}(z_0)} = \frac{h(z_0)}{g'(z_0)}.$$

19 October 9

19.1 Higher-order Poles

We start today with a simple warmup.

Example 19.1. Let $f(z) = \frac{1}{(1-z)^2}$; what is a (Laurent) series representation for f at 0?

There are two such representations: one valid in $B(0,1)$ and one valid in $A(1,\infty)$. To obtain the former representation, differentiate the equality $\frac{1}{1-z} = \sum_{k\geq 0} z^k$ to get

$$\frac{1}{(1-z)^2} = \sum_{k\geq 1} kz^{k-1}.$$

To obtain the second representation, instaed differentiate the equality

$$\frac{1}{1-z} = -\frac{1}{z}\cdot\frac{1}{1-1/z} = -\frac{1}{z}\sum_{k\geq 0}\frac{1}{z^k}$$

to get

$$\frac{1}{(1-z)^2} = \sum_{k\geq 0}(k+1)z^{-k-2}.$$

The previous example showed how to obtain series representations for a series that has a pole of order 2 at some isolated singularity. We can crank this up a notch and generalize.

Example 19.2. Let f be meromorphic with a pole of order n at z_0. What is a formula for $\operatorname{Res}(f; z_0)$?

In this case, we may write

$$f(z) = \sum_{k\geq -n} a_k(z-z_0)^k = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + \sum_{k\geq 0} a_k(z-z_0)^k$$

within some neighborhood around zero. Multiplying this by $(z-z_0)^n$ yields

$$f(z)(z-z_0)^n = a_{-n} + \cdots + a_{-1}(z-z_0)^{n-1} + (z-z_0)^n g(z).$$

Now differentiate this equality $n-1$ times, so that

$$\left(\frac{d}{dz}\right)^{n-1}[f(z)(z-z_0)^n] = (n-1)!a_{-1} + \left(\frac{d}{dz}\right)^{n-1}[(z-z_0)^n g(z)].$$

But now observe that at least one factor of $z-z_0$ will remain on the right hand factor after differentiation, meaning that said derivative tends to zero as $z\to z_0$. It follows that

$$\operatorname{Res}(f; z_0) = a_{-1} = \frac{1}{(n-1)!}\lim_{z\to z_0}\left(\frac{d}{dz}\right)^{n-1}[f(z)(z-z_0)^n].$$

19.2 The Residue Theorem

Now we arrive at, arguably, the second most important theorem in Complex Analysis.

Theorem 19.3 (Residue Theorem)**.** *Let U be an open set; assume that f is analytic in U except at finitely many distinct isolated singularities $z_1,\ldots,z_j$. Further assume that σ is a 0-homologous cycle in U which avoids the singularities $z_1,\ldots,z_j$. Then*

$$\int_\sigma f(z)\,dz = 2\pi i\sum_{k=1}^{j} n(\sigma,z)\operatorname{Res}(f;z_k).$$

Proof. Since σ is compact, we may choose $\varepsilon>0$ small enough that

$$\overline{B(z_k,\varepsilon)} \subset U\setminus\{z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_j$$

for every k between 1 and j, inclusive.

Now for each such k, let σ_k be the cycle consisting of $-n(\sigma, z_0)$ copies of $\partial B(z_k, \varepsilon)$. The crucial claim is that

$$\tilde{\sigma} = (\sigma, \sigma_1, \ldots, \sigma_j)$$

is 0-homologous in $U \setminus \{z_1, \ldots, z_j\}$. Indeed, first note that if $z \notin U$, then

$$n(\tilde{\sigma}, z) = \underbrace{n(\sigma, z)}_{=0} + \underbrace{n(\sigma_1, z)}_{=0} + \cdots + \underbrace{n(\sigma, z)}_{=0} = 0.$$

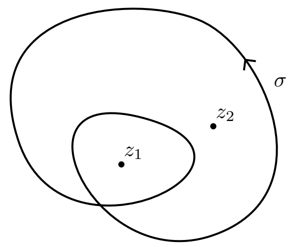
Then remark that for $z = z_k$, we have $n(\tilde{\sigma}, z_k) = n(\sigma, z_k) + n(\sigma_k, z_k) = 0$ by construction.

Thus, the Global Cauchy Theorem (or, more specifically, Corollary 17.4) implies $\int_\sigma f(z)\,dz = 0$, or

$$\begin{aligned} 0 &= \int_\sigma f(z)\,dz - \sum_{k=1}^{j} n(\sigma, z_k) \int_{\partial B(z_k, \varepsilon)} f(z)\,dz \\ &= \int_\sigma f(z)\,dz - \sum_{k=1}^{j} 2\pi i n(\sigma, z_k) \operatorname{Res}(f; z_k). \end{aligned}$$

This completes the proof. □

This result is useful. As an example, consider the contour σ to the right. Then σ is clearly 0-homologous in any open set U containing it, and so the Residue Theorem tells us that



$$\frac{1}{2\pi i} \int_\sigma f(z)\,dz = 2\operatorname{Res}(f; z_1) + \operatorname{Res}(f; z_2).$$

In some sense, this is a generalization of Cauchy's Integral Formula.

19.3 Calculating Integrals using Residues

Somewhat surprisingly, the residue theorem can be used to compute many seemingly-impossible integrals of *real-valued* functions. In some sense, this should not be surprising: we have seen the applicability of complex integration techniques to real-valued integrals already when we computed the Fresnel integral (Example 7.2). However, it turns out that the variety of examples that are now open to us is vast. The following discussion will take us several days.

19.3.1 Application 1: Rational Functions in Sine and Cosine

Suppose we wish to compute an integral of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta,$$

where $R(u, v)$ is a rational function in u and v. The idea here is to let $z = e^{i\theta}$; then as long as we can find an analytic function f such that

$$\int_0^{2\pi} f(e^{i\theta}) i e^{i\theta}\,d\theta = \int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta,$$

we can then exhibit the change of variables $z = e^{i\theta}$ to transform the integral into $\int_{\partial B(0,1)} f(z)\,dz$. Fortunately, such an integral exists: since

$$\cos\theta = \frac{z + z^{-1}}{2} \quad \text{and} \quad \sin\theta = \frac{z - z^{-1}}{2i}$$

whenever $z = e^{i\theta}$, we see that

$$f(z) = R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{1}{iz}$$

works.

As an example, suppose we wish to compute

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{\partial B(0,1)} \frac{dz/iz}{2+\frac{z+z^{-1}}{2}} = \frac{2}{i}\int_{\partial B(0,1)} \frac{dz}{z^2+4z+1}.$$

The function $z \mapsto \frac{1}{z^2+4z+1}$ is meromorphic, with singularities at $z = -2 \pm \sqrt{3}$. Observe that only $z = -2+\sqrt{3}$ lies inside $B(0,1)$. As a result,

$$\int_{\partial B(0,1)} \frac{dz}{z^2+4z+1} = 2\pi i \cdot \operatorname{Res}\left(\frac{1}{z^2+4z+1}; -2+\sqrt{3}\right),$$

which, in light of the remark at the end of Example 18.8, equals

$$2\pi i \cdot \frac{1}{(z^2+4z+1)'|_{z=-2+\sqrt{3}}} = \frac{2\pi i}{(2z+4)_{z=-2+\sqrt{3}}} = \frac{\pi i}{\sqrt{3}}.$$

Thus the original integral equals $\frac{2}{i}\cdot\frac{\pi i}{\sqrt{3}} = \boxed{\frac{2\pi}{\sqrt{3}}}$.

19.3.2 Application 2: Rational Functions without Real Poles

Now suppose we wish to compute an integral of the form $\int_{-\infty}^{\infty} \mathcal{O}(x)\,dx$, where $\mathcal{O}(x) = \frac{p(x)}{q(x)}$ is a rational function with no poles on $\mathbb{R}$ and $\deg q \geq 2 + \deg p$. The key idea here is to recall that

$$\int_{-\infty}^{\infty} \mathcal{O}(x)\,dx = \lim_{r\to\infty}\int_{-r}^{r} \mathcal{O}(x)\,dx.$$

With this in mind, consider the contour $\gamma_R := [-R,R]\cup C_R$, where C_R is the semicircle centered at zero with radius R positioned in the upper half plane. Then the Residue Theorem tells us that

$$2\pi i \sum_{z_j \text{ poles in } \gamma_R} \operatorname{Res}(\gamma_R; z_j) = \int_{\gamma_R} \mathcal{O}(z)\,dz = \int_{-R}^{R} \mathcal{O}(x)\,dx + \int_{C_R} \mathcal{O}(z)\,dz.$$

Now the degree restriction on p and q ensures that $\int_{C_R} \mathcal{O}(z)\,dz$ tends to zero as $R\to\infty$, and so only the real part will be left once we take a limit.

As an example, let us compute $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$. Let $R > 0$ be large, and define γ_R and C_R as before. On one hand,

$$\int_{\gamma_R} \frac{1}{z^4+1}\,dz = \int_{-R}^{R} \frac{1}{x^4+1}\,dx + \int_{C_R} \frac{1}{z^4+1}\,dz.$$

On the other hand, if $R > 1$, then the inside of γ_R contains exactly two of the four isolated singularities of $\frac{1}{z^4+1}$, namely $e^{\pi i/4}$ and $e^{3\pi i/4}$. Therefore

$$\begin{aligned}\int_{\gamma_R} \frac{1}{z^4+1}\,dz &= 2\pi i \operatorname{Res}\left(\frac{1}{z^4+1}; e^{\pi i/4}\right) + 2\pi i \operatorname{Res}\left(\frac{1}{z^4+1}; e^{3\pi i/4}\right) \\ &= \frac{2\pi i}{4z^3|_{z=e^{\pi i/4}}} + \frac{2\pi i}{4z^3|_{z=e^{3\pi i/4}}} = \frac{\pi}{\sqrt{2}}.\end{aligned}$$

Finally, send $R\to\infty$. Observe that

$$\left|\int_{C_R} \frac{1}{z^4+1}\,dz\right| \leq \operatorname{length}(C_R)\cdot \max_{z\in C_R}\left|\frac{1}{z^4+1}\right| = \frac{\pi R}{R^4-1},$$

which tends to zero as $R\to\infty$. Hence

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{1}{x^4+1}\,dx = \lim_{R\to\infty}\int_{\gamma_R} \frac{1}{z^4+1}\,dz = \boxed{\frac{\pi}{\sqrt{2}}}.$$

20 October 11

20.1 Calculating Integrals using Residues (cont.)

The jouney continues.

20.1.1 Application 3: Trigonometric Times a Rational Function

Now suppose we wish to compute an integral of the form $\int_{-\infty}^{\infty} \mathcal{O}(x)\Phi(x)\,dx$, where

- the function $\Phi(x)$ is either $\cos(ax)$ or $\sin(ax)$ for some $a > 0$; and
- the function $\mathcal{O}(x)$ is a rational function of the form $\frac{p(x)}{q(x)}$ with $\deg q \geq \deg p + 2$ and no poles in $\mathbb{R}$.

The idea here is to combine the approaches from the first and second applications: compute $\int_{\gamma_R} \mathcal{O}(z)e^{iaz}$ (where γ_R is the contour from the second example), send $R \to \infty$, and take either a real or an imaginary part.

For example, let us compute $I := \int_{-\infty}^{\infty} \frac{\sin x}{x^2+x+1}\,dx$. In this case, we may write

$$\begin{aligned} I &= \Im\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+x+1}\,dx\right) = \Im\left(\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{x^2+x+1}\,dx\right) \\ &= \Im\left[\lim_{R\to\infty}\left(\int_{\gamma_R} \frac{e^{iz}}{z^2+z+1}\,dz - \int_{C_R} \frac{e^{iz}}{z^2+z+1}\,dz\right)\right]. \end{aligned}$$

To compute the former integral, remark that the function $z \mapsto \frac{e^{iz}}{z^2+z+1}$ has two simple poles, namely at $z = \frac{1}{2}(1 \pm i\sqrt{3})$. However, only $z = \frac{1}{2}(1 + i\sqrt{3})$ appears in inside(γ_R), and in particular only if $R > \frac{1}{2}\sqrt{10}$. Thus

$$\begin{aligned} \int_{\gamma_R} \frac{e^{iz}}{z^2+z+1}\,dz &= 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{z^2+z+1}; \frac{-1+i\sqrt{3}}{2}\right) \\ &= \left.\frac{e^{iz}}{2z+1}\right|_{z=\frac{-1+i\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}e^{-i/2}. \end{aligned}$$

Now the second integral can be bounded in the standard way; in particular,

$$\left|\int_{C_R} \frac{e^{iz}}{z^2+z+1}\,dz\right| \leq \pi R \cdot \max_{z\in C_R}\left|\frac{e^{iz}}{z^2+z+1}\right| \leq \frac{\pi R}{R^2-R-1},$$

which tends to zero as $R \to \infty$. Thus

$$I = \Im\left(\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}e^{-i/2}\right) = \boxed{\frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}\sin\left(\frac{1}{2}\right)}.$$

20.1.2 Application 4: Trigonometric Times a Rational Function (cont.)

In the previous application, we made the assumption that $\mathcal{O}$ has no poles on $\mathbb{R}$. What if it does?

Instead of briefly explaining the theory, we dive straight into an example and compute $I := \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+x+1)}\,dx$. Here, we will compute

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{\{\varepsilon<|x|<R\}} \frac{\sin x}{x(x^2+x+1)} =: \mathrm{P.V.}\int \frac{\sin x}{x(x^2+x+1)},$$

otherwise known as the *principal value* of the integral in question. Taking the imaginary part of this will yield the value of I.

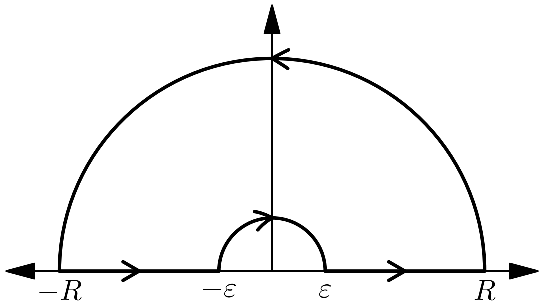
In this vein, consider the contour

$$\gamma_{R,\varepsilon} := [-R, -\varepsilon] \cup (-C_\varepsilon) \cup [\varepsilon, R] \cup C_R,$$

where C_ε and C_R are defined as in the previous examples. (See the diagram below.) Then

$$\int_{\gamma_{R,\varepsilon}} \frac{e^{iz}}{z^2+z+1}\,dz = \int_{\{\varepsilon<|x|<R\}} \frac{e^{iz}}{z^2+z+1}\,dz + \int_{-C_\varepsilon} \frac{e^{iz}}{z^2+z+1}\,dz + \int_{C_R} \frac{e^{iz}}{z^2+z+1}\,dz.$$

The integral on the left hand side equals



$$2\pi i \operatorname{Res}\left(\frac{e^{iz}}{z^2+z+1}; \frac{-1+i\sqrt{3}}{2}\right) = \frac{4\pi}{\sqrt{3}+i\sqrt{3}} e^{-\sqrt{3}/2} e^{-i/2}.$$

Furthermore, the first integral equals $\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+z+1}$ in the limit, while the third we can bound in the same way as before. But how do we deal with the second integral?

The answer lies in the following lemma.

Lemma 20.1. *Suppose f has a simple pole at z_0. Then*

$$\lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} f(z)\,dz = (b-a)i \cdot \operatorname{Res}(f(z); z_0),$$

where $\gamma_\varepsilon : [a,b] \to \mathbb{C}$ is given by

$$\gamma_\varepsilon(\theta) = z_0 + \varepsilon e^{i\theta}.$$

Notice that the integral depends on the values of a and b.

Using the lemma, we see that

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} \frac{e^{iz}}{z^2+z+1} = (0-\pi)i \operatorname{Res}\left(\frac{e^{iz}}{z(z^2+z+1)}; 0\right) = -\pi i.$$

Putting everything together yields

$$\begin{aligned} I = \Im\left(\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\{\varepsilon<|x|<R\}} \frac{e^{iz}}{z(z^2+z+1)}\,dz\right) &= \Im\left(\frac{4\pi}{\sqrt{3}+i\sqrt{3}} e^{-\sqrt{3}/2} e^{-i/2} + \pi i\right) \\ &= \boxed{\pi - \frac{2\pi e^{-\sqrt{3}/2}}{\sqrt{3}}\left[\sin\left(\frac{1}{2}\right) - \cos\left(\frac{1}{2}\right)\right]}. \end{aligned}$$

We now present a proof of the lemma.

Proof. Write $f(z) = \frac{a_{-1}}{z-z_0} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$, and denote the latter sum by $g(z)$. Note that

$$\int_{\gamma_\varepsilon} |g(z)|\,dz \le \operatorname{length}(\gamma_\varepsilon) \cdot \max_{B(z_0,\varepsilon)} |g(z)|,$$

which tends to zero as $\varepsilon \to 0$. Therefore

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} f(z)\,dz = \lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} \frac{a_{-1}}{z-z_0}\,dz &= a_{-1} \int_a^b \frac{1}{\varepsilon e^{i\theta}} \cdot \varepsilon i e^{i\theta}\,d\theta \\ &= i a_{-1} \int_a^b d\theta = (b-a) i a_{-1} = (b-a) i \operatorname{Res}(f(z); z_0), \end{aligned}$$

which is what we wanted. □

We highlight one more example that requires a similar technique to compute.

Example 20.2 (Hilbert Transform)**.** Let $f \in C^\infty(\mathbb{R})$ with "some" amount of decay at ∞. Define the *Hilbert Transform* of f to be the function $Hf : \mathbb{R} \to \mathbb{R}$ given by

$$Hf(x) = \frac{1}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{f(x-t)}{t}\,dt.$$

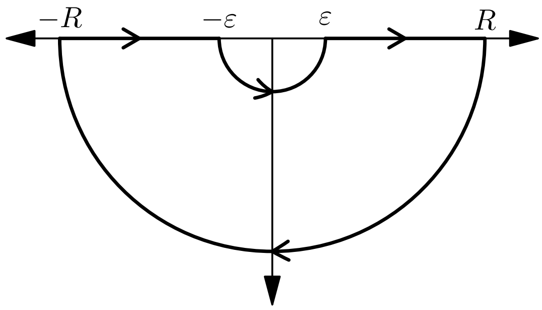
The Hilbert Transform has nice connections to harmonic functions that we will talk about later in the course.

We can compute simple examples now. For example, what is $H(\cos x)$?[7]

To compute this, we can write

$$\begin{aligned} H(\cos x) &= \frac{1}{\pi}\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\{\varepsilon<|t|<R\}} \frac{\cos(x-t)}{t}\,dt \\ &= \frac{1}{\pi}\cdot\Re\left(\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\{\varepsilon<|t|<R\}} \frac{e^{i(x-t)}}{t}\,dt\right) \\ &= \frac{1}{\pi}\cdot\Re\left(e^{ix}\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\{\varepsilon<|t|<R\}} \frac{e^{-it}}{t}\,dt\right). \end{aligned}$$

Now consider the contour $\gamma_{R,\varepsilon}$ defined below; notice that this is similar to the definition of $\gamma_{R,\varepsilon}$ from the previous example, except that now the semicircles lie in the negative halfplane. Then



$$0 = \int_{\gamma_{R,\varepsilon}} \frac{e^{-iz}}{z}\,dz = \int_{\{\varepsilon<|t|<R\}} \frac{e^{-it}}{t}\,dt + \int_{C_R} \frac{e^{-iz}}{z}\,dz + \int_{C_\varepsilon} \frac{e^{-iz}}{z}\,dz.$$

By Lemma 20.1, the third integral approaches $\pi i\,\mathrm{Res}(\frac{e^{-iz}}{z};0) = \pi i$ as $\varepsilon \to 0$. To bound the second integral, parametrize to yield

$$\begin{aligned} \left|\int_{C_R} \frac{e^{-iz}}{z}\,dz\right| &= \left|\int_{-\pi}^{0} \frac{e^{-iRe^{i\theta}}}{Re^{i\theta}}\cdot iRe^{i\theta}\,d\theta\right| \le \int_{-\pi}^{0}\left|e^{-iRe^{i\theta}}\right|\,d\theta = \int_{-\pi}^{0}\left|e^{R\sin\theta}\right|\,d\theta \\ &= \int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \le 2\int_0^{\pi/2} e^{-R\cdot 2\theta/\pi} = \frac{\pi}{R}(1-e^{-R}). \end{aligned}$$

This tends to zero as $R \to \infty$.

Finally, putting everything together yields

$$H(\cos x) = \frac{1}{\pi}\Re(e^{ix}(-\pi i)) = \sin x.$$

[7] Technically, this is $H(\cos(\cdot))(x)$, but this notation is cumbersome.

21 October 14

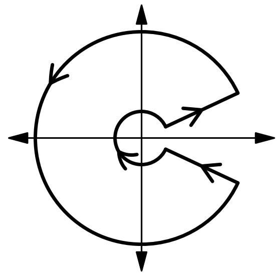
21.1 Calculating Integrals using Residues (even more cont.)

This adventure continues.

21.1.1 Application 5: Power Times a Rational

Now consider an integral of the form $\int_0^\infty x^\rho \mathcal{O}(x)\,dx$, where $\rho \notin \mathbb{Z}$ and where $\mathcal{O}(x)$ is a rational function with no poles on $[0,\infty)$. In this case, consider a branch cut of the function $f(z) = z^\rho \mathcal{O}(z)$ on $[0,\infty)$. This may seem strange at first, but the idea is to exploit the jump discontinuity.

Consider the contour to the right, where the angle made by the sector is 2θ and the radii of the circles are ε and R. Then the idea is to take $\theta \to 0$, and then send $\varepsilon \to 0$ and $R \to \infty$. It may seem as if the limit equals zero because of cancellation, but this is not the case because of the existence of the jump discontinuity. Then, hopefully, the integrals on the circular parts will vanish, and the desired integral is left.



As an example, let's compute $\int_0^\infty \frac{x^\rho}{1+x^2}\,dx$, where $-1 < \rho < 1$ and $\rho \neq 0$. First consider

$$\int_{C_{R,\varepsilon,\theta}} \frac{z^\rho}{1+z^2}\,dz,$$

where $C_{R,\varepsilon,\theta}$ is the contour shown above, and recall that $z^\rho = \exp(\rho(\ln|z| + i\arg z))$, where $\arg z$ is the argument function with image in $(0, 2\pi)$. Then

$$\begin{aligned}\oint_{C_{R,\varepsilon,\theta}} \frac{z^\rho}{1+z^2}\,dz &= 2\pi i\left[\operatorname{Res}\left(\frac{z^\rho}{1+z^2}; i\right) + \operatorname{Res}\left(\frac{z^\rho}{1+z^2}; -i\right)\right] \\ &= 2\pi i\left[\frac{i^\rho}{2i} + \frac{(-i)^\rho}{2(-i)}\right] = \pi\left(e^{i\pi\rho/2} - e^{-i\pi\rho/2}\right).\end{aligned}$$

Now we examine the different components of the above integral. First, observe that

$$\int_{[\varepsilon e^{i\theta}, Re^{i\theta}]} f(z)\,dz = \int_\varepsilon^R f(te^{i\theta})e^{i\theta}\,dt, \quad \text{where} \quad f(te^{i\theta}) = \frac{t^\rho e^{i\theta\rho}}{1+t^2e^{2i\theta}};$$

in particular, $f(te^{i\theta})$ converges uniformly to $\frac{t^\rho}{1+t^2}$ on $[\varepsilon, R]$.[8] Hence the integral converges to $\int_\varepsilon^R \frac{t^\rho}{1+t^2}$ as θ tends to zero. In a similar manner, by writing $e^{i\theta} = e^{i(2\pi-\theta)}$, we see that

$$\int_{[Re^{-i\theta}, \varepsilon e^{-i\theta}]} f(z)\,dz \text{ converges to } -\int_\varepsilon^R \frac{t^\rho e^{2\pi i\rho}}{1+t^2}\,dt$$

as θ approaches zero.

Therefore

$$\begin{aligned}\pi(e^{i\pi\rho/2} - e^{-i\pi\rho/2}) &= \lim_{\theta\to 0^+}\int_{C_{R,\varepsilon,\theta}} f(z)\,dz \\ &= (1 - e^{2\pi\rho})\int_\varepsilon^R \frac{t^\rho}{1+t^2}\,dt + \int_{C_R} f(z)\,dz + \int_{C_\varepsilon} f(z)\,dz.\end{aligned}$$

Finally, observe that

$$\left|\int_{C_R} f(z)\,dz\right| \le \int_{C_R} |f(z)|\,dz \le 2\pi R\cdot\frac{R^\rho}{R^2-1}$$

and

$$\left|\int_{C_\varepsilon} f(z)\,dz\right| \le \int_{C_\varepsilon} |f(z)|\,dz \le 2\pi\varepsilon\cdot\frac{\varepsilon^\rho}{1-\varepsilon^2}.$$

[8]This is precisely why we first send θ to zero and then adjust ε and R, and not the other way around.

Since $-1 < \rho < 1$, both of these limits vanish as $\varepsilon \to 0$ and as $R \to \infty$, and so we obtain

$$\int_0^\infty \frac{t^\rho}{1+t^2}\,dt = \frac{\pi(e^{i\pi\rho/2} - e^{-i\pi\rho/2})}{1 - e^{2\pi i\rho}} = \boxed{\frac{\pi}{2\cos(\frac{\pi}{2}\rho)}}.$$

Before moving on, we note one more example.

Example 21.1. What is the value of the integral $\int_0^\infty \frac{x^\rho \log x}{1+x^2}\,dx$? Well, this does not fall immediately to the techniques we discussed previously, but observe that we can differentiate under the integral sign (with respect to ρ) to get exactly the previous exercise. (Unfortunately, the hypotheses presented in Theorem 6.2 do not exactly apply here, but it is still possible to turn the crank and perform the analysis by hand.)

21.2 Argument Principle

We now turn to a more theoretical application of the Residue Theorem.

Theorem 21.2 (Argument Principle)**.** *Let f be a meromorphic function on a simply connected domain D. Let γ be a simple, closed, counterclockwise contour in D avoiding both the singularities and the zeroes of f. Then*

$$n(f\circ\gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = Z_f - P_f,$$

where Z_f and P_f are the number of zeroes and poles, respectively, of f inside γ.

Proof. To prove the first equality, write

$$\begin{aligned} n(f\circ\gamma, 0) &= \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{dz}{z-0} = \frac{1}{2\pi i}\int_0^b \frac{(f\circ\gamma)'(t)\,dt}{f\circ\gamma(t)} \\ &= \frac{1}{2\pi i}\int_0^b \frac{f'(\gamma(t))\gamma'(t)\,dt}{f(\gamma(t))} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz. \end{aligned}$$

Now, for the second equality, suppose f has a zero of order m or a pole of order $-m$ at z_0¿ Then we may write $f(z) = (z-z_0)^m g(z)$¡ and so

$$\operatorname{Res}\left(\frac{f'}{f}; z_0\right) = \operatorname{Res}\left(\frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)}; z_0\right) = \operatorname{Res}\left(\frac{m}{z-z_0} + \frac{g'(z)}{g(z)}; z_0\right) = m.$$

Summing over all singularities and poles of f yields the desired conclusion. $\square$

Remark. In the special case where f has no poles, we may apply the Argument Principle to $g(z) := f(z) - w$ to get that the number of solutions to the equation $f(z) = w$ – counting multiplicity – is

$$Z_{f-w} = Z_{f-w} - P_{f-w} = n((f-w)\circ\gamma; 0).$$

22 October 16

Midterm I preparation.

23 October 18

No class.

24 October 21

Overslept :(

24.1 The Branched Covering Principle and its Corollaries

The Argument Principle (Theorem 21.2) allows us to explain how analytic functions behave near their zeros.

Corollary 24.1 (Branched Covering Principle)**.** *Let f be analytic and nonconstant on a domain D. Fix $z_0 \in D$, and let $w_0 = f(z_0)$. Denote by m the multiplicity of the function $z \mapsto f(z) - w_0$ at $z = z_0$. Then there is an open set U containing z_0 and an open set V containing w_0 such that $f : U \to V$ is onto and $f : U \setminus \{z_0\} \to V \setminus \{z_0\}$ is m to 1; that is, $f(z) - w$ has m simple zeros for each $w \in V \setminus \{w_0\}$.*

Proof. Without loss of generality suppose $z_0 = w_0 = 0$; in general, apply this special case to the function $\tilde{f}(z) := f(z + z_0) - w_0$.

Since f is nonconstant, f and f' can only have isolated zeros. This means we may choose $r > 0$ such that $f(z) \neq 0$ and $f'(z) \neq 0$ whenever $z \in B^*(0, 2r)$. Now set $\gamma = \partial B(0, r)$, oriented counterclockwise. By the Argument Principle, $n(f \circ \gamma; 0) = m$. Further, denote by V the connected component of $\mathbb{C} \setminus |f \circ \gamma|$ containing zero; Lemma 6.4 tells us that $n(f \circ \gamma; w) = m$ whenever $w \in V$.

Now set

$$U := f^{-1}(V) \cap B(0, r),$$

which is open since f is continuous. The (remark after the) Argument Principle tells us $f : U \to V$ is onto and m to 1, counting multiplicites. However, for $w \in V \setminus \{0\}$, $f(z) - w$ has only simple zeros in U, since $f'(z) \neq 0$ on $U \setminus \{0\}$. This means that $z \mapsto f(z) - w$ has m *distinct* solutions in $U \setminus \{0\}$, implying the result. □

A remark is in order.

Remark. A canonical basic example occurs with the function $f(z) = z^m$: note that $z = 0$ is a zero of mulplicity of order m and that the equation $z^m = w$ has exactly m solutions whenever $w \neq 0$.

In fact, we can use this example to examine the previous theorem in another way. If f has a zero of multiplicity m at 0, we may write $f(z) = z^m g(z)$ where $g(0) \neq 0$; in turn, g does not have zeros on $B(0, r)$ for some $r > 0$, implying that $\log g$ has a branch on $B(0, r)$ (Proposition 8.6).

Now set

$$\varphi(z) = e^{\frac{1}{m}\log(g(z))} \text{ as a branch of } g^{1/m} \text{ on } B(0, r).$$

Then $f(z) = [z\varphi(z)]^m$, which means that f can be written as a composition of functions

$$U \xrightarrow{z\varphi(z)} B(0,s) \xrightarrow{z^m} B(0,s^m).$$

The former function is bijective since $z\varphi(z)$ has a simple zero at $z=0$, while the second function is m to 1. Hence their composition, namely f, must be m to 1 as well.

The previous result is actually quite strong. By stripping away the "m to 1" aspects, we still get a useful result.

Corollary 24.2 (Open Mapping Theorem)**.** *If f is analytic and nonconstant on a domain D, then $f(D)$ is an open set, and hence a domain.*

We can now say something about the existence of inverses of analytic functions.

Corollary 24.3. *Let D be a domain, and suppose f is analytic and one-to-one on D. Then the following hold.*

1. *The derivative f' of f is nonvanishing on D.*
2. *The set $D' := f(D)$ is a domain.*
3. *The function $f^{-1}: D' \to D$ exists and is analytic.*

Proof. First suppose that $f'(z) = 0$ at some point $z \in D$. Then Theorem 24.1 implies that f is either constant or is m to 1 for some $m \geq 2$; either way, we contradict the assumption that f is injective. Hence $f'(z) \neq 0$ on D and the first claim is proven.

The second claim follows from the previous corollary, which states that $f(D)$ is, indeed, a domain.

To prove the third claim, first observe that $f: D \to D'$ is bijective, so its inverse $f^{-1}: D' \to D$ exists. By the Open Mapping Theorem, f takes open sets to open sets, so in fact f^{-1} is continuous. In turn, Theorem 4.1 implies that f^{-1} is differentiable. □

24.2 Rouché's Theorem

Perhaps the most useful application of the Branched Covering Principle is the following result, which allows us to compare the roots of two functions if a certain inequality is satisfied.

Theorem 24.4 (Rouché)**.** *Let f and g be analytic in a simply connected domain D, and let γ be a simple, closed contour in D in the clockwise direction. Suppose that*

$$|f(z) - g(z)| < |f(z)| + |g(z)| \quad \text{for all } z \in |\gamma|. \tag{24.1}$$

Then f and g have the same number of zeros inside γ, counting multiplicites.

Proof. The given inequality (or more specifically, the fact that the inequality is strict) implies that f and g cannot have zeros on $|\gamma|$. Let $h(z) := \frac{f(z)}{g(z)}$; then

$$Z_h - P_h = Z_f - Z_g. \tag{24.2}$$

Furthermore, (24.1) rewrites as $|h(z) - 1| < 1 + |h(z)|$, which implies $h(z) \notin (-\infty, 0]$ for all $z \in |\gamma|$. This means that the image of h cannot wrap around the origin, and so the Argument Principle implies

$$n(h \circ \gamma; 0) = 0 = Z_h - P_h.$$

Plugging this into (24.2) proves the desired result. □

We end with two examples highlighting uses of Rouché's Theorem.

Example 24.5. Let $f(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial, and set $g(z) = a_n z^n$. Observe that, whenever $|z| = R$, we have

$$\begin{aligned} |f(z)| &= \left|a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right| \\ &\leq |a_{n-1}|\,|z|^{n-1} + \cdots |a_1|\,|z| + a_0 \\ &= |a_{n-1}|\,R^{n-1} + \cdots + |a_n|\,R + a_0. \end{aligned}$$

Thus we may choose R so large that $|f(z)| < a_n r^n = |g(z)|$ for all $r \geq R$. In turn, setting $\gamma = \partial B(0,r)$ (oriented counterclockwise) and applying Rouché's Theorem implies that $f(z)$ and $a_n z^n$ have the same number of roots on $\partial B(0,r)$. Sending $r \to \infty$ yields that f and $a_n z^n$ have the same number of roots on $\mathbb{C}$, namely n. We have thus found another proof of the Fundamental Theorem of Algebra.

Example 24.6. We will prove that the function

$$f(z) = \sin(z^2) + 100z^5 + 6$$

has five zeros (counting multiplicity) in the annulus $\{z : \frac{1}{2} < |z| < 2\}$.

On $\partial B(0, \frac{1}{2})$, we have the estimates

$$\left|\sin(z^2)\right| = \frac{\left|e^{iz^2} - e^{-iz^2}\right|}{|2i|} < e^{1/4} \quad \text{and} \quad \left|100z^5\right| = \frac{100}{2^5} = \frac{25}{8}.$$

Let $g(z) = 6$, and note that

$$|f(z) - g(z)| \leq \frac{25}{8} + e^{1/4} < 6$$

on $\partial B(0, \frac{1}{2})$. Therefore f has no zeros on $\overline{\partial B(0, \frac{1}{2})}$.

Likewise, on $\partial B(0,2)$ we have $\left|100z^5\right| = 2^5 \cdot 100$. Now setting $g(z) = 100z^5$, observe that

$$|f(z) - g(z)| \leq e^5 + 6 \ll 2^5 \cdot 100 = |g(z)|$$

for $z \in \partial B(0,2)$, implying $Z_f = Z_g = 5$ in $B(0,2)$. Combining the previous two results proves the claim.

25 October 23

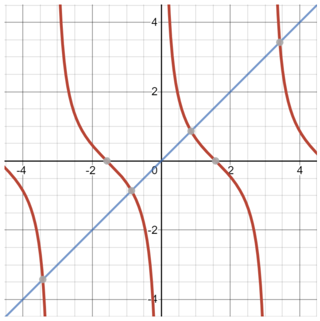
25.1 More Rouché

We start with a more involved application of Rouché.

Example 25.1. We will show that all solutions of the equation $\cot z = z$ are real.

Fix $n \in \mathbb{N}$. Note that on the interval $[-\pi(n+\frac{1}{2}), \pi(n+\frac{1}{2})]$, the equation $\cot x = x$ has exactly $2n+2$ real solutions; one way to see this is to look at the graphs of both functions (shown to the right). We will be done if we can show that the equation $\cot z = z$ has (at most) $2n+2$ solutions in the set

$$Q_n := \{x + iy : |x|, |y| \le \pi(n + \tfrac{1}{2})\}.$$



To approach this, observe that $\cot z = z$ presupposes $\sin z \neq 0$, so we may multiply both sides by $\sin z$ to obtain the equivalent form $\cos z = z \sin z$.

It's easy to check that $z \sin z = 0$ for precisely $2n+2$ values of $z \in Q_n$. Thus, if we can prove

$$|\cos z| < |z \sin z| \quad \text{for all } z \in \partial Q_n, \tag{25.1}$$

Rouché's Theorem tells us that $z \sin z$ and $z \sin z - \cos z$ have the same number of roots inside Q_n. This, when combined with the first paragraph, implies that all the roots in Q_n are real, and so sending $n \to \infty$ yields the desired result.

First, let $z = \pm\pi(n+\frac{1}{2}) + iy$, where $|y| \le \pi(n+\frac{1}{2})$; that is, z lies on the left and right sides of Q_n. Then

$$\begin{aligned} |z \sin z| = |z| \cdot |\sin z| &\ge \pi\left(n + \frac{1}{2}\right) \cdot \left| \frac{e^{\pm i\pi(n+\frac{1}{2}) - y} - e^{\mp i\pi(n+\frac{1}{2}) + y}}{2i} \right| \\ &= \pi\left(n+\frac{1}{2}\right) \cdot \frac{|(-1)^n(\pm i)e^{-y} - (-1)^n(\mp i)e^{y}|}{2} = \pi\left(n+\frac{1}{2}\right) \cdot \frac{e^{-y}+e^{y}}{2}. \end{aligned}$$

Furthermore,

$$|\cos z| = \left|\frac{e^{iz}+e^{-iz}}{2}\right| \le \frac{e^{\Im(z)} + e^{-\Im(z)}}{2} = \frac{e^{y}+e^{-y}}{2}.$$

This establishes (25.1) in this case.

Now let $z = x \pm i\pi(n+\frac{1}{2})$, where $|x| \le \pi(n+\frac{1}{2})$; that is, z lies on the top and bottom sides of Q_n. Then

$$\begin{aligned} |z \sin z| &\ge \pi\left(n+\frac{1}{2}\right) \cdot \frac{1}{2}\left|e^{ix \pm \pi(n+\frac{1}{2})} - e^{-ix \mp \pi(n+\frac{1}{2})}\right| \\ &\ge \pi\left(n+\frac{1}{2}\right) \cdot \frac{1}{2}\left(e^{\pi(n+\frac{1}{2})} - e^{-\pi(n+\frac{1}{2})}\right) > \frac{1}{2}\left(e^{\pi(n+\frac{1}{2})} - e^{-\pi(n+\frac{1}{2})}\right) \ge |\cos z|, \end{aligned}$$

so (25.1) is true in this case as well. We are done.

We conclude our discussion of Rouché's Theorem by discussing a surprising result and its corollary.

Theorem 25.2 (Hurwitz)**.** *Let $(f_n)_{n\ge1}$ be a sequence of analytic and nonvanishing functions converging to f normally on some domain D. Then either $f \equiv 0$ or f is nonvanishing.*

Proof. Assume $f \not\equiv 0$ on D; it suffices to prove that f is nonvanishing.

Observe that, since f is analytic, the zero set of f must be isolated. Now take any simple, closed, and counterclockwise contour γ avoiding all the zeros of f. On the set $|\gamma|$, f is nonvanishing, so there exists $\varepsilon > 0$ such that $|f(z)| \ge \varepsilon$ on $|\gamma|$.

Now recall that $|\gamma|$ is compact, so f_n converges uniformly to f on $|\gamma|$. Thus, we may choose $n > 0$ such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{2} < |f(z)| \quad \text{for all } z \in |\gamma|.$$

Thus, Rouché's Theorem implies that f_n and f have the same number of roots inside γ, namely zero. Since γ was arbitrary, we deduce that f is nonvanishing on all of D. $\square$

As a corollary, we obtain a theorem that segues naturally into the next topic of the course.

Theorem 25.3. *Suppose $(f_n)_{n\geq 1}$ is a sequence of one-to-one functions converging normally to f on some domain D. Then either f is constant or f is itself one-to-one.*

Proof. Assume that f is nonconstant. Given $z_0 \in D$, apply Hurwitz to the set of functions

$$g_n(z) := f_n(z) - f_n(z_0) \quad \text{and} \quad g(z) := f(z) - f(z_0)$$

on D. Since f is nonconstant, $f(z) - f(z_0)$ is not identically zero, so $f(z) - f(z_0)$ is nonvanishing on D. In particular, $f(z) \neq f(z_0)$ for all $z \in D$. Applying this argument to all $z_0 \in D$ establishes injectivity. □

25.2 Angle-Preserving Functions

We now delve into the world of conformal mappings. We start with a definition.

Definition 25.4. Let D be a domain, and suppose $f : D \to \mathbb{C}^2$ is differentiable in the $\mathbb{R}^2$ sense; that is, there exist complex numbers c and d with the property that

$$f(z) = f(z_0) + c(z - z_0) + d(\bar{z} - \bar{z}_0) + E(z),$$

where $\frac{E(z)}{z} \to 0$ as $|z| \to 0$. Fix $z_0 \in D$, and assume there exists $\varepsilon > 0$ uch that $f(z) \neq f(z_0)$ on $B^*(z_0, \varepsilon)$. We say that f *preserves angles* at z_0 if, for any angle $\theta \in [0, 2\pi)$, the limit

$$A(z_0, \theta) := \lim_{r\to 0^+} \left(e^{-i\theta} \cdot \frac{f(re^{i\theta} + z_0) - f(z_0)}{|f(re^{i\theta} + z_0) - f(z_0)|} \right)$$

exists and is independent of θ.

This definition is a bit strange, but it makes more sense in light of the following lemma.

Lemma 25.5. *Let f be one-to-one on D.*

1. *If f' exists at $z_0 \in D$ and $f'(z_0) \neq 0$, then f preserves angles at z_0.*
2. *Conversely, if f is differentiable in the $\mathbb{R}^2$ sense at z_0 with derivative nonzero and f preserves angles at z_0, then f' exists at z_0 and $f'(z_0) \neq 0$.*

Proof. Assume without loss of generality that $z_0 = f(z_0) = 0$.

We begin with the proof of the first part. Since

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{f(z)}{z},$$

we obtain

$$\lim_{r\to 0^+} \frac{f(re^{i\theta})}{re^{i\theta}} = f'(0) \quad \text{and} \quad \lim_{r\to 0^+} \frac{\left|f(re^{i\theta})\right|}{r} = |f'(0)|.$$

Thus, the limit

$$\lim_{r\to 0^+} e^{-i\theta} \cdot \frac{f(re^{i\theta})}{|f(re^{i\theta})|} = \frac{f'(0)}{|f'(0)|},$$

which exists and is independent of θ. This proves the first item.

For the second item, suppose f is differentiable in the $\mathbb{R}^2$ sense at 0, and $f(0) = 0$. This means there exist constants c and d such that

$$f(z) = cz + d\bar{z} + E(z) \quad \text{for all } z \in D.$$

Since f preserves the angles, the limit

$$\lim_{r\to 0^+} \left(e^{i\theta} \frac{cre^{i\theta} + dre^{-i\theta} + E(re^{i\theta})}{|cre^{i\theta} + dre^{-i\theta} + E(re^{i\theta})|} \right) = \frac{c + de^{-2i\theta}}{|c + de^{-2i\theta}|}$$

exists and is independent of θ. (Note that $(\alpha, \beta) \neq (0, 0)$, so the limit actually does exist.) This holds if and only if $\beta = 0$, in which case $\alpha = f_z(0) \neq 0$. □

26 October 25

26.1 Conformal Maps

We begin with a definition.

Definition 26.1. Let $f : D \to \mathbb{C}$ analytic.

1. We say that f is *conformal* if f is one-to-one on D. In this case, $f' \not\equiv 0$ on E, $D' := f(D)$ is a domain, and $f^{-1} : D' \to D$ is also conformal. Furthermore, both f and f^{-1} are angle preserving (Lemma 25.5).
2. Domains D_1 and D_2 are *conformally equivalent* if there exists a conformal, onto map $f : D_1 \to D_2$; f is said to be a *conformal equivalence* from D_1 to D_2.
3. In the case $D_1 = D_2 = D$, any conformal map $f : D \to D$ is called an *automorphism* of D.

A few examples are in order. These will be phrased in terms of propositions, because they are important results in their own right.

Proposition 26.2. *Suppose f is entire and conformal. Then $f(z) = az + b$ for some $a, b \in \mathbb{C}$.*

Proof. Since f is entire, it has a singularity at infinity. Observe that this singularity cannot be removable, since then f is a constant. Suppose now that this singularity is essential. Then, for example,

$$f(B(0,10)^c) \cap f(B(0,1)) \neq \varnothing,$$

so f is not injective.

Thus the singularity at ∞ is a pole of some order, implying f is a polynomial. But f is injective, so the degree of f equals 1. □

Proposition 26.3. *All automorphisms of $B(0,1)$ are of the form*

$$\varphi(z) = e^{-i\theta} \frac{z - c}{1 - z\overline{c}},$$

where $c \in B(0,1)$ and $\theta \in \mathbb{R}$.

Proof. Fix an automorphism $f : B(0,1) \to B(0,1)$, and consider the function

$$\varphi_{f(0)}(z) = \frac{z - f(0)}{1 - z\overline{f(0)}}.$$

Then $\varphi_{f(0)} \circ f : B(0,1) \to B(0,1)$ is an automorphism sending $0 \mapsto 0$.[9] Thus, the Schwarz Lemma (Lemma 9.5) implies that

$$\left|(\varphi_{f(0)} \circ f)(z)\right| \leq |z|$$

for all $z \in B(0,1)$.

Similarly, $(\varphi_{f(0)} \circ f)^{-1} : B(0,1) \to B(0,1)$ is an automorphism sending $0 \to 0$, so the Schwarz Lemma again implies

$$\left|(\varphi_{f(0)} \circ f)^{-1}(z)\right| \leq |z|, \quad \text{or} \quad |z| \leq \left|(\varphi_{f(0)} \circ f)(z)\right|,$$

where the last step is obtained by taking inverses. Thus, we have equality, implying $(\varphi_{f(0)} \circ f)(z) = e^{i\theta} z$ for all $z \in B(0,1)$.

To obtain the desired result, compose with $\varphi_{-f(0)} = (\varphi_{f(0)})^{-1}$. □

[9] This is tricky, but not impossible, to check by hand. It is stated without proof because it has appeared several times in parts of the course not recorded in these notes.

26.2 Möbius Transformations

Let a, b, c, and d be complex numbers with $ad - bc \neq 0$. WE may consider the function $f : \mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{\frac{a}{c}\}$ given by

$$f(z) = \frac{az+b}{cz+d} \quad \text{for all } z \in \mathbb{C}.$$

Note that, by setting $f(\infty) = \frac{a}{c}$ and $f(-\frac{d}{c}) = \infty$, we may extend f to be defined from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

The importance of these functions – called Möbius transformations – is highlighted in the following result.

Proposition 26.4. *A function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an automorphism of $\hat{\mathbb{C}}$ if and only if it is a Möbius transformation.*

Proof. We have already established the ($\Leftarrow$) direction, so it remains to prove the ($\Rightarrow$) one. Let f be an automorphism on $\hat{\mathbb{C}}$. Since f is meromorphic on $\mathbb{C}$, it must be a rational function, i.e. $f(z) = \frac{p(z)}{q(z)}$ for polynomials p and q. If $\deg q \geq 2$, then f has more than one root at ∞, which is a contradiction; thus, $\deg q = 1$. Finally, injectivity of f implies $\deg p = 1$ as well, and so we are done. $\square$

Example 26.5. Suppose g is meromorphic on a domain D, and let f be a Möbius transformation. Then $f \circ g$ is also meromorphic, with poles at the set $g^{-1}(\{-\frac{d}{c}\}) \subset \mathbb{C}$.

26.3 Complex Projective Space

We now give another characterization/perspective of Möbius transforms. As usual, we start with a definition.

Definition 26.6. The *complex projective space* $\mathbb{P}^1(\mathbb{C})$ is defined to be the set of all complex lines through the origin $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ in $\hat{\mathbb{C}}$.

Note that these lines have nice characterizations: they are of the form $\{\lambda\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) : \lambda \in \mathbb{C}\}$, where $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)$ is a nonzero element of $\mathbb{C}^2$.

Elements in $\mathbb{P}^2(\mathbb{C})$ may seem to depend on two parameters, but in fact they only require one. Indeed, for w_1, w_2 nonzero, $\left(\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right)$ if and only if $\frac{z_1}{w_1} = \frac{z_2}{w_2}$; this means we can, without loss of generality, assume $w = 1$. Furthermore, all lines of the form $\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ are equivalent, and $\left(\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z_2 \\ 0 \end{smallmatrix}\right)$ rquires $w_1 = 0$. This means we may write

$$\mathbb{P}^1(\mathbb{C}) = \{\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) : z \in \mathbb{C}\} \cup \{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\},$$

which bears striking similarities to the definition

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

In fact, this identification tells us $\hat{\mathbb{C}} \cong \mathbb{P}^1(\mathbb{C})$.

This change in perspective is not just aesthetic. Any Möbius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $f(z) = \frac{az+b}{cz+d}$, can be identified with the linear transformation $f = T_A$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$; in this way,

$$T_A z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{cases} \left(\begin{smallmatrix} (az+b)/(cz+d) \\ 1 \end{smallmatrix}\right) = \frac{az+b}{cz+d} & z \neq -\frac{d}{c}, \\ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \infty & z = -\frac{d}{c}. \end{cases}$$

Additionally, $T_A(\infty) = A\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$, which equals $\frac{a}{c}$ if $c \neq 0$ and ∞ if $c = 0$.

This identifies Möbius transforms with matrix products, so that

$$T_A \circ T_B = T_{A \circ B} \quad \text{and} \quad (T_A)^{-1} = T_{A^{-1}} \tag{26.1}$$

for all matrices A and B with nonzero determinant. Furthermore, $T_A = T_B$ if and only if $A = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. This means that the set of Möbius transforms can be identified with the set

$$\mathrm{PSL}(z; \mathbb{C}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : ad - bc = 1,\ A \text{ and } -A \text{ identified}\},$$

which is a projective analogue of the special linear group over $\mathbb{C}$.

We close with some remarks.

Remark. Every Möbius transform is a composition of translation, rotation, dilation, and *inversion* (i.e. $z \mapsto \frac{1}{z}$). More specifically,

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}} + \frac{a}{c}.$$

Notice how this collapses to $\frac{a}{c}$ if $ad - bc = 0$.

Remark. Any Möbius transformation f sends circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$. Circles or lines through $-\frac{d}{c}$ will be sent to lines in $\mathbb{C}$, while all other circles or lines will be sent to circles in $\hat{\mathbb{C}}$. These images will not pass through the point $\frac{a}{c}$.

27 October 28

27.1 Möbius Transforms and Circles

Last time, we briefly remarked that circles in $\hat{\mathbb{C}}$ are sent to circles in $\hat{\mathbb{C}}$. Let's make that more concrete.

Example 27.1. Suppose ω_1 and ω_2 are two circles in $\hat{\mathbb{C}}$. Can we find a Möbius transform sending ω_1 to ω_2?

To tackle this problem, we make the following observation: given distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, there is a (unique!) Möbius transformation T with $T(z_k) = w_k$ for all $1 \le k \le 3$. To construct this transform, first define $f : \mathbb{C} \to \mathbb{C}$ via

$$f(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z_2 - z}{z_3 - z}, \tag{27.1}$$

with appropriate modifications whenever z_j is infinite for some j (for example, by taking limits). This is an explicit transformation sending z_1 to 1, z_2 to 0, and z_3 to ∞. Similarly, we may find a Möbius transformation g sending w_1 to 1, ω_2 to 0, and ω_3 to ∞. The desired transform is $T_A := g^{-1} \circ f$.

We now claim this transformation is unique. Suppose T_A and T_B both send z_k to w_k for $k \in \{1, 2, 3\}$. Then the transformation $T_A \circ T_{B^{-1}} = T_{A \circ B^{-1}}$ sends ω_j to ω_j for all j. Hence the matrix $A \circ B^{-1}$ has three distinct eigenvectors, implying $A \circ B^{-1} = \lambda I_2$ for some $\lambda \in \mathbb{C}$. This implies $A = \lambda B$, so the Möbius transforms T_A and T_B are identical.

The result follows by taking $z_1, z_2, z_3 \in \omega_1$ and $w_1, w_2, w_3 \in \omega_2$.

The expression in (27.1) is special enough to warrant its own name.

Definition 27.2. The *cross-ratio* of distinct $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ is

$$[z_1, z_2, z_3, z_4] := \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z_2 - z_4}{z_3 - z_4}.$$

As we have seen, cross-ratios and Möbius transforms play nicely with each other.

- If T_A is a Möbius transformation sending z_1, z_2, and z_3 to 1, 0, and ∞, respectively, then
$$[z_1, z_2, z_3, z_4] = T_A(z_4).$$
- For all $z \in \mathbb{C}$, $[1, 0, \infty, z] = z$.
- For any $z_1, z_2, z_3, z_4 \in \mathbb{C}$,
$$\overline{[z_1, z_2, z_3, z_4]} = [\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}].$$

These properties alone may not be useful enough to justify studying the cross-ratio, but the next property surely will be.

Proposition 27.3. *Let T_B be a Möbius transformation. Then*

$$[z_1, z_2, z_3, z_4] = [T_B(z_1), T_B(z_2), T_B(z_3), T_B(z_4)] \tag{27.2}$$

for all $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$.

Proof. Suppose T_A is the Möbius transformation sending z_1, z_2, and z_3 to 1, 0, and ∞, respectively. Then $T_A(z_4)$ is equal to the cross-ratio $[z_1, z_2, z_3, z_4]$. But then the transform $T_{A \circ B^{-1}}$ sends $T_B(z_1)$ to 1, $T_B(z_2)$ to 0, and $T_B(z_3)$ to ∞, and so

$$[T_B(z_1), T_B(z_2), T_B(z_3), T_B(z_4)] = T_{A \circ B^{-1}}(T_B(z_4)) = T_A(z_4).$$

Thus both sides of (27.2) are equal. $\square$

27.2 Symmetry with Respect to Circles

We can use the cross-ratio to define a useful analogue of reflection for circles.

Definition 27.4. Let Γ be a circle in $\hat{\mathbb{C}}$, and let $z_1, z_2, z_3 \in \Gamma$. We say that the points z and z^* (also denoted $\rho_\Gamma(z)$) are *symmetric* if

$$\overline{[z_1, z_2, z_3, z]} = [z_1, z_2, z_3, z^*].$$

It may not be obvious that z^* exists or whether it depends on z_1, z_2, and z_3. The following examples will hopefully answer those questions. Before continuing, we note that $(z^*)^* = z$ for all $z \in \hat{\mathbb{C}}$.

Example 27.5. If $\Gamma = \mathbb{R} \cup \{\infty\}$, then $z^* = \bar{z}$. Indeed, let $z_1, z_2, z_3 \in \Gamma$ be arbitrary; then

$$\overline{[z_1, z_2, z_3, z]} = [\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z}] = [z_1, z_2, z_3, \overline{z}],$$

so $z^* = \bar{z}$. Notice how the choice of z_1, z_2, and z_3 is arbitrary.

Example 27.6. Now let Γ be an arbitrary line in $\hat{\mathbb{C}}$. We claim that z^* is the reflection of z about Γ, implying that circle symmetry *really is* a generalization of line symmetry. To prove this, let T_B be the translation and rotation taking Γ to $\mathbb{R}$, noting that T_B is thus a Möbius transformation. Then

$$\begin{aligned}\overline{[z_1, z_2, z_3, z]} &= \overline{[T_B(z_1), T_B(z_2), T_B(z_3), T_B(z)]} \\ &= [T_B(z_1), T_B(z_2), T_B(z_3), \overline{T_B(z)}] \\ &= [z_1, z_2, z_3, T_{B^{-1}}\overline{T_B(z)}].\end{aligned}$$

Therefore $z^* = T_{B^{-1}}\overline{T_B(z)}$.

Example 27.7. Finally, suppose $\Gamma = \{z : |z - z_0| = r\}$ is a circle. Then we claim

$$z^* = \frac{r^2}{\overline{z - z_0}} + z_0. \tag{27.3}$$

Indeed, by repeatedly using cross-ratio invariance under Möbius transformation, we see that

$$\begin{aligned}\overline{[z_1, z_2, z_3, z]} = [\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}] &= \left[\frac{\overline{z_1} - \overline{z_0}}{r}, \frac{\overline{z_2} - \overline{z_0}}{r}, \frac{\overline{z_3} - \overline{z_0}}{r}, \frac{\overline{z} - \overline{z_0}}{r}\right] \\ &\overset{(*)}{=} \left[\frac{z_1 - z_0}{r}, \frac{z_2 - z_0}{r}, \frac{z_3 - z_0}{r}, \frac{r}{\overline{z} - \overline{z_0}}\right] = \left[z_1, z_2, z_3, \frac{r^2}{\overline{z - z_0}} + z_0\right].\end{aligned}$$

(In particular, $(*)$ involves the Möbius transformation $z \mapsto \frac{1}{z}$, where the expressions in the first three slots arise from the fact that $|\frac{z_j - z_0}{r}| = 1$.) Recalling the definition of z^* gives (27.3).

Before continuing, we mention one more important relationship between symmetric points and Möbius transforms.

Proposition 27.8. *Suppose T_A is a Möbius transformation sending $\Gamma_1 \subset \hat{\mathbb{C}}$ to $\Gamma_2 \subset \hat{\mathbb{C}}$, and let z and z^* be symmetric with respect to Γ_1. Then $T_A(z)$ and $T_A(z^*)$ are symmetric with respect to Γ_2.*

Proof. Applying T_A to both sides of the equality $\overline{[z_1, z_2, z_3, z]} = [z_1, z_2, z_3, z^*]$ yields

$$\overline{[T_A(z_1), T_A(z_2), T_A(z_3), T_A(z)]} = [T_A(z_1), T_A(z_2), T_A(z_3), T_A(z^*)],$$

whence $T_A(z)$ and $T_A(z^*)$ are symmetric with respect to Γ_2. $\square$

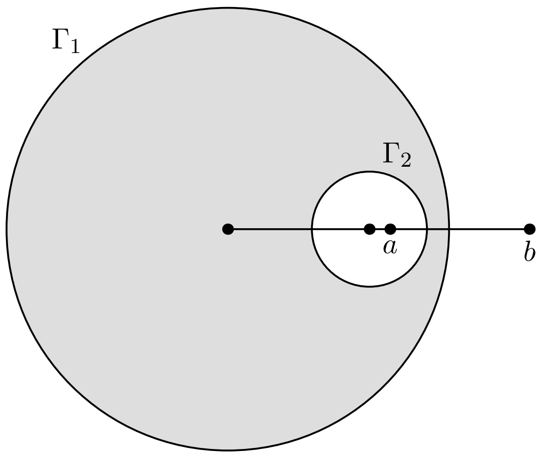
27.3 Conformal Mappings Between Sets

Looking at symmetries can help remove some of the tedium in finding conformal mappings between conformally equivalent sets. Two examples are shown below.

Example 27.9. What is a Möbius transform sending the upper half plane $\{z : \Im(z) > 0\}$ to the unit disc $B(0, 1)$?

Fix z_0 in the upper half plane. The function $f(z) = \frac{z-z_0}{z-\overline{z_0}}$ is a Möbius transformation sending z_0 to 0 and $\overline{z_0}$ to ∞. Since z_0 and $\overline{z_0}$ are symmetric about $\mathbb{R}$, the image $f(\mathbb{R})$ must be a circle for which 0 and ∞ are symmetric. By examining (27.3) with $z^* = \infty$ and $z = 0$, we see that this circle must, in fact, be centered at the origin with radius $r > 0$. But $f(0) = \frac{z_0}{\overline{z_0}}$ has magnitude 1, so in fact $r = 1$. This means that $\mathbb{R}$ is sent to $B(0, 1)$, and since z_0 was sent to the interior of $B(0, 1)$, we deduce the conclusion.

Example 27.10. Let Γ_1 and Γ_2 be two not-necessarily-concentric circles, as shown below. Is there a conformal mapping from the shaded non-concentric annulus below to a concentric annulus?



Consider the ray originating at the center of Γ_1 and passing through the center of Γ_2. We claim there exist points a and b on this ray satisfying

$$b = \rho_{\Gamma_1}(a) = \rho_{\Gamma_2}(a).$$

This is due to a continuity argument. Notice that, as a slides away from the center of Γ_2, $\rho_{\Gamma_2}(a)$ slides from ∞ to inside Γ_1, while $\rho_{\Gamma_1}(a)$ always remains outside Γ_1. Therefore, by the Intermediate Value Theorem, the two points must concide for some a.

Now let $f(z) = \frac{z-a}{z-b}$, which sends a to 0 and b to ∞. Then Proposition 27.8 tells us that

$$\infty = \rho_{f(\Gamma_1)}(0) = \rho_{f(\Gamma_2)}(0),$$

so ∞ and 0 are symmetric about both $f(\Gamma_1)$ and $f(\Gamma_2)$. Thus both circles are centered at zero, implying the annulus is now concentric.

28 October 30

28.1 Power and Exponential Functions as Conformal Maps

We first discuss two different types of conformal maps that will greatly expand the types of sets we can show are conformally equivalent.

Example 28.1. Let $\beta > 0$, and for each $0 < r < 2\pi$ define

$$D_r := \{z \in \mathbb{C} : 0 < \arg(z) < r\},$$

where arg is a branch of the argument function defined on $(0, 2\pi)$. Observe that the function $z \mapsto z^\beta$ has a branch in D_α, namely $z^\beta = \exp(\beta(\ln|z| + i\arg z))$.

Furthermore, if $0 < \beta \le \frac{2\pi}{\alpha}$, then z^β is one-to-one on D_α, and its image is $D_{\alpha\beta}$. So D_α and $D_{\alpha\beta}$ are conformally equivalent. In particular, by taking $\beta = \frac{\pi}{\alpha}$ we deduce that D_α is conformally equivalent to the upper half plane.

Example 28.2. The function $z \mapsto e^z$ is one-to-one on the set $\{z : 0 < \Im z < 2\pi\}$, which means it is also one-to-one on the set

$$S_h := \{z : 0 < \Im z < h\}$$

whenever $0 < h < 2\pi$. Furthermore,

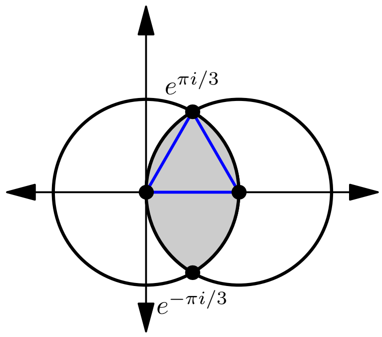
$$e^{S_h} = \{e^{x+yi} : x \in \mathbb{R}, 0 < y < h\} = \{e^x e^{iy} : x \in \mathbb{R}, 0 < h < h\} = D_h,$$

where D_h is defined as in the previous example. Thus, e^z is a conformal equivalence between S_h and D_h.

We now look at some examples of this.

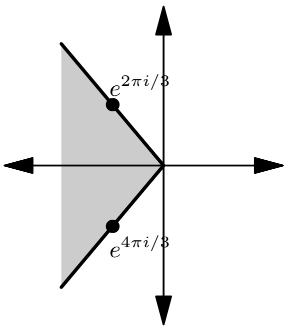
Example 28.3. Let $D = \{z : |z| < 1, |z-1| < 1\}$, which is the set shown below. We wish to find a conformal equivalence between D and the upper half plane.

To do this, first remark that the two circles intersect at the two points $e^{\pm i\pi/3} \in \mathbb{C}$; this can be seen by noticing the equilateral triangle in blue. With this, define the Möbius transform f_1 via



$$f_1(z) = \frac{z - e^{\pi i/3}}{z - e^{-\pi i/3}};$$

then $e^{\pi i/3}$ is sent to 0 while $e^{-\pi i/3}$ is sent to ∞. Since both circles pass through a point sent to ∞, both circles are sent to lines passing through the origin. Furthermore, we may compute $f_1(0) = e^{2\pi i/3}$ and $f(1) = e^{4\pi i/3}$, so the shaded region is sent to the angular region below.



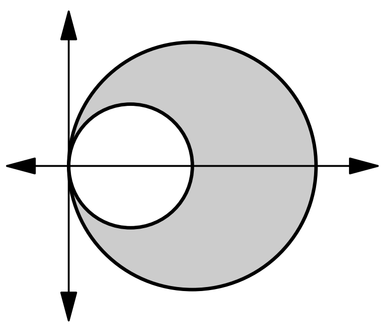
From here, the path is more clear. First, note that the map $f_2(z) = e^{-2\pi i/3}z$ rotates the given region $\frac{2\pi}{3}$ radians clockwise, transforming it to $D_{2\pi/3}$. From there, the transformation $f_3(z) = z^{3/2}$ sends $D_{2\pi/3}$ to the upper half plane. All in all, the transformation

$$f_3 \circ f_2 \circ f_1(z) = \left(e^{-2\pi i/3}\frac{z - e^{\pi i/3}}{z - e^{-\pi i/3}}\right)^{3/2}$$

is a conformal mapping sending D to the upper half plane.

Example 28.4. Let $D = \{z : |z-1| > 1, |z-2| < 2\}$, which is the region shown below. We wish to find a conformal equivalence between D and $B(0,1)$.

Consider some Möbius transform f_1 which maps 0 to ∞, so that both circles are transformed to lines; as an example, take $f_1(z) = \frac{1}{z}$. Observe that $\mathbb{R}$ is sent to $\mathbb{R}$ under f_1, with the points $(2,0)$ and $(4,0)$ sent to $(\frac{1}{2}, 0)$ and $(\frac{1}{4}, 0)$, respectively. Furthermore, the bold circles are orthogonal to the real axis at these points, implying that the images of these circles are orthogonal to the real axis as well. It follows that the circle $\{z : |z-1| = 1\}$ is taken to the line $\{\frac{1}{2} + xi : x \in \mathbb{R}\}$, while the circle $\{z : |z-2| = 2\}$ is taken to the line $\{\frac{1}{4} + xi : x \in \mathbb{R}\}$. In turn, the shaded region is taken to the strip bounded by these lines.



Now composing with $f_2(z) = (z - \frac{1}{4}) i \cdot 4\pi$ transforms this strip into S_π. In turn, the transformation $f_3(z) = e^z$ sends S_π into the upper half plane, and the transformation $f_4(z) = \frac{z-i}{z+i}$ sends the upper half plane to $B(0,1)$. All in all, the function

$$(f_4 \circ f_3 \circ f_2 \circ f_1)(z) = \frac{e^{4\pi i(1/z - 1/4)} - i}{e^{4\pi i(1/z-1/4)} + i}$$

is the desired conformal mapping.

28.2 Joukowsky and Inverse Joukowsky Transforms

We now define a surprisingly useful mapping.

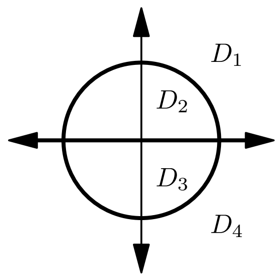
Definition 28.5. The *Joukowsky Transform* is the function $J : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{for all } z \in \mathbb{C} \setminus \{0\}.$$

This new function J has several nice properties.

- The function J is analytic on $\mathbb{C} \setminus \{0\}$.
- For every $z \neq 0$, $J(z) = J(\frac{1}{z})$.
- For every w in the range of J, the solutions to the equation $J(z) = w$ are reciprocals; that is, the solution set is $\{z, \frac{1}{z}\}$ for some $z \in \mathbb{C}$.

These three properties imply that J is conformal on any domain in $\mathbb{C} \setminus \{0\}$ not containing reciprocal points. There are several natural examples of such domains; three of them are the upper and lower half planes, $B(0,1)^*$, and $\mathbb{C} \setminus \overline{B(0,1)}$. These four regions together subdivide $\mathbb{C}$ into four regions, dubbed D_1 through D_4 in the diagram to the right. These allow us to simplify the analysis of J a bit, since $J(D_1) = J(D_3)$ and $J(D_2) = J(D_4)$.



What, exactly, are these images? We may find them by parametrizing a circle; indeed, write

$$J(re^{i\theta}) = \frac{1}{2}(re^{i\theta} + re^{-i\theta}) = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta + \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta$$

and by varying θ we see that the resulting shape is an ellipse whenever $r \neq 1$. (If $r = 1$, then the image is just the line segment $[-1,1]$.) In turn,

$$J(B^*(0,1)) = \bigcup_{0<r<1} J(B(0,r)) = \mathbb{C} \setminus [-1,1].$$

By our previous work, $J(\overline{B(0,1)}) = \mathbb{C} \setminus [-1,1]$ as well.

Notice further we may restrict our attention to half-circles to paint a more precise picture about what J does. Indeed, if we now let $r > 1$ and apply the same reasoning, we see that the region $J(D_1)$ (and hence $J(D_3)$) is the upper half plane, while the region $J(D_4)$ (and hence $J(D_2)$) is the lower half plane. In turn, by writing

$$\text{U.H.P} = D_1 \cup D_2 \cup \{e^{i\theta} : \theta \in (0,\pi)\},$$

we see that

$$J(\text{U.H.P}) = J(D_1) \cup J(D_2) \cup J(\{e^{i\theta} : \theta \in (0,\pi)\}) = C \setminus \{x \in \mathbb{R} : |x| > 1\} =: S_1.$$

Similarly, we may find that $J(B^*(0,1)) = \mathbb{C} \setminus [-1,1] =: S_2$.

Given our transform J, it is natural to ask what the inverse transform J^{-1} is. This is a simple algebra exercise: we have

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right) \quad \text{implies} \quad w = z \pm \sqrt{z^2 - 1},$$

so $J^{-1}(z) = z \pm \sqrt{z^2-1}$, where either choice of sign is appropriate.

On a previous homework assignment, we showed that $\sqrt{z^2-1}$ has branches on S_1 and S_2, which is quite convenient. Indeed, on S_1 we may take the branch of the square root function with $\sqrt{-1} = i$, while on S_2 we may take the branch satisfying $\sqrt{z^2-1} > 0$ whenever $z > 1$. Synthesizing all our previous results, we see that

$$\begin{aligned}
J^{-1}(z) &= z + \sqrt{z^2-1} \text{ maps } S_1 \text{ to U.H.P,} \\
J^{-1}(z) &= z + \sqrt{z^2-1} \text{ maps } S_2 \text{ to } \overline{B^*(0,1)}^c, \\
J^{-1}(z) &= z - \sqrt{z^2-1} \text{ maps } S_1 \text{ to L.H.P, and} \\
J^{-1}(z) &= z - \sqrt{z^2-1} \text{ maps } S_2 \text{ to } B^*(0,1).
\end{aligned}$$

29 November 1

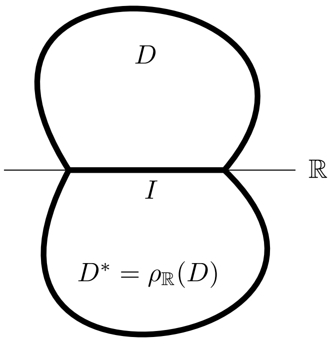
29.1 Schwarz Reflection Principle

The idea of using reflections to extend functions is pervasive in analysis. We explore this idea now.

Example 29.1 (Schwarz Reflection Principle)**.** Let D be a domain as shown in the upper half plane H with part of ∂D lying on an interval $I \subset \mathbb{R}$. Define $D_1 := D \cup I \cup D^*$. Suppose f is analytic on D and continuous on $D \cup I$, and furthermore $f(I) \subset \mathbb{R}$. Then the function $g : D_1 \to \mathbb{C}$ given by

$$g(z) = \begin{cases} f(z) & z \in D \cup I, \\ \overline{f(\bar{z})} & z \in D^* \end{cases}$$

is analytic on D_1.



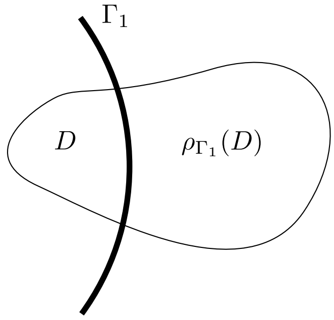
To prove this, note that it is not hard to check that the function $z \mapsto \overline{f(\bar{z})}$ is analytic; this was an old homework problem. Furthermore, $f(I) \subset \mathbb{R}$ implies that g is continuous on D_1. Finally, we may use Morera (Corollary 8.4) to show that g is analytic at each point in I.

Since lines are just circles in $\hat{\mathbb{C}}$, we might suspect that the previous example generalizes to inversion about a circle. This is, indeed, true.

Example 29.2 (Generalized Schwarz Reflection Principle)**.** Let Γ_1 and Γ_2 be two circles in $\hat{\mathbb{C}}$, and let G be a domain symmetric with respect to Γ. Write $G = D \cup I \cup \rho_{\Gamma_1}(D)$, where $I \subset \Gamma_1$. Assume that f is analytic on D and continuous on $D \cup I$, and furthermore $f(I) \subset \Gamma_2$. Then the function $g : G \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} f(z) & z \in D \cup I, \\ \rho_{\Gamma_2}(f(\rho_{\Gamma_1}(z)) & z \in \rho_{\Gamma_1}(D) \end{cases}$$

is analytic in G, except possibly at the points z where $f(\rho_{\Gamma_1}(z))$ is the center of Γ_2. (This is not a problem if Γ_2 is a line in $\mathbb{C}$.)



The proof is simple: find a conformal mapping sending Γ_1 to $\mathbb{R}$ and appeal to the normal Schwarz reflection principle.

We may use the Schwarz reflection principle to prove some interesting facts about entire functions.

Proposition 29.3. *Suppose f is entire, $I \subset \mathbb{R}$ is a nonempty interval, and $f(I) \subset \mathbb{R}$. Then $f(\mathbb{R}) \subset \mathbb{R}$.*

Proof. By the Schwarz reflection principle, the function $g(z) = \overline{f(\overline{z})}$ is entire. Now f and g agree on the set $\mathbb{C} \setminus \mathbb{R}$, which (among other things) contains a limit point. Hence $f(z) = g(z)$ everywhere, implying that $\overline{f(x)} = f(x)$ for all $x \in \mathbb{R}$. In turn, $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. $\square$

A few remarks are in order.

Remark. The use of Schwarz was not strictly necessary – one alternate approach is to Taylor expand f about some point $z_0 \in I$ and note that all derivatives at z_0 are real-valued.

Remark. One might ask whether the following generalization is true: if D is a domain in $\mathbb{C}$, f is analytic on D, and $f(I) \subset \mathbb{R}$ for some interval $I \subset D \cap \mathbb{R}$, is $f(D \cap \mathbb{R}) \subset \mathbb{R}$? Unfortunately, the answer is no. As a specific example, note that the function $z \mapsto \sqrt{z}$ has a branch on $\mathbb{C} \setminus \{iy : y \leq 0\}$ with $\sqrt{1} = 1$. Then $f((0, \infty)) \subset \mathbb{R}$, but $f(\mathbb{R}) \not\subset \mathbb{R}$.

That being said, there are specific cases where this *does* work. As an example, consider the Joukowsky transformation from last lecture. The region D_1 from that lecture (the set of complex numbers in the upper half plane H with magnitude greater than 1) is mapped to H. The three boundary parts of H – $(-\infty, -1)$, $\partial B(0,1) \cap H$, and $(1, \infty)$ – are mapped to $(-\infty, -1)$, $[-1, 1]$, and $(1, \infty)$, respectively.

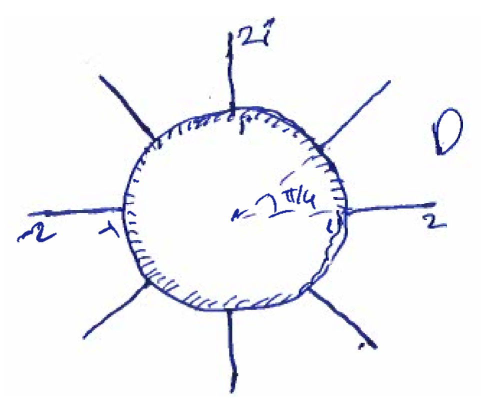
We may now use the Schwarz Reflection Principle to generate analytic continuations of $J|_{D_1}$. For example, reflecting across the boundary of the disc yields a mapping J from H to $\mathbb{C} \setminus (-\infty, -1] \cup [1, \infty)$. Similarly, reflecting across the boundary belonging to the real axis yields a mapping J from $\overline{B(0,1)}^c$ to $\mathbb{C} \setminus [-1, 1]$. This checks with our work from last time.

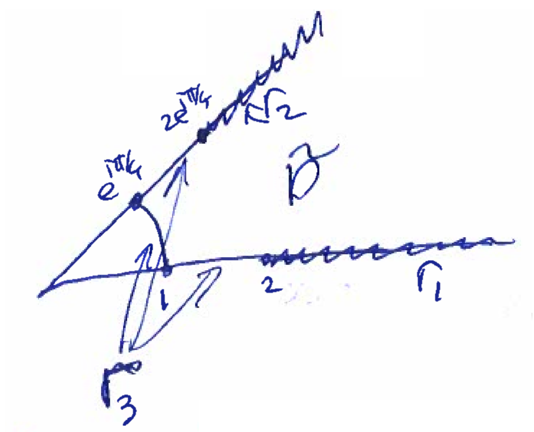
We now move on to the most sophisticated example yet. It shows how the Schwarz Reflection Principle can be used to construct conformal mappings between strange sets.

Example 29.4. In this example, we will find a conformal equivalence between

$$D := \overline{B(0,1)}^c \setminus \bigcup_{j=1}^{8} \{re^{i\frac{\pi}{4}j} : 1 \leq r \leq 2\}$$

and $\overline{B(0,1)}^c$. (Informally, our task is to get rid of the spokes protruding into the complement of the closed unit ball.)





To do this, we will first let

$$\tilde{D} := \{z : |z| > 1, \operatorname{Arg}(z) \in (0, \tfrac{\pi}{4})\},$$

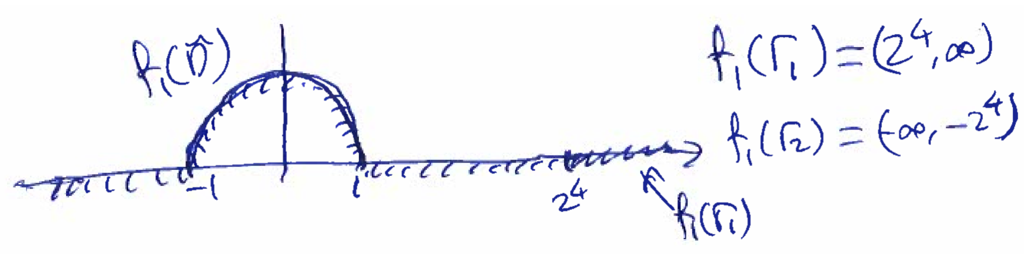
so that the boundary of $\tilde{D}$ can be written as a partition

$$\partial\tilde{D} = (2, \infty) \cup \{re^{i\pi/4} : r \in (2, \infty) \cup \left([1,2] \cup [e^{i\pi/4}, 2e^{i\pi/4}] \cup \{e^{i\theta} : 0 < \theta < \tfrac{\pi}{4}\}\right).$$

Label these regions Γ_1, Γ_2, and Γ_3, as shown in the diagram to the right. We will find a conformal equivalence $f : \tilde{D} \to \tilde{D}$ which is continuous on $\partial\tilde{D}$ and maps Γ_1 to $(1, \infty)$ and Γ_2 to $\{re^{i\pi/4} : r \in (1, \infty)\}$. Then by the general version of the Schwarz Reflection Principle (applied a few times), we obtain the conformal equivalence between D and $\overline{B(0,1)}^c$.

We proceed in steps.

- First unfold $\tilde{D}$ via the mapping $f_1(z) = z^4$; this sends $\tilde{D}$ to D_1 and the regions Γ_1 and Γ_2 to $f(\Gamma_1)$ and $f(\Gamma_2)$, shown below.



- Now consider the Joukowsky transformation $f_2(z) = \frac{1}{2}(z + \frac{1}{z})$; then $f_2(f_1(\tilde{D})) = H$, and the parts Γ_1 and Γ_2 are mapped to (A, ∞) and $(-\infty, -A)$ respectively, where $A = \frac{1}{2}(2^4 + 2^{-4}) = \frac{257}{32}$.
- Rescale via $f_3(z) = z/A$; then $f_3(f_2(f_1(\tilde{D})))$ is still H, but the images of Γ_1 and Γ_2 are now just $(1, \infty)$ and $(-\infty, -1)$.
- Finally, reverse the steps via $f_4(z) = (J^{-1}(z))^{1/4}$; then $f = f_4 \circ f_3 \circ f_2 \circ f_1$ is an automorphism of $\tilde{D}$ which maps Γ_1 to $(1, \infty)$ and Γ_2 to $\{e^{i\pi/4} r : r \in (1, \infty)\}$, as desired.

Remark. In essence, we have taken the circular part $\tilde{\Gamma}_3$ of Γ_3 and mapped it to a small segment on $\partial B(0, 1)$. This means that, via the Schwarz Reflection Principle, we may extend our conformal equivalence f further to the domain $\mathbb{C} \setminus S$, where S is a set of eight slits equally spaced about the boundary of the unit ball.

29.2 Normal families, introduced

In the last few minutes of class, we will lay the foundations for the material we will discuss for the next few days.

Let $K \subset \mathbb{C}$ be compact. Our goal is to understand the set

$$C(K) = \{f : K \to \mathbb{C} \mid f \text{ is continuous}\}.$$

We know already that $C(K)$ is a metric space with the supremum norm as its distance metric. This means we may ask the following question: what are the compact subsets of $C(K)$?

Unfortunately, $C(K)$ is an infinite dimensional metric space, and so the compact sets are not just the closed and bounded sets (like they are in finite dimensions). As an example, the sequence of functions $(f_n)_{n \geq 1}$ given by $f_n(x) = x^n$ is bounded on $C([0, 1])$, but it does not have any conveergent subsequence.

This leads into the concept of a *normal family* of functions in $C(K)$, which we will discuss next time.

30 November 4

30.1 Precompact Subsets of $C(K)$

We start with a definition.

Definition 30.1. We say a set $S \subset C(K)$ is *precompact* (or *relatively compact*) if $\bar{S}$ is compact.

Remark. This definition of S also works in arbitrary measure spaces. It is equivalent to the assumption that every sequence in S has a convergent subsequence (whose limit is not necessarily in S).

As an example, recall that in $\mathbb{R}$ and $\mathbb{C}$ all bounded subsets are precompact. However, this is not true for $C(K)$; see the $f_n(x) = x^n$ example from yesterday. The following theorem gives an equivalent definition of precompactness in $C(K)$.

Theorem 30.2 (Arzela-Ascoli)**.** *Let $K \subset \mathbb{C}$ be compact. Then $S \subset C(K)$ is precompact if and only if S is bounded and equicontinuous. Here equicontinuous means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(z) - f(w)| < \varepsilon \quad \text{for all } f \in S \text{ and all } z, w \in K \text{ with } |z - w| < \delta. \tag{30.1}$$

Proof. The main content of this proof is the ($\Leftarrow$) direction, which we do now. Suppose $S \subset C(K)$ is bounded and equicontinuous, and let $\{f_n\}_{n\geq 1} \subset S$. Our goal is to show that this sequence has a uniformly convergent subsequence converging in K.

Step 1. We will show there exists a subsequence converging *pointwise* on some countable dense subset $\{z_j\}_{j=1}^{\infty}$ of K. First consider the sequence $\{f_n(z_1)\}_{n\geq 1} \subset \mathbb{C}$. This sequence is bounded, so it has a convergent subsequence $\{f_{n_k^1}(z_1)\}_{k\geq 1}$. Now consider the sequence $\{f_{n_k^1}(z_2)\}_{k\geq 1}$; once again, we may extract a convergent subsequence $\{f_{n_k^2}(z_2)\}_{k\geq 1}$. Proceeding inductively, for each $m \geq 1$ w may find a subsequence $\{f_{n_k^m}(z_m)\}_{k\geq 1}$, where $f_{n_k^m}$ converges pointwise on each of $z_1, \ldots, z_m$.

Finally, examine the "diagonal" sequence $\{f_{n_m^m}\}_{m\geq 1}$; this converges pointwise at each z_j, since

$$(f_{n_m^m}(z_j))_{m\geq 1} \text{ is a subsequence of } (f_{n_k^j}(z_j)) \text{ whenever } m \geq j.$$

Step 2. Let $\{f_n\}_{n\geq 1}$ be the sequence of pointwise convergent functions found in the previous step. We claim that $(f_n)_{n\geq 1}$ is uniformly convergent on K, which proves the first direction.

To prove this, let $\varepsilon > 0$. Choose the $\delta > 0$ given by the equicontinuity criterion (30.1) for $\frac{\varepsilon}{3}$. Note that, since $\{z_j\}_{j\geq 1}$ is dense in K, the collection of balls $(B(z_j, \delta))_{j=1}^{\infty}$ is an open covering of K; this means there exists a finite subset $(w_1, \ldots, w_J)$ of the z_j's such that

$$K \subset \bigcup_{j=1}^{J} B(w_j, \delta). \tag{30.2}$$

Now, for each $1 \leq j \leq J$, the sequence $\{f_n(w_j)\}_{n\geq 1}$ converges, so in particular these sequences are all Cauchy. This means there exists N_j such that

$$|f_n(w_j) - f_m(w_j)| < \frac{\varepsilon}{3} \quad \text{for all } m, n > N_j.$$

Now let $N = \max_{1\leq j\leq J} N_j$, and let $m, n > N$ and $z \in K$ be arbitrary. Equation (30.2) implies there exists j such that $z \in B(w_j, \delta)$. Thus,

$$\begin{aligned} |f_n(z) - f_m(z)| \leq |f_n(z) - f_n(w_j)| + |f_n(w_j) - f_m(w_j)| + |f_m(w_j) - f_m(z)| \\ \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, we deduce that the sequence $\{f_n\}_{n\geq 1}$ is uniformly Cauchy, and so it is uniformly convergent as desired.

The other direction is not important for our use, but we shall go over it for the sake of completeness. Fix some precompact set $S \subset C(K)$. We already know S is bounded, so let us assume for the sake of contradiction that S is precompact but not equicontinuous. This means there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exist $z, w \in K$ and $f \in S$ such that $|z-w| < \delta$ but $|f(z)-f(w)| \geq \varepsilon$. Apply this with $\delta = \frac{1}{n}$ to get sequences $\{f_n\}$, $\{z_n\}$, and $\{w_n\}$ such that

$$|z_n - w_n| < \frac{1}{n} \quad \text{while} \quad |f_n(z_n) - f_n(w_n)| \geq \varepsilon.$$

Now observe that the sequence $z_1, \dots$ has a subsequence $\{z_{n_j^1}\}_{j\geq 1}$ converging to $z \in K$. From here, the sequence $\{w_{n_j^1}\}_{j\geq 1}$ has a subsequence $\{w_{n_j^2}\}_{j\geq 1}$ converging to $w \in K$, and passing *yet again* to the sequence $\{f_{n_k^2}\}_{j\geq 1}$ yields a further subsequence $\{f_{n_k}\}_{k\geq 1}$ converging to some $f \in C(K)$.

But now

$$|f_{n_k}(z_{n_k}) - f_{n_k}(w_{n_k})| \leq |f_{n_k}(z_{n_k}) - f(z_{n_k})| + |f(z_{n_k}) - f(w_{n_k})| + |f(w_{n_k}) - f_{n_k}(w_{n_k})|,$$

which tends to zero as $k \to \infty$. In particular, for some $k \geq 0$, the difference is less than ε, which is a contradiction. □

30.2 Normal Families

We now have the tools to discuss normal families of continuous functions.

Definition 30.3. Let $\mathcal{F}$ be a family of functions defined on a domain $D \subset \mathbb{C}$. We say that D is a *normal family* if every sequence of elements of $\mathcal{F}$ has a subsequence converging normally in D.

With Arzela-Ascoli, we may prove an important sufficient condition for $\mathcal{F}$ to be a normal family.

Theorem 30.4. *Let D be a domain. If $\mathcal{F}$ is a family of continuous functions on D that is both bounded and equicontinuous on D, then $\mathcal{F}$ is a normal family.*

Proof. Consider an *exhaustion of D by compacta*, that is, a sequence of compact sets $\{K_j\}_{j\geq 1} \subseteq D$ such that $K_j \subset K_{j+1}^\circ \subset D$ for each j and $\bigcup_{j\geq 1} K_j = D$.[10] By Arzela-Ascoli, we know that $\mathcal{F}$ is precompact on each K_n.

Now let $\{f_j\}_{j\geq 1}$ be a sequence of functions in $\mathcal{F}$. We repeat the diagonal argument from before. Take a subsequence $\{f_{n_k^1}\}_{k\geq 1}$ of this sequence which converges uniformly on K_1. From this, we may construct a subsequence $\{f_{n_k^2}\}_{k\geq 1}$ converging on K_2, and, proceeding inductively, we may construct nested subsequences $\{f_{n_k^m}\}_{k\geq 1}$ converging uniformly on K_m for each m. Then the sequence $\{f_{n_m^m}\}_{m\geq 1}$ converges uniformly on each compact set K_n.

Finally, let $K \subset D$ be compact. Since $\bigcup_{n=1}^\infty K_n^\circ$ is an open cover for K, there exists some $n \in \mathbb{N}$ for which $K \subset K_n$. Thus $\{f_{n_m^m}\}_{m\geq 1}$ converges uniformly on K, and since K was compact, we deduce the desired result. □

Next, we give a different criterion for $\mathcal{F}$ to be a normal family that builds off the previous one.

Theorem 30.5 (Mantel)**.** *Let D be a domain, and let $\mathcal{F}$ be a family of analytic functions on D. Suppose that, on each compact subset K of D, there exists $M = M(K)$ such that $|f(z)| \leq M$ for all $z \in K$ and $f \in \mathcal{F}$. Then f is a normal family.*

Proof. By the previous result, it suffices to check that $\mathcal{F}$ is equicontinuous on each compact set $K \subset D$.

Fix such a compact set K, and let

$$r := \frac{1}{2}\min(\operatorname{dist}(K, \partial D), 1) > 0.$$

For each $s \in \mathbb{R}$, let $K_s := \{z \in \mathbb{C} : \operatorname{dist}(z, D) \leq s\}$ be the closure of the s-neighborhood of K. Notice that $\mathcal{F}$ is bounded on K_r by some constant M, since K_r is a compact subset of D. Now let $z \in K_{r/2}$ be arbitrary. Then there exists $w \in K$ with $B(w, \frac{r}{2}) \subset K_r$, and so the derivative estimates (Theorem 8.8) imply

$$|f'(z)| \leq \frac{M}{r - |w-z|} \leq \frac{2M}{r} \quad \text{for all } f \in \mathcal{F}.$$

[10]For a explicit construction, set

$$K_n = \{z \in D : |z| \leq n, \operatorname{dist}(z, \partial D) \geq \tfrac{1}{n}\}.$$

In other words, the family $\{f' : f \in \mathcal{F}\}$ is uniformly bounded by some constant C in $C(K_{r/2})$.

Finally, fix z and w in K. If $|z - w| > \frac{r}{2}$, then

$$|f(z) - f(w)| = \left| \int_{[z,w]} f'(\xi)\, d\xi \right| \leq C\, |z - w|\,.$$

So $\mathcal{F}$ is uniformly *Lipschitz*, which implies it is uniformly convergent as well. □

Finally, we state a massive generalization of Montel's Theorem that Erdogan finds pretty cool.

Theorem 30.6 (Big Montel)**.** *Suppose $\mathcal{F}$ is a family of analytic functions on D satisfying the following properties:*

- *there exist distinct complex numbers a and b with $f(z) \notin \{a, b\}$ for every $z \in D$ and $f \in \mathcal{F}$;*
- *there exists some $z_0 \in D$ such that $\mathcal{F}$ is bounded at z_0.*

Then $\mathcal{F}$ is a normal family.

31 November 6

31.1 Riemann Mapping Theorem

Our goal for today is to prove the following theorem. Its proof will synthesize many previous results, from both the past few days and from the rest of the course.

Theorem 31.1 (Riemann Mapping Theorem)**.** *Any simply connected domain $D \neq \mathbb{C}$ is conformally equivalent to $B(0,1)$. Moreover, given $z_0 \in D$, we can find an equivalence with $f(z_0) = 0$ and $f'(z_0) > 0$. These additional conditions determine f uniquely.*

Proof. The proof proceeds in steps.

Step 1. We will first prove there exists a conformal map f *into* (but not necessarily onto) $B(0,1)$ with $0 \in \mathrm{Range}(f)$.

Since $D \neq \mathbb{C}$, there exists some $a \in \mathbb{C} \setminus D$. Then $z - a \neq 0$ on D, and D is simply connected, so the function $z \mapsto \log(z-a)$ has a branch on D (Proposition 8.6). In turn, $z \mapsto \sqrt{z-a} = e^{\frac{1}{2}\log(z-a)}$ has a branch on D; let $g(z)$ be this branch. Note that g is conformal and one-to-one on its image.

Now let $w \in \mathrm{Range}(g)$. By the Open Mapping Theorem (Corollary 24.2), there exists some $r > 0$ such that $B(w,r) \subset \mathrm{Range}(g)$. Observe that no complex number ξ and its negative can simultaneously be in the range of g (why?). Thus $B(-w,r) \not\subset \mathrm{Range}(g)$, i.e. $|z - w| \geq r$ for all $z \in \mathrm{Range}(g)$.

In turn, we may consider the conformal map $h : \mathbb{C} \setminus \{w\} \to \mathbb{C}$ given by $h(z) = \frac{r/2}{z+w}$; then $h \circ g(z) \in B(0,1)$ for all $z \in D$.

Finally, to guarantee that 0 is an element of the range, we may fix $\alpha \in \mathrm{Range}(g)$ and compose $h \circ g$ with the disc automorphism $\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$.

Step 2. By the previous step, we may assume without loss that D is a simply connected domain in $B(0,1)$ with $0 \in D$. We will show that, under these conditions, D is conformally equivalent to $B(0,1)$.

Consider the family of functions

$$\mathcal{F} \coloneqq \{f : D \to B(0,1) \mid f \text{ is conformal and } f(0) = 0.$$

Clearly $\mathcal{F}$ is nonempty since the identity function lies in $\mathcal{F}$. We further note that, since the range of any $f \in \mathcal{F}$ is contained in the unit disc, the family $\mathcal{F}$ is uniformly bounded by 1. Hence, by Mantel (Theorem 30.5), $\mathcal{F}$ is a normal family.

Now consider the set $\mathcal{S} \coloneqq \{|f'(0)| : f \in \mathcal{F}\}$. Observe that, by either derivative estimates or the Schwarz lemma, the set $\mathcal{S}$ is bounded, so it has a supremum s. By the definition of supremum we can find a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{F}$ such that $|f_n'(0)| \to s$ as $n \to \infty$. This sequence has a normally convergent subsequence, so we can assume (by e.g. relabeling) that f_n converges normally to some function F.

We now collect several facts about F.

- By Weierstrass (Theorem 10.6), we know that F is analytic on D and $|f'(0)| = s$.
- Note that F is nonconstant, and recall that each f_n is conformal and therefore one-to-one. Hence Theorem 25.3 implies that F is also one-to-one.
- Since $|f_n| < 1$ for each $n \in \mathbb{N}$, sending $n \to \infty$ tells us that $|F| \leq 1$. However, we can say more: the Open Mapping Theorem implies the range of F is open, so actually $|F| < 1$.

Combining these three bullets yields $F \in \mathcal{F}$.

We now make the surprising claim that $F : D \to B(0,1)$ is surjective, which will complete Step 2 of the proof. To show this, assume for contradiction that F is not onto, so there exists $\alpha \in B(0,1) \setminus \mathrm{Range}(F)$. Composing F with the function $\varphi_\alpha = \frac{z-\alpha}{1-\bar{\alpha}z}$ yields a function $\varphi_\alpha \circ F : D \to B(0,1)$ with $0 \notin \mathrm{Range}(F)$. In turn, $\sqrt{z}$ has a branch h on the range of $\varphi_\alpha \circ F$.

Now set

$$G \coloneqq \varphi_{h(-\alpha)} \circ h \circ \varphi_\alpha \circ F.$$

Observe that F is one-to-one on D, and hence $G \in \mathcal{F}$. Furthermore, we can "invert" both sides of the above equality to get

$$F = \underbrace{\varphi_{-\alpha} \circ h^{-1} \circ \varphi_{-h(\alpha)}}_{=:H} \circ G,$$

where $h^{-1} : \mathbb{C} \to \mathbb{C}$ is given by $h(z) = z^2$. In particular, H is *not* one-to-one and sends 0 to 0. Hence the Schwarz Lemma (Lemma 9.5) implies that $|H'(0)|$ is *strictly* less than 1 (or else $H(z) = ze^{i\theta}$ for some θ, which is one-to-one). Thus

$$|F'(0)| = |H'(G(0))| \cdot |G'(0)| = |H'(0)| \cdot |G'(0)| < |G'(0)|,$$

which contradicts the fact that F is the maximizer of S in $\mathcal{F}$. This concludes the proof of the first part of Theorem 31.1.

Step 3. Now fix some $z_0 \in D$. Since $F : D \to B(0,1)$ is a conformal equivalence, $\varphi_{F(z_0)} \circ F$ is a conformal equivalence with z_0 mapped to zero. Furthermore,

$$(\varphi_{F(z_0)} \circ F)'(0) \neq 0,$$

since otherwise it would fail to be one-to-one at zero. In particular, translate $\varphi_{F(z_0)} \circ F$ so that it equals 0 at $z = 0$; then the multiplicity of zero is at least two, allowing us to invoke the Branched Covering Principle (Corollary 24.1). Hence there exists $\theta \in \mathbb{R}$ with $(e^{i\theta}\varphi_{F(z_0)} \circ F)'(0) > 0$, and so $e^{i\theta}\varphi_{F(z_0)} \circ F$ is the desired conformal equivalence.

Step 4. Finally, assume f_1 and f_2 are conformal equivalences from D to $B(0,1)$ with $f_1(z_0) = f_2(z_0) = 0$ and with both $f_1'(z_0)$ and $f_2'(z_0)$ positive. Observe that the function $\varphi := f_1 \circ f_2^{-1}$ is a disc automorphism with $\varphi(0) = 0$. Furthermore, we may compute

$$\varphi'(0) = (f_1 \circ f_2^{-1})'(0) \overset{(*)}{=} \frac{f_1'(f_2^{-1}(0))}{f_2'(f_2^{-1}(0))} = \frac{f_1'(z_0)}{f_2'(z_0)} > 0.$$

Now observe that, since φ is a disc automorphism, we know that $\varphi(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ for some $\alpha \in \mathbb{C}$ and some $\theta \in \mathbb{R}$. From $\varphi(0) = 0$ we obtain $\alpha = 0$, and so actually $\varphi(z) = e^{i\theta}z$. But also we know $\varphi'(0) > 0$, so $e^{i\theta} = 1$, implying $\varphi(z) = z$. In turn, $f_1 = f_2$, which is what we were after. □

32 November 8

32.1 Modulus of Conformal Maps

Our goal for today is to discuss an invariant associated with conformal maps. We begin with two preliminary definitions.

Definition 32.1. Let D be a domain, and $\rho : D \to [0, \infty)$ be a continuous function (sometimes called a *density*).

1. Given a piecewise-smooth contour γ, the *length with respect to ρ of the path γ* is defined to be
$$\ell_\rho(\gamma) := \int_a^b |\gamma'(t)|\, \rho(\gamma(t))\, dt.$$
2. Given a set $G \subset D$ with $\vec{x} \mapsto \chi_G(\vec{x})$ Riemann-integrable in $\mathbb{R}^2$, the *area of G with respect to ρ* is defined to be
$$A_\rho(G) := \iint_G \rho(x, y)^2\, dx\, dy.$$

We now define the conformal modulus of a conformal map.

Definition 32.2. Let E, F, and G be subsets of a domain D.

1. The "triplet" $[E, F : G]$ is called a *configuration* of D if E and F are pairwise disjoint subsets of G.
2. The set $\Gamma[E, F : G]$ is the set of contours $\gamma \subset G$ with $\gamma(0) \in E$ and $\gamma(1) \in F$.
3. A density ρ is *admissible* if $\ell_\rho(\gamma) \geq 1$ for every contour $\gamma \in [E, F : G]$. Write $\mathrm{Adm}[E, F : G]$ as the set of admissible densities.
4. The *conformal modulus* of the configuration $[E, F : G]$ is
$$M[E, F : G] := \inf\{A_\rho(G) : \rho \in \mathrm{Adm}[E, F : G]\}.$$
By convention, this infimum is ∞ if $\mathrm{Adm}[E, F : G]$ is empty.

This definition, at first, seems extremely unwieldy. While it feels arbitrary, the next theorem shows that it is useful.

Theorem 32.3. *Let $f : D \to \mathbb{C}$ be a conformal map, and suppose $[E, F : G]$ is a configuration in D. Then $[f(E), f(F) : f(G)]$ is a configuration in $f(D)$ and*
$$M[E, F : G] = M[f(E), f(F) : f(G)].$$

Proof. It suffices to prove that the left hand side is at most the right hand side; the other inequality can be obtained by looking at f^{-1}.

Let $\tilde{\rho} \in \mathrm{Adm}[f(E), f(F) : f(G)]$ be arbitrary. Define $\rho : D \to [0, \infty)$ via
$$\rho(z) := |\tilde{\rho}(f(z))| \cdot |f'(z)|\,.$$
We claim that $\rho \in \mathrm{Adm}[E, F : G]$. To prove this, consider any contour connecting E to F in G. Then $f \circ \gamma$ connects $f(E)$ to $f(F)$ in $f(G)$, so
$$\begin{aligned} 1 \leq \ell_{\tilde{\rho}}(f \circ \gamma) &= \int_a^b |(f \circ \gamma)'(t)|\, \tilde{\rho}(f \circ \gamma(t)) \\ &= \int_a^b f'(\gamma(t)) \tilde{\rho}(f \circ \gamma(t))\, |\gamma'(t)| = \ell_\rho(\gamma). \end{aligned}$$
Since γ was arbitrary, we deduce that ρ is admissible.

Now
$$M[E, F : G] = \inf\{A_\rho(G) : \rho \text{ admissible}\} \leq A_\rho(G) \text{ for any fixed } \rho.$$
So
$$A_\rho(G) = \iint_G (\rho(z))^2\, dx\, dy = \iint_G (\tilde{\rho}(f(z)))^2\, |f'(z)|\, dx\, dy. \tag{32.1}$$

Now recall the multivariable chain rule: if $f : A \to B$ is a diffeomorphism, and $h : B \to \mathbb{R}$, then

$$\int_B h(y)\,dy = \int_A h \circ f(x)\,|\det(Df(x))|\,dx,$$

where $Df(x)$ is the *Jacobian* matrix. In our case, the diffeomorphism is $f(x,y) = \binom{u(x,y)}{v(x,y)}$, so

$$|\det(Df(x))| = \left|\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\right| = |u_x v_y - v_x u_y| \overset{(*)}{=} u_x^2 + v_x^2 = |f'(x)|^2.$$

(Here $(*)$ is a result of the Cauchy-Riemann equations (2.1).) Hence the integral in (32.1) is equal to $\int_{f(G)} \tilde{\rho}(x+iy)^2\,dx\,dy = A_{\tilde{\rho}}(f(G))$.

Hence, given $\tilde{\rho} \in \mathrm{Adm}[f(E), f(F) : f(G)]$, we have shown $M[E,F:G] \le A_{\tilde{\rho}}(f(G))$. Taking the infimum over all admissible $\tilde{\rho}$ yields the desired inequality. $\square$

Until now, all our work concerning the modulus of a conformal map has been theoretical. Let's do an example.

Example 32.4. Let $E = \overline{B(0,r_0)}$ and $F = B(0,r_1)^c$, where $0 < r_0 < r_1 < \infty$. We claim that

$$M[E,F:\mathbb{C}] = \frac{2\pi}{\log(r_1/r_0)}.$$

First, let $\gamma(r) = re^{i\theta}$, for $r \in [r_0, r_1]$. If ρ is any admissible density,

$$1 \le \ell_\rho(\gamma) = \int_{r_0}^{r_1} \rho(re^{i\theta})\left|e^{i\theta}\right|dr = \int_{r_0}^{r_1} \rho(re^{i\theta})\,dr. \tag{32.2}$$

By Cauchy-Schwarz, we may bound

$$A_\rho(\mathbb{C}) = \iint_{\mathbb{C}} \rho^2(z)\,dx\,dy = \int_0^{2\pi}\int_0^\infty \rho^2(re^{i\theta})r\,dr\,d\theta$$

to obtain

$$\begin{aligned}\int_{r_0}^{r_1} \rho(re^{i\theta})\,dr = \int_{r_0}^{r_1} \frac{\rho(re^{i\theta})\sqrt{r}}{\sqrt{r}} &\le \sqrt{\int_{r_0}^{r_1} \rho^2(re^{i\theta})\,r\,dr}\cdot\sqrt{\int_{r_0}^{r_1}\frac{1}{r}\,dr} \\ &= \sqrt{\int_{r_0}^{r_1} \rho^2(re^{i\theta})\,r\,dr}\cdot\sqrt{\frac{1}{\log(r_1/r_0)}}.\end{aligned}$$

Combining this inequality with (32.2) yields $\int_{r_0}^{r_1} \rho^2(re^{i\theta})r\,dr \ge \frac{1}{\log(r_1/r_0)}$, and so

$$A_\rho(\mathbb{C}) \ge A_\rho(B(0,r_1)\setminus B(0,r_0)) = \int_0^{2\pi}\int_{r_0}^{r_1}\rho^2(re^{i\theta})r\,dr \ge \frac{2\pi}{\log(r_1/r_0)}.$$

Hence $M[E,F:G] \ge 2\pi/\log(r_1/r_0)$.

Now, given $\varepsilon > 0$, let

$$\rho(z) = \begin{cases} (|z|\log(r_1/r_0))^{-1} & r_0 \le |z| \le r_1, \\ 0 & |z| > r_1 + \varepsilon \text{ or } |z| < r_0 - \varepsilon, \\ g(z) & \text{otherwise,} \end{cases}$$

where $g(z)$ is some continuous function that makes ρ continuous. We claim that ρ is admissible. To prove this, given γ connecting E to F, take real numbers a and b such that $|\gamma(t)| \in (r_0, r_1)$ whenever $r \in (a,b)$. Then

$$\begin{aligned}\ell_\rho(\gamma) &\ge \left(\log\left(\frac{r_1}{r_0}\right)\right)^{-1}\int_a^b \frac{1}{|\gamma(t)|}\cdot|\gamma'(t)|\,dt \\ &\overset{(*)}{\ge} \left(\log\left(\frac{r_1}{r_0}\right)\right)^{-1}\int_a^b (\log|\gamma(t)|)'\,dt = 1.\end{aligned}$$

In particular, to obtain the inequality $(*)$ we may write

$$\begin{aligned}(\log|\gamma(t)|)' = \left(\frac{1}{2}\log|\gamma(t)|^2\right)' &= \frac{(\gamma(t)\overline{\gamma(t)})'}{2|\gamma(t)|^2} \\ &= \frac{2\Re(\gamma'(t)\overline{\gamma(t)})}{|\gamma(t)|^2} \le \frac{|\gamma'(t)|\,|\gamma(t)|}{|\gamma(t)|^2} = \frac{|\gamma'(t)|}{|\gamma(t)|}.\end{aligned}$$

Finally, write

$$\begin{aligned}A_\rho(\mathbb{C}) = \iint_{r_0-\varepsilon<|z|<r_1+\varepsilon} \rho^2(z)\,dz &\le \iint_{r_0-\varepsilon<|z|<r_1+\varepsilon} \frac{dx\,dy}{|z|^2\log(r_1/r_0)^2} \\ &= \int_0^{2\pi}\int_{r_0-\varepsilon}^{r_1+\varepsilon} \frac{1}{r}\cdot\frac{dr\,d\theta}{(\log(r_1/r_0))^2} = 2\pi\frac{\log\frac{r_1+\varepsilon}{r_0-\varepsilon}}{(\log(r_1/r_0))^2}.\end{aligned}$$

Now send $\varepsilon \to 0$.

In the last minute or two, we record two small propositions regarding the modulus of conformity. We will use these on Monday.

Proposition 32.5. *Suppose $[E_1, F_1 : G_1]$ and $[E_2, F_2 : G_2]$ are configurations for which $E_1 \subset E_2$, $F_1 \subset F_2$, and $G_1 \subset G_2$. Then $M[E_1, F_1 : G_1] \le M[E_2, F_2 : G_2]$.*

Proposition 32.6. *Suppose $\overline{B(z_1, r_1)}$ and $\overline{B(z_2, r_2)}$ are two disjoint closed balls in $\mathbb{C}$. Then*

$$M[\overline{B(z_1,r_1)}, \overline{B(z_2,r_2)} : \mathbb{C}] < \infty.$$

33 November 11

33.1 Caratheodory-Osgood

We start today with an application of the conformal modulus.

Theorem 33.1. *Let $f : B(0,1) \to D$ be a conformal equivalence, where D is a domain whose boundary ∂D is a simple closed contour. Then f can be extended to $\tilde{f} : \overline{B(0,1)} \to \overline{D}$, where $\tilde{f}$ is one-to-one, onto, and continuous, and furthermore $\tilde{f}^{-1} : \overline{D} \to \overline{B(0,1)}$ is continuous.*

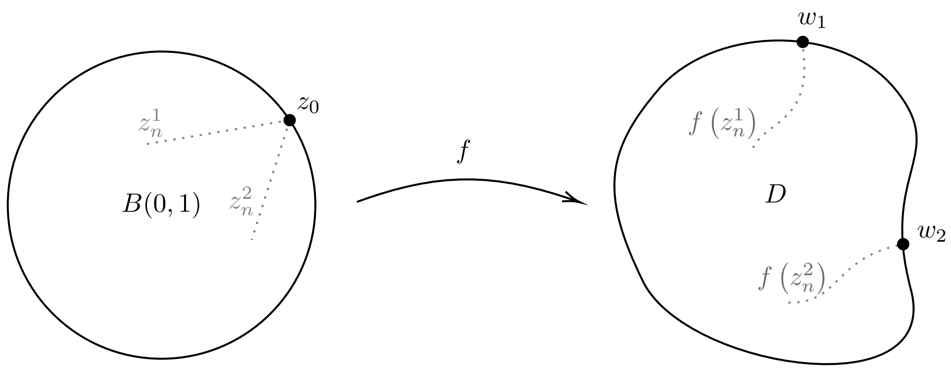
Remark. If D is unbounded, replace $\overline{D}$ with $\hat{D}$, the closure of D in $\hat{\mathbb{C}}$. Also, note we may replace $B(0,1)$ with H by composing with the Möbius transformation $z \mapsto \frac{z-i}{z+i}$; this will be useful later.

Proof. We will only provide a proof of the main claim: for $z_0 \in \partial B(0,1)$, the limit $\lim_{z \to z_0} f(z)$ exists. Then, we may set

$$\tilde{f}(z) := \begin{cases} f(z) & \text{if } z \in B(0,1), \\ \lim_{\xi \to z} f(\xi) & \text{if } z \in \partial B(0,1); \end{cases}$$

this turns out to work.

Assume for the sake of contradiction that the limit does not exist. This means there exist $w_1 \neq w_2$ in ∂D and sequences $\{z_n^1\}_{n \geq 1}$ and $\{z_n^2\}_{n \geq 1}$ such that $z_n^j \to z_0$ as $n \to \infty$, but $f(z_n^j) \to w_j$, where $j \in \{1, 2\}$.



Since ∂D is piecewise smooth, we may choose $0 < r < \frac{1}{2}|w_1 - w_2|$ such that $D \cap B(w_1, r)$ and $D \cap B(w_2, r)$ are connected.[11] Now choose N such that $f(z_n^j) \subset B(w_j, r)$ for all $n \geq N$. Let γ_1 and γ_2 be continuous curves passing through the points $f(z_n^1)$ and $f(z_n^2)$ respectively (for $n \geq N$), and consider $E := f^{-1}(\gamma_1)$ and $F := f^{-1}(\gamma_2)$. (So γ_1 and γ_2 are the curves corresponding to the dotted lines in the diagram to the right, while E and F are the curves corresponding to the dotted lines in the diagram to the left.) Then Propositions (32.5) and (32.6) imply

$$M[\gamma_1, \gamma_2 : D] \leq M[\overline{B(w_1, r)}, \overline{B(w_2, r)} : \mathbb{C}] < \infty.$$

We will be done if we can show that $M[E, F : B(0,1)] = \infty$, which is a contradiction since modulus is invariant under conformal mapping.

Choose ε small enough so that $\partial B(w_j, \varepsilon)$ intersects both E and F. For $0 < s < \varepsilon$, choose $\theta_1(s)$ and $\theta_2(s)$ in $[0, 2\pi]$ so that

$$z_0 + se^{i\theta_1(s)} \in E \quad \text{and} \quad z_0 + se^{i\theta_2(s)} \in F.$$

Now recall that

$$M[E, F : B(0,1)] = \inf\{A_\rho(B(0,1)) : \rho \text{ is admissible}\}.$$

[11]This follows from the fact that, near z_0, the boundary of D looks "flat"; I believe this can be made rigorous by viewing D as a smooth manifold with boundary.

Given an admissible ρ, define $\gamma_s(\theta) = z_0 + se^{i\theta}$ for $\theta \in [\theta_1(s), \theta_2(s)]$. We know that, since ρ is admissible,

$$\begin{aligned} 1 \le \ell_\rho(\gamma_s(\theta)) &= \int_{\theta_1(s)}^{\theta_2(s)} \rho(z_0 + se^{i\theta}) s \, d\theta \\ &\le s\sqrt{\int_{\theta_1(s)}^{\theta_2(s)} 1 \, d\theta} \cdot \sqrt{\int_{\theta_1(s)}^{\theta_2(s)} \rho^2(z_0 + se^{i\theta}) \, d\theta} \le s\sqrt{2\pi} \cdot \sqrt{\int_{\theta_1(s)}^{\theta_2(s)} \rho^2(z_0 + se^{i\theta}) \, d\theta}. \end{aligned}$$

Since this holds for all $s \in (0, \varepsilon)$, we deduce that

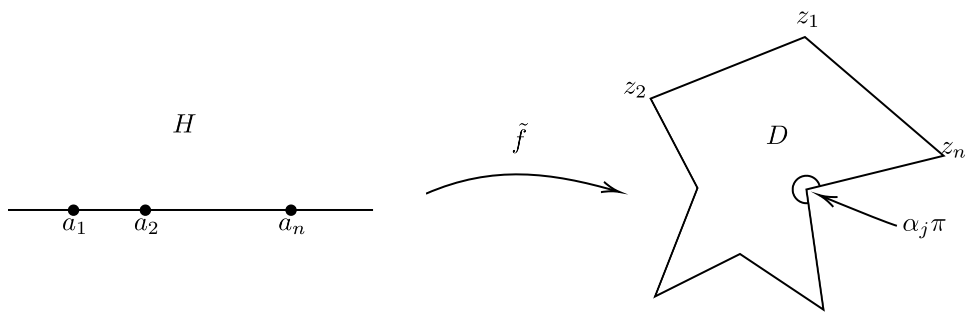
$$A_\rho(B(0,1)) \ge \int_0^\varepsilon \int_{\theta_1(s)}^{\theta_2(s)} \rho^2(z + se^{i\theta}) \, d\theta \, ds \ge \int_0^\varepsilon s \cdot \frac{1}{2\pi s^2} \, ds = \infty.$$

We are done. □

33.2 Schwarz-Cristoffel Maps

Many of the theoretical results concerning conformal maps are non-constructive – they tell us *when* conformal equivalences exist, but they give no indication as to what those maps are. In general, this question is difficult. There is an important special case, though, that allows us to construct a conformal equivalence between the upper half plane H and a particular kind of domain D.

Let D be a polygonal domain whose boundary is simple. By the Riemann Mapping Theorem, there exists a conformal equivalence $f : H \to D$. Caratheodory-Osgood tells us this extends to $\tilde{f} : \hat{H} \to \bar{D}$. Denote by $z_1, \ldots, z_n$ the corners of the domain D, and for each $1 \le j \le n$ set $a_j = f^{-1}(z_j)$; without loss of generality we may let $a_1 < a_2 < \cdots < a_n$. Further, let $\alpha_j\pi$ be the inner angle at the corner z_j for each j.



The crucial claim is that

$$f'(z) = A(z - a_1)^{\alpha_1 - 1} \ldots (z - a_n)^{\alpha_n - 1},$$

where A is some constant and the branches on the functions $z \mapsto z^{\alpha_j - 1}$ are principal; after this, we may deduce

$$f(z) = A \int_1^z (\xi - a_1)^{\alpha_1 - 1} \ldots (\xi - a_n)^{\alpha_n - 1} \, d\xi + B,$$

where A and B are constants. (If $a_n = \infty$, we drop the last term.)

To prove this, without loss let $a_n < \infty$, and proceed in steps.

Step 1. Define $g := \frac{f''}{f'}$. We claim g can be extended to an analytic function on $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$. To see this, note by the Schwarz Reflection Principle (Example 29.1), for each $0 \le i \le n$ we can extend f to F_j defined on $H \cup H^* \cup (a_j, a_{j+1})$, where by convention we set $a_0 = -\infty$ and $a_{n+1} = \infty$. In particular, $F_j(z) = \rho_j(f(\bar{z}))$, where ρ_j denotes reflection across the line passing through z_j and z_{j+1}.

In general, these F_j functions are not necessarily identical; this is where g comes in. More specifically, note that for every $j \ne k$ we have

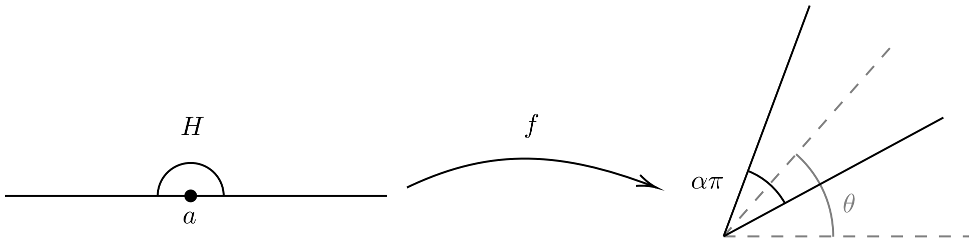
$$F_j \circ F_k^{-1} = (\rho_j \circ f \circ \rho) \circ (\rho_k \circ f \circ \rho)^{-1} = \rho_j \circ \rho_k^{-1} = \rho_j \circ \rho_k.$$

Since the composition of reflections $\rho_j \circ \rho_k$ is an affine transformation on $\mathbb{C}$, we deduce $F_j \circ F_k^{-1}(z) = az + b$ for some constants a and b. Hence $F_j = aF_k + b$, and so

$$\frac{F_j''}{F_j'} = \frac{aF_k''}{aF_k'} = \frac{F_k''}{F_k'}.$$

Thus the extensions of g along each side and interval agree.

Step 2. We claim that g has simple poles at each a_j with residues $\alpha_j - 1$. To prove this, fix j, and let $a = a_j$ and $\alpha = \alpha_j$ for ease of typesetting.



Define

$$\psi(z) := \left[e^{i\theta}(f(z) - f(a))\right]^{1/\alpha},$$

where θ is the argument of the ray which bisects the interior angle of the corner $f(a)$ (see above). Then the image of ψ is the right half plane, so we may extend ψ to $B(a, \varepsilon)$ by reflection; such a reflection is both one-to-one and analytic. Note further that $\psi(a) = 0$, and since ψ is one-to-one it must be a simple zero. Thus $\psi(z) = (z - a)h(z)$, where $h(z)$ is a nonzero analytic function on $B(a, \varepsilon)$.

Setting these expressions equal to each other yields

$$\psi^\alpha(z) = (z - a)^\alpha h^\alpha(z) = e^{-i\theta}(f(z) - f(a)),$$

and hence

$$f(z) = (z - a)^\alpha h^\alpha(z) e^{i\theta} + f(a).$$

Finally, we may compute $f'(z)$ and $f''(z)$ whenever $z \in H \cap B(a, \varepsilon)$; doing so yields

$$\frac{f''}{f'} = \frac{1}{z - \alpha}\left[\frac{\alpha(\alpha - 1)h^\alpha + 2\alpha(h^\alpha)'(z - a) + (z - a)^2(h^\alpha)''}{\alpha h^\alpha + (z - a)(h^\alpha)'}\right].$$

Miraculously, this expression is analytic and equals $\alpha - 1$ at $z = a$. Thus the pole is simple with residue $\alpha - 1$, as desired.

We will finish this example next time.

34 November 13

We continue with our investigation into Schwarz-Cristoffel maps.

Step 3. We further claim that $g(z) \to 0$ as $z \to \infty$. To prove this, define $f_1(z) := f(-\frac{1}{z})$; then $f_1 : H \to D$. Steps 1 and 2 imply that the function $g_1(z) := \frac{f_1''(z)}{f_1'(z)}$ is analytic on $\mathbb{C}$ except at some simple poles. Now a bit of computation reveals that

$$g_1(z) = \frac{f(-\frac{1}{z})''}{f(-\frac{1}{z})'} = \frac{\frac{1}{z^4} f''(-\frac{1}{z}) - \frac{2}{z^3} f'(-\frac{1}{z})}{\frac{1}{z^2} f'(-\frac{1}{z})} = -\frac{2}{z} + \frac{1}{z^2} g\left(-\frac{1}{z}\right).$$

Since g_1 has at most a simple pole at zero, $\left|g_1(-\frac{1}{z})\right| \le C\,|z|$ for some constant C. Thus $g(z) \to 0$ as $z \to \infty$.

Step 4. Now consider the function

$$h(z) := g(z) - \sum_{j=1}^{n} \frac{\alpha_j - 1}{z - a_j}.$$

Step 2 tells us that h is entire, while Step 3 implies it vanishes at ∞; thus, $h(z)$ is identically zero and

$$g(z) = \sum_{j=1}^{n} \frac{\alpha_j - 1}{z - a_j} \quad \text{for all } z \in H.$$

Finally, observe that $\log f'$ is a primitive for g while $\sum_{j=1}^{n} (\alpha_j - 1) \operatorname{Log}(z - a_j)$ is a primitive for the right hand side. Since H is convex, we deduce that

$$\log f'(z) = \sum_{j=1}^{n} (\alpha_j - 1) \operatorname{Log}(z - a_j) + C \text{ for some constant } C,$$

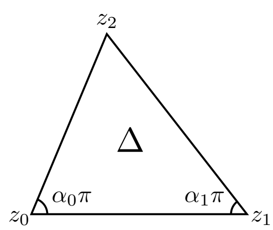
and hence

$$f'(z) = e^C \prod_{j=1}^{n} (z - a_j)^{\alpha_j - 1}.$$

We are done.

We now show how to use these maps in practice.

Example 34.1. Let Δ be the triangle in the complex plane with vertices at z_0, z_1, and z_2; further let the angles of Δ at z_0 and z_1 be $\alpha_0\pi$ and $\alpha_1\pi$, respectively. Recall by the Riemann Mapping Theorem there exists some conformal map f sending H to T_A. Denote by $a_j \in \partial H$ the preimage of z_j under the extension $\hat{f} : \overline{H} \to \overline{\Delta}$ for each j; we may assume without loss that $a_0 < a_1 < a_2$.



Now consider the Möbius transform T_A sending 0 to a_0, 1 to a_1, and ∞ to a_2. Observe that T_A is an order-preserving Möbius transform fixing the real axis, and thus it must also fix H. In turn, $g := f \circ T_A$ is also a conformal map from H to Δ, and its extension $\hat{g} : \overline{H} \to \overline{\Delta}$ sends 0 to z_0, 1 to z_1, and ∞ to z_2. From this, we may use Schwarz-Cristoffel to deduce that

$$g'(z) = A z^{\alpha_0 - 1} (z - 1)^{\alpha_1 - 1}. \tag{34.1}$$

We finish by describing the explicit structure of g. First remark that $g(0) = z_0$, so integrating (34.1) yields

$$g(z) = \int_{[0,z]} g'(\xi)\, d\xi + z_0 = A \int_{[0,z]} \xi^{\alpha_0 - 1} (\xi - 1)^{\alpha_1 - 1}\, d\xi + z_0.$$

Furthermore, we have $g(1) = z_1$, so

$$z_1 = g(1) = A \int_{[0,1]} \xi^{\alpha_0 - 1} (\xi - 1)^{\alpha_1 - 1}\, d\xi + z_0,$$

whence

$$A = \frac{z_1 - z_0}{\int_{[0,1]} \xi^{\alpha_0 - 1} (\xi - 1)^{\alpha_1 - 1}\, d\xi}.$$

All in all,

$$g(z) = \boxed{(z_1 - z_0)\frac{\int_{[0,z]} \xi^{\alpha_0 - 1}(\xi - 1)^{\alpha_1 - 1}\,d\xi}{\int_{[0,1]} \xi^{\alpha_0 - 1}(\xi - 1)^{\alpha_1 - 1}\,d\xi} + z_0}.$$

Example 34.2. Let S be a square with vertices z_0, z_1, z_2, and z_3 in clockwise order. Note that, by the Riemann Mapping Theorem, there exists some conformal mapping $f : H \to S$.

As in the previous example, we may assume (via composition by a suitable Möbius transform) that the preimages of z_0, z_1, and z_2 are 0, 1, and ∞, respectively. The previous example details how to find f; it remains to extend this definition to include z_3 as well. It seems natural to construct z_3 by reflecting the map f across the diagonal $[z_0, z_2]$. However, naïvely this does not work, since doing so leads to a map with an incorrect domain.

Instead, let $D = \{x + yi : x > 0, y > 0\}$ be the positive quadrant of the complex plane, so that the conformal mapping $s(z) = z^2$ sends D to H. *Now* we may use Schwarz Reflection Principle on the composition $f \circ s : D \to S$ to obtain a conformal mapping from H to S as desired. In particular, the preimage of z_3 is the reflection of 1 across the imaginary axis, i.e. -1.

34.1 Construction of Entire Functions with Prescribed Poles

We now go in a different direction and talk about other ways to construct functions with various properties.

The first idea we will discuss is that of the Mittag-Leffler construction. Recall that f has a pole at z_0 of order m if and only if there exists $r > 0$ such that, on $B^*(z_0, r)$, we may write

$$f(z) = \sum_{k=-m}^{-1} a_k (z - z_0)^k + \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

as a Laurent series expansion of f about z_0. We call the former sum the *singular* or *principal* part and the latter sum the *analytic* part. Note that the principal part can be written as a polynomial in $\frac{1}{1-z_0}$ with zero constant term.

We now ask whether it is possible to construct functions with *given principal parts.* The answer is "yes".

Theorem 34.3 (Mittag-Leffler). *Let $E = \{z_j\} \subset \mathbb{C}$ be a discrete subset of $\mathbb{C}$. For each $z_j \in E$, let $q_j(z) = p_j(\frac{1}{1-z_j})$, where p_j is a polynomial with $p_j(0) = 0$, be a given principal part of z_j. Then there exists a meromorphic function f on $\mathbb{C}$ with poles exactly on E and principal part q_j at z_j.*

Proof. If E is finite the proof is easy: simply take $f(z)$ to be the sum of all $q_j(z)$ terms. If E is infinite, however, we have to be a bit more careful, because *a priori* the sum may not converge normally.

Since E is a discrete subset of $\mathbb{C}$, we may assume

$$|z_1| \le |z_2| \le \cdots \le |z_n| \le \cdots$$

with $z_n \to \infty$ as $n \to \infty$. The idea will be to write

$$f(z) = \sum_{j=1}^{\infty} (q_j(z) - T_j(z)),$$

where $T_j(z)$ is an entire "correction" term chosen so that the sum converges normally on $\mathbb{C} \setminus E$.

For each j with $z_j \ne 0$, observe that the Taylor series of q_j centered at $z = 0$ converges normally on $B(0, |z_j|)$ (why?), so it converges *uniformly* on $B(0, \frac{1}{2}|z_j|)$. Hence e.g. there exists $N_j > 0$ such that, if T_j consists of the first N_j terms of the Taylor series, then

$$|q_j(z) - T_j(z)| \le \frac{1}{j^2} \quad \text{for all } z \in B(0, \tfrac{1}{2}|z_j|). \tag{34.2}$$

With this, we claim that the series

$$f(z) := q_1(z) + \sum_{j=2}^{\infty} (q_j - T_j)(z)$$

converges normally on $\mathbb{C}$. (We ignore the q_1 term because it is possible that $z_1 = 0$, in which case the Taylor series is not well-defined.)

Fix a compact set K, and choose $R > 0$ such that $K \subset B(0, R)$. By assumption, there exist only finitely many j with $|z_j| < 2R$. In turn, we may write

$$f(z) = q_1(z) + \sum_{|z_j|<2R} (q_j - T_j)(z) + \sum_{|z_j|\geq 2R} (q_j - T_j)(z).$$

Since

$$\left| \sum_{|z_j|\geq 2R} (q_j - T_j)(z) \right| \leq \sum_{|z_j|\geq 2R} |(q_j - T_j)(z)| \leq \sum_{|z_j|\geq 2R} \frac{1}{j^2} < \infty,$$

the tail converges uniformly on $B(0, R)$. The former sum is analytic on $B(0, R) \setminus E$ with the prescribed poles and singular parts; sending $R \to \infty$ gives us the desired. □

34.2 Cotangent Series Representation

Mittag-Leffler is not extremely useful as a black box; instead, we may use its key ideas to construct series representations of various analytic functions. We illustrate these ideas in the next example, which is very important in analytic number theory.

Example 34.4. Let $f(z) = \pi \cot(\pi z)$. Recall/observe that the poles of f occur exactly on $\mathbb{Z}$, and each pole is simple with residue 1. (The latter can be seen by writing $f(z) = \frac{\pi \cos(\pi z)}{\sin(\pi z)}$ and appealing to Example 18.8.) Thus, in the context of Mittag-Leffler, $f(z)$ is meromorphic, $E = \mathbb{Z}$, and $q_j = \frac{1}{z-j}$ for each $j \in \mathbb{Z}$.

When $j \neq 0$, observe that

$$q_j(z) = \frac{1}{z - j} = \frac{-1/j}{1 - z/j} = -\frac{1}{j} + O\left(\frac{1}{j^2}\right);$$

i.e. the first term of the Taylor expansion of q_j about $z = 0$ is $-\frac{1}{j}$. Furthermore, whenever $0 < R < \frac{1}{2}|j|$, we have

$$\left| \frac{1}{z - j} + \frac{1}{j} \right| = \left| \frac{z}{(z - j)j} \right| \leq \frac{R}{\frac{1}{2}|j| \cdot |j|} = \frac{2R}{j^2}.$$

Thus, as in the proof of Mittag-Leffler, we may write

$$g(z) = \frac{1}{z} + \sum_{j\neq 0} \left(\frac{1}{z - j} + \frac{1}{j} \right) \tag{34.3}$$

and observe that the sum converges on $\mathbb{C} \setminus \mathbb{Z}$ and has the same poles and principal parts as f.

Now observe that (34.3) can be rewritten as

$$g(z) = \frac{1}{z} + \sum_{j=1}^{\infty} \left(\frac{1}{z - j} + \frac{1}{z + j} \right) = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2},$$

which again converges normally on $\mathbb{C} \setminus \mathbb{Z}$. In turn, the function

$$h(z) = \pi \cot(\pi z) - \frac{1}{z} - \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}$$

is entire. Next time, we will see that it is bounded and tends to zero as $z \to 0$; this implies that $h(z) \equiv 0$, whence

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}. \tag{34.4}$$

Furthermore, we may differentiate both sides of this equality to obtain

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z - n)^2}. \tag{34.5}$$

Both of these formulas are pretty important.

35 November 15

35.1 Cotangent Series Representation (cont.)

We continue with the example from last time.

Example 35.1. We substantiate the claims from last time, namely that $h(z)$ is bounded and tends to zero as $z \to 0$. To prove the first claim, first observe that h is one-periodic – that is, $h(z+1) = h(z)$ for all z – since both $\pi\cot(\pi z)$ and the sum are. This means that $|h| \le C$ for some constant C on some horizontal strip, say $\{|y| \le 10\}$.

Now assume $|x| \le \frac{1}{2}$ and $|y| > 10$; note that the first assumption is possible due to one-periodicity. First remark that

$$|\pi\cot(\pi z)| = \pi\frac{\left|e^{i\pi z} + e^{-i\pi z}\right|}{\left|e^{i\pi z} - e^{-i\pi z}\right|} \le \pi \cdot \frac{e^{\pi|y|} + e^{-\pi|y|}}{e^{\pi|y|} - e^{-\pi|y|}},$$

which is bounded since it tends to 1 as $|y| \to \infty$. Furthermore, we may compute

$$\begin{aligned}\left|z^2 - j^2\right| &= \left|x^2 - y^2 - j^2 + 2ixy\right| \ge \left|x^2 - y^2 - j^2\right| \\ &= |y|^2 + j^2 - |x|^2 \ge |y|^2 + j^2 - \tfrac{1}{4} \ge |y|^2 + \left(j - \tfrac{1}{2}\right)^2.\end{aligned}$$

Therefore, when combined with the bound $2|z| \le 3|y|$ (which holds since $|x|$ is small and $|y|$ is large), we get

$$\begin{aligned}\left|\frac{1}{z} + \sum_{j=1}^{\infty}\frac{2z}{z^2 - j^2}\right| &\le \frac{1}{10} + \sum_{j=1}^{\infty}\frac{3|y|}{|y|^2 + (j - \frac{1}{2})^2} \\ &\le \frac{1}{10} + 3\left(\frac{|y|}{|y|^2 + \frac{1}{4}} + \int_{1/2}^{\infty}\frac{|y|}{|y|^2 + s^2}\right) \\ &= \frac{1}{10} + 3\left(\frac{|y|}{|y|^2 + \frac{1}{4}} + \tan^{-1}(2|y|)\right) \le \frac{31}{10} + \frac{3\pi}{2} < \infty.\end{aligned}$$

In turn, the sum is bounded on $\mathbb{C}$, so h is bounded on $\mathbb{C}$; Liouville then implies h is constant.

To determine exactly which constant, we determine the behavior of h as $z \to 0$. Observe that

$$\pi\cot(\pi z) = \frac{\pi\cos(\pi z)}{\sin(\pi z)} = \frac{\pi(1 + O(z^2))}{\pi z + O(z^3)} = \frac{1}{z} + O(z)$$

and

$$-\frac{1}{z} - \sum_{j=1}^{\infty}\frac{2z}{z^2 - j^2} = -\frac{1}{z} - 2\sum_{j=1}^{\infty}\frac{2}{z^2 - j^2} = -\frac{1}{z} + O(z),$$

where the last equality comes from the fact that the inner sum approaches $\frac{\pi^2}{3}$ as $z \to 0$. Therefore

$$h(z) = \left(\frac{1}{z} + O(z)\right) + \left(-\frac{1}{z} + O(z)\right) = O(z),$$

and hence $h(z) \to 0$ as $z \to 0$. In particular, $h(0) = 0$, so $h(z) \equiv 0$ identically, and we may conclude (34.4).

Remark. The equalities (34.4) and (34.5) give rise to some quirky identities. For example, setting $z = \frac{1}{2}$ in (34.5) yields

$$\pi^2 = \sum_{n\in\mathbb{Z}}\frac{1}{(\frac{1}{2} - n)^2} = \sum_{n\in\mathbb{Z}}\frac{4}{(1 - 2n)^2} = \sum_{n=0}^{\infty}\frac{8}{(2n+1)^2},$$

whence $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

35.2 Convergent Products

The next construction we will discuss is the so-called *Weierstrass construction.* Instead of making sure functions have prescribed *poles*, we want to make sure these functions have prescribed *zeros*. This means that we must multiply entire functions instead of add them, and so a discussion on convergence of products is in order.

Theorem 35.2. *Let $\{a_n\}_{n=1}^\infty \subset \mathbb{C}$ be a sequence of complex numbers with $\sum_{n=1}^\infty |a_n| < \infty$. Then the sequence*

$$\alpha_N \coloneqq \prod_{n=1}^N (1+a_n)$$

converges to some $\alpha \in \mathbb{C}$, denoted $\prod_{n=1}^\infty (1+a_n)$. Furthermore, $\alpha = 0$ if and only if $a_n = -1$ for some n.

Proof. We first note that the inequality $1 + x \le e^x$ yields

$$\left|\prod_{n=1}^N (1+a_n)\right| \le \prod_{n=1}^N (1+|a_n|) \le \prod_{n=1}^N e^{|a_n|} = e^{\sum_{n=1}^N |a_n|} < e^{\sum_{n=1}^\infty |a_n|} < \infty \tag{35.1}$$

for all $N \ge 1$; this bound will be useful later.

Now let $M > N \ge 1$ be arbitrary. Write

$$|\alpha_M - \alpha_N| = \left|\prod_{n=1}^M (1+a_n) - \prod_{n=1}^N (1+a_n)\right| = |\alpha_N| \left|\prod_{n=N+1}^M (1+a_n) - 1\right|. \tag{35.2}$$

But further remark that for any sequence of complex numbers $b_1, \dots, b_k$,

$$\left|\prod_{n=1}^k (1+b_n) - 1\right| = \left|\sum_{\varnothing \ne S \subseteq [k]} \prod_{j \in S} b_j\right| \le \sum_{\varnothing \ne S \subseteq [k]} \prod_{j\in S} |b_j| = \prod_{n=1}^k (1+|b_n|) - 1.$$

This means that

$$\left(\prod_{n=N+1}^M (1+|a_n|) - 1\right) \le e^{\sum_{n=N+1}^M a_n} - 1 \le \left(\sum_{n=N+1}^M |a_n|\right) e^{\sum_{n=1}^\infty |a_n|},$$

where the last inequality is due to the bound $|e^w - 1| \le |w|\,|e^w|$, which is a direct consequence of the Mean Value Theorem. Therefore (35.2) can be upper bounded by

$$|\alpha_N| \left(\sum_{n=N+1}^M |a_n|\right) e^{\sum_{n=1}^\infty |a_n|} \overset{(35.1)}{\le} e^{2\sum_{n=1}^\infty |a_n|} \sum_{n=N+1}^M |a_n|,$$

which implies that $\{\alpha_n\}_{n\ge1}$ is Cauchy and thus convergent. This completes the first part of the theorem.

We now move to the second part. If $a_N = -1$ for some N, then the product is clearly zero, since $\alpha_n = 0$ for all $n \ge N$. Now assume $a_n \ne -1$ for any $n \in \mathbb{N}$. This means we may define $\beta_N \coloneqq \prod_{n=1}^N (1+a_n)^{-1}$ for each $N \in \mathbb{N}$.

We claim that the sum $\sum_{n=1}^\infty \frac{|a_n|}{|1+a_n|}$ converges. Indeed, note that absolute convergence implies $a_n \to 0$, so there exists $M \ge 1$ such that $|a_m| \le \frac{1}{2}$ for all $m \ge M$. Therefore $|1 + a_m| \ge \frac{1}{2}$ for all $m \ge M$, which implies

$$\sum_{n=1}^\infty \frac{|a_n|}{|1+a_n|} \le \sum_{n=1}^{M-1} \frac{|a_n|}{|1+a_n|} + \frac{1}{2}\sum_{n=M}^\infty |a_n| < \infty,$$

as requested.

Therefore the sequence

$$\beta_N = \prod_{n=1}^N \left(1 - \frac{a_n}{1+a_n}\right) \quad \text{converges to some } \beta \in \mathbb{C}.$$

But $1 = \alpha_N \beta_N$ for all N, so sending $N \to \infty$ yields $1 = \alpha\beta$. In particular, $\alpha \ne 0$. ☐

We remark that if the a_n terms are actually functions of x, and the convergence of the series $\sum_{n=1}^\infty |a_n(x)|$ is normal/uniform, then the convergence of the product $\prod_{n=1}^\infty (1+a_n(x))$ is also normal/uniform. This yields the following corollary.

Corollary 35.3. *Suppose $\{f_n\}$ is a sequence of analytic functions on some domain D. If the sum $\sum_{n=1}^\infty |f_n(z)-1|$ converges normally on D, then the product $\prod_{n=1}^\infty |f_n(z)|$ converges normally to an analytic function f on D. Moreover, $f(z_0)=0$ if and only if $f_n(z_0)=0$ for some n.*

35.3 Construction of Entire Functions with Prescribed Zeroes

We now lead into the inquiry posed at the beginning of this section. Before diving into the general case, we present an important example in which the naïve construction works.

Example 35.4. Consider the function $f(z)=\sin(\pi z)$; recall that f has simple zeros on $\mathbb{Z}$. With this in mind, we may define

$$g(z) \coloneqq z\prod_{n=1}^\infty \left(1-\frac{z^2}{n^2}\right).$$

Observe that, on any compact set $K\subset\mathbb{C}$, we may choose $C_K>0$ such that $|z|\le C_K$ for all $z\in K$, which then implies

$$\sum_{n=1}^\infty \left|\frac{z^2}{n^2}\right| \le C_K^2\sum_{n=1}^\infty \frac{1}{n^2}<\infty.$$

Thus the product $g(z)$ is entire and has simple zeros on $\mathbb{Z}$.

Since both $g(z)$ and $\sin(\pi z)$ have simple zeros on $\mathbb{Z}$, the quotient $\frac{\sin(\pi z)}{g(z)}$ is entire and nonvanishing. Thus, $\log(\frac{\sin(\pi z)}{g(z)})$ has a branch on $\mathbb{C}$, which means there exists some entire function $h(z)$ with

$$\sin(\pi z)=e^{h(z)}z\prod_{n=1}^\infty\left(1-\frac{z^2}{n^2}\right).$$

What, exactly, is $h(z)$? To resolve this, we exploit (34.4) in a clever way. Write

$$\pi\cot(\pi z)=\frac{[\sin(\pi z)]'}{\sin(\pi z)}=\frac{\left[e^{h(z)}z\prod_{n=1}^\infty(1-\frac{z^2}{n^2})\right]'}{e^{h(z)}z\prod_{n=1}^\infty(1-\frac{z^2}{n^2})}=h(z)+\frac{1}{z}+\frac{\left[\prod_{n=1}^\infty(1-\frac{z^2}{n^2})\right]'}{\prod_{n=1}^\infty(1-\frac{z^2}{n^2})},$$

where in the last step we use the fact that e.g. $\frac{(fg)'}{fg}=\frac{f'}{f}+\frac{g'}{g}$. Now, since the product $\prod_{n=1}^\infty(1-\frac{z^2}{n^2})$ converges normally, we may write

$$\begin{aligned}\frac{\left[\prod_{n=1}^\infty(1-\frac{z^2}{n^2})\right]'}{\prod_{n=1}^\infty(1-\frac{z^2}{n^2})}&=\lim_{N\to\infty}\frac{\left[\prod_{n=1}^N(1-\frac{z^2}{n^2})\right]'}{\prod_{n=1}^N(1-\frac{z^2}{n^2})}\\&=\lim_{N\to\infty}\sum_{n=1}^N\frac{-2z/n^2}{1-z^2/n^2}=\sum_{n=1}^\infty\frac{2z}{z^2-n^2}.\end{aligned}$$

Therefore

$$\pi\cot(\pi z)=h'(z)+\frac{1}{z}+\sum_{n=1}^\infty\frac{2z}{z^2-n^2}\overset{(34.4)}{=}h'(z)+\pi\cot(\pi z),$$

so $h'(z)\equiv 0$ and h is a constant.

Finally, note that taking the equality $\sin(\pi z)=e^h z\prod_{n=1}^\infty(1-\frac{z^2}{n^2})$, dividing both sides by z, and sending $z\to 0$ yields $\pi=e^h$. Thus, we obtain the product representation

$$\sin(\pi z)=\pi\prod_{n=1}^\infty\left(1-\frac{z^2}{n^2}\right),$$

which is Euler's Formula.

It is finally time to unveil the Weierstrass construction.

Theorem 35.5 (Weierstrass). *Let $E = \{z_j\}_{j=1}^\infty \subset \mathbb{C}$ be a discrete set. Assume that for all $z_j \in E$, a multiplicity m_j is given. Then there exists an entire function F with zero set E and multiplicities m_j at z_j. Furthermore, if F and G are two such functions, then there exists some entire h with $F = e^h G$.*

Proof. Write the elements of the set E as $|z_1| \le |z_2| \le \cdots$, where we repeat each z_j exactly m_j times. Let ν be the integer such that $z_1 = \cdots = z_\nu = 0$ but $z_{\nu+1} \neq 0$. We would like the product

$$F(z) = z^\nu \prod_{n=\nu+1}^\infty \left(1 - \frac{z}{z_n}\right)$$

to work; unfortunately, as in the proof of Mittag-Leffler, this product does not necessarily converge. Thus, we need to insert some correction terms, and we will do this via multiplication by an exponential. This way, we will not add any extra zeros to our product.

Instead, let

$$F(z) = z^\nu \prod_{n=\nu+1}^\infty f_n(z), \quad \text{where} \quad f_n(z) = \left(1 - \frac{z}{z_n}\right) \exp\left[\sum_{\ell=1}^{T_n} \frac{1}{\ell}\left(\frac{z}{z_\ell}\right)^n\right],$$

where the series is the (beginning of the) Taylor expansion of $-\operatorname{Log}(1 - \frac{z}{z_n})$. Note that

$$\begin{aligned}\left|\left(1 - \frac{z}{z_n}\right)\exp\left[\sum_{\ell=1}^{T_n}\frac{1}{\ell}\left(\frac{z}{z_n}\right)^\ell\right] - 1\right| &= \left|\exp\left[\operatorname{Log}\left(1 - \frac{z}{z_n}\right) + \sum_{\ell=1}^{T_n}\frac{1}{\ell}\left(\frac{z}{z_n}\right)^\ell\right] - 1\right| \\ &= \left|\exp\left[-\sum_{\ell=T_n+1}^\infty \frac{1}{\ell}\left(\frac{z}{z_n}\right)^\ell\right] - 1\right|.\end{aligned}$$

Now the inequality $|e^w - 1| \le |w|\, e^{|w|}$ implies that, whenever $|z| < \frac{1}{2}|z_n|$ and T_n is sufficiently large, the above expression can be upper bounded by

$$\left|\sum_{\ell=T_n+1}^\infty \frac{1}{\ell}\left(\frac{z}{z_n}\right)^\ell\right| \exp\left(\left|\sum_{\ell=T_n+1}^\infty \frac{1}{\ell}\left(\frac{z}{z_n}\right)^\ell\right|\right) \le \frac{1}{n^2} \cdot e^{\sum_{\ell=1}^\infty \frac{1}{\ell 2^\ell}} = \frac{C}{n^2}$$

for some constant C. Therefore, as in the proof of Mittag-Leffler, $\sum_{n=1}^\infty |f_n(z) - 1|$ converges normally on $\mathbb{C}$, and hence the product converges normally on $\mathbb{C}$ as well. Citing Proposition 35.3 yields the first part of the theorem.

For the second part of the theorem, use again the fact that $\frac{F}{G}$ is entire and nonvanishing, which implies that $\log \frac{F}{G}$ has a branch. $\square$

36 November 18

36.1 Construction of Entire Functions with Prescribed Zeroes (cont.)

We start with an example showing how the correction terms in Weierstrass work; this example will be important later.

Example 36.1. We will construct an entire function with simple zeros on $-\mathbb{N} = \{-1, -2, \ldots\}$.

Our first guess is that $f(z) = \prod_{n=1}^\infty (1 + \frac{z}{n})$ is the desired function; however, this product does not converge normally since $\sum_{n=1}^\infty \frac{1}{n}$ diverges. Instead, we add a correction term and set

$$f(z) := \prod_{n=1}^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

We claim that this is well-defined – that is, the product converges normally to a function with simple zeroes on $-\mathbb{N}$. To prove this, let $f_n(z) := (1 + \frac{z}{n})e^{-z/n}$. In order to estimate $f_n(z) - 1$, consider the function $h(w) := (1 + w)e^w$. Then

$$\begin{aligned}
|f_n(z) - 1| &= \left|h\left(\frac{z}{n}\right) - h(0)\right| = \left|\int_0^{z/n} h'(w)\, dw\right| \leq \int_0^{z/n} |we^w|\, dw \\
&\leq \left|\frac{z}{n}\right| \cdot \max_{[0,z/n]} |w|\, e^{|w|}\, dw = \frac{|z|^2}{n^2} e^{|z|/n} \leq |z|^2 e^{|z|} \cdot \frac{1}{n^2}.
\end{aligned}$$

The function $z \mapsto |z|^2 e^{|z|}$ is bounded on compact sets, and so the sum $\sum_{n=1}^\infty |f_n(z) - 1|$ converges normally. This proves the claim.

It turns out that Mittag-Leffler and Weierstrass are valid on any open set D. We present an example here.

Example 36.2. Let $\{w_j\}_{j=2}^\infty$ be a sequence of complex numbers with norm 1. Find a meromorphic function on $B(0, 1)$ with simple poles on $z_j := (1 - \frac{1}{j})w_j$ having residue 1.

As before, the naïve construction $f(z) := \sum_{j=2}^\infty \frac{1}{z - z_j}$ does not work because the sum does not converge uniformly. Thus, we will need to correct. To do this, write

$$\frac{1}{z - z_j} = \frac{1}{z - w_j + \frac{w_j}{j}} = \frac{1}{z - w_j} \cdot \frac{1}{1 + \frac{w_j}{j(z - w_j)}}.$$

Whenever, say, $\left|\frac{w_j}{j(z-w_j)}\right| \leq \frac{1}{2}$, this may be expanded to yield

$$\frac{1}{z - w_j}\left(1 - \frac{w_j}{j(z - w_j)} + O\left(\frac{1}{j^2(z - w_j)^2}\right)\right).$$

Thus, whenever $j \geq 2\,|z - w_j|^{-1}$, we have the asymptotic

$$f_j(z) := \frac{1}{z - z_j} - \frac{1}{z - w_j} + \frac{w_j}{j^2(z - w_j)^2} = O\left(\frac{1}{j^2(z - w_j)^2}\right).$$

It turns out this is sufficient for the sum $\sum_{j=2}^\infty f_j(z)$ to converge normally on $B(0, 1) \setminus \{z_j\}_{j=2}^\infty$. Indeed, if $K \subset B(0, 1)$ is compact, there exists $\varepsilon > 0$ with $|z| < 1 - \varepsilon$. Thus $|z - w_j| > \varepsilon$, and so for $j \geq \frac{2}{\varepsilon}$ the terms in the tail bound like $\frac{C\varepsilon}{j^2}$ and sum to something finite.

36.2 Gamma Function

Let $G(z) := \prod_{n=1}^\infty (1 + \frac{z}{n})e^{-z/n}$ be the function from Example 36.1. We know already that G is entire with simple poles on $-\mathbb{N}$. We will spend the rest of the day analyzing the properties of G and some related functions.

First recall that

$$\sin(\pi z) = \pi z \prod_{n=1}^\infty \left(1 - \frac{z^2}{n^2}\right) = \pi z \prod_{n=1}^\infty \left(1 + \frac{z}{n}\right) e^{-z/n} \left(1 - \frac{z}{n}\right) e^{z/n} = \pi z G(z) G(-z). \tag{36.1}$$

This will be useful later.

Now remark that $G(z-1)$ has simple zeroes on $-\mathbb{N}\cup\{0\}$. However, we know $zG(z)$ does as well, so there exists some entire function $\gamma(z)$ with $G(z-1) = zG(z)e^{\gamma(z)}$.

To compute γ, first remark that $G(0)=1$, so $G(1)e^{\gamma(1)} = G(0) = 1$ as well. Hence

$$\begin{aligned}\gamma(1) &= \ln\left[\frac{1}{G(1)}\right] = \ln\left[\prod_{n=1}^{\infty}\frac{1}{1+\frac{1}{n}}e^{1/n}\right] = \lim_{N\to\infty}\ln\left[\prod_{n=1}^{N}\frac{n}{n+1}e^{1/n}\right] \\ &= \lim_{N\to\infty}\ln\left[\exp\left(\sum_{n=1}^{N}\frac{1}{n}\right)\cdot\frac{1}{n+1}\right] = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n} - \ln(N+1).\end{aligned}$$

This is a constant γ, known in the literature as the *Euler-Mascheroni constant.*

Lemma 36.3. *For all z, $\gamma(z)\equiv\gamma(1)=\gamma$. That is, $\gamma(z)$ is a constant.*

Proof. Fix $z\notin\mathbb{Z}$. Observe that

$$\begin{aligned}\frac{G'(z)}{G(z)} &= \frac{\left[\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-z/n}\right]'}{\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-z/n}} = \sum_{n=1}^{\infty}\frac{\left[(1+\frac{z}{n})e^{-z/n}\right]'}{(1+\frac{z}{n})e^{-z/n}} \\ &= \sum_{n=1}^{\infty}\left(\frac{\frac{1}{n}e^{-z/n}}{(1+\frac{z}{n})e^{-z/n}} - \frac{\frac{1}{n}(1+\frac{z}{n})e^{-z/n}}{(1+\frac{z}{n})e^{-z/n}}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right).\end{aligned}$$

Thus $\frac{G'(z-1)}{G(z-1)} = \sum_{n=1}^{\infty}(\frac{1}{z-1+n}-\frac{1}{n})$. But on the other hand, we may use the recursion $G(z-1) = zG(z)e^{\gamma(z)}$ to yield

$$\frac{G'(z-1)}{G(z-1)} = \frac{(zG(z)e^{\gamma(z)})'}{zG(z)e^{\gamma(z)}} = \frac{1}{z}+\gamma'(z)+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right).$$

Setting these equal to each other and combining terms yields $\gamma'(z)=0$, so $\gamma(z)$ is a constant. ☐

This means that $G(z-1) = zG(z)e^{\gamma}$ for all z, which leads into an important definition.

Definition 36.4. The *Gamma function* $\Gamma:\mathbb{C}\setminus\{0,-1,\ldots\}\to\mathbb{C}$ is defined via

$$\Gamma(z) := \frac{1}{zG(z)e^{\gamma z}}.$$

This definition is somewhat unnatural, but it has the benefit of being entirely self-contained. It also allows us to deduce some properties of Γ quite easily.

- $\Gamma(z)$ has simple poles on the set $-\mathbb{N}\cup\{0\}$.
- $\Gamma(z)$ has no zeros, since $zG(z)e^{\gamma(z)}$ is never equal to infinity.
- Via the recursion for G, we can deduce

$$\Gamma(z+1) = \frac{1}{(z+1)G(z+1)e^{\gamma z}e^{\gamma}} = \frac{1}{G(z)e^{\gamma(z)}} = z\Gamma(z).$$

  Combining this with $\Gamma(1)=1$ yields the relation $\Gamma(n+1)=n!$ whenever n is a positive integer. Thus, in some sense Γ is an analytic continuation of the factorial function to $\mathbb{C}\setminus\{0,-1,\ldots\}$.
- By turning (36.1) into a relationship concerning Γ, we get

$$\sin(\pi z) = \frac{-\pi}{z\Gamma(z)\Gamma(-z)} = \frac{\pi}{\Gamma(z)\Gamma(1-z)}. \tag{36.2}$$

To finish today, we prove an alternate characterization of Γ in certain situations.

Proposition 36.5. *For every $z\in\mathbb{C}$ with $\Re(z)>0$, we have*

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt.$$

Proof. First note that

$$\int_0^\infty t^{z-1}e^{-t}\,dt = \lim_{n\to\infty}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n\,dt.$$

One way to see this is to note that $\chi_{[0,n]}(t)(1-\frac{t}{n})^n \le e^{-t}$ for all n, at which point we may appeal to the Dominated Convergence Theorem. (For an alternate explanation not relying on measure theory, see Palka.)

Now focus on the inner integral. By making the substitution $t = ns$, we see that

$$\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n\,dt = \int_0^1 n^z s^{z-1}(1-s)^n\,ds,$$

which, after n applications of Integration by Parts, equals

$$\frac{n^z\cdot n!}{z(z+1)\dots(z+n-1)}\int_0^1 s^{z+n-1}\,ds = \frac{n^z\cdot n!}{z(z+1)\dots(z+n)}.$$

Now we unpack the product and write

$$\begin{aligned}\frac{n^z\cdot n!}{z(z+1)\dots(z+n)} &= \frac{n^z}{z}\prod_{k=1}^n\frac{k}{z+k} = e^{z\ln n}z\prod_{k=1}^n\frac{1}{1+\frac{z}{k}} \\ &= \frac{e^{z\ln n}}{z}\left(\prod_{k=1}^n\frac{1}{(z+\frac{k}{n})e^{-z/k}}\right)\left(\prod_{k=1}^n e^{-z/k}\right) \\ &= \prod_{k=1}^n\frac{1}{(z+\frac{k}{n})e^{-z/k}}\cdot\exp\left(z\left(\ln n-\sum_{k=1}^n\frac{1}{k}\right)\right).\end{aligned}$$

Finally, we see the fruits of our labor: as n tends to infinity, the product converges to $\frac{1}{zG(z)}$, whle the sum inside the exponential converges to $-\gamma z$. Thus,

$$\lim_{n\to\infty}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n\,dt = \frac{1}{zG(z)e^{\gamma z}} = \Gamma(z),$$

and we are done. $\square$

37 November 20

Did not attend lecture today for personal reasons.

37.1 Riemann Zeta Function

Today will revolve around the following function.

Definition 37.1. For $\Re(z) > 1$, the *Riemann zeta function* is defined as

$$\zeta(z) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Observe that the $\Re(z) > 1$ condition is necessary since

$$|n^z| = \left|e^{z \ln n}\right| = e^{\Re(z \ln n)} = n^{\Re(z)};$$

in turn, the series is summable precisely on the set $\{z : \Re(z) > 1\}$.

The zeta function has surprising connections to number theory, in part because it has a nice representation in terms of primes.

Proposition 37.2. *For every complex number z with $\Re(z) > 1$,*

$$\zeta(z) = \prod_{p \text{ prime}} (1 - p^{-z})^{-1}.$$

Proof. Fix a prime P. We claim that

$$\begin{aligned}\prod_{\substack{p \text{ prime} \\ p \leq P}} (1 - p^{-z})^{-1} &= \prod_{\substack{p \text{ prime} \\ p \leq P}} (1 + p^{-z} + p^{-2z} + \cdots) \\ &= \sum_{n=1}^{P} \frac{1}{n^z} + O\left(\sum_{n>P} \left|\frac{1}{n^z}\right|\right).\end{aligned}$$

The first line is straightforward. To see the second line, observe that, by unique prime factorization, each term $\frac{1}{n^z}$ in the expansion of the first line appears at most once. Furthermore, since the product runs over all primes at most P, every integer from 1 to P appears in this expansion. Combining both observations proves the claim, and the proposition follows by sending P to ∞. □

This factorization highlights two facts about the primes and the zeta function.

- For any z with $\Re(z) > 1$, $\zeta(z) \neq 0$. This is because none of the terms $(1 - p^{-z})^{-1}$ in the product vanish.
- Conversely, plugging in $z = 1$ and inverting both sides yields $\prod_{p \text{ prime}} (1 - \frac{1}{p}) = 0$. Notice that we have a product of nonzero numbers that equals zero; it follows that $\sum_{p \text{ prime}} \frac{1}{p}$ diverges.

37.2 Analytic Continuation of Zeta

As the title suggests, we now attempt to extend ζ to (almost) all of $\mathbb{C}$. Recall first that, through the substitution $t \mapsto nt$

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt = n^z \int_0^\infty e^{-nt} t^{z-1}\, dt.$$

Dividing both sides by n^z yields $n^{-z} = \int_0^\infty e^{-nt} t^{z-1}$. Summing over all n and appealing to the Dominated Convergence Theorem then gives

$$\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{z-1}\, dt = \int_0^\infty \frac{1}{e^t - 1} t^{z-1}\, dt. \tag{37.1}$$

We will obtain an analytic continuation of $\zeta(z)$ by studying this integral.

Remark that, since $\frac{1}{e^z-1}$ has a simple pole at $z=0$ with residue 1, $\frac{1}{e^t-1}-\frac{1}{t}$ is bounded near zero. This means we may write

$$\zeta(z)\Gamma(z)=\int_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}\right)t^{z-1}\,dt+\int_0^1\frac{1}{t}t^{z-1}\,dt+\int_1^\infty\frac{t^{z-1}}{e^t-1}\,dt=:I+II+III.$$

Note that I is analytic on $\Re(z)>0$ since the integral converges normally there (this is due to the previous remark). Furthermore, II equals $\frac{1}{z-1}$ via a simple computation. Finally, note that the integral III is entire since the integral converges normally on $\mathbb{C}$. Thus, since $\frac{1}{\Gamma(z)}$ is entire, the expansion

$$\zeta(z)\Gamma(z)=\int_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}\right)t^{z-1}\,dt+\frac{1}{z-1}+\int_1^\infty\frac{t^{z-1}}{e^t-1}\,dt$$

yields an analytic continuation of $\zeta(z)$ to the set $\{z:\Re(z)>0,z\neq 1\}$; the equality $\Gamma(1)=1$ implies that $\zeta(z)$ has a simple pole at 1 with residue 1. Furthermore, note that whenever $0<\Re(z)<1$, $\frac{1}{z-1}=-\int_1^\infty t^{z-2}\,dt$. Hence

$$\begin{aligned}\zeta(z)\Gamma(z)&=\int_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}\right)t^{z-1}\,dt-\int_1^\infty\frac{1}{t}t^{z-1}\,dt+\int_1^\infty\frac{t^{z-1}}{e^t-1}\,dt\\&=\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}\right)t^{z-1}\,dt\end{aligned}$$

whenever $0<\Re(z)<1$.

We can repeat this argument by pulling out one more term of the Laurent series of $\frac{1}{e^t-1}$ about $t=0$. Indeed, remark that

$$\frac{1}{e^t-1}=\frac{1}{t}-\frac{1}{2}+O(t)\quad\text{as } t\to 0,\tag{37.2}$$

which means we may perform similar computations and write

$$\begin{aligned}\zeta(z)\Gamma(z)&=\int_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)t^{z-1}\,dt-\frac{1}{2}\int_0^1 t^{z-1}\,dt+\int_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}\right)t^{z-1}\,dt\\&=\int_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)t^{z-1}\,dt-\frac{1}{2z}+\int_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}\right)t^{z-1}\,dt.\end{aligned}$$

This gives an analytic continuation of $\zeta(z)$ to $-1<\Re(z)<1$; in particular, the $O(t)$ term in (37.2) allows us to step down to $-1<\Re(z)$ in the first integral. (Notice that the singularity at $z=0$ is removable since Γ has a simple pole at $z=0$.) Finally, on $-1<\Re(z)<0$, we may write $-\frac{1}{2z}=\frac{1}{2}\int_1^\infty t^{z-1}\,dt$; substituting this into the above equality yields

$$\zeta(z)\Gamma(z)=\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)t^{z-1}\,dt\quad\text{whenever }\ -1<\Re(z)<0.\tag{37.3}$$

One might suspect that we can repeat this ad infinitum to get analytic continuations of ζ to almost all of $\mathbb{C}$, but it turns out we can do better. Note the surprising computation

$$\frac{1}{e^t-1}+\frac{1}{2}=\frac{1}{2}\cdot\frac{e^t+1}{e^t-1}=\frac{i}{2}\cot\left(\frac{it}{2}\right)=\frac{1}{t}+2t\sum_{n=1}^\infty\frac{1}{t^2+4\pi^2n^2},$$

where the last line follows from (34.4) and some algebraic manipulation. Thus, whenever $-1<\Re(z)<0$, (37.3) yields[12]

$$\begin{aligned}\zeta(z)\Gamma(z)&=2\int_0^\infty\sum_{n=1}^\infty\frac{t^z}{t^2+4\pi^2n^2}\,dt=2\sum_{n=1}^\infty\int_0^\infty\frac{t^z}{t^2+4\pi^2n^2}\,dt\\&=2\sum_{n=1}^\infty(2\pi n)^{z-1}\int_0^\infty\frac{t^z}{t^2+1}\,dt=2(2\pi)^{z-1}\zeta(1-z)\int_0^\infty\frac{t^z}{t^2+1}\,dt.\end{aligned}$$

[12]The observant analyst may question why we may swap the order of the integral and sum here. This involves some Dominated Convergence similar to that of (37.1); in fact, the steps that follow show one way to do the bounding.

Now recall that way back in Section 21.1.1 we showed that

$$\int_0^\infty \frac{t^x}{t^2+1}\,dt = \frac{\pi}{2\cos(\frac{\pi}{2}x)} \quad \text{whenever } -1 < x < 0.$$

This equality extends to $-1 < \Re(z) < 0$ since both sides are analytic on that domain. Therefore the previous equality rewrites as

$$\zeta(z)\Gamma(z) = (2\pi)^{z-1}\zeta(1-z)\frac{\pi}{\cos(\frac{\pi}{2}z)}.$$

Finally, recall from (36.2) that

$$\Gamma(z) = \frac{\pi}{\Gamma(1-z)\sin(\pi z)} = \frac{\pi}{\Gamma(1-z)\cdot 2\sin(\frac{\pi}{2}z)\cos(\frac{\pi}{2}z)}.$$

Thus, substituting and clearing denominators *finally* yields the equality

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\tfrac{\pi}{2}z)$$

whenever $-1 < \Re(z) < 0$, which is known as *Riemann's functional equation.*

In particular, note that the right hand side is analytic on the set $\{z : \Re(z) < 0\}$, meaning that it extends $\zeta(z)$ analytically to $\mathbb{C} \setminus \{1\}$, with a simple pole of residue 1 at $z = 1$.

37.3 Zeros of the Zeta Function

We end today with a quick discussion on the zeros of $\zeta(z)$.

Since $\Gamma(1-z)$ has simple poles at $z = 1, 2, 3, \ldots$, $\zeta(1-z)\sin(\frac{\pi}{2}z)$ has simple zeros at $z = 2, 3, \ldots$, stemming from the fact that $\zeta(z)$ is analytic and nonzero on $\{z : \Re(z) > 1\}$. But now observe that $\sin(\frac{\pi}{2}z) = 0$ whenever z is an integer integer, so the zeros of $\zeta(1-z)$ must be the zeros that remain, i.e. $z = 3, 5, 7, \ldots$. Thus $\zeta(z)$ has simple zeros at $z = -2, -4, -6, \ldots$. The function $\zeta(z)$ has no other zeros outside $\{z : 0 \le \Re(z) \le 1\}$.

It is natural to ask, then, what the zeros of $\zeta(z)$ are inside this strip. The answer is, surprisingly (or unsurprisingly!) unknown. The *Riemann Hypothesis* asserts that the zeros of $\zeta(z)$ inside this strip all lie on the line $\{z : \Re(z) = \frac{1}{2}\}$. We know that there are infinitely many zeros lying on this line, and that there are no zeros on the lines $\{z : \Re(z) = 0\}$ and $\{z : \Re(z) = 1\}$. But not much is known otherwise, which has led to one of the biggest unsolved problems in mathematics.

38 November 22

38.1 Approximation by Polynomials

Let $K \subset \mathbb{C}$ be a compact set. Recall that we can rig the space of continuous functions $C(K)$ on K with the metric $d(f,g) := \sup_{z\in K} |f(z) - g(z)|$; this allows us to possibly approximate functions in $C(K)$ with functions from some "nicer" class.

For example, let $P(K)$ be the closure of the subspace of polynomials in $C(K)$. Is $P(K) = C(K)$? Unfortunately, if K° is nonempty, the answer is no: we know via the Weierstrass Approximation Theorem (Theorem 10.6) that the uniform limit of analytic functions on the open set K° must also be analytic.

We may then ask for something slightly weaker: is

$$P(K) = \{f \in C(K) : f \text{ analytic on } K^\circ\}? \tag{38.1}$$

Unfortunately, this is not always true either.

Example 38.1. Take $K = \partial B(0,1)$. Recal that for any polynomial $p(z)$, $\int_K p(z)\,dz = 0$. Hence, if $\{p_n\}_{n=1}^\infty$ is a sequence of polynomials converging to some function f, we must have $\int_K f(z)\,dz = 0$ as well. This means that the function $f(z) = \frac{1}{z}$ – which has integral $2\pi i$ – cannot be approximated uniformly by polynomials.

It turns out that (38.1) *is* true whenever $\mathbb{C} \setminus K$ is connected; this was proven by Mergelyan in 1951. However, the proof is beyond the scope of this course. We instead opt to prove something weaker.

Theorem 38.2 (Runge)**.** *Let K be a compact set with $C \setminus K$ connected. If f is analytic on an open set containing K, then there exists a sequence $\{p_n\}_{n=1}^\infty$ of polynomials converging to f uniformly on K.*

Proof. The proof proceeds in steps.

Step 1: $P(K)$ is an algebra. That is, if f and g are elements of $P(K)$, then so are $f+g$, fg, and αf, where $\alpha \in \mathbb{C}$. The first and last results are easy. For the second, fix sequences of polynomials $p_n \to f$ and $q_n \to g$. Observe that, by uniformity, $d(p_n, f) \le 1$ for all n sufficiently large, implying $|p_n| \le |f| + 1$ for all n sufficiently large. Now use the Triangle Inequality to write

$$|fg - p_nq_n| \le |f - p_n|\,|g| + |g - q_n|\,|p_n| \le |f - p_n|\,|g| + |g - q_n|\,(|f| + 1),$$

which tends to zero uniformly as $n \to \infty$.

Step 2: Special Rational Functions are in $P(K)$. Define the set

$$S := \{a \notin K : \tfrac{1}{z-a} \in P(K)\}.$$

We claim that $S = \mathbb{C} \setminus K$; that is, every function of the form $\frac{1}{z-a}$ for $a \notin K$ can be approximated by polynomials. This part itself can be divided into a few substeps.

Step 2.1. We first show that if $|a| > M := \sup_{z\in K} |z|$, then $a \in S$. Indeed, note that

$$\frac{1}{z-a} = \frac{1}{-a} \cdot \frac{1}{1 - \frac{z}{a}} = -\frac{1}{a} \sum_{n\ge 0} \left(\frac{z}{a}\right)^n.$$

The sequence of partial sums is thus a sequence of polynomials which converges uniformly on K.

Step 2.2. We now claim that if $a \in S$ and if

$$|b - a| < d(a, K), \tag{38.2}$$

then $b \in S$. We start analogously to the previous step by writing

$$\frac{1}{z-b} = \frac{1}{(z-a) - (b-a)} = \frac{1}{z-a} \cdot \frac{1}{1 - \frac{b-a}{z-a}} = \frac{1}{z-a} \sum_{n\ge 0} \left(\frac{b-a}{z-a}\right)^n.$$

The sequence of partial sums again converges uniformly on K. Thus, given $\varepsilon > 0$, there exists some N such that

$$\left| \frac{1}{z-a} - \underbrace{\sum_{n=0}^{N} \frac{(b-a)^n}{(z-a)^{n+1}}}_{(*)} \right| < \varepsilon.$$

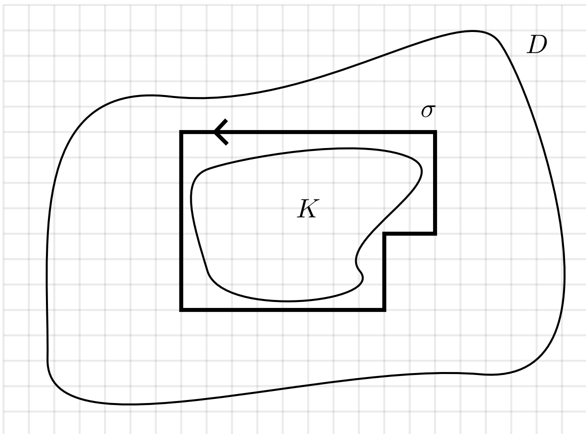
But recall that $P(K)$ is an algebra, so by Step 2.1, $\frac{1}{(z-a)^{n+1}}$ is in $P(K)$ for each n. Thus $(*)$ is also in $P(K)$, and the result follows by the Triangle Inequality and sending ε to zero.

Step 2.3. The previous step implies that S is open. Furthermore, the "uniform" bound (38.2) yields that $(\mathbb{C} \setminus K) \setminus S$ is open too; it tells us that, if b is *not* in $\mathbb{C} \setminus K$, then $B(b, \frac{d(b,K)}{2})$ is *not* a subset of $\mathbb{C} \setminus K$. Thus, since $\mathbb{C} \setminus K$ is connected and S is nonempty, $(\mathbb{C} \setminus K) \setminus K = \varnothing$. This completes Step 2.

Step 3: More Rational Functions are in $P(K)$. We now claim that all rational functions with poles outside K are in $P(K)$. But this is easy: write any such function as the product of rational functions from Step 2 and use the fact that $P(K)$ is an algebra.

Step 4: Approximation by Rational Functions. We now show that any function f which is analytic on an open set containing K may be approximated by rational functions with poles outside K. This will complete the proof.

Let D be the open set on which f is analytic. We first claim that there exists a cycle $\sigma \in D \setminus K$ such that σ is 0-homologous on D and $n(\sigma, z) = 1$ for any $z \in K$. To prove this, set $0 < \delta < \frac{1}{1000} d(K, \partial D)$, and tile the plane with squares of side length δ. Now consider the squares which intersect K, and define σ to be the counterclockwise boundary of these squares after canceling common edges. This works.



Now Cauchy's Integral Formula yields

$$f(z) = \frac{1}{2\pi i} \int_{\sigma} \frac{f(w)}{z-w}\, dw. \tag{38.3}$$

As a function of w, $\frac{f(w)}{z-w}$ is singular outside K. It suffices to prove that, for any smooth γ with $|\gamma| \in D \setminus K$, $\int_\gamma \frac{f(w)}{z-w}\, dw$ can be approximated by rational functions with poles outside D. To prove this, we need a lemma concerning Riemann integration.

Lemma 38.3. *Suppose $g : \mathbb{R} \to \mathbb{R}$ satisfies $|g'(t)| \le C$ on $[a, b]$. Then for all $\varepsilon > 0$, there exists $\delta = \delta(C, \varepsilon) > 0$ so that, for any partition $a = t_1 < \cdots < t_n = b$ of $[a, b]$ with mesh less than δ,*

$$\left| \int_a^b g(t)\, dt - \sum_{k=1}^{n} g(t_k)(t_k - t_{k-1}) \right| < \varepsilon.$$

Since γ can be represented as a function from $[a, b] \to D$, (38.3) rewrites as

$$f(z) = \int_a^b \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}\, dt.$$

Let $g(t)$ be the integrand. Then $g'(t)$ is bounded by a constant C depending only on $d(|\gamma|, K)$, γ, and f; in particular, it does not depend on z. Thus the lemma guarantees a partition $a = t_1 < \cdots < t_n = b$ satisfying

$$\left| \int_a^b \frac{f(\gamma(t))\gamma'(t)}{\gamma(t) - z}\, dt - \sum_{k=1}^n \frac{f(\gamma(t_k))\gamma'(t_k)}{\gamma(t_k) - z}(t_k - t_{k-1}) \right| < \varepsilon.$$

Thus we have approximated the integral by some rational function with poles outside K, which completes Step 4 and hence the proof. □

To finish today, we present a nontrivial application of Runge.

Example 38.4. We will show there exists a sequence of polynomials $\{p_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} p_n(z) = \begin{cases} 1 & \text{if } \Re(z) > 0, \\ 0 & \text{if } \Re(z) \le 0. \end{cases}$$

To do this, let

$$A_n := \{x + yi : -n \le x \le \tfrac{1}{n}, |y| \le n\} \quad \text{and} \quad B_n := \{x + yi : \tfrac{2}{n} \le x \le n, |y| \le n\},$$

and set $K_n := A_n \cup B_n$. Then K_n is a compact set with $\mathbb{C} \setminus K_n$ connected.

Since A_n and B_n are disjoint closed sets, for each n there exists some analytic function f_n which equals 1 on a $\frac{1}{100n}$-neighborhood of B_n and which equals 0 on a $\frac{1}{100n}$-neighborhood of A_n. Then f_n is analytic on a $\frac{1}{100n}$-neighborhood of K_n, so there exists some polynomial p_n with $|f_n - p_n| < \frac{1}{n}$ on K_n. Now send n to ∞.

39 December 2

39.1 Harmonic Functions: the Basics

The remainder of the course will be devoted to so-called *harmonic functions*, which are, in some sense, a real analogue of analytic functions.

Definition 39.1. Fix an open set D, and let $h : D \to \mathbb{C}$ be a function with continuous second order partial derivatives.[13]

1. The *Laplacian* of h is defined via
$$\Delta h := h_{xx} + h_{yy}. \tag{39.1}$$
2. We say h is *harmonic* on D if $\Delta h = 0$ on D.

Example 39.2. The function $h(x, y) = \log(x^2 + y^2)$ is harmonic on $\mathbb{C}$. This is because

$$h_x = \frac{2x}{x^2+y^2} \quad \text{and hence} \quad h_{xx} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}.$$

Similarly $h_{yy} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$, and so $h_{xx} + h_{yy} = 0$.

Example 39.3. The function $h(x, y) = x$ is harmonic since both second partial derivatives are zero. However, $h(x, y)^2 = x^2$ is *not* harmonic, since $\Delta h = 2$. This tells us that the product of two harmonic functions is not necessarily harmonic, which is different from what occurs in the analytic case.

The connection between analytic and harmonic functions is displayed in the next theorem.

Theorem 39.4. *Let $D \subset \mathbb{C}$ be a domain.*

1. *If f is analytic on D, then $\Re(f)$ and $\Im(f)$ are harmonic on D.*
2. *If h is harmonic on D, and if D is simply connected, then there exists some analytic function f on D satisfying $h = \Re(f)$. Moreover, f is unique up to an imaginary constant.*

Proof. We begin with the proof of the first part. First suppose $f = u + iv$ for some real functions u and v. Since f is analytic, both u and v are C^∞, which means that e.g. $(v_x)_y = (v_y)_x$. Therefore

$$u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy},$$

and hence $\Delta u = 0$. Analogously we deduce $\Delta v = 0$, which completes the proof of the first part.

Now we proceed with the second part. First we show uniqueness. Let $h = \Re(f)$ for some analytic function f. This means $f = h+ik$ for some (harmonic) function k, so the Cauchy-Riemann equations tell us that
$$f' = h_x + ik_x = h_x - ih_y.$$
This means that f' is uniquely determined by h, so f is as well, up to some constant. This constant must be imaginary.

Conversely, fix h harmonic, and define $g := h_x - ih_y$. Then g has continuous partial derivatives, and furthermore
$$(\Re(g))_x = h_{xx} = -h_{yy} = (\Im(g))_y.$$
Analogously, $(\Re(g))_y = -(\Im(g))_x$, and so g is analytic. Thus, since g is simply connected, Proposition 17.6 tells us g has a primitive on D, say $F = u + iv$.

Finally, observe that

$$h_x - ih_y = g = F' = u_x - iu_y, \quad \text{so} \quad \nabla(u - h) = 0.$$

This means that $u = h + c$ for some $c \in \mathbb{R}$, whence $h = \Re(F - c)$. □

The next example shows that the assumption that D is simply connected is necessary.

[13] In what follows, this assumption is not necessary, but it turns out that there is no harm in assuming this beforehand.

Example 39.5. Let $f(z) = \log|z| = \frac{1}{2}\log(|z|^2)$. Example 39.2 implies that f is a harmonic function on $\mathbb{C} \setminus \{0\}$. However, $\log|z| = \Re(\operatorname{Log} z)$, and $\operatorname{Log} z$ is the only such function by uniqueness (which does not require that D be simply connected!); this is a contradiction because Log is not harmonic on $\mathbb{C} \setminus \{0\}$.

Example 39.3 showed that the product of two harmonic functions need not be harmonic. The next proposition (whose proof we omit as an exercise) establishes necessary and sufficient conditions for this to hold.

Proposition 39.6. *Suppose h and k are harmonic and nonconstant on a simply connected domain D. Then hk is harmonic on D if and only if $f := h + ick$ is analytic for some $c \in \mathbb{R}$.*

Finally, before continuing on to some more involved properties of harmonic functions, we record a few corollaries of Theorem 39.4.

Corollary 39.7. *Let D_1, and D_2 be open sets, and suppose $h : D_2 \to \mathbb{R}$ is harmonic.*

1. *The function h is C^∞ on D_2.*
2. *If $f : D_1 \to D_2$ is analytic, then the function $h \circ f : D_1 \to \mathbb{R}$ is harmonic.*

Proof. Since being analytic or harmonic is a strictly local property, it suffices to show the result when D is an open ball.

For the first part, since D is simply connected, there exists some analytic function f with $h = \Re(f)$. But f is C^∞, and hence by definition h is also C^∞.

For the second part, once again write $h = \Re(g)$ for some analytic function g. Then $g \circ f$ is analytic and $h \circ f = \Re(g \circ f)$, so $h \circ f$ is harmonic. □

39.2 Extensions of Analytic Results to Harmonic Functions

Since harmonic functions are so closely related to analytic functions, we might expect that many of the same results port over. It turns out this is mostly the case.

First recall that, if f and g are analytic functions on D with $f = g$ on some nondiscrete set $E \subset D$, then $f = g$ on D. Unfortunately, this same result does not port over to harmonic functions.

Example 39.8. Let $h(x, y) = e^x \sin y = \Im(e^{x+yi})$. Then h is harmonic on $\mathbb{C}$. However, h is zero on the real axis yet is not identically zero.

Despite this, a slightly weaker statement is fortunately true.

Corollary 39.9. *Let h and k be harmonic functions on some simply connected domain D. Suppose that $h = k$ on some open ball in D. Then $h = k$ in D.*

Proof. Note that $h = k$ if and only if $h - k = 0$, so we may assume without loss that $k = 0$. This means that $h = 0$ on some open ball B contained in D. Since D is simply connected, there exists some analytic function g on D with $h = \Re(g)$. Hence

$$g'(z) = h_x(z) - ih_y(z) = 0 - i \cdot 0 = 0 \quad \text{for all } z \in B,$$

so $g'(z) = 0$ in D. This implies that g is equal to some constant on D, and this constant must be purely imaginary. □

Our next result extends the Cauchy Integral Formula to harmonic functions.

Theorem 39.10 (Mean Value Property). *If h is harmonic on an open set D and $\overline{B(z_0, r)} \subset D$, then*

$$h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})\, d\theta. \tag{39.2}$$

Proof. Since $\overline{B(z_0, r)} \subset D$, there exists $r_0 > r$ with $B(z_0, r_0) \subset D$; in turn, we may write $h = \Re(f)$ for some f analytic on $B(z_0, r_0)$. Now Cauchy's Integral Formula applied to f yields

$$\begin{aligned} f(z_0) = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(w)}{w - z_0}\, dw &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \cdot rie^{i\theta}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \end{aligned}$$

Taking the real part of both sides yields (39.2). ☐

Our third result shows that the Maximum Modulus Principle ports over to harmonic functions as well. In fact, this result is *stronger* than the one for analytic functions, since we no longer have to worry about taking a modulus.

Theorem 39.11 (Maximum Modulus for Harmonic Functions)**.** *Let h be harmonic on a domain D.*

1. *If h has a local maximum or local minimum in D, then h is constant on D.*
2. *Suppose further that D is bounded and that h is continuous on $\overline{D}$. Then h obtains its maximum and minimum on ∂D.*

Proof. The proof behaves exactly the same as the proof of the Maximum Modulus Principle (Theorem 9.3); in particular, using the language of the remark which follows it, h and $-h$ are both (sub)harmonic. ☐

The Maximum Modulus Principle allows us to strengthen the Mean Value Property somewhat.

Corollary 39.12. *Suppose h and k are harmonic on a bounded domain D and continuous on $\overline{D}$. If $h = k$ on ∂D, then $h = k$ on all of D.*

Proof. The function $f := h - k$ is identically zero on ∂D, and so by Theorem 39.11, it must be identically zero on D as well. ☐

Finally, we port over Liouville to harmonic functions. Once again, we may strengthen the corresponding result from analytic functions a bit.

Theorem 39.13 (Liouville for Harmonic Functions)**.** *Let h be harmonic on $\mathbb{C}$ and either bounded above or bounded below. Then h is constant.*

In particular, we allow h to be bounded above but not bounded below, for instance.

Proof. Suppose h is bounded above by some constant M; the proof for bounded below is similar. Since $\mathbb{C}$ is simply connected and h is harmonic, there exists some entire function k with $g := h + ik$ entire. Now set $f := e^g$. Then

$$|f| = \left|e^{h+ik}\right| = \left|e^h e^{ik}\right| = e^h \le e^M.$$

Thus, Liouville tells us that f is constant, so h is constant as well. ☐

40 December 4

40.1 The Dirichlet Problem on $B(0,1)$

Today, we will focus on solving the following problem: given $f : \partial B(0,1) \to \mathbb{R}$ continuous, does there exist $h : \overline{B(0,1)} \to \mathbb{R}$ that is harmonic on $B(0,1)$ and continuous on $\overline{B(0,1)}$ which satisfies $h|_{B(0,1)} = f$? It turns out h exists and is unique, and this extension – called the *harmonic extension* of f – is unique. In fact, by combining this result with the Riemann Mapping and Caratheodory-Osgood theorems, we deduce the result whenever $B(0,1)$ is replaced by a domain whose boundary is a simple closed contour.

Fortunately for us, uniqueness is easy and follows by Corollary 39.12. Existence is where things get interesting. Before continuing, note that, in polar coordinates, the conditions rewrite as

$$\Delta h = h_{rr} + \tfrac{1}{r}h_r + \tfrac{1}{r^2}h_{\theta\theta} = 0 \quad \text{and} \quad h(1,\theta) = e^{i\theta}.$$

We will look for "separable" solutions of the form $h(r,\theta) = \varphi(r)\psi(\theta)$. Plugging this in to the first equation yields

$$\varphi''(r)\psi(\theta) + \tfrac{1}{r}\varphi'(r)\psi(\theta) + \tfrac{1}{r^2}\varphi(r)\psi''(\theta) = 0,$$

which implies

$$\frac{r^2\varphi''(r) + r\varphi'(r)}{\varphi(r)} = -\frac{\psi''(\theta)}{\psi(\theta)}.$$

Suppose both equalities are equal to a constant λ; then

$$r^2\varphi''(r) + r\varphi'(r) - \lambda\varphi(r) = \psi''(\theta) + \lambda\psi(\theta) = 0.$$

To analyze the θ equation, we split into three cases.

- First suppose $\lambda = 0$, so that $\psi(\theta) = a\theta + b$. The condition $h(1,\theta) = e^{i\theta}$ implies that ψ must be 2π-periodic, and so $a = 0$. This means that $\psi(\theta) = 1$ up to multiplication by a constant.
- Now suppose $\lambda = m^2 > 0$. Then the solutions to the equation are $\psi(\theta) = e^{\pm im\theta}$. The condition that ψ is 2π-periodic implies $m \in \mathbb{N}$.
- Finally, suppose $\lambda = -m^2 < 0$. Then $\psi(\theta) = ae^{m\theta} + be^{-m\theta}$. There are no 2π-periodic solutions other than the trivial solution.

Similarly, to analyze the r equation, we split into two cases.

- First suppose $\lambda = 0$. Then $r^2\varphi''(r) + r\varphi'(r) = 0$. Observe that $\varphi = 1$ (and in particular any constant function) is a solution. Otherwise, φ' is nonconstant, so we may divide to yield

  $$-\frac{1}{r} = \frac{\varphi''(r)}{\varphi'(r)} = (\log\varphi'(r))',$$

  which has solutions $\varphi(r) = A\log r + B$. No such solution is periodic, and so the only possible solution is $r = 1$.
- Now suppose $r = m^2 > 0$. (Note that r is real, so this is the only other case we need to consider.) We may consider solutions of the form $\varphi(r) = r^\alpha$; plugging this in and dividing through by r^α yields $\alpha^2 = m^2$, so $\alpha = \pm m$. However, r^{-m} is not bounded as $r \to 0$, so the only possible bounded solution is $\varphi(r) = r^m$.

All in all,

$$h(r,\theta) \in \left\{r^m e^{im\theta}, r^m e^{-im\theta}, 1 : m \in \mathbb{N}\right\}.$$

This means that we may consider solutions for h of the form

$$h(r,\theta) = \sum_{m=-\infty}^{\infty} r^{|m|}e^{im\theta}a_m. \tag{40.1}$$

Note that all our computations are formal: we discover what a possible solution might be and prove that our math is rigorous later.

Plugging $r = 1$ in (40.1) yields

$$f(e^{i\theta}) = h(1,\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta};$$

this means that a_m must be the m^{th} Fourier coefficient of f. As a reminder, we may compute

$$\int_{-\pi}^{\pi} e^{im\theta}e^{-in\theta} = \begin{cases} 2\pi & \text{if } m = n \in \mathbb{Z}, \\ 0 & \text{if } m \neq n \in \mathbb{Z}. \end{cases}$$

This means $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}$, since multiplying by $e^{-in\theta}$ and integrating forces all other Fourier coefficients to vanish. All in all,

$$\begin{aligned} h(r,\theta) &= \sum_{m=-\infty}^{\infty} r^{|m|}e^{im\theta} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\eta})e^{-im\eta}\,d\eta \\ &\overset{(*)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\eta}) \sum_{m=-\infty}^{\infty} r^{|m|}e^{im(\theta-\eta)}\,d\eta \\ &=: f * P_r(\theta). \end{aligned}$$

(The equality in $(*)$ is, again, formal.) Here P_r is the *Poisson kernel*, defined by

$$P_r(\theta) \coloneqq \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} r^{|m|}e^{im\theta} \quad \text{for } 0 \le r < 1.$$

It turns out that all our manipulations have led to the correct answer.

Theorem 40.1. *Let $f : \partial B(0,1) \to \mathbb{R}$ be continuous. Define*

$$h(z) = h(re^{i\theta}) = f * P_r(\theta) \quad \textit{for } 0 \le r < 1 \quad \textit{and } \theta \in \mathbb{R},$$

and further set $h(e^{i\theta}) = f(e^{i\theta})$ (that is, force h and f to be equal on the boundary). Then h is a harmonic solution to the Dirichlet problem.

40.2 The Poisson Kernel

In order to prove this theorem, we need to establish a few properties of the function P_r we have just defined. These properties are listed below.

- Writing $z = re^{i\theta}$, we have

$$\begin{aligned} 2\pi P_r(\theta) &= 1 + \sum_{m=1}^{\infty} z^m + \sum_{m=1}^{\infty} \overline{z}^m = 1 + \left(\frac{1}{1-z} - 1\right) + \left(\frac{1}{1-\overline{z}} - 1\right) \\ &= \frac{1-|z|^2}{1 - z - \overline{z} + z^2} = \frac{1-r^2}{1 - 2r\cos\theta + r^2}. \end{aligned}$$

  This form for P_r is good for computation. Furthermore, we may write

$$P_r(\theta) = 2\Re\left(\frac{1}{1-z}\right) - 1 = \Re\left(\frac{1+z}{1-z}\right);$$

  since $\frac{1+z}{1-z}$ is analytic, we deduce that $P_r(\theta)$ is harmonic.
- Since

$$1 - 2r\cos\theta + r^2 = (1 - r\cos\theta)^2 + (r\sin\theta)^2 \ge 0, \tag{40.2}$$

  we deduce that $P_r(\theta) > 0$ for all θ and r with $0 \le r < 1$.
- Observe that, for fixed $r \in (0,1)$, the sum $\sum_{m=-\infty}^{\infty} r^{|m|}e^{im\theta}$ converges uniformly in θ. This means that

$$\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta}\,d\theta = 1.$$

- Fix $\delta > 0$. If $\delta \le |\theta| \le \pi$, then

$$-1 \le \cos\theta \le \cos\delta < 1.$$

  This means we may write

$$P_r(\theta) \le \frac{1}{2\pi} \cdot \frac{1-r^2}{1 - 2r\cos\theta + r^2} = \frac{1}{2\pi} \cdot \frac{1-r^2}{(1-r)^2 + 2r(1-\cos\delta)},$$

  implying that $P_r(\theta)$ tends to zero as $r \to 1^-$ uniformly on $[-\pi,\pi] \setminus [-\delta,\delta]$.

The last three bullets imply that the sequence of functions $\{P_r\}_{r\to 1^-}$ forms an *approximate identity.* This yields the following result for free, but in the intersect of being self-contained we will prove it from first principles.

Lemma 40.2. *Let $f : \partial B(0,1) \to \mathbb{R}$ be continuous. Then*

$$f * P_r(\theta) \to f(e^{i\theta}) \quad \text{as } r \to 1^-,$$

and furthermore this convergence is uniform in θ.

Proof. Write

$$\begin{aligned}
\left|f * P_r(\theta) - f(e^{i\theta})\right| &= \left|\int_{-\pi}^{\pi} f(e^{i(\theta-\eta)})P_r(\eta)\,d\eta - f(e^{i\eta})\right| \\
&= \left|\int_{-\pi}^{\pi} f(e^{i(\theta-\eta)})P_r(\eta)\,d\eta - f(e^{i\eta})\int_{-\pi}^{\pi} P_r(\eta)\,d\eta\right| \\
&= \left|\int_{-\pi}^{\pi} (f(e^{i(\theta-\eta)}) - f(e^{i\eta}))P_r(\eta)\,d\eta\right| \\
&\le \int_{-\pi}^{\pi} \left|(f(e^{i(\theta-\eta)}) - f(e^{i\eta}))P_r(\eta)\right|\,d\eta \\
&= \int_{-\delta}^{\delta} \left|(f(e^{i(\theta-\eta)}) - f(e^{i\eta}))P_r(\eta)\right|\,d\eta \\
&\qquad + \int_{[-\pi,\pi]\setminus[-\delta,\delta]} \left|(f(e^{i(\theta-\eta)}) - f(e^{i\eta}))P_r(\eta)\right|\,d\eta.
\end{aligned}$$

Label these integrals as I and II, respectively. To bound I, we use the fact that f is continuous on the compact set $\partial B(0,1)$, which implies f is actually uniformly continuous. This means that

$$I \le 2\pi \cdot \sup_{|\eta|<\delta} \left|f(e^{i(\theta-\eta)}) - f(e^{i\theta})\right|,$$

which goes to zero uniformly as δ tends to zero. To bound II, we instead note that

$$\begin{aligned}
II &\le \int_{[-\pi,\pi]\setminus[-\delta,\delta]} \left[\left|f(e^{i(\theta-\eta)})\right| + \left|f(e^{i\eta})\right|\right] P_r(\eta)\,d\eta \\
&\le 2 \sup_{z\in\partial B(0,1)} |f(z)| \cdot \int_{[-\pi,\pi]\setminus[-\delta,\delta]} P_r(\eta)\,d\eta;
\end{aligned}$$

by the fourth bullet point, this also goes uniformly to zero regardless of δ. Combining both bounds proves the claim. □

41 December 6

41.1 The Poisson Kernel (cont.)

Recall our previous discussion of the Poisson Kernel. We proved last time that $f * P_r$ converges uniformly in θ to f as $r \to 1^-$, provided f is continuous. The only thing left to check is that h is the desired extension.

Lemma 41.1. *The function h is harmonic on $B(0,1)$ provided that f is continuous.*

Proof. Write

$$\begin{aligned} h(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\eta})P_r(\theta-\eta)\,d\eta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\eta})\Re\left(\frac{1+re^{i(\theta-\eta)}}{1-re^{i(\theta-\eta)}}\right)d\eta \\ &= \frac{1}{2\pi}\Re\left(\int_{-\pi}^{\pi} f(e^{i\eta})\frac{1+re^{i(\theta-\eta)}}{1-re^{i(\theta-\eta)}}\,d\eta\right) = \frac{1}{2\pi}\Re\left(\int_{-\pi}^{\pi} f(e^{i\eta})\frac{e^{i\eta}+z}{e^{i\eta}-z}\,d\eta\right). \end{aligned}$$

Consider the function g defined by $g(z) = \int_{-\pi}^{\pi} f(e^{i\eta})\frac{e^{i\eta}+z}{e^{i\eta}-z}\,d\eta$. Fix a triangle $\Delta \subset B(0,1)$. Note that there exists $C > 0$ such that $\left|e^{i\eta}-z\right| < \delta$ for all η, so the integrand is uniforly bounded above; in turn, so is g. Thus, we may write

$$\int_\Delta g(z)\,dz = \int_\Delta\int_{-\pi}^{\pi} f(e^{i\eta})\frac{e^{i\eta}+z}{e^{i\eta}-z}\,d\eta\,dz = \int_{-\pi}^{\pi} f(e^{i\eta})\int_\Delta \frac{e^{i\eta}+z}{e^{i\eta}-z}\,dz\,d\eta.$$

As the function $z \mapsto \frac{e^{i\eta}+z}{e^{i\eta}-z}$ is analytic on $B(0,1)$, the inner integral is zero, and hence $\int_\Delta g(z)\,dz = 0$. Since Δ was arbitrary, we deduce by Morera's Theorem (Corollary 8.4) that g is analytic on $B(0,1)$¿ In turn, h is the real part of an analytic function, implying that h is harmonic. □

Remark. We can generalize this and solve the Dirichlet problem on any ball $B(z_0,R)$ relatively easily. Indeed, write $z = z_0 + re^{i\theta}$ for $0 \le r < R$; then, if f is given on $\partial B(z_0,R)$, we may write

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0+Re^{i\eta})\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\eta)+r^2}\,d\eta =: f(z_0+R\cdot) * P_{R,r}(\theta), \tag{41.1}$$

where

$$P_{R,r}(\theta) = \frac{R^2-r^2}{R^2-2Rr\cos\theta+r^2} = \Re\left(\frac{R+z}{R-z}\right).$$

The proof is exactly the same.

41.2 Applications of Harmonic Extensions

The existence of a unique solution to the Dirichlet problem implies that harmonic functions, just like analytic ones, are "one-dimensional". In particular, while we have previously shown that harmonic functions give us information along the boundary of a ball, we can also go the other way around and use information along the boundary of a ball to deduce information about h. This philosophy lies at the core of the next few examples, which further serve to port analytic results to the harmonic setting.

Our first application is, in some sense, the converse of the Mean Value Property for harmonic functions.

Corollary 41.2. *Let h be real-valued and continuous on a domain D. If h has the Mean Value Property – that is,*

$$h(z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(z_0+re^{i\theta})\,d\theta \quad \text{whenever } \overline{B(z_0,r)} \subset D,$$

then h is harmonic.

Proof. It suffices to prove the result inside any ball B that is compactly contained in D. By compact containment, we may harmonically extend h from ∂B into B. Let u be the resulting harmonic extension. Then $u - h$ is zero on ∂B, and furthermore $u - h$ has the mean value property. Thus, $u - h$ is subharmonic, implying that $u - h \le 0$ on B. Similarly, $h - u$ is subharmonic on B, and so $h = u$ in B. In particular, h is harmonic. □

Our second result shows that convergence of harmonic functions is just as nice as convergence of analytic functions is.

Corollary 41.3. *Normal limits of harmonic functions are harmonic.*

Proof. Suppose $\{h_n\}_{n\geq 1}$ is a sequence of functions converging normally to h on a domain D. Note that h is continuous on each compact subset (since the uniform limit of continuous functions is continuous), and so h is continuous on D. Thus, whenever $\overline{B(z_0, R)} \subset D$,

$$h(z_0) = \lim_{n\to\infty} h_n(z_0) = \lim_{n\to\infty} \int_{-\pi}^{\pi} h_n(z_0 + Re^{i\theta})\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} h(z_0 + Re^{i\theta})\, d\theta.$$

Thus h satisfies the Mean Value Property, implying that h is harmonic. □

Our next result places bounds on h in a manner similar to what the derivative estimates for analytic functions yield.

Corollary 41.4 (Harnock's Inequality)**.** *Let $h : \overline{B(z_0, R)} \to \mathbb{R}$ be harmonic in the interior of the ball, continuous, and nonnegative. Then for all $z \in B(z_0, R)$,*

$$h(z_0)\frac{R - |z - z_0|}{R + |z - z_0|} \leq h(z) \leq h(z_0)\frac{R + |z - z_0|}{R - |z - z_0|}.$$

Proof. Recall that (41.1) tells us

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0 + Re^{i\eta})\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \eta) + r^2}\, d\eta.$$

Since both h and $P_{r,R}$ are nonnegative, we may write

$$\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \eta) + r^2} \leq \frac{R^2 - r^2}{(R - r)^2} = \frac{R + r}{R - r},$$

and by similar computations we also deduce that $\frac{R-r}{R+r}$ is a lower bound for the quotient. Therefore

$$h(z) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} h(z_0 + re^{i\eta})\frac{R + r}{R - r}\, d\eta = h(z_0)\cdot\frac{R + r}{R - r},$$

which proves the upper bound. Similar computations occur for the lower bound. □

Our fourth result derives an analogue of Mantel's Theorem to harmonic functions.

Theorem 41.5. *Let $h_1 \leq h_2 \leq \cdots$ be a series of pointwise increasing harmonic functions on a domain D. Then either $h_n \to \infty$ normally on D or h_n converges to a harmonic function h normally on D.*

Proof. We may assume each h_n is nonnegative by replacing h_n with $h_n - h_1$. This means that $h_n(z)$ converges to $h(z) \in [0\infty]$ for each $z \in D$. Now set

$$A \coloneqq \{z \in D : h(z) = \infty\} \quad \text{and} \quad B \coloneqq \{z \in D : h(z) < \infty\}.$$

We will show that A and B are both open, which implies that, since D is connected, one of these two sets must be empty.

Fix $z_0 \in D$ and take $R > 0$ such that $\overline{B(z_0, R)} \subset D$. Recall that Harnack's Inequality yields

$$\frac{R - |z - z_0|}{R + |z - z_0|}h_n(z_0) \leq h_n(z) \leq \frac{R + |z - z_0|}{R - |z - z_0|}h_n(z_0)$$

for every $z \in B(z_0, R)$ and every $n \geq 1$. If $|z - z_0| \leq \frac{R}{2}$, then the above inequality can be weakened to $\frac{1}{3}h_n(z_0) \leq h_n(z) \leq \frac{1}{3}h_n(z_0)$.

Thus,

- If $h_n(z_0) \to \infty$, then $h_n(z) \to \infty$ uniformly on $B(z_0, \frac{R}{2})$;
- if $h_n(z_0) \to h(z_0) < \infty$, then the sequence $\{h_n\}$ is bounded uniforly on $B(z_0, \frac{R}{2})$.

This implies that both A and B are open as desired.

Finally, we case. If $D = A$, then the convergence to ∞ is uniform on all compact subsets of D, and so $h_n \to \infty$ normally on D. If $D = B$, then we may instead note that Harnack's Inequality yields

$$h_m(z) - h_n(z) \le 3(h_m(z_0) - h_n(z_0)) \quad \text{whenever } z \in B(z_0, \tfrac{R}{2}).$$

Thus $\{h_m\}$ is uniformly Cauchy on every compact subset, so the convergence is indeed normal and h is harmonic. $\square$

Our last result is a different form of Cauchy's Integral Formula that results from our work on harmonic functions.

Theorem 41.6 (Schwarz Formula). *Let f be analytic on a domain containing $\overline{B(0,R)}$. Then for all $z \in B(0,R)$,*

$$f(z) = i\Im(f(0)) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re(f)(Re^{i\eta})\frac{Re^{i\eta}+z}{Re^{i\eta}-z}\,d\eta. \tag{41.2}$$

Proof. Observe that the equality holds if we take the real part of both sides since $\Re(f)$ is harmonic. It follows that the difference between the left-hand and right-hand sides is purely imaginary, hence constant. By plugging in $z = 0$, we see that this constant equals $i\Im(f(0))$. $\square$

42 December 9

42.1 More on Schwarz

Recall last time that we derived equation (41.2). We start today by deriving a corollary of this equality.

Corollary 42.1 (Borel-Caratheodory). *Let f be analytic on $\overline{B(0,R)}$, and suppose $\Re(f) \leq A$ on $\partial B(0,R)$. Then*
$$|f'(z)| \leq \frac{2r}{R-r}A + \frac{R+r}{R-r}\,|f(0)|\,,$$
where $r = |z| < R$.

Proof. Begin by writing
$$\begin{aligned}
f(z) - f(0) &= \frac{1}{2\pi}\int_0^{2\pi} \Re(f)(Re^{i\eta})\frac{Re^{i\eta}+z}{Re^{i\eta}-z}\,d\eta - \Re(f(0)) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Re(f)(Re^{i\eta})\frac{Re^{i\eta}+z}{Re^{i\eta}-z}\,d\eta - \frac{1}{2\pi}\int_0^{2\pi}\Re(f)(Re^{i\eta})\,d\eta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Re(f)(Re^{i\eta})\frac{2z}{Re^{i\eta}-z}\,d\eta.
\end{aligned}$$

At this point, taking the absolute value of both sides is not optimal because we would still have absolute value signs on the $\Re(f)(Re^{i\eta})$ term that prevent us from applying the Mean Value Property for harmonic functions. Instead, we use the fact that $\Re(f) \leq A$ inside $B(0,R)$. Indeed, note that whenever $z/R \in B(0,1)$,
$$\begin{aligned}
\int_0^{2\pi} \frac{1}{e^{i\eta}-z/R}\,d\eta = \int_0^{2\pi}\frac{ie^{i\eta}}{ie^{i\eta}(e^{i\eta}-z/R)}\,d\eta &= \int_{B(0,1)}\frac{d\xi}{i\xi(\xi - z/R)} \\
&= \frac{R}{iz}\int_{B(0,1)}\left(\frac{1}{\xi - z/R} - \frac{1}{\xi}\right)d\xi = \frac{R}{iz}(2\pi i - 2\pi i) = 0.
\end{aligned}$$

Thus $\int_0^{2\pi}\frac{A\cdot 2z}{Re^{i\eta}-z}\,d\eta = 0$, and so
$$f(z) - f(0) = \frac{1}{2\pi}\int_0^{2\pi}(\Re(f)(Re^{i\eta}) - A)\frac{2z}{Re^{i\eta}-z}\,d\eta.$$

Now $\Re(f)(Re^{i\eta}) - A$ is nonpositive, and so taking the absolute value of both sides yields
$$\begin{aligned}
|f(z) - f(0)| &\leq \frac{1}{2\pi}\int_0^{2\pi}(A - \Re(f)(Re^{i\eta}))\left|\frac{2z}{Re^{i\eta}-z}\right|d\eta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}(A - \Re(f)(Re^{i\eta}))\frac{2r}{R-r}\,d\eta \\
&= A\cdot\frac{2r}{R-r} - \frac{2r}{R-r}\Re(f(0)).
\end{aligned}$$

Therefore
$$\begin{aligned}
|f(z)| \leq |f(z) - f(0)| + |f(0)| \leq A\cdot\frac{2r}{R-r} + \frac{2r}{R-r}\,|\Re(f(0))| + |f(0)| \\
\leq A\cdot\frac{2r}{R-r} + \frac{R+r}{R-r}\,|f(0)|\,,
\end{aligned}$$

and we're done. □

42.2 Zeros of Analytic Functions

In previous lectures, we have discussed properties of zeros of analytic functions; namely, that the zero set is discrete whenever the function is nonconstant. Our final goal of the course is to prove something stronger in the case when the domain D is $B(0,1)$. The crux of this argument is the following result.

Theorem 42.2 (Poisson-Jensen Formula)**.** *Say $f(z)$ is analytic and not identically zero on a domain containing $\overline{B(0,R)}$. Denote the zeros of this function by the sequence $\{a_j\}$, where we repeat roots in the sequence according to multiplicities. Then*

$$\ln|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f(Re^{i\eta})\right| \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\eta)+r^2}\,d\eta + \sum_{|a_j|<R} \ln\left|\frac{R(z-a_j)}{R^2-\overline{a_j}z}\right|. \tag{42.1}$$

Proof. We proceed in steps.

Step 1. First suppose f has no zeros on $\overline{B(0,R)}$, where $R>0$. Then f is positive on $B(0,R+\varepsilon)$ for some $\varepsilon>0$, and hence $\log f$ is analytic on this larger ball. Then $\ln|f| = \log f$ is harmonic, and so the result we want to prove is just (41.1).

Step 2. Now suppose f has zeros on $B(0,R)$ but not on $\partial B(0,R)$. Fix $k\in\mathbb{N}$, and suppose $|a_k|<R$. Then the function $g_k(z) = \frac{R(z-a_k)}{R^2-\overline{a_k}z}$ is analytic on $B(0,R)$, and furthermore $|g_k(z)|=1$ whenever $|z|=R$. Thus, g_k is a Möbius transformation from $B(0,R)$ to $B(0,1)$. Furthermore, $g_k(a_k)=0$.

Thus, we may define g via

$$g(z) = \frac{f(z)}{\displaystyle\prod_{j:|a_j|<R} \frac{R(z-a_j)}{R^2-\overline{a_j}z}}.$$

This function g has no zeros on $\overline{B(0,R)}$, and furthermore $|g(z)|=|f(z)|$ for all $z\in\partial B(0,R)$ since each term in the product has magnitude 1 for those z. Thus, we may apply Step 1 to g to obtain

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\eta})\right| P_{R,r}(\theta-\eta)\,d\eta &= \frac{1}{2\pi}\int_0^{2\pi}\ln\left|g(Re^{i\eta})\right| P_{R,r}(\theta-\eta)\,d\eta \\ &= \ln|g(z)| = \ln\left|\frac{f(z)}{\prod_{j:|a_j|<R}\frac{R(z-a_j)}{R^2-\overline{a_j}z}}\right| \\ &= \ln|f(z)| - \sum_{j:|a_j|<R}\ln\left|\frac{R(z-a_j)}{R^2-\overline{a_j}z}\right|.\end{aligned}$$

Moving the logarithm terms to the other side proves the theorem in this case.

Step 3. Finally, suppose f has some zeros on $\partial B(0,R)$, say $b_1,\dots,b_m$. Consider the function

$$g(z) = \frac{f(z)}{\prod_{j=1}^m (z-b_j)}.$$

Then g has no zeros on $\partial B(0,R)$ since all zeros on the boundary cancel out. In turn, we may apply Step 2 to g to see t hat

$$\begin{aligned}\ln|f(z)| - \sum_{j=1}^m \ln|z-b_j| = \ln|g(z)| &= \frac{1}{2\pi}\int_0^{2\pi}\ln\left|g(Re^{i\eta})\right| P_{R,r}(\theta-\eta)\,d\eta + \sum_{|a_j|<R}\ln\left|\frac{R(z-a_j)}{R^2-\overline{a_j}z}\right| \\ &= \frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\eta})\right| P_{R,r}(\theta-\eta)\,d\eta + \sum_{|a_j|<R}\ln\left|\frac{R(z-a_j)}{R^2-\overline{a_j}z}\right| \\ &\qquad - \sum_{j=1}^n \frac{1}{2\pi}\int_0^{2\pi}\ln\left|Re^{i\eta}-b_j\right| P_{R,r}(\theta-\eta)\,d\eta.\end{aligned}$$

We will be done if we can show that

$$\ln|z-b_j| = \frac{1}{2\pi}\int_0^{2\pi}\ln\left|Re^{i\eta}-b_j\right| P_{R,r}(\theta-\eta)\,d\eta. \tag{42.2}$$

To prove this, let $\varepsilon>0$. Observe that the function $z\mapsto z-b_j(1+\varepsilon)$ has no zeros in $B(0,R)$; thus, by Step 1,

$$\ln|z-b_j(1+\varepsilon)| = \frac{1}{2\pi}\int_0^{2\pi}\ln\left|Re^{i\eta}-b_j(1+\varepsilon)\right| P_{R,r}(\theta-\eta)\,d\eta.$$

Now send $\varepsilon \to 0$. The left hand side converges to $\ln|z - b_j|$ by continuity. For the right hand side, write $b_j = Re^{i\beta}$, so that

$$\begin{aligned}\left|\int_0^{2\pi} \ln\left|Re^{i\eta} - b_j\right| P_{R,r}(\theta-\eta)\, d\eta\right| &\le \frac{R+r}{2\pi(R-r)}\int_0^{2\pi} \ln\left|R(e^{i\eta} - e^{i\beta})\right|\, d\eta \\ &\le C + \int_0^{2\pi} \ln\left|e^{i(\eta-\beta)-1}\right|\, d\eta.\end{aligned}$$

To upper bound this integral, note that

$$\ln\left|e^{ix} - 1\right| = \frac{1}{2}\ln\left|e^{ix}-1\right|^2 = \frac{1}{2}\ln(2 - 2\cos x),$$

and observe that via two applications of L'Hopital we may deduce

$$\lim_{x\to 0}\left|\frac{\ln(1-\cos x)}{\ln x}\right| = 2.$$

Thus the integrability of the singularity at $\eta = \beta$ is equivalent to the integrability of $\ln x$ near $x = 0$, which is indeed true. Thus, we may appeal to the Dominated Convergence Theorem to show that the limit of the right hand side as $\varepsilon \to 0$ is the right hand side of (42.2). We are done. ☐

With this result, we may now answer the query presented at the beginning of this section.

Corollary 42.3. *Suppose $f \not\equiv 0$ is analytic and bounded on $B(0,1)$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of zeros repeated according to multiplicities. Then*

$$\sum_{n=1}^{\infty}(1 - |z_n|) < \infty. \tag{42.3}$$

Before we prove this corollary, we make a motivating remark.

Remark. Suppose $f(0) \neq 0$. Then plugging $z = 0$ into (42.1) yields

$$\ln|f(0)| = \sum_{j:|z_j|<R}\ln\left|\frac{a_j}{R}\right| + \int_0^{2\pi}\frac{1}{2\pi}\ln\left|f(Re^{i\eta})\right|\, d\eta. \tag{42.4}$$

Exponentiating both sides then gives

$$|f(0)|\prod_{j:|a_j|<R}\frac{R}{|a_j|} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\ln\left|f(Re^{i\eta})\right|\, d\eta\right). \tag{42.5}$$

This gives us size information about $|a_j|$, which will be the key to proving our corollary.[14]

Proof. Let $M := \sup|f| < \infty$. Assume the sequence of zeros is infinite, else the problem is trivial.

Without loss of generality suppose $f(0) \neq 0$, else replace $f(z)$ with $f(z)/z^m$, where m is the multiplicity of 0 at f (cf. Theorem 13.8). Since the zero set of f is discrete, we may reorder the zeros in order of magnitude. Let $n(R)$ be the number of zeros less than a given radius $R < 1$. Then (42.5), when combined with the fact that f is bounded, yields that

$$|f(0)|\prod_{j=1}^{n(R)}\frac{R}{|a_j|} \le \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\ln(M)\, d\eta\right) = M.$$

Now fix N, and take R close to 1 so that $n(R) \ge N$. Then

$$|f(0)|\prod_{j=1}^{N}\frac{R}{|a_j|} \le |f(0)|\prod_{j=1}^{n(R)}\frac{R}{|a_j|} \le M,$$

[14] As an unrelated remark, note that (42.4) implies that, whenever f is analytic, $\log|f|$ is subharmonic.

so $\prod_{j=1}^N |a_j| \geq \frac{R^N |f(0)|}{M}$. Sending $R \to 1$ yields $1 \geq \prod_{j=1}^N |a_j| \geq \frac{|f(0)|}{M}$. This means that the sequence $\alpha_N \coloneqq \frac{R^N |f(0)|}{M}$ is a monotonically decreasing and uniformly bounded sequence of positive real numbers, implying that the product converges to some nonzero number.

We claim this implies $\sum_{n=1}^\infty (1-|a_n|)$ is finite. To prove this, write

$$\prod_{j=1}^N |a_j| = \prod_{j=1}^N |1-(1-|a_j|)| \leq \prod_{j=1}^N \exp(-(1-|a_j|)) = \exp\left(-\sum_{j=1}^N (1-|a_j|)\right).$$

Sending $N \to \infty$ yields

$$\exp\left(\sum_{j=1}^\infty (1-|a_j|)\right) \leq \frac{1}{\prod_{j=1}^\infty |a_j|} < \infty,$$

which gives the desired conclusion. □

Remark. This is a pretty strong theorem – its converse is also true! Suppose $\sum_{j=1}^\infty (1-|a_j|) < \infty$, where $a_1, \ldots$ is a sequence of complex numbers with $|a_j| < 1$ for all j. One can consider the *Blaschke product*

$$g(z) = z^m \prod_{n=1}^\infty \frac{\overline{a_n}}{|a_n|} \cdot \frac{a_n - z}{1-\overline{a_n} z}$$

and show that g is bounded and analytic on $B(0,1)$ with zeros at precisely the right places.

In turn, suppose f is a given analytic function with zeros as above. Construct g as in the previous equation. Then f/g has no zeros, so $\log(f/g)$ has a branch on $B(0,1)$. So $f(z)/g(z) = e^{h(z)}$ for some entire function h, implying

$$f(z) = e^{h(z)} z^m \prod_{n=1}^\infty \frac{\overline{a_n}}{|a_n|} \cdot \frac{a_n - z}{1-\overline{a_n} z}.$$

This describes all bounded analytic functions on $B(0,1)$.

43 December 11

Final exam review.