Math 5440 Aaron Fogelson
Fall, 2013

Math 5440 Problem Set 5 – Solutions

1: (Logan, 2.2 # 3) Solve the outgoing signal problem

$$u_{tt} - c^2 u_{xx} = 0 \quad x > 0, \quad -\infty < t < \infty,$$

and

$$u_x(0,t) = s(t), \quad -\infty < t < \infty,$$

where $s(t)$ is a known signal.

The general solution to the wave equation is $u(x,t) = F(x-ct) + G(x+ct)$. We seek a wave outgoing from $x = 0$ to $x > 0$, so we set $G \equiv 0$, and have $u(x,t) = F(x-ct)$. To satisfy the BC, we compute $u_x(x,t) = F'(x-ct)$ and see that $s(t) = u_x(0,t) = F'(-ct)$. Hence $F'(z) = s(-z/c)$, and

$$\begin{aligned} F(z) &= \int_0^z s\left(\frac{-z'}{c}\right) dz' + A \\ &= \int_0^{-z/c} s(y)(-c\,dy) + A \\ &= -c\int_0^{-z/c} s(y)dy + A. \end{aligned}$$

It follows that $u(x,t) = F(x-ct) = -c\int_0^{\frac{ct-x}{c}} s(y)dy + A = A - c\int_0^{t-x/c} s(y)dy$. It is easy to check that the BC is satisfied. $u_x(x,t) = -cs(t-x/c)(-1/c) = s(t-x/c)$, so $u_x(0,t) = s(t)$ as required. ∎

2: (Logan, 2.3 # 1) Show that the Cauchy problem for the backward diffusion equation

$$u_t + u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = f(x)$$

is unstable by considering the solutions

$$u(x,t) = 1 + \frac{1}{n} e^{n^2 t} \sin(nx)$$

for large n.

The solution to the Cauchy problem with $u(x,0) = 1$ for all x is $u(x,t) = 1$ for all x and $t > 0$. The function $u(x,t) = 1 + \frac{1}{n} e^{n^2 t} \sin(nx)$ is the solution to the Cauchy problem for the initial condition $u(x,0) = 1 + \frac{1}{n} \sin(nx)$. The maximum difference between the initial functions is $1/n$ which gets smaller and smaller as n grows. The maximum difference at time $t > 0$ between the solutions to the two Cauchy problems is $\frac{1}{n} e^{n^2 t}$ which grows unboundedly as n grows. Thus at any time $t > 0$ the ratio of the maximum change in the solution to the maximum difference in the initial data, namely, $e^{n^2 t}$ can be made arbitrarily large by choosing n sufficiently large. The Cauchy problem is not stable for the backward diffusion problem. ∎

3: (Logan, 2.3 # 2) Let $u = u(x, y)$. Is the problem $u_{xy} = 0$ for $0 < x, y < 1$, on the unit square $\Omega = [0, 1] \times [0, 1]$, where the value of u is prescribed on the boundary $\partial\Omega$ of the square, a well-posed problem? Discuss.

Integrate the PDE once to get $u_x(x, y) = f'(x)$ for some arbitrary function $f(x)$. Integrate again to get $u(x, y) = f(x) + g(y)$ for arbitrary functions $f(x)$ and $g(y)$. Let $u_L(y)$, $u_R(y)$, $u_B(x)$, and $u_T(x)$, denote the specified values of u on the left, right, bottom, top sides of $\partial\Omega$, respectively. Since $u(x, y)$ is supposed to satisfy the boundary conditions, we have

$$u_L(y) = u(0, y) = f(0) + g(y), \quad 0 < y < 1,$$

and

$$u_R(y) = u(1, y) = f(1) + g(y), \quad 0 < y < 1.$$

So we need that both $g(y) = u_L(y) - f(0)$ and $g(y) = u_R(y) - f(1)$ which cannot happen unless $u_L(y) = u_R(y) - f(1) + f(0)$, that is, unless $u_L(y)$ differs from $u_R(y)$ by a constant independent of y. Since this is not true in general, there would not exist a solution to the problem in general so it is not well-posed. ∎

4: (Logan, 2.3 # 3) Consider the two Cauchy problems for the wave equation with different initial data:

$$u_{tt}^{(i)} = c^2 u_{xx}^{(i)}, \quad 0 < t < T,$$

with

$$u^{(i)}(x,0) = f^{(i)}(x), \;\; u_t^{(i)}(x,0) = g^{(i)}(x), \quad -\infty < x < \infty,$$

for $i = 1, 2$ where $f^{(1)}(x), f^{(2)}(x), g^{(1)}(x), g^{(2)}(x)$ are given functions. If for all x, we have

$$|f^{(1)}(x) - f^{(2)}(x)| \leq \delta_1, \quad |g^{(1)}(x) - g^{(2)}(x)| \leq \delta_2,$$

show that $|u^{(1)}(x,t) - u^{(2)}(x,t)| < \delta_1 + \delta_2 T$ for all x and for $0 < t < T$. What does this mean in regard to stability?

The solutions to the respective Cauchy problems are

$$u^{(1)}(x,t) = \frac{1}{2}\left\{f^{(1)}(x-ct) + f^{(1)}(x+ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^{(1)}(s)ds,$$

and

$$u^{(2)}(x,t) = \frac{1}{2}\left\{f^{(2)}(x-ct) + f^{(2)}(x+ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^{(2)}(s)ds.$$

Subtracting the first of these equations from the second we get

$$\begin{aligned} u^{(2)}(x,t) - u^{(1)}(x,t) &= \frac{1}{2}\left(f^{(2)}(x-ct) - f^{(1)}(x-ct)\right) \\ &+ \frac{1}{2}\left(f^{(2)}(x+ct) - f^{(1)}(x+ct)\right) \\ &+ \frac{1}{2c}\int_{x-ct}^{x+ct} (g^{(2)}(s) - g^{(1)}(s))ds. \end{aligned}$$

For any x and $0 \leq t \leq T$,

$$\begin{aligned} |u^{(2)}(x,t) - u^{(1)}(x,t)| &\leq \frac{1}{2}|f^{(2)}(x-ct) - f^{(1)}(x-ct)| \\ &+ \frac{1}{2}|f^{(2)}(x+ct) - f^{(1)}(x+ct)| \\ &+ \frac{1}{2c}\int_{x-ct}^{x+ct} |g^{(2)}(s) - g^{(1)}(s)|ds \\ &\leq \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1 + \frac{1}{2c}2ct\delta_2 \\ &\leq \delta_1 + \delta_2 T. \end{aligned}$$

The difference in the solutions is less than a prescribed tolerance $\epsilon > 0$ whenever the differences in the initial data are small enough, e.g., $\delta_1 < \frac{\epsilon}{2}$ and $\delta_2 < \frac{\epsilon}{2T}$. This problem is stable. ∎

5: (Logan, 2.4 # 1) Solve the problem

$$u_t = ku_{xx}, \quad x > 0,\ t > 0,$$

$$u_x(0,t) = 0, \quad t > 0,$$
$$u(x,0) = \phi(x), \quad x > 0,$$

with an insulated boundary condition by extending ϕ to all of the real axis as an even function. The solution is

$$u(x,t) = \int_0^\infty [G(x-y,t) + G(x+y,t)]\phi(y)dy.$$

First note that the solution to the IVP $u_t = ku_{xx}$, $-\infty < x < \infty$, $t > 0$, $u(x,0) = f(x)$, $-\infty < x < \infty$ is an even function of x if $f(x)$ is even. To see this consider

$$\begin{aligned}
u(-x,t) &= \int_{-\infty}^{\infty} G(-x-y,t)f(y)dy \\
&= \int_{\infty}^{-\infty} G(-x+y',t)f(y')(-dy') \\
&= \int_{-\infty}^{\infty} G(-x+y',t)f(y')dy' \\
&= \int_{-\infty}^{\infty} G(x-y',t)f(y')dy' = u(x,t).
\end{aligned}$$

In the last line, we used that G is an even function of its first arguement. Smooth even functions have zero slope at $x = 0$, i.e., $u_x(0,t) = 0$. So we solve our semi-infinite domain problem by extending the initial data to $-\infty < x < \infty$ as an even function. Let

$$F(x) = \begin{cases} \phi(x) & x > 0 \\ \phi(-x) & x < 0. \end{cases}$$

The solution to this IVP is

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} G(x-y,t)F(y)dy \\
&= \int_0^\infty G(x-y,t)\phi(y)dy + \int_{-\infty}^{0} G(x-y,t)\phi(-y)dy \\
&= \int_0^\infty G(x-y,t)\phi(y)dy - \int_{\infty}^{0} G(x+y',t)\phi(y')dy' \\
&= \int_0^\infty \{G(x-y,t) + G(x+y,t)\}\, \phi(y)dy.
\end{aligned}$$

∎

6: (Logan, 2.4 # 2) Find a formula for the solution to the problem

$$u_t = k u_{xx}, \quad x > 0, \quad t > 0,$$

$$u(0,t) = 0, \quad t > 0, \qquad u(x,0) = 1, \quad x > 0.$$

Sketch the graph of several solution profiles with $k = 0.5$.

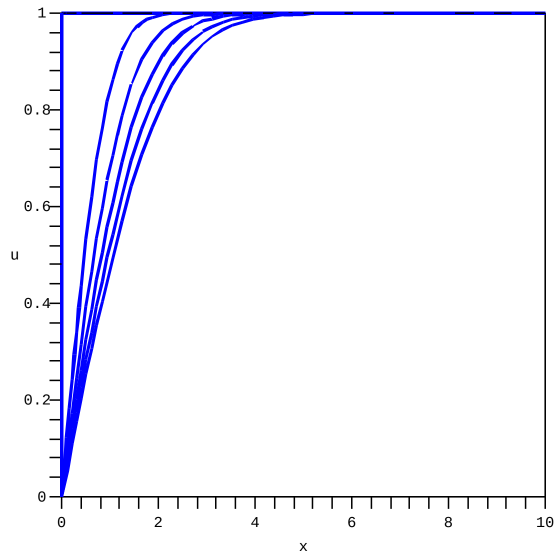
$$\begin{aligned}
u(x,t) &= \int_0^\infty \{G(x-y,t) - G(x+y,t)\}\, 1 dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x-y)^2/4kt} dy - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x+y)^2/4kt} dy.
\end{aligned}$$

Letting $s = (y-x)/\sqrt{4kt}$ in the first integral and $r = (y+x)/\sqrt{4kt}$ in the second integral we obtain

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^\infty e^{-s^2} ds - \frac{1}{2}\frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^\infty e^{-r^2} dr \\
&= \frac{1}{2}\left\{ \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds + \frac{2}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^0 e^{-s^2} ds \right. \\
&- \left. 1 + \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-r^2} dr \right\} \\
&= \frac{1}{2}\left\{ \frac{2}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^\infty e^{-s^2} ds + \text{erf}\left(\frac{\text{x}}{\sqrt{4\text{kt}}}\right) \right\} \\
&= \frac{1}{2}\left\{ \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^\infty e^{-s'^2} (-ds') + \text{erf}\left(\frac{\text{x}}{\sqrt{4\text{kt}}}\right) \right\} \\
&= \frac{1}{2}\left\{ 2\text{erf}\left(\frac{\text{x}}{\sqrt{4\text{kt}}}\right) \right\} \\
&= \text{erf}\left(\frac{\text{x}}{\sqrt{4\text{kt}}}\right).
\end{aligned}$$

To check this solution, note that $u(0,t) = \text{erf}(0) = 0$ for $t > 0$, and

$$u(x,0) = \lim_{t \to 0^+} \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-r^2} dr = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-r^2} dr = 1.$$



∎

Figure 0.1: Plot of solution profiles at times 0,0.5,1.0,1.5,2.0, and 2.5 with k=0.5.

7: (Logan, 2.4 # 3) Find the solution to the problem

$$u_{tt} = c^2 u_{xx}, \quad x > 0, \quad t > 0,$$

$$u(0,t) = 0, \quad t > 0, \quad u(x,0) = xe^{-x},\ u_t(x,0) = 0, \quad x > 0.$$

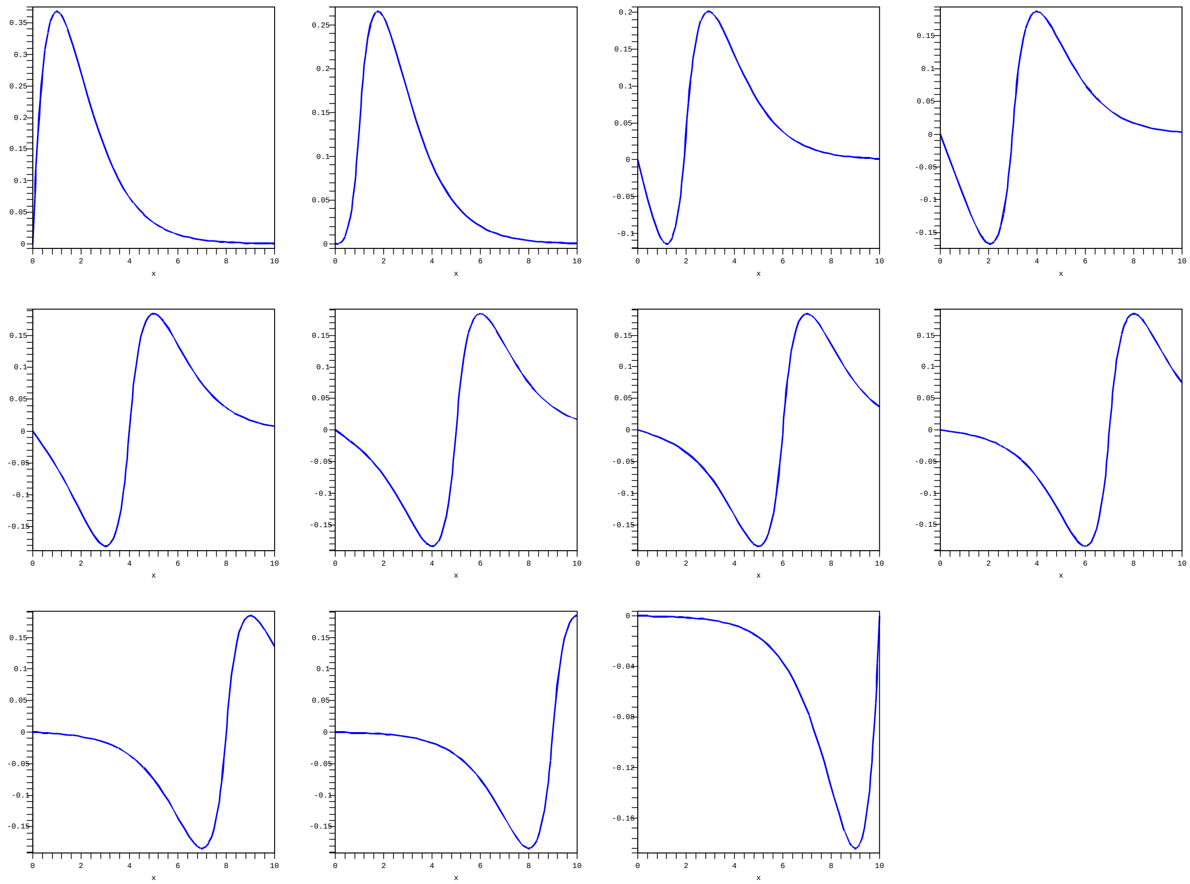
Pick $c = 0.5$ and sketch several time snapshots of the solution surface to observe the reflection of the wave from the boundary.

In order to satisfy the boundary condition automatically we extend the problem to the entire real line by extending the initial functions as odd functions. The solution for the pure IVP for the wave equation with odd initial data is an odd function of x, so it vanishes at $x = 0$. Using the D'Alembert solution for the extended problem and then rewriting it completely in terms of the values of the initial displacement $f(x)$ and initial velocity $g(x)$ for $x > 0$, we find (as in the book and in class) that

$$u(x,t) = \begin{cases} \frac{1}{2}\left\{f(x-ct) + f(x+ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, & x - ct > 0 \\ \frac{1}{2}\left\{f(ct+x) - f(ct-x)\right\} + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)ds, & x - ct < 0. \end{cases}$$

For the current problem $f(x) = xe^{-x}$ and $g(x) = 0$, so

$$u(x,t) = \begin{cases} \frac{1}{2}\left\{(x-ct)e^{-(x-ct)} + (x+ct)e^{-(x+ct)}\right\}, & x - ct > 0 \\ \frac{1}{2}\left\{(ct+x)e^{-(ct+x)} - (ct-x)e^{-(ct-x)}\right\}, & x - ct < 0. \end{cases}$$



■

Figure 0.2: Plot of solution surface for $0 \leq x \leq 10$ and $0 \leq t \leq 10$.

8: (Logan, 2.5 #1) Write a formula for the solution to the problem

$$u_{tt} - c^2 u_{xx} = \sin(x), \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = u_t(x,0) = 0, \quad -\infty < x < \infty.$$

Graph the solution surface when $c = 1$.

$$\begin{aligned}
u(x,t) &= \int_0^t \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin(s) ds d\tau \\
&= \frac{1}{2c} \int_0^t \{\cos(x - c(t-\tau)) - \cos(x + c(t+\tau)\} d\tau \\
&= \frac{\sin(x - c(t-\tau))}{c}\Big|_0^t + \frac{\sin(x + c(t-\tau))}{c}\Big|_0^t \\
&= \frac{1}{2c^2} \{\sin(x) - \sin(x - ct) + \sin(x) - \sin(x + ct)\}.
\end{aligned}$$

■

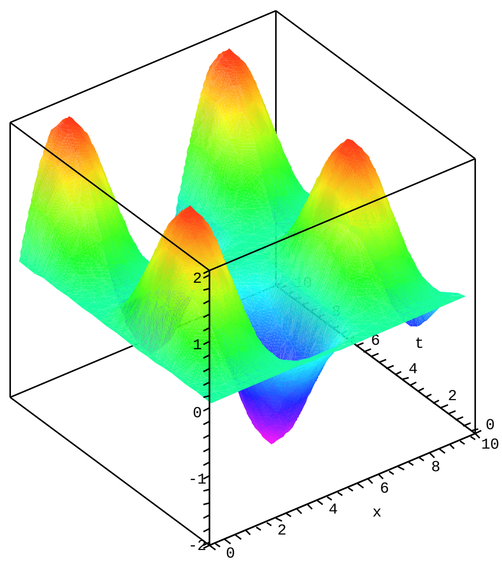


Figure 0.3: Plot of solution surface for $0 \le x \le 10$ and $0 \le t \le 10$.

9: (Logan, 2.5 # 3) Using Duhamel's principle, find a formula for the solution to the initial value problem for the convection problem

$$u_t + cu_x = f(x,t), \quad -\infty < x < \infty,\ t > 0; \quad u(x,0) = 0 \quad -\infty < x < \infty.$$

Hint: Look at the problem

$$w_t(x,t;\tau) + cw_x(x,t;\tau) = 0, \quad -\infty < x < \infty,\ t > 0; \quad w(x,0;\tau) = f(x,\tau) \quad -\infty < x < \infty.$$

Solve the problem

$$u_t + 2u_x = xe^{-t}, \quad -\infty < x < \infty,\ t > 0; \quad u(x,0) = 0 \quad -\infty < x < \infty.$$

The solution to the IVP for $w(x,t;\tau)$ is

$$w(x,t;\tau) = f(x - ct, \tau).$$

The Duhamel principle formula for the solution to the nonhomogeneous problem is therefore

$$u(x,t) = \int_0^t f(x - c(t-\tau), \tau)d\tau.$$

It is easy to check that this solves the PDE:

$$u_t = \int_0^t f'(x - c(t-\tau), \tau)(-c)d\tau + f(x - c(0), t),$$

and

$$u_x = \int_0^t f'(x - c(t-\tau), \tau)d\tau.$$

So,

$$u_t + cu_x = f(x,t)$$

as desired. The solution to $u_t + 2u_x = xe^{-t}$ with $u(x,0) = 0$ is

$$\begin{aligned} u(x,t) &= \int_0^t (x - 2(t-\tau))e^{-\tau}d\tau \\ &= (x-2t)(1-e^{-t}) + 2(1 - e^{-t} - te^{-t}). \end{aligned}$$

■