Math 90 Course Pack

Summer 2012
Instructor: Yolande Petersen

DO NOT BUY THESE NOTES IF YOU HAVE A DIFFERENT INSTRUCTOR

<u>Inside</u>:

- **Lecture Notes Outline with writing space for your notes**
- **Syllabus**
- **Homework Assignments for each section**
- **Example Test Problems (Petersen style only) for each chapter**
- **Final Exam Review Problems & Answers**

<u>How to assemble your notebook</u> (3-ring binder needed):

The lecture notes pages are in reverse order, upside down, and punched on the "wrong" side for a reason! My notes read like a book, with the printed side on the left and handwritten extra notes on the right. To make your notebook look like mine:

1. If there is a staple, remove it.
2. Separate the "upside down", single-sided section from the "rightside up" double-sided section.
3. KEEPING ALL PAGES IN A STACK IN THE SAME ORDER, flip the whole stack of "upside down" pages so that you have the blank back of page 1 on top, holes on the left. Insert these pages into the binder. When you turn the first page, page 1 will be on your left. It should look like a book, with the printed page on the left and the blank page for writing extra notes on the right.
4. Insert the "right-side up" pages into the binder, as you normally would.

Effort was made to minimize the number of pages printed to reduce your cost, while leaving enough space for your notes to be arranged in an orderly way. If you don't like this arrangement, feel free to assemble the pages however you like.

<u>Mrs. Petersen's website</u>: http://peterseny.faculty.mjc.edu

Before you take this class, you may find it helpful to read the document "Teaching Style and Educational Philosophy" to decide whether this instructor is a good match for you. You can find it at the above web address, with the link on the home page under the "What To Expect" heading.

Chapter 2

2.5 Inequalities

3 forms of inequalities:

Algebraic	Graph	Interval Notation
•		
•		
•		

Goal in solving: Isolate x

Caution 1: If you multiply or divide by a negative number ___________________

Caution 2: Subtracting is not the same as multiplying by a negative #.

2.6 More Inequalities/Problem Solving

Fractions

Compound Statements

1. "and" -- Statements using "and" are called conjunctions.

   Equivalent statements:
   A and B
   $A \cap B$ (A intersect B)

   Example: A playing card is red and a king

2. "or" -- Statements using "or" are called disjunctions

   Equivalent statements:
   A or B
   $A \cup B$ (A union B)

   Example: A playing card is red or a king

Ex

2.7 Equations and Inequalities with Absolute Value

Definition of Absolute Value:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

|a| is also defined as the distance between 0 and a on a number line - useful definition.

Property of absolute value equations

If k is positive, then |x| = k is has 2 possible solutions:

1)

2)

Caution: Don't confuse this with x = |k|

Example

Property of absolute value inequalities – LESS THAN case

If k is positive $|x| < k$ is equivalent to:

1)

AND

2)

Note: Distance less than – "close in"

Example

Property of absolute value inequalities – GREATER THAN case

If k is positive, $|x| > k$ is equivalent to:

1)

OR

2)

Note: Distance greater than – "far out"

Example

<u>3.4 – 3.7 Factoring - Quick Review</u>

<u>3.4 Factoring: Greatest Common Factors</u> – reverse distributive law
Goal: Take out the largest amount possible from every term

<u>Eyeball Method</u> – works most of the time
1.

2.

<u>Ex a</u>

<u>Prime Factorization Method</u> – best when GCF isn't easily "eyeballed"
1.

2.

<u>Ex b</u>

<u>Factoring by grouping</u> - most common for
<u>Ex c</u>

<u>3.5 Factoring: Difference of Squares & Sum/Difference of Cubes</u>

1. $f^2 – s^2$

2. $f^3 + s^3$

3. $f^3 – s^3$

Cautions:

3.6 Factoring trinomials

If factorable, a trinomial factors to 2 binomials:

$x^2 \pm bx \pm c =$

Our job:

Some FOIL examples

$(x + 1)(x + 2) = x^2 + x + 2x + 2 =$

$(x – 3)(x – 5) = x^2 -5x -3x + 15 =$

$(x – 4) (x + 3) = x^2 + 3x – 4x – 12 =$

$(x – 2)(x + 5) = x^2 + 5x – 2x – 10 =$

Observe

1. If c is positive

2. If c is negative

3. If 2 signs same, b is

4. If 2 signs different, b is

Example

Example – when x^2 term has coefficient $\neq$ 1

3.7 Solving – set each factor = 0

4.1 – 4.4 Rational Expressions - Quick Review

Rational Expression –
Zeroes in the denominator are to be avoided. (Try $5 \div 0$ on your calculator)
Zeroes in the denominator are called

Ex a

Canceling Rule
Common factors (multiplied) can be canceled
Common terms (added or subtracted) cannot be canceled

Canceling opposite factors:

Multiplying – Factor first, then cancel

Dividing – Keep 1st fraction same, invert & multiply 2nd fraction

Factoring by grouping – common for 4 terms with no GCF

Adding & Subtracting Rational Expressions
Case 1: (easy) If same denominators, keep denominator, add numerators
Case 2: If different denominators:

1. Factor
2. Find the LCD -- Eyeball if possible. If not easy to eyeball:
    a. Break down each denominator into prime factors (with exponents)
    b. Write each prime factor to the highest power possible
    c. The product is the LCD
3. Write new fractions with same denominator by building each to the LCD

4.5 Dividing Polynomials

Division by a Monomial - separate terms and cancel

Division by Polynomials
Long division - Recall arithmetic

Procedure- Dividing by Polynomials
1. Divide the 2 leading terms. Put quotient piece above.
2. Multiply by the divisor
3. Subtract
4. Repeat until terms are used up
5. Write remainder over divisor

Synthetic Division - a shortcut

Requirements:

1. Divisor is a binomial (2 terms only)
2. The binomial is linear (no exponents), with no missing terms.
   It looks like:

   Divisors that can't use synthetic division look like:

Procedure:

1. Write coefficients of polynomial
2. Write the divisor number in front, taking the opposite sign
3. Bring down the first coefficient
4. Multiply the first coefficient by the divisor number, add to next coefficient
5. Repeat until all numbers are used
6. Rewrite the polynomial using the numbers as coefficients, reducing the highest power by 1 degree.

4.6 Solving Fractional Equations/Applications

Goal: Use LCD to multiply and get rid of fractions

Caution: Check equation for bad points

Proportions

Cross multiplication: If $\frac{a}{b} = \frac{c}{d}$, then ad = bc

4.7 More Solving & Applications of Fractional Equations

Some problems require factoring of LCD

Solving for a Specified Variable (procedure)

1. Get rid of denominators (multiply by LCD or cross multiply)
2. Get all terms with desired variable on one side, all other terms on other side
3. Factor out the desired variable
4. Divide by "junk"

Applications

1. Distance = rate X time

2. Work = rate X time, but rate is often calculated via indirect interpretation

5.1 – 5.6 Radicals – Quick Review

5.1 Integer Exponents (Rules)

1. $b^m \cdot b^n = b^{m+n}$

2. $(b^m)^n = b^{mn}$

3. $(ab)^n = a^n b^n$

4. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

5. $\frac{b^n}{b^m} = b^{n-m}$

6. $b^o = 1$

7. $b^{-n} = \frac{1}{b^n}$

Ex

5.2 Roots and Radicals

Goal: No perfect square factors inside radical

Ex

Variables w/exponents

Perfect squares have exponents divisible by

Perfect cubes have exponents divisible by

Caution: Don't confuse a base number with an exponent

5.3 Combining (Adding/Subtracting) Radicals

Caution: $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$

(Don't confuse addition with multiplication rules)

Ex

5.4 Multiplying & Dividing Radicals

$\sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y}$ and $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

Ex

Rationalizing the Denominator

Fractions inside radicals and square roots in the denominator are not considered simplified.

1. For monomials, multiply top & bottom by the piece needed to "fill the pie"

2. For binomials, use conjugate (same 2 terms, except 2^{nd} term has opposite sign)

5.5 Solving Radical Equations – undo square root by squaring both sides

<u>6.1 Complex Numbers</u>

Definition: $i = \sqrt{-1}$

$i^2 = -1$

Note: Powers of i are cyclic

$i = i$

$i^2 =$

$i^3 =$

$i^4 =$

A complex number can be expressed as: $a + bi$

<u>Adding & Subtracting Complex Numbers</u> – group real & imaginary parts separately (similar to like terms)

<u>Products of Complex Numbers</u>

Note: Sq. roots of negative numbers must be converted to i before multiplying, since the rule $\sqrt{x} \cdot \sqrt{y} = \sqrt{x \cdot y}$ is only true for real numbers inside the radical

Correct:

Incorrect:

Quotients of Complex Numbers

Simplified expressions should not have i in the denominator. To get rid of i:

1. For monomials, multiply the numerator and denominator by

2. For binomials multiply the numerator and denominator by

6.2 Quadratic Equations

Quadratic equation: ______ degree
Standard form of quadratic equation: $ax^2 + bx + c = 0$

4 main methods of solving:

1. Factoring –
2. Square Root Method –
3. Complete the Square –
4. Quadratic Formula –

Factoring – best when polynomial looks easy to factor

Radical equations sometimes produce quadratic equations that can be factored. Checking is mandatory.

Square Root Property -- works best for equations that look like $(\text{stuff})^2$ = number
Property: $x^2 = a$ has 2 soutions: $x = \sqrt{a}$ or $x = -\sqrt{a}$
Short hand: $x = \pm\sqrt{a}$

Pythagorean Theorem

30 – 60 - 90 Triangles

The sides of a 30 – 60 – 90 triangle have the following relationship:

45 – 45 – 90 Triangles

The sides of a 45 – 45 – 90 triangle have the following relationship:

6.3 Complete the Square

Procedure – Complete the Square

1. Get equation in form: $x^2 + bx = c$
2. Find the needed number (nn) to complete the square

   $$nn = \left(\frac{b}{2}\right)^2$$

3. Add it to both sides
4. Factor the perfect square. Write it as $\left(x \pm \frac{b}{2}\right)^2$
5. Take square root of both sides
6. Solve for x

6.4 Quadratic Formula

The equation $ax^2 + bx + c = 0$ has solutions(s):

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{for } a \neq 0$$

A tool for checking solutions:

1. The sum of the 2 roots is $-\frac{b}{a}$
2. The product of the 2 roots is $\frac{c}{a}$

Discriminant: $b^2 - 4ac$ (inside of radical)

1. If $b^2 - 4ac > 0$

2. If $b^2 - 4ac = 0$

3. If $b^2 - 4ac < 0$

6.5 More Quadratic Equations/Applications

Choosing a method:

1. Factoring –

2. Square root –

3. Complete the square –

4. Quadratic formula –

Which method would you use to solve the following?

$3x^2 - 25x + 11 = 0$

$x^2 - 8x + 15 = 0$

$x^2 - 8x - 397 = 0$

$(7x + 2)^2 = 12$

Application of Quadratic Formula and Fractional Equations

You have probably seen simple interest problems where A = P(1 + r), where A is the amount earned on a principal, P, at an interest rate, r. If you solve for P, the equation looks like:

$$P = \frac{A}{1+r}$$

Problem:

Gloria wants to get a BA degree in 3 years. After paying for her first year up front, she has $11,000 left to pay for the last 2 years. Each year costs $6000, and payment is due at the beginning of each year. At what interest rate must she invest to earn enough interest to cover the costs? The formula for investments at 2 annual payments, with interest compounded annually is:

$P = \frac{A}{1+r} + \frac{A}{(1+r)^2}$; P = principal invested, A = amount paid out, r = interest rate

Solution:

From our data, P = 11,000 and A = 6000. Plugging into the formula:

$$11{,}000 = \frac{6000}{1+r} + \frac{6000}{(1+r)^2}$$

To get rid of fractions, use the LCD: $(1 + r)^2$

$(11000)(1 + r)^2 = 6000(1 + r) + 6000$

$(11000)(1 + 2r + r^2) = 6000 + 6000r + 6000$

$11000 + 22000r + 11000r^2 = 12000 + 6000r$

$11000r^2 + 16000\, r - 1000 = 0$

$1000(11r^2 + 16r - 1) = 0$

By the quadratic formula, $r = \dfrac{-16 \pm \sqrt{16^2 - 4(11)(-1)}}{2(11)}$ = 0.06 or -1.5

So the interest rate needed is .06 = 6%.

Business majors' note:

When calculating interest we commonly ask the question, "If I invest P dollars for t years at r% interest, how much will I have at the end?"

But sometimes the question needs to be asked in reverse. For example, "If I want to have an income of A dollars each year paid out over t years, how much Principal do I have to invest originally at r% interest?" This kind of calculation is called Net Present Value (NPV).

6.6 Quadratic (and higher) Inequalities

A quadratic inequality looks like $x^2 + x - 6 < 0$.

Procedure for solving quadratic (& higher) inequalities

1. Write the inequality as an equation and solve – break points
2. Use the break points to divide the number line into regions
3. Test a point in each region
4. Graph the solution
5. Write the solution in interval notation

7.1 Rectangular (Cartesian) Coordinate System
Analytic (coordinate) geometry – making connections between algebraic equations and graphs

Linear Equations have 2 common final forms
1. $Ax + By = C$ (General Form)
2. $y = mx + b$ (Slope-Intercept Form)

A solution of an equation is an ordered pair that gives a true equation.
Finding a solution of an equation:

Intercepts – separate points with 2 coordinates
x – intercept: Point where graph hits x-axis, where

y – intercept: Point where graph hits y-axis, where

To graph linear equations (lines)

- You need at least

- Generally, the easiest points to graph are

7.2 Graphing Linear Inequalities – 2 variables

Procedure - Graphing 2-dimensional inequalities

1. Graph the inequality as if it were an equation, keeping in mind
    a. Use solid line for $\geq$ or $\leq$
    b. Use dotted line for $>$ or $<$
2. Test a point on one side of the line. DO NOT choose a point on the line!
3. Shade on the true side

7.3 Distance and Slope

Distance Formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Slope – the slant or steepness of a line

$$\text{slope} = \frac{\text{vertical (y) change}}{\text{horizontal (x) change}}$$

$\text{slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are 2 points on a line

7.4 Determining the Equation of a Line

"Find the equation of the line that…"

Point – Slope Form – useful tool for intermediate work, but not for final answer

$(y - y_1) = m(x - x_1)$, where

- m = slope
- (x_1, y_1) = coordinates of a point
- x & y remain as variables

To get an answer in General Form ($Ax + By = C$):

1. Get rid of fractions
2. Get x and y terms on one side, number on other side
3. Get x positive

Graphing a Line using Slope-Intercept Form (anchor and count)

1. Use b to anchor a point on the y axis
2. Use m to determine how many vertical and horizontal units to count for another point.

Using y = mx + b to quickly find slope

1. Isolate y on the left
2. Look at the number in front of x. This is the slope.

Parallel and Perpendicular Lines

(Given 2 lines with slopes m_1 and m_2)

1. The lines are parallel if their slopes are

2. The lines are perpendicular if their slopes are

7.5 Graphing Non-Linear Equations
Non-linear equations – don't look like Ax + By = C
Some examples:

They tend to be unpredictable. Use symmetry to make efficient choices.

y-axis symmetry
- a mirror-image reflection with y-axis as mirror.
- replacing x with -x results in an equivalent equation

Ex a Which of the following are y-axis symmetrical?

Equation	Test	Sym?
$y = 3x + 2$		
$y = x^2 + 4$		
$y = x^2 + 2x + 4$		
$y = x^4 + 2x^2$		
$y = x^3$		

x-axis symmetry
- a mirror-image reflection with x-axis as mirror.
- replacing y with -y results in an equivalent equation

Ex b Which of the following are x-axis symmetrical?

Equation	Test	Sym?
$y = 3x + 2$		
$y = x^2 + 4$		
$y^2 = x + 4$		
$y = x^4 + 2x^2$		
$x = y^3$		

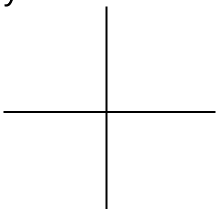
Origin symmetry
- reflected twice (about both x and y axes)
- replacing both x with -x and y with -y results in an equivalent equation

Ex b Which of the following are origin symmetrical?

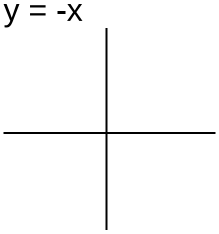
Equation	Test	Sym?
$y = x^2$		
$x = y^2$		
$y = 1/x$		
$y = x^3$		

Some basic graphs to memorize:

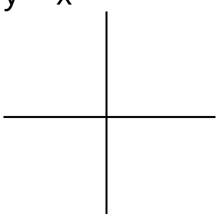
$y = x$



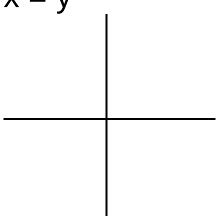
$y = -x$



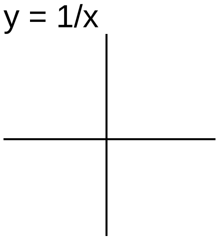
$y = x^2$



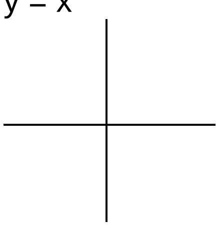
$x = y^2$



$y = 1/x$

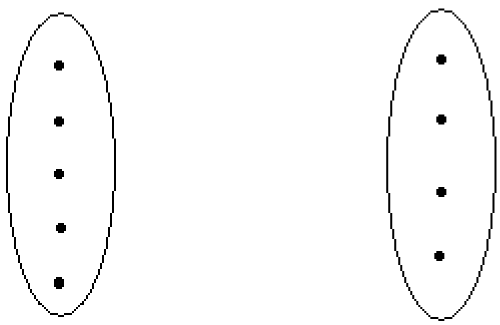


$y = x^3$



8.1 Concept of a Function

function – a relation connecting a set of inputs (x) to a set of outputs (y) where each input has one and only one output.
Note: It is OK for an output to be produced by more than 1 input.
Example – a vending machine



Notation
y = f(x) -- the value of a function for a certain value of x

Deciding if a relation is a function

1. Ordered pairs - No different outputs for same input

2. Graphs - Vertical line test - If any vertical line cuts the graph twice (or more), it's not a function

3. Equations – Odd powers test – If all powers of y are odd, it is a function

8.2 Linear Functions

Linear functions look like $f(x) = ax + b$

Some special linear functions:

Constant Function: $f(x) = k$, where k is a fixed number

Identity Function: $f(x) = x$

Ex a The profit earned from selling candy is represented by the linear function

$P(x) = 0.25x - 3$ where $P(x)$ is the profit, and x is the # of candies sold.

Piecewise functions - use different formulas for different regions

Ex A phone company charges $35/month, with 500 free minutes. After 500 minutes, $0.10 is charged for each additional minute.

8.3 Quadratic Functions

Linear functions (recall):

- $f(x) = ax + b$
- The shape is

- $f(x) = x$ is a "basic" graph through the origin. All other lines are the "basic" graph shifted and/or tilted

Quadratic functions:

- $f(x) = ax^2 + bx + c$ for $a \neq 0$
- The shape is

- $f(x) = x^2$ is a "basic" graph through the origin. All other parabolas are the "basic" graph shifted (vertically or horizontally), flipped, and/or stretched

Direction of Opening

For $f(x) = ax^2 + bx + c$

Vertical Translation

$f(x) = x^2 + k$ shifts $f(x) = x^2$ up by k units (down if k is negative)

Ex

Check: (long way)

x	f(x)
-2	
-1	
0	
1	
2	

Stretching/Squashing

$f(x) = ax^2$ makes $f(x) = x^2$ taller (narrower) for $|a| > 1$
shorter (wider) for $|a| < 1$

Ex

Check: (long way)

x	f(x)
-2	
-1	
0	
1	
2	

Horizontal Translation

$f(x) = (x - h)^2$ moves $f(x) = x^2$ to the right for h positive
to the left for h negative

Note: The subtraction symbol is built into the formula and is expected. So
$(x - 3)$ means h is positive
$(x + 3) = (x - (-3))$ means h is negative

Ex

Check: (long way)

x	f(x)
-2	
-1	
0	
1	
2	

In summary, to plot a parabola, we need:

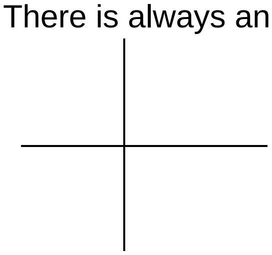
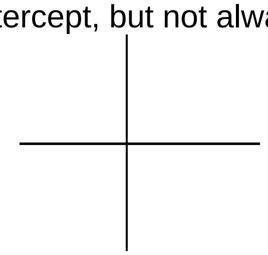
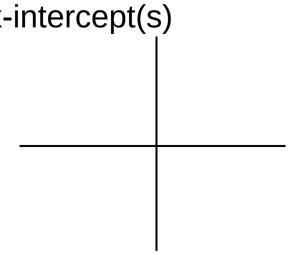
1. Direction of opening, up or down (look for sign of "a")
2. Vertical and horizontal translation (look at k = vertical shift and h = horizontal shift)
3. Width of parabola (plot one other point)

8.4 More on Quadratic Functions

Alternative way to graph a parabola

1. Find the direction of opening using the sign of "a"
2. Find the vertex coordinates (x, f(x)) using the formula
   a) Find x: $x = \frac{-b}{2a}$
   b) Find f(x) by plugging the value of x you found into the original equation
      (or if you prefer, use formula, $f(x) = \frac{4ac - b^2}{4a}$)
3. Graph another point to find width

Intercepts – good for extra points on parabola
f(x)-intercept (or y-intercept) - let x = 0
x-intercept(s) – let f(x) = 0
There is always an f(x) intercept, but not always x-intercept(s)

Maximum and Minimum -- Think:

Vertex formula: $x = \frac{-b}{2a}$

$f(x) =$

(sometimes f(x) is not even needed)

Ex A profit function is represented by the equation:
$P(x) = -x^2 + 120x - 2700$, where x is the number of items sold. Find

a) The # of items to sell to maximize the profit.

b) The maximum profit

8.5 Transformations of Curves

Curves to know

1. $f(x) = x$

2. $f(x) = x^2$

3. $f(x) = x^3$

4. $f(x) = x^4$

5. $f(x) = |x|$

6. $f(x) = \sqrt{x}$

7. $f(x) = 1/x$

Vertical Translation

$f(x) + k$ shifts $f(x)$ up k units

$f(x) – k$ shifts $f(x)$ down k units

Horizontal Translation

$f(x - h)$ shifts $f(x)$ right by k units

$f(x + h)$ shifts $f(x)$ left by k units

x-axis Reflection

$- f(x)$ reflects $f(x)$ across the x-axis (above/below) y-axis Reflection

$f(-x)$ reflects $f(x)$ across the y-axis (left/right)

Vertical Stretching

For $c > 1$, $c \cdot f(x)$ stretches $f(x)$ vertically

For $0 < c < 1$, $c \cdot f(x)$ shrinks $f(x)$ vertically

Successive Transformations

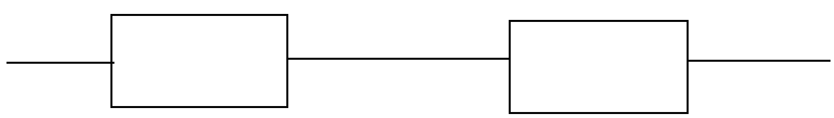
1. Do shrinking, stretching, and reflections first
2. Do translations (vertical and horizontal) last

8.6 Combining Functions

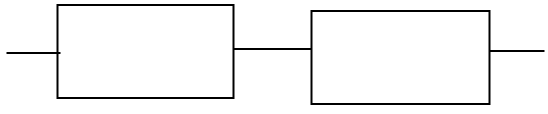
Addition, subtraction, multiplication, and division all work as expected

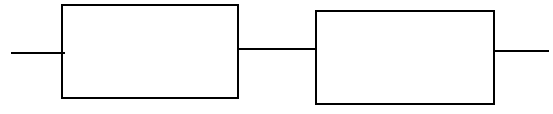
Composition of functions

Notation: $f \circ g(x) = f(g(x))$



Note: Composition is not necessarily commutative (order matters!)

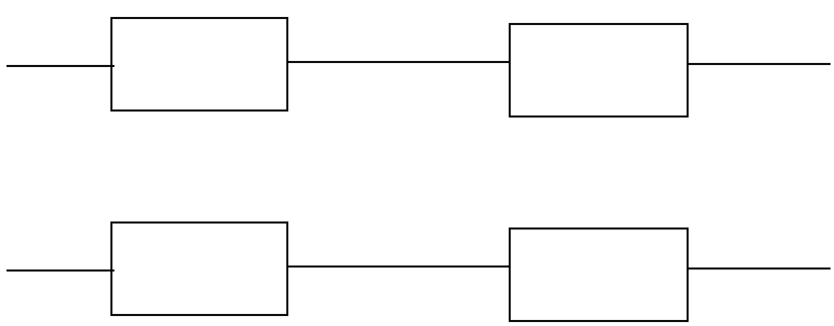




Inverse Functions

$f(x)$ and $g(x)$ are inverses if and only if (iff)

1) $f \circ g(x) = x$ and
2) $g \circ f(x) = x$



8.7 Direct and Inverse Variation

Direct relationship - 2 quantities go up together

Inverse relationship - one goes up as the other goes down

Goal: Get one of 2 formulas, and find a number value for k, to write final formula

- $y = kx$ (direct) OR
- $y = \frac{k}{x}$ (inverse)

Direct Variation

Typical procedure (not all parts are always asked for)

1. Write a generic equation, using "direct" or "inverse" to connect the quantities in the correct relationship. Use k, which represents "varies" or "proportional"
2. Find a number for k. Write a specific equation, exchanging the number for k in the previous equation.
3. Use the specific equation to find the new data point, plugging in the new numbers and solving.

10.1 Exponents and Exponential Functions

1. Product rule: $b^m \cdot b^n = b^{m+n}$
2. Power to a power: $(b^m)^n = b^{mn}$
3. Power of a product: $(ab)^n = a^n b^n$
4. Power of a quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
5. Quotient rule: $\frac{b^n}{b^m} = b^{n-m}$
6. Zero rule: $a^o = 1$
7. Negative exponent: $a^{-n} = \frac{1}{a^n}$
8. " " $\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$
9. Fractional exponent: $a^{1/n} = \sqrt[n]{a}$
10. " " $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Ex a

Equal Exponent Property

For $b>0$, $b \neq 1$, m and n real numbers

$b^n = b^m$ if and only if $n = m$

Exponential Function

$f(x) = b^x$ (b > 0, b $\neq$ 1 is the exponential function with base b

Ex Graph $f(x) = 2^x$

x	f(x)
-2	
-1	
0	
1	
2	

Ex Graph f(x) =

x	f(x)
-2	
-1	
0	
1	
2	

Note: When b> 1, f(x) is an increasing function
When 0<b<1, f(x) is a decreasing function

Transformations – similar to before

10.2 Applications of Exponential Functions

Basic formula:

$V = V_o b^t$, where V = current value, V_o = original value, b = base, t = time

Recall: Function is increasing if b >1, function is decreasing if b is a fraction

Exponential Decay/Depreciation (decrease)

Ex a If a car originally costs $20,000, how much is it worth after 1 year? After 3 years? After 10 years?

Half Life - The amount of time it takes for a value to decrease to half the original value.

From above, the half life is calculated:

Half Life Formula

$$Q = Q_o\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

, Q = current quantity, Q_o = original quantity, h = half life, t = time

Ex

Compound Interest - A type of growth (increase)

1. Compounded Annually
   $A = P(1 + r)^t$, where A = current amount, P = Principal (original amount), r = interest rate (% converted to decimal), t = time
2. Compounded More Than Annually
   $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where n = number of compoundings per year
3. Compounded Continuously
   $A = Pe^{rt}$, where e is the natural number ≈ 2.718

Law of Exponential Growth/Decay

If the base is stated explicitly, use that base. If no other base is explicitly stated, it is understood that the base is "e"

$Q(t) = Q_o e^{kt}$, $Q(t)$ = quantity that changes with time, Q_o = original quantity, $e \approx 2.718$, k = rate of growth/decay, t = time

Growth: If $k > 0$, $Q(t)$ grows exponentially over time

Decay: If $k < 0$, $Q(t)$ decays exponentially over time

Ex

When the base is "e" ≈ 2.718, the base is always >1. However, for different bases, decay can be caused when the base < 1.

Growth: If b

Decay:

10.3 Inverse Functions

2 kinds of relations:

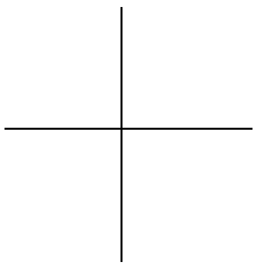
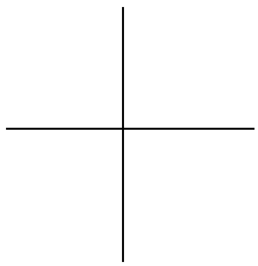
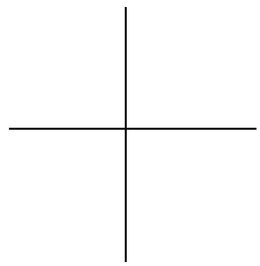
Function

1. Never 2 outputs for same input
2. Passes vertical line test – no vertical line cuts graph twice

One-to-One

1. Never has 2 inputs produce same output
2. Passes horizontal line test – no horizontal line cuts graph twice.

Examples of relations:

One-to-one functions have the properties

- Each f(x) has only one x associated with it, and each x has only one f(x)
- If $f(x_1) = f(x_2)$, then $x_1 = x_2$
- They are invertible (they have an inverse)

Inverse functions

The functions f(x) and g(x) are inverse functions if:

Ex

Finding the inverse of a function

We use the notation $f^{-1}(x)$ = inverse of f(x)

By definition of inverse function:

Procedure for finding $f^{-1}(x)$

1. Let y = f(x)
2. Exchange x and y
3. Isolate y
4. Replace y with $f^{-1}(x)$

Ex

Increasing and Decreasing Functions

1. f is increasing on an interval if $f(x_1) < f(x_2)$ when $x_1 < x_2$
2. f is decreasing on an interval if $f(x_1) > f(x_2)$ when $x_1 < x_2$
3. f is constant on an interval if $f(x_1) = f(x_2)$ for every x_1 and x_2

10.4 Logarithms

Sometimes we "create" operations to provide an inverse for an operation we already have (when we need to isolate x)

Classic example

For an exponents, how do we isolate x?

$e^x = 5$, $x = ?$

Try $e^1 = 2.718$

$e^2 =$

We create a new operation, the logarithm, to be the inverse of e^x

By definition:

Logarithms can be created for exponents of any base

Equivalent formulas:

Conversion of log form to exponent form

Ex a Convert the following to exponent form:

Conversion of exponent form to log form

Ex b Convert the following to log form

Properties of Logarithms

1. $\log_b b = 1$
2. $\log_b 1 = 0$
3. $b^{\log_b r} = r$
4. $\log_b b^r = r$

Equivalent Properties of Exponents

1. $b^1 = b$
2. $b^o = 1$

Ex

More Properties of Logs

5. $\log_b r \cdot s = \log_b r + \log_b s$
6. $\log_b \frac{r}{s} = \log_b r - \log_b s$
7. $\log_b r^p = p\log_b r$

Equivalent Properties of Exponents

5. $b^r b^s = b^{r+s}$
6. $\frac{b^r}{b^s} = b^{r-s}$

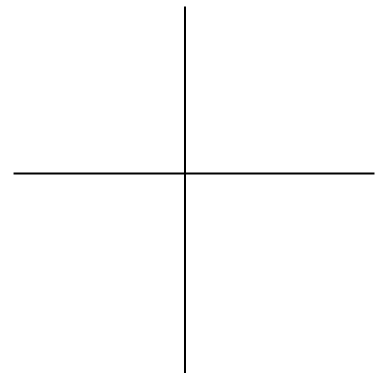
Ex

10.5 Logarithmic Functions

$f(x) = \log_b x$ is the log function with base b. It has a graph.

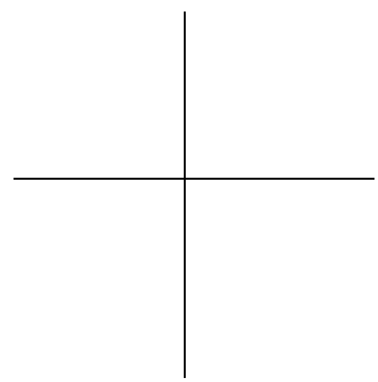
<u>Ex a</u> Graph f(x) =

x	f(x)



<u>Ex b</u> Graph f(x) =

x	f(x)



Are these 2 functions inverses?

Transformations of log functions – shifting and reflections still apply

1. $f(x) = -2 + \log_2 x$
2. $f(x) = \log_2(x – 3)$
3. $f(x) = - \log_2 x$
4. $f(x) = \log_2(-x)$

Common Logarithms – base 10

$\log_{10}x = \log x$ (by definition, when no base is shown, assume base 10)

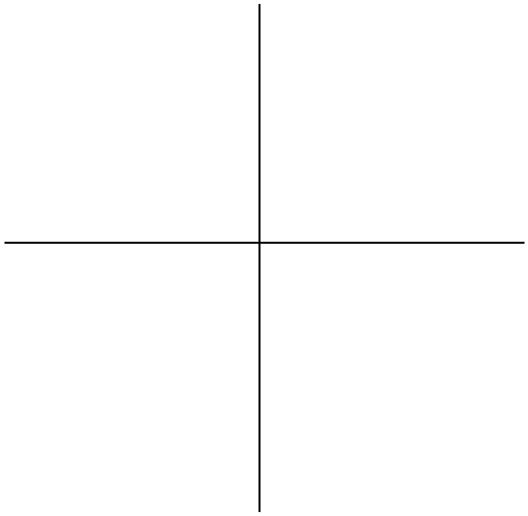
Ex

Natural Logarithms – base e

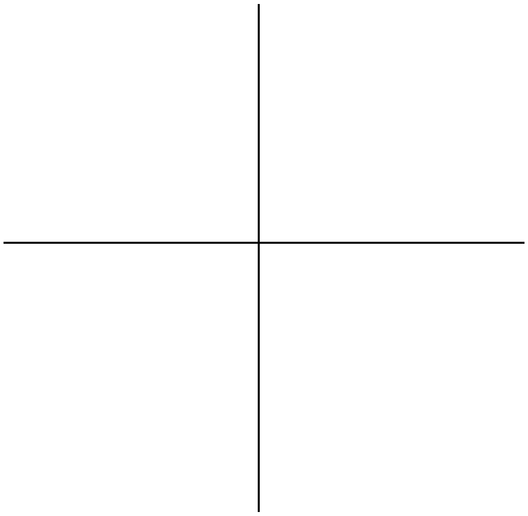
$\log_e x = \ln x$ (by definition) Recall: $e \approx 2.718$

Graphs of log functions

$f(x) = \log x$



$f(x) = \ln x$



10.6 Solving Exponential Equations and Log Equations

Logarithmic Property of Equality

$x = y$ and $\log_b x = \log_b y$ have same solution

Ex a

Solving Log Equations

Type A - Mix of log and non-log terms

1. Combine all log terms into one log expression. Get everything else on other side of equation.
2. You now have $\log_b r = t$. Convert the log equation into an exponent equation.
3. Solve.

Type B - All terms have logs

1. Get everything on left hand side into 1 log term.
2. Get everything on right hand side into 1 log term.
3. Set expressions inside the logs equal to each other
4. Solve

Richter Numbers

$R = \log \frac{I}{I_o}$ where I_o is the reference intensity

Ex A tsunami was estimated at 8.0 intensity. The actual intensity was 9.1. How many times greater was the actual wave than the estimated wave?

Simplifying with base other than 10 or e

Change of base property

$\log_a r = \dfrac{\log_b r}{\log_b a}$ (Choose as the new base)

More Applications

11.1 Systems of 2 Linear Equations with 2 Variables

Systems of 2 equations have 3 possible outcomes:

1. 2 lines intersect at 1 point

   The system: $y = 3x$
   $y = -x + 4$

2. 2 lines never intersect

   The system: $y = x - 1$
   $y = x + 3$

3. 2 lines lie on top of each other

   The system: $y = x$
   $3y = 3x$

Procedure for Substitution method

1. Isolate one variable in one equation
2. Plug that variable into other equation
3. Solve for the remaining variable
4. Solve for the other variable. Write as an ordered pair
5. Check (optional)

Elimination by Addition Method

1. Write both equations in general form $Ax + By = C$
2. Choose which variable to eliminate
3. Make coefficients have same amount, opposite sign, by multiplying one or both equations
4. Add both equations to eliminate one variable
5. Solve for remaining variable
6. Solve for other variable. Write ordered pair
7. Check

11.2 Systems of 3 Linear Equations – 3 variables

An example of an equation in 3 variables:

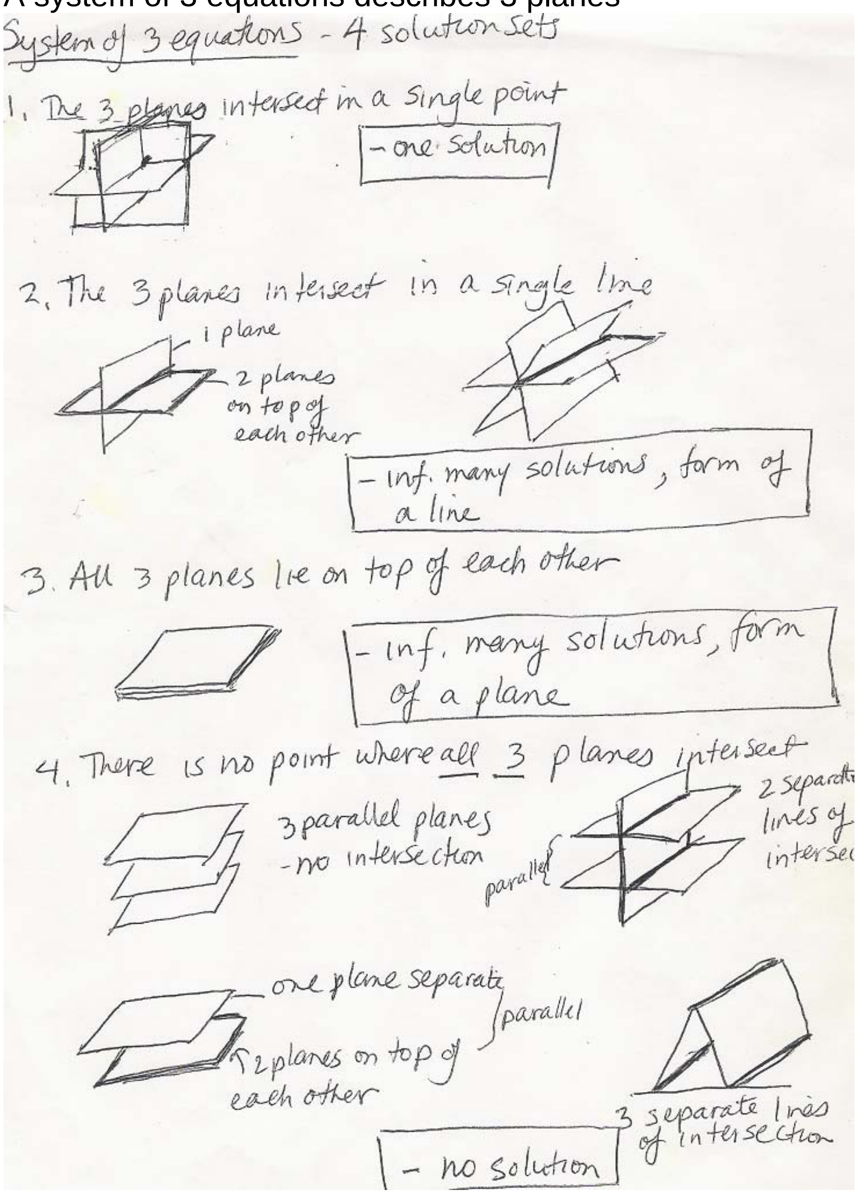
This describes a plane, not a line.

A solution to this is an ordered

All solutions lie on the plane.

Some examples of solutions to this plane:

A system of 3 equations describes 3 planes



We usually hope for the first case. Goal: Solve for x, y, and z

Elimination by Addition

1. Choose which variable to eliminate - use an equation with coefficient = 1
2. Multiply that equation and add to another row
3. Replace the added to row with the new coefficients

11.3 Matrix Approach to Solving Linear Equations

A matrix is an array of numbers in rows and columns

Any system of equations can be represented as an augmented matrix.

$2x - 3y + 4z = 10$

$5y - 2z = -16$

$3z = 9$

Note: It's much easier to solve a system with 3 zeroes below the diagonal. Our goal: Get matrix in one of 2 forms:

Triangular Form

Reduced Row-Echelon Form

To get zeroes where we want, we can do the following operations without changing the solutions:

1. Exchange rows
2. Multiply or divide any row by a real number
3. Replace any row by adding a multiple of another row to that row

Ex a Solve the system

Solving a 3-D System Using Triangular Form

Goals: 1) Get 1's on the diagonal
2) Get zeroes in every spot below the diagonal

Procedure:

1. Get "1" in position row 1, col 1 (r1c1). Divide by coefficient if necessary.
2. Use the 1 to put zeroes below the diagonal in col 1.
3. Get "1" in r2c2. Divide if necessary.
4. Use the 1 to put zeroes in below the diagonal in col 2.
5. Get "1" in r3c3. Divide if necessary
6. Solve for z.
7. Plug z into Equation 2, and solve for y.
8. Plug z and y into Equation 1, and solve for x.

Example: Solve the system:

$$\begin{aligned} x + y + 6z &= -9 \\ 3x + 5y - 8z &= 23 \\ 5x + y + 3z &= 13 \end{aligned}$$

Form the matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & -9 \\ 3 & 5 & -8 & 23 \\ 5 & 1 & 3 & 13 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & -9 \\ 0 & 2 & -26 & 50 \\ 0 & -4 & -27 & 58 \end{array}\right] \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & -9 \\ 0 & 1 & -13 & 25 \\ 0 & -4 & -27 & 58 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & -9 \\ 0 & 1 & -13 & 25 \\ 0 & 0 & -79 & 158 \end{array}\right] \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & -9 \\ 0 & 1 & -13 & 25 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Solve the system:

$z = -2$
$y - 13z = 25$; $y - 13(-2) = 25$; $y = -1$
$x + y + 6z = -9$; $x - 1 + 6(-2) = -9$; $x = 4$

Solution: $(4, -1, -2)$

Solving a 3-D System Using Row-Echelon Form

Goals: 1) Get 1's on the diagonal
3) Get zeroes in every spot below the diagonal
4) Get zeroes in every spot above the diagonal

Procedure:

1. Get "1" in position row 1, col 1 (r1c1). Divide by coefficient if necessary.
2. Use the 1 to put zeroes below the diagonal in col 1.
3. Get "1" in r2c2. Divide if necessary.
4. Use the 1 to put zeroes in below the diagonal in col 2.
5. Get "1" in r3c3. Divide if necessary
6. Use the "1" in r3c3 to put zeroes in col 3 above diagonal
7. Use the "1" in r2c2 to put zeroes in col 2 above diagonal
8. x,y, and z appear naturally on the RHS of the augmented matrix

Example: Solve the system:

$$\begin{aligned} x + y + 6z &= -9 \\ 3x + 5y - 8z &= 23 \\ 5x + y + 3z &= 13 \end{aligned}$$

Form the matrix:

$$\begin{bmatrix} 1 & 1 & 6 & | & -9 \\ 3 & 5 & -8 & | & 23 \\ 5 & 1 & 3 & | & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 6 & | & -9 \\ 0 & 2 & -26 & | & 50 \\ 0 & -4 & -27 & | & 58 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 6 & | & -9 \\ 0 & 1 & -13 & | & 25 \\ 0 & -4 & -27 & | & 58 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 6 & | & -9 \\ 0 & 1 & -13 & | & 25 \\ 0 & 0 & -79 & | & 158 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 6 & | & -9 \\ 0 & 1 & -13 & | & 25 \\ 0 & 0 & 1 & | & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & | & 3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

Solution is x = 4, y = -1, z = -2 (4,-1,-2)

11.4 Determinants

For a 2 X 2 matrix called A

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ has a determinant

det A = |A| = $a_{11}a_{22} - a_{21}a_{12}$

Note the difference between a matrix and a determinant:
A matrix is

A determinant is

Ex a

Expansion and calculation of a 3 X 3 determinant

A determinant can be expanded along any row or column.
It is strategic to choose the row or column with the most zeroes.
Each term the product of the following parts:

1. The element, a_{ij} (row I, column j)
2. The sign
3. The minor, m_{ij}

How to calculate the sign

How to calculate the minor

Properties of Determinants (different from matrices)

1. If any row or column has only zeroes, $|A| = 0$
2. Exchanging any 2 rows or columns changes the sign of the determinant
3. If any 2 rows or columns are identical, $|A| = 0$
4. Multiplying (or dividing) any row or column by k multiples (or divides) the determinant by k
5. Adding the product of k times any row to another row does not change the determinant

11.5 Cramer's Rule

(2 dimensional) Given the system:

$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

D =

D_x =

D_y

The system has the solution $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$

(3-dimensional case) Given the system

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

Similar to before,
$D =$ $D_x =$

$x = D_x/D$

Chapter 13 – Conic Sections (quadratics with x^2, y^2, or both squared)

1. Circles
2. Parabolas
3. Ellipses
4. Hyperbolas
5. Solving systems (multiple types)

13.1 Circles

Equation for circle with center at origin:

distance from points (x,y) to origin:

Standard form of circle with center at (h,k):

General form: $x^2 + y^2 + Dx + Ey + F = 0$

Other descriptions that tell us h, k, and r

1. A <u>tangent</u> to a circle is a line touching the circle at one point. It touches the circle at the center and is perpendicular to a line drawn from the center to the point.
2. The <u>perpendicular bisector</u> of a chord across a circle contains the center
3. Any 3 points not on a straight line (<u>non-collinear</u>) determine a circle.

Ex

13.2 Parabolas

A new definition of parabola - the set of all points such that each point is equidistant from a fixed point, F (the focus) and a fixed line, d (the directrix).

Formulas for parabolas with vertex at origin

1. Vertical

2. Horizontal

Formulas for parabolas with vertex at (h,k)

1. Vertical

2. Horizontal

Finding Vertex, Focus & Graph of a Parabola (Procedure)

1. "Package up" the equation to look like:

   $(y - k)^2 = 4p(x - h)$ OR

   $(x - h)^2 = 4p(y - k)$
2. Find the vertex and plot it.
3. Find p
4. Decide direction of opening
5. Use p to place the focus (count p units from the vertex). Get the coordinates.
6. Plot another point to get the width.

13.3 Ellipses

An ellipse has 2 foci (F_1 and F_2).

Ellipse (definition) - The set of points whose sum of distances from F_1 and F_2 is constant.

Standard form of ellipse (center at origin):

Horizontal Ellipse

- Has vertices at (a, 0), (-a, 0)
- Has semivertices at (0, b), (0, -b)
- major axis length = 2a
- minor axis length = 2b
- foci at (c, 0) (-c, 0)
- $c^2 = a^2 – b^2$

Vertical Ellipse

- Has vertices at (0, b), (0, -b)
- Has semivertices at (a, 0), (-a, 0)
- major axis length = 2b
- minor axis length = 2a
- foci at (0, c) (0, -c)
- $c^2 = b^2 – a^2$

Ex

13.4 Hyperbolas

Hyperbola - The set of points whose difference of distances from 2 foci, F_1 and F_2 is constant.

Standard form of a Horizontal Hyperbola with Center at Origin (Transverse axis on x-axis):

- a and b are used to draw an imaginary box
- asymptotes are lines drawn thru diagonals (we won't do formulas of these)
- Vertices at (a, 0), (-a, 0)
- Foci at (c, 0) , (-c, 0)
- $c^2 = a^2 + b^2$

Standard form of a Vertical Hyperbola with Center at Origin (Transverse axis on y-axis):

- Vertices at (0, b), (0, -b))
- Foci at (0, c) , (0, -c)
- $c^2 = a^2 + b^2$

Quick trick for telling if a hyperbola is vertical or horizontal:

Hyperbolas with center at (h, k)

Summary of Shapes

Shape	Standard Form ("useful" form)	Landmarks	General Form ("pretty" form)
	$y = mx + b$ Ex: $y = 3x + 5$		$Ax + By = C$ $3x - y = 5$
	$4p(y - k) = (x - h)^2$ Ex: $4(y - 3) = x^2$		$Ax^2 + Cx + Dy + E = 0$ $x^2 - 4y + 12 = 0$
	$4p(x - h) = (y - k)^2$ Ex: $8(x - 3) = (y - 2)^2$		$By^2 + Cx + Dy + E = 0$ $y^2 - 4y - 8x + 28 = 0$
	$(x - h)^2 + (y - k)^2 = r^2$ Ex: $(x+ 4)^2 + (y - 6)^2 = 25$		$x^2 + y^2 + Cx + Dy + E = 0$ $x^2 + y^2 + 8x - 12y + 43 = 0$
	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ Ex: $\frac{(x-2)^2}{4} + \frac{y^2}{25} = 1$		$Ax^2 + By^2 + Cx + Dy + E = 0$ (coefficients of x^2 & y^2 same sign) $25x^2 + 4y^2 - 100x + 8y - 96 = 0$
	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Ex: $\frac{x^2}{9} - \frac{(y+1)^2}{4} = 1$		$Ax^2 - By^2 + Cx + Dy + E = 0$ (coefficients of x^2 & y^2 opposite signs) $4x^2 - 9y^2 - 18y - 45 = 0$
	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ Ex: $\frac{(y+4)^2}{4} - \frac{(x-1)^2}{4} = 1$		$By^2 - Ax^2 + Cx + Dy + E = 0$ (coefficients of x^2 & y^2 opposite signs) $x^2 - y^2 - 2x - 8y - 11 = 0$

Identifying Quadratic (and Linear) Equations

For each equation, tell what shape it is (line, circle, parabola, ellipse, hyperbola).

1. $\frac{(y-3)^2}{4} - \frac{x^2}{9} = 1$ ________________________

2. $x^2 - 6x + 4y^2 + 8y - 4 = 0$ ________________________

3. $x^2 - 6x - 4y^2 + 8y - 4 = 0$ ________________________

4. $x^2 - 8x + y^2 + 2y + 5 = 0$ ________________________

5. $x - y = 9$ ________________________

6. $x^2 - y = 9$ ________________________

7. $x^2 - y^2 = 9$ ________________________

8. $x^2 + y^2 = 9$ ________________________

9. $x^2 + 9y^2 = 9$ ________________________

10. $\frac{(y+1)^2}{16} + \frac{(x-2)^2}{9} = 1$ ________________________

11. $4x^2 + 8y^2 = 32$ ________________________

12. $x = 16y^2 - 32$ ________________________

13.5 Solving Non-linear Systems – 2 variables

2 types of methods of solving systems

1. Graphing – gives a general impression of solutions
2. Algebraic methods: Substitution or Elimination – give precise solutions

Both types will be required for this section

14.1 Arithmetic Sequences

A sequence is a string of numbers that follow a pattern

Arithmetic Sequences

An arithmetic sequence has a common difference between terms (number added to each term to get to the next term). We call this difference "d".

Formula for the general term:

Sums of Arithmetic Sequences - The Karl Gauss Problem (1784)

Formula for the sum of an arithmetic sequence:

14.2 Geometric Sequences

A geometric sequence has a common ratio between terms (number multiplied by each term to get the next)

Formula for the general term of a geometric sequence:

Sum of Geometric Sequences

Formulas for the sum of a geometric sequence:

Sum of an Infinite Sequence

Some sequences add up to a fixed number as n approaches infinity. Even though there are an infinite number of terms, they add up to a finite number

Formula for the sum of an infinite sequence:

Repeating Decimals (how to convert to fractions)

A terminating decimal can be converted by a fraction by dividing the number by 10^n, where n is the number of decimal places.

To convert repeating decimals, use multiply method or infinite sequences.

14.3 More Problem Solving

We will use the formulas for sequences to solve word problems

Arithmetic: $a_n = a_1 + (n - 1)d$

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Geometric: $a_n = a_1 r^{n-1}$

$$Sn = \frac{a_1 r^n - a_1}{r - 1} = \frac{a_n r - a_1}{r - 1}$$

$$S_\infty = \frac{a_1}{1 - r}$$

2 important questions to ask before you begin a word problem

Chapter 15 – Combinatorics, Counting, Binomial Theorem

15.1 Fundamental Principle of Counting

Fundamental Counting Principle

The number of ways 2 independent tasks can be accomplished (in sequence) is xy, where x = number of ways of accomplishing the first task

y = number of ways of accomplishing the second task

15.2 Permutations & Combinations

Factorials

1!

2!

3!

4!

n!

Note: 0! =

Permutations - ordered arrangements

Ex

P(n,r) – ways of ordering n elements r ways

P(n,r) = n(n-1)(n-2)…..

Note: $P(n, r) = \frac{n!}{(n-r)!}$

Combinations - unordered choosings (subsets)
C(n, r) – ways of choosing r unordered elements from n possible elements

$$C(n,r) = \frac{n!}{(n-r)!\ r!} = \frac{P(n,r)}{r!}$$

C(n,r) is also written as $\binom{n}{r}$ and called "n choose r", so

$$\binom{n}{r} = \frac{n!}{(n-r)!\ r!}$$

Choosing multiple groups from the same set:

$$\frac{n!}{r_1!\ r_2!\ r_3!\ \ldots\ r_i!}$$

15.6 Binomial Theorem

Taking a binomial to a power:

$(x + y)^0 =$

$(x + y)^1 =$

$(x + y)^2 =$

$(x + y)^3 =$

$(x + y)^4 =$

Patterns:

1. For each polynomial powers of x

   powers of y

2. Coefficients (numbers)

How do we get the coefficients?

Pascal's Triangle

Binomial Coefficients

$\binom{n}{r}$ = the binomial coefficient

Binomial Theorem

$(x+y)^n =$