Math 210 Final Project

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Math and Mondrian

The Golden Ratio, often referred to as Phi, is one of the most ubiquitous irrational numbers known to man. The properties of the Golden Ratio are found if a line is divided into two segments in such a way that the ratio of the total length to the length of the longer segment is the same as the ratio of the length of the longer segment to the length of the shorter segment. Numerically, this ratio equals $(1+\sqrt{5})/2 \approx 1.618$. The ancient Greeks were the first to believe that the Golden Ratio was the most aesthetically pleasing, and we have seen many examples of the ratio in art, architecture, music, and in nature. Whether or not the Golden Ratio's appearance in our world is by chance or by man's necessity to declare order in the world by assigning specific numbers that make sense to him, its ubiquity is unquestioned. The works of Piet Mondrian, a Dutch painter of the early 20th century, have been scrutinized for these reasons pertaining to the Golden Ratio. Mondrian heavily contributed to the De Stijl ("The Style") art movement and created a non-representational form, which he termed Neo-Plasticism. His works are primarily composed of simple geometric shapes, primarily rectangles, and primary colors. The Golden Rectangle often appears in his paintings; the height-to-width ratios of specific rectangles approximately equal 1.618. It is true that Mondrian's works exhibit rectangular figures with ratios near the Golden Ratio. However, the debate as to whether or not the artist considered the ratio in any conscious way is left unsettled, and provides a context for a mathematical analysis for an idea that is seemingly not mathematically testable. This paper will examine Mondrian's work from a more mathematical perspective, and will use probability theory to gain deeper insight into the reasons for the Golden Ratio's existence in his art.

One Mondrian painting particularly piqued our interest – *Composition with Gray and Light Brown.* The painting is comprised of 124 different rectangles, which include both individual rectangles and larger rectangles made up of smaller individual rectangles. To ensure we counted correctly, we assigned a number to each individual rectangle, and then recorded the larger conglomerate rectangles by the numbers of the smaller rectangles within it. We then proceeded to measure the heights and widths of all 124 rectangles, enabling us to calculate all height-to-width ratios, width-to-height ratios, and the natural logs of both of these ratios. We calculated the natural logs of these ratios because we recognized that the natural log of any number is equal to the negative of the natural log of its multiplicative inverse – allowing for a more symmetric distribution function if there were in fact peaks at certain values.

We created two of our own random Mondrian paintings, complete with simple rectangular figures. It is evident that there were not nearly as many total rectangles in our random Mondrian painting compared to the actual one. Nonetheless, we followed the same procedure as previously described, calculated the same values, and graphically constructed distribution functions.

After having found our ratios (and the logs of those ratios, to help clarify our findings), we need a probability density curve to describe what distribution we *should* see for a truly random Mondrian-esque painting:

Lets model a “random” Mondrian painting with a square of side-length one, in which vertical and horizontal lines are drawn at random. If we pick two vertical lines defined by the points (x_1, x_2), and two horizontal lines defined by (y_1, y_2) within the square of side 1, we can define a rectangle with width x_2-x_1 and height y_2-y_1. The ratio of height to width of an rectangle within (and including) the unit square in which we are drawing these lines is therefore:

$$r=\frac{|y_2-y_1|}{|x_2-x_1|}$$

This is the information we got from taking measurements of our various images. We then took the log (ln(r)) of r, to make our ratio-plots more symmetric and more easily comparable.

This information is not sufficient, alone; we need to find the probability density function, $f_{\ln(r)}(u)$, of the logs of our ratios.

To find the probability density function of the logs of our ratios, we must first find the probability functions for the individual height ($|y_2$-$y_1|$) and width($|x_2$-$x_1|$) of a randomly chosen rectangle.

The distribution function describing the width of the rectangle, $F_x(t)$ is the probability that the width($|x_2$-$x_1|$) is less than or equal to t. Each uniformly distributed random variable(x_2, x_1, y_2, y_1) that defines the height and width of the rectangle has a density function $f(t)$, which is 1 if $t\in[0,1]$ and 0 otherwise. The joint probability density function for x_2 and x_1, $f_{x1,x2}(t_1,t_2)$, is therefore also 1 for $0 \leq t_2$, $t_1 \leq 1$ and 0 otherwise. So $F_x(t)$ is all (t_1, t_2) $t\in[0,1]$ whose difference is less than t; all widths in the “canvas” of our Mondrian painting whose length is less than a given t. Looking at our canvas, we can model the area covered by the lines $\leq t$ as the complement to the area covered by two equilateral right triangles of height (1-t) at the corners (0,1) and (1,0) of the canvas. This leaves us with a strip of a constant width of *t* that spans the diagonal of the canvas. The area of the two triangles is 2*½ $(1-t)^2$, making the area of the strip left on the canvas:

$$1-(2*½ (1-t)^2) = 1- (1-t)^2 = 2t-t^2.$$

This describes the probability of finding a rectangle with width *t*($F_x(t)$), which is the area under the probability density curve describing the widths of all the rectangles on the canvas, $f_x(t)$. To find $f_x(t)$, we need to take the derivative of $F_x(t)$:

$$f_x(t) = \frac{d}{dt} F_x(t) = 2-2t$$

The density function for the height($|y_2-y_1|$), $f_y(t)$ is the same as $f_x(t)$, making the joint density function for a given height and width:

$$h_{(x,y)}(t_1, t_2) = f_x(t_1) * f_y(t_2) = (2-2t_1)*(2-2t_2)$$

where the function is 1 if $0 \le t_2, t_1 \le 1$, and 0 otherwise.

We can now use this joint density function to find the density function for a specific ratio of height/width, r:

$$F_r(w) = Prob(y/x \leqslant w) = Prob(y \leqslant xw) = \int_{A(w)} h_{(x,y)}(t_1, t_2)\, dt_1\, dt_2 = \int_{A(w)} (2-2t_1)(2-2t_2)\, dt_1\, dt_2$$

where A(*w*) is the subset of points on the canvas for which the height is less than or equal to the width multiplied by the given ratio ($t_2 \le wt_1$).

Lets first consider the situation in which the ratio is less than or equal to 1; in this case, A(*w*) is bounded by the t_1 axis, $t_2 = wt_1$ and the line $t_1 = 1$:

$$F_r(w)=\int_{t_1=0}^{t_1=1}\int_{t_2=0}^{t_2=wt_1}(2-2\mathrm{t}_1)(2-2\mathrm{t}_2)\,dt_2\,dt_1$$
$$\int_{t_1=0}^{t_1=1}(2-2\mathrm{t}_1)\left(\int_{t_2=0}^{t_2=wt_1}(2-2\mathrm{t}_2)\,dt_2\right)dt_1$$
$$\int_{t_1=0}^{t_1=1}(2-2\mathrm{t}_1)(2wt_1=w^2t_1^2)\,dt_1$$
$$\int_{t_1=0}^{t_1=1}(4wt_1-2w^2t_1^2-4wt_1^2+2w^2t_1^3)\,dt_1$$
$$2w-\frac{2w^2+4w}{3}+\frac{w^2}{2}$$
$$\frac{2w}{3}-\frac{w^2}{6}\,if\ 0\leqslant w\leqslant 1$$

When w ≥ 1, $F_r(w)$ = Prob($y/x \le w$) = 1- Prob($x/y < w^{-1}$) =

$$F_r(w^{-1})=\frac{2w^{-1}}{3}-\frac{w^{-2}}{6}\,if\ w\geqslant 1.$$

Showing:

$$F_r(w)=1-\left(\frac{2w^{-1}}{3}-\frac{w^{-2}}{6}\right)=1-\frac{2w^{-1}}{3}+\frac{w^{-2}}{6}\,if\ w\geqslant 1.$$

If we replace the ratio of height to width with its log, for symmetry's sake, we end up with u = ln(w), and $e^u = w$. If we now replace w in our equations, we end up with:

$$\frac{e^{2u}}{6}\,of\ w=e^u\,lel$$
$$F_{\ln(r)}(u)=\frac{2w}{3}-\frac{w^2}{6}=\frac{2e^u}{3}-¿$$

and

$$F_{\ln(r)}(u)=1-\frac{2w^{-1}}{3}+\frac{w^{-2}}{6}=\frac{2e^{-u}}{3}+\frac{e^{-2u}}{6}\,of\ w=e^u\geqslant 1$$

If we now take the derivative of $F_{\ln(r)}(u)$, we get the density function for the logs of the ratios (ln(r)), which is what we've been looking for:

$$f_{\ln(r)}(u)=\frac{2e^{-|u|}}{3}-\frac{e^{-2|u|}}{3}$$

We can model what the probability of any given ratio of height to width of random rectangles in a Mondrian *should* be for a “random” Mondrian that is a unit square, using this function.

What about rectangular Mondrian paintings (aka most Mondrian paintings)? Lets suppose we now have a rectangle for a canvas, in which the height, R, is >0 (instead of 1). To translate what we've found for the unit square to this rectangle, we can use the map $(x, y) \to (x, Ry)$, replacing r with Rr. We can now adjust the random Mondrian painting's density function to satisfy:

$$\int_{w=a}^{w=b} f_{\ln(Rr)}(t)\,dt = Prob(a \leqslant \ln(Rr) \leqslant b)$$

$$Prob(a - \ln(R) \leqslant \ln(Rr) \leqslant b - \ln(R))$$

$$\int_{w=a-\ln(R)}^{w=b-\ln(R)} f_{\ln(r)}(t)\,dt$$

implying

$$f_{\ln(Rr)}(t) = f_{\ln(r)}(t - \ln(R)) \; for\, all\, t$$

if we apply this to our original probability density function, we arrive at:

$$f_{\ln(Rr)}(u) = \frac{2e^{-|u-\ln(R)|}}{3} - \frac{e^{-2|u-\ln(R)|}}{3}$$

which can be used as a model of what distribution of rectangle ratios we *should* see in a rectangular Mondrian painting, if the ratios are truly random.

Determining the Significance of Our Results:

Our mathematical analysis gives us clear evidence that Mondrian had a natural inclination towards the Golden Ratio in his painting. The natural logs of the height-to-width ratios in his painting peak at approximately -.5, and the natural log of the Golden Ratio is approximately -0.48. To further support Mondrian's intentional use of the Golden Ratio, our findings from our random Mondrian paintings show that we have peaks at approximately –1 and 0.8: not nearly as close to the intended value of +/-0.48. Our derived probability distribution function, used to model the probability that a random rectangle has a particular height-to-width ratio, predicts that the peak in height-to-width ratios should lie at the value of the height-to-width ratio of the entire canvas. Therefore, the model predicts that the peak in Mondrian's actual painting should approximately equal -0.63757 (height-to-width ratio of Mondrian's canvas) – not consistent with our actual findings of -0.5.

The mean of the logs of the ratio in a random Mondrian painting with the same height to width ratio as our painting should be

$$\text{Ln}(R) = \text{Ln}(.5285) = -.6375.$$

The Standard Deviation for a painting with 124 rectangles is

$$\sqrt{(5/N)}/(2) = \sqrt{(5/124)}/(2) = .1004$$

The mean of the logs of the ratios that we calculated through our analysis of the actual Mondrian painting is -.1479.

We applied these values to the normal distribution curve to calculate the cumulative probability that given a certain standard deviation, a number would fall between the mean of the random Mondrian painting and mean that we calculated.

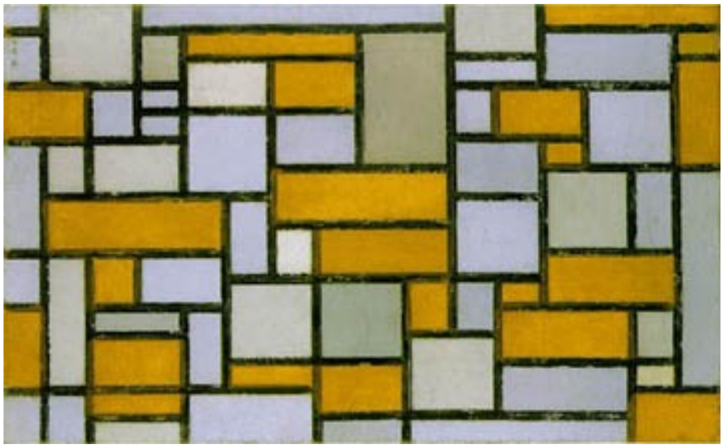
We got a very high probability of > .9997 which we had expected given that we were 4.876 standard deviations away from the expected mean. This number we had to multiply by a factor of 2 in order to account for the fact that half of the area normal curve was left out in our previous calculation.

This is our z- value.

We then put this value of 1.99994 in the equation $1 - \mathrm{erf}(z/\sqrt{2})$ to get the odds that something would be this far from the expected value. Our answer, .00467 or .467 % indicates that the value is significant. (any values less than 5% would be considered significant)

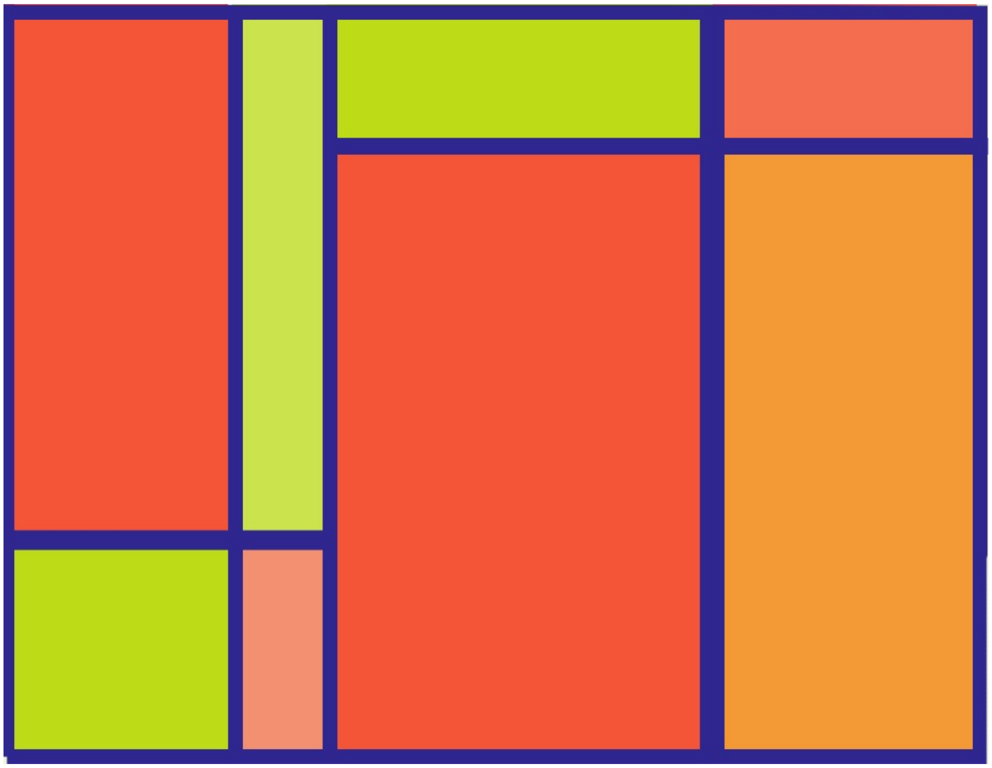
Based on this analysis, we can say with some degree of accuracy that Mondrian was not being random with his drawings due to the fact that the average ratio is so far from what the expected value should be. We can say that there is a 99.53% probability that he was not being random with the size of the rectangles in his work.

<u>Images and Figures</u>

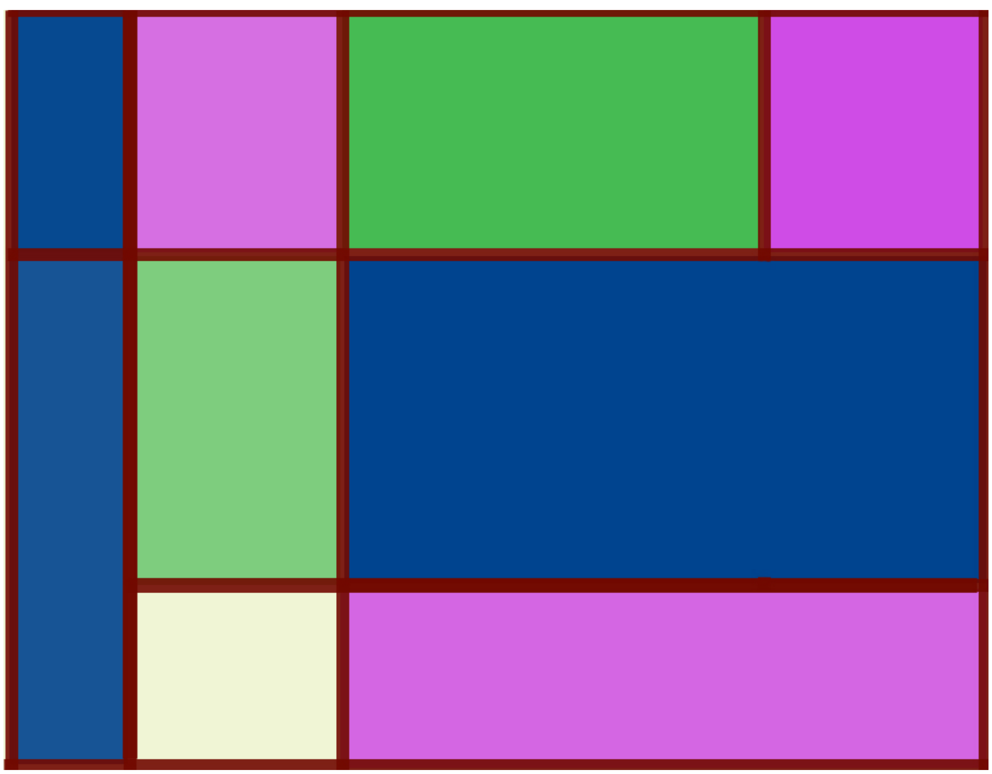


Composition with Gray and Light Brown

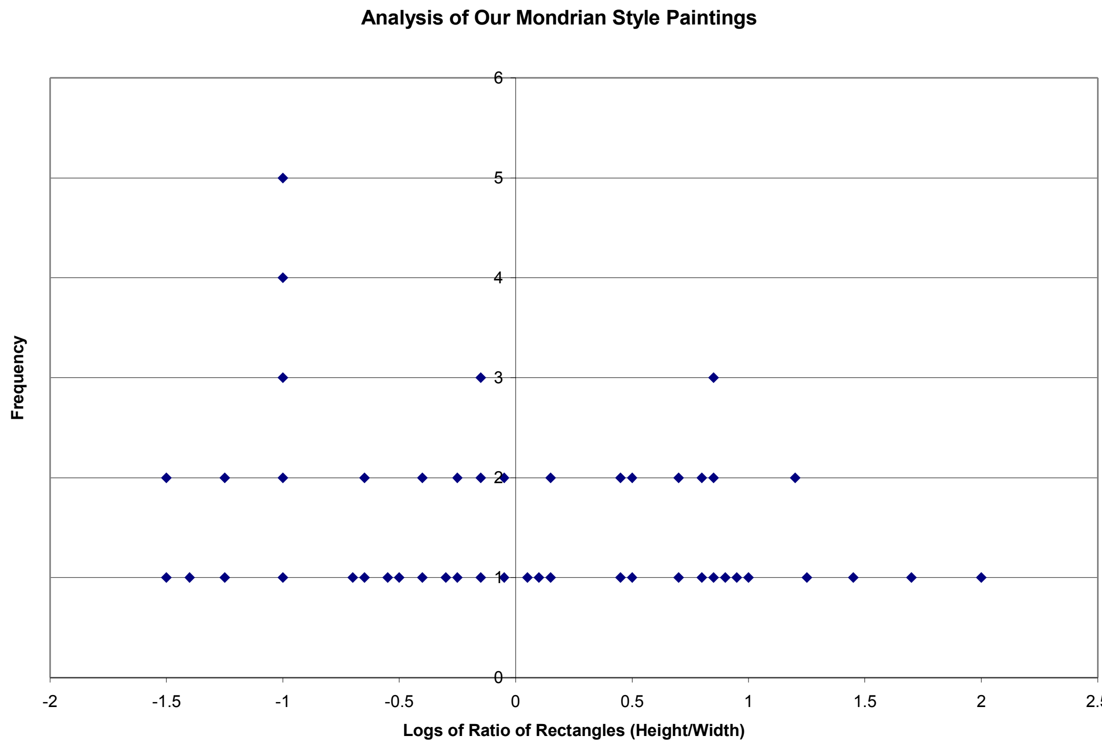
by Piet Mondrian



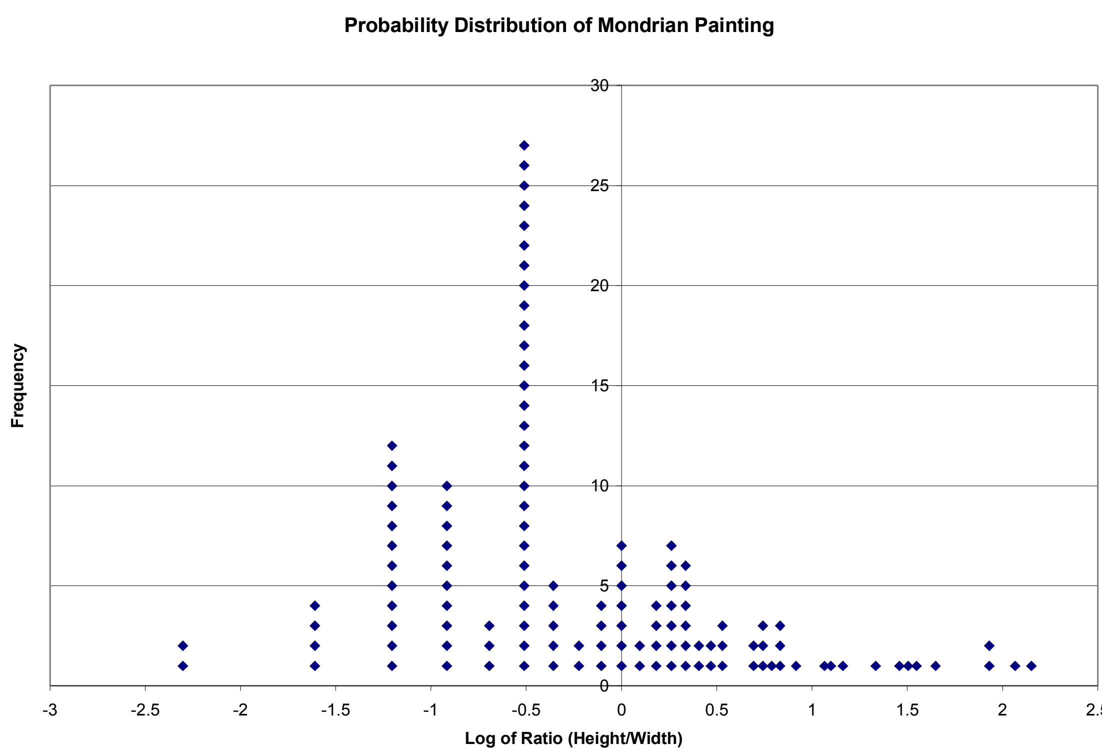
"Random" Mondrian 1



"Random" Mondrian 2



Analysis of Our Mondrian Style Paintings
Frequency
Logs of Ratio of Rectangles (Height/Width)
0
1
2
3
4
5
6
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5



Probability Distribution of Mondrian Painting
Frequency
Log of Ratio (Height/Width)
0
5
10
15
20
25
30
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5