MATH QUEST 5

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MATH QUEST 5

PREFACE

Welcome to Math Quest Level 5. As you work through the book, you will develop a foundation for problem solving in Logic, Algebra, Geometry, Counting, and Number Theory. The book began with a rich collection of problems developed primarily by Cody Patterson, a Mathworks student himself who taught in the program for many years. The problems were then collected and edited by Sam Baethge. Sam is a past head of the American Regions Math League Texas Team, and long time teacher in the Mathworks programs as well.

The authors, along with Cody and Sam, include math education graduate students Sonalee Bhattacharyya, Nama Namakshi, Christina Starkey, and Christina Zunker, who wrote content narratives and pedagogical guides to support the teaching, and Texas State Mathematics Department faculty Terry McCabe, Hiroko Warshauer, and Max Warshauer who supported the graduate students, provided content materials from their Math Exploration curriculum, and guided the project. We met as a team to develop explorations and activities that introduce the key concepts needed to work increasingly more challenging problems.

1 Logical Reasoning

SECTION 1.1 QUESTIONING TECHNIQUES

OBJECTIVES

- Develop techniques for asking good questions when problem solving
- Critically reason about problems and determine exactly what the problem is asking

This chapter is an introduction to logic, questioning techniques, and strategies as a way to think about mathematical problem solving. You will have opportunities to build connections through the flow of ideas and employ visual models to expand your reasoning skills.

1. A traveler comes to a fork in the road which leads to two villages. In one village the people always tell lies, and in the other village the people always tell the truth. The traveler needs to conduct business in the village where everyone tells the truth. A man from one of the villages is standing in the middle of the fork, but there is no indication of which village he is from. The traveler approaches the man and asks him one question. From the villager's answer, he knows which road to follow. What did the traveler ask?

2. Classmates Harry, Ignacio, Julia, Karen, and Leanne are comparing their heights. Place the students in order from tallest to shortest based on this information:

   The tallest student is a girl.

   Ignacio is shorter than all three girls.

   Harry is not the shortest student in the group.

   Leanne is not the tallest student in the group, but there are at least two students who are shorter than she is.

   Julia is the third-tallest student in the group.

3. A ferry boat driver must take a goat, a wolf, and a cabbage across a river. However, he can only haul one item at a time. Also, the wolf will eat the goat if the two are left alone together, and the goat will eat the cabbage if the two are left alone together. How can the ferry driver get all three across to the other side of the river safely?

4. Mrs. Langtree was tutoring two of her students, Richard and Selina, after school. When she noticed that her yardstick was missing she knew that one of them had taken it. She also knew that one of them always told the truth and the other always lied but couldn't remember who lied and who told the truth. She asked the students who had taken her yardstick and got these replies:

   Richard: Selina took the yardstick.

   Selina: The liar took the yardstick.

   Based on this information determine who told the truth and who took the yardstick.

5. Cody has five students in his class and each of them always tells the truth or always lies. Cody, frustrated, asks, "Isn't there anyone who won't lie to me?" He gets these replies:

   Alicia: None of us are truth-tellers.

   Brandin: Exactly one of us is a truth-teller.

   Candyce: Exactly two of us are truth-tellers.

   Denice: Exactly three of us are truth-tellers.

   Emily: Exactly four of us are truth-tellers.

   Who are the liars?

SECTION 1.2 STRATEGIES

OBJECTIVES

- Explore and develop varying strategies for solving problems

INTRODUCTORY ACTIVITY 1:

Break up into small groups of 3 or 4 students. Each group gets a single 10-sided die (or a set of cards Ace – 9). Each group should have five distinct (different) numbers either as a result of rolling their die or by picking 5 cards from their card deck. One person in the group records the five digits so that everyone can refer to these numbers. Using the numbers only once, each group must solve the problem below.

Using the letters A, B, C, D, and E represent the distinct digits that you found above, arrange the digits as ABC and DE such that the product is as large as possible.

1. Find a positive integer N such that:

   N is a four-digit number

   The units digit of N is twice the hundreds digit

   The tens digit of N is twice the thousands digit

   The thousands digit of N is twice the units digit

2. Suppose that we are given the numbers 7, 12, and 14. We multiply two of the numbers and then add the third. If the sum is a multiple of ten, what is it?

3. Melissa has a digital clock that gives the time in military (24 hour) format. When she goes to sleep the clock reads C:AB. When she awakens 11 hours and 27 minutes later it reads AB:BA. At what time did she go to sleep?

4. (Challenge) Find a ten-digit positive integer that has these properties: The left most digit is the number of times zero appears in the number, the next digit is the number of times one appears in the number, the next digit is the number of times two appears in the number, and so on until the last digit is the number of times nine appears in the number.

5. (Challenge) In a certain competition, each of six contestants competes in each of four events. For each event a judge awards points as follows: 10 points for first place, 6 points for second place, 3 points for third place and 1 point for fourth place. After the four events are completed the contestant with the highest point total is the winner.

   a. If there is no tie for first place what is the lowest possible score for the winner?

   b. If there is no tie for any place what is the lowest possible score for the winner?

6. (Challenge) On an eight-by-eight chessboard, how many squares of any size have a side that lies along the edge of the chessboard?

SECTION 1.3 DIVISIBILITY RULES

OBJECTIVES

- Understand divisibility rules and use them to solve problems
- Reason about problems and use known facts to reduce the number of possible solutions

Recall: Digits, Bases (esp. base 10), Military Time (24-hrs), Time Conversions (60 minutes in an hour, 24 hours in a day, etc.)

In this section, we use an important concept called divisibility that is related to but not exactly the same as division. As an example, consider the numbers 12 and 4. We say that 12 is divisible by 4 to mean that 12 can be written as 4 times an integer which in this case is 3. In other words, $12 = 4 \times 3$.

Another way of stating that 12 is divisible by 4 is to say that 4 divides 12. Notice that if we divide 12 by 4 we have a quotient 3 and a remainder 0. The important part of divisibility is that when we take two integers say n and d, with d not equal to zero, that n is divisible by d or that d divides n if when n is divided by d we get an integer quotient q and a remainder $r = 0$.

We would say that 12 is not divisible by 7 because 12 divided by 7 gives a quotient of 1 with remainder 5. In other words, we cannot write 12 as 7 times an integer.

Any number N can be written as multiples of the powers of 10.

$$\begin{aligned} ABCD &= A \cdot 1000 + B \cdot 100 + C \cdot 10 + D \\ &= ABC \cdot 10 + D \\ &= AB \cdot 100 + CD \\ &= A \cdot 1000 + BCD \end{aligned}$$

EXAMPLE:

Expand the number 5,234 using the powers of 10.

$$\begin{aligned} A \cdot 1000 + B \cdot 100 + C \cdot 10 + D &= ABCD \\ &= 5234 \\ 5 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + 4 &= 5234 \end{aligned}$$

Answer: Expanding 5,234 gives us $5 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + 4$.

Introductory Activity: What are all the divisibility rules and why do they work?

Divisibility Rules

By	Rule	Proof/Reasoning
2		
3		
4		
5		
6		

By	Rule	Proof/Reasoning
7		
8		
9		
10		

1. Find the smallest positive integer greater than 123456789 that is:
   a. divisible by 3.

   b. divisible by 4.

   c. divisible by 8.

   d. divisible by 9.

2. Find the greatest nine-digit positive multiple of 8 that contains each of the digits 1 through 9.

3. Find a three-digit integer ABC, where A, B and C represent different digits, such that $\frac{ABC}{CBA} = \frac{5}{6}$

2 Number Theory

SECTION 2.1 PRIMES AND NUMBERS OF DIVISORS

OBJECTIVES

- Discover and develop strategies for solving number theory problems dealing with divisors
- Articulate patterns students notice about products of divisors of natural numbers, numbers of divisors of perfect squares, and other number theory topics, with a focus on proper notation and terminology
- Use the formula for the number of divisors of a positive integer to solve related problems

Number theory focuses on examining properties and relationships of the positive whole numbers or the natural numbers. Many questions arise in number theory, some of which are still unanswered.

To solve the problems in this chapter, simple trial and error or guessing and checking may not be sufficient. If we have a method or strategy to help us, the problem may be done much quicker than trying to guess and check. We also try to generalize the patterns that we notice into theorems by asking questions such as, "Will this result hold true all the time? What are some conditions of this result being true?". Once we are confident that we have thought about all the possibilities, we write our findings, being careful to include all of the conditions and possibilities. This statement is called a theorem, and becomes accepted by other mathematicians when we can show them that it must be true.

Recall: Definition of Prime Numbers, Divisors of a Number, Factoring Natural Numbers, And Prime Factorizations

What makes a number a prime number? How would you explain the definition of a prime number to a friend? With your neighbor, write out a definition for prime numbers.

How are prime numbers related to factors and divisors of numbers? In this section, you will explore different properties of prime numbers and divisors.

1. The number 979 has a prime factor greater than 20. What is this factor? Explain the method you used to determine your answer.

2. How many distinct ordered triples (p, q, r) of positive prime numbers are there such that $p < q < r$, and $p + q + r$ is a prime between 20 and 30?

3. What is the product of the positive integer divisors of the number 30?

4. Which positive integers have an odd number of divisors? Is there a pattern?

NUMBERS OF DIVISORS

Activity: What are some ways in which you can determine the number of divisors 20 has? Write down at least two different ways to determine the number of divisors 20 has.

For these problems, we will investigate the question: How many positive integer divisors does a particular positive integer have? We will also explore how we can use the number of positive integer divisors to help us solve other problems.

One way to find the number of positive integer divisors of a natural number N is to find the **prime factorization** of N by writing N as a product of prime numbers raised to whole number exponents. We can give these prime numbers names, such as p_1, p_2, etc.

For example, we can find the prime factorization of 75 by writing $75 = 31 \times 52$. Let $p_1 = 3$ and $p_2 = 5$. We can also give names like a_1, a_2, etc. to the exponents used. Because 3 was raised to the 1st power, we can write $a_1 = 1$ and because 5 was raised to the 2nd power, we write $a_2 = 2$. We do this for each of the k possible prime factors of the number N.

Once we have written out our prime factorization and identified the primes and exponents, we are ready to calculate the number of positive integer divisors for any number N. The number of divisors of a number N is found by adding one to each exponent and then multiplying together the sums. Can you see why? Explain.

In other words:

The number of positive integer divisors of N is $(a_1 + 1)(a_2 + 1)\ldots(a_k + 1)$.

The prime factorization of 75 is $75 = 31 \cdot 52$, thus the number of positive integer divisors of 75 is $(1 + 1)(2 + 1) = 2 \times 3 = 6$.

How many positive integer divisors would the number 168 have?

5. Find the smallest positive integer that has exactly 9 positive divisors.

6. Find the smallest positive integer that has exactly 10 positive divisors.

7. a. There is only one integer less than 2004 that has exactly 25 positive divisors. What is it?

   b. How do the exponents for the positive integer in this problem compare to the exponents for the positive integer in #4? Do you notice a similarity? Why might this be?

8. Let N be the smallest positive integer that has exactly 100 positive divisors. How many digits does N have?

SECTION 2.2 DIVISIBILITY

OBJECTIVES

- Discover and develop strategies for solving number theory problems dealing with divisors, divisibility, and place value
- Apply concepts of divisibility rules and computing the sum of first n natural numbers from chapter 1 to solve problems
- Reason about problems and use known facts to reduce the number of possible solutions

Recall: Definition of divisibility, Divisors of a Number, Factoring Natural Numbers, Prime Factorizations

DEFINITION: DIVISIBILITY
Suppose that n and d are integers, and that d is not 0. The number n is divisible by d if there is an integer q such that $n = d \cdot q$. Equivalently, d is a factor of n and n is a multiple of d.

1. If M is a multiple of 11 that is less than 300 and N is a multiple of 15 that is less than 700, then what is the greatest possible value of $M + N$?

2. Suppose that A, B, C, and D, are digits, (not necessarily distinct), so that $A + C = B + D = 9$, and that the four-digit number ABCD is divisible by 13. What is the largest possible value of the three-digit number BCD?

3. Produce a three-by-three square array with the following properties:

   - All digits are nonzero and the digit in the center square is 7.
   - The three digits in each row, read left to right, form a three-digit perfect square.
   - The three digits in each column, read top to bottom, form a three-digit perfect square.

4. Find all positive integers n, $n < 50$, such that the sum $1 + 2 + 3 + \ldots. + n$ has a units digit of 0. For the same conditions find all n so that the units digit is 3.

5. Find digits A, B, and C so that the seven-digit number A23B57C is divisible by 792.

SECTION 2.3 CONGRUENCES

OBJECTIVES

- Discover and develop strategies for solving number theory problems using congruences
- Use Modulus to help solve application problems.

Recall: Definition of Prime Numbers, Prime Factorizations, and Greatest Common Divisor.

What is the definition of a prime number?

What is the definition of the Greatest Common Divisor (GCD) of two natural numbers?

What is the Greatest Common Divisor of two or more prime numbers?

CONGRUENCES

How are integers related to each other? This section will show you a new way to relate a number to several other numbers; it's called congruence. In geometry we call shapes congruent if they have the same shape and the same size. We use this word with numbers and say that a number x is congruent to a number y modulo n if a certain condition holds. This section will not only show how we define a congruence relation between numbers, but how these relations can be useful.

We say that two integers x and y are congruent modulo n, where n is a positive integer greater than 1, if the difference $x - y$ is divisible by n. We have a special notation for congruence, $\equiv$. Notice that it looks like an equal sign with an additional bar; 3 bars instead of just 2.

Example: We write a congruence statement: $12 \equiv 8 \pmod{2}$

We read this as "12 is congruent to 8 mod 2". To show this is a true congruence statement we say that $12 - 8 = 4$, which is divisible by 2.

1. Based on our definition, determine if each of the following are true or false.

   a. $3 \equiv 9 \pmod{3}$

   b. $5 \equiv 16 \pmod{3}$

   c. $18 \equiv {}^{-}2 \pmod{5}$

   d. ${}^{-}5 \equiv {}^{-}30 \pmod{7}$

   e. $172 \equiv 25 \pmod{7}$

2. For each of the following statements, written in (mod n), find a value for x so that $0 \leq x < n$,

   a. $37 \equiv x \pmod 6$

   b. $69 \equiv x \pmod{11}$

   c. $405 \equiv x \pmod{45}$

   d. $^{-}7 \equiv x \pmod 5$

   e. $^{-}49 \equiv x \pmod 3$

WORKING IN MODS

Let's look at an alternate way to state the relationship of numbers using mods.

Definition: For integers N, k, m and r, we say $N \equiv r \pmod{m}$ if $N = k \cdot m + r$.

For instance, if $N \equiv 1 \pmod{3}$ then $N = 3k + 1$ for integer values of k. Thus, …$^{-}2$, 1, 4, 7, …. are equivalent when working in mod 3.

Consider the powers of 10 which are the place values for our base 10 number system. In mod 11 we have 100, 102, 104, …. $\equiv$ 1 (mod 11) and 101, 103, 105, …. $\equiv$ $^{-}1$ (mod 11). Then A + 10B + 100C + 1,000D + …. Can be written in mod 11, as A – B + C – D + …. And the number is divisible by 11 if A – B + C – D + … is divisible by 11.

Example:

How many ways can you purchase some items costing \$3.00 each and some items costing \$7.00 each with a total cost of \$95.00?

3. Solve each equation for its positive integer solutions and write your answers in the form $(\boldsymbol{x}, \boldsymbol{y})$.
   a. $13\boldsymbol{x} + 3\boldsymbol{y} = 100$

   b. $7\boldsymbol{x} + 4\boldsymbol{y} = 64$

   c. $7\boldsymbol{x} + 5\boldsymbol{y} = 51$

   d. $11\boldsymbol{x} + 8\boldsymbol{y} = 137$

   e. $11\boldsymbol{x} + 4\boldsymbol{y} = 202$

4. Suppose Farmer Jones buys some sheep, goats, and baby chicks. Each sheep costs \$8.00, each goat costs \$6.00, and each baby chick costs \$0.50. Farmer Jones buys 100 animals for \$100. How many of each did he buy?

5. Suppose that A, B, C, and D, are digits, (not necessarily distinct), so that $A + C = B + D = 9$. Find all four-digit numbers ABCD is divisible by 13. Try solving this problem using what you have learned about mods!

SECTION 2.4 GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLE

OBJECTIVES

- Understand and use the greatest common divisor to solve problems
- Understand and use multiples of numbers
- Understand and use the least common multiple to solve problems

Vocabulary: Multiples, Euclidean Algorithm, GCD, LCM, Relatively Prime, Factor Tree, Prime Factorization, Integer

One useful strategy to find the least common multiple (LCM) and the greatest common divisor (GCD) of a set of integers is to find the prime factorization of the integers and use them to determine the two numbers.

Example: Find the prime factorization of 42 and 36

$42 = 2 \times 21 = 2 \times 3 \times 7$

$36 = 2 \times 18 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$

A factor tree can also be used to find the prime factorization.

To find the GCD of two numbers, find all the prime factors that the two numbers have in common and multiply them. The numbers above have one 2 and one 3 in common. Therefore GCD(42, 36) = 6 where 6 is the largest number that divides both 42 and 36. For the LCM, choose the largest power of each prime appearing in either factorization list. Therefore LCM(42, 36)=$2^2 \times 3^2 \times 7 =$ 252 which is the smallest number that is divisible by both 42 and 36.

1. A certain fast food chain sells chicken nuggets in boxes of 6 or 9. Is it possible to order exactly 20 chicken nuggets? Explain how you arrived at your answer.

2. To make things easier, the chain has now added a box of 20 nuggets. Is it possible to order exactly 77 nuggets, exactly 91 nuggets, or exactly106 nuggets? What is the largest number of nuggets you cannot order?

3 Algebra

SECTION 3.1 SEQUENCES

OBJECTIVES

- Recognize and describe patterns in sequences
- Understand the definition of a sequence and write recursive formulas for finding terms in a sequence
- Become comfortable using sequence notation
- Understand the idea of arithmetic progressions and use it to find terms in an arithmetic sequence

Recall: Exponent rules for multiplication and division

This chapter uses algebra to formalize and extend the pattern recognition and translating mathematical scenarios to equations that we examined in Chapter 1. You will have a chance to make sense of application problems, manipulate variables, and find sums of a sequence.

1. Is it possible to write the numbers 1, 2, …, 10 in a list so that each pair of adjacent numbers differ by either 3 or 5? (As an example the list: 1, 4, 7, 10, 5, 8, 3, 6, 9, 2 would not suffice because 9 and 2 differ by 7.)

In mathematics, we often talk about ordered lists of numbers, called sequences. In a sequence, the order that we write the numbers (called terms) is very important. For example, even though A = 1, 3, 5, 7, 9, 11 and S = 11, 7, 9, 1, 5, 3 both use the same numbers, they are considered two different sequences because the terms are not in the same order.

When working with sequences, it is crucial to understand the notation that helps us talk about which sequence and which term in the sequence we are talking about. For example, let us look at sequence A from above:

A = 1, 3, 5, 7, 9, 11, … What would be the next term in this sequence?

In sequence A, the first term is 1, the second term is 3, the third term is 5, and so on. It can get tedious to have to write out "the first term", "the second term", "the one hundredth term", etc. each time. To shorten the amount of writing we have to do, we use a letter with a subscript to talk about particular terms in the sequence.

For example, our sequence A can be written as A = $a_1, a_2, a_3, \ldots$, where the subscripts 1, 2, 3, … tell us which term in the sequence we are talking about. So a_2 refers to the second term in the sequence A, a_{15} refers to the fifteenth term in the sequence A, and so on. How would you write the 32nd term in the sequence? What about the nth term in the sequence?

2. For each of the four given sequences, try to find a pattern that is true for all the terms of the sequence. Some sequences may have more than one solution or there may be more than one way to describe the pattern.

   a. 2, 9, 16, 23, 30, 37, 44, 51, …

   b. -2, 3, 8, 13, 18, 23, 28, …

   c. 3, 6, 12, 24, 48, 96, …

   d. 2, 6, 18, 54, 162, …

   e. 1, 1, 2, 3, 5, 8, 13, 21,…

   f. 2.5, 4, 7, 13, 25, 49, 97, 193,…

   g. 1, 5, 14, 30, 55, 91, 140, 204,…

   h. 5, 13, 37, 109, 325, 973, 2917, 8749, …

3. (Challenge) A sequence of integers is defined as having 1 and 2 as the first two terms and in that order, and all the remaining terms of the sequence are products of the two previous terms. What is the 9th term of this sequence?

4. Five real numbers a, b, c, d, and e are in arithmetic sequence in the given order. If $e - b = 20$, compute the value of $2b - c - d$.

5. (Challenge) Suppose a 3-by-4 rectangle is divided into 1-by-1 squares. A diagonal is drawn. How many of the 1-by-1 squares does the diagonal pass through? ("Pass through means the diagonal goes through the interior of the square). What if the rectangle is 5-by-7? What if the rectangle is 3-by-5? What if the rectangle is 6-by-10? Do you see any patterns based on the dimensions of the rectangles?

SECTION 3.2 SUMMATIONS

OBJECTIVES

- Develop and use formulas for finding sums of sequences

Intro Activity: Compute the value of 1 + 2 + 3 + 4 + 5 + … + 95 + 96 + 97 + 98 + 99 + 100. Rather than just adding the terms together, try to explore how else you might calculate the sum.

What if we wanted to sum the numbers 1 to 149; 1 to n? Can we extend the pattern we recognized last time to a formula?

1. a. Compute each of the following: 1, 1 + 3, 1 + 3 + 5, 1+ 3 + 5 + 7.

   b. Write out the sum of the first 10 odd integers. Can you predict what the 10th term is? What is the sum?

   c. What would be the last term in the sum of the first 50 odd integers? Can you predict what the sum is?

   d. What pattern do you see? Can you find the sum of the first n odd integers? How would you write that sum?

   e. Can you visually represent this sum using squares?

2. a. Compute each of the following: 2, 2 + 4, 2 + 4 + 6, 2 + 4 + 6 + 8

   b. Write out the sum of the first 8 even integers. What is the sum?

   c. Write out the sum of the first 20 even integers. What is the sum?

   d. What is the 100th even integer? Write out the sum of the first 100 even integers (you might want to use …) What is the sum?

   e. How do you find the nth even integer? Using the pattern you noticed from parts a - d, what is the sum of the first n even integers?

   f. How does this formula compare to the formula for the sum of the first n positive integers?

3. Compute the value of $(1 + 2 + 3 + \ldots + 99 + 100) - 2(1 + 3 + 5 + \ldots + 99)$.

4. Compute the sum of 1 + 2 – 3 + 4 + 5 – 6 + 7 + 8 – 9 ….. + 97 + 98 – 99

SECTION 3.3 VARIABLE MANIPULATION

OBJECTIVES

- Work with multiple variables in an equation
- Manipulate equations to solve for unknown quantities
- Reason about values of expressions

Recall: Solving Linear Systems of Equations, Distributive Property and Factoring

1. Let x and y be real numbers such that $x - y = 3$ and $xy = 11$. What is $x^2 + y^2$?

2. Let x and y be real numbers such that $x + y = 7$ and $x^2y + xy^2 + x + y = 42$. What is the value of $x^2 + y^2$?

3. Suppose that m, n and k are integers such that $m^2 + k = n^2 - k = 65$. What is the greatest possible value of n^2?

SECTION 3.4 RATIO AND RATE APPLICATIONS

OBJECTIVES

- Understand and use the distance formula to solve application problems
- Solve application problems involving ratios, percentages, and rates
- Translate written problems and situations into mathematical symbols and equations in order to solve them

Recall: Solving Equations, Using Variables to Represent Unknown Quantities, Distance Formula, Operations with Fractions

1. Max has a large bag of pennies, nickels, and dimes with a total value of $48.00. If the ratio of pennies to nickels to dimes is exactly $4:2:1$, then how many coins does he have?

2. Wednesday Emily was three minutes late for the bus to school because it took her 20% more time to get ready then it did on Tuesday. Thursday she only took 10% longer than she did on Tuesday and she was two minutes early for the bus. Assume Emily woke up at the same time both days. How many minutes did it take her to get ready on Tuesday?

3. The ratio of the sum of a number and its reciprocal to the positive difference between the number and reciprocal is $3:2$. If the number is greater than 1, what is the number?

Distance formula: distance = rate · time; $d = rt$

4. (challenge) Juanita made a trip from San Marcos to Austin and back. On the way to Austin she drove at an average speed of M miles per hour. However on the way back she had to make a detour that increased the distance by 20%. In addition the traffic was slower and she only averaged $\frac{2}{3}M$ miles per hour. In terms of M, what was her average speed for the round trip?

5. (Challenge) A swimming pool has two pumps and a drain. One pump, working by itself, can fill the pool in 8 hours. The other pump, working by itself can fill the pool in 10 hours. The drain can empty the pool in 6 hours. Pat turns on both pumps but fails to notice that the drain is also open. How long will it take to fill the pool? How much water is wasted measured in units of one full pool?

6. (Challenge) Curtis runs a mile at a steady rate of 8 miles per hour. Cody begins running the mile at 9 mph but slows to 7 mph at some point. Curtis completes the mile in one minute less than Cody.

   a. How long did it take Curtis to run the mile? Can we represent this in minutes instead of hours?

   b. How long did it take Cody to run the mile?

   c. What fraction of the mile did Cody run at 9 mph?

7. (Challenge) Pyro Pete lit two candles at the same time. The candles are initially the same length but burn at different rates. Ten minutes after lighting them one candle is twice as long as the other. Five minutes later one candle is five times as long as the other. After this, how long will it take for each candle to burn completely?

4 Counting

SECTION 4.1 RULE OF SUM AND RULE OF PRODUCT

OBJECTIVES

- Count using the rule of product
- Count using the rule of sum
- Represent problems using a tree diagram

Carefully counting the number of ways one can do something is a basic idea in mathematics. Counting with precision uses the key idea of breaking down a problem into simpler steps. We then count the number of ways to do something by looking at these simpler pieces. Let's begin with an example to see how to break a problem down into simpler steps.

PROBLEM 1:

Suppose that you are asked to write down two letters taken from the set ***S*** = {A, B, C, D, E}. How many ways can you do this?

Class Discussion: The first thing to observe is that it is not clear what this problem means. What are some ambiguities that can occur?

Let's revise the question to make the problem a little more precise:

PROBLEM 1A:

Suppose you are asked to "arrange" two letters from the set ***S*** = {A, B, C, D, E} in some order. We allow for the letters to be the same or different. How many ways can this be done?

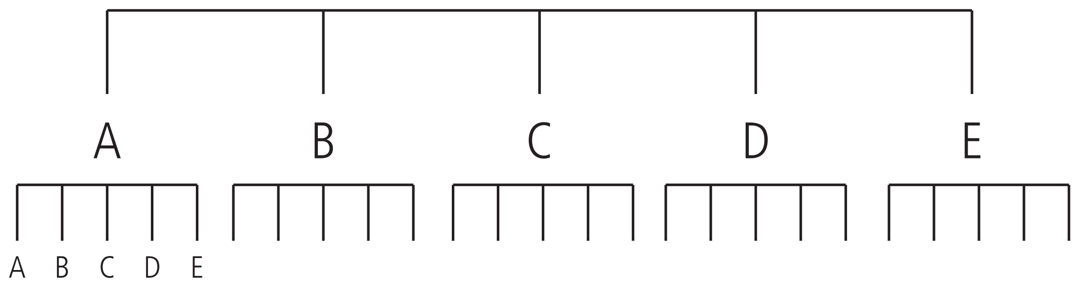
Let's explore different ways of organizing our problem:

1. Make a table:

	A	B	C	D	E
A		AB			
B	BA				
C					
D					
E					

Note that the arrangement AB in row 1 is different from the arrangement BA in row 2.

2. Make a tree diagram:



The first branch represents the choices for the first letter. The next connecting branch represents the choices for the second letter.

Another way to think of this problem is to think of arranging the two letters as doing two operations;

Operation 1: Pick the first letter

Operation2: Pick the second letter

We can represent these operations with a tree diagram.

In general, tree diagrams can be used to represent the different possibilities from doing successive operations. We state this more formally as the **rule of product**.

DEFINITION: RULE OF PRODUCT
If one operation can be done in r ways and a second operation can be done in s ways, then there are $r \cdot s$ ways to first do operation 1 **and** then do operation 2.

For our example, there are 5 ways to pick the first letter.

There are also 5 ways to pick the second letter, because we allow the second letter to be any letter from our original set. Hence there are $5 \cdot 5 = 25$ ways to arrange 2 letters from S.

We call an arrangement of 2 letters from set S a word of length 2 using the letters in S, even though it might not be a real word. When we write down a word, letters can be repeated. Some examples of words would be AB, BA, BB, etc.

On the other hand, sometimes we may have two operations but only want to do one or the other, but not both. For example, consider the problem:

PROBLEM 2:

How many ways are there to choose a letter from the alphabet **or** to choose a digit from 0-9?

In this example, what are the operations?

Operation 1: ____

Operation 2: ____

In this example, how many ways can each operation be done?

Operation 1 can be done in ____________ ways.

Operation 2 can be done in ____________ ways.

So there are 26 + 10 = 36 ways to do either operation 1 or operation 2. We state this formally as the rule of sum:

DEFINITION: RULE OF SUM
If one operation can be performed in ***r*** ways and a second operation can be performed in ***s*** ways (and there is no overlap in the results), then there are ***r*** + ***s*** ways to do operation 1 **or** operation 2, but not both.

EXERCISES

1. How many five-digit positive integers have five distinct digits?

2. How many "words" of length five can be formed from the letters A, B, and C?

3. A flag is to be designed with 13 horizontal stripes colored red, white or blue, subject to the condition that two adjacent stripes may not be the same color. In how many ways can this be done?

DEFINITION: FACTORIALS
$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

EXAMPLE:

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, or the expanded form. More simply, $6! = 720$.

PROBLEM 3:

Write the expanded form of 8!

4. There are 7 kindergarteners in a class. How many ways can these 7 kindergarteners line up in a straight line for snack?

5. How many five-digit positive integers have five distinct digits if no digit can be 0? How can we represent this answer using factorials?

SECTION 4.2 PERMUTATIONS AND COMBINATIONS

OBJECTIVES

- Count the number of r-permutations of an n-set S.
- Count the number of r-combinations of an n-set S.

Let's start with a set containing 5 elements, say S = { a, b, c, d, e}. We call S a 5-set, and write $|S|$ = 5, which says that the number of elements in set S equals 5. So we are saying that the absolute value of a set is the number of elements in that set, or the "size" of S. consider the problem:

PROBLEM 1:

How many ways are there to pick two distinct elements from S?

Again, the question is does the order of choosing the elements matter? Let's make this explicit by rewording Problem 1 as:

PROBLEM 1A:

How many ways are there to arrange two distinct elements from S?

Now it is clear that the order of the elements we pick matters, since we are asking how many ways there are to "arrange" the elements. We can solve this problem using our rule of product from the previous section:

In this example, what are the operations?

Operation 1: ____

Operation 2: ____

In this example, how many ways can each operation be done?

Operation 1 can be done in ____________ ways.

Operation 2 can be done in ____________ ways.

It follows that there are $5 \cdot 4 = 20$ ways to arrange 2 distinct elements from S.

In general, we can begin with a set having ***n*** elements, and ask how many ways are there to "arrange" ***r*** distinct elements from ***S***?

DEFINITION: ARRANGEMENT
An arrangement of ***r*** distinct elements from an ***n***-set ***S*** is called an ***r***-permutation of ***S***.

For our set ***S*** = {a, b, c, d, e}, write down all of the 2 permutations of ***S***. (There should be 20 of these total) Represent these ***r***-permutations using a tree diagram.

DEFINITION:
The number of ***r***-permutations of an ***n***-set ***S*** is denoted by ***P***(***n***, ***r***).

PROBLEM 2:

How many 2-permutations of the 5-set ***S*** are there?

SOLUTION:

As seen above, there are 20 different ways to arrange 2 elements from ***S***. $P(5, 2) = 5 \cdot 4 = 20$.

When the order of the elements does not matter we call this a **selection** (instead of an arrangement.) Consider the following problem:

PROBLEM 3:

How many ways are there to select two distinct students from a set ***S*** of 10 students who will then shake hands? (This is the famous hand-shake problem.)

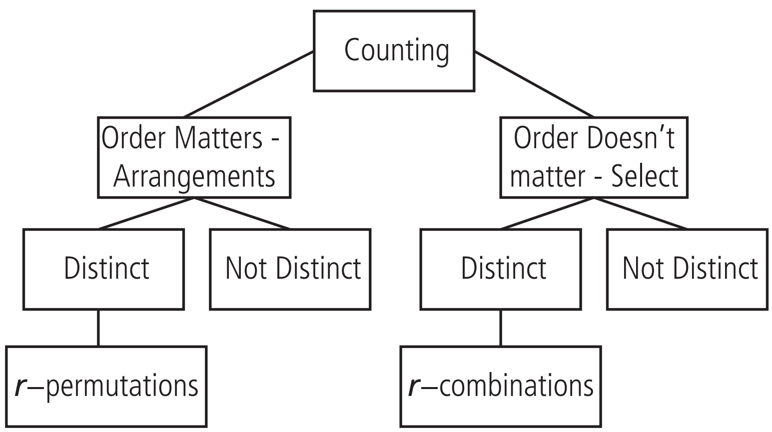
Class Discussion: What is the difference between this and an r-permutation? After all the people selected must be distinct (meaning different)?

DEFINITION: R-COMBINATION OF S
A selection of r distinct elements from an n-set S is called an r-combination of S. An r-combination of an n-set is just a selection of r different elements from S.

We can use these two rules to analyze many different counting problems that arise in a variety of settings. We also have some definitions.

Suppose S is a set with n distinct elements. An r-permutation of S is an arrangement of r distinct elements from S. We write $P(n, r)$ to be the number of r-permutations of the set S. Notice that in making a r-permutation, the order of the arrangement matters.

An r-combination of S is a selection of k distinct elements from S. We write $C(n, r)$ to be the number of r-combinations of the set S. When we make an r-combination, the order that the objects are selected does not matter. Refer to the following diagram:



EXAMPLE:

How many "words" of length five can be formed from the letters A, B, and C?

Step 1. Describe in words what the 3 and 5 represent.

There are 5 spots to fill

There are 3 letters to fill them with

Step 2. Illustrate as below:

$$\underline{\quad 3 \quad} \cdot \underline{\quad 3 \quad} \cdot \underline{\quad 3 \quad} \cdot \underline{\quad 3 \quad} \cdot \underline{\quad 3 \quad}$$

Step 3. 3^5 words can be made.

EXTENSION ACTIVITY:

What if there were n positions and only r things to fill those positions with? How many ways could one fill those n with r objects?

Notice how order mattered here, but the objects placed into the positions need not be different. How many five-digit integers have five distinct digits if no digit can be 0?

Represent this answer using factorials.

Hom many n letter words can you make using r letters? Explain.

1. How many ways are there to select a president, vice-president, and secretary from a group of five people?

2. How many committees of three can be selected from a group of five people? What is different about this exercise and exercise 1?

3. How many different results are possible for first, second, and third place in a race that has 10 entrants?

4. Max has seven boys and seven girls in class and wants to choose two of them to do a special project. He insists that both must be boys or both must be girls. How many different pairs are possible?

5. How many ways are there to divide eight people into two teams of four people each?

6. (Challenge) If no three diagonals of a convex decagon intersect at the same point inside the decagon how many points of intersection are in the interior of the decagon?

7. (Challenge) How many ways are there to create a five character password using the digits 0-9, capital letters A-Z, and lowercase letters a-z, given the following:
   a. with no restrictions?

   b. If there must be at least one lowercase letter?

   c. If there must be at least one digit?

   d. If there must at least one uppercase letter and one digit?

   e. If there must be at least one uppercase letter, one lower case letter, and one digit?

8. (Challenge) A postman has to deliver four letters to four houses. One letter is addressed to each house. The postman decides to give every house the wrong letter. How many ways are there for him to achieve this?

SECTION 4.3 COUNTING PROBLEMS WITH A TWIST

OBJECTIVES

- Count number of arrangements when the objects are not distinct
- Count the number of selections when the objects are not distinct using inter-cell partitions
- Count number of ways to do operations that use both the rule of sum and rule of product

When we write down an r-permutation of an n-set, we are basically writing down a word of length r, where all of the letters are different.

PROBLEM 1:

How many words of length 2 can you make using the letters a, b, c, d, e if each of the letters in the word are different?

SOLUTION:

There are 5 choices for the first letter. After we choose the first letter, there are 2 choices for the second letter. So there are $5 \cdot 4 = 20$ total words.

PROBLEM 2:

How many words of length 2 can you make using the letters a, b, c, d, e if the letters need not be distinct?

SOLUTION:

In this case, there are 5 choices for the first letter, and 5 choices for the second letter, so we get $5 \cdot 5 = 25$ total words.

Now let's count the number of ways of "selecting" 2 distinct elements from our set. In this case we can apply the rule of product in a backdoor kind of way. We call the number of such selections $\boldsymbol{C}(5, 2)$ which is the number of 2-combinations of a 5-set. Note that each selection of 2 elements corresponds to 2-permutations, since the 2 elements can be arranged in $2! = 2$ ways. Thus we can count the number of 2-permutations of a 5 set by doing 2 operations:

1. Select 2 elements from the set. This can be done in $\boldsymbol{C}(5, 2)$ ways.
2. Arrange those 2 elements in some order. This can be done in $2!$ ways.

In total, $\boldsymbol{P}(5, 2) = \boldsymbol{C}(5, 2) \cdot 2!$ In other words, the number of ways of selecting two elements, $C(5,2) = \frac{20}{2!}$.

To summarize: Each $\boldsymbol{r}$ combination of an $\boldsymbol{n}$-set $\boldsymbol{S}$ corresponds to $\boldsymbol{r}!$ permutations. Thus $\boldsymbol{C}(5, 2) = \frac{\boldsymbol{P}(5,2)}{\boldsymbol{r}!}$.

For example, the words AC and CA are two distinct permutations. However, these correspond to the same combination. $\boldsymbol{C}(5, 2) = 5 \cdot 4 \div 2 = 10$ total combinations. Let's practice these ideas with a few problems.

These problems are either more challenging or involve different concepts than previously discussed (such as circular permutations). Some of these will help to emphasize that we cannot just apply one formula to all problem situations.

1. How many ways can a five-member committee be chosen from 10 women and 4 men:
   a. With no restrictions?

   b. If there are to be more men than women on the committee?

   c. If a particular man and woman cannot serve on the committee together?

2. How many four-digit integers contain at least one repeated digit?

3. How many three-digit positive integers are there:
   a. With no restriction?

   b. With exactly three different digits?

   c. With exactly two different digits?

   d. With no different digits?

   e. With at least one even digit?

4. How many ways are there to order the letters of AXIOM so that all three vowels occur before the letter X?

5. A fair coin is tossed 8 times. How many outcomes are possible:
   a. In all?

   b. With exactly three heads?

   c. With at most four heads?

   d. With a head on the fifth and seventh toss?

6. A school has three clubs; the MC, the SC and the CC. There are 18 members in the SC, 34 members in the CC and 25 members in the MC. There are 13 who are members of both the SC and CC, 5 who are members of both the SC and MC, and 17 who are members of both the CC and MC. There are 2 students who are members of all three clubs.

   a. How many students are members of the SC and CC but not the MC?

   b. How many students are members of the CC but no other club?

   c. How many students belong to at least two of the clubs?

7. The following list contains 500 fractions. How many of these fractions can be reduced to lower terms?

$$\frac{201}{6}, \frac{202}{6}, \frac{203}{6}, \ldots, \frac{698}{6}, \frac{699}{6}, \frac{700}{6}$$

8. How many paths are there from the point (0, 0) to the point (4, 5) using only one unit moves, either to the right or upward. How many of these paths are possible if we are not allowed to pass through (1, 4)?

9. How many ways can 8 students be seated around a circular table?

10. Suppose we have an n-by-n board consisting of unit squares, each of which contains a light that can be turned on or off. Begin with all the lights turned off:

    a. How many squares are lit if we light all the squares on the outer border?

    b. How many squares are lit if we light all of a row and all of a column?

    c. Assume $n > 7$. Light a 3-by-5 rectangle and a 2-by-7 rectangle. What is the smallest possible number of squares that are lit?

11. Consider the 9 points in the coordinate plane whose coordinates come from the set {-1, 0, 1}. How many triangles are there in the coordinate plane that have their vertices at three of these points?

12. How many ways are there to select 12 donuts from the set of donuts A, B, C?

SECTION 4.4 PROBABILITY

OBJECTIVES

- Find the sample space for an experiment.
- Count the number of outcomes in the sample space
- Find the number of elements in an event ***E*** (which is a certain set of outcomes)
- Compute the probability of event ***E***

When you think about the concept of the probability of a certain outcome, you may think about the chance that the outcome will occur or the likelihood of it occurring. In that way, we examine problems that ask for the probability that some outcome will occur. In the problems that follow, we assume that all the outcomes in our experiments are equally likely. In other words, for coin flips, we assume the coin will be equally likely to be heads as tails. Now in a real case of flipping the coin 6 times, the coin may actually be heads 4 times and tails only 2. We say that the theoretical probability is that the coin should come up half the time heads and half the time tails.

ACTIVITY:

Roll 2 dice (red and green) and determine the list of all possible outcomes called the **sample space** (it might be helpful to use either a tree diagram, grid, or ordered pairs).

How many outcomes are there:

a. With no restriction?

b. Where the green die lands on 2?

c. Where the sum of the dice is 2?

d. Where the sum of the dice is 7?

The probability of an event E in an experiment with sample space S is given by:

$$P(E) = \frac{|E|}{|S|}$$

Where $|E|$ is the number of outcomes where E occurs and $|S|$ is the number of outcomes where S occurs.

1. Suppose we toss a fair coin five times. What is the probability that the coin lands on heads three successive times but not four successive times?

2. Suppose we toss two dimes and two quarters. We are allowed to keep the coins that come up heads. What is the probability that we get to keep at least 20 cents?

3. Samantha has a bag containing five red stones and three blue stones. She draws one stone at random from the bag and places it in a tray. She then draws another stone at random and adds it to the tray.
   a. What is the probability that both stones are red?

   b. What is the probability that both stones are blue?

   c. What is the probability that one stone is red and the other blue?

   d. What is the sum of your three answers? Why?

5 Geometry

SECTION 5.1 AREAS OF TRIANGLES AND QUADRILATERALS

OBJECTIVES

- Compute area using standard and non-standard methods
- Develop area formulas

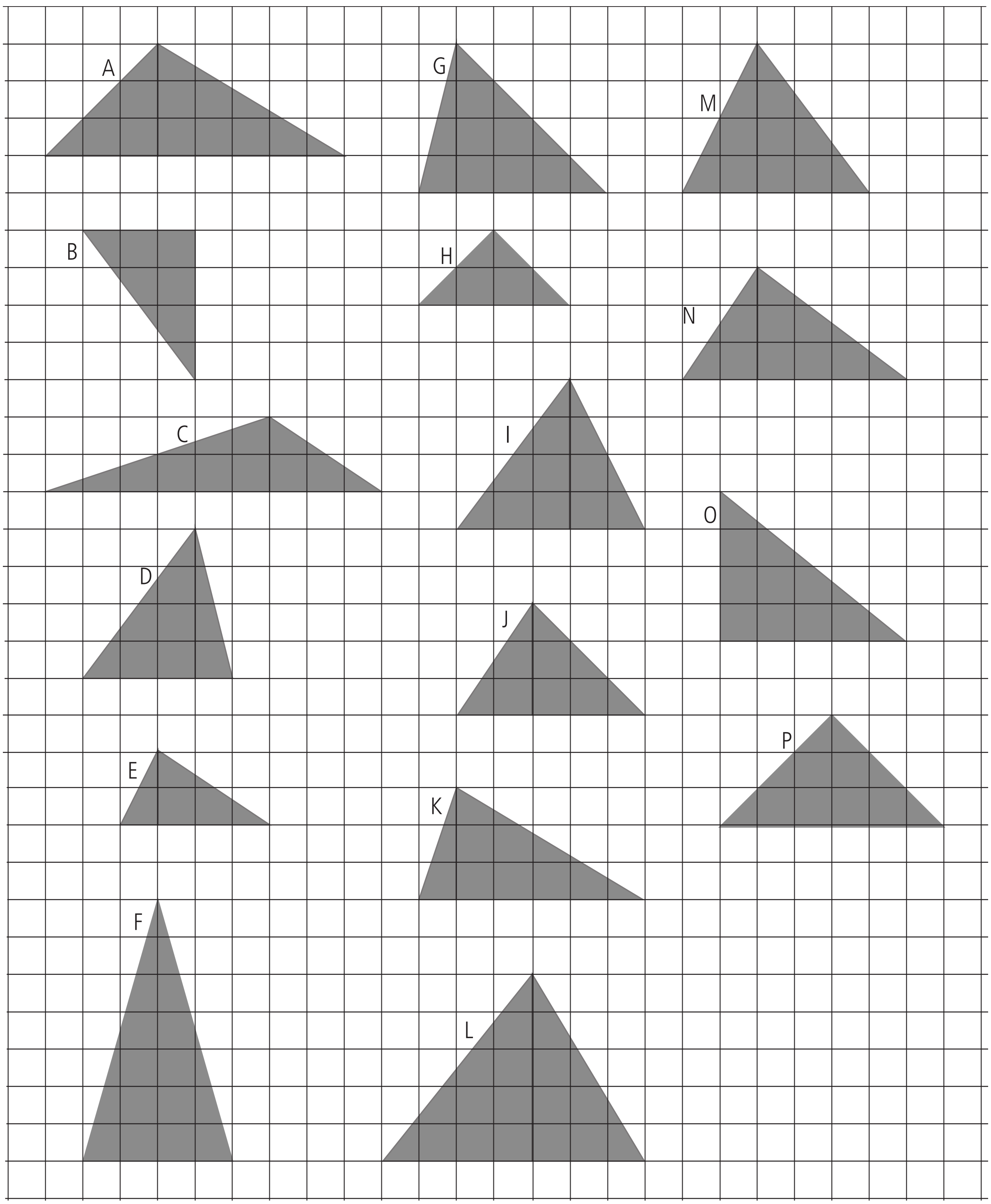
This chapter focuses on various topics in geometry. The first section covers areas of triangles, quadrilaterals, circles, and irregular figures using standard and non-standard methods. The second section introduces the Pythagorean theorem, properties of right triangles, equilateral triangles, and isosceles triangles. The third section covers the concept of scaling and similarity, properties related to the lengths and areas of similar figures, and extending pattern recognition and counting in the context of geometry. A review of basic definitions for each section can be found in the Appendices at the end of the workbook.

Recall: Area of Rectangles, Triangles

In mathematics, we study the properties of geometric figures and measurements related to the figures. We then work to create formulas that describe these measurements. Because we know how to compute the area and perimeter of rectangles and some of us also recall the formula to find the area of triangles, let's try to use our previous knowledge to help figure out the following problems.

ACTIVITY:

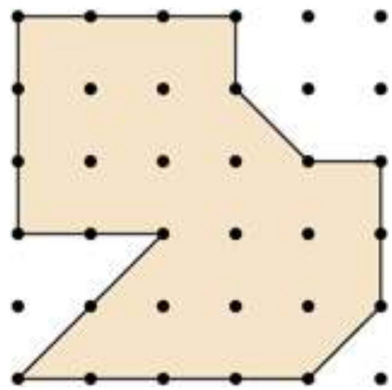
Calculate the areas of the triangles below where each box is 1 square unit. Notice that only one side of each triangle is on a grid line. Is it possible to find the area of these triangles without using the formula? Give it a try.



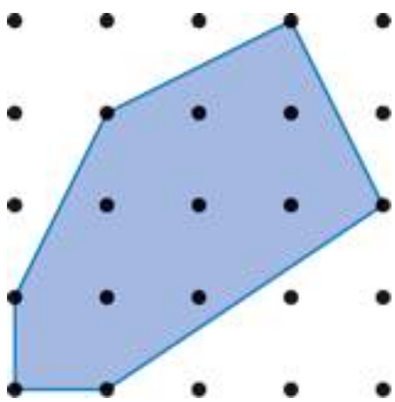
ACTIVITY:

Use your knowledge of finding areas of rectangles, triangles, and the concept of enclosing or dividing up a figure into a familiar bigger (smaller) figures to find out the areas of the following figures.

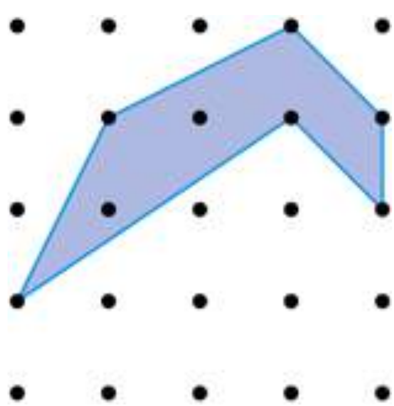
a.



b.

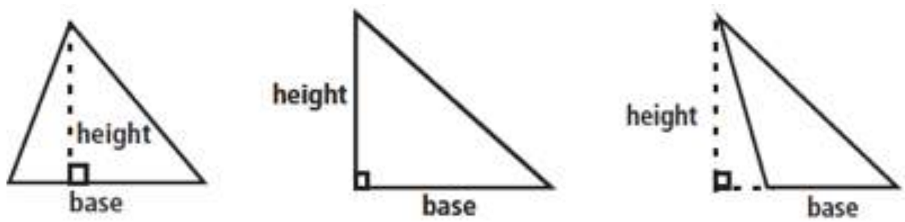


c.



FORMULA: AREA OF A TRIANGLE
$A = \frac{1}{2}\, b \cdot h$ or $A = \frac{1}{2} bh$ or $A = \frac{bh}{2}$

Be careful in identifying the base and the height of the triangle. The **base**, b, must be a side of the triangle, and the **height**, or **altitude**, h, must be perpendicular to the base, or an extension of the base, and be drawn from the vertex opposite the base.

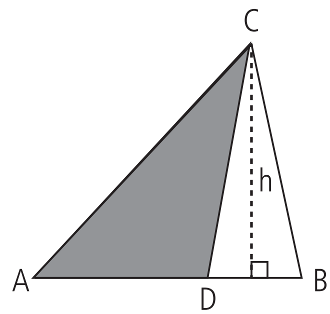


1. Find the indicated attributes in the following:
   a. Suppose that $\triangle ABC$ has base AB = 5 units and has the area of 7.5 sq. units. What is the height of $\triangle ABC$?

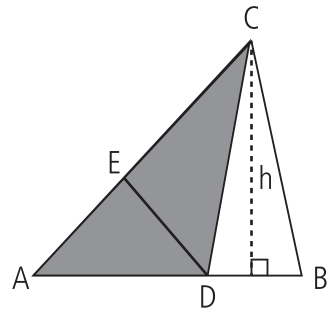
   b. Suppose that $\triangle ABC$ has the height 9 units and has the area of 54 sq. units. What is the base of $\triangle ABC$ associated with this height?

   c. Suppose that $\triangle ABC$ has sides AB = 13 units, BC = 16 units, and AC = 8.6 units. The area of the triangle is 56 sq. units. If we used BC as our base what would be the associated height? Find the corresponding heights associated with the other two bases.

2. Suppose that $\triangle ABC$ has $AB = 10$ units. The area of the triangle is 40 sq. units. A point D is marked on side AB such that $AD = 6$ units. What is the area of $\triangle ADC$?



3. Consider the triangle in the previous problem. This time point E is marked on AC such that $AC = 12$ units and $AE = 5$ units. What is the area of $\triangle ADE$?



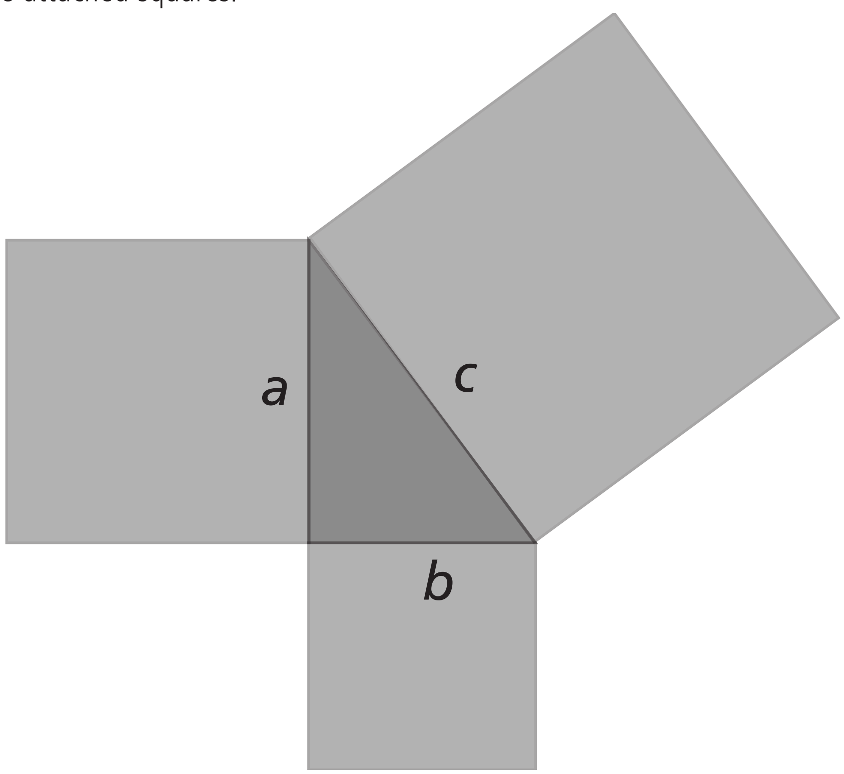
SECTION 5.2 RIGHT TRIANGLES

OBJECTIVES

- Reviewing the Pythagorean Theorem
- Solving problems by applying the Pythagorean Theorem

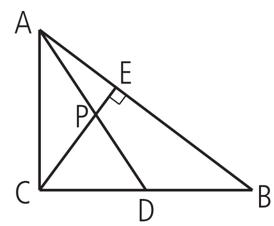
We say a triangle is a **right triangle** if it has two sides that meet at a perpendicular or right angle. We call the two perpendicular sides of a right triangle the **legs** of the right triangle. The ancient Greeks called the diagonal side of a right triangle (side opposite to the right angle) the **hypotenuse**. Which side of the right triangle is the longest? Is it always this case in a right triangle?

Looking at the triangle below, what is the area of each square? Express the relationship between the areas of the attached squares.



You might have noticed how the three areas relate. The area of the square attached to the hypotenuse is equal to the sum of the areas of the other two squares attached to the legs. This can be written as $a^2 + b^2 = c^2$, where c is the length of the hypotenuse and a and b are the lengths of the legs. This formula is called the Pythagorean formula or the Pythagorean Theorem.

1. In the figure below, $AC = 6$ units, $BC = 8$ units and angle $ACB = 90^0$. Segments CD and BD are equal. Segment CE is an altitude of $\triangle ABC$.



a. How many hypotenuses can you find in the figure above? For each hypotenuse that you found complete the following table for its associated parts. The first row is done for you as an example for the hypotenuse AB.

Triangle Name	Leg 1	Leg 2	Hypotenuse	Pythagorean Relationship
$\triangle ACB$	AC	CB	AB	$(AC)^2+(CB)^2=(AB)^2$

b. Compute the following:

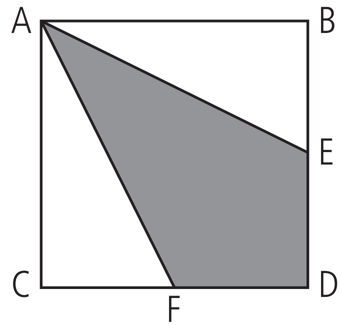
i) The length of AB.

ii) The length of AD.

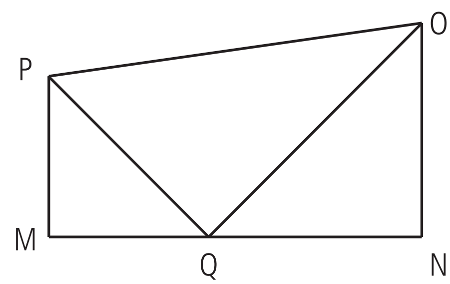
iii) The length of CE.

iv) The length of AE.

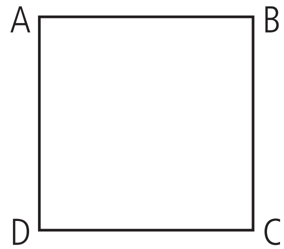
2. In the diagram below, *ABCD* is a square and *E* and *F* are midpoints of *BC* and *CD* respectively. What is the ratio of the shaded area *AEDF* to the area of square *ABCD*?



3. In quadrilateral ***MNOP***, segments ***PM*** and ***ON*** are both perpendicular to segment ***MN***. Point ***Q*** lies on side ***MN*** so that $\triangle$***PQM*** = $\triangle$***OQN*** = 45°. If ***MQ*** = 6 and ***NQ*** = 8, what is the area of quadrilateral ***MNOP***?

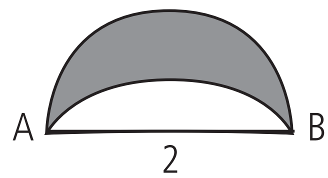


4. Let $ABCD$ be a square with sides of 8 units, and let M and N be the midpoints of AB and BC respectively. Compute the area of quadrilateral $AMNC$.



5. What is the area of a triangle in the coordinate plane with vertices at (2, 2), (5, 0) and (3, 5)?

6. (Challenge) In the diagram below, AB has a length of 2 units. Two arcs are drawn from A to B. The outer arc has a measure of 180° and the inner arc has a measure of 90°. What is the area of the shaded region?



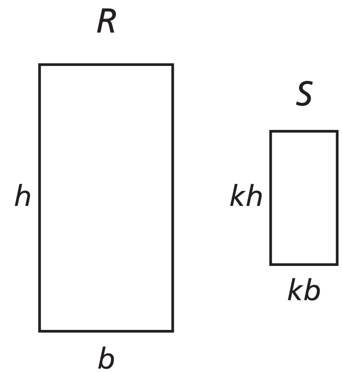
SECTION 5.3 SCALING AND SIMILARITY

OBJECTIVES

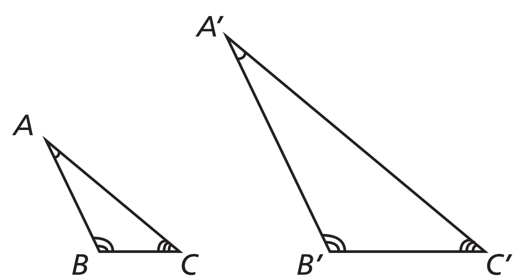
- Investigating similar triangles and the concept of scaling.
- Discovering selected properties related to lengths and areas of similar figures

In this section we will explore the ideas of scaling and similarity.

If a rectangle R has dimensions b and h, two positive numbers, and rectangle S has dimensions kb and kh, where k is also a positive number, then k is called **scale factor** from R to S. This definition also holds for the dimensions of other shapes such as triangles and squares.



Two triangles are **similar** when their corresponding angles have equal measure and their corresponding sides have the same ratio.



THEOREM: TRIANGLE SIMILARITY THEOREM
If two triangles have corresponding angles of the same measure, then the ratios of their corresponding sides are the same. Conversely, if two triangles have corresponding sides with the same ratio, then the triangles’ corresponding angles have equal measure.

By the triangle similarity theorem, $\triangle ABC$ is similar to $\triangle A'B'C'$ or $\triangle ABC \sim \triangle A'B'C'$.

The similarities of the triangles implies that:
$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} = k$, where k is the scale factor from $\triangle ABC$ to $\triangle A'B'C'$.

When we say that two shapes are similar, corresponding sides must be labeled in the same order. For instance, we do not say that $\triangle ABC$ is similar to $\triangle A'C'B'$ because the corresponding ratios $\frac{AB}{A'C'} = \frac{AC}{A'B'} = \frac{BC}{C'B'}$ are not equal.

In general, we can have similarity in any type of polygon.

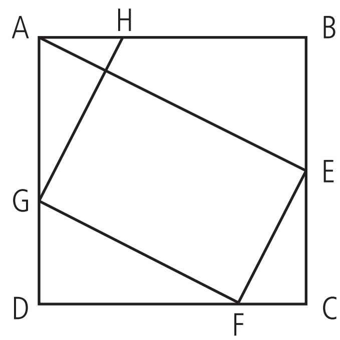
THEOREM: POLYGON SIMILARITY THEOREM
Two polygons are similar when their corresponding angles have equal measure and their corresponding sides have the same ratio.

1. Triangle ABC is similar to $\triangle DEF$. If $AB = 4$ units, $BC = 5$ units, $AC = 6$ units, and the perimeter of $\triangle DEF$ is 80 units, what are the lengths of the sides of $\triangle DEF$?

2. Two parallel lines, k and m, are 18 units apart. Points A and B are located on line k such that AB = 8 units. Points C and D are located on line m such that CD = 28 units. The points are positioned so that lines AD and BC intersect at a point P between lines k and m. Compute the distances from P to k and from P to m.

3. Given the information from the previous problem observe that lines AC and BD also intersect at a point Q that is not between k and m. Compute the distance from Q to k and from Q to m.

4. In square $ABCD$, points E, F, G and H lie on sides BC, CD, DA and AB respectively such that angles AEF, EFG, and FGH are all right angles. If E is the midpoint of side BC, what is the value of the ratio $\frac{AH}{AB}$?



6. (Challenge) What is the radius of the inscribed circle of a right triangle whose sides have lengths of 5, 12, and 13?