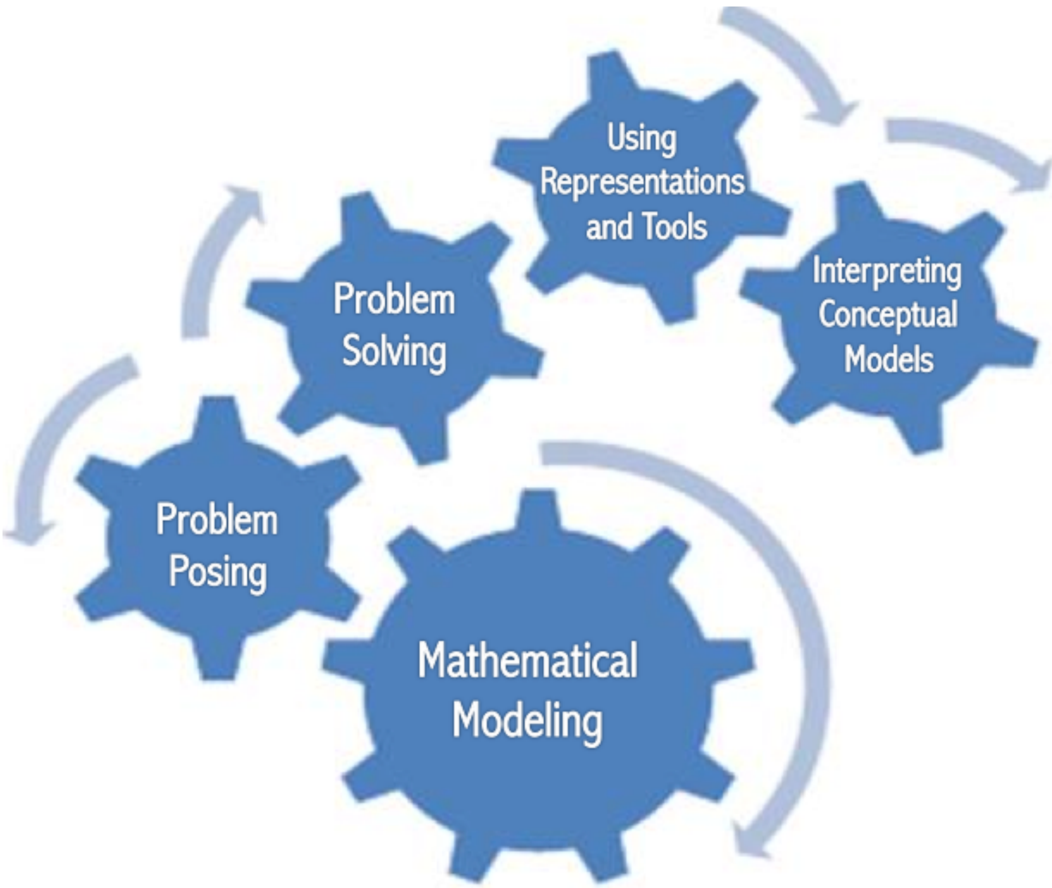
MATH MODELING TOOLKIT

APPENDICES: Modeling Mathematical Ideas Toolkit available on
http://modelmath.onmason.com

MMI Toolkit 1.0 Developing Strategic Competence through Modeling Math Ideas



Developing Strategic Competence through Modeling Mathematical Ideas include the application of mathematics for 1) problem solving; 2) problem posing; 3) mathematical modeling; 4) the flexible use of representational models, tools, technology and manipulatives to solve problems and communicate mathematical understanding; and 5) the deep understanding of conceptual models critical to understanding a specific mathematics topic.

MMI Toolkit 1.1 Promoting Mathematical Practices

Common Core mathematical practices (NGA Center and CCSSO, 2010, p. 6) with Question prompts for encouraging mathematical practices (Suh & Seshaiyer , 2014)

Mathematical Practices	Questioning prompts
(MP1) Make sense of problems and persevere in solving them.	Does the problem make sense? What do you need to find out? What information do you have? What strategies are you going to use? Does this problem require you to use your number sense, spatial reasoning, and/or logical reasoning? What can you do when you are stuck?
(MP2) Reason abstractly and quantitatively.	What do the numbers in the problem mean? What is the relationship among the numbers in the problem? How can you use number sense to help you check for reasonableness of your solution? What operations or algorithms are involved? Can you generalize the problem using symbols?
(MP3) Construct viable arguments and critique the reasoning of others.	How can you justify or prove your thinking? Do you agree with your classmate's solution? Why or why not? Does anyone have the same answer but a different way to explain it? How are some of your classmates strategies related and are some strategies more efficient than others?
(MP4) Model with mathematics.	How is this math concept used in a real world context? Where have you seen similar problems happening in everyday life? Can you take a real world problem and model it using mathematics? What data or information is necessary to solve the problem? How can you formulating a model by selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables?
(MP5) Use appropriate tools strategically.	What tools or technology can you use to solve the problem? Are certain manipulatives or representations more precise, efficient, and clear than others? How could you model this problem situation with pictures, diagrams, numbers, words, graphs, and/or equations? What representations might help you visualize the problem?
(MP6). Attend to precision.	What specific math vocabulary, definitions, and representations can you use in your explanation to be more accurate and precise? What are important math concepts that you need to include in your justification and proof to communicate your ideas clearly?
(MP7) Look for and make use of structure.	What patterns and structures do you notice in the problem? Are there logical steps that you need to take to solve the problem? Is this problem related to a class of problems (i.e. multi-step, work backwards, algebraic, etc.)? Can you use a particular algorithmic process to solve this problem?
(MP8) Look for and express regularity In repeated reasoning.	Do you see a repeating pattern? Can you explain the pattern? Is there a pattern that can be generalized to a rule? Can you predict the next one? What about the last one?

MMI Toolkit 1.2 Performance-based Assessment

<table>
<tr><td colspan="2">Performance based assessment: The Classic Handshake Problem</td></tr>
<tr><td colspan="2">NCTM: Algebra (Mathematical Problem Solving)
Students will apply mathematical concepts and skills and the relationships among them to solve problem situations of varying complexities</td></tr>
<tr><td colspan="2">1. If everyone at your table shakes hands with everyone else, how many handshakes would there be?

2. If everyone in your class shakes hands with everyone else, how many handshakes would there be?

3. What if there were 100 people in the room?

4. At a birthday party, each guest shakes hands with every guest. If 190 different handshakes take place, how many guests were at the party?</td></tr>
<tr><td>Anticipated Students Response and performance</td><td>Tools & Technology</td></tr>
<tr><td>Developing An Algebraic Habit of Mind
(Driscoll, 1999)
1. Abstracting from computation
2. Doing and undoing
3. Building a rule from patterns</td><td>Manipulatives
http://completecenter.gmu.edu/java/handshake/index.html</td></tr>
<tr><td>Signposts for evaluation:
Did the student use:
☐ pictures, charts, graphs, or t-tables with supporting explanation
☐ a written explanation with detailed sentences
☐ the equation or number sentence
☐ the answer (Is the answer reasonable? Why or why not?)
☐ the solution to relate to other situations</td><td>Teacher Notes:
(1) Understanding-

(2) Computing-

(3) Applying-

(4) Reasoning-

(5) Engaging-</td></tr>
</table>

MMI Toolkit 2. UCARE Rubric to assess mathematics proficiency

Student name	Comments (Supporting Evidence)
Understanding (Conceptual Understanding)	
o Understands the problem and task o Makes connection to similar problems o Uses and connects models and multiple representations	
Computing (Procedural Fluency)	
o Proper use of algorithm o Accurate computation o Flexibility in computation	
Applying (Strategic Competence)	
o Formulates and carries out a plan o Can pose similar problems o Can solve problem using appropriate math and strategies	
Reasoning (Adaptive Reasoning)	
o Justifies responses logically o Reflects on and explains procedures o Explains concepts clearly	
Engaging (Productive Disposition)	
o Tackles difficult tasks o Perseveres o Shows confidence in one's ability o Collaborates/Shares ideas	

Overall Assessment:

MMI Toolkit 3.0 Self Reflection on 21st Century Skills (Partnership for 21st Century Skills, 2011)

Critical Thinking & Creativity

Reason Effectively/ Use Systems Thinking

- Use various types of reasoning (inductive, deductive, etc.) as appropriate to the situation

Make Judgments and Decisions

- Effectively analyze and evaluate evidence, arguments, claims and beliefs
- Analyze and evaluate major alternative points of view

Solve Problems

- Solve different kinds of non-familiar problems in both conventional and innovative ways
- Identify and ask significant questions that clarify various points of view and lead to better solutions

Work creatively with others

- *Demonstrate* originality and inventiveness in work and understand the real world limits to adopting new ideas

Collaborate with Others

- Demonstrate ability to work effectively and respectfully with diverse teams. Assume shared responsibility for collaborative work, and value the individual contributions made by each team member

Communication

- Articulate thoughts and ideas effectively using oral, written and nonverbal communication skills in a variety of forms and contexts;
- Use communication for a range of purposes (e.g. to inform, instruct, motivate and persuade (Partnership for 21st Century Skills, 2010)

Prompts for students to self-assess and peer-assess after a problem-solving task

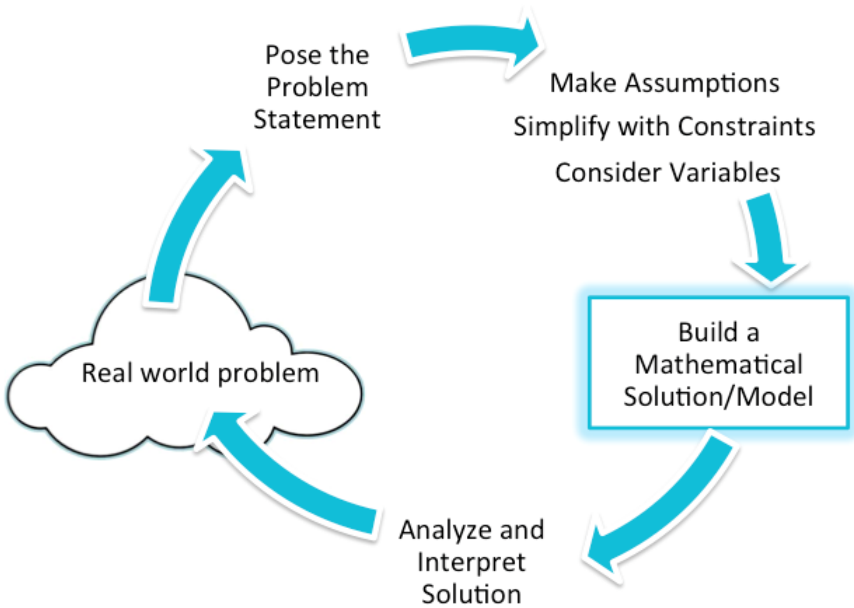
Assessing your 21st Century Learning Skills in Mathematics				
Prompts to assess your 4 Cs Contribution	**Critical Thinking:** How did you solve this problem in new ways linking what you know?	**Creativity:** What new approaches did you consider to solve this problem or did you invent a strategy that was efficient?	**Communication:** Did you share thoughts, questions and solutions?	**Collaboration:** How did you work together to each a goal, using your knowledge, talents and skills?
Self assessment				
Peer Group				

MMI Toolkit 3.1 The Math Modeling Cycle

Mathematical Modeling in the Elementary Grades

Mathematical Modeling involves posing mathematical problems in authentic real life contexts that are relatable to students' personal interests, knowledge and skills.

Mathematical Modeling enables students to use mathematics to help make decisions (i.e., optimize, predict and determine the meaningful solutions to the problem).



Math Modeling Process- Modified from the SIAM- Moody's Mega Math Challenge website)

1. **Pose the Problem Statement: Is it real-world and does it require math modeling? What mathematical questions come to mind?**

2. **Make Assumptions, Define, and Simplify: What assumptions do you make? What are the constraints that help you define and simplify the problem?**

3. **Consider the Variables: What variables will you consider? What data/information is necessary to answer your question?**

4. **Build Solutions: Generate solutions.**

5. **Analyze and Validate Conclusions: Does your solutions make sense? Now, take your solution and apply it to the real world scenario. How does it fit? What do you want to revise?**

6. **Present and Justify the Reasoning for Your Solution**

MMI Toolkit 3.2 Planning for Math Modeling and Debriefing after a Lesson Study

Phases of the Mathematical Modeling Lesson		
Launch	**Explore**	**Summarize**
1. Posing the Problem Statement: Is it real-world and does it require math modeling? What mathematical questions come to mind?	2. Making Assumptions to Define, and Simplify the Real World Problem: What assumptions do you make? What are the constraints that help you define and simplify the problem. 3. Considering the Variables: What variables will you consider? What data/information is necessary to answer your question. 4. Building Solutions: Generate solutions.	5. Analyzing and Validating their Conclusions: Does your solutions make sense? 6. Now, take your solution and apply it to the real world scenario. How does it fit? What do you want to revise?
Iterative process allows students to move back and forth across this process as they revise and refine their thinking.		

Pre-observation
• What important mathematical ideas and competencies will the task and context afford you as you engage your students in this mathematical modeling process?
Post Observation
• How did students actually engage with mathematical ideas in this lesson? How did mathematical modeling support students use of math ideas, tools and reasoning to answer questions about a contextual situation?
Next Steps
• How will leverage students' mathematical ideas, misconceptions revealed, and mathematical opportunities that presented itself in this lesson to build on future lessons?

MMI Toolkit 3.3 Points for Evaluation of the Mathematical Modeling Process

Defining a Problem Statement:
- Are the problems that students posing mathematical in nature?
- Can the problems be posed in a way that allows students to predict, optimize, and make decisions on solutions to the problem?
- Will the students get to the mathematical goals set for the lesson?

Making Assumption and Constraints:
- How do students discuss and determine the assumptions and constraints to define the problem?
- Are their assumptions close to reality? What are the basis for their assumptions?
- How much reasonableness do they demonstrate?

Considering the Variables
- What variables do students define?
- As students consider variables to mathematize or build a solution to the problem, are they considering the appropriate variables?
- Are the quantifiable variables?
- How do they use data and information to build a solution?

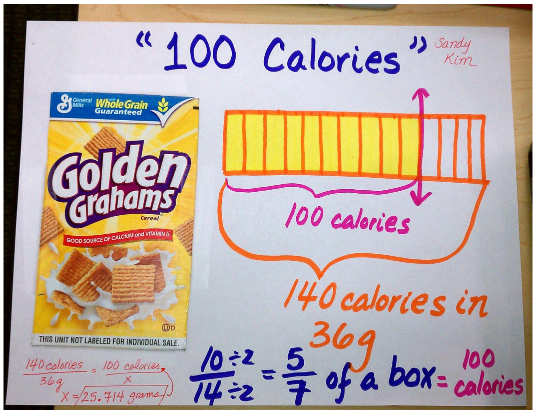
Build a solution
- What problem solving strategies are they using as they build their solution?
- Do students have opportunities to use mathematical modeling to make decisions, predict, optimize, predict and determine a solution to a problem**?**
- Do students have opportunities to model with tools, diagrams, graphs, tables, or equations? Are connections made between multiple representations? What features of the representation(s) do they focus on?

Analyzing and making conclusions:
- How do students learn and practice justifying their solutions and/or explaining their mathematical thinking?
- In what ways are they making sense of the results: Ways in which students are prompted to connect solutions back to the problem situation.

Applying their solution to the real world scenario
- In what ways are students able to connect the solutions back to the problem scenario?
- How does it fit? Do they see ways they may want to revise/refine their thinking?

MMI Toolkit 4. Sharing Math Happenings

Share a math happening!



OBJECTIVE: Share a real-life event (math happening)
and pose a question that can be answered using the information given in the story. Illustrate the number sentence by drawing a picture.

MATERIALS: real world math materials, artifacts, and or manipulatives to engage students

BACKGROUND INFORMATION: Math happenings occur daily in all of our lives. The math happening lessons serve as a framework for teaching many mathematical concepts within the context of real-life math events. The teacher's role in the math happening lesson is:

- to encourage students to share stories about events that actually happen to them
- to interpret, translate, and represent these stories mathematically, using appropriate materials
- to introduce other math concepts for which students are ready.

PROCEDURE:
The teacher begins by grouping students on the floor and asking them to tell a real-life situation that involved math. This can be done by using Plan A, Plan B, or Plan C.

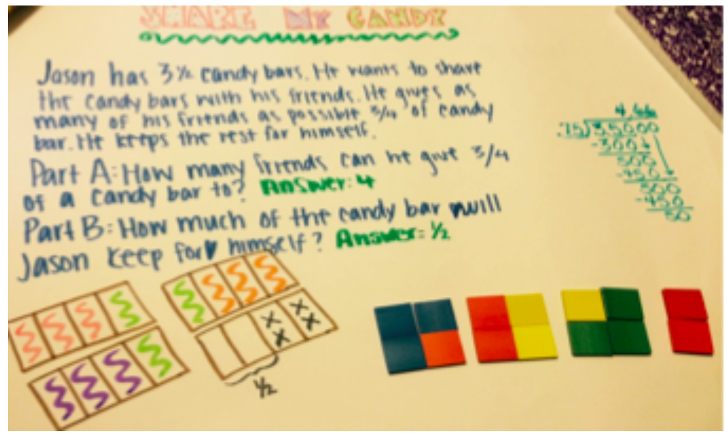
- Plan A - What math happened to you? Tell us about it. (Generate questions about given data.)
- Plan B - Tell me what you did last night, yesterday, or this weekend. (Listen to the event. Probe to gain enough information to make a math story and ask a question.)
- Plan C - Math happened to me. Let me tell you about it. (Tell the story. Say what you're trying to find out. Ask the question.)

After a story has been shared, the teacher and students can model the story using real objects.

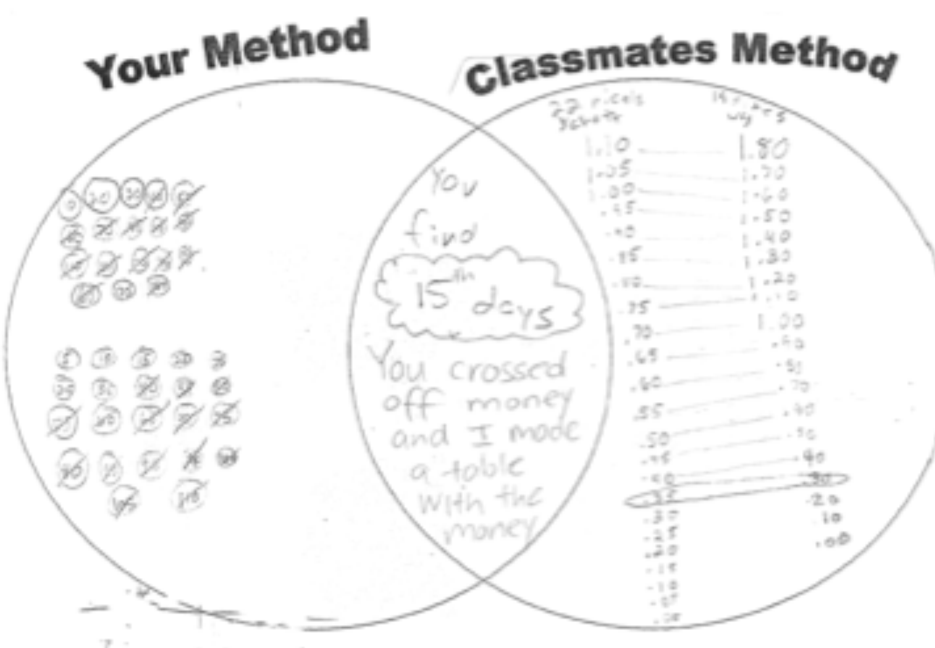
For example, if the story is about setting the table, plastic tableware should be used to act out the story. If real objects are unavailable, representative objects can be used (straws instead of tableware). The teacher then writes a math sentence on a sentence strip to go with the story. (As the "math happening" becomes more of a classroom routine, the <u>students</u> can begin to write the math sentence.) The answer to the math sentence should also be written. The student who shares the math story can later illustrate the story that corresponds to the number sentence. After the teacher has modeled many stories (over time) using materials, all students can represent problems with semi-concrete materials and drawings.

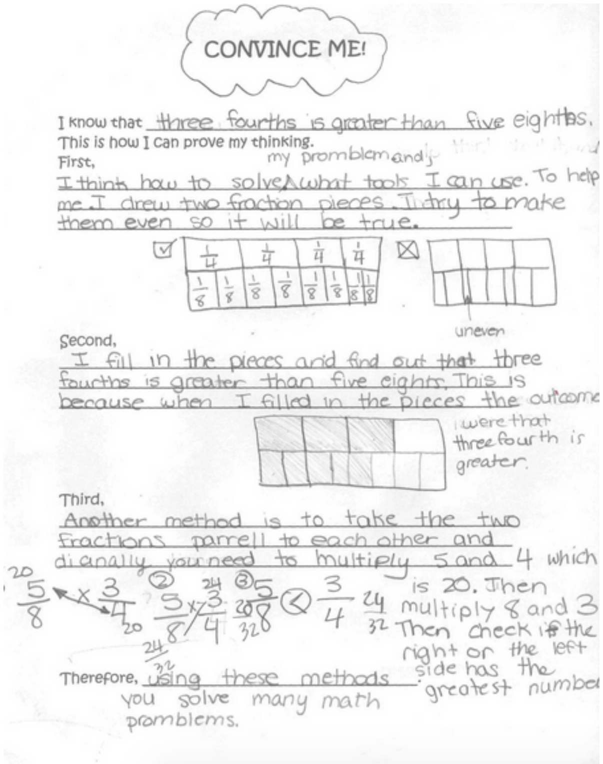
MMI Toolkit 5 Visible Thinking Strategies

<u>Poster Proofs-</u> This is a visible thinking strategy that requires students to display their work publicly. Similar to publishing writing, we want our mathematicians to publish their work so that others can critique their reasoning.



<u>Venn Compare</u>- This visible thinking strategy allows for students to sit side by side and compare each other's strategies and discuss similarities and differences in their solution strategies. This encourages students to appreciate multiple representations and evaluate strategies for efficiency and clarity in their representational thinking




CONVINCE ME!

I know that three fourths is greater than five eighths.
This is how I can prove my thinking.
First,
I think how to solve my promblem and what tools I can use. To help me. I drew two fraction pieces. I try to make them even so it will be true.

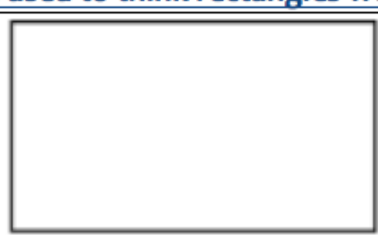
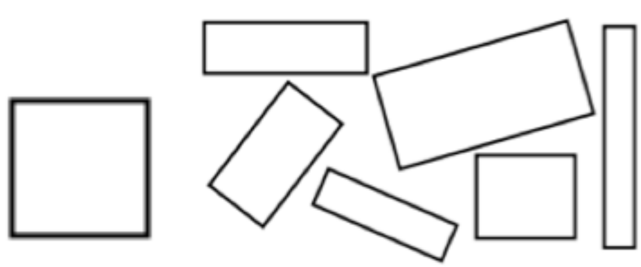
Second,
I fill in the pieces and find out that three fourths is greater than five eighths. This is because when I filled in the pieces the outcome were that three fourth is greater.

Third,
Another method is to take the two fractions parrell to each other and diganally you need to multiply 5 and 4 which is 20. Then multiply 8 and 3 Then check if the right or the left side has the greatest number

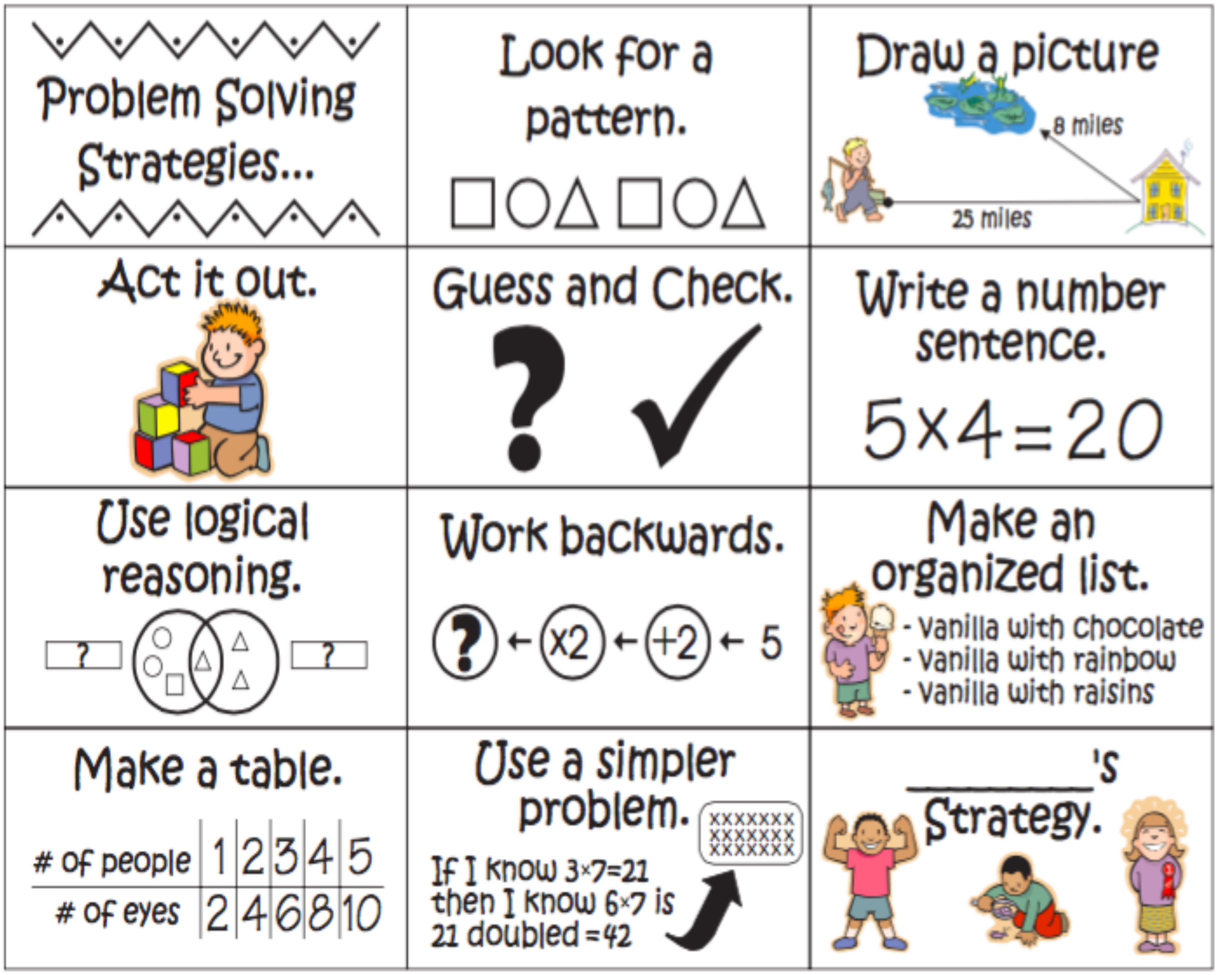
Therefore, using these methods you solve many math promblems.

<u>Convince Me</u>- This visible thinking strategy provides opportunities for diverse strategies to be shared in class and evaluate the number of students who also had similar misconceptions and solutions. First the teacher will sort the number of solutions that students came up with on their own. Next, the teacher can ask students with different answers to convince the other group using several different examples.

<u>I used to think… Now I think…</u> This visible thinking strategy provides an opportunity to share what their pre-conceived ideas were before the lesson and then students are able to share what they think after the lesson. An example of this strategy is asking students to draw all rectangles and after the lesson, some may learn that squares are rectangles as well.

<u>I used to think rectangles were</u>	<u>Now I think rectangles are</u>
	

MMI Toolkit 6 Nameable Problem Solving Strategy Cards



MMI Toolkit 7 Family of Problems

The Ichiro Problem

Adapted from the Lesson Study by:
Linda Gillen, Candice Ives, Steve Klarevas, and Rae Perry
Math 610: Number Systems and Number Theory for K-8 Teachers

The Task

It has been one month since Ichiro's mother has entered the hospital. Ichiro decided to pray with his younger brother at a local temple every morning so that she will get better soon. There are 18 ten-yen coins in Ichiro's wallet and just 22 five-yen coins in the younger brother's wallet. They decided to take one coin from each wallet every day, put them in the offertory box, and continue to pray until either wallet becomes empty. One day when they were done with their prayer, they looked into each other's wallets. The amount of money in the younger brother's wallet was greater than Ichiro's amount of money. When this happened, how many days had it been since they started their prayers?

Big Ideas

- Relationships and patterns
- Solving linear equations (or inequalities) both algebraically and graphically
- Slope as a rate of change and y-intercept as an initial amount
- Writing equations of lines in slope-intercept form

Standards of Learning for Grades 3-4-5

3.19 The student will recognize and describe a variety of patterns formed using numbers, tables, and pictures, and extend the patterns, using the same or different forms.

4.15 The student will recognize, create, and extend numerical and geometric patterns.

4.16a The student will recognize and demonstrate the meaning of equality in an equation.

5.17 The student will describe the relationship found in a number pattern and express the relationship.

Standards of Learning for Grades 6-7-8

7.12 The student will represent relationships with tables, graphs, rules, and words.

7.13a The student will write verbal expressions as algebraic expressions and sentences as equations and vice versa.

7.13b The student will evaluate algebraic expressions for given replacement values of the variables.

8.14 The student will make connections between any two representations (tables, graphs, words, and rules) of a given relationship.

8.15a The student will solve multistep linear equations in one variable with the variable on one and two sides of the equation.

8.16 The student will graph a linear equation in two variables.

The Ichiro Problem (Continued)

Adapted from the Lesson Study by:
Linda Gillen, Candice Ives, Steve Klarevas, and Rae Perry
Math 610: Number Systems and Number Theory for K-8 Teachers

Standards of Learning for Algebra I

A.4 The student will solve multistep linear equations in two variables, including

b) justifying steps used in simplifying expressions and solving equations, using field properties and axioms of equality that are valid for the set of real numbers and its subsets;

d) solving multistep linear equations algebraically and graphically;

f) solving real-world problems involving equations.

Graphing calculators will be used both as a primary tool in solving problems and to verify algebraic solutions.

A.5 The student will solve multistep linear inequalities in two variables, including

a) solving multistep linear inequalities algebraically and graphically;

b) justifying steps used in solving inequalities, using axioms of inequality and properties of order that are valid for the set of real numbers and its subsets;

c) solving real-world problems involving inequalities.

A.6 The student will graph linear equations and linear inequalities in two variables, including

a) Slope will be described as rate of change and will be positive, negative, zero, or undefined;

b) writing the equation of a line.

A.7 The student will investigate and analyze function (linear and quadratic) families and their characteristics both algebraically and graphically, including

f) making connections between and among multiple representations of functions including concrete, verbal, numeric, graphic, and algebraic.

Process Goals

- Problem Solving and Reasoning – Students will examine relationships and patterns and use their understanding of slope as a rate of change and y-intercept as an initial amount in the form y=mx+b to determine both algebraically and graphically the relationship between the money in two brothers' wallets.
- Connections and Representations – Students will recognize and use mathematical connections to extend or generalize patterns. Students will use abstract or symbolic representation to record their findings and solve the problem.
- Communication – Students will justify their findings and present their results to the class with precise mathematical language.

The Ichiro Problem (Continued)

Adapted from the Lesson Study by:
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Related Task – Buying mp3s

You have decided to use your allowance to buy an mp3 purchase plan. Your friend Alex is a member of i-sound and pays $1 for each download. Another one of your friends, Taylor, belongs to Rhaps and pays $13 a month for an unlimited number of downloads. A third friend, Chris, belongs to e-musical and pays a $4 monthly membership fee and $0.40 a month per download. Each friend is trying to convince you to join their membership plan. Under what circumstances would you choose each of these plans and why?

Related Task – Carlos' Cell Phone

Carlos is thinking of changing cell phone plans so he is comparing several different plans.

Plan 1) Pay as you go plan $0.99 per minute
Plan 2) $30 monthly fee plus $0.45 per minute
Plan 3) $40 monthly fee plus $0.35 per minute
Plan 4) $60 monthly fee plus $0.20 per minute
Plan 5) $100 monthly fee for unlimited minutes

What will Carlos need to consider to make his decision? How can Carlos figure out which plan is best for him?