Math S1201
Calculus 3
Chapters 12.1 – 12.4

Summer 2015

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Outline

- CH 12.1 Coordinate Systems
  - Representing points in space
  - Distance between points
  - Metrics
  - Representing a set of points
- CH 12.2 Vectors
  - Definition & Interpretation
  - Operations and manipulating vectors
  - Proving properties
  - Special vectors (basis, unit)

Outline

- CH 12.3 Dot Product
  - Definition & Interpretation
  - Orthogonality
  - Scalar & vector projection
- CH 12.4 Cross Product
  - Definition & Interpretation
  - Matrix and determinant
  - Direction and length of cross-product vector
  - Parallel vectors
  - Scalar triple product
- Applications to physics

Guiding Eyes (12.1)

A. How do you represent a point in space?

B. What is the distance between two points?

C. How do you represent a set of points?

How do you represent a point in space?

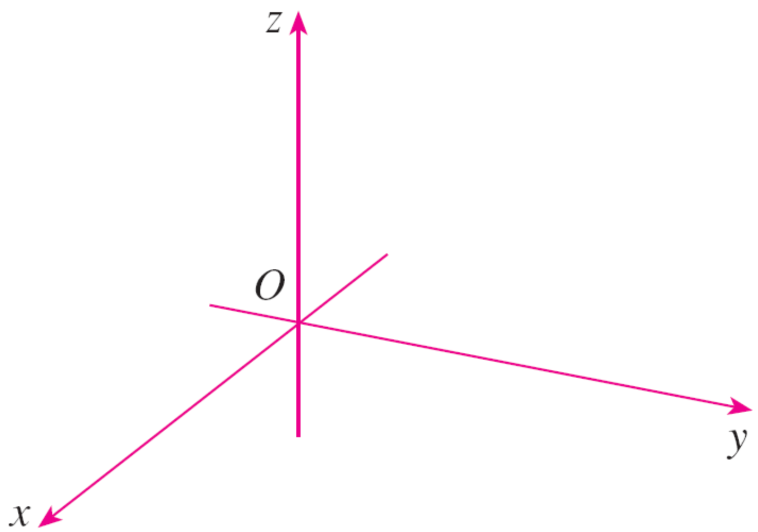
- On a line (1-dimensional): single number ‘a’, x-coordinate
- In a plane (2D): ordered pair (a,b), x & y coordinates
- In space (3D): ordered triple (a,b,c), x, y, z coordinates

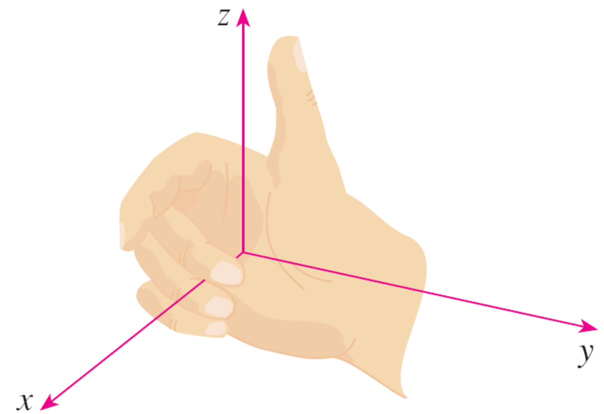
Q. How does the representation generalize to n-dimensional (ND) space?

Answer:

1) Choose a fixed point – the origin
2) Choose perpendicular lines – the coordinate axes
3) Determine orientation (direction) of axes – right hand rule for z-axis

Note: In 2D, axes divide plane into quadrants. In 3D, coordinate planes divide space into octants.


z
O
y
x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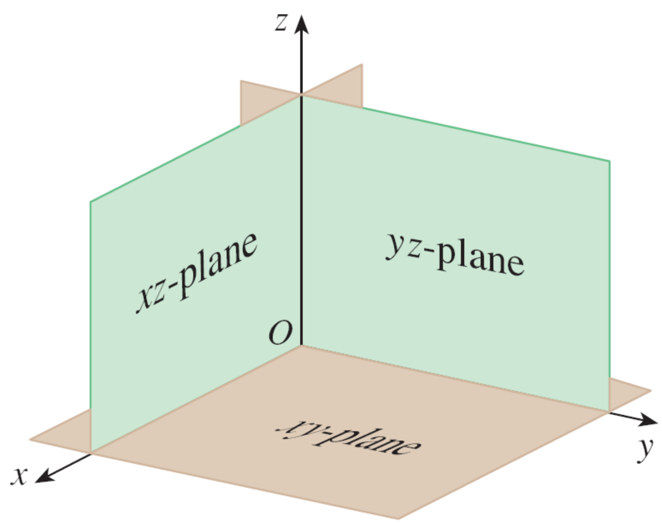

z
xz-plane
yz-plane
O
xy-plane
x
y

How do you represent a point in space?

Q. What is required of our representation?
A. we wish to represent each point in space as a unique ordered n-tuple (triple).

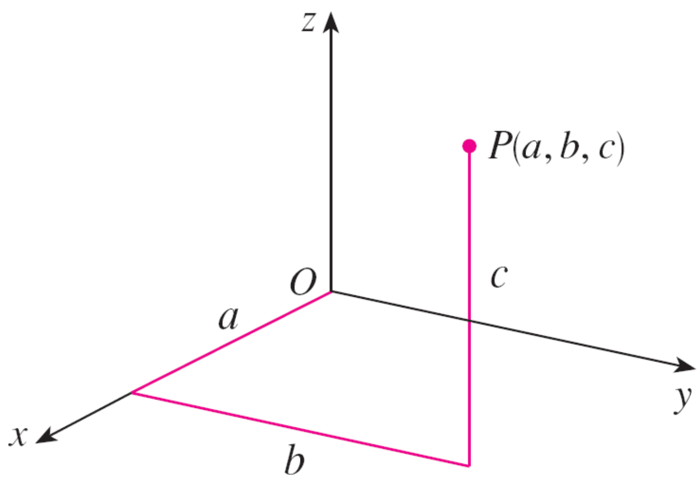
Concept: this one-to-one correspondence between points in space and ordered triples is called a three-dimensional rectangular coordinate system. $\mathbb{R}^3$

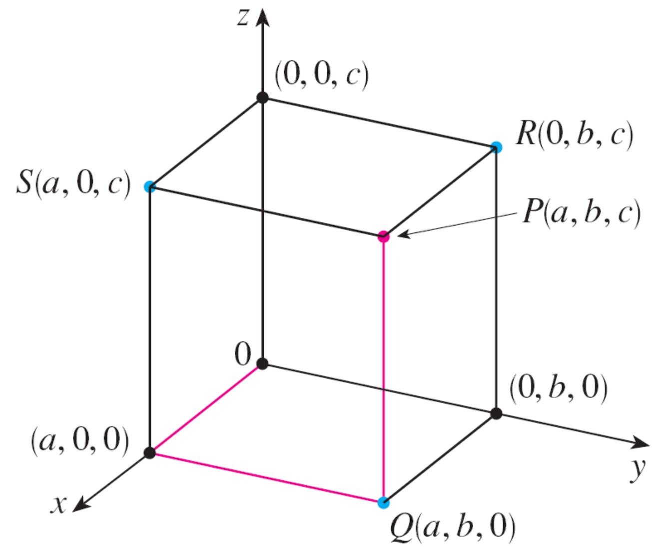
Concept: Cartesian Product of n sets ($A_1 \times A_2 \times \ldots\ldots \times A_n$) is the set of ordered n-tuples ($a_1, a_2, \ldots\ldots, a_n$) where $\forall$i, $a_i \in A_i$

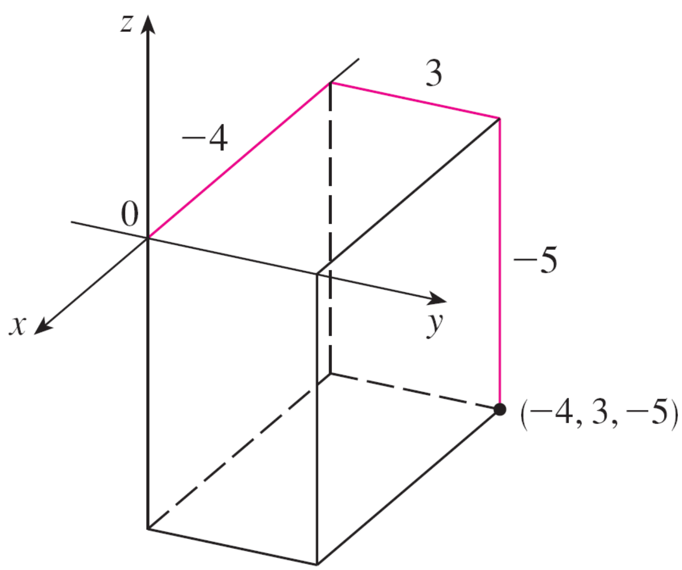
Note: we usually work with real numbers, our sets are denoted $\mathbb{R}$

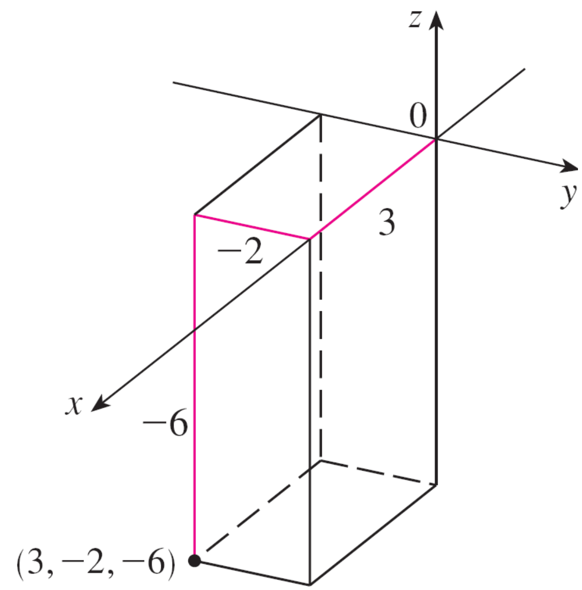
$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Concept: projection of point onto a (coordinate) plane


z
P(a, b, c)
O
c
a
x
b
y


z
(0, 0, c)
R(0, b, c)
S(a, 0, c)
P(a, b, c)
0
(0, b, 0)
(a, 0, 0)
y
x
Q(a, b, 0)


z
3
−4
0
−5
x
y
(−4, 3, −5)


z
0
y
3
−2
x
−6
(3, −2, −6)

What is the distance between two points?

Q. What is the distance between two points on a plane?

A. $$d\left(P_1(x_1,y_1),P_2(x_2,y_2)\right)=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$$

Q. What is the distance between two points in space?

A. Use Pythagorean Theorem twice

Distance Formula in Three Dimensions The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Q. Can we generalize the notion of distance?

Q. How does the distance formula generalize to ND space?

What is the distance between two points?

Q. Can we generalize the notion of distance?

A. A valid distance function (metric) defines the distance for each pair of elements in the space (set), and satisfies 3 properties:

$$(1)\ d(x,y)=0 \iff x=y$$

$$(2)\ d(x,y)=d(y,x)$$

$$(3)\ d(x,z)\le d(x,y)+d(y,z)$$

Concept: metric space is a set equipped with a distance function (metric)

Q. How does the distance formula generalize to ND space?

A. Use summation notation

$$d\left(P_1,P_2\right)=\sqrt{\sum_{k=1}^{n}\left(x_k-y_k\right)^2}$$

Concept: Euclidean space is a metric space with the distance function above. It is denoted by $\Re^n$

How do you represent a set of points?

- A curve in R^2 (equation in x,y)
- A surface in R^3 (equation in x,y,z)

Examples: (C is any constant)

$$(1)\ z = C$$

$$(2)\ y = C$$

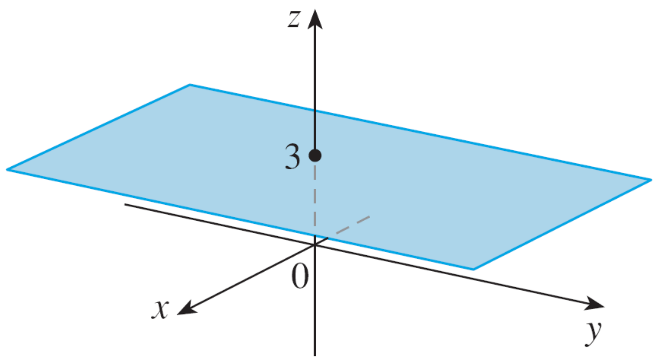
$$(3)\ x = y$$

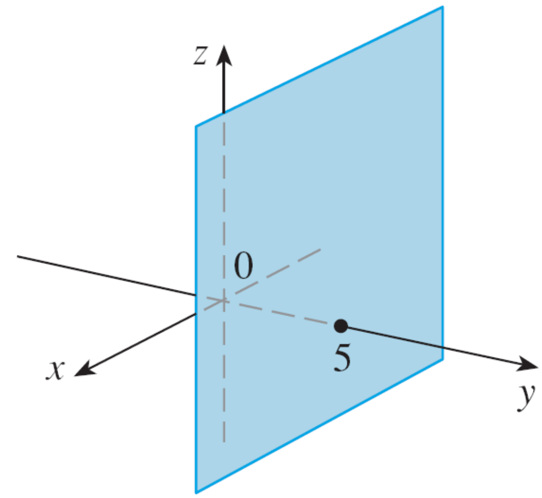
$$(4)\ (x - c_1)^2 + (y - c_2)^2 = r^2$$

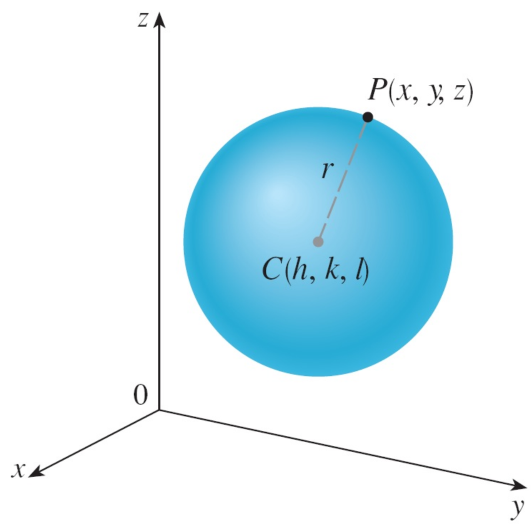
$$(5)\ (x - c_1)^2 + (y - c_2)^2 + (z - c_3)^2 = r^2$$

$$(6)\ C_1 \leq x^2 + y^2 + z^2 \ \ \&\ \ x^2 + y^2 + z^2 \leq C_2$$

Note: equation of circle (4) in 3D is a cylinder!
Note: equation of sphere (5) can be derived from distance formula.
Note: in example (6) we represent a region with inequalities


z
3
0
x
y


z
0
5
x
y


z
P(x, y, z)
r
C(h, k, l)
0
x
y

Guiding Eyes (12.2)

A. Why do you need vectors?

B. How do you manipulate vectors?

C. How do you prove properties (axioms) of vectors?

D. Vectors with special (specific) meaning

Why do you need vectors?

We need to indicate a quantity (object) with both magnitude and direction.
Notation: we specify and initial and terminal points, and use an arrow $(\vec{v})$ or boldface (**v**) to indicate a vector.

Concept: vector (linear) space is a collection of objects called vectors together with two operations (addition and scalar multiplication) which must satisfy certain properties (axioms).

Physical interpretation: forces
Geometric Interpretation: displacements in space (provides visualization to an abstract object)
Algebraic interpretation: n-tuples

Concept: equivalent (equal) vectors have the same length and direction regardless of (invariant to) position

How do you manipulate vectors?

Algebraic representation using components: $\mathbf{a} = \langle a_1, a_2 \rangle, \mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Concept: the particular representation of an arbitrary vector from the origin is called the position vector.

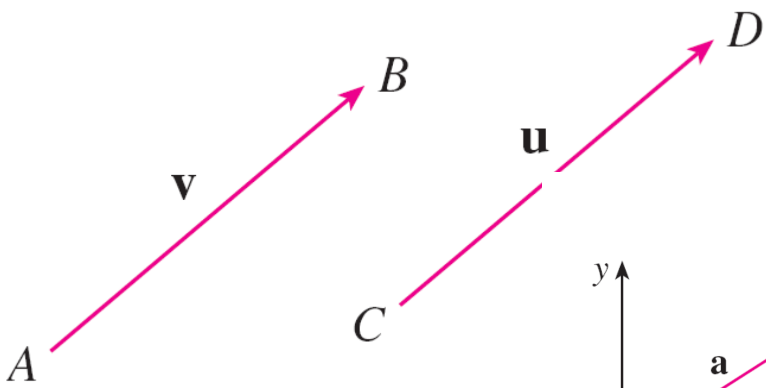
Problem: given two points P & Q, find the vector represented by the directed line segment from P to Q.
Solution: subtract components of initial point from terminal point.

Concept: vector addition is the "sum" of two vectors.
Algebraic definition: add components of vectors
Geometric definition: triangle law

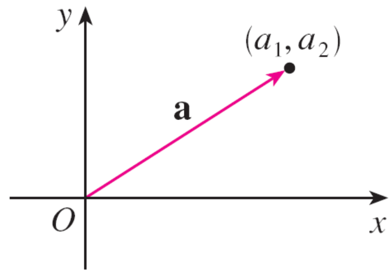
Concept: magnitude (length) of vector is the length of its representation
Algebraic definition: use distance formula $|\mathbf{a}| \equiv \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

How do you manipulate vectors?

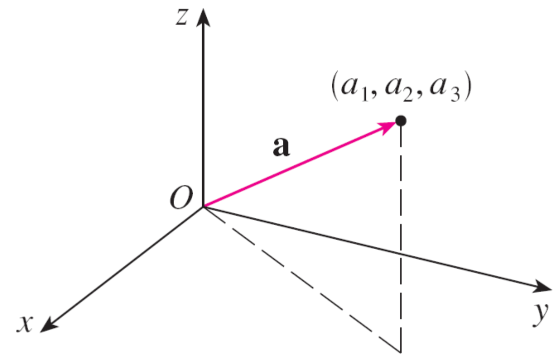
Concept: scalar multiplication adjusts the length of a vector and changes its direction if scalar is negative
Algebraic definition: multiply components by a scalar (number)
Geometric definition: adjust length, reverse direction



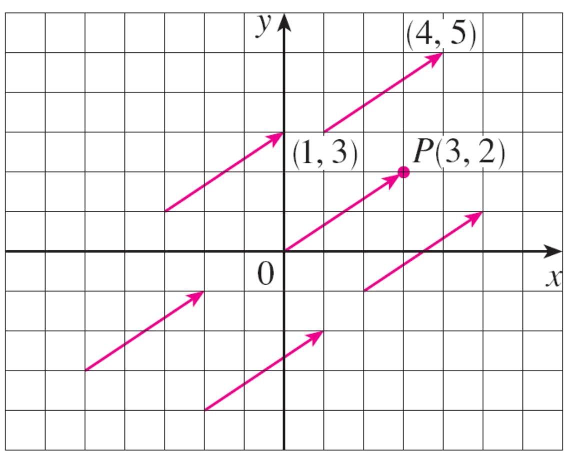
$\mathbf{v} = \overrightarrow{AB}$

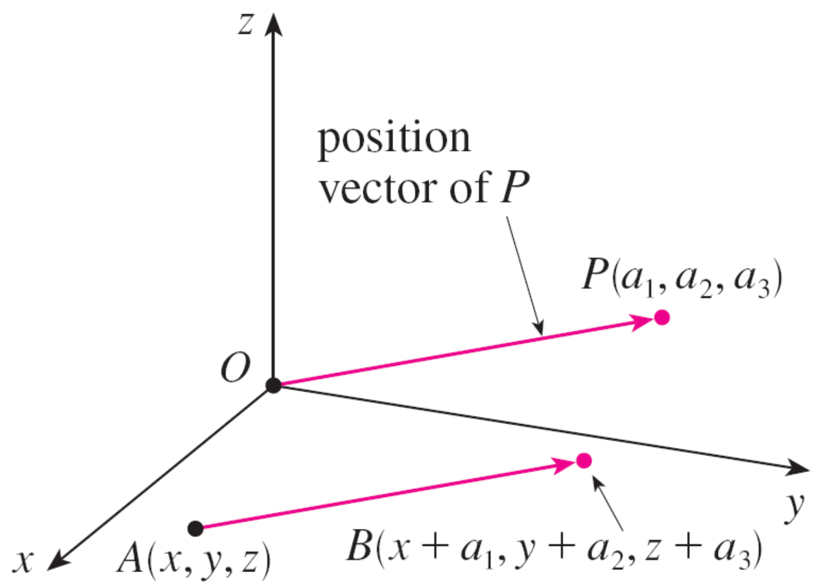


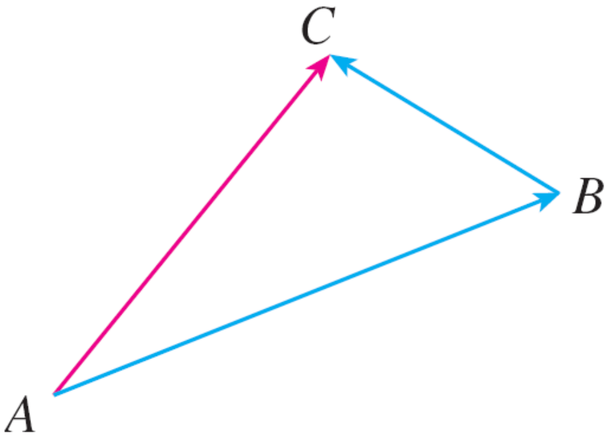
$\mathbf{a} = \langle a_1, a_2 \rangle$



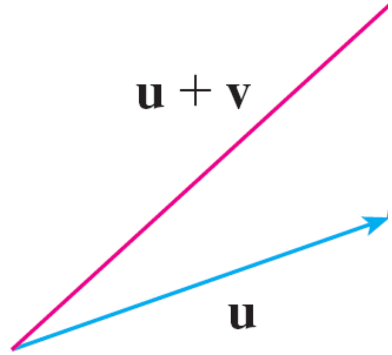
$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

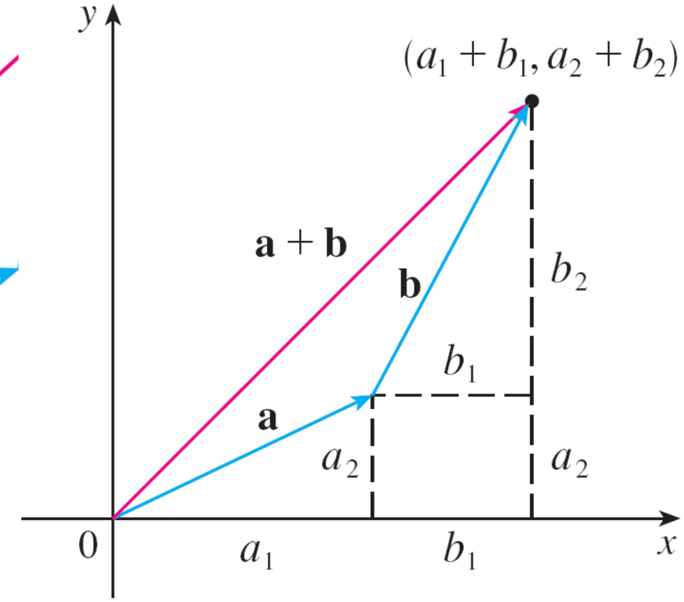


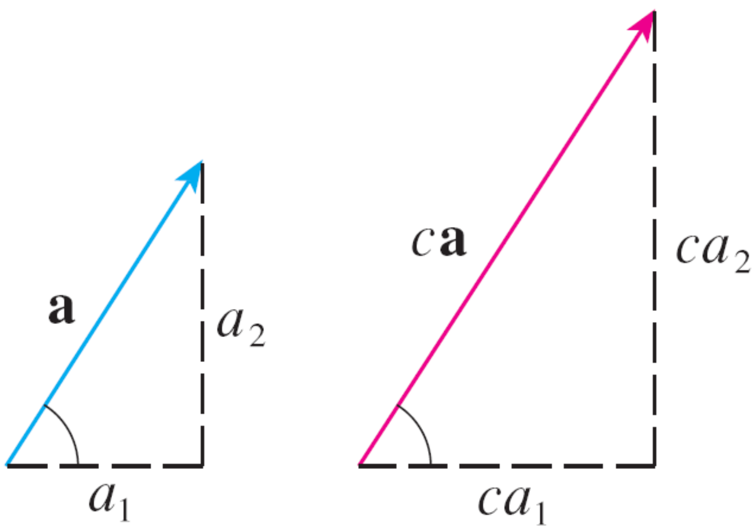


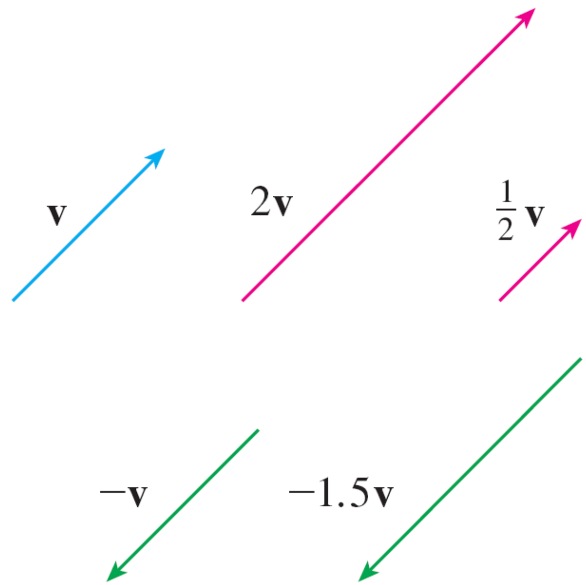


$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$







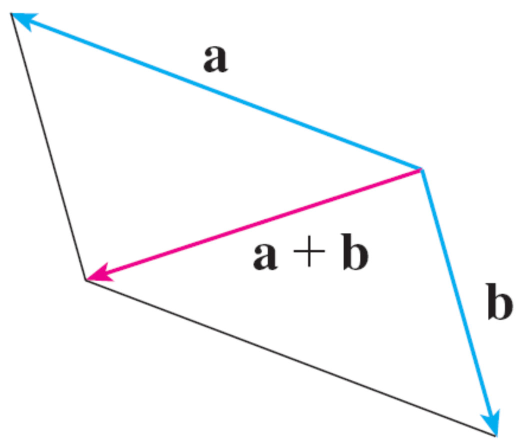


How do you prove properties (axioms) of vectors?

Problem: given an axiom, prove that it is valid (correct).
Solution: suppose you are given an equality (equation), you need to demonstrate that the right-hand-side (RHS) is exactly the same thing as the left-hand-side (LHS).

Example: commutative property $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
Geometric proof: parallelogram law
Algebraic proof:

(1) Re-write LHS using components of vectors – your starting point
(2) Re-write RHS using components – your "target"
(3) Algebraic manipulation: manipulate components using general algebra/arithmetic rules to make LHS similar to RHS



Properties of Vectors If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Special vectors

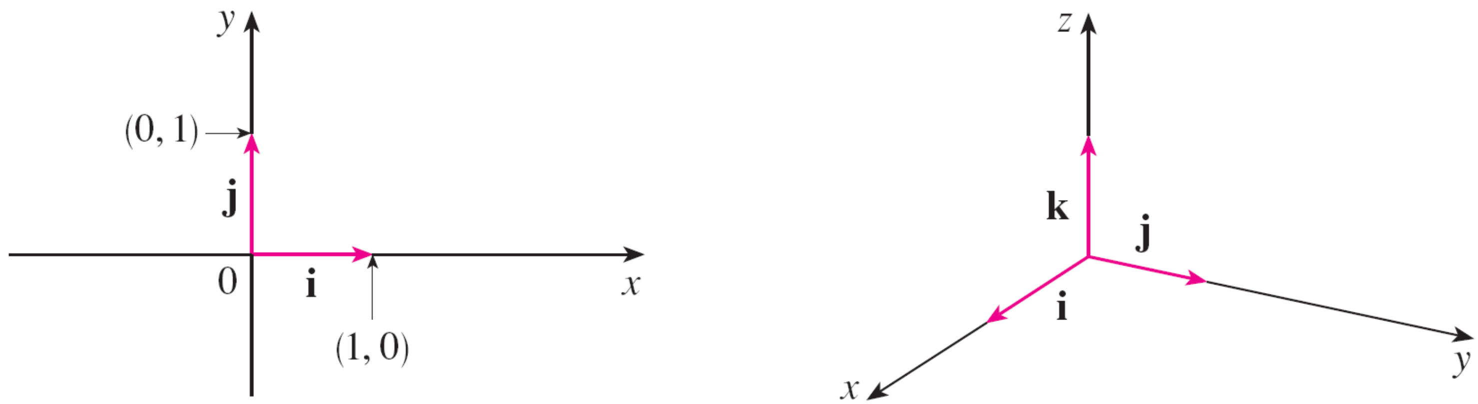
Concept: zero vector is the only vector without direction

Concept: a basis or (set of basis vectors) is a set of linearly independent vectors where every other vector is linearly dependent on this set

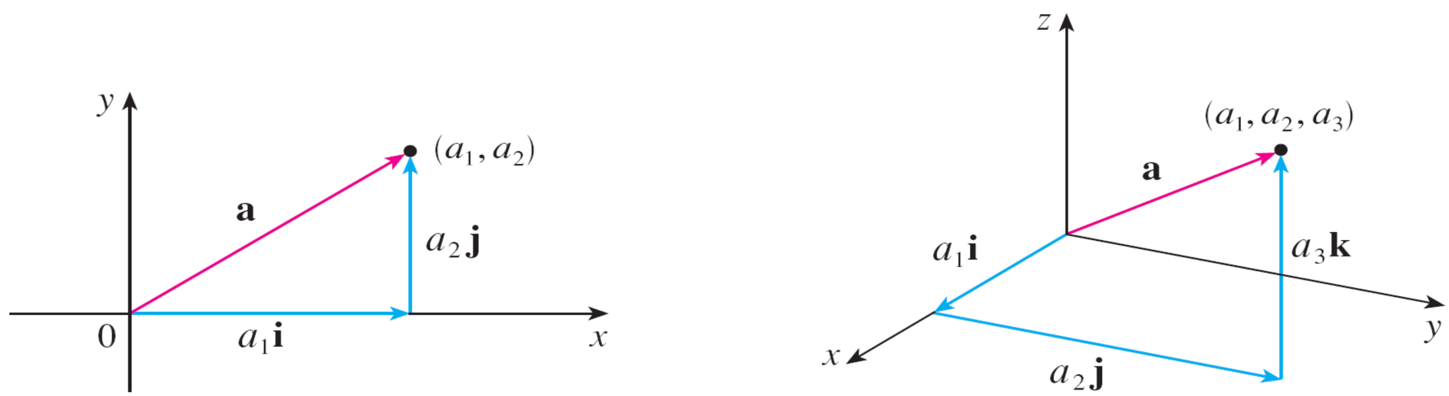
Concept: a standard (canonical) basis is a set of vectors of length one that point in the directions of the positive coordinate axes

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

Concept: a unit vector is a vector whose length is 1. $\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$



(a) (b)



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ (b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Figure 18

Guiding Eyes (12.3)

A. Can we multiply vectors by vectors?

B. We can “project” points, can we project vectors?

Can we multiply vectors by vectors?

Concept: scalar / inner / dot product is a function defined for every pair of elements (objects) in the space.
Note: inner product can be used to define the length (norm) and angle between vectors

Algebraic definition:

$$\mathbf{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$|\mathbf{a}| \equiv \|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{a_1 a_1 + a_2 a_2 + a_3 a_3} = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

Geometric definition:
Observe: angle between vector and itself is zero
Observe: angle between orthogonal vectors is 90
Observe: if vectors point in opposite directions

$$1)\ \mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\| \|\mathbf{a}\|$$

$$2)\ \mathbf{a} \perp \boldsymbol{b} \Rightarrow \mathbf{a} \cdot \boldsymbol{b} = \mathbf{0}$$

$$3)\ \mathbf{a} \cdot \boldsymbol{b} = -\|\mathbf{a}\| \|\boldsymbol{b}\|$$

Can we “guess” the formula from observations?

Can we multiply vectors by vectors?

Dot Product formula: $\mathbf{a} \cdot \boldsymbol{b} = \|\mathbf{a}\| \|\boldsymbol{b}\| \cos\theta$

Proof:

1) Use Law of Cosines (generalization of Pythagorean Th to any triangle).
2) Use properties of dot product to rewrite LHS
3) Simplify equation

$$\|\mathbf{a} - \boldsymbol{b}\|^2 = \|\mathbf{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\|\mathbf{a}\|\|\boldsymbol{b}\|\cos\theta$$

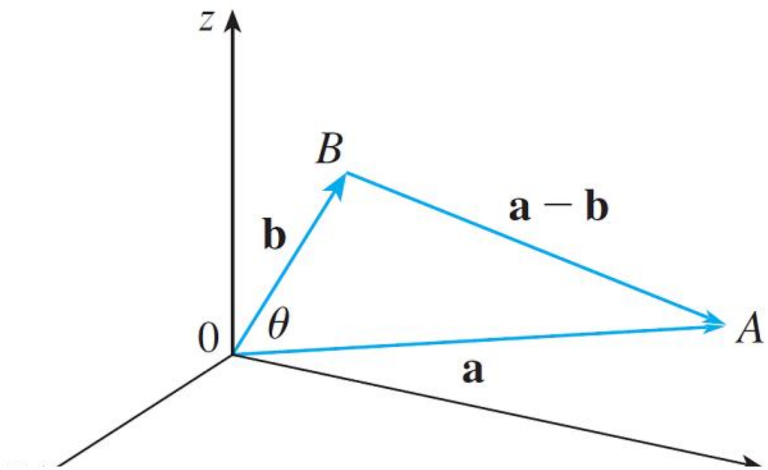
$$\|\mathbf{a} - \boldsymbol{b}\|^2 = \|\mathbf{a}\|^2 - 2(\mathbf{a} \cdot \boldsymbol{b}) + \|\boldsymbol{b}\|^2$$

$$-2\mathbf{a} \cdot \boldsymbol{b} = -2\|\mathbf{a}\|\|\boldsymbol{b}\|\cos\theta$$

$$\mathbf{a} \cdot \boldsymbol{b} = \|\mathbf{a}\|\|\boldsymbol{b}\|\cos\theta$$

Problem: find the angle between two vectors (**a** & **b**).
Solution: 1) Compute the magnitude of each vector, and their dot product
2) Apply formula for angle, and use arccos (inverse of cosine function)

Problem: show that two vectors (**a** & **b**) are perpendicular (or not).
Solution: compute dot product, if zero then yes.



2 **Properties of the Dot Product** If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

6 **Corollary** If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

7 Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

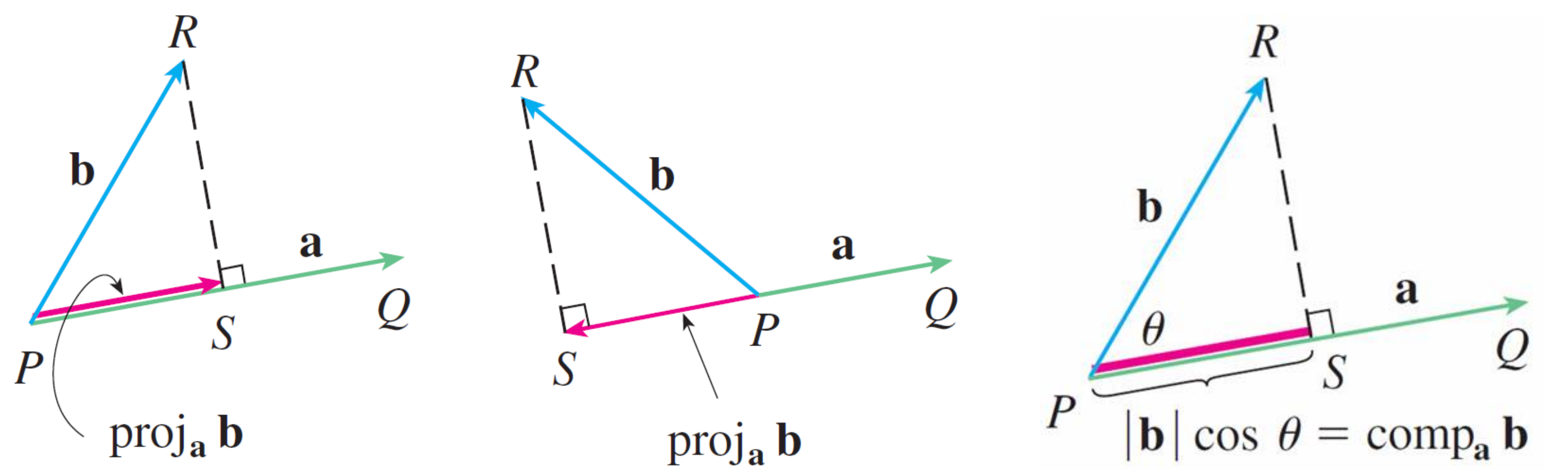
Can we project vectors?

What do we get when we project a vector **b** onto vector **a**?
Concept: vector projection is a vector component (of **b** in the direction of **a**)
Concept: scalar projection is the signed magnitude of the vector projection

$$\boldsymbol{b}_{sp} = \|\boldsymbol{b}\|\cos\theta = \|\boldsymbol{b}\|\frac{\mathbf{a}\cdot\boldsymbol{b}}{\|\mathbf{a}\|\|\boldsymbol{b}\|} = \frac{\mathbf{a}\cdot\boldsymbol{b}}{\|\mathbf{a}\|}$$

$$\boldsymbol{b}_{vp} = \left(\frac{\mathbf{a}\cdot\boldsymbol{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \mathbf{a}\left(\frac{\mathbf{a}\cdot\boldsymbol{b}}{\|\mathbf{a}\|^2}\right)$$

Problem: find the scalar and vector projection of vector **b** onto vector **a**.
Solution: compute the dot product, the magnitude of **a**. To get vector projection, multiply result by unit vector in direction of a.



Scalar projection of **b** onto **a**: $$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

Vector projection of **b** onto **a**: $$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$$

Guiding Eyes (12.4)

A. Given two vectors, how do you find a vector which is perpendicular to both?

B. What is the direction and length of this vector?

C. What is the significance of "multiplying" more than two vectors (triple products)?

How do you find a perpendicular vector?

Problem: given two nonzero vectors **a, b**, find a nonzero vector **c**. perpendicular to both.

Solution:

1) By definition of dot products
2) Rewrite using components
3) Eliminate one variable
4) Obvious solution (?)
5) Substitute back
   obtain missing variable.

$$1)\ \mathbf{a}\cdot\boldsymbol{c}=0 \qquad \boldsymbol{b}\cdot\boldsymbol{c}=0$$

$$2)\ \mathbf{a}\cdot\boldsymbol{c}=a_1c_1+a_2c_2+a_3c_3=0$$

$$\boldsymbol{b}\cdot\boldsymbol{c}=\mathrm{b}_1c_1+b_2c_2+b_3c_3=0$$

$$3)\ a_1b_3c_1+a_2b_3c_2+a_3b_3c_3=0$$

$$a_3\mathrm{b}_1c_1+a_3b_2c_2+a_3b_3c_3=0$$

$$\Longrightarrow a_1b_3c_1-a_3\mathrm{b}_1c_1+a_2b_3c_2-a_3b_2c_2=0$$

$$\left(a_1b_3-a_3\mathrm{b}_1\right)c_1+\left(a_2b_3-a_3b_2\right)c_2=0$$

$$4)\ c_1=\left(a_2b_3-a_3b_2\right) \qquad c_2=-\left(a_1b_3-a_3\mathrm{b}_1\right)$$

$$5)\ c_3=\left(a_1b_2-a_2b_1\right)$$

How do you find a perpendicular vector?

Definition follows from derivation:

4 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

How can we "remember" this result?

Concept: a matrix (rectangular array of numbers) determinant gives the scaling factor and the orientation induced by the mapping represented by the matrix.

Note: cross product formula is the determinant of a symbolic 3by3 matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \left\langle a_2 b_3 - a_3 b_2, -\left(a_1 b_3 - a_3 \mathrm{b}_1\right), a_1 b_2 - a_2 b_1 \right\rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

What is the direction & length of cross-product vector?

Q. In which direction does the cross-product vector point?
A. Right-hand rule

Q. What is the length of the cross product vector?
A. The following formula is courtesy of "black magic"

9 Theorem If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

10 Corollary Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

What is the length of cross-product vector?

$$\mathbf{a} \times \mathbf{b} = \left\langle a_2 b_3 - a_3 b_2, -\left(a_1 b_3 - a_3 \mathrm{b}_1\right), a_1 b_2 - a_2 b_1 \right\rangle$$

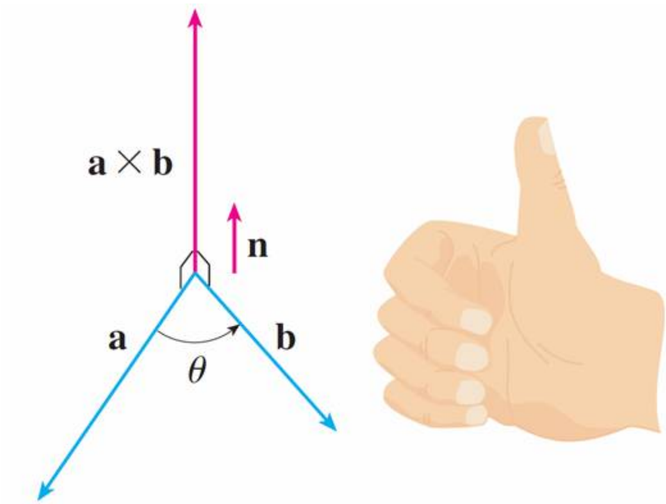
9 Theorem If θ is the angle between **a** and **b** (so $0 \leqslant \theta \leqslant \pi$), then

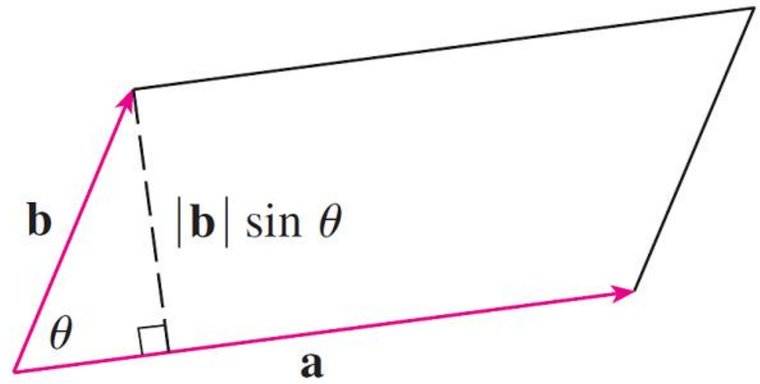
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Algebraic derivation:

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|^2 &= (|\mathbf{u}||\mathbf{v}|)^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= (|\mathbf{u}||\mathbf{v}|)^2 - (|\mathbf{u}||\mathbf{v}| \cos \theta)^2 \\ &= (|\mathbf{u}||\mathbf{v}|)^2 - (|\mathbf{u}||\mathbf{v}|)^2 \cos^2 \theta \\ &= (|\mathbf{u}||\mathbf{v}|)^2 (1 - \cos^2 \theta) \\ &= (|\mathbf{u}||\mathbf{v}|)^2 \sin^2 \theta. \end{aligned}$$

*Geometric*ally: area of parallelogram determined by vectors a & b.





11 Theorem If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Related Problems

Problem: given three points P, Q, R, find a vector perpendicular to the plane containing all three points.

Solution:

1) Choose one point (P), determine two vectors (PQ, PR)
2) Compute cross product of vectors

Problem: given the vertices of a triangle, compute its area.

Solution: area of parallelogram is twice the area of a triangle.

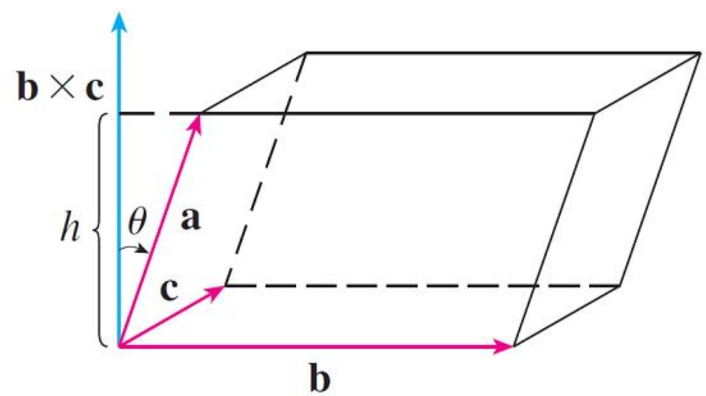
1) Solve previous problem (above)
2) Compute magnitude of cross product and multiply by ½.

Triple Products

Concept: scalar triple product is defined as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
Concept: co-planar vectors lie in the same plane (scalar triple product is 0).
Concept: vector triple product is defined as $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

*Geometric*al interpretation: volume of a parallelepiped is the magnitude of a scalar triple product.
Derivation:
1) Area of parallelogram = magnitude of cross product $A = |\mathbf{b} \times \mathbf{c}|$
2) Height = $h = |\mathbf{a}||\cos\theta|$
3) Volume = Area * Height = $V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
4) Observe that volume is an expression of a dot product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



Applications (Physics Problem 1)

Problem: weight (**w**) hangs from two wires. Find the tensions ($\mathbf{T}_1$ and $\mathbf{T}_2$) and their magnitude in both wires.

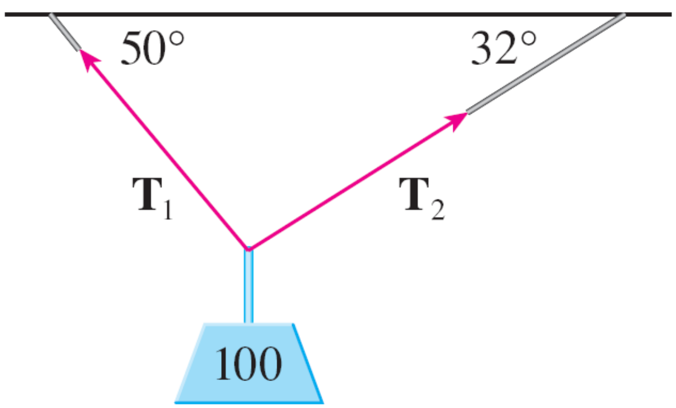
Solution:

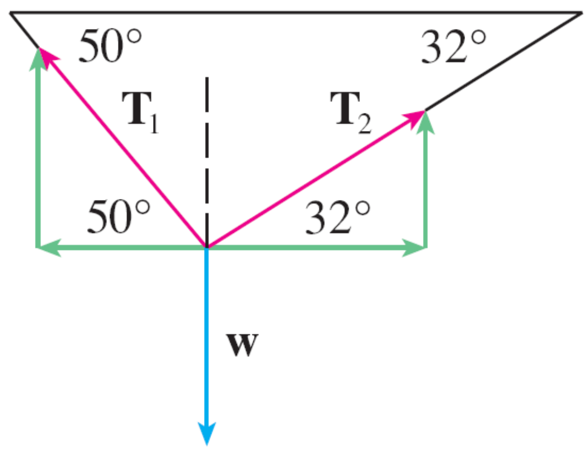
1) Express $\mathbf{T}_1$ and $\mathbf{T}_2$ as vectors (i.e., in terms of their horizontal and vertical components) – need to choose origin of coordinate system!

2) The sum of the tensions ($\mathbf{T}_1 + \mathbf{T}_2$) counterbalances (equality) the weight **w** so we must have an equation.

3) Solve for components in equation, obtain magnitude of tensions.

4) Substitute back to obtain the tension vectors





Applications (Physics Problem 1)

Step 1)

$$\mathbf{T}_1 = -|\mathbf{T}_1|\cos 50^\circ\,\mathbf{i} + |\mathbf{T}_1|\sin 50^\circ\,\mathbf{j}$$
$$\mathbf{T}_2 = |\mathbf{T}_2|\cos 32^\circ\,\mathbf{i} + |\mathbf{T}_2|\sin 32^\circ\,\mathbf{j}$$

Step 2)

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\,\mathbf{j}$$

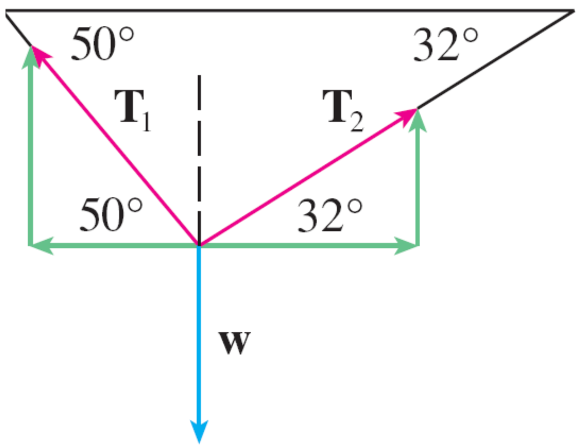
Step 3)

$$-|\mathbf{T}_1|\cos 50^\circ + |\mathbf{T}_2|\cos 32^\circ = 0$$
$$|\mathbf{T}_1|\sin 50^\circ + |\mathbf{T}_2|\sin 32^\circ = 100$$

$$|\mathbf{T}_1|\sin 50^\circ + \frac{|\mathbf{T}_1|\cos 50^\circ}{\cos 32^\circ}\sin 32^\circ = 100$$

$$|\mathbf{T}_1| = \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ}$$

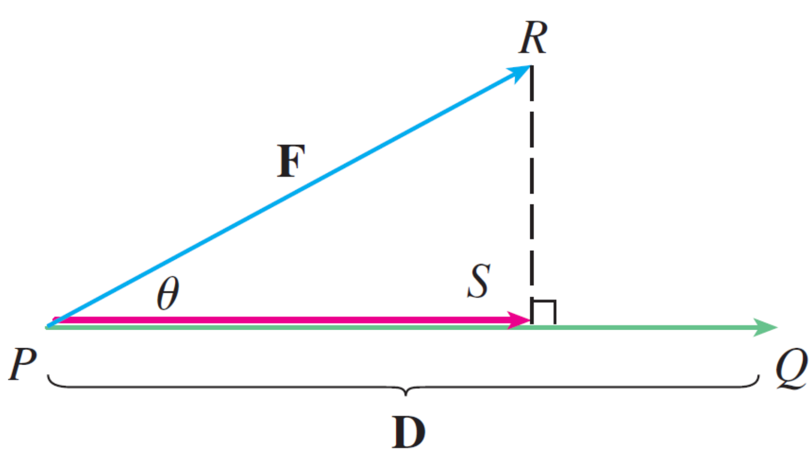
$$|\mathbf{T}_2| = \frac{|\mathbf{T}_1|\cos 50^\circ}{\cos 32^\circ}$$

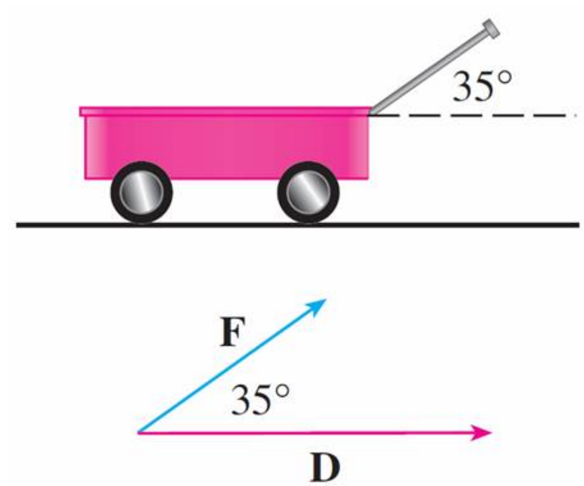


Applications (Physics Problem 2)

Problem: a wagon is pulled a distance **D** along a horizontal path by a constant force of **F**. The handle of the wagon is held at an angle of $\theta°$ above the horizontal. Find the work **W** done by the force.

Solution:
1) The **work** done by a force is defined to be the product of the component of the force along the path and the distance moved. $W = (|\mathbf{F}| \cos \theta)\,|\mathbf{D}|$
2) $W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}|\,|\mathbf{D}| \cos \theta$





Applications (Physics Problem 3)

Problem: a bolt is tightened by applying a force **F** to a wrench of length **M.** Find the magnitude of the torque about the center of the bolt.

Solution:

1) The **torque** $\boldsymbol{\tau}$ measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. It is defined to be the cross product of the position and force vectors (relative to the origin). $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$

2) The magnitude of the torque vector is $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$