MATH TRAILS

Mary Margaret Shoaf
Henry Pollak
Joel Schneider

Executive Director, COMAP: Solomon Garfunkel
Production Manager: George Ward
Cover and Design: Daiva Kiliulis
Production: Tim McLean, Pauline Wright

The Consortium for Mathematics and Its Applications (COMAP)

175 Middlesex Turnpike, Suite 3B
Bedford, MA 01730

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It was not quite a typical Sunday dinner. Robert's sister Debbie who was an engineer with a local chemical company had joined the Johnson family—Jean, Robert, and their children Sally and Tom. The conversation around the table came to the math course in school. "As I see it," said Tom, "there are three troubles with math: I don't like it, none of the rest of you ever liked it, AND it's of absolutely no use. Oh, I don't mean you, Aunt Debbie," he added when he saw the frown on her face, "You like the stuff, I know, and you've told me before how much it helps you with your work. But it does absolutely nothing for any of the rest of us except waste our time and make us miserable."

"You've mentioned this before, Tom," said Debbie, "and some of the other people at the factory have told me about similar sentiments in their families. I've been thinking about it, and I want to try something when we all take our stroll after dinner. Let's see if anything that reminds us of math shows up as we walk along." Now it was Tom's turn to frown a little. "Don't worry, Tom, this won't spoil our digestion—or our walk. Nothing to memorize, no right or wrong answers, no tests, we're just going to keep our eyes open."

"I see you've planted your begonias along the edge of the driveway, Bob," said Debbie. "Why did you use this pattern?"

"Well, the Garden Center sold them by the dozen, so I bought two dozen," Bob answered. "The strip is a little narrow, and six rows of four each didn't quite fit. Then I tried eight rows of three and that fit, but it didn't look right. Too regular, looked like a box! But by having seven rows, alternating three and four plants, it looked a lot better."

"I like that pattern too, Dad," Sally chimed in, "but I'd have liked it even better if you'd started with the 4 instead of the 3. You know, like the stars on the flag."

"Would that have worked?" asked Debbie.

"Sure, why not," said Sally. "4, 3, 4, 3, 4, 3, 4 instead of 3, 4, 3, 4, 3, 4, 3."

"But that takes 25 plants, not 24," Debbie observed. "Would any 'bigger, smaller…, bigger' pattern have worked with two dozen plants?"

"Enough of this," said Tom, "Let's get moving!"

"Where's your car, Debbie?" wondered Jean. Debbie told her that there wasn't a parking space in front of the house, and she had left it around the corner. "But there's usually space for three cars in front of our house. What happened?"

"Well, you see the two cars parked here. There's a lot of space between them, but not enough for another car. Of course, if they had marked parking spaces like they do downtown, this wouldn't happen."

"But those downtown spaces are angle parking, which sure wouldn't look right here. And the parking in the public lot isn't parallel either, it's at right angles to the curb."

“I had hoped you would bring up all the different possibilities,” said Debbie. “Why do you suppose there are so many different patterns of parking?”

As they walked by the Jorgensons, they noticed the spruce in front of the house. It had been planted just two years ago. “It certainly has grown since it was planted. I wonder how much?” They had fun discussing that question. Bob recalled that it was Tom’s height when they first saw it and even though Tom had grown since then, the tree had grown a lot more. How tall was it now? Jean suggested that Tom stand next to the tree, and that they measure the two shadows. If Tom’s shadow were twice his height, then the tree’s shadow would be twice the tree’s height! They hadn’t brought anything to measure shadows, but they could estimate pretty well. Someone suggested that you really needed only the ratio of the two shadows! After some discussion everyone agreed with that. They paced the shadows and seemed pretty pleased with themselves.

They had no trouble finding more good questions after that. A motorcycle came roaring by—how fast do you suppose it was going, and how fast was the dog going that was chasing it? That new front door at the Brown’s: The glass above it looked like a semicircle but was it really? How far apart were the dashed white lines in the middle of the main road they crossed, and how might someone have decided on the spacing? What’s the pattern of the spirals on those pinecones overhanging the sidewalk? The street seemed to go uphill for a stretch, and they wondered what the grade was. How would you estimate that? Bob suggested letting a ball roll down, and seeing how long it took to go 20 paces. They thought this should give them an idea of the grade. The steeper it was the less time the ball should take, but they weren't exactly sure what to do next to figure it out.

Tom told Aunt Debbie that this had been an interesting walk—a lot better than he thought it was going to be! He had been surprised by all the mathematical ideas that had come up as they strolled through the neighborhood. But he added that he wouldn't have thought of any of these by himself! It was Aunt Debbie's knack and experience in seeing math every place she looked that had made the difference. He didn't think they could ever do anything like this without her.

Debbie replied that what they had done was, in a sense, walk a math trail with her acting as a trail guide. It was like a nature walk with a ranger in the nearby state park, where the ranger kept telling them what to look for and answered their questions. But did she really have to be there in person? Sally said that she had been on a different kind of nature walk, one where she picked up a printed trail guide at the beginning. There were numbered stops on the trail, and the trail guide pointed out special features at the stops, and trees and plants to look for along the way. She said it was different from a group walking with a ranger. On the one hand there was no one to ask about some unexpected observation, but on the other hand she could proceed at her own pace and follow up on some animal tracks she hadn't seen before. Couldn't you have a math trail like that?

Debbie asked them to think about the various questions they had considered. Suppose you wrote them down in a trail guide—would it work? Bob said that he would not want to have everybody in a trail walking group stop, dicuss, and trample all over his begonia bed; but that you could come to the same question in the public garden downtown! The steepness of a grade, the height of a tree or building or statue that stands out, and the geometric shapes that are part of many structures are permanent and public. You could make them trail stops and prepare questions that would start discussions about them. You could also be prepared to look for patterns of parking, or a dog chasing a motorcycle, or somebody's hat blowing off as you walk between stops. The trail guide could ask you to keep these things in mind. The main thing was that they had found a lot of interesting questions to think about. They were using mathematical thinking in a carefree non-threatening environment and they were having fun. It seemed to them that you could do this either with your own Aunt Debbie or with a trail guide that everybody's Aunt Debbie had prepared for them.

This book is meant to provide ideas for the Aunt Debbies of this world and examples of the kinds of questions that they might find, as well as thoughts about where and how to actually lay out a trail and organize participation in it. There are three parts: An overall discussion of the purposes and the organization of a math trail; examples, with lots of pictures, of trails in various settings; and more detailed discussion of some of the mathematics that is likely to arise. Have fun!

INTRODUCTION

A *mathematics trail* is a walk to discover mathematics. A math trail can be almost anywhere—a neighborhood, a business district or shopping mall, a park, a zoo, a library, even a government building. The *math trail map* or guide points to places where walkers formulate, discuss, and solve interesting mathematical problems. Anyone can walk a math trail alone, with the family, or with another group. Walkers cooperate along the trail as they talk about the problems. There's no competition or grading. At the end of the math trail they have the pleasure of having walked the trail and of having done some interesting mathematics. Everyone, no matter what age, gets an "I Walked the Math Trail" button to wear.

This book is a guide to blazing a math trail. We'll review the history of math trails and discuss their attributes. We'll also discuss practical issues of organization and logistics in setting up and maintaining a math trail. We'll discuss mathematical issues in choosing and describing problems and tasks along a trail. And we'll also describe a variety of specific examples of trails and of problems.

BACKGROUND AND HISTORY

Math trails fit very nicely into the ideas of popularization of mathematics and of informal mathematics education that have been increasingly recognized as valuable adjuncts to improving mathematics education in the schools. The *NCTM Principles and Standards for School Mathematics* (2000) and *Curriculum Standards* (1989) call for recognizing broad characteristics of mathematics as Communication, Connections, Reasoning, and Problem Solving. Math trails are a medium to experience mathematics in all of these dimensions. Math trails anticipated the *NCTM Standards and Principles*. They exemplify a worldwide collection of projects that aim to popularize mathematics through out-of-school activities. By providing opportunities for doing mathematics out of school, these projects extend time spent thinking about math and math problems. They also tend to connect back into school. Many trailblazers are school people, and teachers often take advantage of the existence of trails by including them in their instructional programs. All of this makes for a stronger mathematics education program in general (Blane, 1989).

The earliest math trails appeared in England and in Australia. In 1985, Dudley Blane and his colleagues blazed a trail (Blane and Clarke, 1985; Blane and Jaworski, 1989, 114–116) around the center of Melbourne as a holiday-week activity for families. The trail's mathematical ideas included investigating a circular pattern of bricks in the pavement (to discover the invariance of *pi*), studying the timetables in a train station, looking at the reflection of a cathedral in a pond (to estimate its height), trying to estimate the speed of water rushing down a spillway, counting the number of windows in a wall of a skyscraper, and looking for patterns in the numbers of post office boxes.

Australian mathematics educators constructed many more trails based on a variety of themes and venues, including preparing for prospecting in a gold rush town, acting as an apprentice keeper in a local zoo, and working on the ship works and sailing boats in a historical nautical village. Each of the Australian trails had a brochure that contained thought provoking, mathematically oriented questions. In many cases, the questions had no single correct answers as such. The tens of thousands of Australians who walked these trails attested to their popularity. Many walkers returned for a second round accompanied by their families. Because of the strong demand for Blane's Melbourne trail, the organizers maintained it for several months longer than the planned one week.

Like any good idea, the idea of a math trail has spread and people have adapted it. Carole Greenes of Boston University (Massachusetts) created a historical mathematics trail in Boston centered on the Common and the Public Garden. Unlike Blane's Melbourne trail, walkers on Greenes' trail followed a human guide who knew the historical and mathematical aspects of the trail and who could give hints and suggestions to walkers who got stuck on a task or idea. Kay Toliver, an award-winning New York City schoolteacher, leads her students on walks while guiding them to discover mathematics in their school neighborhood. Student walkers do not write their ideas and solutions on paper, but informally discuss their discoveries on the spot and then take the discussion back to the classroom. Florence Fasanelli, Fred Rickey, and Richard Torrington developed an elaborate math trail that takes advantage of The Mall in Washington. It provides an opportunity for the thousands of people who visit The Mall every year to include a mathematical dimension to their sightseeing. These successful math trails show that the idea is robust and malleable enough to meet the needs and imagination of trailblazers in many different situations.

CHARACTERISTICS OF MATH TRAILS

We'll describe here a basic model for a math trail. Of course, we offer the model for you to adapt to match the interests and needs of you, the trailblazer, and your walkers.

- **Math trails are for everyone.** Everyone studies (or studied) math in school. Everyone uses math. Math trail problems should be interesting and accessible to people at all levels of age and experience. We aim for the widest possible participation. Trail walkers discuss how to approach a problem and they compare their thinking. Talking about the mathematics helps to bring it to life and to build confidence in one's abilities. Math trails respond to the variety of the walkers with variety in the math problems, both in level of difficulty and in type of mathematics. Easier and harder problems blend together or alternate. Arithmetic, geometry, measurement, estimation, and other mathematical topics all appear. The aim is for everyone to feel the accomplishment of contributing to the problem solving. A math trail is not just for math lovers or A-students!

- **Math trails are cooperative, not competitive.** In the spirit of the *NCTM Curriculum Standards* the emphasis is on talking about and doing mathematics. The purpose of the math trail problems is to bring attention to the processes for formulating and solving problems, not to find single correct solutions. While an individual might walk a trail with pleasure and profit, the orientation of trails is toward families and other groups.

- **Math trails are self-directed.** Such trails are ready when a walker is ready. There is no time limit.

- **Math trails are voluntary.** An important characteristic of our model is that walking a math trail is entirely voluntary. This idea is in keeping with the general principles of the popularization of mathematics in general. You can turn off or tune out a television show. There are other things besides mathematics to see in a museum. It's not likely that anyone can be forced to solve a puzzle. A popular presentation of mathematics must first attract and keep its participants. If a particular bit of mathematics or the setting is not attractive or interesting, it won't work for a math trail. In particular, a trailblazer has to remember that the trail is meant primarily for people who don't usually do math consciously, people whose memories of school math may not be all that positive. Math trails are not made for specialists. What works for a person who already likes math a lot, may not work for someone who doesn't like it or for someone who is not confident about doing it.

- **Math trails are opportunistic.** We believe that "math is everywhere." Trailblazers can prove the assertion by taking advantage of their locale: neighborhood street, business district, parking lot, college campus, shopping mall, park, zoo, library, grocery store, clothing store, and more. Any public place that allows safe walking is ripe with math problems for an imaginative trailblazer.

- **Math trails are temporary.** Places change. Permanent trails require maintenance and continuing time and energy and, perhaps, miss new opportunities. Rather than maintain this year's trail, blaze a new trail next summer as a way to bring back satisfied walkers and use the novelty to attract new walkers.

BLAZING A TRAIL

Trailblazing is straightforward and lots of fun in itself. It can take as much time and energy as you have available. Our model opts for less rather than more on the part of the trailblazers. The project should be fun for you as well as for the trail walkers.

- **Location.** Where do you want to locate a trail: neighborhood street, business district, parking lot, college campus, shopping mall, park, zoo, grocery store, clothing store, or some other interesting place? Walk the venue looking for math problems from which to choose. Collect many more than you will use, so that you can pick and choose for good balance among the final lot. Snapshots and sketches will be useful in your planning, although you will probably make several visits to the site. Keep notes on your ideas so you can sort out the pictures when they come back from the developer. Be alert for problems that involve elements of local culture or history, as well as the physical attributes of the site. Work with a scale map of the venue, noting the specific location of each problem. Your stops on the final trail route will probably occur in a different sequence or route than you used in collecting the ideas.

- **Length.** Walking distance, walking time, and the number of problems all affect the length of a math trail. One mile will do for many people, although you might design a two-part trail to give a longer option. Two hours is the limit of most people's attention for problem solving, even in the most attractive locale. Consider including a sit-down break midway in distance and time. This is a chance for the group to talk about what they've been doing, as well as to catch their collective breath. Spacing consecutive trail stops more than 10 minutes apart risks losing a walker's attention. The time required at each trail stop depends on the walkers as well as the problem.

- **Trail guide.** Prepare a scale map of the trail including all appropriate landmarks and features, as well as clear notations to the trail stops. Describe the problems for each stop, leaving room for trail walkers to write, sketch, and record their thinking and solutions. Make note of the tools that a trail walker might need for each problem. Keep the overall list simple. Paper, pencil, an eraser, a watch, and a hand calculator are what you can expect most people to have readily available. Include an address to encourage trail walkers to send comments to about the trail. This will be helpful to you in thinking about the success of the trail. We'll provide samples of several trail guides below.

- **Mathematics.** Sketch out the math problems that you discover. Organize them on your site map. Including two or three problems at a trail stop helps to hold a walker's interest and increases the ratio of math time to walking time. Posing the problems at different levels and with different mathematical focuses will help to achieve a good variety in the overall trail. Don't overemphasize estimation or geometry or arithmetic or any one topic, especially not arithmetic, which will come up in many of the problems anyway. Include problems with ideas to follow up at home. Make the problems independent of one another so that trail walkers will be encouraged by a fresh problem at each stop, regardless of how well they understand earlier problems. Aim for problems that are not like the problems that students are currently solving in school. Novel mathematics will help demonstrate the value of studying mathematics. Problems that arise naturally from the situation are best, although incidental problems are also fun. You will find that blazing a math trail is itself a good math challenge.

Our emphasis throughout this book is on creating a math trail with a written trail guide for the use and enjoyment of individuals or groups at their own pace. We do not expect that you or anyone else will be acting as leader for an organized walk. However, you may very well choose to act as leader for a group of trail walkers as part of the process of *creating* the trail. They may help you in finding interesting problems and with making the choice on how detailed and prescripted an individual question in the guide might be. It is not easy to decide, for example, when to ask the trail walkers for several different ways of estimating how fast, how high, or how many, and when to write a series of one-step questions that will lead to the same result. In Part 2 we have taken the opportunity of describing settings in a wide variety of ways, from detailed and prescriptive to open-ended to asking you to consider the situation and choose a formulation. If it sounds at times as if we were leading a group on an unwritten walk, that's part of the thinking about the trail guide.

The creation of a written trail guide to be followed without a leader also affects the kind of questions you can ask. You have to ask about features of the landscape, the architecture, the animals, and the stores that you know are going to be there for a while! Here is a building structure with lots of line segments in the design. It's interesting to ask how many different triangles you can find, especially if they may be mutually overlapping and inclusive. But if an advertising sign gives just as promising an opportunity, you have to ask if it's likely to be there for a while or is it in danger of being replaced next week. Estimating the height of a building or statue you can't climb on can be done in more than one way. Asking the participants to use shadows won't succeed on a cloudy day, but it is great fun when it works. Looking for tessellations and deciding what makes them different is usually occasioned by walls or pavements or sides of buildings, and they are always present. A pattern of stars on a flag is permanent, but there are parts of the country where the same pattern in a flowerbed may disappear seasonally in the park perhaps, but not in the mall!

Such problems are naturals for the stops on your trail. While walking from place to place on the math trail, other questions and problems may arise. You can decide if you are walking level or perhaps going uphill or down, and you can think about this in several different ways. You can look for special polygons (examples of n-gons with n up to perhaps 12, convex or otherwise), or special polyhedra (examples with different numbers of faces, convex or concave), or special curves and surfaces (circles, ellipses, ovals, spheres, and parts of these). There are also problems that might be fun but are not predictable: Perhaps special numerical properties of license plates of passing cars, or the probability of meeting an unusually large number of consecutive persons of the same sex coming the other way. A trail guide could suggest keeping an eye open for such opportunities even though they don't always occur.

ORGANIZING A MATH TRAIL PROJECT

Any project takes organization and planning. In addition to blazing the trail, a math trail project will require some supporting work to see that it functions smoothly.

- **Trailblazers.** An individual might set up a trail, but teamwork is fun and the result can be stronger for having contributions from several points of view. Including a math user (engineer, business person, etc.,) can help prevent a school-like tone to the problems. And including someone on the team who is not a heavy math user can be a good reality check by representing the majority of the public you will want to reach.

- **Sponsorship.** Is there a club or other such group that will adopt the math trail project? Of course, a school group might initiate a project, but there are many other community groups that could step forward, such as service clubs, scout troops, fraternal organizations, among others. Sponsorship might include either the money or a mechanism for raising the money to cover the costs of the project. The trailblazers will prepare the master copy of the trail guide. Printing or duplication of the trail guide is a cost. You might translate the guide into other languages depending on the make-up of your community. Distributing the trail guide is a problem to solve. You might consider approaching local merchants, the library, or some other facility close to the trail. Publicity will be helpful in alerting the community to the existence of the math trail and in telling them how to get a trail guide. Sponsors can be helpful in securing publicity. Consider putting signs in store windows or on community bulletin boards or an announcement in the local newspaper. A sponsor could be helpful in case permissions are necessary for entries into premises or otherwise. In any event, you might alert shops or other businesses in the vicinity of a trail stop of the likelihood of groups of people coming by, talking about mathematics or otherwise displaying unusual mathematical behavior! Even if you have a non-school sponsor, your local school can be helpful in publicizing the math trail with students. A sponsor might offer a reward for finishing the trail. A badge or simple certificate are nice symbols of participation as well as suggestions that others walk the trail themselves. You will need to arrange for a place where the successful trail walkers can receive their badges or certificates.

- **Evaluation.** You and your sponsors will want to know if your trail is a success. You'll want to agree on how to define success. The best definition is probably one that is simple. Track the number of trail guides distributed. Include an address for people to send a note after walking the trail. Make a few visits to the trail when you might anticipate a large number of walkers. Observe them and perhaps ask a few questions to gauge their reactions and to provide you with some anecdotes. Anything much more elaborate is likely to overburden the trailblazers.

- **History.** Math trails are ephemeral. Capturing your experience can be useful to others and interesting to those who think about popularization of mathematics. Think about sending a brief letter concerning your math trail and your trail guide to COMAP. Be sure to include your interesting anecdotes, both from the trailblazers and from the trail walkers.

We give four examples of trails in particular situations: A park playground, a city, a zoo, and a shopping mall. The playground and the shopping mall are each a single actual location, the city and the zoo trails are composites drawn from several locations. For each trail stop you will find one or more pictures, and then ideas and questions suggested by the pictures. The trail in the park playground is more specific than the others, in order for you to see examples of the development from what you see to questions and ideas to actual items for a trail guide.

RECREATIONAL MATHEMATICS IN THE PARK

Parks are popular places for many people, including family groups. In this chapter we will take a tour of a local park to blaze a math trail in the park playground. We'll discuss our thinking as we spot opportunities to highlight some math, identify questions to ask, and note choices to make in settling on a final guide design. We'll also include a draft of the trail guide for each stop.

THE FIRE ENGINE

Figure 1.

Our park has an attractive red fire engine. Children of all ages have fun climbing aboard the fire engine, pretending to be the driver or one of the firefighters rushing to a fire across town. The fire engine certainly is large, but how large is it? Here's a chance for walkers to estimate some measurements using their hands and feet, although a tape measure would help move things along more quickly.

Think about the size of the tires, the length of the truck or various pieces, or the height of the seat from the ground. Walkers could record measurements on the trail guide in order to compare their results. They might also try estimating some of the lengths before measuring.

As an extension of this exploration, trail walkers can compare their numerical results when they divide the height of the tire into its circumference. While we would not expect everyone to get exactly the same quotient, we would expect the trailers to obtain similar results. This is a neat way to let everyone discover that π did not just "pop" out of the sky! It (π) really does have a true and valid meaning in life and mathematics. There will be other opportunities during our visit at the park to explore π.

GUIDE TO STOP 1—The Fire Truck

This truck is bigger than most cars on the road, but how big is it? Even the tires are large. Each of you estimate how long your hand is. Now use your hand to measure the height of a tire and then measure the distance around the outside of the tire. Compare your answers. Are they different? Talk about why.

Divide the height of the tire into its circumference. Is this number familiar? Compare your results.

Name:
Tire height:
Circumference:
Quotient:
Truck length:
Bumper length:
Seat height:

How long is the truck? How long is the front bumper? The driver's seat is high off the road, how high is it? Why is the seat so high? Try estimating the height first and then measuring. Is it easier to estimate longer or shorter lengths?

TILING BLOCKS

This gate has a grid of square tiles, light on one side and dark on the other. The tiles are mounted so that they rotate on a vertical wire. Rotating them generates a great variety of patterns and that affords a variety of counting problems. The most direct problem is to estimate or count the number of square tiles. A more complex problem is to count the number of different patterns. You might want to ask exactly what you mean by different.

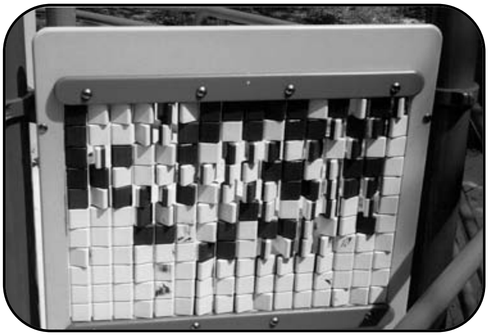


Figure 2.

GUIDE TO STOP 2—Tile Grid

See how each small square tile rotates independently so that you can turn either its light face or its dark face to the outside. Count the number of rows of tiles. Count the number of columns. How many individual tiles are there in the grid?

Rows	
Columns	
Tiles	

Each time you turn a tile, you change the pattern. One pattern has all the light faces showing. The opposite pattern has all the dark faces showing. There are many, many more patterns mixing light and dark tiles.

Look at the four tiles in the corner that make a 2 x 2 square. How many different patterns can you make with them?

How many patterns can you make with a 3 x 3 set of tiles?

2 x 2 patterns	
3 x 3 patterns	
4 x 4 patterns	
5 x 5 patterns	
Total patterns	

How many patterns can you make on the whole board? The number is very large! Think about 4 x 4 and 5 x 5 arrays first and see if you can spot a pattern.

SHAPES AND NUMBERS BOARD

This panel on the carousel is not only decorative, but also an instructive opportunity for very small children to recognize numbers and simple shapes. This can be a prompt for the youngest of the trail walkers to look for numbers and shapes along the trail.

Figure 3.

GUIDE TO STOP 3—Shapes and Numbers

Look at all of the shapes and numbers on the board. Name them and trace them with your finger. As you walk through the playground today, see if you can find all of these shapes and numbers. Keep a tally for each of the shapes and numbers. At the end of your walk, check the tallies to see which shape you found most often and which number you found most often.

	How many?
0	
1	
2	
3	
4	
5	
6	

	How many?
7	
8	
9	
Square	
Triangle	
Rectangle	
Circle	

CHIMES

Chimes have always fascinated children and adults alike. Windchimes catch the breeze and play beautiful musical notes. Other chimes, such as the brightly colored panel containing eight chimes pictured here, need to be struck with something like a stick in order for us to hear musical notes.

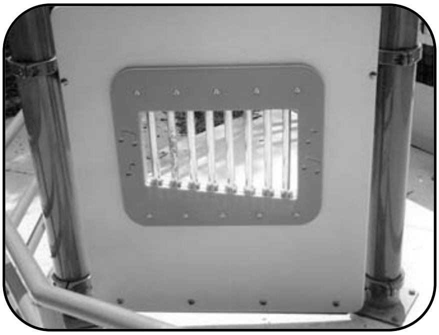


Figure 4.

GUIDE TO STOP 4—Chimes

What a terrific place to stop and have some fun with musical chimes! Are all the eight chimes the same length?

Hit each of the chimes starting from the shortest to the longest, left-to-right. Do all of the chimes make the same sound? Which chime has the lowest sound? Which has the highest sound? How do you think the length of the chime and the pitch of the tone are related?

GAME BOARD

Game boards can be found throughout many playgrounds, recreational picnic areas, and zoos. This game board in our playground is made up of Xs and Os. Each of the nine faces can come up either as an X or an O or a blank. The game board is in an arrangement of a 3 X 3 grid.

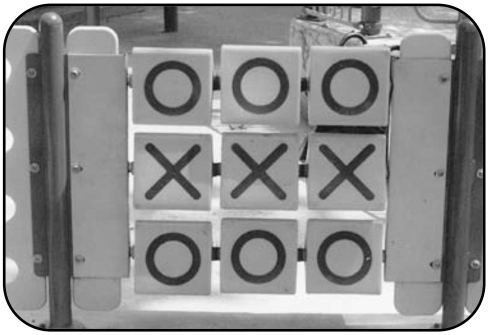


Figure 5.

SLIDES

Slides are loads of fun for everyone! We never seem to outgrow the thrill of coming down a slide—the steeper and faster, the more fun! One of the things you learn in mathematics has to do with the slopes of lines. We can combine the fun and thrill of a slide with the concept of the slope of the slide. Once the walkers are comfortable finding the slope of the first slide, it is a good idea to have them find other slides or ramps in the playground and compare their slopes.

GUIDE TO STOP 5—The Game Board

See how each square face of the game board rotates independently so that you can turn an X or an O or a blank to the outside.

Work with a partner to see how many different ways you can arrange the Xs and Os on the game board grid with no blanks. How is this like Stop 2?

How many different ways can you arrange the Xs and Os if the nine squares were lined up in a row instead of in a grid?

	Team 1	Team 2
Game 1		
Game 2		
Game 3		
Game 4		

What if you decided that the Xs and Os must alternate? How many different ways can this be done?

Choose teams and play some games of tic-tac-toe.

GUIDE TO STOP 6—Straight Slides

Measure and record the height of the slide at its tallest point and at its lowest point. Subtract the lowest from the highest and record your results in the numerator of the fraction below.

$$\frac{\text{height of tallest point} - \text{height of lowest point}}{\text{distance from tallest point to end of slide}} = \frac{\qquad}{\qquad} =$$

Now measure and record how far it is along the ground from the tallest point of the slide to the end of the slide. Record this answer in the denominator of the fraction above.

This fraction is called the average slope of the slide.

Divide the first result by the second result.

Some slides are not perfectly straight, but rather have a slight bend toward the end of the slide. This bend acts as a brake for children, slowing them down before they come to the end of the slide. Once trail walkers are comfortable finding the slope of a straight slide, have them tackle finding the slope of a slide with a bend in it.

Figure 6.

GUIDE TO STOP 6—Slides With Bends

Discuss ways of defining the slope of this slide. There might be more than one suggestion for defining the slope. How close are the different slopes? Why are there different answers? If you see other slides or ramps, find their slopes and compare them.

GUIDE TO STOP 6—Slides

How long does it take to get down each of the different slides you find on the playground? Suppose your friend weighs more than you. Would she/he get down the slide faster than you? Record your findings in the following table.

Time to get down each slide	Walker 1	Walker 2	Walker 3
Slide 1			
Slide 2			
Slide 3			
Slide 4			

GUIDE TO STOP 6—Slides

If you roll a ball down the slide, how far does it land from the end of the slide?

What do you think affects this distance?

Distance from the end of the slide
Slide 1
Slide 2
Slide 3
Slide 4

SWINGS

Have you ever noticed how children run over eagerly to a swing set once they spot it? Many times they will also screech with delight as they run toward the swings. Adults also enjoy swinging! There just seems to be something relaxing and carefree about swinging.

Figure 7.

GUIDE TO STOP 7—Swings

Now walk over and watch the children as they swing. Do some of the children need to be pushed by someone else in order to go higher? Why? Take turns swinging with your friends. How would you describe the motion of the swings? How do you make yourself swing higher? Why does the swing eventually stop?

Some trail walkers could work the with the following situation using the motion of the swing or "damping."

GUIDE TO STOP 7—Swings

At this stop along the math trail you are to decide if the motion of the swing or "damping" is constant. You will need two friends to help you out with this activity. Have one of your friends sit in the swing. Stand behind your friend and mark the position in the sand from where you will let the swing go. Another friend should stand to the side of the swing to mark the distance the swing will travel. Now bring the swing back to your marked position in the sand and let go. The friend on the side should place marks in the sand to indicate the distance that the swing travels on each successive swinging motion. After the swing comes to a stop (or near-stop), repeat the swinging and distance measuring activity again using the same person in the swing and letting the swing go from the same position. Repeat once more and see if you can make a conjecture about your observations.

GUIDE TO STOP 7—Swings

Now explore another activity at the swings. You will need a stopwatch or a watch with a second hand, and two friends for this exploration. Do you remember what a period is? It is the time it takes for one back-and-forth motion of the swing. Have a friend sit in one of the swings. Stand behind your friend and bring the swing back and start your friend going in the swing. Have another friend time 10 back-and-forth swings and divide that time by 10. Do this several times giving your friend a different amount of push on each trial. Does this affect the period?

Repeat the whole process using different distances from which to start the swing. Does this affect the period?

	$\frac{\text{Time for 10 back-and-forth swings}}{10}$
Trial 1	
Trial 2	
Trial 3	
Trial 4	

GUIDE TO STOP 7—Swings

Find some other swings in the playground with different length chains. Try the same experiment using these swings and see if the length of the chain affects the period of the swing.

	$\frac{\text{Time for 10 back-and-forth swings}}{10}$
Swing 1	
Swing 2	
Swing 3	
Swing 4	

COUNTING THE LEGS OF A CATERPILLAR

One of the best parts about a math trail is that there is always some mathematics that everyone can do—no matter how young they may be. Playground designers will often include opportunities for younger children to try out their skills at counting.

Figure 8.

> **GUIDE TO STOP 8—Counting the Legs of a Caterpillar**
>
> Count the number of legs on the left. How many are there? ______________
>
> Count the number of legs on the right. How many are there? _____________
>
> How do you tell the number of legs on the right without actually counting them?_____________

THE SNACK BAR

After several stops on the math trail, walkers should be hungry and thirsty! This provides an excellent opportunity for some mental arithmetic.

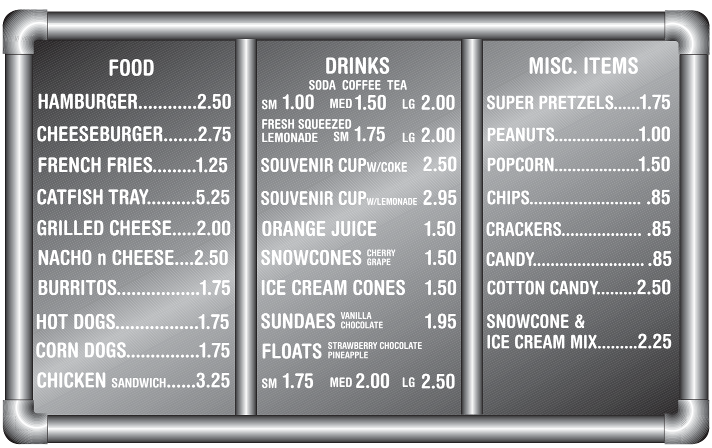


Figure 9.

GUIDE TO STOP 9—The Snack Bar

Suppose you have $6.00 and you want to buy a drink and a snack. Using the menu on page 25, what are three possible combinations of drink and snack that you could buy? But wait! Is there any sales tax on the food? Estimate the sales tax. Do you have enough money to pay for your food? Record your choices and the total for each in the table below.

Food selection	Cost	Estimated tax	Estimated total cost	Estimated change
Combination 1				
Combination 2				
Combination 3				

A clerk at the snack bar earns $5.50 per hour. How many hours must the clerk work to earn $75.00 before taxes?

THE TRAIN STATION

Many playgrounds and recreational parks offer trains to ride around the area. Children and adults alike enjoy these train rides. If the playground you are using for a math trail offers train rides, it is an excellent opportunity to enjoy some mathematics along with a pleasant trip around the playground.

Figure 10.

GUIDE TO STOP 10—The Train Ride

Suppose the stationmaster tells you that the average speed of the train is 5 mph. Time the length for one complete ride on the train. About how long is the train track?

Figure 11.

Suppose the train stops in the station for 5 minutes before departing for the next tour of the park. What is the probability that the train will be in the station when you arrive? If the train is gone when you arrive, what is the probability that you will wait less than 5 minutes for the next train?

Now find the stationmaster and ask how long the train rests at the train stop before taking off on another trip and the average speed of the train. Answer each of the questions in the previous paragraph for *your* train ride.

CIRCLES

As you plan your math trail at the playground keep an eye out for circles. There could be circles within game boards, as edging around a circular garden within the playground, or as part of the construction of the playground equipment. Circles are the place to look for π.

Figure 12.

GUIDE TO STOP 11—Circles

As you walk around the playground watch for circles. Recall from the stop at the fire truck your estimate of the length of your hand. When you find a circle, use your hand to measure its circumference and diameter. Record your results in the table to the right.

Circumference of circle	Diameter of circle	$\frac{\text{Circumference}}{\text{Diameter}}$
Circle 1		
Circle 2		
Circle 3		
Circle 4		
Circle 5		
Circle 6		

As you come upon other circles during your math trail adventure, find the circumference and diameter of each and record the measurements in the table. Then find the ratio of each circle's circumference divided by its diameter and record this result in the table.

Divide the circumference of each circle by its diameter. Do you see a pattern? Where did similar questions come up before on the walk? Compare the ratios of circumference divided by diameter from the fire truck to the ratios you have just obtained. You observed then that the ratio of the circumference of a circle to its diameter is a constant! You may recall that this number is called pi and is written as π. It is approximately equal to 3.14.

GUIDE TO STOP 11—Circles

Now that we have found the circumferences and diameters of various circles as well as the ratio of the circumference to the diameter for each circle, let's find the area of each circle. Use the formula:

Area of a circle = π (radius)^2, where radius = $\frac{1}{2}$ diameter

	Area of a circle = π (radius)^2, where radius = $\frac{1}{2}$ diameter
Circle 1	
Circle 2	
Circle 3	
Circle 4	
Circle 5	
Circle 6	

GUIDE TO STOP 11—Circles

See if you can find a circle within another circle with the same center. You may recall that these are called concentric circles.

Find the area of each of the two circles and record your results in the table below.

What is the probability that the coin will land in the smaller, inner circle?

	First set of circles	Second set of circles	Third set of circles
Area of smaller circle (A)			
Area of larger circle (B)			
B – A			
# of times coin landed in smaller circle			
# of times coin landed in larger circle			
Probability that coin lands in smaller circle			

THE JUMPING POLE

Figure 13.

Some playgrounds have a long, round pole that is held up by supports and is parallel to the ground. Children call it a jumping pole or, perhaps, a snake. It is different-looking from much of the other play equipment on the playground and children sometimes need to watch other children play with it before they get the knack of playing on it themselves. Two children can sit on the snake and push upward with their feet as they are bouncing up and down. A third, or perhaps a fourth, child can also get on each end of the snake and push up and down. As one or more children sit on the snake and friends push up and down on the ends a wavy, up-and-down movement occurs. It always looks like so much fun and children certainly love to play on it!

GUIDE TO STOP 12—The Jumping Pole

Now let's head over to the jumping pole for some fun. Sit on the jumping pole and push upward with your feet. See how you go up and down as the snake moves in a wavy type motion. Now have a friend stand at one end of the pole and push up and down. Describe the ride when your friend gives the pole just a little push. What happens if your friend gives the end a harder push? How does the distance you travel up and down depend on how your friend pushes on the end of the pole? Now have a taller or heavier friend take your place on the pole and you do the pushing at one end of the pole. How does this affect the motion of the pole? See what happens to the pole's up-and-down motion if two people get on the pole and two more do the pushing. What happens if the two people doing the pushing push down on the pole at the same time? Try the same thing, but this time push one end down and the other end up. How does this action affect the motion of the pole?

Time to try something different with the jumping pole. Have everyone get off. Let one person push down hard on one end of the pole to start the wavy motion. What is the distance between the two high points of the waves?

Have a person stand at each end of the pole. Then have each person push down on their end of the pole at the same time. What happens to each wave?

LADDERS OF DIFFERENT SHAPES

It is also fun to look for ladders of different designs and shapes on the math trail. Ladders do not always go straight up-and-down. Some can be curved and each ladder could have a different angle of elevation.

Figure 14.

GUIDE TO STOP 13—Ladders Of Different Shapes

Describe in your own words the shape of this unusual ladder. Is it possible that this ladder is part of a circle? How would you decide if it is really part of a circle or not? Now look at the rungs of the ladder. Are *they* pieces of circles? Describe how you went about making your decision.

Math trails are a truly fun activity for the entire family! One particular family of trailblazers includes everyone in the activity—even the family's Labradors! Penny and Maggie first pose for their picture at the end of the playground slide. That being taken care of, they get down to the business of estimating the slope of the slide by taking several trips up and down the slide to get a bird's-eye view from all angles!

Figure 15.

Figure 16.

Figure 17.

They compare notes before they announce their estimated slope of the slide. Maggie prefers to check the slope estimate by actually coming down the slide while her mother, Penny, opts for the stairway!

SOME REFLECTIONS FOR THE TRAILBLAZER

You have just finished your first playground math trail! Just look at all the mathematics you found around you in the playground! Now that you are a true “Math Trailer,” can you find even more mathematics in your playground?

As the creator of the math trail guide, you can judge how much detail your trail walkers would want to put into each stop. Take a moment to consider three playground items that are almost certain to be part of any playground: a seesaw, a swing, and a slide.

With a seesaw, trail walkers can consider the question of what makes it balanced—or unbalanced! Before anyone sits on the seesaw, it is probably more or less in balance. If two children of equal weight sit at an equal distance from the point of balance (the fulcrum), the seesaw will stay in balance. If children of unequal weight sit on opposite sides of it, the heavier child must sit closer to the fulcrum in order for the seesaw to be balanced. A good mathematics question to ask here would be “How much closer?” If child A weighs 40 pounds and child B weighs 80 pounds, should child A sit twice as far from the fulcrum as child B? Some of your trail walkers may come to that conclusion right away. However, it might be fun for them to consider this question in a little more detail. Why is there a problem? Because the seesaw itself has weight and often, for reasons of strength, is actually pretty heavy. Say each half of the board itself weighs 50 pounds. What is the *moment* on the left side of the board? You may want to remind your trail walkers that *moment is weight times distance to the fulcrum*. You can think of the 50 pounds of board as being concentrated at the midpoint on the left, and the distance is the distance from that midpoint to the fulcrum. On the right side you have another 50 pounds concentrated at the midpoint of the right part of the board. This means that the moments are equal. Add the moment of the left side to the moment of the child sitting on the left end of the seesaw and then add the moment of the right side to the moment of the child sitting on the right end of the seesaw. Now let's say that the whole system, board plus children, is in balance. So you set the two sums equal to each other. Then the moments due to the board itself are equal, and when you subtract them, the moments of the children by themselves have to balance! So it works after all! Did we worry you for just a minute?

On the swings you can discover that, unlike the seesaw, the weights of the children don't matter! That's quite a surprise! The period of the oscillation doesn't depend on the weight of the child—only the push needed to make her start swinging does! Secondly, the period doesn't seem to depend on the initial displacement either—and you probably don't want to get into the mathematics of that—but the walkers can perhaps explore if a swing at the end of a longer rope has a longer period. How much longer? Any guesses on that?

A slide may be trickier because the friction between the slide and the person, or favorite pet, going down has so much to do with how long it takes to get to the end. But you might send balls of different sizes and weights down the slide and see that the time it takes for the balls to get to the end doesn't depend on the weight of the ball. Also it may not differ that much from a child, or a pooch, as long as they have the confidence to let go and not try to brake. How far from the end of the slide does the ball or the child land on the ground? What does *that* depend on? Is it how heavy the child is, how far the upper end of the slide is from the ground, or how far the lower end is off the ground? How long did it take to get from the top of the slide to the ground? What does this time depend on? Some slides don't go straight down, but may spiral a bit. As long as there is not a lot of friction, what effect would such twisting have?

RECREATIONAL MATHEMATICS AROUND TOWN

Most of the time when we walk around town doing errands and exploring new stores we neglect to see the mathematics surrounding us. Villages, towns, small cities, and neighborhoods in larger cities all offer wonderful opportunities for trail walkers to have fun with mathematics! We illustrate the possibilities with samples from Toronto, Ontario; Waco, Texas; Paris, France; New York City, New York; and Summit, New Jersey. In each instance, we'll look at a variety of mathematical questions that are inherent in the setting. Aside from working out these possibilities, a trailblazer will need to pick and choose which questions to use in order to achieve a trail that balances interest, time, and mathematics.

THE FOUNTAIN IN A PARK IN TORONTO, ONTARIO

Many communities feature lovely gardens and fountains. Here's a fountain in Toronto, Canada. Geometry and measurement play obvious roles in the design and layout of both the structural elements and the garden plantings. In this case, symmetry, tessellations, patterns in the paving stones, and comparisons among the dimensions are all open to discussion. The pool of this fountain looks like a rectangle with a quarter circle cut out of each corner. Shrubs form a border along the edges of the pool, perhaps to encourage people to keep their distance from the water. Altogether the esthetic impression of the fountain, pool, and plantings is attractive and pleasing because of the geometry inherent in the design.

Figure 18.

ACTIVITIES

- Identify the geometric shapes in this structure.
- Estimate the number of shrubs planted around the fountain. How does this number compare with the perimeter of the fountain?
- Assuming that the bottom of the fountain is level with the ground, estimate the depth of the pool and then estimate the number of gallons of water that the fountain can hold.

ACTIVITIES

- Estimate the perimeter of the darker gray brick rectangle.
- How does this perimeter compare with the perimeter of the pool?

Figure 19.

The brick mason laid an interesting pattern of bricks at each corner of the fountain and this contributes to the artwork. The darker gray bricks add a finishing touch that defines the rectangle enclosing the structure.

INCIDENTAL SCULPTURE

Many cities and towns have sculptures or statues in various places about the community. These might be used to inform passersby about some local history or simply to add a pleasant and charming atmosphere. Some cities, like Chicago, Illinois, have used non-permanent sculpture exhibits throughout the town and actually created a 'trail' for automobiles to follow in order to see the changing exhibits. Waco, Texas, has used this idea of non-permanent sculptures and created a 'traveling' sculpture that appears in different parts of town on an unannounced schedule. Passersby never know what part of town the sculpture will appear in the next week! If it were stationary, you might put in the center of town or as near to the center as possible. What do you mean by center? The trail walkers might enjoy this discussion. If you have a region shaped something like an S, no matter if it is irregular, you can think of two points in it as far apart as possible. The line segment joining them would be a diameter of S, and its midpoint the center of the shape. It would have the property that the maximum distance from anywhere in the region to that point would be as small as the choice of point could make it.

Sculptured cows sitting so quietly and peacefully on the town green are always eye catchers, particularly for younger children. All of the cows in Figure 20 are lying down on the grass and facing different directions.

Figure 20.

ACTIVITIES

- Decide if the cows are solid structures or if they are hollow.
- How did you come to your decision?
- Are the cows life-sized or smaller or larger than in real life?
- Are all the sculptures the same and just placed differently on the grass or is each individual cow truly different from each of the other cows?

If your exhibit traveled to two locations, you can imagine dividing the region into two pieces, each with a center, and making the maximum distance to the center of each piece as small as possible. And so on for more locations.

STREET INCLINES

Have you ever walked along a street that sloped down or sloped up? Of course you have! Very few towns are built on flat land and that means the streets rise and fall. Even so, the buildings along the streets have floors that are level and walls that are vertical. Architects compensate for the slope of a street. Another interesting observation on a street that goes downhill or uphill has to do with perspective.

ACTIVITIES

- Does the street slope to the right or to the left?
- Stand in the middle of the sidewalk and look down the sidewalk. What do you notice about the width of the sidewalk as you look farther and farther down the sidewalk?

- Now turn around and look up the sidewalk. Does the same thing happen to the width of the sidewalk as before when you were looking down the street?

GEOMETRIC SHAPES WITHIN BUILDINGS

Architects use geometric shapes and combinations of geometric shapes to enhance new buildings *and* additions to older buildings. If you were to travel to Europe, you would find a beautiful example of this in the newest addition to the Louvre Museum in Paris, France.

Figure 21.

Many people call this new addition the Pyramid, because the glass structure has a pyramid shape. The water fountains are also in the shape of a triangle.

ACTIVITIES

- What shape is the base of this pyramid?
- What is the name for a pyramid with a base shaped like this one?
- Do all pyramids have the same shape base?
- What shape is one of the faces of the pyramid?
- What shape is each panel of glass?
- Look for other geometric shapes within the face itself.
- Is the fountain triangle equilateral, isosceles, or scalene?

ACTIVITY

- Find as many geometric shapes as you can.

This is the view inside the Louvre directly beneath the Pyramid. This glass structure, too, consists of different geometric shapes.

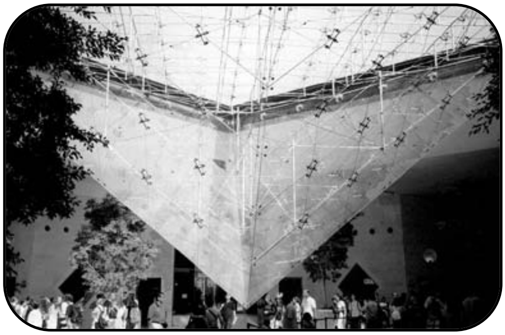


Figure 22.

TILINGS

Sidewalks, road- or pathways, and floor tiling can contain all kinds of interesting and beautiful mathematics. The foyers of hotels and public buildings are wonderful places to have trail walkers look for geometric shapes. Many of these floor tilings are large and colorful—meant to catch the public's eye not only through the contrasting colors and the use of different geometric shapes, but also by setting the design into the white floor tiles with a different angle. You can guide them in seeing a circle inscribed within a square.

ACTIVITIES

- Devise a method to determine if the round shape in the center of this tiling is a circle and if the shape surrounding it is a square.
- Identify the geometric shapes in this floor tiling.

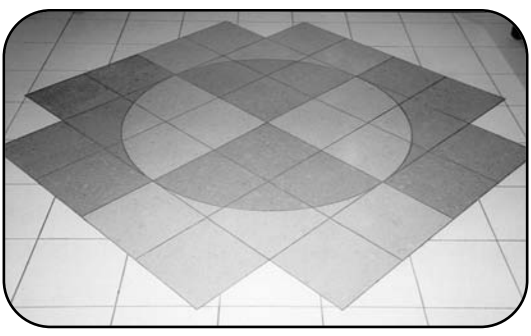


Figure 23.

- Estimate the area of the square and use that estimate to help estimate the area of the inscribed circle.
- Estimate the area of the entire tiling.

The second picture shows a tiling design that appears in a sidewalk in the downtown area of a small town, Summit, New Jersey. Trail walkers will notice that the pattern repeats itself.

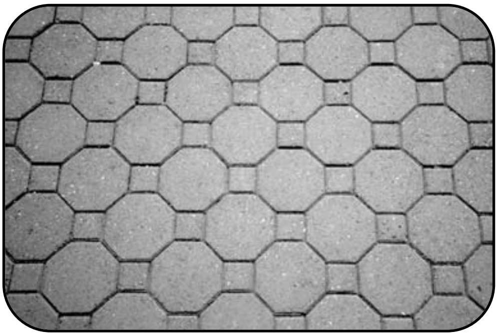


Figure 24.

ACTIVITIES

- What two geometric shapes are used over and over to make the sidewalk?
- Were these two shapes laid down in a particular order?
- Is it necessary for the order of tiles to be constant so that the pattern continually repeats itself?

This tiling, with two octagons and a square meeting at each vertex (mathematically referred to as an (4,8,8) tiling), is an example of what is called a semi-regular tiling. There are eleven of these tilings. They occur in so many math trails contexts that we have a section devoted to deriving and discussing them on pp. 88–99 of this book. The derivation is in a very leisurely and elementary fashion, and as a trailblazer you may be able to adapt parts of it when the occasion arises.

The trail walker may want to discuss the (4,8,8) tiling further. It looks as if the octagons occupy most of the area being tiled. How much is 'most'? If the square has unit side so that its area is 1, what is the area of the octagon? Trail walkers may first estimate it by saying "Well, it looks like it would hold four of five of those little squares." They may come up with various ways of actually computing the area: One nice way to do it is to imagine 'filling out' the octagon into a square by adding four little isosceles right triangles in the corners. The hypotenuse of such a triangle is 1, its side is $1/\sqrt{2}$, so that its area is $\frac{1}{4}$. The side of the octagon augmented to be a square is then $1 + \sqrt{2}$, so that its area is $(1 + \sqrt{2})^2$. The area of the original, unaugmented, octagon is then $(1 + \sqrt{2})^2 - 1 = 2 + 2\sqrt{2}$, or about 4.8.

This might be taken a bit further. Does this mean that throughout an entire tiling, the ratio of area covered by octagons to area covered by squares is about 4.8 to 1? That looks reasonable, but how would you be sure? (This wouldn't be true in *every* tiling. If you look at Figure 72, a (3,3,4,3,4) tiling, found on p. 93, a triangle has slightly less than half the area of a square, but the overall impression is of more area covered by triangles that by squares!) One way to look at this is to imagine making a strange new polygon—call it a "rattle"—by combining a square with the octagon to its immediate right. You can then imagine the plane tiled with rattles. Since each rattle is 4.8 to 1 octagon, so is the whole plane! (In order to do this trick in Figure 25, you would have to attach *two* triangles to each square!)

Another way to estimate the relative area of the square and the octagon is to toss some small objects like pennies or pebbles onto the pattern and count how many land in a square and how many land in an octagon. What if a penny crosses an edge? Then you count where the center of the coin would be. The trail walker would obtain a so-called Monte Carlo estimate of the relative area. This method of estimating areas of irregular-shaped regions relies on the role of chance or random processes and, thus, was named after the European city famous for its casinos.

Another interesting activity with tilings is to think about all the motions of the plane that would take the tiling onto itself. Can you move any vertex onto any other in such a way that the tiling lands exactly on itself? Does the fact that all vertices look alike imply that you can always do this?

Some tilings use only one geometric shape. Have your trail walkers use their imagination to picture a sidewalk repeating only one geometric shape other than a square or a rectangle. Is it possible to repeat the pattern continuously to construct the sidewalk? (Hint: Think about the edges of the sidewalk and those places where the sidewalk meets another sidewalk, roadway, or simply ends.)

ACTIVITY

- What geometric shape was used for the tiling in this sidewalk?

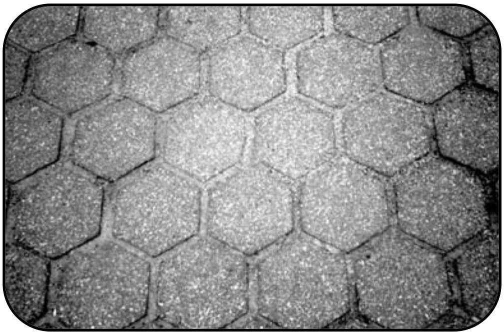


Figure 25.

The fourth example of a sidewalk tiling is very interesting! Quite different from the previous examples your trail walkers may have encountered! How do you suppose the brick masons planned this walkway?

Figure 26.

ACTIVITY

- Think of yourself as the brick mason who laid this walk. What type of rules do you think he/she would have needed to follow in order for the pattern to fit together without any gaps? Answer: They are probably aesthetic, not mathematical rules.

Examples of beautiful tilings can be found everywhere. Driveways and roadways in front of public and private buildings such as museums and hotels are often laid in patterns.

A beautiful example of a tiling used on a roadway outside a museum is shown in the next picture. Notice the graceful arcs made by the bricks! These arcs make a fan-like pattern that repeats itself. Is it possible that the arc is really part of a circle? Have the group decide if this is true or false.

ACTIVITY

- These graceful arcs made out of bricks are in a fan-like pattern that repeats itself. Work with your group to decide if this arc is really part of a circle.

Figure 27.

Here is another roadway pattern using arcs. At first glance, this tiling might appear to be the same pattern that the trail walkers have examined in other roadways but just in a different colored brick. Encourage the group to decide amongst themselves if the roadway tilings are the same pattern or not. Then have the trail walkers make a list of those characteristics that the tilings have in common and those in which they are different.

ACTIVITIES

- Make a list of those characteristics that the tilings using arcs have in common.
- Make a second list of those characteristics that the tilings do not have in common.

Figure 28.

Public buildings and hotels, in particular, use sculptures to enhance the foyers. The title of this sculpture in the lobby of a large hotel is Parabola. Some of your trail walkers would easily understand why the artist chose the title Parabola for this particular sculpture. Walking around the Parabola will give the walkers an opportunity to see the different three-dimensional effects of the glass and the different angles at which the glass panels are positioned.

Figure 29.

Figure 30.

Figure 31.

ACTIVITIES

- Find the lines or planes of symmetry for the Parabola sculpture.
- It is possible that there is more than one line of symmetry. As you walk around the sculpture point out possible lines or planes of symmetry for others in your group. Then together discuss each observation and decide which are lines of symmetry and which are not.

Companies use billboards to attract the attention of passersby and encourage them to use and buy their products and services. The bid for consumer dollars has led many advertisers to use flashier, more colorful, and cleverer billboards. This particular example of a billboard advertises three different products—not just one! But how is this done? Have trail walkers spot an example of a rotating billboard to help them see how this can happen. The long, narrow and flat, vertical slats of many moving/changing billboards have a triangular cross section making it possible to advertise three different products on one board. A motor connected to this billboard changes the billboard's face about every seven seconds. It would be interesting to investigate the ideal time for a moving billboard to stay on each face in order for a passerby to read and reflect on all three faces of the billboard.

Figure 32.


Figure 33.

ACTIVITIES

- What is the geometric shape of a cross section of each of the vertical slats used in the billboard?
- How many seconds does it take for all three advertisements on the billboard to be seen by the passersby?
- Is this enough time? What would be the ideal time that you would recommend for each message to be viewed and read by passersby?

As trail walkers go about the town have them look for circular shapes used in different ways. One beautiful example of a circle would be the wheels of an antique gold carriage on display at The Mews in London.

ACTIVITIES

- How many spokes are radiating out from the center of the wheel?
- The spokes break the circle into how many sections?
- If the sections are approximately equal in size, how many degrees are in each section of the circle?

Figure 34.

Circles add grace and elegance to doorways and windows. Encourage the group to discuss ways to determine if they are, indeed, circles or parts of circles.

Figure 35.

ACTIVITY

- As you walk about town look for examples of circles. Keep a log of the circles that you find, where they were on your walk, and their different uses.

Figure 36.

RECREATIONAL MATHEMATICS AT THE ZOO

One of the most exciting and fun math trails can be found by walking through a zoo. Since many cities and towns of all sizes have a zoo, it is a neat place to have fun while at the same time seeing all of the different animals and exploring some interesting mathematics.

Many zoos have elaborate entrances to catch the attention of passersby. The magnificent entrance gate to the Berlin Zoo in Germany is one of the prettiest in the world. Before paying the admission fee to your zoo math trail, walkers can take turns standing by one of the elephants or other statues that might be a part of the entrance gate. They can estimate the height of the statue by comparing its height to their own. Geometric shapes are used within the design of an entrance gate and throughout the zoo. Finding and naming the different shapes is a particularly fun activity for the younger members of your group who might be just learning the basic geometric shapes. They love to race about the zoo to see if they can find a new shape and be the first to point it out to other trail walkers. Other examples of geometry such as angles can be found throughout the zoo.

Figure 37.

ACTIVITIES

- Use one person's known height as a gauge to help you estimate the height of the entrance gate.
- Find as many geometric shapes as possible within the entrance gate area. As you continue your walk through the zoo, locate other geometric shapes and note how they are used.
- The very top of the Berlin Zoo gate *appears* to be part of a circle. How much of a circle do you think is used to form the top of the gate?
- The large lantern is suspended above the entrance gate by three heavy wires forming two angles. Use them to initiate a discussion of the symmetries of the gate. In fact, the center wire, if extended, is a line of __________________ for the entire gate.
- Explore different ways to estimate the size of each of the two angles. What would be the measure of the large angle? Test your method of estimating these angles to see if it would work for all angles? As a group explore different ways to estimate an angle (see p. 109).

Encourage the trail walkers to keep a keen eye as they walk through the zoo. Mathematics can "jump out" from all directions! Trail walkers can even find mathematics surrounding a little girl washing her hands at a water fountain. The water flows from the mouth of the bear into a container on the ground. The rim of the reservoir is not a circle, so the reservoir cannot be half of a sphere. Walkers may need to review their geometric shapes to help them determine the name for the shape of the rim of the reservoir and then find the name for the three-dimensional figure that has this shape.

ACTIVITIES

- What is the shape of the water reservoir?
- Estimate how many minutes it would take to fill the reservoir. Then time the filling to see how close you came to the actual number of minutes.

Figure 38.

Another interesting place to look for mathematics within a zoo is a waterfall. Encourage trail walkers to think of different ways to estimate how fast the water is flowing.

ACTIVITIES

- Discuss different ways to estimate how fast the water is flowing.
- How much water is flowing?

Figure 39.

ACTIVITIES

- If the Los Angeles Zoo is 9682 kilometers away from the Berlin Zoo, how many miles would that be?
- Of the many different ways to travel from Berlin to Los Angeles, which one do you think is 9682 kilometers?

Figure 40.

The walkways within many zoos have figures of animals drawn on them or embedded in tiles. There are many interesting questions for walkers to consider about the size of the animals. To estimate the height of the giraffe or rhinoceros or another animal, one walker could lie down on the tiles beside the animal and the other walkers could estimate how much taller or shorter the animal is compared to that person's height.

ACTIVITIES

- Estimate the number of white tiles it takes to outline the giraffe. Then estimate the number of gray tiles inside the giraffe.
- Estimate the number of white tiles it takes to outline the rhinoceros. Then estimate the number of gray tiles inside the rhinoceros.
- Estimate the height of the giraffe and the rhinoceros. One way to do this is to have one of the trail walkers lie down on the tiles beside the animal and then the other walkers can estimate how much taller or shorter the animal is compared to their friend's height. Another way would be to pace off the height of each animal.

Figure 41.

Figure 42.

Nearly all zoos, no matter their size, have giraffes because they are a favorite of children and adults alike! Children love to try to guess the height of each of the giraffes or compare the heights of a male giraffe to a female giraffe or the height of a baby giraffe to its mother. Encourage trail walkers to devise methods to estimate the heights of the different giraffes. This particular giraffe is standing next to a large pillar with gray horizontal lines. Walkers could estimate which gray horizontal line the giraffe would come closest to if she were standing up straight.

ACTIVITY

- Estimate the height of the giraffe. If the group comes up with more than one way, compare the height results to see if they are approximately the same.

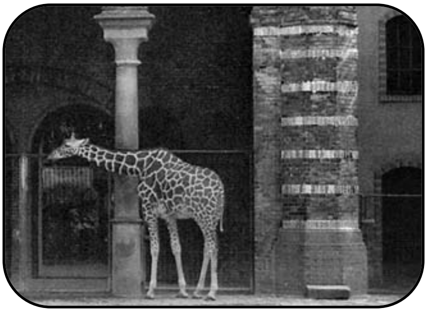


Figure 43.

This would also provide an excellent opportunity for members of the group to talk about proportionality. Encourage trail walkers to think about the proportionality of the giraffe or another animal in comparison with its surroundings. Have them also explain how grid paper helps them to estimate the proportionality of the animals they are drawing.

ACTIVITIES

- Use grid paper to draw a scaled picture of the giraffe.
- As each of you works on your drawing, talk to each other and explain how you are estimating the height of the animal and then representing this on your grid paper.
- Most likely each of you will have a different sized giraffe on your grid paper. How can this be possible when each of you is drawing a correctly proportioned giraffe?

Children really love to watch baby giraffes! The fact that they are very small compared with their mothers and fathers really amazes kids, because the baby is so big when compared with members of the trail walkers group or even other animals! This picture shows Jenny with her new baby boy. He was born on Saturday, June 20, 1998, at the Cameron Park Zoo in Waco, Texas.

Figure 44.

ACTIVITIES

- The baby giraffe weighed 150 pounds and was 6 feet tall when he was born. His mother weighs one ton and is 19 feet tall. Are these numbers consistent?
- Find a young giraffe or another baby animal with its mother. Estimate the weight and height of the baby using what you know about the mother's weight and height. Then ask a zoo attendant how much the baby weighs and how tall it is. How close was your estimate to the correct weight and height?

Monkeys love to swing from ropes and swings made of different materials, delighting children and adults alike with their games and many different facial expressions! Some of the rope formations are really interesting works of mathematical art! Heavy roping has been used to make a net for the monkeys to jump into. They can also climb up the sides of the net. In this case each rope, those going from left-to-right *and* those going from front to back, is shaped very much like a parabola.

ACTIVITY

- Find various different geometric shapes within the ropes and tires used as swings by the monkeys.

Figure 45.

Sidewalks within many zoos in the warmer, southern part of the United States often have canopies to provide shade for visitors. In this zoo beautiful, brightly multi-colored, fabric panels are laced together with rope to follow the curved path of the sidewalk.

ACTIVITIES

- Estimate the width and length of each colored panel and then count the total number of panels used to cover the walkway.

- Approximately how many square yards were needed to make the entire canopy?

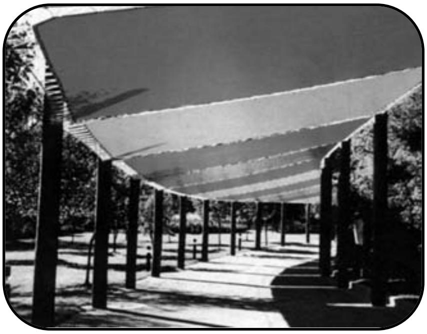


Figure 46.

Many zoos have started to incorporate different versions of popular children's games along their pathways. One such popular game looks like tic-tac-toe except there are pictures of animals, reptiles, and fish instead of Xs and Os. For this particular game board each of the nine squares has two faces.

By placing the nine blocks in a certain order six *real* animals can be made.

Figure 47.

ACTIVITIES

- How many different pictures of *make-believe* animals can you make on this game board?
- Help each other design a game that could be laid out on one of the zoo's sidewalks.

Walking around a zoo always makes people hungry! Have the trail walkers find a snack bar and look at the menu.

ACTIVITIES

- Each trail walker has $8 to spend on food for the entire day. Decide if you want to use all of your $8 at lunch or if you want to save some money for an afternoon snack or drink. How many different lunch orders can you choose with the amount of money you have decided to use?
- Is there a sales tax? If so, how much will it add to your total bill?
- Now how much money do you have left for an afternoon snack or drink?

Figure 48.

Remember throughout the walk to encourage walkers to hunt for geometric shapes. They might even find a star inscribed within a circle!

ACTIVITIES

- Name each of the many different geometric shapes you can find within the circle.
- Estimate the diameter of the circle and then find the area.
- Work with your fellow trail walkers to devise a plan to help you find the area within the star.

Figure 49.

- Once you have found the area of the circle and the star, how would you find the area between the circle and the star? What is that area?

Larger and smaller zoos alike often supply each visitor with a map of the zoo, or there may be a large map of the zoo mounted near the entrance and at numerous places within the zoo. These maps can suggest many explorations in graph theory. Perhaps the most natural question is to ask for the shortest path through the zoo that will visit every exhibit. One might prefer never to pass by any exhibit more than once, but in many zoos this will not be possible. For example, if there is a path to a dead end, like the Herpatorium in the picture, you will have to retrace that path to get back to Pulse Point 2.

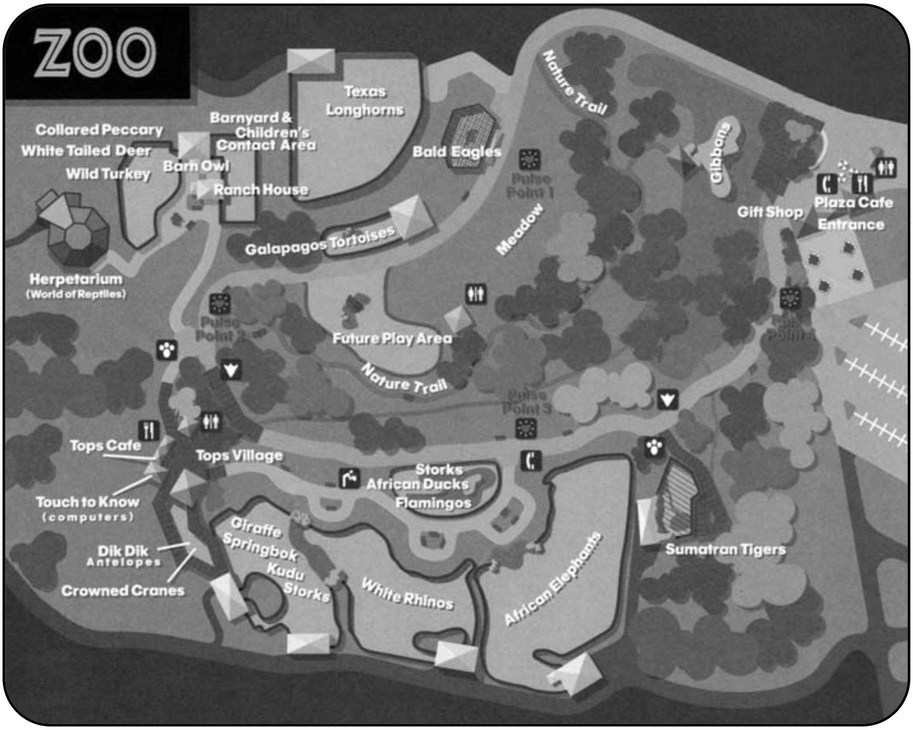


Figure 50.

Is there a path that will allow you to get to every exhibit once and only once? Certainly there cannot be a dead end side trip, but even if there are no dead ends, can it always be done? Absent such a path, what then? How about a path through the zoo that repeats as little as possible? Can you find that?

Many zoos are too large for every exhibit to be seen within the time frame the trail walkers may have available.

ACTIVITIES

- Make a list of the exhibits you really want to see.
- Now use the map of the zoo to look for a path that doesn't retrace any of its sections.

You might decide that the trail walkers can do only so much walking and no more. Now how much can you get to see?

In some zoos, there are special opportunities for helping with time and fatigue. Both the San Diego Zoo and the Bronx Zoo in New York have a skyride that can help visitors avoid a lot of retracing of steps, provide a rest in the middle of the tour, and help get walkers to some very interesting areas in a lot less time. Some zoos also have a tour bus where visitors can ride and see a lot of the area and exhibits. How do you suppose the management picks the route for these tours? Some zoos, especially in hilly terrain, also have escalators to take you from the bottom of a valley to the top. They, too, can provide a brief rest period in the middle of your visit. All of these situations provide excellent opportunities for trail walkers to explore mathematics on a fun-filled outing with family or friends.

Have your group plan a visit to one of these faraway zoos and lay out their own tours. If the zoo does not have a train or trolley path, have the trail walkers plan one for the zoo, discussing where the train or trolley would travel and recommending stopping places for picking up or letting off passengers.

RECREATIONAL MATHEMATICS IN A MALL

Shopping malls provide wonderful opportunities to blaze a math trail and discover that mathematics can show up when you least expect it. We cannot resist, however, bringing to your attention a good math trail question for mall goers before they start: Why did you come to the mall? The first answer may well be, "because things are so much cheaper." Before they accept this as fact, you should have them figure the cost of driving to the mall, and the value of their time spent in getting and being there at the mall. Well, what does it cost to drive? The Federal Government, and many employers, nowadays use about $0.38 per mile as the cost of driving a car when all factors are considered. If the mall is 20 miles away, that's a round trip cost of about $15. Will that much money be saved on a trip to the mall?

How about the value of your time? If it takes an hour to drive back and forth, what could you have done with that time? Could you have done something around the house or apartment that's been on your "to-do" list a long time? Could you have earned some more money by doing part-time work from home? This may, of course, be the wrong way for you to look at going to the mall. Perhaps it is the only place nearby where a particular purchase can be made. If that is the reason, then there is no argument. On the other hand, many people go to the mall to meet friends, see a new movie, buy those new sneakers, experience the virtual reality of a video game, or even ride a roller coaster. You are not going there to save money. You are going for relaxation and leisure-time values. A little mathematical thinking makes that quite clear.

As shoppers walk through a mall, usually they are not conscious of the mathematics surrounding them. Of course, they use mental math to estimate the cost of a sweater that is on sale to make sure they have enough money to buy it and to compare prices between stores. But do they actually give much thought to mathematics not related to purchases? Chances are the answer to that question is a definite "No!" Keep in mind that each mall has its own unique examples of mathematics.

Every store at the mall works hard at attracting customers. One way to do this is by advertising sales in the local newspaper. This is a good way to bring people to the mall with the intention of coming to that particular store. Stores also want to attract the passerby who had *no* prior plan to visit the store. This can be done by an eye-catching display in front of the store and in each of the store's windows. One such example is illustrated by a display in front of a sporting goods store: A clear hexagonal-shaped column filled with tennis balls. A word of advice: This example presents the opportunity for a very large amount of mathematics. There is a real danger of losing the trail walker by doing too much in one spot. We shall mention a variety of possible topics, and at a trail stop like this, you will have to pick and choose.

Figure 51.

A first series of questions might concern the shape and size of this column. Ask walkers to compare the shapes of slices through the column at the top and at the bottom: They will probably decide that a slice has the same shape all along the column, but not the same size. Each slice is hexagonal in shape, and all the hexagons look the same because, while they are all regular and therefore similar, the lengths of the sides are different. The walkers may not know or remember the word 'similar,' but they will have the idea. You can take this further if you want: What exactly is meant by 'size' of the column? The top of the column has a different diameter than the bottom of the column. Trail walkers might be asked to decide what they want to mean by the diameter of the column: Since the column is vertical, they will naturally come to cross sections at different heights, so it's the diameter of a cross section that we are talking about. They may want to think about what a diameter should mean. The trail walkers might decide that they know for sure what they mean by the diameter of a circle and that the diameter of the cross section is the diameter of the smallest circumscribing or the largest inscribing circle. Once this is decided, have the trail walkers estimate both of these diameters.

ACTIVITIES

- Look at this clear hexagonal-shaped column filled with tennis balls. Estimate the number of tennis balls in the display column.
- Suppose you could make two or three parallel horizontal cuts through the column. What is the shape of each 'slice'? Compare the shapes of the slices.

Another series of questions could lead to estimating the volume of the column. To estimate the volume of the column, start with the height—a more standard trail-walking problem. The areas of the top and bottom hexagonal cross section can be estimated—perhaps by observing that they *almost* fill the circumscribed circles whose diameters the walkers have estimated. The walkers may wish to estimate volume as average area times height. But what does average area mean? As the group of walkers discusses how to do this, they might use the average of the top and bottom cross sectional areas as a good estimate of the average cross sectional area. It is possible they will decide that the average area should be weighted towards one end or the other. If so, which end should they choose?

ACTIVITIES

- Work together to come up with a plan to determine the size of the column. How would you go about measuring the column?
- Estimate the volume of the column. Devise a plan that would help you to come up with a good estimate.

A further natural series of questions might concern the balls inside the column. Many tennis balls touch the plexiglass surface. Let's first stick to balls touching the surface. Trail walkers can pick out one particular tennis ball that touches the flat plexiglass surface and count the number of tennis balls touching that ball that are also touching the plexiglass surface. This should be done several times in order to make a conjecture about the maximum number of tennis balls that can touch one tennis ball. If we think of balls on a flat surface, and of great circles of the balls parallel to the surface, then we are really trying to see how many circles of equal size can touch a given circle of that same size. What happens if the problem is taken from a flat two-dimensional surface to three dimensions? Ask the walkers to think about how many tennis balls of one size can touch a given tennis ball of the same size. Without using the terms two-dimensional and three-dimensional, you can move the walkers' discussions and explorations of two-dimensional ideas into those of three dimensions. In this way the three-dimensional ideas become natural extensions of two-dimensional ideas.

ACTIVITIES

- Find a tennis ball that touches the plexiglass surface and is surrounded by other balls also touching the surface with as small amount of gaps as possible. Count the number of balls it takes to totally surround that tennis ball. Choose another tennis ball and do the same. Make a conjecture about the number of tennis balls that can touch one target tennis ball and also the plexiglass.

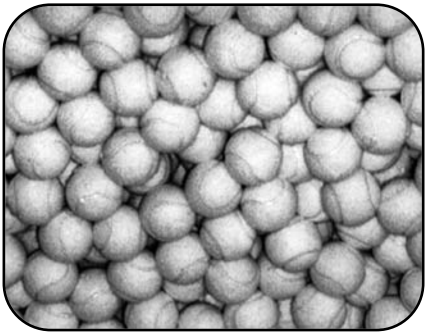


Figure 52.

- Suppose you could climb into the column and find a tennis ball *totally surrounded* by other tennis balls. What is the largest number of balls that can touch one central ball? Choose another similar ball and do the same. Try this several times before making a new conjecture about this number of tennis balls.

The *geometric* questions that arose from the column of tennis balls lead to many interesting mathematical possibilities. In laying out the trail guide, you might center on the geometry as we have done. You might alternatively choose to start with the question of estimating the *number* of tennis balls inside the display. This in turn could lead to successive steps of estimating the volume and the proportion of the volume occupied by tennis balls. Questions about the shape of the column and its diameter would then arise from trying to estimate the volume of the column. The close packing of the balls becomes important when the walkers try to see what fraction of the volume the balls occupy.

You will need to choose the setting into which you put this series of questions. You may feel that estimating the number of tennis balls is just the right level of both familiarity and challenge to catch the trail walkers' attention. However, if you already have too many arithmetic questions, it might be better to focus on the geometry. This could later lead into estimating the number of balls in the column. This idea of using tennis balls is interesting, but you can look at the same questions with a gumball machine.

Another way to draw shoppers into a store, other than displays, is the store's sign. If you look at a row of stores in the mall you will notice that each sign is different. Trail walkers can ask the question why. One reason is that this leads to a type of eye-catching uniqueness for each different store. National store chains have the same sign for each of their stores in malls throughout the U. S. because shoppers become accustomed to that particular store's sign and seek it out at the different malls they visit.

ACTIVITY

- If you look at a row of stores in the mall you will notice that each sign is different. Have you ever asked yourself why this is true? Talk with your fellow trail walkers and see if you can come up with some reasons as to why this may be true.

Figure 53.

Figure 54.

Consider the spacing between letters in a particular store's name. Are the letters evenly spaced? The problem of spacing letters in any type of printing or font is called kerning. Old-fashioned typewriters used a uniform size for the block each letter occupied, no matter whether the letter was an *l* or a *w* or the number was a *1* or a *5*. On the other hand, many fonts on today's computers have built-in automatic spacing adjustment. This problem of spacing leads to another type of trail exploration. How is the letter spacing in a sign decided? What makes it look right? Trail walkers would need to decide what they mean by the word 'right.' Some walkers might enjoy trying to formulate in a precise way what makes letter spacing look good to them. Others might argue that this is a purely aesthetic question and that such an attempt at a mathematical formulation is artificial and irrelevant. Is it possible that some stores have signs that actually are *not* appealing and eye-catching in their design? If you see such a possibility, it might make a good trail stop.

ACTIVITIES

- Look at the spacing between the letters of a particular store's name. Are the letters evenly spaced? Select the sign of another store and check to see if those letters are evenly spaced.
- How do you think the spacing between letters in each of the signs was decided upon? What makes it look right?

The floors of malls are great places to look for interesting tiling designs that can help to explore mathematics and geometry! Some tiling designs are made up of concentric circles. The walkers might estimate the diameter and circumference of each of the different circles. But that activity should bring up another question: Is everyone using the same 'gauge' or unit of measurement? You should lead the members of the group into explaining how they are estimating these measurements and have them decide as a group how to use these different estimates to obtain one 'best' estimate. It might be that one method of estimating is better than another method or that taking the average of all the different estimates is best. A similar format of questions can be used when asking the group to estimate the area of each of the circles and then comparing the area of the smallest and largest circles.

ACTIVITIES

- Estimate the diameter and circumference of each of the different circles.
- Discuss how the estimates were arrived at. Did everyone use the same gauge for estimating?

Figure 55.

- Is one method better than the others? Decide as a group how you might use the different estimates to obtain one best estimate.
- Compare the diameter and circumference of the smallest and largest circles.
- Repeat this activity for the area of each circle and compare the areas of the smallest and largest circles.
- Make a list of the geometric shapes that can be found within the circles.

Another floor design involves a circle surrounding a star. The trail walkers could find the various regularities and patterns in this medallion.

Figure 56.

ACTIVITIES

- What geometric shapes could have been used to make the star in Figure 56?
- Estimate the diameter, circumference, and area of this circle.
- Estimate the area of the star inside the circle.
- What percentage of the area of the circle does the star cover?
- Do the points nearest the periphery of the circle look equally spaced? If so, how many degrees are there between consecutive points?
- How many degrees are in each point of the star?

Water fountains are great gathering and meeting places at malls—children are always fascinated by the water and the sound it makes, they love to throw coins into the water, and it is a nice place to sit and relax for a while before continuing on the journey of the mall. People of all ages like to look down on the water fountain from a staircase or floor above it. They look at the shape the water makes as it shoots out and upward and then comes back downward and splashes into the water in the fountain's pool. Children will sit on the floor near water fountains, look at the water and the arcs made by the water, and listen to the soothing sound.

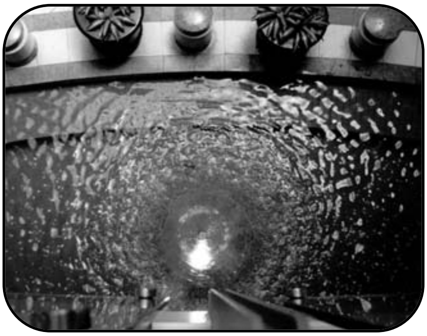


Figure 57.

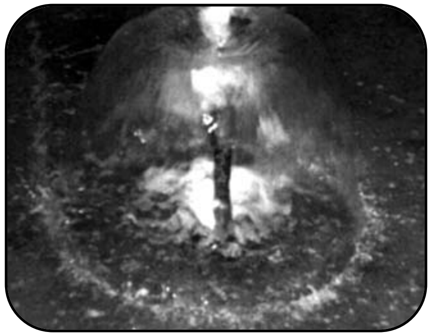


Figure 58.

ACTIVITIES

- The water shoots out of a vertical pipe in various directions at the same angle with the vertical. Describe what shape the water makes as it comes out of the pipe, reaches a peak, and then falls.
- Estimate the amount of money on the floor of the fountain.

ACTIVITY

- Decide if the pairs make up squares. Discuss the various techniques you used to come to a conclusion.

Figure 59.

ACTIVITIES

The pattern is more complex.

- What geometric shape is used to make up this tiling? There are several correct answers to this question. Discuss why this is true.
- Pick a point of intersection on the tiles and count the number of hexagons that meet at that point. Use this result to help you decide how many degrees there are in each angle of a regular hexagon.

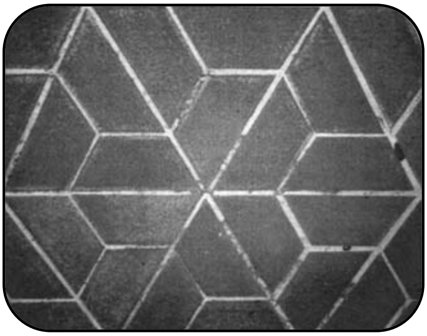


Figure 60.

- How many total degrees are there in a regual hexagon? Is this result the same for a non-regular hexagon? Are you surprised by the answer?
- Now find as many different quadrilateral, pentagonal, and heptagonal shapes as you can in the tiles.

On the upper floors of many malls there are open areas surrounded by brass railings where shoppers can look down on other shoppers and stores on the lower floors. In the mall used in this illustration, the brass railing is in the shape of an oval (two parallel lines with semi-circles at each end). The walkers may not agree on a definition of oval. This is an example where the meanings of a word in everyday English and in mathematics may not agree. Some walkers may wish to call this an oval, while others, and you, may not. Would you call it an oval if it were a running track around a football field?

Figure 61.

Figure 62.

ACTIVITY

- Go to the open atrium area on one of the floors of the mall. What is the shape of the brass railing? Explain how you came to your decision.

Throughout most malls there are benches for people to sit on. Many of the benches are straight, but sometimes they are curved. The mall designers probably did this so that the curve of the bench would follow the curve of the railing. An interesting question for trail walkers to consider would be how would a carpenter make a curved bench like this one? The pattern of the wood of what had been the rings of the tree suggest whether the curved bench was made by cutting the shape from the wood or by soaking the wood until it is very wet, bending it, keeping it in the desired shape, and then allowing the wood to dry completely.

Figure 63.

Figure 64.

ACTIVITY

- See if you can find a curved bench in the mall. How do you think the carpenter would make a curved bench like this one?

See if you can find an empty store in the local mall. Have trail walkers estimate the square footage of the empty store. Then they could think about the monthly rent per square foot, and what the monthly cash flow would have to be to make a profit.

ACTIVITIES

- Estimate the number of square feet in this empty store.
- Go to the mall management office and ask about the monthly rental of the store. How much does it cost to rent each square foot of the store each month?
- Is rent the only cost the store must pay each month? If not, estimate the amount of these other monthly expenditures.
- How much money would the store need to take in every month in order to break even? How much money would be needed for the store to make a profit?

A mall is a great place to find all kinds of mathematics at work! You have now taken a short walk around one mall and discovered some mathematics. Now go and see if you can discover even more on your own.

PARKING

As you are walking along, you see cars parked along the curb or in a parking lot in front of a supermarket or a neighborhood group of stores. There are a lot of questions you might ask about the parking. For example,

1) On most streets, parking is often parallel parking, while in parking lots it is often angle parking. If the street is very wide, you might also see angle parking there. Why is parallel parking common on normal width streets, but angle parking more likely on wide boulevards? What do you think? Typical comments that people might make is that angle parking, if the angle is not too big, is easier for the driver than parallel parking. You can go right in without pulling up parallel to the car in front of the space and then backing in, so why isn't *all* parking angle parking if it's that much easier? Is it perhaps because of the less efficient use of space? First of all, cars parked at an angle use up more of the width of the street than parallel parked cars. How much more? If you imagine the car to be roughly a rectangle, say length L and width W, the parallel parking uses up a width W from the width of the street. Is that all? Well, maybe an extra few inches since one doen't always back perfectly to the curb. (Also, if you have a passenger who has some difficulty getting out of the car, you may have to stay a bit further from the curb so the passenger can comfortably step onto the street first before stepping up to the sidewalk.) If you park at an angle, how much extra width do you take from the street because the rear end of the car sticks out a little? See if you can figure that out.

Now, back to angle parking and another aspect of it. When you park at an angle you leave a little triangle-shaped area empty. Where is it? Its edges are the front bumper of the car, the white line (or the edge next to the car), and the sidewalk. A further triangle *behind* the car is also wasted if it isn't marked in any way, like the one in front, but does take something more away from the usable width of the street. How big are these triangles in front and in back of the car? How would you find out? Use a formula? Measure? Guess? Those are all OK.

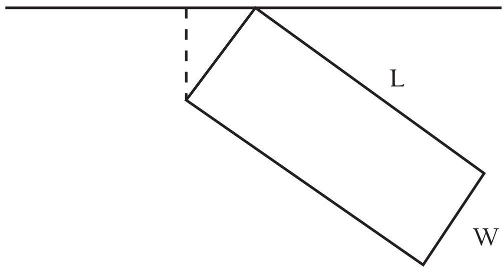


Figure 65.

Hey, wait a minute! If the angle in the angle parking is big enough, these triangles you're talking about are awfully small. In fact, sometimes thay aren't there at all! Suppose you pull in at right angles (or you can say perpendicular) to the curb. There is no wasted triangle. You see here an important point about thinking mathematically: You have to be precise. It is true that there are little wasted triangles for most angle parking, but not when the angle is 0° or 90°. You have to account for such exceptions—some people call them "limiting cases"—when you are being thorough.

What are the advantages and disadvantages of perpendicular parking? Disadvantages: You are taking more width away from the street—this time you take L away rather than W plus a little bit. Also, perpendicular spaces are harder to get into than most angle spaces: You can't go straight in, but have to pull up at an angle behind the next car, back up, and then go in. This maneuver needs a pretty wide street.

Advantage: You can get more cars into a given length of parking frontage. How much more? Is it *really* worth the trouble? Figure it out. For a typical car, is L as much as twice W? What do you think? If it were twice, what would you do to the number of parking spaces by having perpendicular rather than parallel parking? (In a sample of one, W was a little more than 6', and L a bit more than 15'.)

Let's get back to angle parking that isn't either parallel or perpendicular. When you see angle parking on a wide street, about how big an angle do you think it is? How does someone decide at what angle to the curb to paint the white lines? It's perhaps a balance between making it easy to get in and out, and the waste of space caused by the little triangles. At what angle do you think the little triangles are biggest? The answer turns out to be 45°. What tends to be the angle of angle parking? A guess is that, in many places, angle parking tends to be at about 30°. This brings up an interesting point. If you are out on a math trail, how do you estimate an *angle* anyway? We typically know various "rules of thumb" for estimating *length*. You might know the length of your foot or your arm or the length of your stride. You know how to use shadows. But what do you know that would allow you to estimate an angle? Not many people have a technique for that. Please look in the section called Estimation on pp. 108–111 of this book for a nice method.

Mathematical Note: One way to see how far a car parked at an angle sticks out into the road is to develop a formula. If the acute angle between the long side of the car and the curb is θ, then the furthest distance of the front of the car from the curb is $W \cos \theta$. From that point, the car sticks a further $L \sin \theta$ into the street. So how much space does the car take from the width of the street? $W \cos \theta + L \sin \theta$. So if L were 15', W were 6', and θ were 30° then the car would stick out about 12.7'. This is well over the 6' if it were parallel parked, but less than the 15' of perpendicular parking.

For what value of θ does the car stick out the most, and at what angle would that happen? It turns out that the most the car ever sticks out is a little over 16', and this happens if θ is about 68°. So at worst it sticks out *more* than its length! Did you expect that?

Let's finish with a nice, open-ended, debatable question, how can the traffic department pick θ? There are lots of ways of thinking about this, but here's one: It's a tradeoff. If you make θ large, so that cars are close to perpendicular to the curb, then you use up a lot of the width of the street, but you save on how much curb you use up. If, on the other hand, you have parallel parking, you take as little as possible from the width of the street, but you use an awful lot of curb. If the car is 6' wide and 15' long and you do perpendicular parking, your spaces are probably about 8' wide because you need about 2' for opening the door. (Why didn't we say 10', for opening doors on *both* sides?) How far does the car stick out into the street? It sticks out about 15'. So perpendicular parking uses 8' of curb and 15' of width. Parallel parking will use about 17' of curb (to allow 2' between cars), but very little more than 6' of width (with parallel parking, an open door doesn't run into another car). At a 60° acute angle between car and curb, you would use a little more than 9' of curb (a bit more than perpendicular parking) and stick out about 16'. What, more than perpendicular parking? Yes, because at worst you stick out the length of a diagonal of our standard car, which is a bit more than 16'. Here's a little table for a car in the shape of a rectangle 6' long, 15' wide, and doors that need 2' to open.

Angle	Length of curb	Street width used
90° (perpendicular)	8'	15'
60°	9'	16'
45°	11.3'	16.3'
30°	16'	14.4'
0° (parallel)	17'	12' at a minimum

What you see is that *most* values of the angle use a lot of street width and only parallel parking conserves street width. What you find in many communities is that you have parallel parking on most streets because width is scarce, and angle parking on wide enough streets in shopping areas because curb space is scarce.

2) We have mentioned that, in most places, white lines typically mark parking spaces. Why are parking spaces marked? Oh come on, don't be so naïve, so that people can put parking meters next to them and make money for the town! But that's not true in a mall or supermarket parking lot. They don't have meters, but they still have marked parking spaces. One purpose might be to channel the traffic so that cars don't run into each other. Another purpose might be to use space more efficiently. If you allow people to park anywhere, they will leave an empty space between cars that is too small for another car, but way more than safety requires. Let's think about this last one. If people were to be allowed to park anywhere there was enough room, how much space would they waste?

It so happens that when you start to look at this question, it's not easy. Let's begin by simplifying it. Let's not imagine a parking lot because it is two-dimensional and people could conceivably park in any direction, a horrendous complication we don't need. Let's go to a more rural town, or maybe a spa, where making money with parking meters isn't a big deal. A good image to have in mind, for example, would be the main street in Saratoga Springs, NY, which is a nice wide street with (when last seen) neither meters, nor marked parking spaces. You just parallel park anywhere there is enough room!

How can you analyze this? When you consider parallel parking, you want to have a minimum space between cars. So this time let L represent the length of a car plus a couple of feet around it for safety and a little maneuvering room. Imagine that all cars are the same length. That isn't true in the real world. Also some drivers are better parallel parkers than others and may feel more comfortable getting into a smaller space. These additional complications are real, of course, but enough of the essence of the problem remains without them. Let's try an example. Say that the length of the block available for possible parking is 300 feet. The sign says "No parking within 50 feet of the corner" so subtract 50 feet at each end for safety. We are left with 200 feet for cars and we'll make each space 20 feet long. So, if you have marked spaces you can park 10 cars. Now suppose a car comes along; parks in a space that's S feet long, where S is at least 20; and leaves the rest of the space, namely S – 20 feet, divided randomly between space in front and back of the car. How much of S – 20 will be wasted? If S < 40, then all of it will be (why?) but it's not *this* car's fault. A previous car loused things up by leaving a space that's too large but still holds only one car. But if S > 40, then the present car has a chance either to waste parking frontage or to conserve it. What are the possibilities? Try some cases and see.

So how would you find out what is likely to happen without marked parking spaces? It turns out that this is a tough problem even for professional mathematicians. (It was first solved by the Hungarian mathematician A. Renyi in 1958, and is still a subject of active research.) You might try a few very simple cases like S between 40 and 60, and then use these to make some guesses. Or, if you get too interested, you could perhaps simulate the situation: Use random numbers to locate the front, or the middle, of the next car, and see how many cars you can place before every remaining space is less than 20 feet. The answer is that, on a long block, about 25% of the space will be wasted on the average.

3) While we're at the subject of meters, you might want to think about whether meters really do pay. The simplest question might be whether a meter pays for itself during its lifetime. How much money is a meter likely to take in during one day—or week or month or year; how much does a new meter cost, how long does it last? That's a simple form of the question. You might go on from there. If something goes wrong with a meter, is it easily repaired or do you have to junk it? What's most likely to go wrong anyway? Perhaps people putting in a wrong coin or a slug and thereby jamming the meter, or perhaps a car bumping into it. The oldest author remembers 50 years ago, when he was a student taking the East Boston ferry. They had a turnstile to enter. He tried a nickel—too big. He grumbled and tried a dime, which got stuck. At this point a worker came over, gave him a dirty look, took the dime out and gave it back to him—and told him it cost a penny! Even then, one didn't think that *anything* cost a penny!

Now we can make the problem a little more complicated, and a little more realistic. If you don't hire any meter readers, for example, drivers might just ignore the meters. So you need people to visit the meters periodically and write out parking tickets if the meter hasn't been fed sufficiently. How many people? Enough so that a driver will not feel it's worth the risk to leave the meter unfed. An itinerary for the meter reader should not be *too* regular (people might learn to outguess it), but the reader does need to visit every meter every so often. How would *you* make out a route and a schedule for a meter reader? How many would you hire? When you take their salaries and benefits into account, does it still pay the city to have parking meters? What else does the system within which the meter readers operate include? Do the fines they levy balance their loaded salaries and the costs of the associated judicial system?

4) Maybe one more math trail question about parking. As you walk around and observe the traffic, can you tell if the town has *enough* parking? In some places this is a subject of hot debate. "If we had more parking, not so many people would go to the mall."

   "If we had more parking, more people from nearby communities would come here to shop."

   "If we had more parking, people from other communities would come here just to take the train, cause all kinds of traffic problems, pollute the environment, and make this a less desirable place to live." So there are lots of ways to argue. What would you suggest as a way to tell if the community had enough parking?

Some people will tell you that they just aren't going to shop here because they can never find a parking space. You could take data of that kind, but that's a bigger, long-term project. The receipts from each parking meter will tell you something about how much it is used but you don't have that information right now. Here's one thing you can do immediately: When a car pulls out of a parking space, time how long it takes before someone else pulls in. In one town, the average figure was only about 20 seconds! Would you take that as a clear indication of a shortage of spaces, at least in that part of town? Do you have a better measure to suggest?

SUPERMARKETS

If the math trail takes you into a supermarket, there are many interesting items you might want to consider. Let's talk about a few of them.

1) You have a choice of several size packages of a particular item. Which size should you get? One way of looking at this, of course, simply deals with the price. Which is cheaper? Usually, but not quite always, the smaller package is cheaper. So what? We shouldn't be looking at the total price, but the price *per unit weight,* say per pound. Nowadays you no longer have to try to figure this out, the label under the item on the shelf is supposed to tell you. (You should check that the label takes into account the temporary change in price if the item is on sale.) You still have to think a little more: If the larger package is cheaper per pound, as it often is, will you be able to use it up before it gets stale or spoils? Will there be someone in the family for whom it is too heavy to lift?

The computation can get trickier if you have a coupon that is valid for either size you are considering and gives you the same discount. The price per pound that's written underneath the shelf may not be the best way to look at it any more. The smaller item may well have become cheaper by pound when you subtract the refund from the coupon. But the problem is different if the coupon is for something you use a lot.

You have only one coupon! Maybe you should just buy the larger item at the price reduced by the coupon, because if you buy the smaller one, you'll have to get a large one pretty soon anyway.

2) Where do you look for things on the shelves? There are items that people tend to run out of unexpectedly, that lead to a quick trip to the store. What, for example? Let's say milk. You have probably noticed that in many stores, milk is quite a long way from the door and the cash registers. Why is that? Because something like 30% of all supermarket purchases represent impulse buying, something you didn't intend to get when you came in the store. The further you have to go when you come into the store, the more likely you are to buy something else too. So milk and hamburger tend to be a long walk.

Supermarket managers may complain that this is unfair criticism. These items require refrigeration, and the sections in which food is kept chilled are naturally at the walls. That's the way one builds supermarkets. Also the refrigerated storage for meats is often at the back of the store, and the meat counters will be close to that. All that is true, but there is also another phenomenon. We are beginning to see frozen food sections in the middle of the store rather than against the walls. In the summer, these help with the cooling of the store as well as of the products. In the winter, the heat removed from the cold storage can be used to help heat the store itself—in which case you might like this heat source to be more in the middle of the store. All we're saying is that refrigeration may no longer be a good excuse for shelving milk and hamburger far away.

Let's return to the locations of items on the shelves. Think of the cereal aisle. How do you suppose they decide at what height to put a particular product? Why do high-fiber cereals tend to be on the top or the bottom shelf, while the Soggies, which are the latest kiddie rage, show up in their glorious multicolored packages on the middle shelf? A possible explanation is that cereals that are advertised on Saturday morning children's television are put at the height of a child seated in the carriage. That way the child will see it, ask (a euphemism for whine) for it, and there you have one more impulse buy.

You may want to think about the following question, not just for the supermarket, but also for a 5-and-10 (if there still are any), and for any good-sized store. How much of one item should you display? If you display too little you will keep running out, and restocking the shelves in the middle of the day may be a nuisance. But space is valuable, and if you display too much you are wasting space needed for some additional product. On the other hand, even if the item is huge, you have to show at least *one!* What's a good balance? Do you use extra space for items you sell a lot of or items that you *wish* you sold a lot of, but don't? Some stores use the most prominent display areas—say in front of the store opposite the cash registers—for weekly specials. Why? Perhaps people get a little angry if they come in for an advertised special and can't find it. Better to put those items where people can see them right away, and then let them shop further in a happy frame of mind.

3) Many supermarkets have express lanes for people with n packages or less. What is the value of n in the store in which you are at the moment? Is it 10 packages? What other values of n have you seen? Various supermarkets across the country seem to allow anywhere from 5 to 15 packages in the express lane. If the number varies that much, maybe people don't really know what it should be. How many packages do you think should be allowed in an express lane?

How do you think about a question like that? One way would be to figure out first why you have an express lane at all. That's easy, to keep customers happy. Yes, but how do you do that? You might want to minimize the average waiting time at the checkout position. Average, eh? Average over what, average wait per customer, or average wait per package? Does it make a difference? In the first statement, you are saying that all customers are created equal. A long wait for any of them is equally bad. The second statement says that customers with a lot of packages can wait longer than customers with just a few; the wait *per package* might still be equal. There are other possible criteria. You might say that it's really *long* waits that are a problem. In that case you want to minimize maximum waiting time. Maybe your consumer research has shown that people are willing to wait ten minutes; after that, they shove the carriage into a corner and go elsewhere. So what you want to do is to minimize the probability that the wait exceeds ten minutes—or whatever number you believe. How do all these criteria square with the following argument? If you restrict the number of packages in any lane, there will be times when under your rule that lane is idle, but would be in use if there was no restriction. Therefore an express lane must increase waiting time. Under what assumptions would this actually be true?

Once you know what the waiting line is supposed to accomplish, you can then take some data. How many people get in line with how many packages? How does the time to go through the line vary with the number of packages? Do four packages take twice as long as two? No, because there is a setup time, which is the time to get ready for the next customer, no matter how many packages s(he) has. But once you take that into account the time may be linear, at least until the point at which you need two bags instead of one.

One suggestion that has been made is that it takes a *person* to staff an express lane. Allow as many packages in the express lane as will keep one person as fully occupied as you can. Don't allow a queue to grow at that counter, but don't allow much idle time either. That's the most efficient way to use the checker! Here's a clever idea—think of the problem from the point of view of the store management, not of the customer. The customer enters the solution if the number of packages this rule allows is too small. In that case, get a second express lane.

One supermarket we have seen disables all express lanes at the busiest time, namely Friday night. What do you suppose is their idea of the purpose of an express lane?

BUILDINGS

Almost any size locality has a fairly tall building. It may be a church, or a big old house, or an apartment or an office building. It's fun to estimate how tall this building is. How many ways can you think of doing that? If it's a sunny day, shadows are a natural thought. Say the shadow of the building is 30' tall, how tall is the building? Well, you have to know *something* else! Say you are 5' tall and your shadow is 3'. Then the shadow of the building is ten times your shadow, and therefore the height of the building would be ten times your height. So you would estimate the height of the building at 50'.

Another way to think about this would be to count floors. If you see 6 floors from ground level on up, and you guess that the ceilings are about 8 feet from the floor, you might guess 48 feet for the height of the building. What adjustments might you make to this first estimate? First of all, the 8 feet is only a guess. Also, there is some distance from the ceiling at one level to the floor of the next, and we have five of those gaps to account for. There is also a roof of some thickness. Maybe the main floor is built for the kind of business that needs higher ceilings than the upstairs offices. All in all if any of these thoughts apply, you might adjust your estimate to 55 feet.

If it's a church, trying to argue about floors won't get you very far. You might still be able to guess the height of a big window, and estimate how many times the height of the window is to the height of the church. What else might you try?

As you walk through the town, you might look for different geometric shapes. Windows in many buildings are in the form of rectangles; in churches you often see circles, or rectangles surmounted by semicircles or surrounded by other, fancier shapes. Sometimes you see square windows. On some pavements, you find triangles and hexagons and octagons. When a piece of pavement has been cut out for a tree, what shapes might you see there? Can you find a pentagon somewhere?

Brick buildings and brick walls offer some nice possible stops for a math trail. There are many possible patterns in which bricks are laid, and they have both practical and aesthetic features that you can examine. Let's begin by looking at two patterns, both are seen frequently in brick walls.

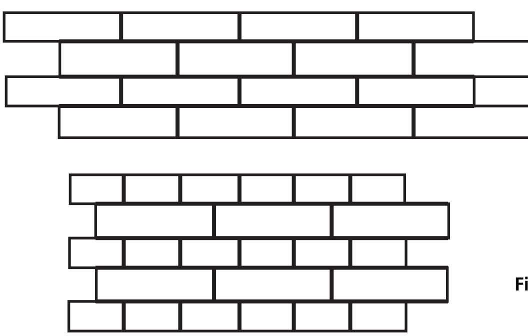


Figure 66.

Each of these is called a "bond," which means an arrangement of bricks in a wall in such a manner as to prevent any collection of adjacent vertical joints, which would be very bad for the strength of the wall. When a brick is laid lengthwise, as every brick is in the first pattern, it is called a "stretcher." A brick laid so that the end is facing you, as every other row is in the second pattern, is called a "header." What do you notice about the dimensions of a stretcher and a header?
The thickness, of course, is the same, but the stretcher is a tiny bit longer than twice a header. The most common bricks have dimensions 2" x 3 3/4" x 8". This has the effect that two headers, when laid, are exactly as long as one stretcher. How? This happens because there is an extra 1/4" (or is it 1/2"?) of mortar in between the bricks. This 2 to 1 ratio is the basis of innumerable patterns for laying bricks. The first of the two patterns above is called "running stretcher bond" and the second is called an "English bond."

It is interesting to find out what movements of the brick patterns bring the pattern exactly on top of itself. In the running stretcher bond, if you move the full length of a brick to the right (or left), you come back onto the same pattern. If you move *two* rows up or down, you come back onto the same pattern. If you move only one row up, you also have to move half a brick sideways to come back onto the same pattern. A motion that brings you back onto the same pattern is called a "symmetry" of the pattern. So moving one brick length horizontally, or two brick widths vertically, or one width vertically and half a length horizontally at the same time are all symmetries of the pattern. Are there any others? Well, you could imagine drawing a horizontal line through the middle of a row of bricks and taking a mirror image across that line. You might call that "reflecting" across that line. For what vertical lines would the same thing be true? Also, there might be points where you can rotate the whole pattern around them by some amount and come back to the same pattern. Can you see any?

Try these same questions for the other pattern. A way of distinguishing brick patterns might be to examine their collection of symmetries. By the way, if you follow one symmetry by another, will you get yet another symmetry? One consequence of this, if it's true, is that there are an awful lot of symmetries of a brick pattern. Not in the real world, of course, in which real walls begin and end (and aren't made that precisely anyway) but in the geometric abstraction of a brick wall with which we have been playing.

A HIKE IN THE COUNTRY

There are many different kinds of walks and terrain, so all we can do is give some examples of ideas. If the state park or the national forest you are in has provided a trail map, then you have the opportunity to map out the route you want to follow and the time it should take. You can use time and distance estimates, features on the map as well as in the terrain itself, to keep track of where you are and what you should be looking for next. A good map also allows you to judge the desirability of a particular hike: The views you will have, access to drinking water, the changes in elevation, and the amount of backtracking you may have to do. Some people try to combine features such as these into an index of expected pleasure for a particular route.

Many trails follow brooks or rivers some of the way, and it is fun to estimate the speed of flow. Someone will perhaps estimate a distance from where you are standing to a reference point, and then time a twig or leaf that you've thrown in; that's one way to do it. You may observe that very few flowing bodies of water are straight and a larger stream that has not been regulated is likely to have a large number of meanders. Why do you suppose rivers meander, and what determines the amount of bending (someone may show off and say “curvature” or “radius of curvature”)? Basically rivers bend because a straight river is not stable: A little departure from straightness tends to lead to more departure from straightness. Often, human attempts to straighten a river lead to the water moving too fast and causing unexpected erosion downstream. Rivers seem to want to meander.

You pick up a leaf and ask what tree it came from. The group may agree it’s from a maple or an oak or a beech or a dogwood. The question is, how do we tell? We look for particular patterns in the shape or the color. If a leaf has lots of ins and outs then it can be from a maple or an oak, but not from a dogwood or beech, which have leaves that are pretty much convex. To make the difference precise: For example, if you take two spots on a beech leaf and imagine drawing a line segment between them, it will always be on the leaf, while this is far from true on an oak or maple. That’s one way of making the notion of convex precise. How can you tell an oak leaf from a maple leaf? The maple leaf has a lot of pointed ‘ends’ while the oak leaf ends are more rounded. How would you write down a sequence of choices that would permit a decision among a larger variety of different leaves?

If trail walkers enjoy the leaf question, you can try a scheme in a similar spirit for the bark of trees or for animal tracks or for wild flowers or birdcalls.

If you walk by some farmer’s fields surrounded by fences or stonewalls, you might wonder how big the field is. Farmers in Brazil are said to estimate the area of fields with four more-or-less straight sides (quadrilaterals is the fancy term) by taking the average length of each pair of opposite sides and finding the product of these average lengths. What do you think of this method? Does it give a pretty good guess much of the time? Do you think it tends to give too big or too small an answer? When does it give the 'right' answer by our understanding of what we mean by area? Remember, of course, that our notion of area is what our mathematical types have developed over the centuries, and other people have every right to define a notion of area to suit their convenience and utility.

One thing you often see on a farm is an old wagon wheel. If the condition is at all good, it will have a series of spokes that connect the outer circle to an inner polygon. The interior of the polygon will be the hole through which an axle used to go. If you see such a wheel, how many spokes does it have? If you imagine the outer rim made up of arcs that end at the spokes, how many such arcs are there? If the inner polygon has edges that end at the spokes, how many such edges are there?

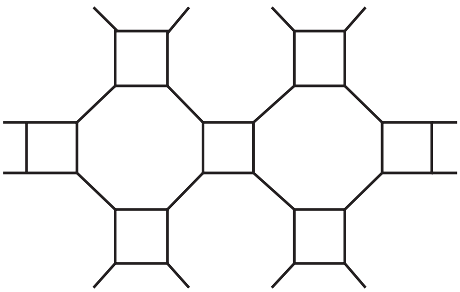
As an example, suppose the wheel has 8 spokes. Then the number of arcs comprising the outer rim is also 8, and so is the number of sides of the inner polygon. Now imagine the simplest possible line drawing of the wheel—not very realistic, I admit—consisting of an outer circle made up of 8 arcs, 8 line segments representing the spokes, and 8 edges of the inner octagon. On this line drawing there are 16 places where line segments or arcs meet. We call these "vertices." There are 24 line segments, or arcs—call them "edges." The piece of paper on which you made the line drawing is divided into how many regions? Well, inside the circular rim there are 8 sections bounded by spokes and one inner polygon where the axle would go. Outside the circular rim there is one region, the rest of your piece of paper. This makes 10 regions. If you take vertices minus edges plus regions, you get 16 – 24 + 10 = 2. With a different number of spokes, you'd still get 2—try it! For any line drawing you make with vertices, edges, and regions in the plane, you'll get 2! This is known as Euler's formula.

TILINGS

We are going to do a very leisurely, purely arithmetic, classification of semi-regular tilings. We present this in case the question comes up in your trail planning and you choose to pursue it. The more rapid algebraic classification is quicker and more familiar, but may be too much concentrated mathematics all at once. Of course, ours may be just as bad!

On many possible math trails, we will see tilings on the floor or the walls or the pavement. Each individual tile is typically in the shape of a polygon made out of clay, plastic, linoleum, or wood, or else designed into the cement as if the pattern was made of individual tiles. There are tilings that are not regular in any sense, for example, a walkway made of rectangular stones of various sizes. We are not talking about those. What distinguishes the bathroom or floor tilings? It looks like the whole tiling has the same pattern that repeats itself many times and could go on indefinitely, at least as long as the money and the supply of tiles holds out. (There may be a border in some different pattern, but we are ignoring that.) It also looks like every corner where the tiles come together looks just like every other corner. It also appears, in many cases, but by no means all, that every tile is a *regular* polygon.

People have standard names for these things. If the pattern could go on forever in all directions, if every corner looks like every other corner, if every tile is a regular polygon, and if all these polygons have the same shape and size, we call it a *regular* tiling. The ‘honeycomb’ pattern of hexagons that is so typical of bathroom floors, and the squares laid end to end on bathroom walls are examples of regular tilings. On the other hand: If the pattern could go on forever in all directions, if every corner looks like every other corner, if every tile is a regular polygon, *but* more than one kind of regular polygon is allowed, we call it a *semi-regular* tiling. Very likely you have seen some of these. Perhaps the most common consists of squares and octagons. It looks like this:



$(4,8^2)$

Figure 67.

You can see this can extend indefinitely, that every tile is a regular polygon, but that more than one polygon is involved. We *do* need to check that all the corners are alike: Each corner is indeed a corner of a square and of two octagons, so this is a semi-regular tiling.

At this point, ask yourself what other regular and semi-regular tilings you have seen. Make a list. This is not a trivial point, by the way. HOW does one make a list? When do you want to call two tilings the same and when do you want to call them different? What we hope you discover as you make a list is that you need a *system,* some way of listing possibilities that tries to make sure you don't miss anything. We will now show you one such system.

A START: 6 POLYGONS AT A CORNER

The basis of our system is the fact that every corner must look like every other corner. Let's start by asking *how many* polygons come together at a corner. Well, the measures of the angles that meet at a corner must add up to to 360°. How large is each angle? Well the smallest angle you can find in any regular polygon is in an equilateral triangle, where each angle measures 60°. On the other hand, since each component figure *is* a regular polygon, each angle measures *less* than 180°. This says that there can be no more than six regular polygons meeting at a corner (if they are all equilateral triangles), and that there have to be at least three—because two numbers each less than 180 cannot add up to 360. OK, there have to be six, five, four, or three polygons meeting at a corner, and if it's six, they have to be six identical equilateral triangles. So that's a regular, not just a semi-regular tiling. You can get it, for example, by taking the honeycomb tiling of regular hexagons and dividing each hexagon into six triangles (Figure 79).

5 POLYGONS AT A CORNER

Now, let's roll up our sleeves and continue to work systematically. What angles do you get in regular polygons? The formula from geometry is: If the regular polygon has n sides—we call it an n-gon—each angle measures, in degrees, $180(n - 2)/n$. Make a table of these angle measures for n up to 12. So how can you get *five* polygons at a corner? After the triangle, in which every angle measures 60°, the next regular polygon is a square with each angle measuring 90°. Can you get 360 by adding five numbers each of which is either a 60 or a 90? Just one 90 and four 60s gives 330, which is too small. Two 90s and three 60s gives 360—there's one! Three 90s and two 60s gives 390, which is too big, and more than three 90s is even worse. So we have found a new possible semi-regular tiling made up of three triangles and two squares at each corner.

Is there such a tiling? Yes, probably you found one when you were trying out possibilities. You take a strip as long as you feel like drawing it, and cut it into squares.

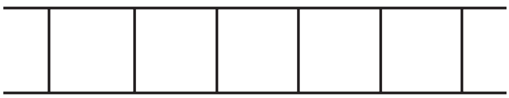
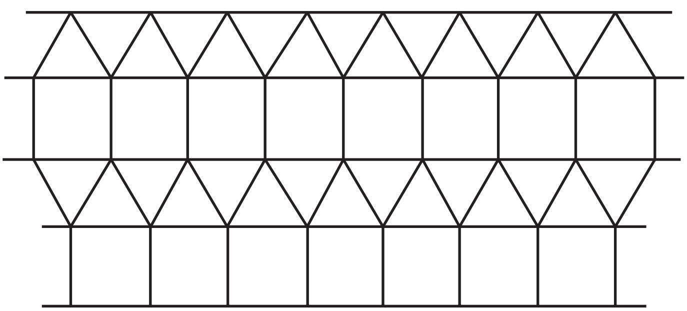


Figure 68.

Then you attach a strip of equilateral triangles above and below the strip of squares, and another strip of squares above and below that. You can continue alternating strips of squares and equilateral triangles as long as you like, so that you do indeed have a tiling. Is it semi-regular? Yes, at each corner there are two squares and three equilateral triangles, just as we said.



$(3^3,4^2)$

Figure 69.

Can you make a tiling with five polygons and use any regular pentagons? An angle in a regular pentagon measures 108°, which would leave 252° for four more angles. No combination of measures of angles from equilateral triangles, squares, or regular pentagons (60, 90, or 108) can add up to 252, so no such tiling is possible.

Can you use five polygons if one is a regular hexagon? The angle of a hexagon is 120°, and this plus four 60° angles makes 360°. So a semi-regular tiling with four equilateral triangles and one regular hexagon meeting at each corner is a possibility. Can it be drawn? Yes, see below.

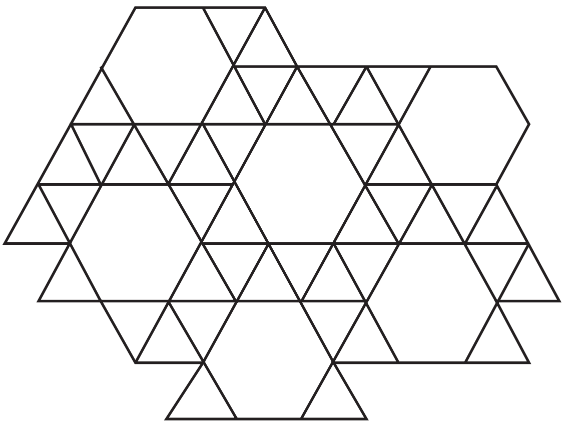

$(3^4,6)$

Figure 70.

This finishes tilings with five polygons meeting at each corner. Well, almost. There is one more point we have to consider. Let's summarize what we know about semi-regular tilings: If 6 polygons meet at each corner; every polygon is an equilateral triangle. If 5 polygons meet at each corner, we have seen that they could be three triangles and two squares, or they could be four triangles and a hexagon. We have also seen tiling of each kind. What we *haven't* said is that these two examples are the *only* possible tilings with these particular polygons. Could there perhaps be a different looking tiling where three triangles and two squares meet at each corner?

Isn't that just like a mathematician! Just when you are beginning to get into the process of learning about a problem, and even enjoying it, here they come and try to rain on our parade—make life more complicated! Why? The trouble is that as mathematicians we are trained to be as scrupulously careful and honest as we can be. If you haven't looked into the question of whether there is only one tiling with three triangles and two squares at each corner, we haven't done the best job we can! So all right, is there only one?

Obviously we are going to do more than just count polygons. What can we say about the *order* in which we find the three triangles and the two squares as we go around a corner? In our tiling, the two squares are adjacent to each other, and then the three triangles are adjacent to each other. Could the two squares be separated?

Well, separated by *what?* Triangles. How many? As you go all the way around a corner, if the two squares are not adjacent then they must be separated by one triangle and then by two triangles. The corner has to look like this:

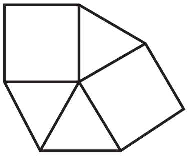
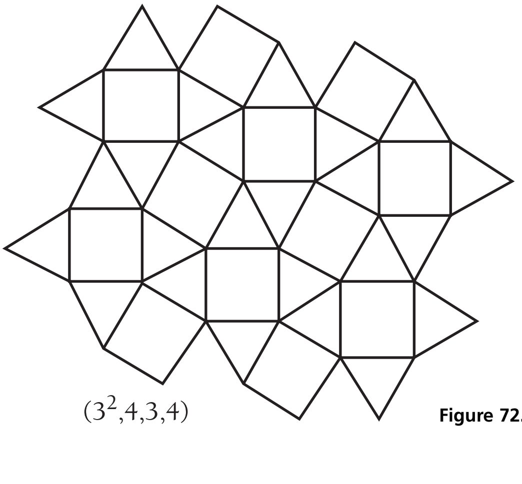


Figure 71.

Can you continue this tiling indefinitely? It doesn't look very promising, but all you can do is try. Are you constrained in how such a tiling would continue? For example, the two edges of the upper left square have to be edges of triangles to the west and north. These triangles will have a square between them, and so forth. Yes, you can continue such a tiling indefinitely; see right.



$(3^2,4,3,4)$

Figure 72.

So there *are* two different tilings that use three triangles and two squares at each corner. The previous one, where three triangles are adjacent and the two squares are adjacent, can be written like this: (3,3,3,4,4)—or, more compactly, $(3^3,4^2)$. The tiling we have just seen, where the squares are separated, can be written $(3^2,4,3,4)$.

Our other semi-regular tiling with five polygons, which we can now write as $(3^4,6)$, is unique. How do you know? Because after the first corner, every move is forced since every corner must look the same.

4 POLYGONS AT A CORNER

Had enough? Oh, but you can't quit yet. We still have to look at four polygons and three polygons at each corner. Again, we need a *system*. Let's look at four polygons at each corner. What's the smallest angle? We want to see how many triangles there will be at each corner, because triangles have the smallest angle, namely 60°. First suppose there is *no* triangle. Then the smallest angle measure is 90°. But there are four angles that add up to 360° and the smallest is 90°! So each one is 90°, there are four squares, and we have the familiar all-square regular tiling, which we call (4^4). Good.

Next, let's try exactly *one* triangle. That says one angle measures 60° and every other angle is at least 90°. How can you do that? If you take one square, you have now accounted for 60° + 90° = 150°, and that leaves 210° to go with *two* polygons. So one must have an angle below half of 210°, i.e., 105° and the other above 105°, but there are no regular polygons with angles between 90° and 108°. Therefore the smaller one must be 90° and the larger one 120°. We have a triangle, two squares, and a hexagon; we have learned to write that as $(3,4^2,6)$ if the squares share an edge, and (3,4,6,4) if they don't. Are these possible?

(3,4,6,4) with the squares separated is drawn on the right, but $(3,4^2,6)$ can't be done. Why not? Try it and you'll see. Suppose it was possible: Draw such a corner, at *y:*

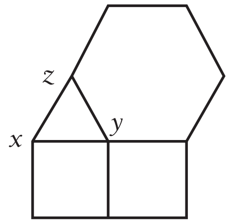
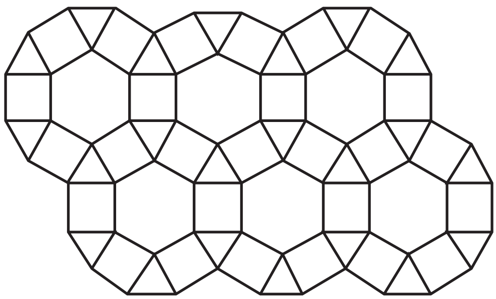


Figure 73.

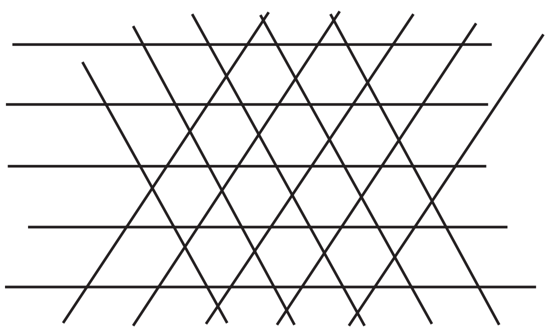
Now what will you do at *z* to the northwest of *y*? You must have two squares there, and they will be adjacent. But now you have two squares at *x* that are *not* adjacent and your attempt at a semi-regular tiling has failed. Here, now is the (possible) (3,4,6,4):



(3,4,6,4)

Figure 74.

Let's get back to our *system*. We have taken care of zero or one triangle at a corner, what about two? If you have two 60° angles, you have 240° left to go, which means either two 120° angles or one 90° and one 150° angle. The first is a $(3^2,6^2)$ or a (3,6,3,6), while the second is a $(3^2,4,12)$ or a (3,4,3,12). Quickly you'll see that $(3^2,6^2)$ can't be done (just as we did above, it can't continue very far) and (3,6,3,6) is the following:



(3,6,3,6)

Figure 75.

What about two 3s, a 4, and a 12? You will find when you try it that neither $(3^2,4,12)$ nor (3,4,3,12) (either the two 3s share an edge or they don't) can be continued very far. Neither of them can make a semi-regular tiling.

So we have finished the case of four polygons meeting at a corner. The possibilities that have succeeded are (4^4), which is regular, and $(3,4^2,6)$ and $(3^2,6^2)$, which are semi-regular.

3 POLYGONS AT A CORNER

Now we need *three* angles of regular polygons that add up to 360°. There are a lot of cases and the lazy thing to do is to get rid of many of these cases before we end up working too hard.

Let's suppose that one of the three regular polygons is an equilateral triangle. We are going to draw a figure and explore a little. Our intention is to find out more about what the other two polygons have to be, something more than just that the angles must add up to 300°. So we start out drawing the beginnings of a figure:

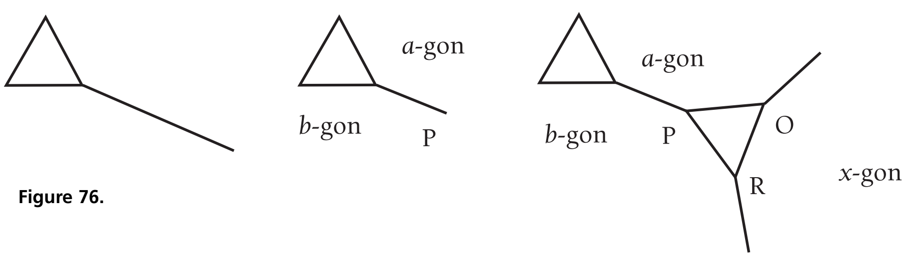


Figure 76.

We draw an equilateral triangle and another edge of tiling coming out of one of the vertices. On each side of the edge will be a regular polygon, one with *a* sides on one side of that edge and one with *b* sides on the other. Wait a minute! If this is to be a semi-regular tiling, that edge coming out is too long—*much* too long. It must be the same length as the sides of the triangle. After all, each side is a regular polygon. We correct and expand our figure. At P, the end of that edge sticking out, another triangle has to begin, because all vertices have to look alike.

So we have drawn a triangle at P, with its other vertices at Q and R, and there must be one additional edge coming out of each of Q and R. Now the question is: What is the outer polygon whose edge is the line segment between Q and R? Because of the corner at Q, it must be a *b*-gon, and because of the corner at R, it must be an *a*-gon! So the only possibility is that $a = b$! If you have three polygons at each corner of a semi-regular tiling and one is a triangle, the other two must have the *same* number of sides! That fact is going to save us a lot of work. It means that if one polygon is a triangle, so that it takes care of 60°, there are only 300° left to go so the other two polygons must each have angles of 150°, which means they are 12-gons (dodecagons, if you prefer). So we have only one candidate for a semi-regular tiling with three polygons at each corner, one a triangle: $(3,12^2)$.

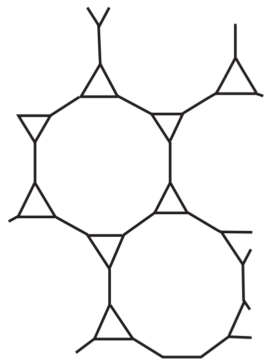
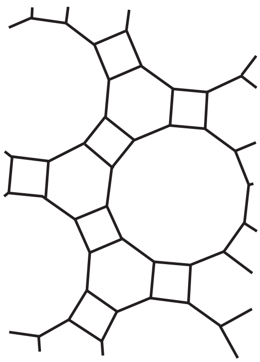

$(3,12^2)$

Figure 77.

Does one of the polygons have to be a triangle, clearly not? If all three polygons were the same then each would have 120° angles. That would make for a tiling of three hexagons meeting at each corner—the regular bathroom tiling pattern. But if three angles add up to 360° and they are not all alike, at least one has to be smaller than 120°. That means the smallest angle, if it is not 60°, is either 108° (from a regular pentagon), or 90° (from a square). By exactly the same argument we used with the triangle, if one figure were a pentagon then the other two would have to be regular polygons with the same number of sides. This means that 252° have to be shared by the two, so each would have to have angles of 126°. But 126° does not appear on our list of possible angles in a regular polygon, so this case is out. This leaves the possibility that one is a square. The other two angles would have to add up to 360° – 90° = 270°. One way to do this is with two 135°-angles, which means two 8-gons (octagons). See Figures 24 and 67. We saw this tiling at the beginning as our first example of a semi-regular tiling. The other possibility would be a (4,6,12) tiling with angles of 90°, 120°, and 150°. This one can be done, see the figure at right.



(4,6,12)

Figure 78.

Are we finished? Wait just a minute, says the mathematician, if you can have (4,8,8), you could have (4,*x*,*y*) with $x < 8$ and $y > 8$. What is less than 8? Well 5, 6, and 7. We've *got* the 6, that is the (4,6,12). How about the other two? Ah, but they would have an *odd* number in them (5 or 7), and we saw that each of these can only work if the other two were equal, and neither (5,*x*,*x*) nor (7,*x*,*x*) is possible.

Finally, there is the regular (6^3).

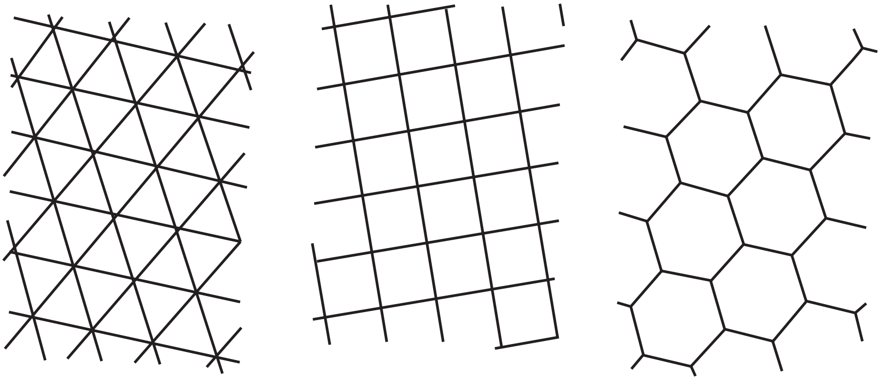


Figure 79.

Yes, we *are* done. We have found *eleven* semi-regular tilings of which in fact three are regular, and the other eight only semi-regular. Here they are:

(3^6) $(3^3,4^2)$; $(3^4,6)$; $(3^2,4,3,4)$ (4^4); $(3,4,6,4)$; $(3,6,3,6)$
$(3,12^2)$; $(4,8^2)$; $(4,6,12)$; (6^3).

AMERICAN FLAGS

On many math trails, you will have occasion to see an American flag. You can ask how many stripes and how many stars it contains. The answer is 13 stripes and 50 stars. Why those numbers? Because there are 13 original states and 50 current states. How are these stripes and stars arranged? The stripes come simply on top of one another, but the stars are in a more complicated pattern. There are nine rows of stars, alternatively containing 6 and 5 stars, with 6-star rows at both the top and the bottom. Does this pattern have 50 stars? Yes: 6 + 5 + 6 + 5 + 6 + 5 + 6 + 5 + 6 = 50. Why did people choose this particular pattern? Well, what others could you have? Take some time to think about that and come up with some alternatives. You could have just 10 rows of 5 stars each, or 5 rows of 10 stars each. You could have 11 rows that alternated between 5 and 4 stars, with 5-star rows at the top and bottom. Of course, you could have 25 rows of 2 stars each, but that would look pretty ridiculous!

How do we know it would look ridiculous? When does one pattern look better than another? Why didn't they use 10 rows of 5 or 5 rows of ten? Let's have a look and see if we can figure out what looks best. First we have the existing pattern and then some alternatives.

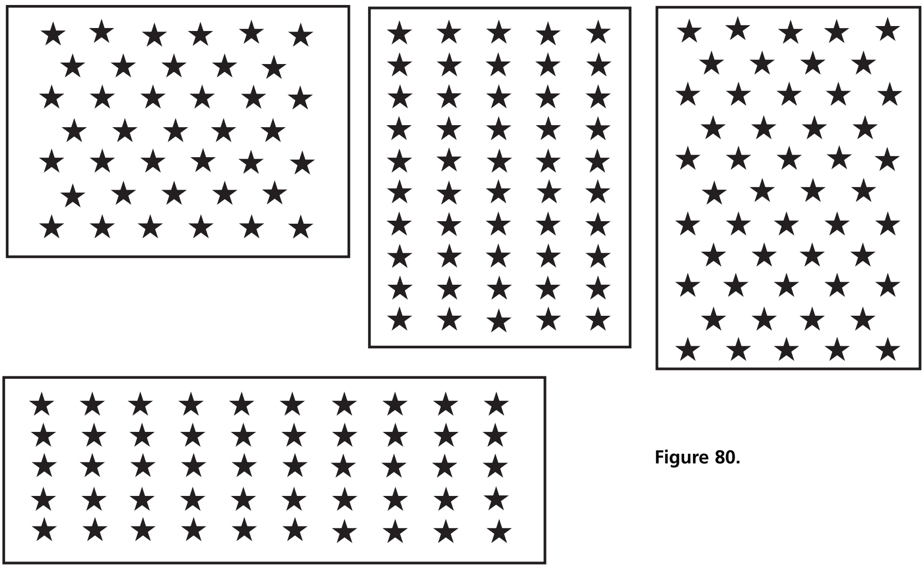


Figure 80.

We won't draw 25 rows of 2 stars each!

I think we can see which one is going to look best. We want a pattern that's as nearly square as possible (the rows of stars on a real flag are closer together than the ones in the drawings). We also like patterns that are rectangular or nearly so. What does 'nearly so' mean? It means that alternate rows may differ by one star, with the rows interlaced like the current pattern for 50 stars. It looks like the way people prefer to plant begonias or impatiens: A rectangular pattern is not quite as pleasing to the eye as one that interlaces, but any of these is better than one that is not nearly square.

Well, what about a pattern like our current 6,5,6,5,6,5,6,5,6; but where the top and bottom rows have the *smaller* rather than the *larger* number of stars? That might be a nice alternative pattern for 50 stars if the length and width are not too different. Well, try it. Can you get 50 stars that way? You will find that you *can't*—it can't be done! How do you know? Just wait a bit and we'll show you. In the meantime just try it. Maybe you'll discover why it can't be done.

But let's notice that it doesn't *always* fail, it only fails with 50 stars. Try 32 stars, for example. You can have 4 rows of 8; that's the rectangular pattern. You can have 5,4,5,4,5,4,5—call this the "out-in" pattern because the rows that go further out include both the first and last rows. You can also have 6,7,6,7,6—that's the "in-out" pattern, which is not possible with 50 stars. In fact, here is the amazing result:

- It is possible to have all three patterns—like 32.
- It is possible to have just rectangular and out-in—like 50.
- It is possible to have just rectangular and in-out—like 52.
- It is possible to have just in-out and out-in—like 47.
- It is possible to have *only* rectangular—like 30
- It is possible to have *only* out-in—like 29
- It is possible to have *only* in-out—like 31.
- **But at least one of them must always work. There is no number that has *none* of the three.**

Let's look at 29, for which only the out-in is possible. There is no rectangular pattern and no in-out pattern with 29 stars. What is the out-in pattern? It is 10,9,10. It won't be pretty, that's true, because it is too long with too few rows. The only other choice is 2,1,2,1,2, …, 1,2, which is even worse. Now, let's get more realistic and up-to-date about the United States: What would happen if we added one more state and had to have a 51-star flag? A rectangular pattern would be possible: We could have three rows of 17 stars each. From our previous findings we know that wouldn't look very good. *And there is neither an out-in nor an in-out pattern for 51 stars!* What a catastastrophe, as Jimmy Durante used to say. What to do? See if you can come up with something, which is of necessity, a little different from anything we have discussed, but that is at least pleasing to the eye and close to being a square. Here is one:

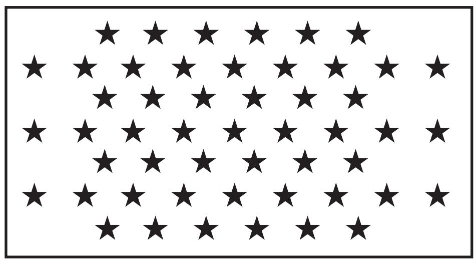


Figure 81.

What is the mathematics behind all this? How did we get the examples given above and how do we know that at least one of the three patterns must always work? Let's start with the rectangular patterns. For example, for 50 stars we found a 5 by 10 or a 10 by 5 pattern. This means that 50 has been written as a product of two whole numbers larger than 1, i.e., 50 is a *composite* number not a *prime* number. It's clear that a rectangular pattern will exist whenever the number of stars is composite. The pattern may not be pretty, like the 3 by 17 possibility for 51. We have discovered that pretty patterns have roughly the same number of rows as they have stars in any given row.

When is there an in-out pattern and how do we find it? Take twice the number and add 1: There will be an in-out pattern whenever this new number is composite! Let's think about this further. Call the original number N, then this new number is $2N + 1$. Now $2N + 1$ will always be an odd number. If $2N + 1$ is composite, you can write $2N + 1$ as a product of two odd numbers, say $2a + 1$ and $2b + 1$. Then N will have an in-out pattern with $2a + 1$ rows, and b and $b + 1$ stars in alternate rows. Two questions: How do you know this is true, and how do you discover such a truth, anyway? One way to know it's true is by algebra: If $2N + 1 = (2a + 1)(2b + 1)$, then $N = 2ab + a + b$. But this can be written as $N = (a + 1)b + a(b + 1)$. What does this say? That N stars can be made into a pattern of $(a + 1)$ rows of b stars each, and a rows of $(b + 1)$ stars each. If you alternate these rows, you have an in-out pattern!

What about an out-in pattern? This will exist whenever $2N – 1$ is composite. Why? $2N – 1$ is again odd. Let $2N – 1 = (2c – 1)(2d + 1)$. Then $N = 2cd + c – d$, which in turn equals $N = (d + 1)c + d(c – 1)$. So N stars can be made as a pattern of $(d + 1)$ rows of c stars each and d rows of $(c – 1)$ stars each, that is an out-in pattern.

It is impossible to resist a 'pure' mathematical note at this point. For what N would *both* the out-in and the in-out pattern be impossible? This will happen when both $2N – 1$ and $2N + 1$ are prime numbers. These would be so-called twin primes (like 101 and 103 in the case of 51 stars). One of the famous unsolved problems in mathematics is to decide whether there are a finite number of pairs of twin primes.

How do you discover, as distinct from prove, when the in-out pattern is possible? A way that often works is to use pictures, that is, geometry. Suppose there is an in-out pattern, let's say with N stars altogether. The top row has b stars, the next row $(b + 1)$, etc., the last row has b stars, and the number of rows is a. Draw it.

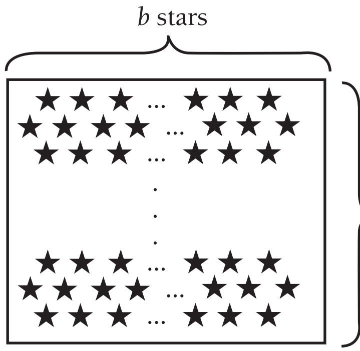


Figure 82.

Original pattern

Now draw a second pattern next to it that is almost but not quite the same. It has the same number of rows, namely *a*, but this time the top row has (b + 1) stars, the second row has *b* stars, the third row (b + 1) stars, etc., with the last row again having (b + 1) stars. How many stars are in this new second pattern? There is one more than in the first pattern, which is N + 1. So the *combined* pattern has 2N + 1 stars. Each row in the combined pattern has the same number of stars, namely ($2b$ + 1), and therefore 2N + 1 will have been drawn as a pattern of *a* rows of ($2b$ + 1) stars each. But that says that 2N + 1 is a composite number!

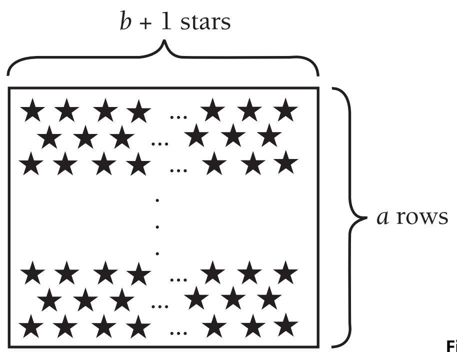


Figure 83a.

Second pattern

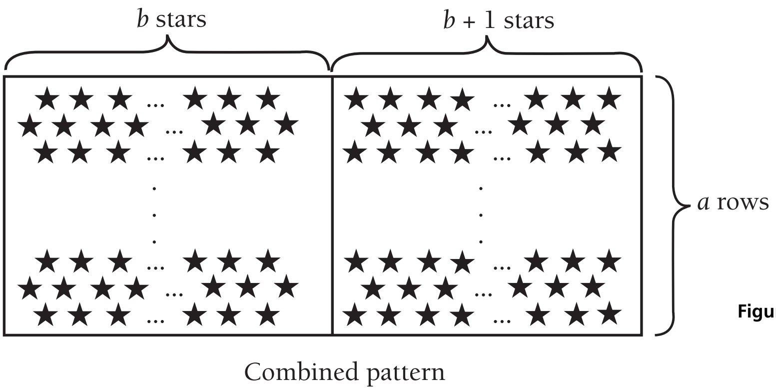


Figure 83b.

What did historical American flags look like? The flag with 20 stars had 4 rows of 5 stars, the flag with 28 stars had 4 rows of 7 stars, and the flag with 48 stars had 6 rows of 8 stars. The flag with 34 stars, as of April 12 1861, presented a problem. The number N = 34 is composite, but 2 times 17 would not be at all attractive. The number 2N – 1 = 67 is prime so no out-in pattern is possible, and 2N + 1 = 69 is composite but 3 times 23 would lead to three rows of 11, 12, and 11 stars respectively. Instead they chose five rows of 7,7,6,7,7 stars. Could they have done better? They could have tried five rows of 8,5,8,5,8 that is similar to the pattern for 51 stars suggested earlier. Would it have looked better? That decision is up to you.

One more question: Why is at least *one* pattern of rectangular in-out and out-in always possible (although it may be too long to be pretty)? We are saying that at least one of the numbers N, 2N – 1 and 2N + 1 must be composite! In fact, we claim that one of them must be divisible by 3! Why? Well 2N – 1, 2N, and 2N + 1 are consecutive numbers, so that exactly one of them must be divisible by 3. If either 2N – 1 or 2N + 1 is divisible by three then it is composite. If, on the other hand, it happens to be 2N that is divisible by 3 then N must be divisible by 3—and therefore composite. After all, if you divide a number (2N) by 2, and 2N is a multiple of 3, then N is a multiple of 3.

MOVING VANS

It wouldn't be surprising if you saw a moving van while you were walking a math trail. There are many mathematical questions you can ask concerning the process of moving. How much can a moving van hold? What are some of the principles used to pack a moving van? What are the standard sizes of the boxes that movers use and why were these sizes chosen? A question with which many of us have struggled at home, on which opinions may differ greatly, and one that can be attacked either mathematically or instinctively, concerns moving objects (usually furniture) within the house. Can you or can you not get the couch down the stairs and around the corner without scratching anything? How should you get the legs through? A simplified two-dimensional model of this type of question was the subject of serious research some years ago. Given a corridor with an L-shape with the same width (call it w) in both legs, what is the largest object you can get around the corner?

The most obvious answer is a square that is w on each side. You push it to the end of the corridor where it is ready to go in the other direction. The area you got through is w^2. Can you get any more through? Yes, imagine a semi-circle with radius w and its diameter along the inner wall. Push the semi-circle until the center is at the inner corner C. Now pivot around C. Because the radius is w and the center is now at C, it will turn into the other leg of the corridor. How big is this? The answer is about $1.57w^2$, which is quite an improvement. Maybe you can get a little more by cutting out a bit of a hole beginning at the previous center, lengthening the object a bit, keeping the width of the 'ring' at no more than w, and sliding the pivot point. Will it work? What does this do for you?

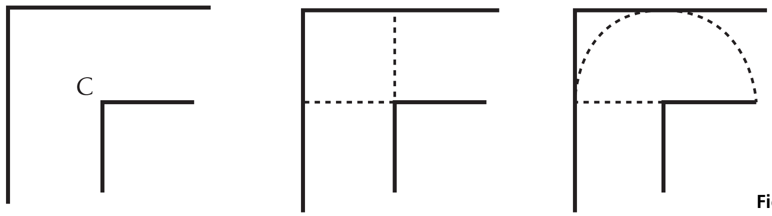


Figure 84.

Suppose the floor inside the moving van is 90” by 120”. If you have boxes that measure 30” by 36” (forget the height for a minute), how many can you get in one layer in the van? Try it and see. After you have thought about this for a while, you might argue that the *area* of the floor is 10,800 square inches and the area of a box is 1080 square inches. If you are lucky in how they fit, you will be able to get in 10 boxes, but you’ll certainly never get *more* than 10. It’s no trouble to get in eight boxes—put four of the 30” widths next to each other to make 120”, and 2 columns of this will fit easily into 90” with 18” to spare. Change the directions and you can fit in nine boxes: Three 30” widths fit into 90” and three 36” lengths fit into 120”. But 10 boxes won’t fit, even though 10 times 1080 square inches makes 10,080 square inches. The trouble is that while 30” divides into both 90” and 120” evenly, 36” doesn’t divide either one.

There’s more to this kind of story, however. Suppose the space was longer and narrower: 66” by 180”. The boxes are still 30” by 36”. Total floor area is 11,880 square inches and each box is still 1080 square inches. Eleven of them would fit into the area, but two columns 30” wide would accommodate 10 only! And yet you can pack 11 boxes in this time: Fill one column with five boxes next to each other the long way and fill the second column with six boxes adjacent the other way. But it is not true that either 30 or 36 divides 66!

HA!

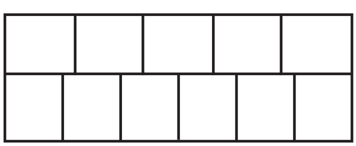


Figure 85.

Try this one: A truck's inside floor area is 8' by 15', and the boxes are 2.5' by 3'. What's the largest number of boxes you could possibly have in one layer? (Answer is 120/7.5 = 16.) Show that it is possible to carry out this packing of 16 boxes.

What are some standard size boxes that people use for moving things, and how big is the inside of various trucks and vans? What other principles can you think of that are involved in loading a truck? Have fun!

ESTIMATION

There are many times in thinking about a math trail when you want to be able to estimate something: A certain distance, a time, an angle, an area, or velocity. What might people think of in such a situation? Here are a few samples.

Distance: On the one hand distance is a quantity that is difficult because there is such a large range of measurements (from fractions of an inch to many miles) in which you might be interested. On the other hand, distance is easy because of the many different methods of estimating available. Many people can show by using their thumbs and index fingers the distance of a centimeter or an inch; they may know the distance of the span between the tips of their thumbs and little fingers when their hands are spread out; they may know the length of their arms; and many people know their height (especially useful for estimating shadows). For longer distances, it may be useful to know that the derivation of the word 'mile' is from the Latin *mille* meaning 'a thousand.' A Roman soldier was trained to a standard of *milia passuum,* a thousand paces or double steps per mile. Counting a hundred double steps at a fairly good pace is a good estimate for 0.1 mile. (A Roman mile was actually a little less, about 8%, than our familiar English mile. Thus the Roman pace was about 5 feet.) For estimations of much greater distances, it is worth remembering that the circumference of the Earth is about 25,000 miles, so that a time zone at the equator is about 1000 miles wide. You can compute how much less is the width of a time zone at your latitude.

Time: Because many people carry watches, this is not likely to be hard to estimate. Some people can count a minute pretty accurately by saying "a thousand one, a thousand two" etc., at a rate they have practiced. Often they can estimate a minute this way to within a few seconds.

Angle: Most people have no idea how to estimate an angle. Here is one way to prepare yourself for angle estimation: Go into the right-angle corner of a room and put your right arm, say, straight out in front of you, palm up as if you were a police officer stopping traffic. Line up the right edge of the right palm with the wall on your right. The left edge of the palm should then be lined up with some particular spot in front of you. Move the right edge of the hand so that it lines up with that same spot and notice where the left edge is now. Move the right edge of your palm to a new spot. Repeat this process until your palm touches or crosses the left wall. If it has taken you exactly 10 palm positions to get from one wall to the other then your hand is 90/10 or 9 degrees wide. If the middle of the 10th palm position is to the left wall, then your palm is about 90/9.5 or about 9 1/2 degrees wide. An angular width of a palm at arm's length of about 9 or 10 degrees is pretty common.

This is pretty handy in guessing, for example, a possible identity for a distant object when you have a view, a map, and another object that is clear on both the map and in the view.

Area: People often estimate the area of a figure by thinking of that figure as made up of equal-size squares and then estimating the number of squares involved. If the edge of the figure is irregular, you may have to think of an outside fragment in the square that's being counted as compensating for an inside fragment that looks about the same and is not being counted. It takes about three or four small shoe soles to make a square foot. For curved edges it may help to know about a few circular figures: A quarter is about (not quite) an inch in diameter and an eyeglass lens is about five centimeters. An old-fashioned frisbee has an area of about 60 square inches and a cookie tin lid around 40 to 45 square inches.

Velocity: Velocity is distance divided by time. Sometimes what makes it tricky to estimate velocity is that distances tend to be measured in feet or miles, and time is measured in seconds or hours. If you observe something moving for a little time, you tend to measure in feet and seconds, but your instincts for velocity may be based on miles per hour. By the time most people have fussed with the conversion, they tend to have lost all interest with the problem. The handy conversion to remember is that 60 miles per hour equals exactly 88 feet per second. For estimations you can make it 90 feet per second if you prefer: A typical lively walking speed of four miles per hour is the same as six feet per second.

While you are walking a math trail, a skateboarder or cyclist may pass you and this presents an opportunity to estimate how fast such a person is going. How do you do this? Here is one way: Start measuring time from the moment the person passes you and pick a landmark (tree, bush etc.,) ahead that you will both pass. The cyclist gets there in time t_c, where t is time (measured in seconds), and c stands for cyclist. The time that you, as a walker, get to the same landmark is designated t_w. Both of these are known. Also there is the *unknown* velocity of the cyclist v_c, and your known walking velocity v_w. How do we proceed from here? Well, the distance *from* the place where you were when you started to count *to* the landmark is equal to $t_c v_c$ and also to $t_w v_w$. Therefore

$$t_c v_c = t_w v_w.$$

Since you know t_c, t_w, and v_w, you can solve for v_c, which is what you wanted to know!

Maybe you prefer a picture:

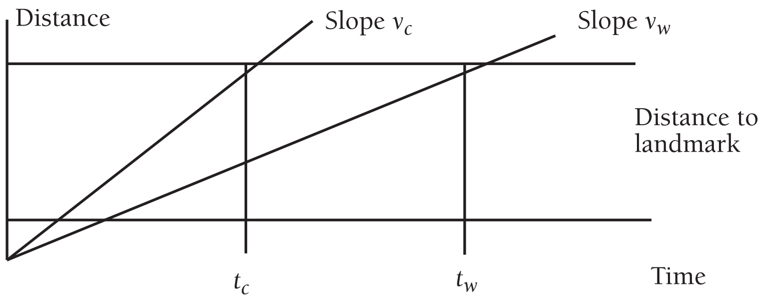


Figure 86.

The same mathematictics can be used if you are on a four-lane highway and want to estimate the speed of a car that passes you in what looks like inordinate haste.

FINALE

We remember hearing John Conway give a lecture recently. He wondered out loud how to end it. He said that the way to finish was to stop. So he did.

REFERENCES

p. 7 Dudley Blane (1989), “Mathematics Trails”, ICMI Conference on The Popularization of Mathematics, Leeds University, 1989.

p. 7 Blane, D. C. and Clarke, D. (1984), “A Mathematics Trail Around the City of Melbourne”, Monash Mathematics Education Centre, Monash University.

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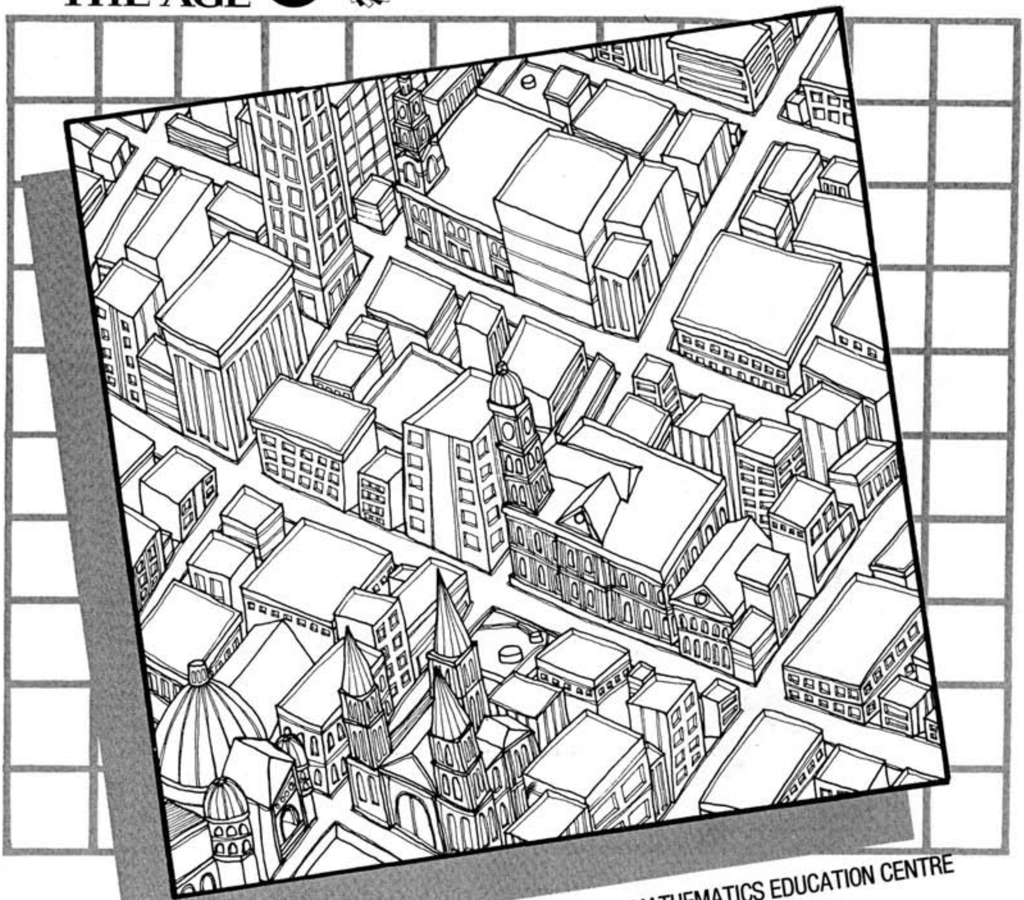
A MATHEMATICS TRAIL AROUND THE CITY OF MELBOURNE

THE AGE CRA ANZAAS FESTIVAL OF SCIENCE



PRODUCED BY STAFF AND STUDENTS OF THE MONASH MATHEMATICS EDUCATION CENTRE

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- The Age
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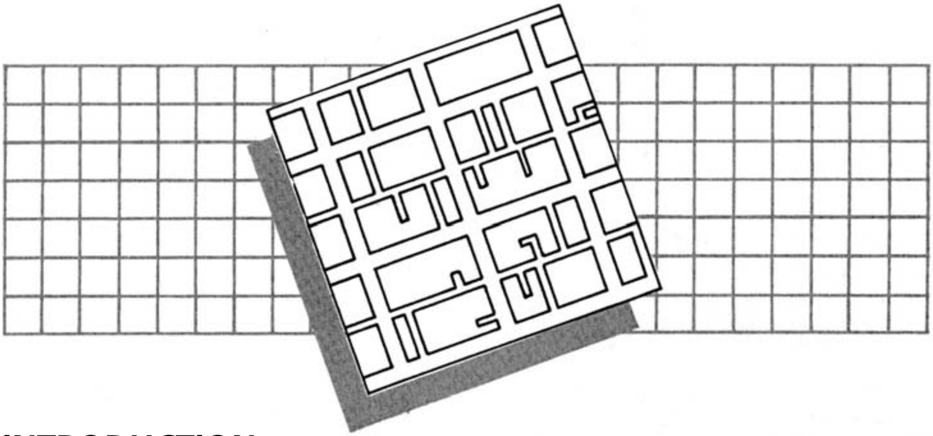
David Grundy	Arthur Tsingoidas
Philip Knowles	John Tucci

INSTRUCTIONS

1. Follow the Trail Map and Instructions contained in this booklet starting at the State Bank Centre.
2. An answer sheet will be provided at the State Bank Centre on completion of the Trail.
3. Successful completion is defined as making a good attempt at all the questions (excluding "Challenge Questions") up to and including those at Flinders Street Station. Please complete the rest if you have the energy!
4. Participants must return to the State Bank Centre to establish their "Successful Completion" and for the award of badges and certificates.
5. "Trails" will not be issued after 2.30 pm and participants must be back at the State Bank Centre by 4.30 pm on any particular day.
6. Please be aware of Road Safety at all times.

REQUIREMENTS

1. The main requirements are a pencil and preferably something to rest the booklet on when writing.
2. A calculator, although not essential, will be helpful, especially for the "Challenge Questions." It is quite possible, however, to complete the basic trail without the help of a calculator.



INTRODUCTION

The idea of producing a “Mathematics Trail Around the City of Melbourne” as part of the ANZAAS Festival of Science has grown out of some earlier work carried out by the authors for the 1984 Mathematics Association of Victoria (MAV) Conference at Monash University. The concept of producing Mathematics Trails has been explored as a way of developing an appreciation and enjoyment of mathematics in everyday situations, usually to complement work in the classroom. There is no record of one having been produced for parents and families and indeed there have been very few for children. This trail may, therefore, be a world first and one of the last great challenges!

By not tying this trail to any particular age range or to any subsequent work in school, some design difficulties arose which we hope we have overcome. The trainee mathematics teachers from Monash University who helped to design and prepare the material, the children and the teachers who trialled it, and the authors have all enjoyed themselves and have learned something more, both of mathematics and of Melbourne. We hope you do the same.

Happy Maths Trailing!

Dudley Blane & Doug Clarke

A. STATE BANK CENTRE

From the State Bank Centre cross Bourke St and then Elizabeth St carefully so that you are on the corner diagonally opposite the State Bank Centre and look back at it.

1. Without calculating write down a quick *estimate* of the number of small windows you can see in the side of the tower facing you.

A1

2. Now calculate the number of windows.
(**Hint:** Count how many there are in each row and the number of floors.)

A2

3. A window cleaner takes 1/4 hour to clean each window. If the cleaner works for 8 hours each day, 5 days per week, how many weeks will it take to clean all the small windows on the building?

A3

B. GENERAL POST OFFICE (GPO)—OUTSIDE

Walk about 50 metres up Elizabeth St (north) and find the stamp machines outside the GPO at the top of the steps. The one on the left takes coins up to 50c, (1, 2, 5, 10, 20, 50). To be able to purchase the lowest value stamp provided by this machine, calculate:

1. the greatest number of coins you could use

B1

2. the least number of coins you need to use

B2

3. In how many different ways could you put coins to the value of 10c into the machine?

B3

4. On your left beside the machine you will see details of posting times. If you post a letter to Perth on a Tuesday at 5.00 pm, would you expect it to arrive on the next working day?

B4

5. If you post a letter to Sydney at the same time, will that one arrive on the next working day?

B5

C. GENERAL POST OFFICE—INSIDE

Enter the GPO building from Elizabeth Street. On your right is the *Daily Weather Report*. From this:

1. What is the rainfall for Melbourne so far this year, in millimetres?

C1

2. Is this above or below average for the past 129 years?

C2

3. Estimate the *average* monthly rainfall so far this year.

C3

4. If rain continues to fall at the same rate for the rest of the year, estimate the total rainfall for this year.

C4

5. What might cause your answer to be different from the actual answer?

C5

The *River Report* is also on this wall. From the information given calculate:

6. The height of the Murray River at Bringenbrong. (NB. Use another town on the Murray River if the Bringenbrong figures are not available.)

C6

7. How far, in metres, does the Murray have to rise at the town you used before it floods?

C7

CHALLENGE QUESTION

If the change in the height of the river for the past 24 hours continued at the same rate for some time, calculate how long it would be before the river

a. floods (if change is positive)

CQa

b. dries up (if change is negative)

CQb

Moving further into the GPO from the entrance, you will see hundreds of small private mail boxes on your left. Look at the top five rows on *Board A*. If we use a co-ordinate system with 1 to 36 along the bottom and 1 to 4 up the left hand edge, then Box 5A is at position (1, 1).

8. What box is at (17, 3)?

C8

9. Give the co-ordinates of Box 27A.

C9

Leave the GPO by the same door and walk down Elizabeth Street and enter the Bourke Street Mall. Re-enter the GPO through the Mall entrance. In the passage there is a plaque on your right.

10. Assuming that Victoria is 150 years in 1985, calculate how old the State of Victoria was when the Postal Hall was established?

C10

Behind you is a painting of the only English Captain who came to Australia and failed to take a wicket or score a run.

11. Write down his name.

C11

D. BOURKE STREET MALL

Go down the steps into the Bourke Street Mall and walk past the David Jones Store. Move to the front of "K-K-K-Katies", *watching carefully for trams.* Find a circular pattern in the paving.

1. How wide across the middle is the smallest circle in *brick widths?* (This is called the *diameter* (D).) D =

D1

2. Using the same units, calculate the distance around the outside of the circle (the circumference (C).) C =

D2

3. Divide C by D and give an approximate answer.

$C \div D =$

D3

4. Move out *three* rings and take the same measurements for the bigger circle. $D =$

D4

5. Work out C divided by D again $C \div D =$

D5

6. Do you think all the circles would have approximately the same value for $C \div D =$

D6

7. Write down what we call this value and the Greek letter used to represent it.

D7

E. SWANSTON STREET AND THE CITY SQUARE

1. Cross the Mall at the lights and then cross Swanston Street. Walk down Swanston Street towards Flinders Street Station. As you walk, try to locate the exact single spot from where the photographs shown on the next page were taken. (It is within this 300-metre stretch of pavement.) Write down the name of the object you find at this position.

E1

Walk past the Town Hall, cross Collins Street and you will come to the City Square. On the ground 5 metres in front of the Directory Board is a Surveyor's Plaque.

2. What name do we give this shape?

E2

3. How old is the Victorian Institute of Surveyors?

E3

Look for the statue of Burke and Willis and the stream of water beside it. Time how long it takes an object (e.g., a leaf) to float down from the statue to where the waterfall branches into two parts.

4. How far did it float, measured in bricks?

E4

5. How long did it take in seconds?

E5

6. Calculate its speed in bricks per second. (**Hint:** Divide the numbers of bricks by the number of seconds.)

E6

7. Is this faster or slower than your normal walking speed?

E7

8. How did you work this out?

E8

Return to the Directory Board. Locate the Amphitheatre and the Reflecting Pool on the diagram. Move to the Amphitheatre.

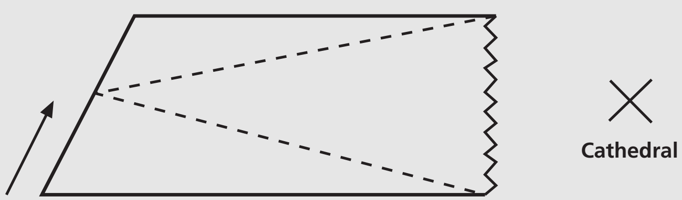
9. Look up. What shapes made by the roof struts can you see? Write down as many names as you can.

E9

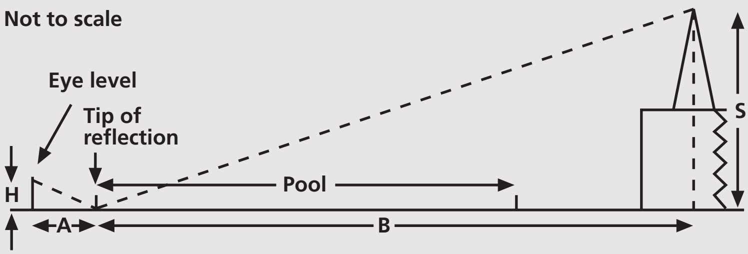
Walk to the Reflecting Pool

CHALLENGE QUESTION

Notice how the spine of St. Paul's Cathedral is reflected in the pool. Similar triangles can help us to make a rough calculation of the height of the spire as follows:



Stand so that the tip of the cathedral spire is reflected exactly at the centre of the edge of the pool furthest from the cathedral spire. Looking from the side we could draw a diagram as shown below.



We say that these two triangles are *similar*. That is, one is an enlarged version of the other. Now measure in your normal paces the distances A, B, and H. (H is your own height but with some ingenuity you will be able to work out your height in this way.)

c. A =

CQc

d. B =

CQd

e. H =

CQe

Because the triangles are similar:

$S \div B = H \div A$ so $S = (H \div A) \times B$

f. From this calculate S, in paces S =

CQf

g. By estimating the length of your pace (e.g., 1/2 metre, 3/4 metre) calculate the approximate height of the spire.

CQg

F. THE PLAZA

Leave the City Square, walk past St. Paul's Cathedral and cross Flinders Street at the lights. On the corner above the footpath there is a Plaza. Climb the steps to reach it.

Look back towards the Cathedral (North) at the Cathedral Spires and try to positions yourself so that the cross on the front left hand spire completely covers the cross on the larger rear spire. Without leaving this spot, turn around 180° and face the 150th Anniversary Arch.

1. If you draw an imaginary line from the top of the Arts Centre Tower straight down, which of the towns listed on the Anniversary Arch do you come to first?

F1

Rotate anti-clockwise until you see a tall building in the distance with two blue stars on it. This is called Nauru House.

2. How many sides does it have?

F2

3. What do we call a shape with this number of sides?

F3

Remaining on the Plaza, move to the corner nearest the road intersection. Look down on the intersection with the traffic lights.

F4

4. Count the number of trams that pass through the intersection in five minutes and write down your answer.

F5

5. If this rate was maintained for one hour, how many trams would this be?

F6

6. How many would pass through between 5.00 am and 11.00 pm?

G. FLINDERS STREET STATION

Cross Swanston Street and move into the foyer of Flinders Street Station. From the timetables find out:

1. What time the first train leaves Flinders Street for Werribee on a weekday?

G1

2. When does it arrive at Werribee?

G2

3. How long does the trip take?

G3

4. If you have an appointment in Melbourne at 6.00 pm, at what time would the last train that you could catch to keep your appointment leave Werribee?

G4

H. FLINDERS STREET

Leave the Station and walk along Flinders Street (West) towards Elizabeth Street until you come to the Station Booking Office. Stand with your back to the "BOOKING Suburban Country Interstate" sign. Look across Flinders Street at the Council for Adult Education building.

1. When was it built?

H1

2. When was it re-built?

H2

3. How long was it between the two constructions?

H3

4. If the building had continued to be re-built at the same intervals of every (   ) years. How many times would it have been re-built by 1987?

H4

Continue along Flinders Street to the intersection with Elizabeth Street and cross the road to the Commonwealth Bank building. Look back up Flinders Street where you will see a "CLEARWAY" sign. From this calculate:

5. How many hours and minutes each weekday cars are not permitted to park here?

H5

I. ELIZABETH STREET AND EMBANK ARCADE

Walk up Elizabeth Street (North) and stop at the corner of Elizabeth Street and Flinders Lane (West) where you will see the Rialto Building in the distance.

1. Estimate the number of floors this building has.

I1

2. How tall do you think it is in metres?

I2

Now cross Elizabeth Street and Flinders Lane and then walk up Elizabeth Street (North) until you reach Embank Arcade. Turn into the Arcade and walk through.

3. What must you beware of as you walk through the Arcade?

I3

At the North end of the Arcade there is an interesting mirrored ceiling. Move onto the footpath under the mirrored ceiling and above you is another 'you' looking down at you. This is your reflection.

4. Calculate, in metres, how high your 'reflected feet' appear to be above the ground. (**Hint:** The blocks on the wall may be a help in measuring.)

I4

J. COLLINS STREET AND THE STOCK EXCHANGE

Walk about 10 metres further along Collins Street (West) where you will see a public seating area. The back of this area has some concrete pipes set up as plant pots.

1. How many more of the *widest* pot would fill up the tiled area, assuming the seats were not there?

J1

Continue along Collins Street in the same direction until you arrive at the Melbourne Stock Exchange. Go in (quietly) through the doors and on your right you will see a pendulum clock on the wall.

J2

2. Through what fraction of a complete circle does the pendulum swing?

J3

3. What is the name we give to the shape that is used as a weight on the pendulum?

J4

4. Look at the Stock Exchange "Coat of Arms" next to the clock. What is the motto of the Exchange?

J5

5. As part of the badge there are a number of gold and silver balls. Which are there more of?

Move further into the building and look at Currency Exchange Rates in the window of the New Zealand Bank and calculate:

J6

6. How much would the bank give you in United States Dollar Notes for 100 Australian Dollars?

7. How many New Zealand Dollars for the same amount?

J7

8. Based on the information given, how do you think the bank makes a profit from currency exchange?

J8

K. QUEEN STREET

Leave the Stock Exchange and walk up Collins Street (West) and cross Queen Street and then cross Collins Street. In front of you (in Queen Street) you will see a clock with Roman numerals.

1. What is 9 in Roman numerals?

K1

2. If this was a 24 hour clock: What would the hour hand point to at 1800 hours (in Roman numerals)?

K2

3. What time would it be when the hour hand is halfway between XXI and XXII hours?

K3

4. Under the clock are a number of parking meters. If a car is parked at 9.47 am on a weekday, what time must it leave by?

K4

Continue to walk up Queen Street (North) until you come to the RACV Building. This building has two sets of doors onto Queen Street.

5. Calculate the difference in height between the two entrances. (**Hint**: Look at the blocks on the wall.)

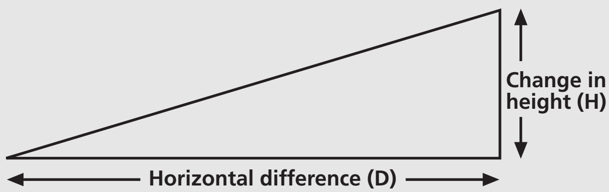
K5

6. Using the same units of measurement, calculate the horizontal distance between the two entrances.

K6

CHALLENGE QUESTION

The gradient of a slope is found by calculating the difference in height (H) divided by the horizontal distance (D):



h. Using your answers for (H) and (D) calculate the gradient of the hill between Little Collins Street and Bourke Street. H ÷ D =

CQh

Walk further up the hill to the “Top Deck Flight and Travel Centre” and look at the details of international flights given in the window.

7. If you wanted to go to London and back but need to visit a friend in Auckland both ways, how much extra money would you need to spend on fares by visiting Auckland twice instead of travelling to London via Singapore?

K7

L. BOURKE STREET

Walk up to Bourke Street, cross Queen Street and walk down (East) Bourke Street until you come to the Colorcode Discount Bookstore.

1. If the book you want had a "Pink" sticker last time you visited the shop, but today had an "Orange" sticker and normally costs $10 before discount, how much would you have saved if you bought it last time you visited?

L1

2. For a joke suppose someone had stuck all five stickers on the most expensive book in the shop (normal price $100). How much would it appear to cost after discount?

L2

CHALLENGE QUESTION

i. If the discounts were applied one at a time in the previous case, what would be the final price?

CQi

STATE BANK CENTRE

Go into the State Bank Centre and ask for an answer sheet to check your answers.

Make a note of the time

How long did the Trail take you?

TOTAL TIME

WELL DONE!!

COMMENTS: