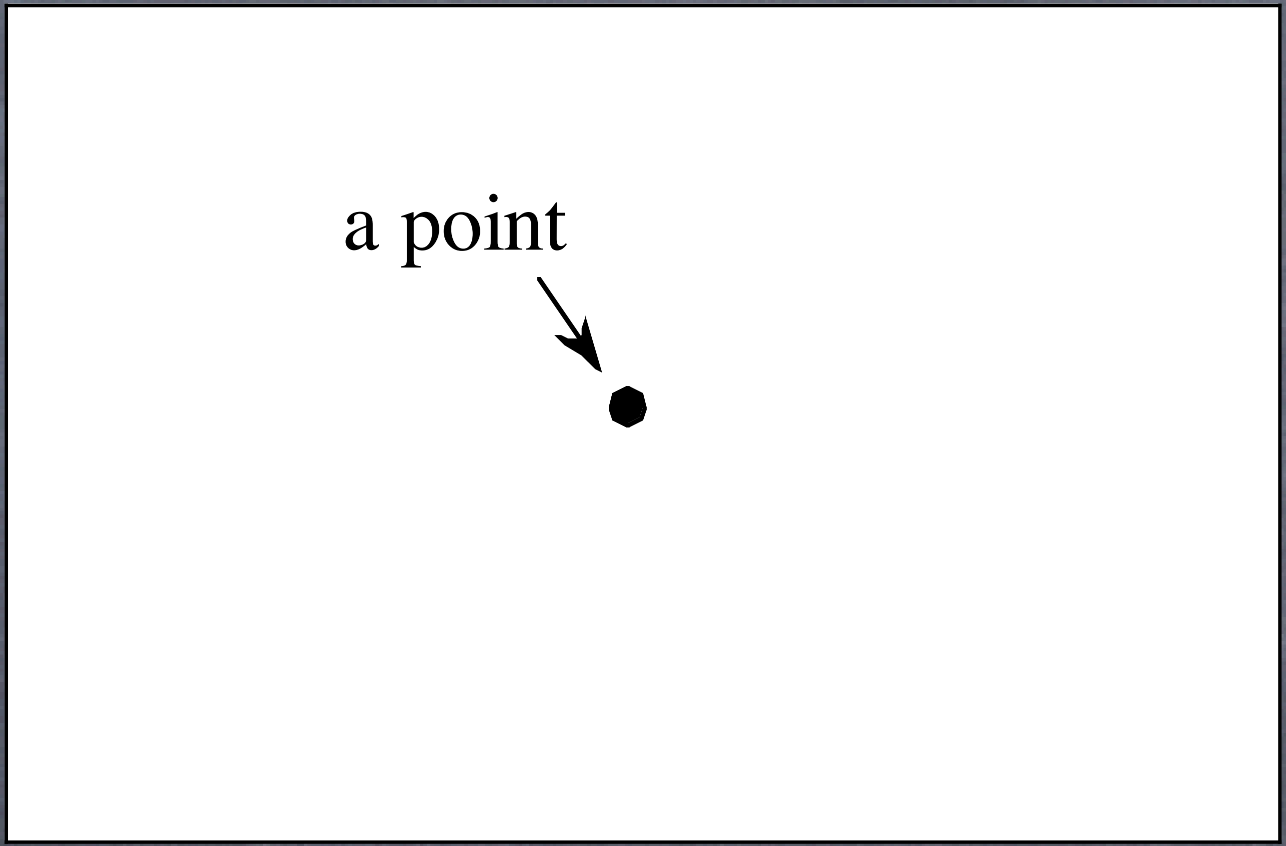
Mathematical Methods in Origami

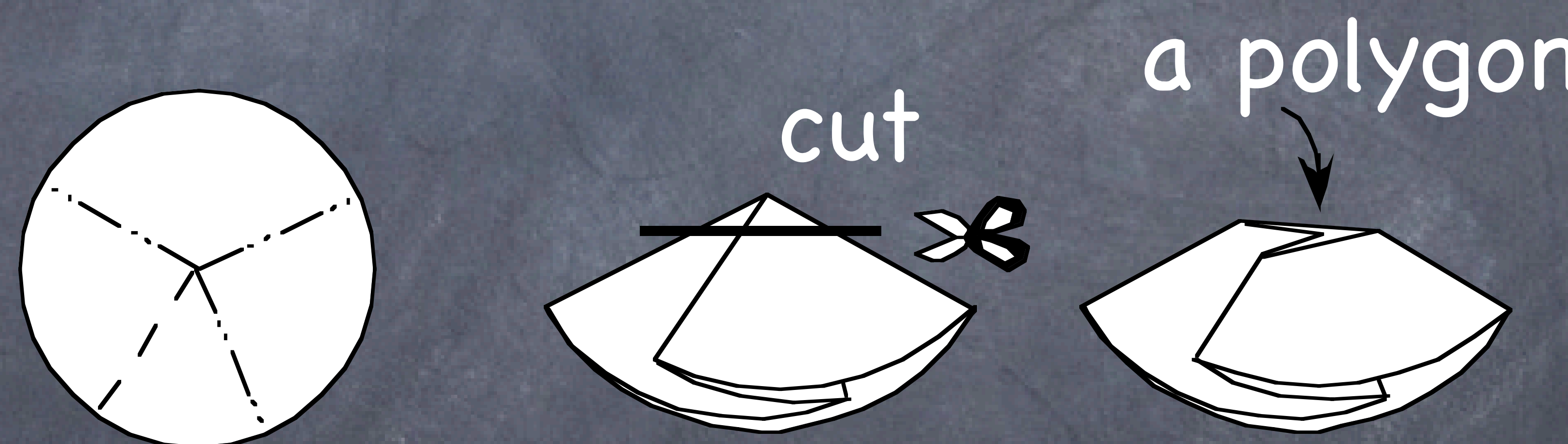
University of Tokyo, Komaba
Day 2 Slides, Dec. 17, 2015
Thomas C. Hull, Western New England University

Folding Exercise



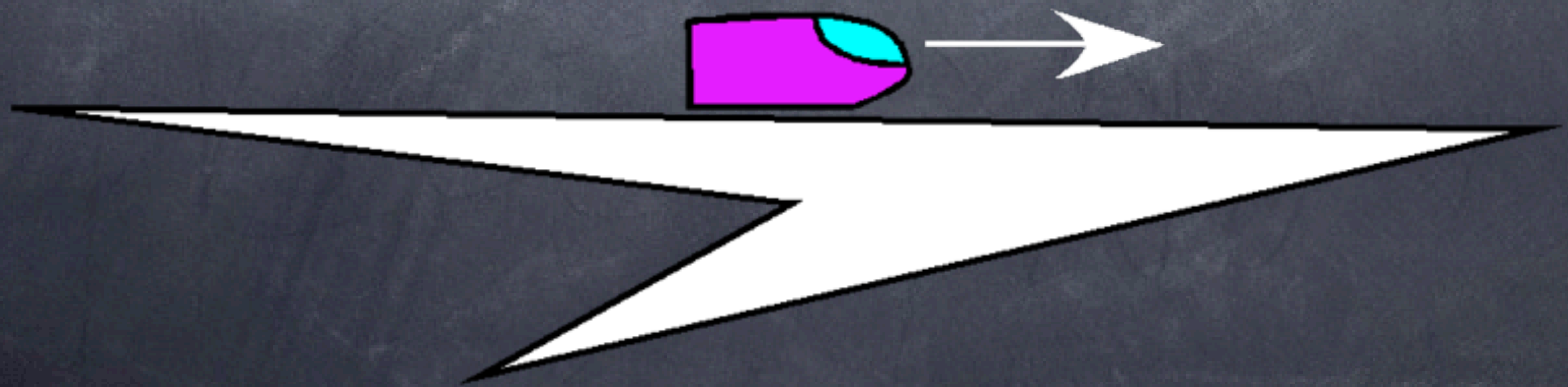
Maekawa's Theorem: Let v be a vertex in a flat origami crease pattern. Let M and V be the number of mountain and valley creases at v, respectively. Then $M - V = \pm 2$.

Proof: Fold our vertex flat and cut it off, to reveal a polygonal cross-section.



Maekawa's Theorem: Let v be a vertex in a flat origami crease pattern. Let M and V be the number of mountain and valley creases at v, respectively. Then $M-V = \pm 2$.

Proof: Imagine a monorail train traveling clockwise around the cross-section.

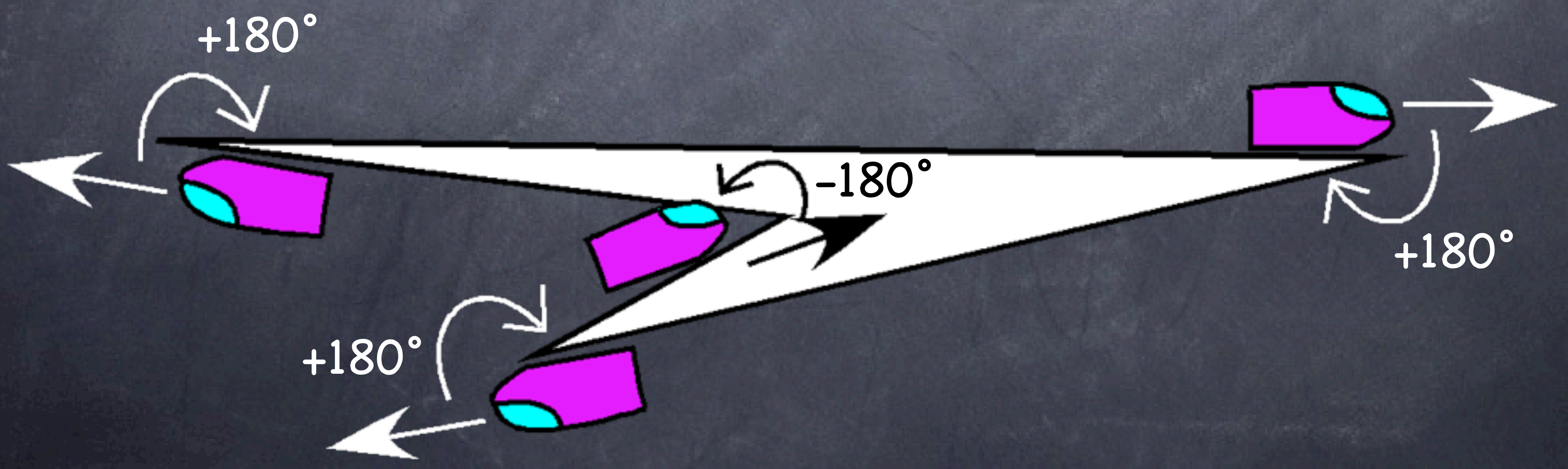


Maekawa's Theorem: Let v be a vertex in a flat origami crease pattern. Let M and V be the number of mountain and valley creases at v, respectively. Then $M - V = \pm 2$.

Proof: Imagine a monorail train traveling clockwise around the cross-section.

Every time it comes to a M -> rotates by 180°

Every time it comes to a V -> rotates by -180°

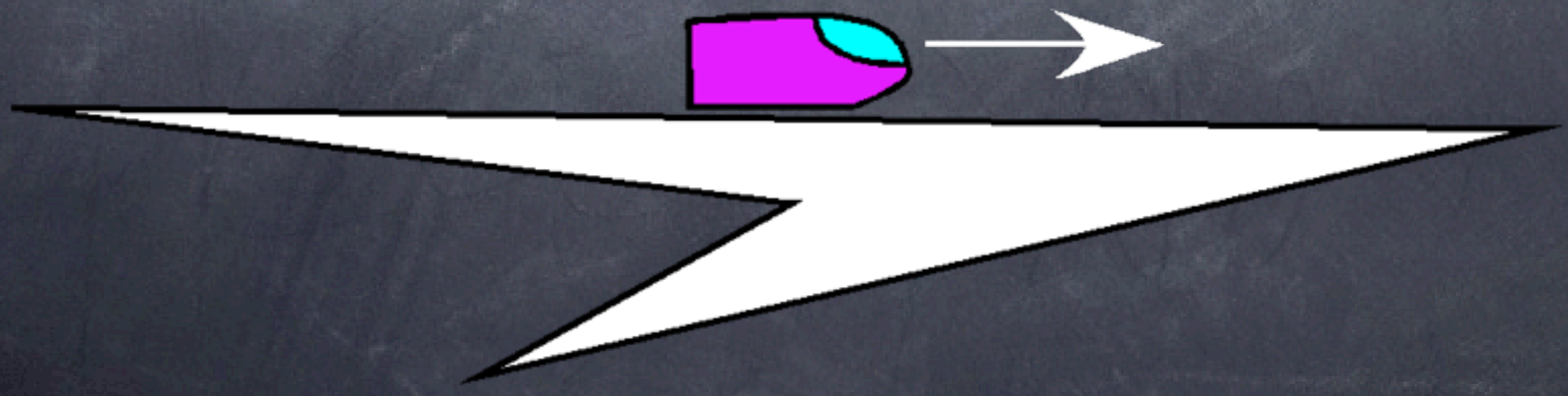


Maekawa's Theorem: Let v be a vertex in a flat origami crease pattern. Let M and V be the number of mountain and valley creases at v, respectively. Then $M - V = \pm 2$.

Proof: So we have...

$$180\,M - 180\,V = 360^\circ \text{ (one full circle turn)}$$

So $M - V = 2$.



Corollary: Every vertex in a flat origami crease pattern has even degree!
(That is, an even number of creases.)

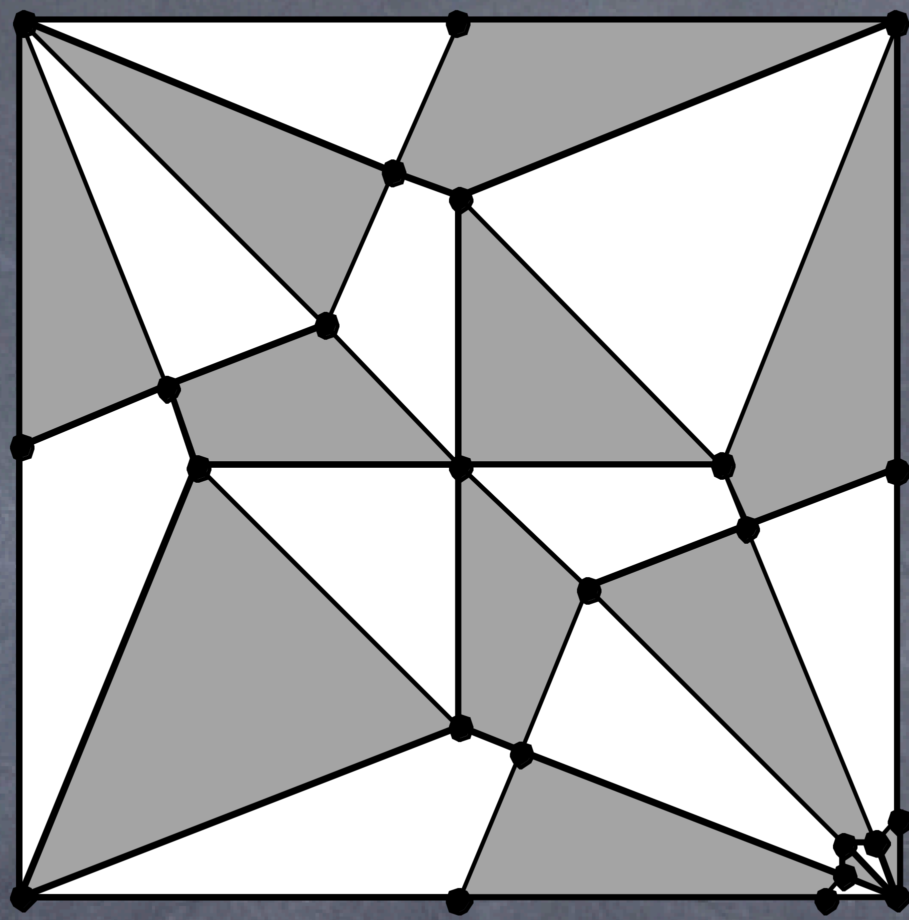
Proof: number of creases = M + V

$$= M - V + V + V$$

$$= \pm 2 + 2V$$

$$= 2(V \pm 1) = \text{an even number!}$$

Corollary: Every flat origami crease pattern is two-face colorable!



Corollary: Every flat origami crease pattern is two-face colorable!

A more rigorous proof:

Pick any face f of the crease pattern, and let f' be any other face. Let $\mu : C \to \{-\pi, \pi\}$ be our MV-assignment. Draw any curve γ from a point in f to a point in f' that crosses the creases (in order) $l_1, ..., l_k$. Now let

$$Q(f') = \frac{1}{\pi} \sum_{i=1}^{k} \mu(l_i) \mod 2$$

(This just = 0 if γ crosses an even # of creases and = 1 if it crosses an odd # of creases.)

Corollary: Every flat origami crease pattern is two-face colorable!

A more rigorous proof:

Pick any face f of the crease pattern, and let f' be any other face. Let $\mu : C \to \{-\pi, \pi\}$ be our MV-assignment. Draw any curve γ from a point in f to a point in f' that crosses the creases (in order) $l_1, ..., l_k$. Now let

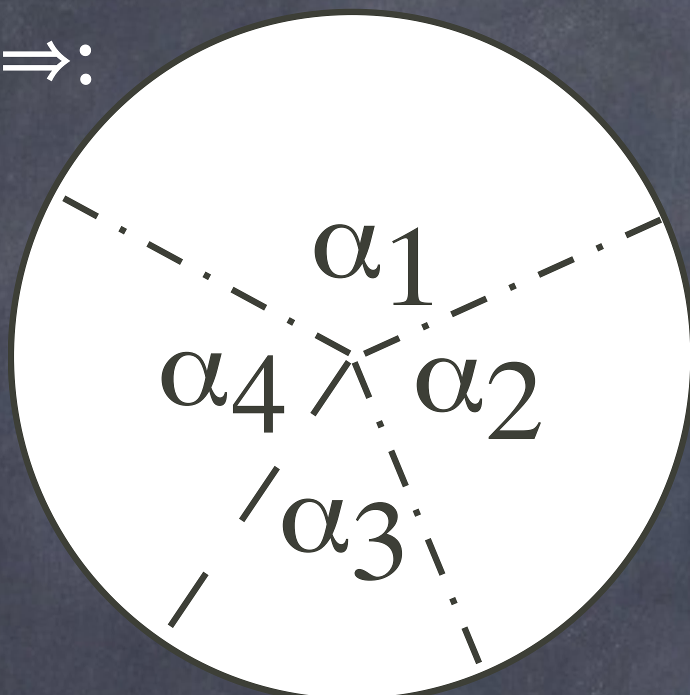
$$Q(f') = \frac{1}{\pi} \sum_{i=1}^{k} \mu(l_i) \mod 2$$

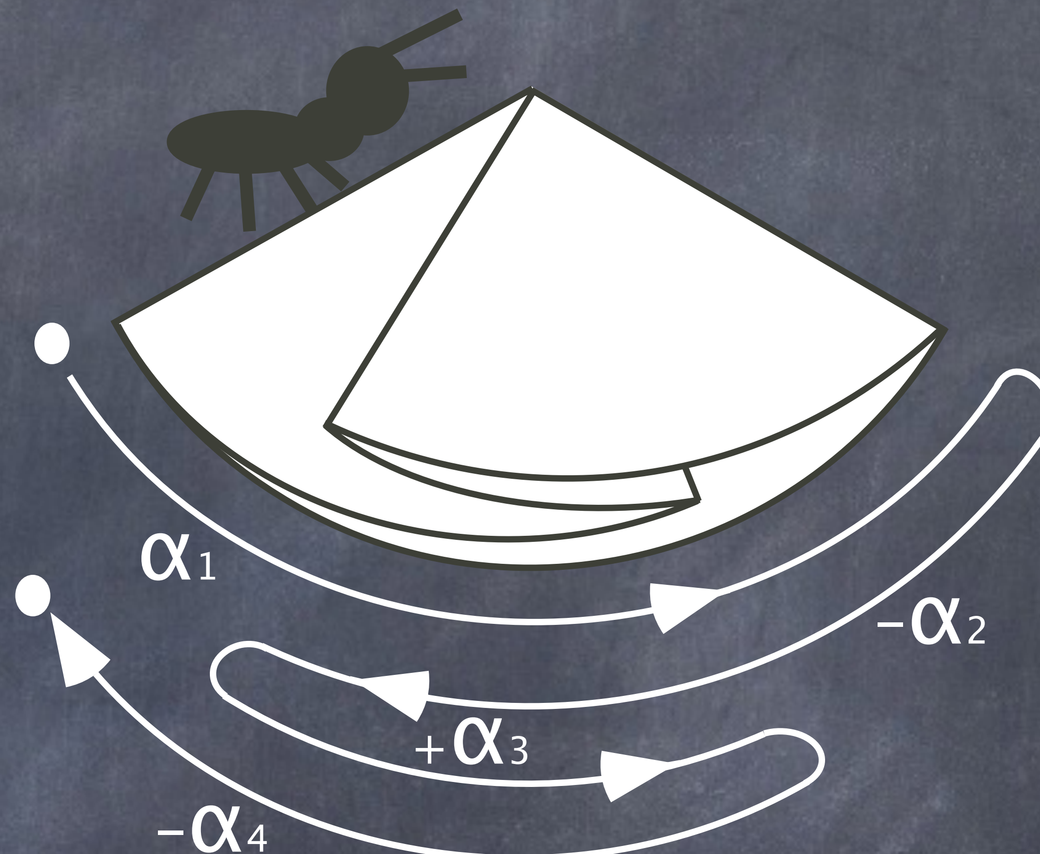
Then our two coloring is:

Color face f' grey if $Q(f') = 0$ and white if $Q(f') = 1$

Kawasaki's Theorem: Let v be a vertex in an origami crease pattern. Then v folds flat if and only if the sum of the alternate angles about v is 180°.

Proof of ⇒:





So... $\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = 0$

add to this $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 360°$

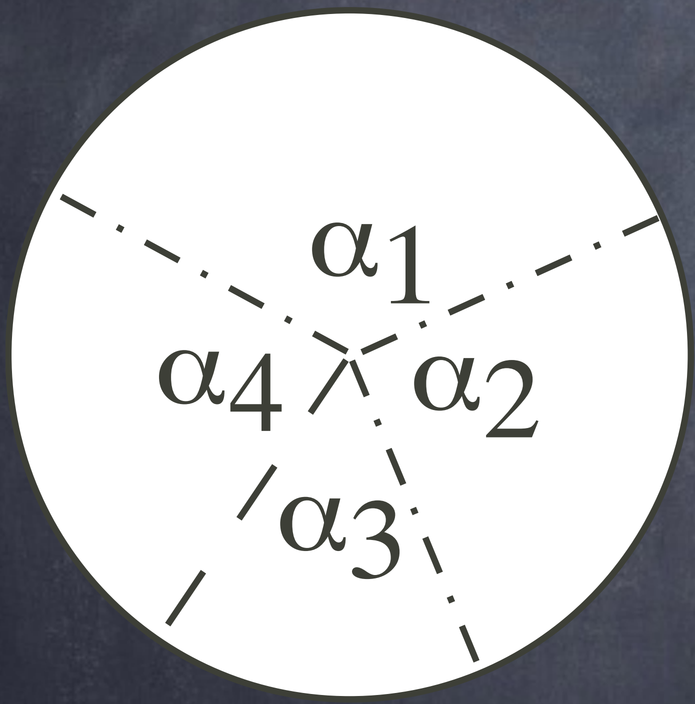
and you get $2\alpha_1 + 2\alpha_3 = 360°$

or $\alpha_1 + \alpha_3 = 180°$

Kawasaki's Theorem: Let v be a vertex in an origami crease pattern. Then v folds flat if and only if the sum of the alternate angles about v is 180°.

Proof of ⇒:

Note that every folded crease is reflecting part of the paper.



Let $R(l_i)$ = reflecting the plane about l_i.
Then $R(l_1)R(l_2)R(l_3)R(l_4) = I$.

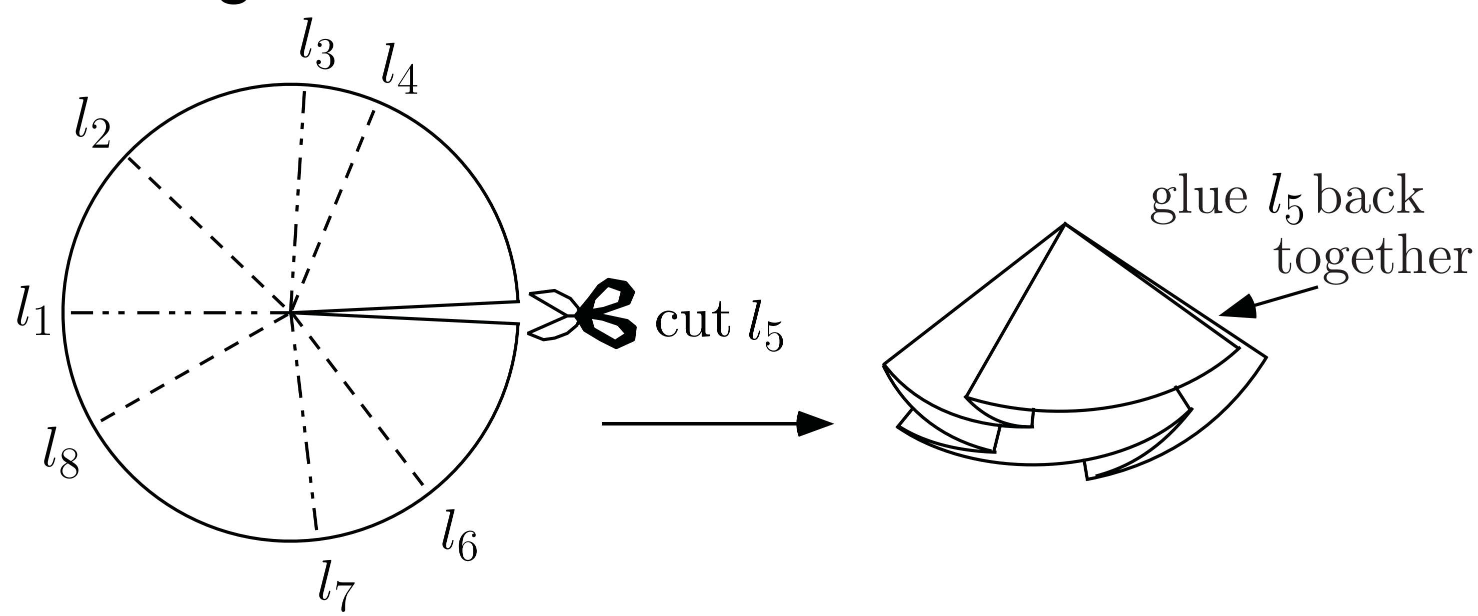
The product of 2 reflections is a rotation by twice the angle in between them...

So... $2\alpha_1 + 2\alpha_3 = 360°$
or $\alpha_1 + \alpha_3 = 180°$

Flat vertex folds

- **Kawasaki's Theorem:** A collection of creases meeting at a vertex are flat-foldable if and only if the sum of the alternate angles around the vertex is π.

Proof of ⇐ : Cut along one crease and make the others alternate MVMVMV…

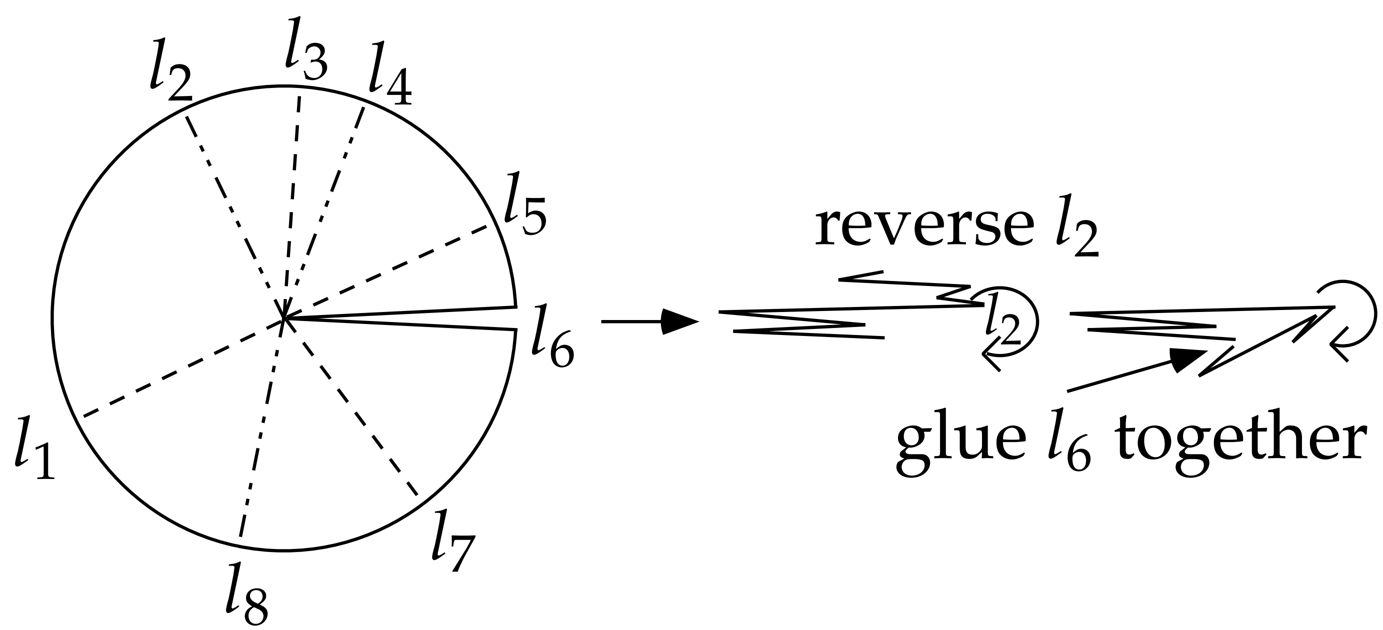


since $\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \ldots - \alpha_{2n} = 0$, the cut edges will line up after we fold the creases. So glue them back together! Uh … unless there's stuff in the way…

Flat vertex folds

- **Kawasaki's Theorem:** A collection of creases meeting at a vertex are flat-foldable if and only if the sum of the alternate angles around the vertex is π.

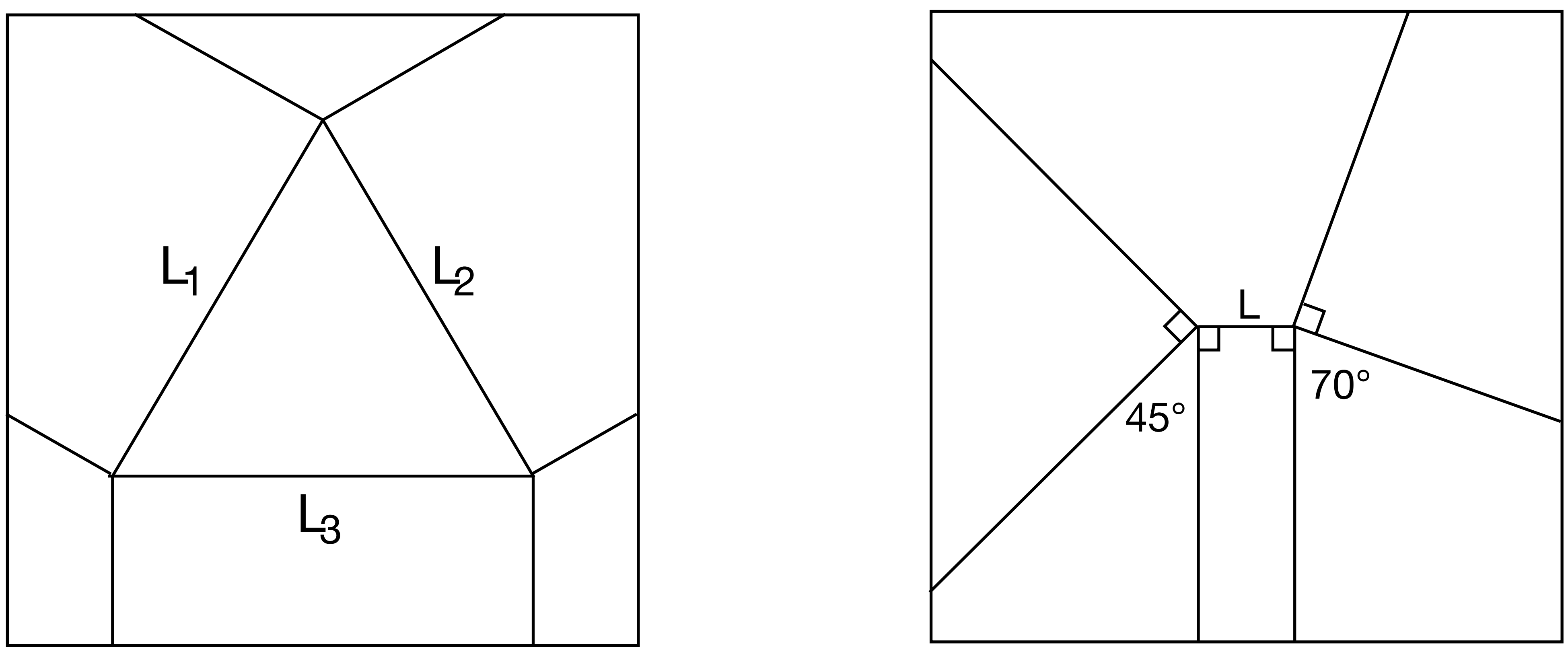
Proof of $\Leftarrow$: Cut along one crease and make the others alternate MVMVMV…



If layers of paper are in the way, then reverse the right-most crease and then glue. ☺

Generalizing ... can cause problems

- Kawasaki's Theorem (sufficiency part) does not generalize to larger crease patterns.



Determining if a given crease pattern is flat-foldable is NP-hard (Bern & Hayes, 1996)

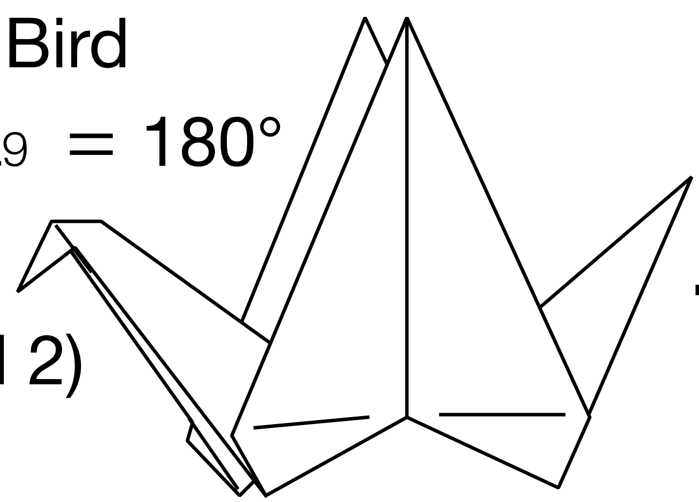
Generalizing ... can be cool

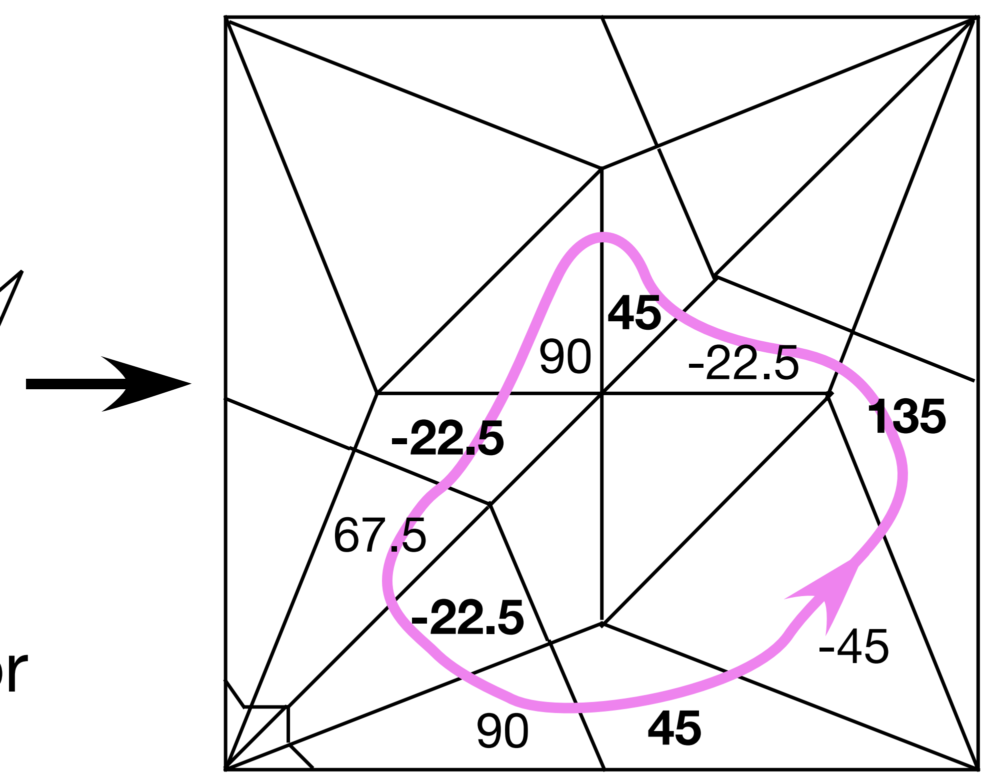
- **Justin's Theorem:** Given any flat origami model, let R be a simple, closed, vertex-avoiding curve drawn on the crease pattern that crosses creases c_1, c_2, c_3, ..., c_{2n}, in order. Let $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ be the angles between these crease lines (determined consistently), and let M and V be the number of mountain and valley creases among c_1, ..., c_{2n}. Then
$$\alpha_1 + \alpha_3 + \ldots + \alpha_{2n-1} = \alpha_2 + \alpha_4 + \ldots + \alpha_{2n} = \frac{M-V}{2}\pi \pmod{2\pi}$$

- Example: The Flapping Bird

Here $\alpha_1 + \alpha_3 + \ldots + \alpha_9 = 180°$

and so $\frac{M-V}{2} = 1 \pmod 2$

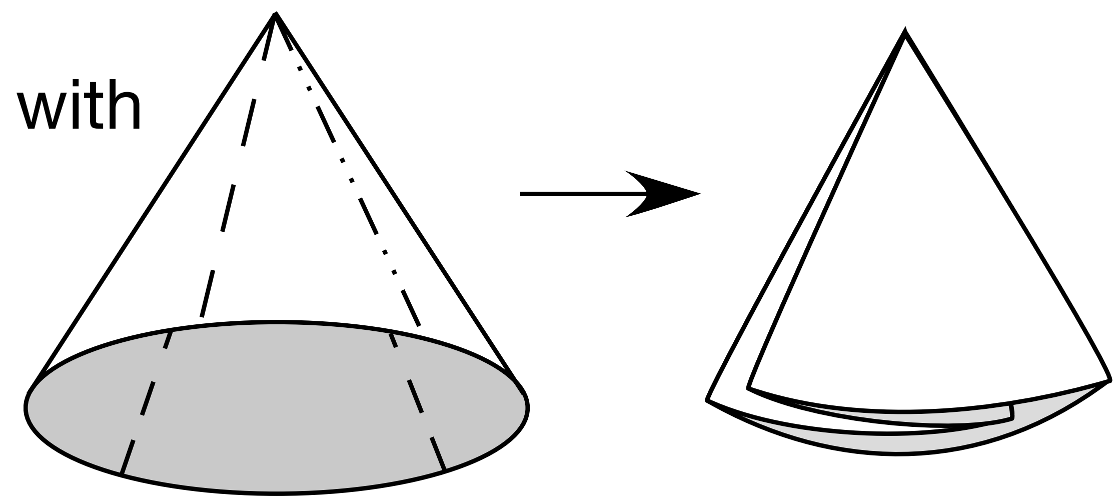




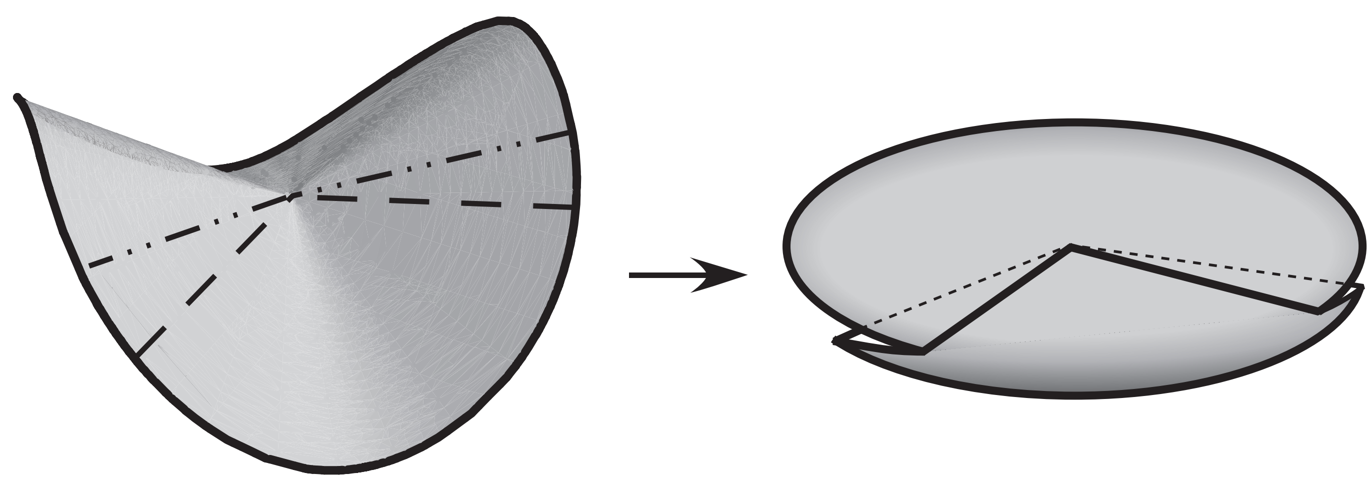
For the flapping bird, we have M = 8, V = 2 for this curve, so M – V = 6, which works.

Generalizing ... to Folding Vertex Cones!

- If cone angle $A \leq 2\pi$, then Kawasaki holds with $\alpha_1 + \alpha_3 + \ldots + \alpha_{2n-1} =$ $\alpha_2 + \alpha_4 + \ldots + \alpha_{2n} = A / 2$ and Maekawa still hods.



- If cone angle $A > 2\pi$, then both Kawasaki and Maekawa can fail!



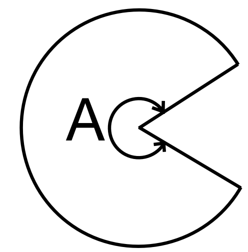
Hyperbolic paper!

Here $\alpha_1 - \alpha_2 + \alpha_3 - \ldots - \alpha_{2n} = 2\pi$ and

$M - V = 0$.

Generalizing ... to Folding Vertex Cones!

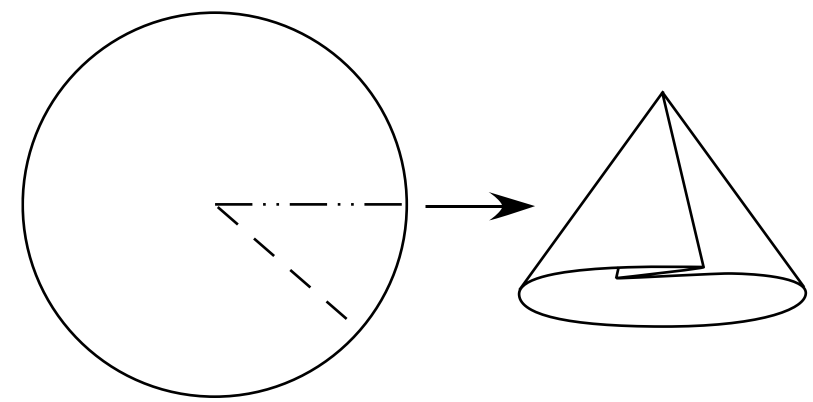
- Let C_A denote a cone with cone angle A.



- Consider a flat vertex fold to be a mapping between two cones.

$$\mu : C_A \longrightarrow C_B$$

- If A = 2π, then this is just folding paper into a cone.



- μ can be one of two types, determined by the alternating sum of the angles:

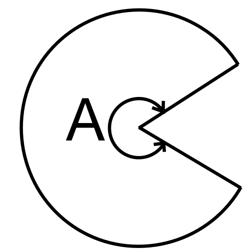
If $\alpha_1 - \alpha_2 + \alpha_3 - \ldots - \alpha_{2n} = 0$ then the image of μ is a sector of a disc. (μ is a **pointy map**)

If $\alpha_1 - \alpha_2 + \alpha_3 - \ldots - \alpha_{2n} = B$ then the image of μ is another cone with cone angle B < A. (μ is a **cone map**, or a **folded disc**)

Justin's Theorem tells us that M – V also captures this information, depending on whether (M – V)/2 = 0 or 1 (mod 2) around a closed curve about the vertex.

Generalizing ... to Folding Vertex Cones!

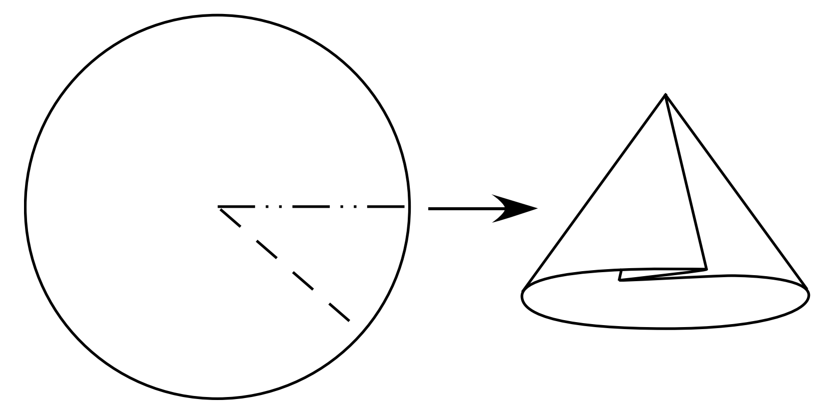
- Let C_A denote a cone with cone angle A.



- Consider a flat vertex fold to be a mapping between two cones.

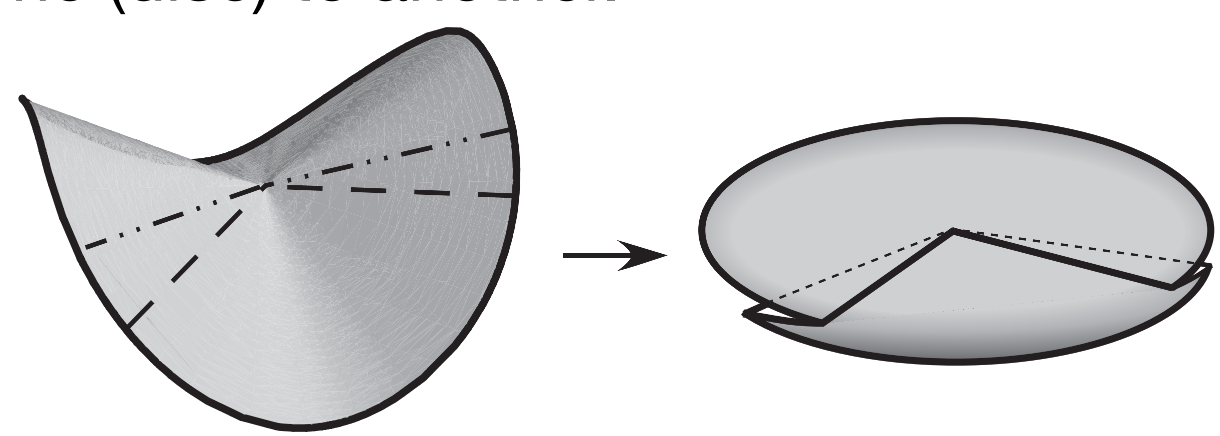
$$\mu : C_A \longrightarrow C_B$$

- If A = 2π, then this is just folding paper into a cone.



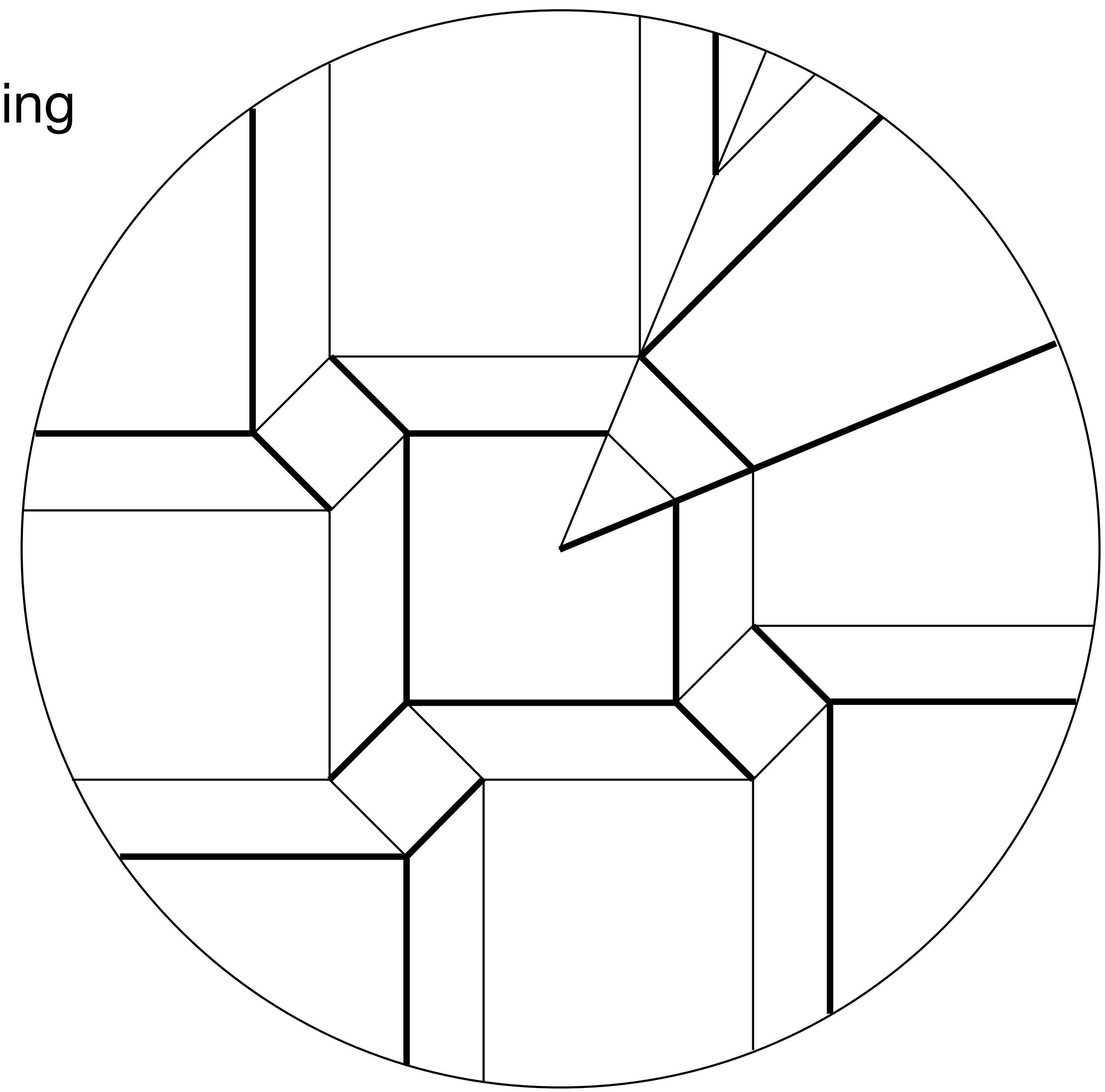
- μ can be one of two types, pointy map or cone map.

- So folding hyperbolic paper to a flat disk is not really violating Maekawa or Kawasaki, it's just a map from one cone (disc) to another.



Generalizing ... to Folding Cones?

- What if we consider “cone folds” with many vertices in the crease pattern?
- For example, should the the following tessellation be considered a “flat origami”?
- If so, how far do we go?



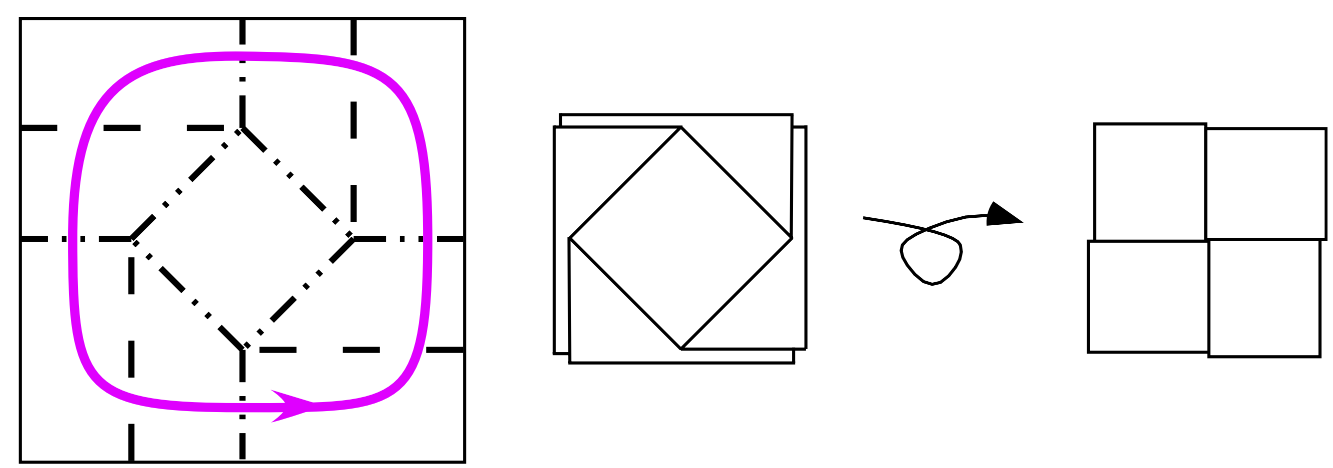
Generalizing pointy and cone maps?

- Given a multiple-vertex flat fold $\mu : C_{2\pi} \longrightarrow C_{2\pi}$
  Let R be a simple, closed, vertex-avoiding curve drawn in the crease pattern.

Square Twist:

Along R, $\dfrac{M - V}{2} = 0 \pmod 2$

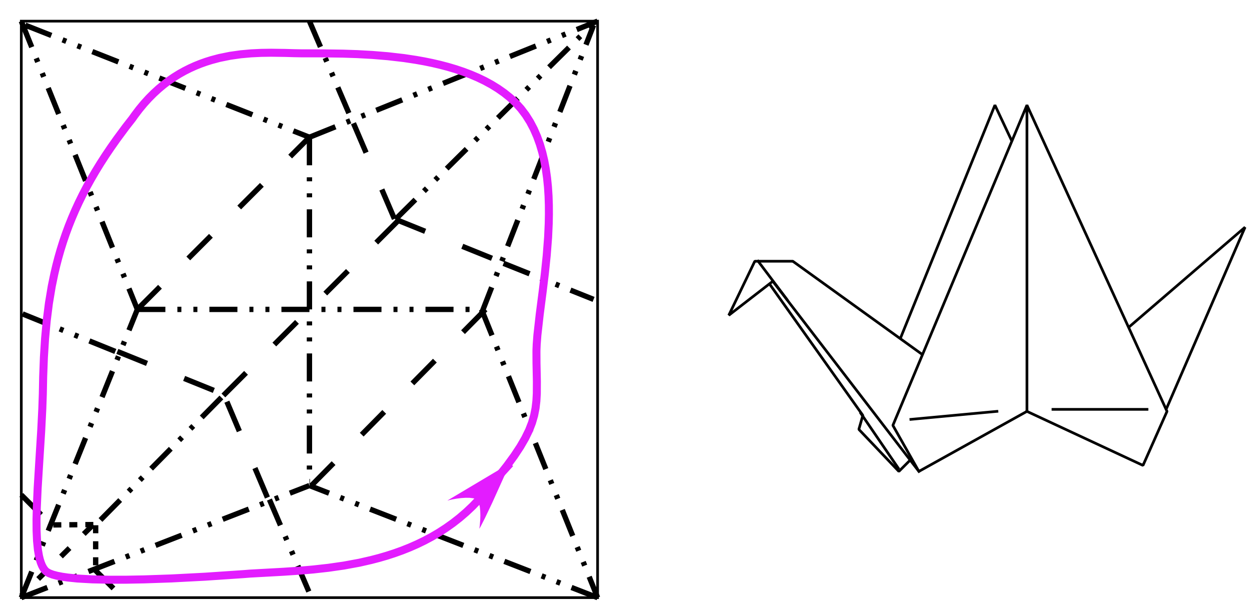
This behaves like a cone/disc map.



Crane: Along R, M=13, V=3

Along R, $\dfrac{M - V}{2} = 1 \pmod 2$

This behaves like a pointy map.

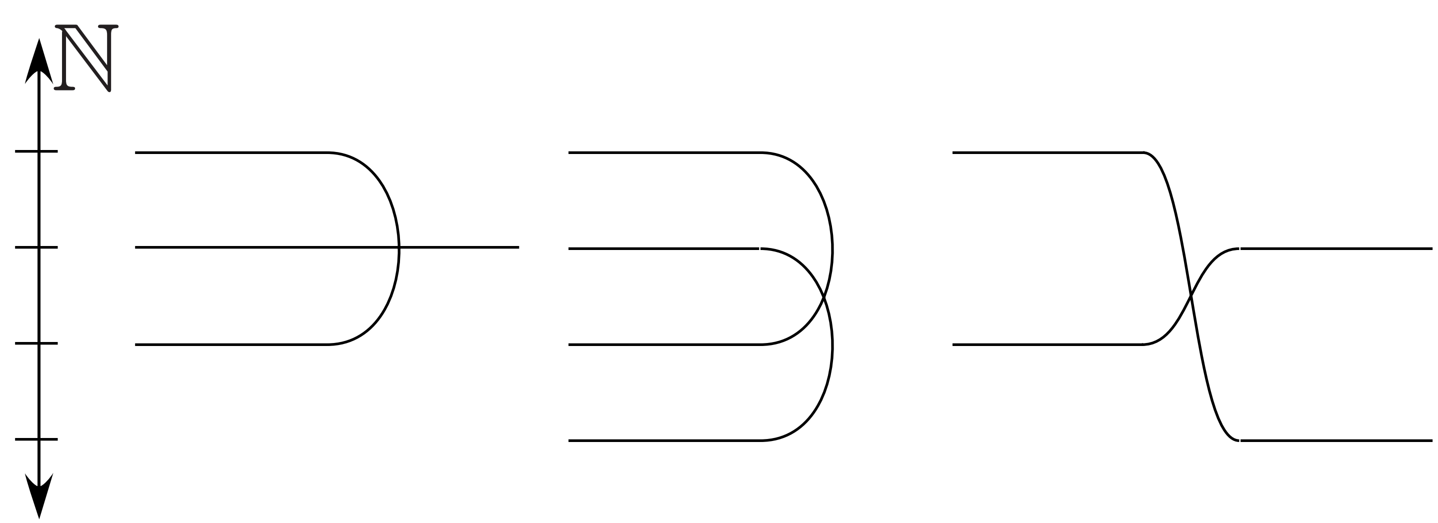


But wait ... how do we *define* flat origami?

A **crease pattern** is a plane graph embedding G = (V, E, F) on a closed region P (which we may assume is simply connected).

A **flat origami** is a crease pattern (P,G) together with
• a map $\mu : P \longrightarrow S$ where S is a zero-curvature surface (the **fold map**)
• a map $L : F \longrightarrow \mathbb{N}$ indicating the **layer order** of the faces (open polygons)
• a map $eg : E \longrightarrow \mathbb{N} \times \mathbb{N}$ indicating the layers each edge straddles (the **glueing map**),
such that

(i) μ is continuous and $\mu|_f$ is an isometry for each f in F.
(ii) The image $\mu(P)$ together with L and eg do not force Justin's **crossing conditions:**

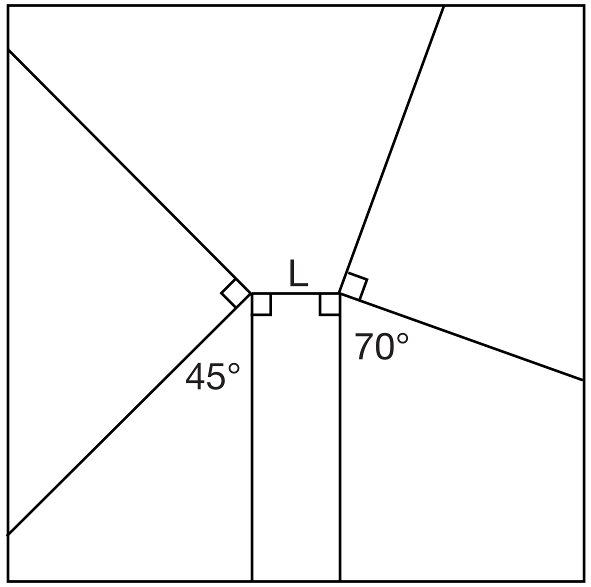


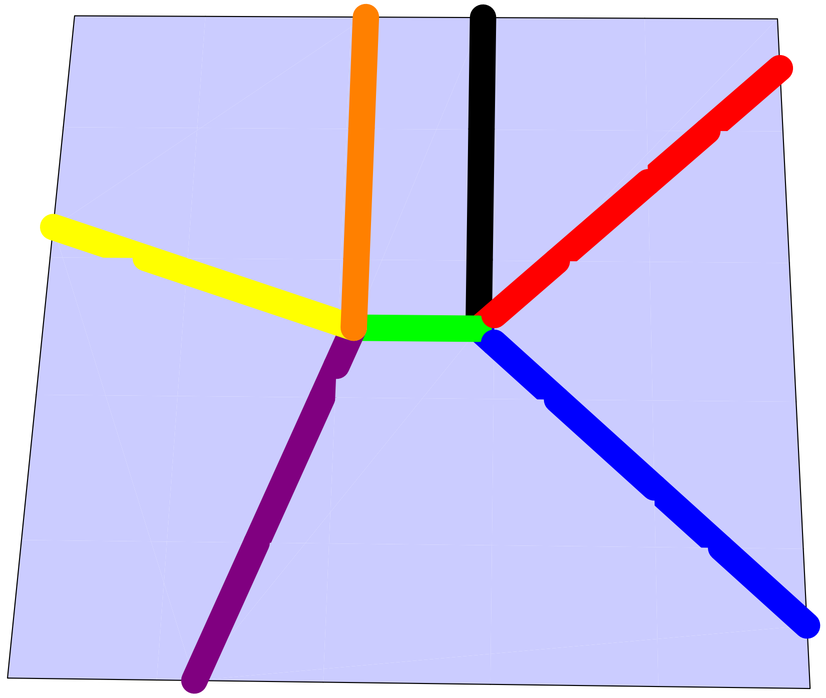
But wait ... how do we *define* flat origami?

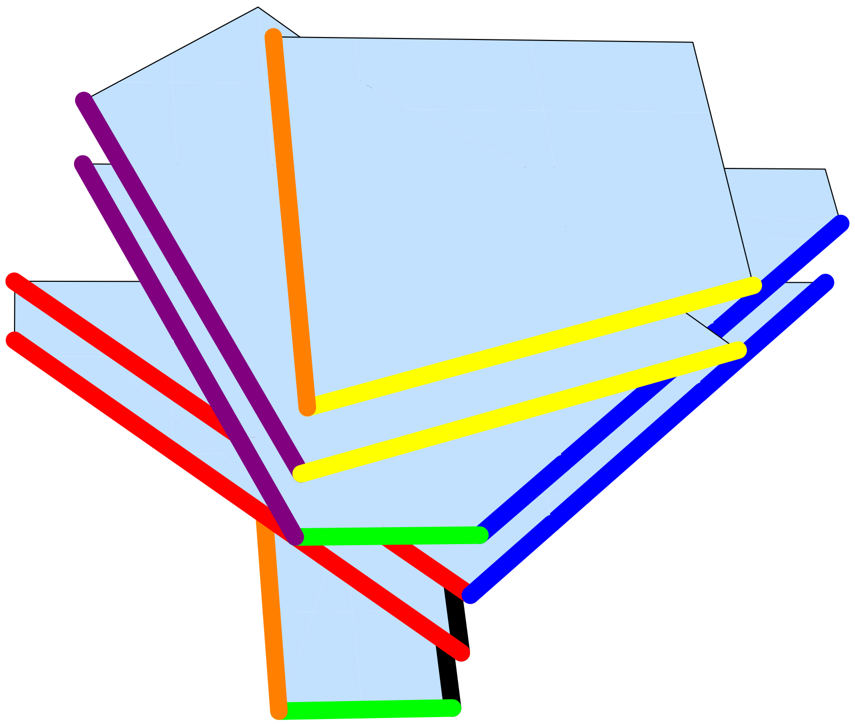
- Example: Our impossible, 2-vertex fold from before:

  Why can’t this fold flat?

  Let’s look at μ with a possible layering assignment:





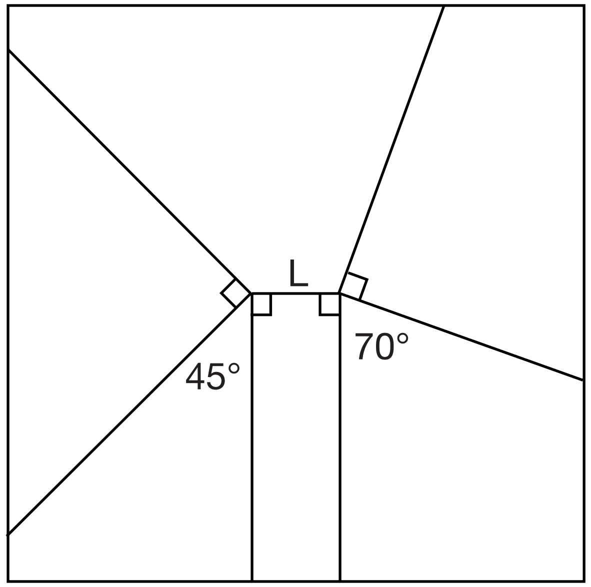


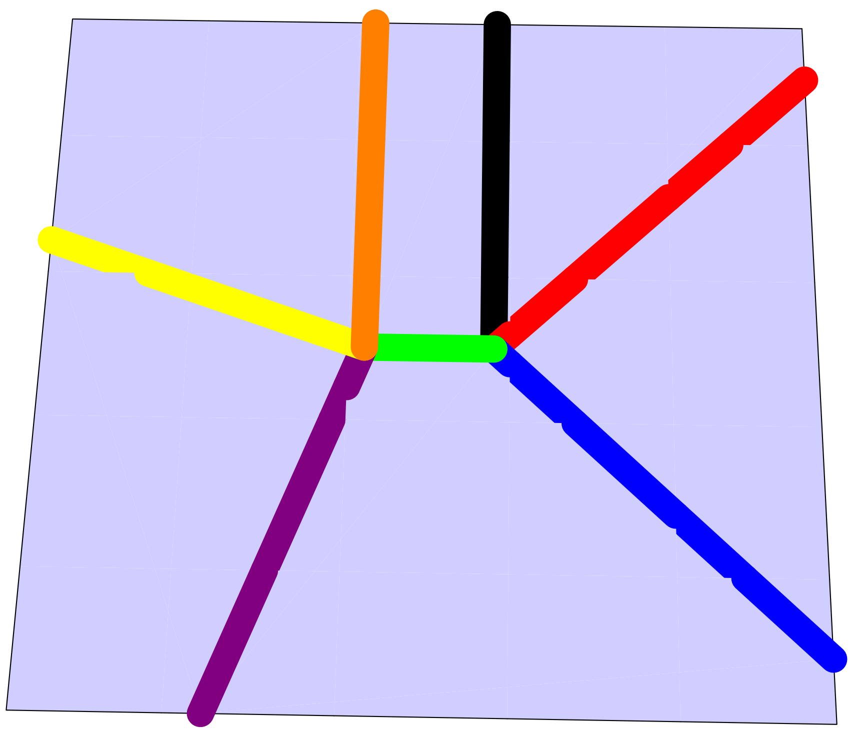
But wait ... how do we *define* flat origami?

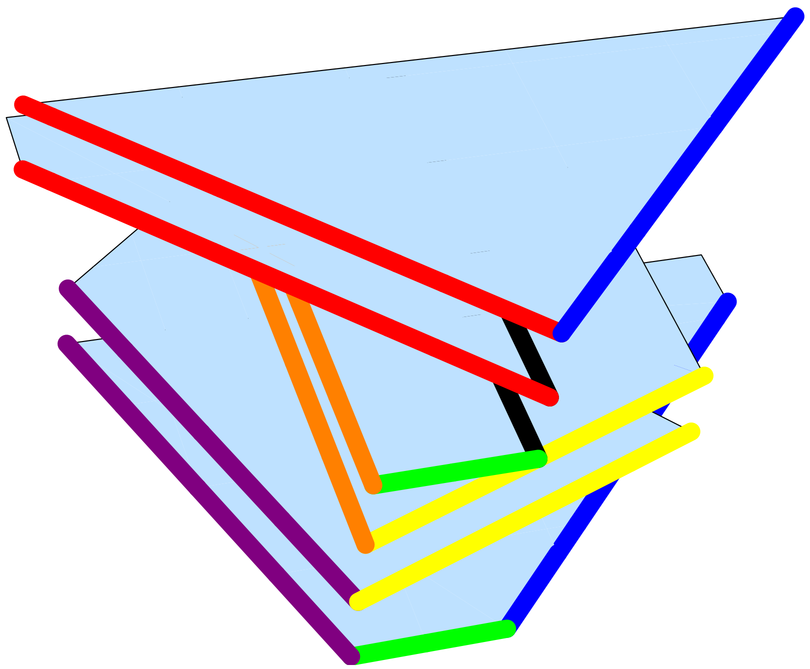
- Example: Our impossible, 2-vertex fold from before:

  Why can't this fold flat?

  Let's look at μ with a possible layering assignment:



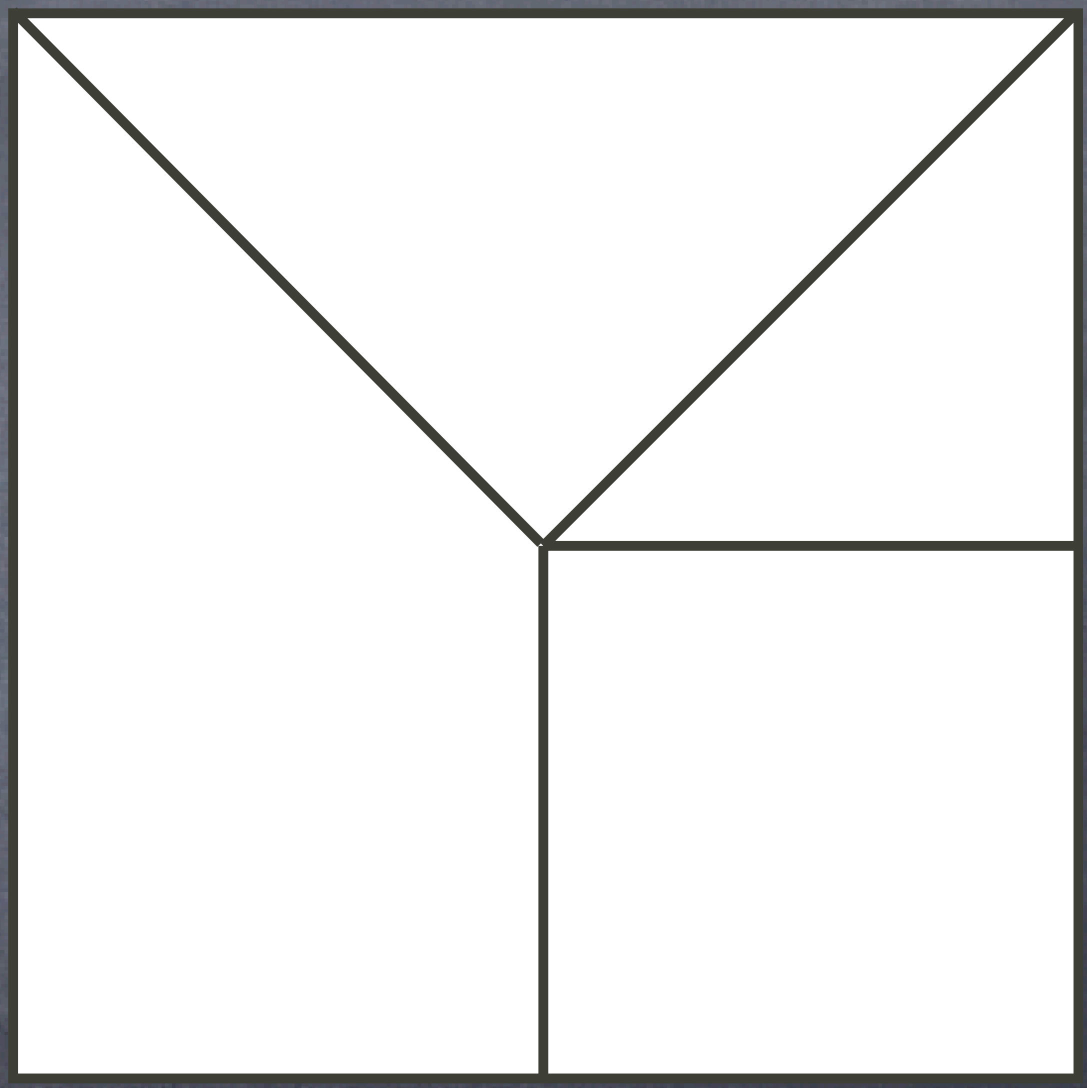




Other layering orders are possible, but all will force a non-crossing condition to be violated at some edge.

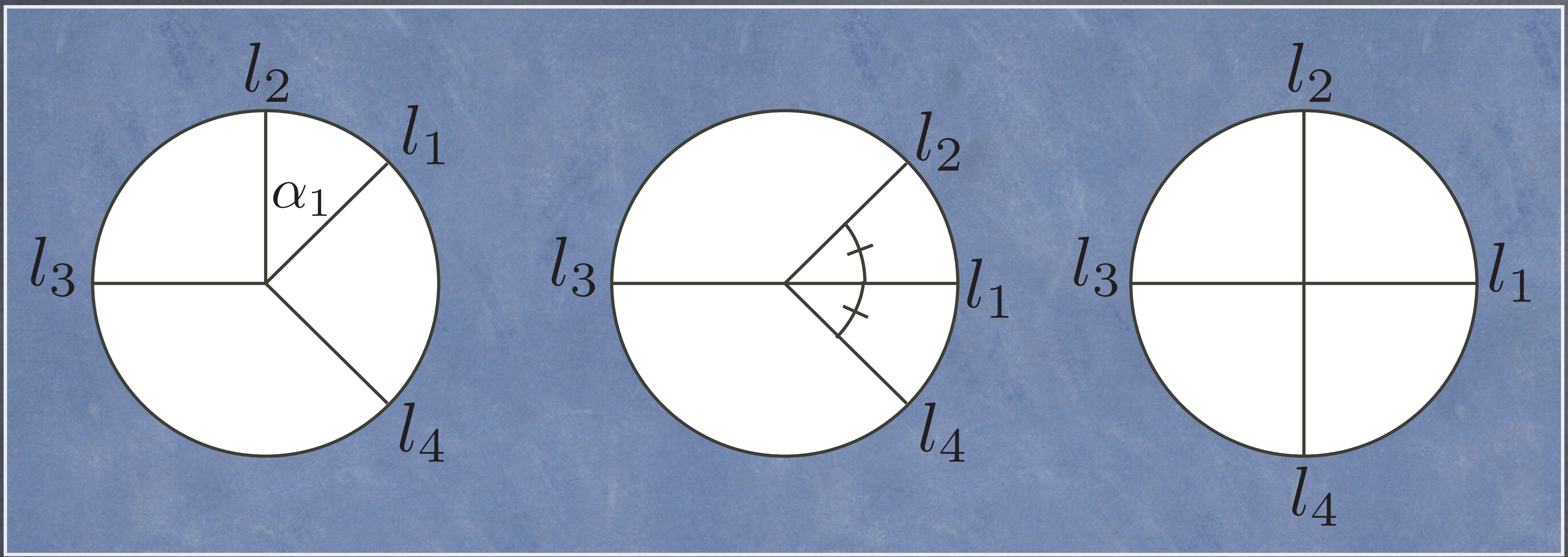
Another Activity

Make these creases.
How many ways can it fold flat?
That is, how many **different** MV assignments can you make?



What about other vertices of degree 4?

Degree 4 flat vertex folds

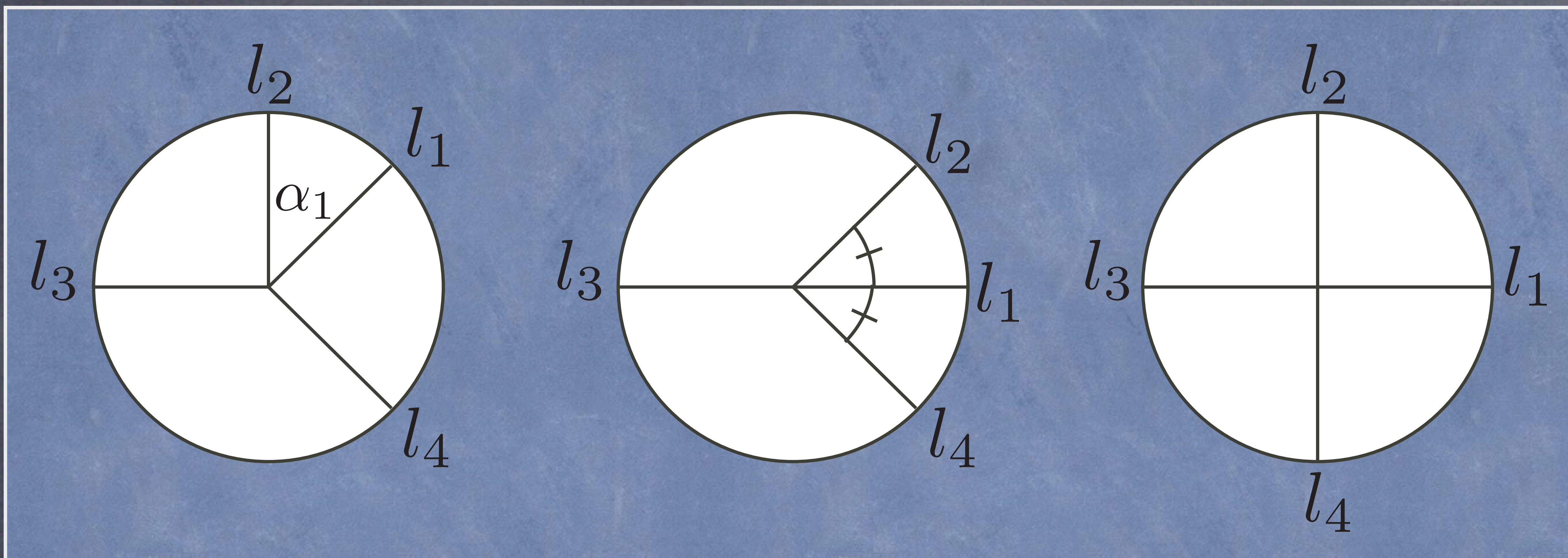


C(v) = 4 C(v) = 6 C(v) = 8

where C(v) = the number of valid MV assignments the vertex v can have.

Think of v as a vector of angles, v= (α_1, α_2, ..., α_{2n})

Degree 4 flat vertex folds



C(v) = 4 C(v) = 6 C(v) = 8

Theorem: For any flat-foldable vertex v= (α_1, α_2, ..., α_{2n}),

$$2^n \leq C(\alpha_1, \ldots \alpha_{2n}) \leq 2\binom{2n}{n-1}$$

$$2^n \leq C(\alpha_1, ...\alpha_{2n}) \leq 2\binom{2n}{n-1}$$

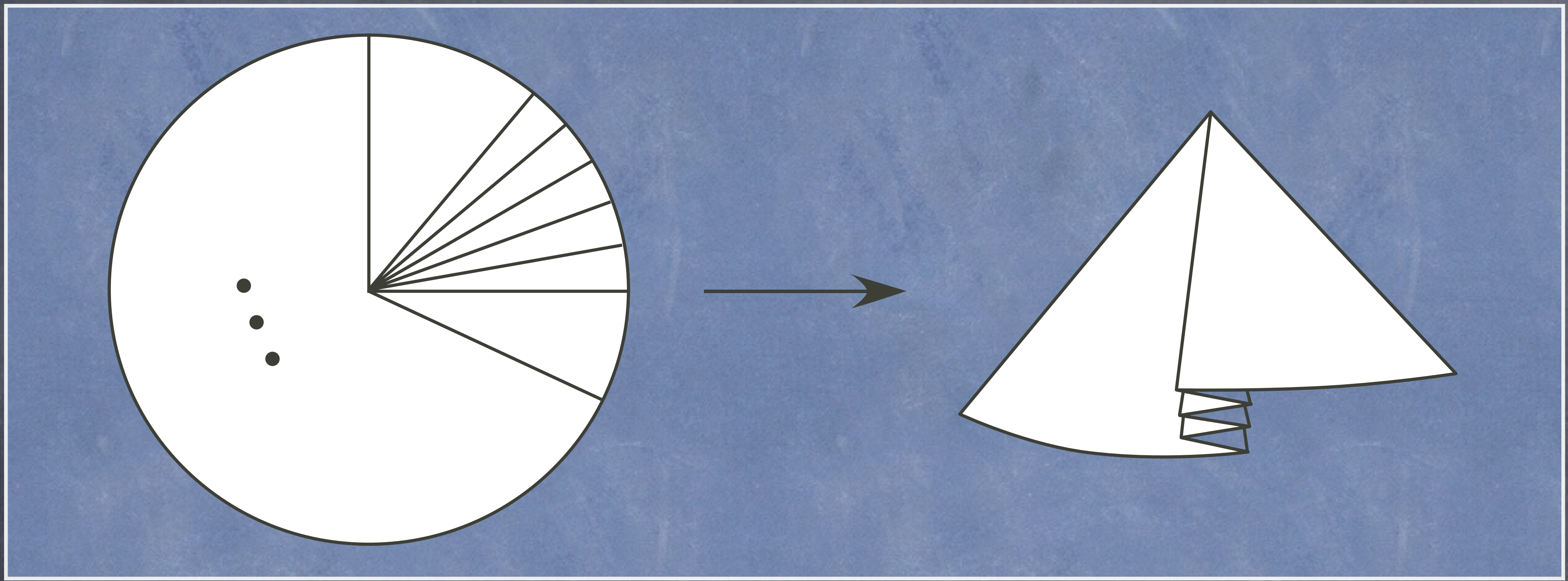
Question: What values can $C(\alpha_1, \alpha_2, ..., \alpha_{2n})$ attain between these bounds?

$C(\alpha_1, ... , \alpha_4) \in \{4, 6, 8\}$

$C(\alpha_1, ... , \alpha_6) \in \{8, 12, 16, 18, 20, 24, 30\}$

$C(\alpha_1, ... , \alpha_8) \in \{16, 24, 32, 36, 40, 48, 54, 60, 70, 72, 80, 90, 112\}$

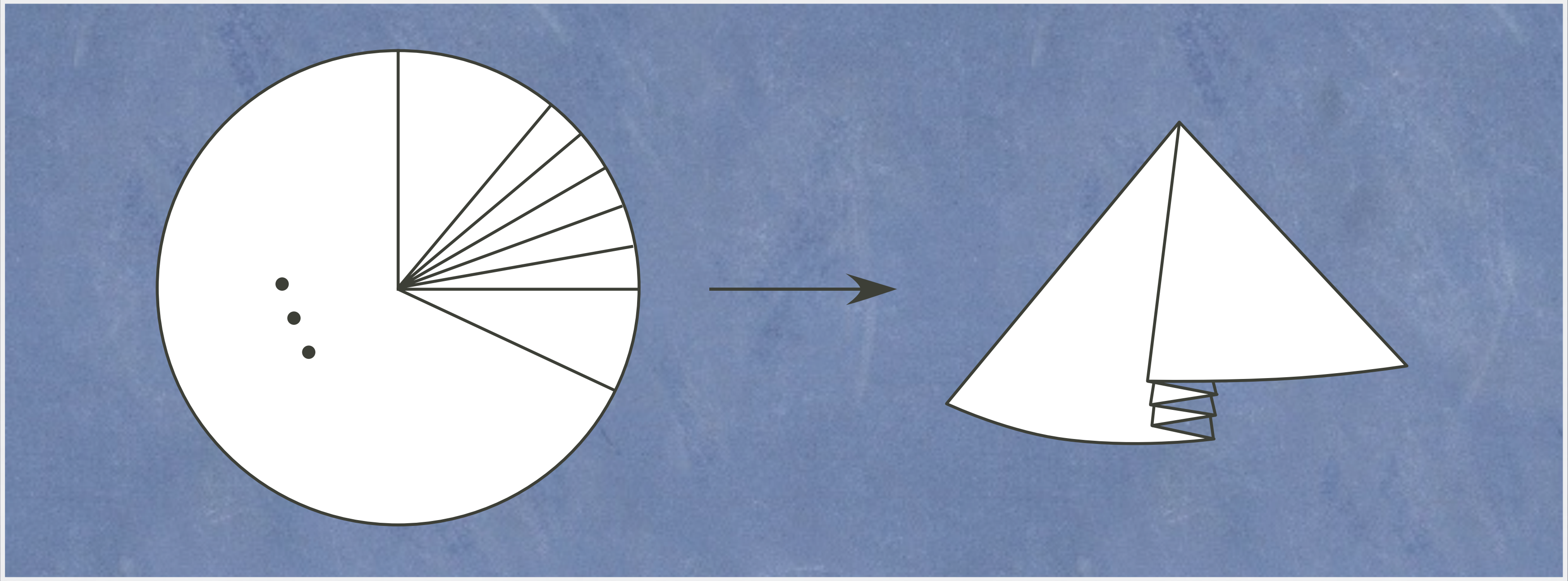
How do we compute these numbers?
With recursion!



Equal angles in a row, surrounded by larger angles.

5 equal angles use 6 creases needing 3Ms and 3Vs.

Here we have $\binom{6}{3}$ ways to fold those angles flat.



In general, suppose we have a sequence of k equal angles in a row: $\alpha_i = \alpha_{i+1} = \ldots = \alpha_{i+k-1}$, $\alpha_{i-1} > \alpha_i$, $\alpha_{i+k} > \alpha_{i+k-1}$

Then if k is odd we have

$$C(\alpha_1, ..., \alpha_{2n}) = \binom{k+1}{\frac{k+1}{2}} C(\alpha_1, ..., \alpha_{i-2}, \alpha_{i-1} - \alpha_i + \alpha_{i+k}, \alpha_{i+k+1}, ..., \alpha_{2n})$$

and if k is even then

$$C(\alpha_1, ..., \alpha_{2n}) = \binom{k+1}{\frac{k}{2}} C(\alpha_1, ..., \alpha_{i-1}, \alpha_{i+k}, ..., \alpha_{2n})$$

Let A_n = the number of different values that $C(\alpha_1, \ldots, \alpha_{2n})$ can attain.

A_n : 1, 3, 7, 13, 24, 39, 62, 97, 147, 215, 312, 440, 617, 851 1161, ...

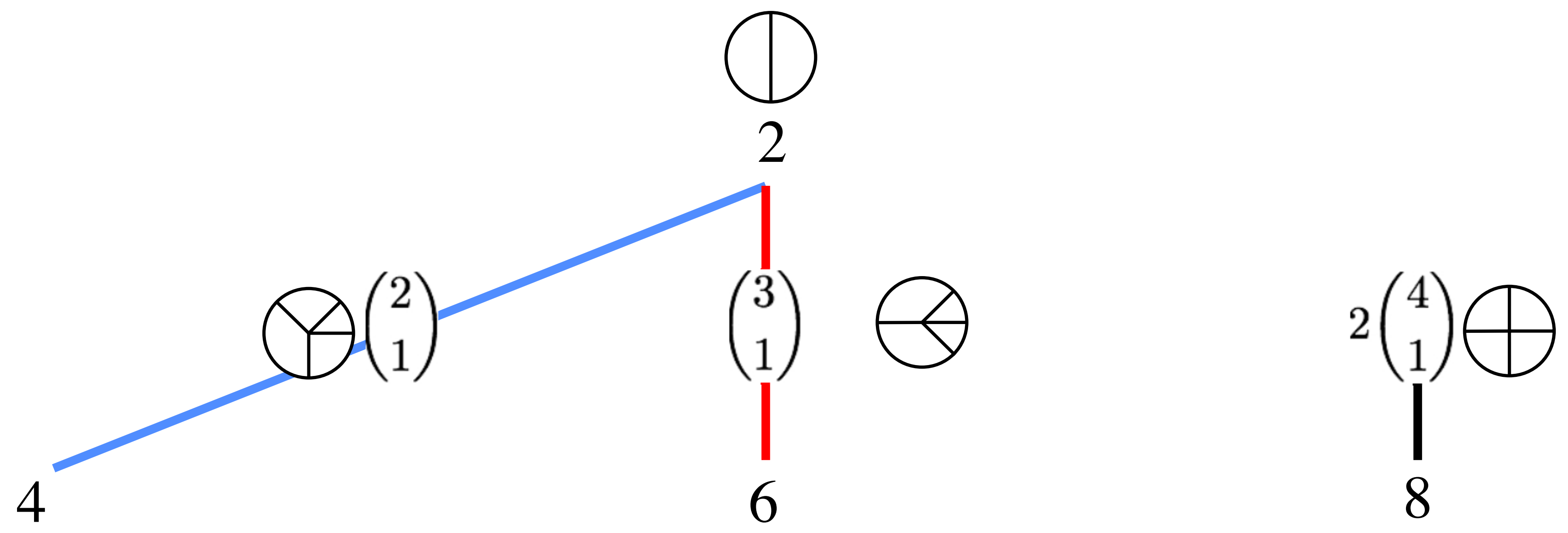
sequence A156209

This sequence is ~~not~~ in the Online Encyclopedia of Integer Sequences.

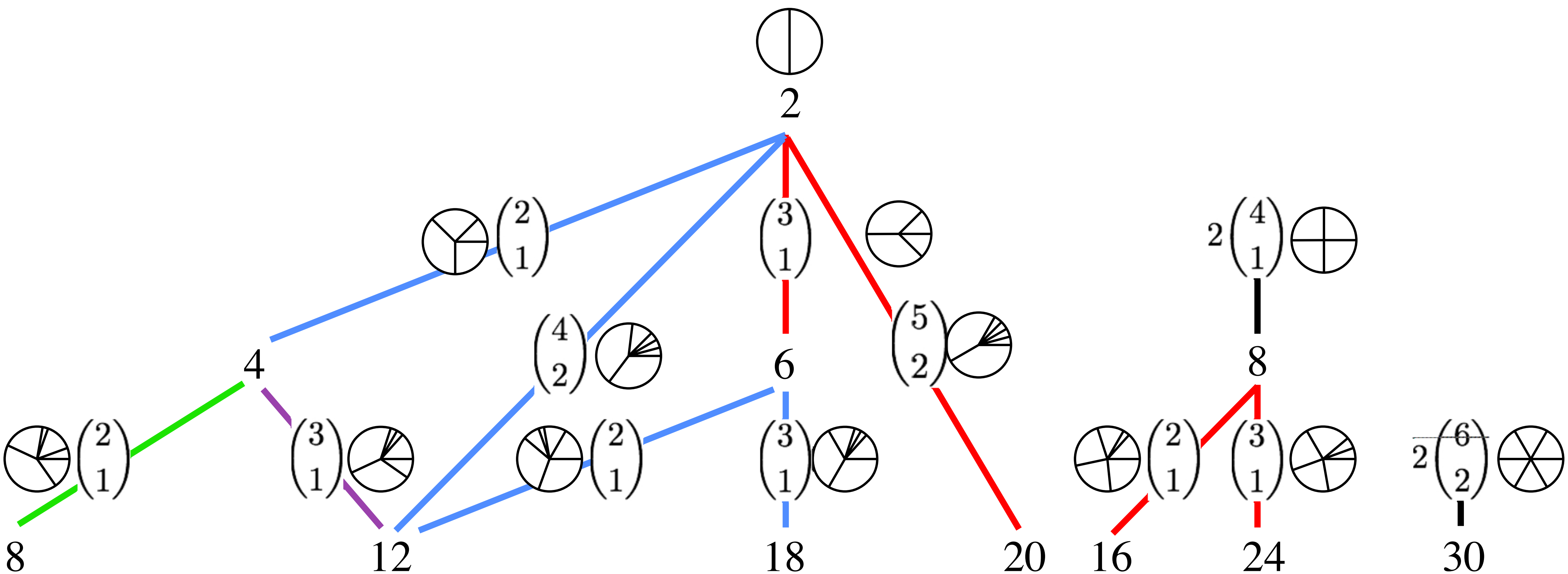
Finding a closed formula for A_n might be hopeless, since we don't know the prime factorizations of

$$\binom{2n}{n} \text{ and } \binom{2n+1}{n}.$$

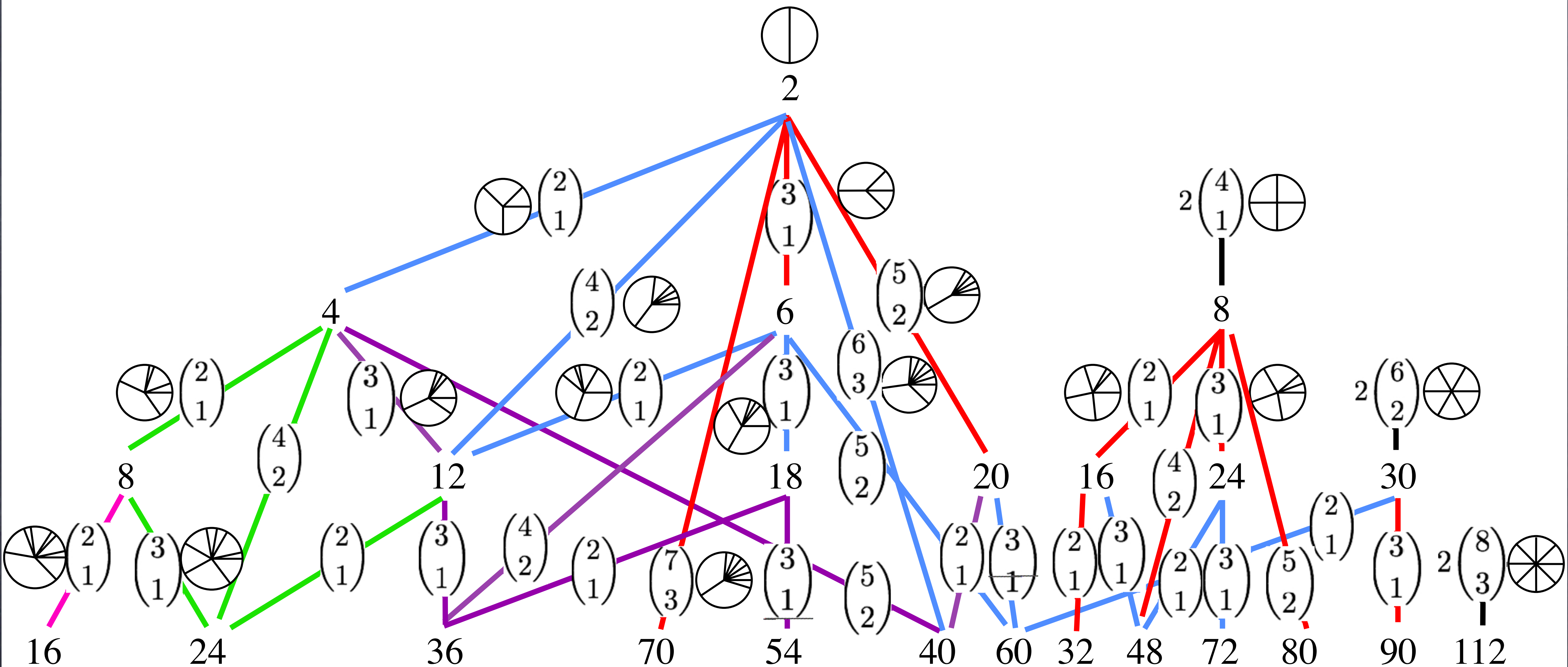
Recursive Tree for $C(\alpha_1, \ldots, \alpha_{2n})$



Recursive Tree for $C(\alpha_1, \ldots, \alpha_{2n})$

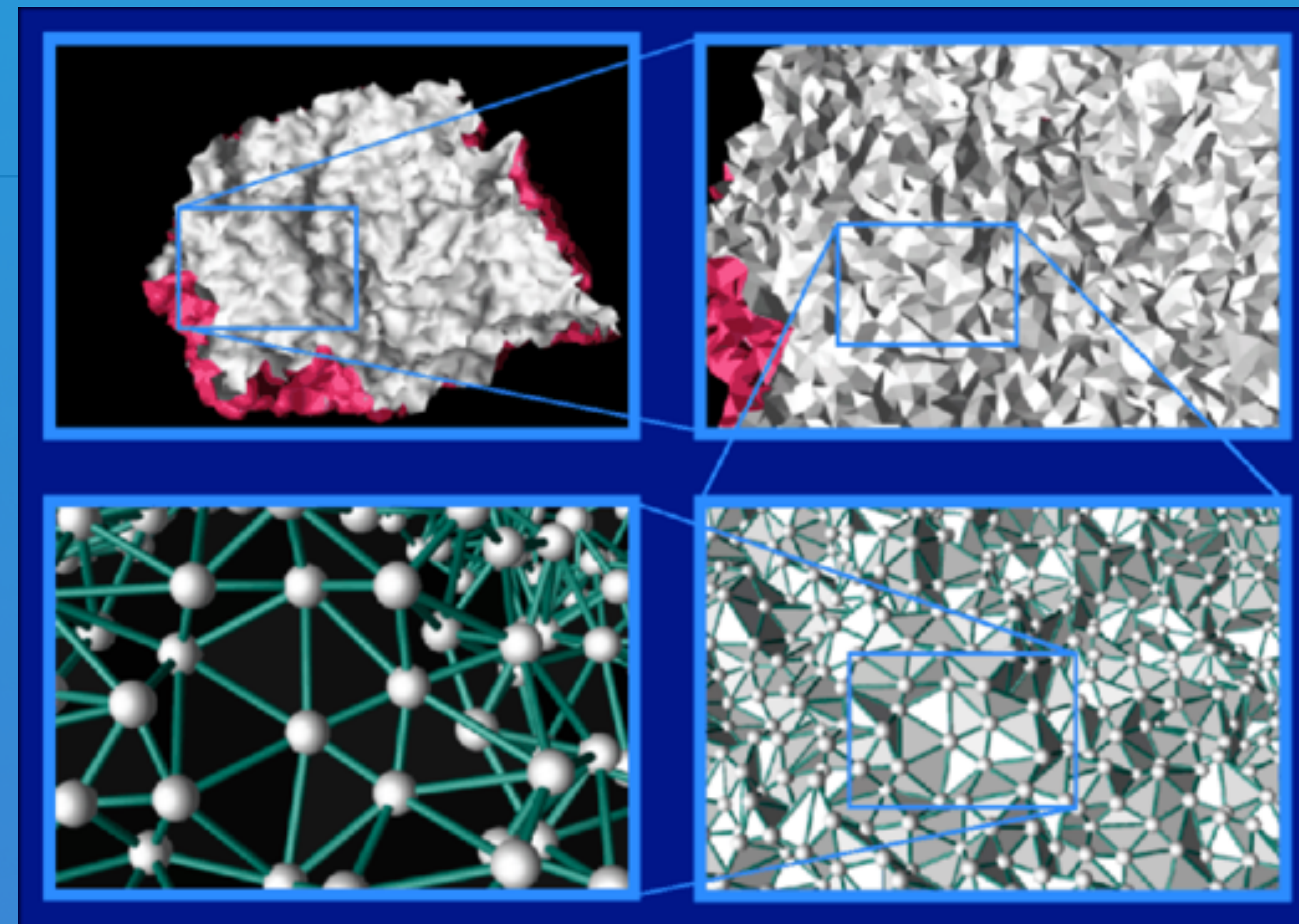


Recursive Tree for C(α_1, ... , α_{2n})



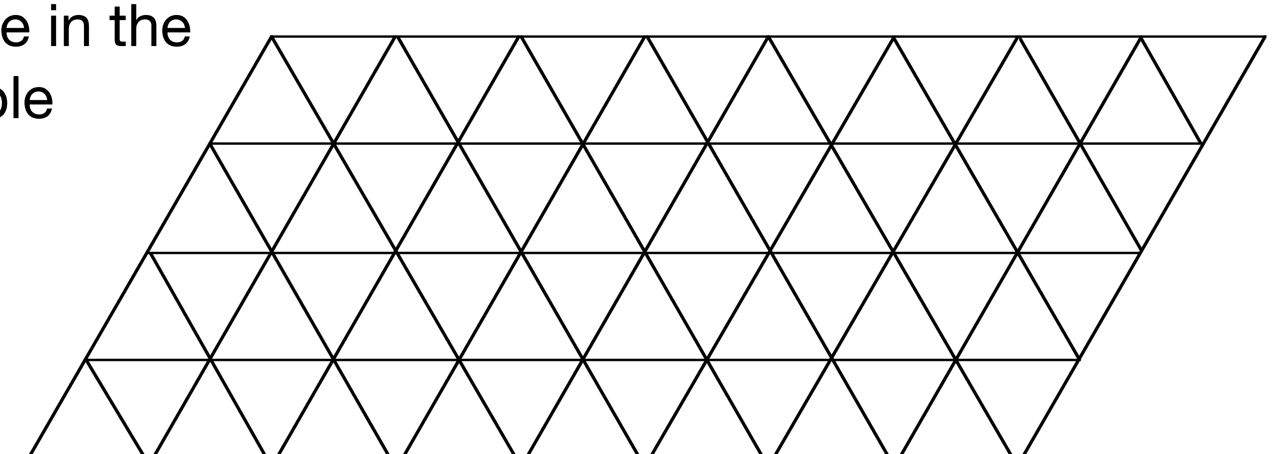
Another counting question

- Some physicists and physical chemists are very interested in **polymer membrane folding.**



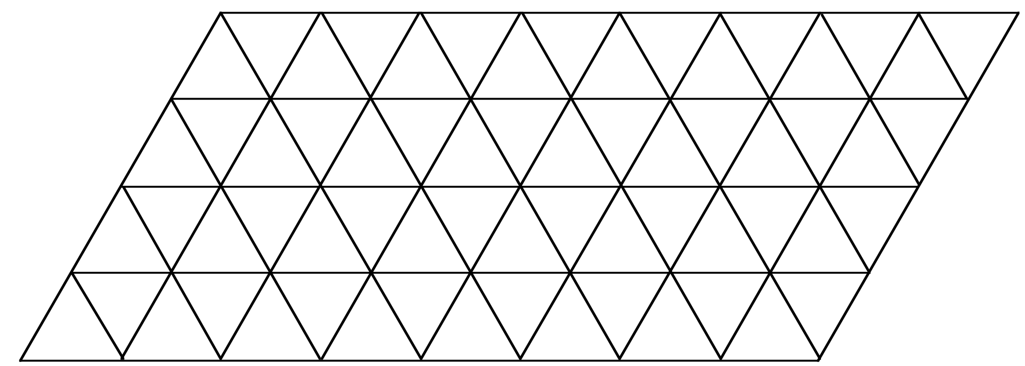
Source: IBM Almaden Research Center

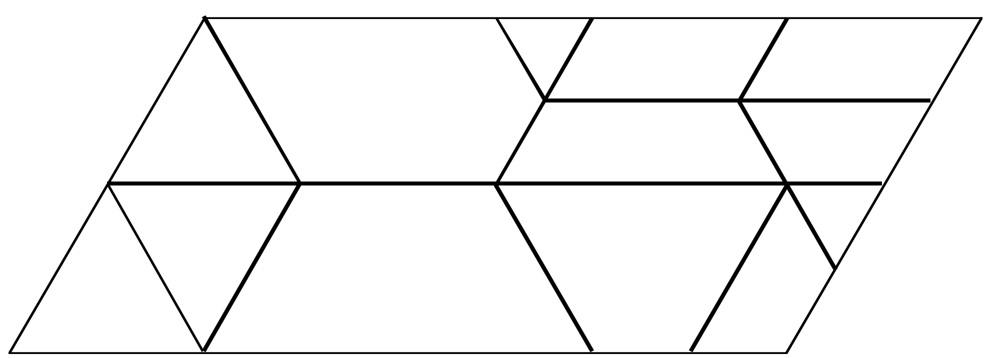
- **Key question:** Given a regular lattice in the plane, how many different flat-foldable crease patterns can you make using only the lattice for crease lines?

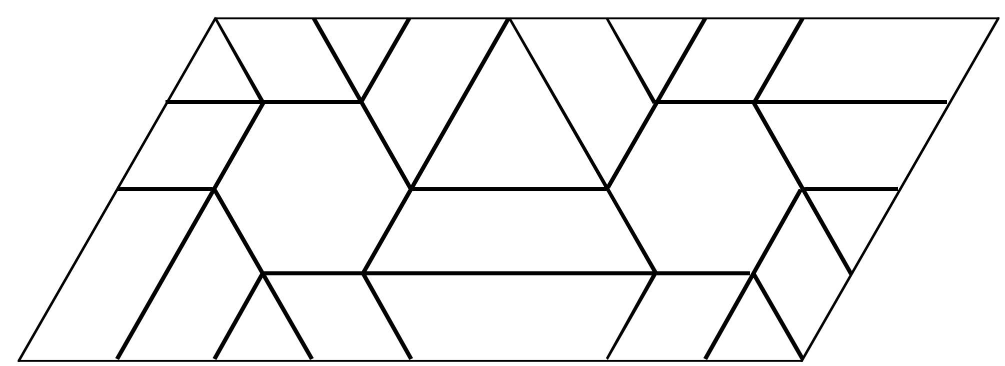


Another counting question

- **Key question:** Given a regular lattice in the plane, how many different flat-foldable crease patterns can you make using only the lattice for crease lines?

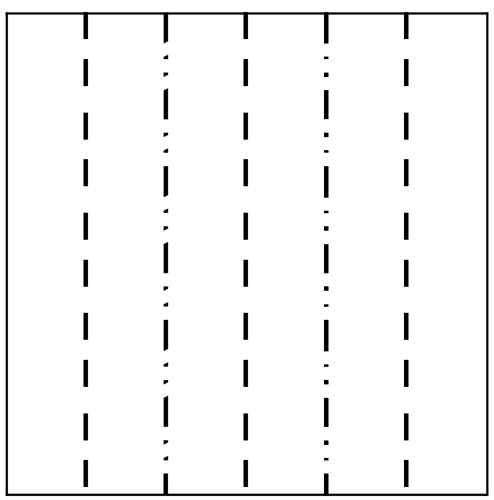


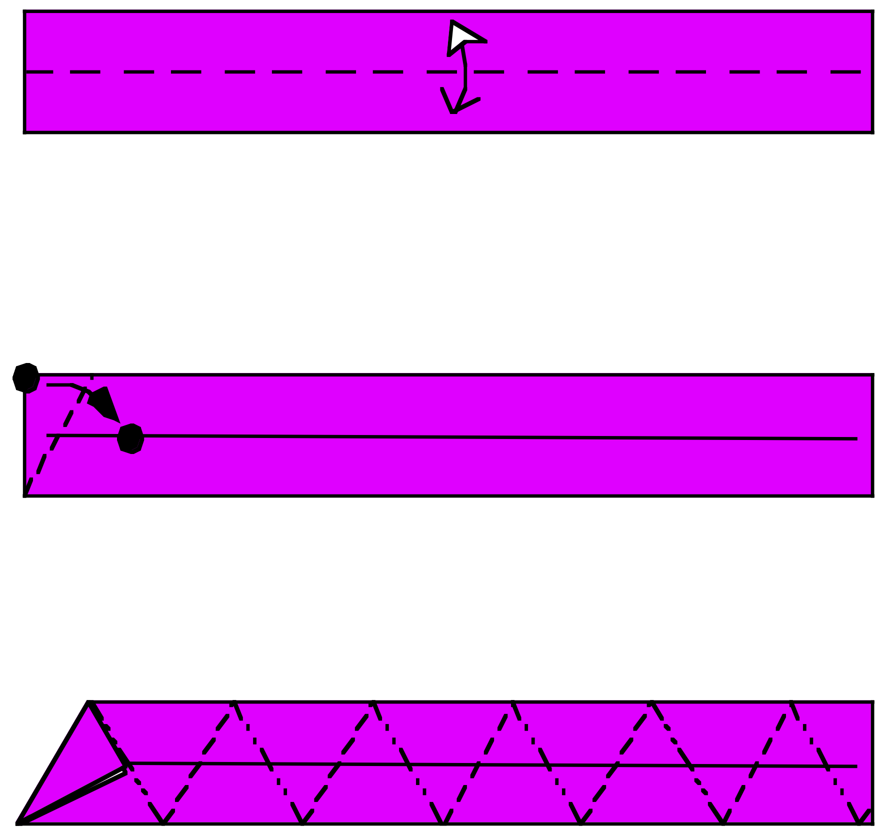




Another counting question

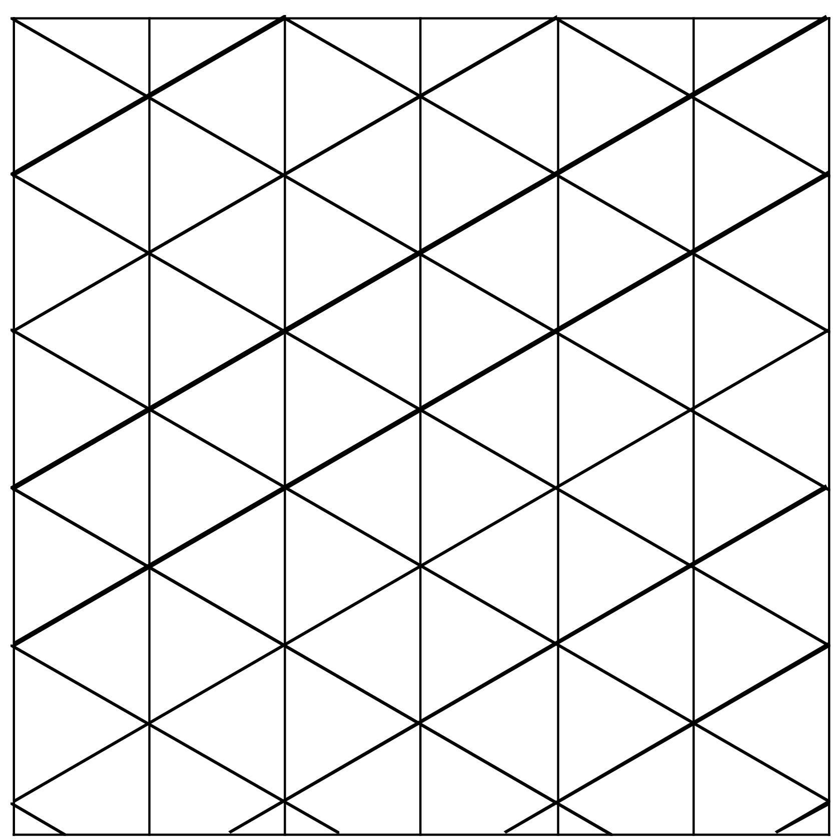
- **Activity:** Let's fold a hexagon twist!

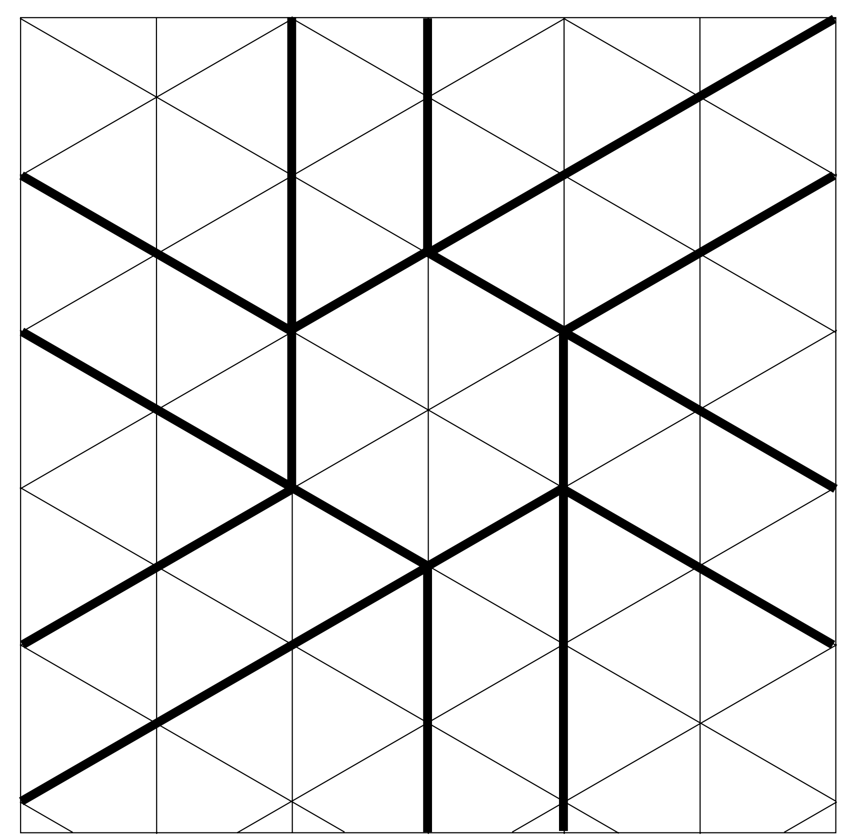




Another counting question

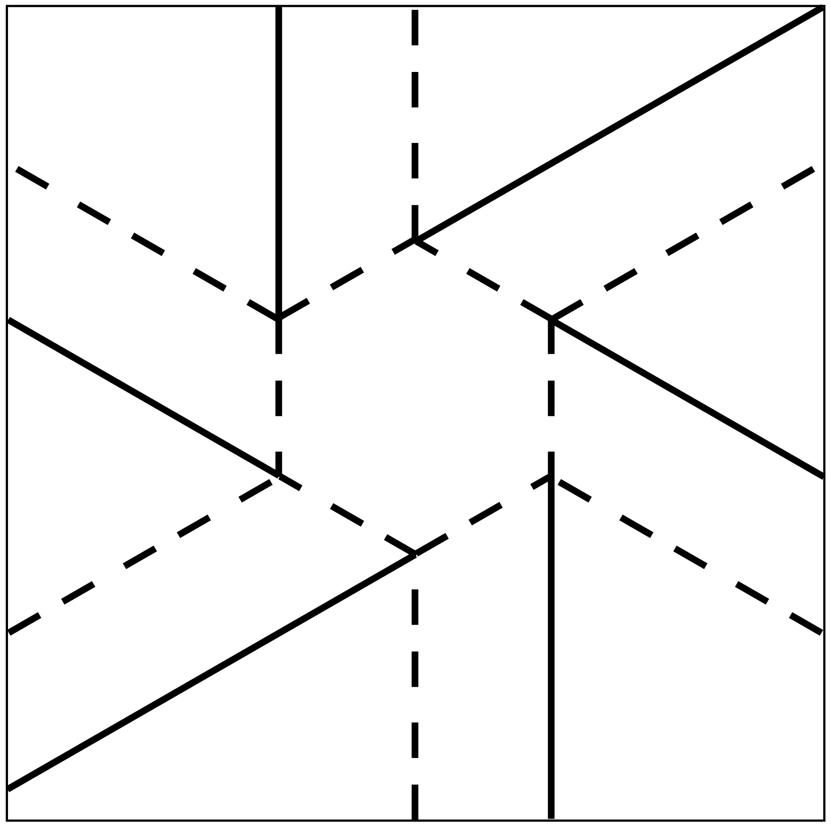
- **Activity:** Let's fold a hexagon twist!





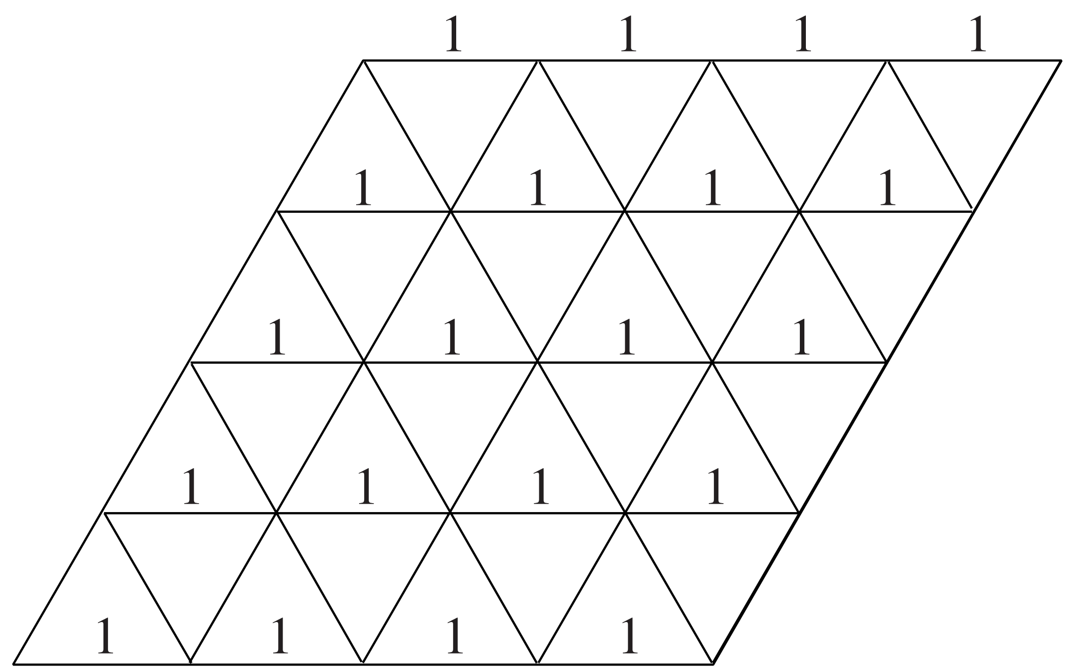
Another counting question

- **Activity:** Let's fold a hexagon twist!



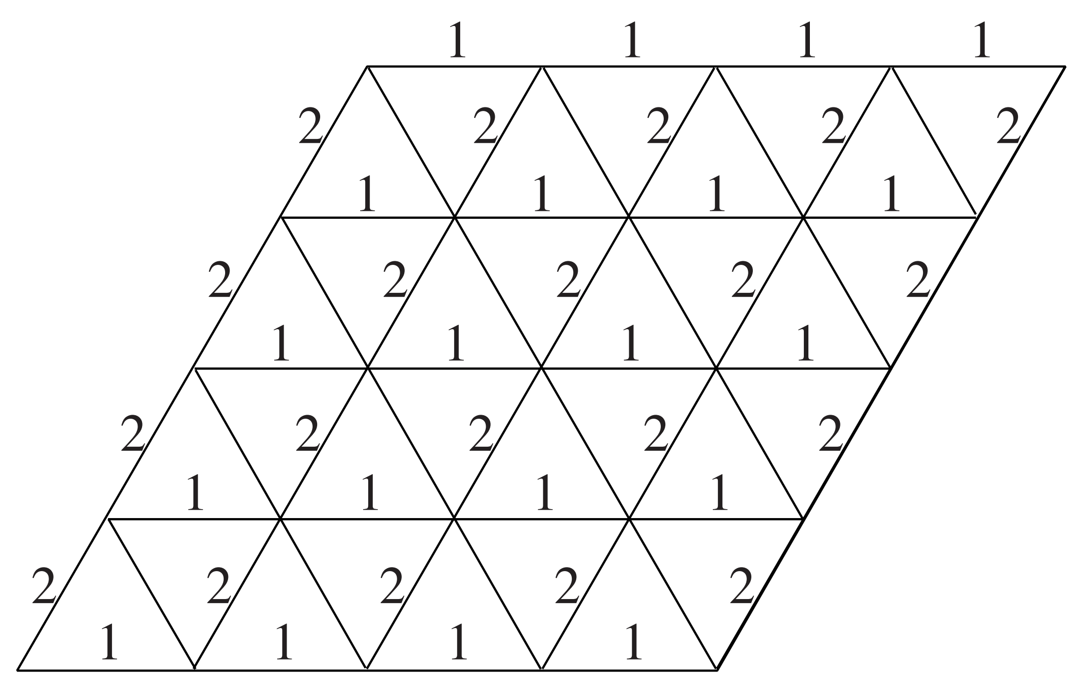
Another counting question

- **Grünbaum coloring** of the triangle lattice



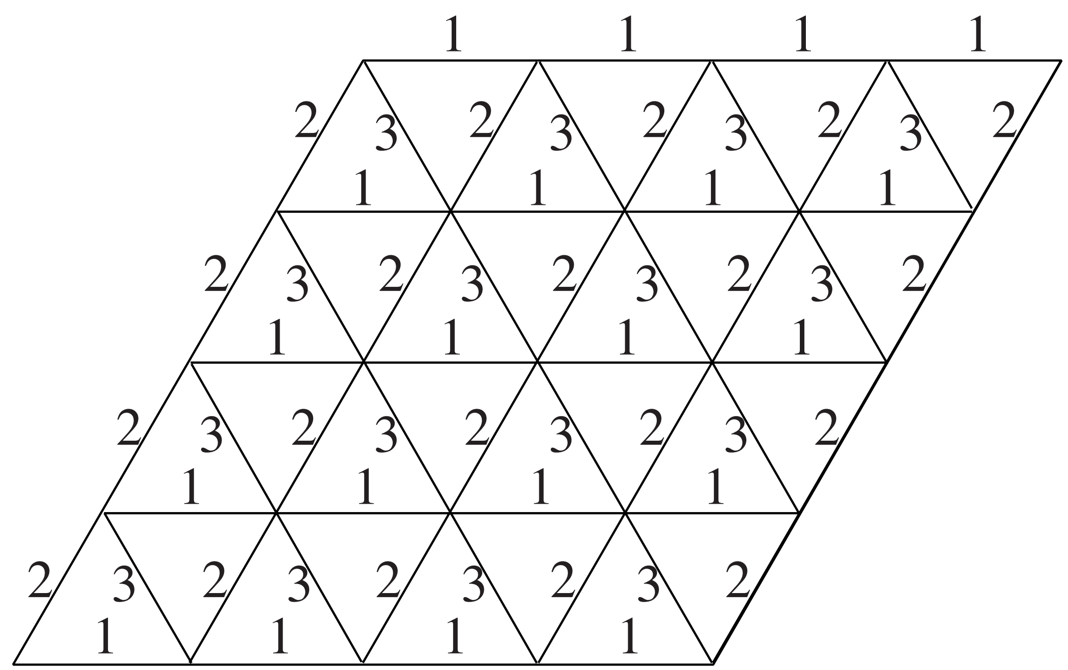
Another counting question

- **Grünbaum coloring** of the triangle lattice



Another counting question

- **Grünbaum coloring** of the triangle lattice



Every triangle must have all three colors around it. This coloring shown is the canonical Grünbaum coloring.

Another counting question

- **Bijection!!!!** (due to Philippe Di Francesco)

Different Grünbaum colorings
of the triangle lattice

Different flat-foldable
crease patterns
of the triangle lattice

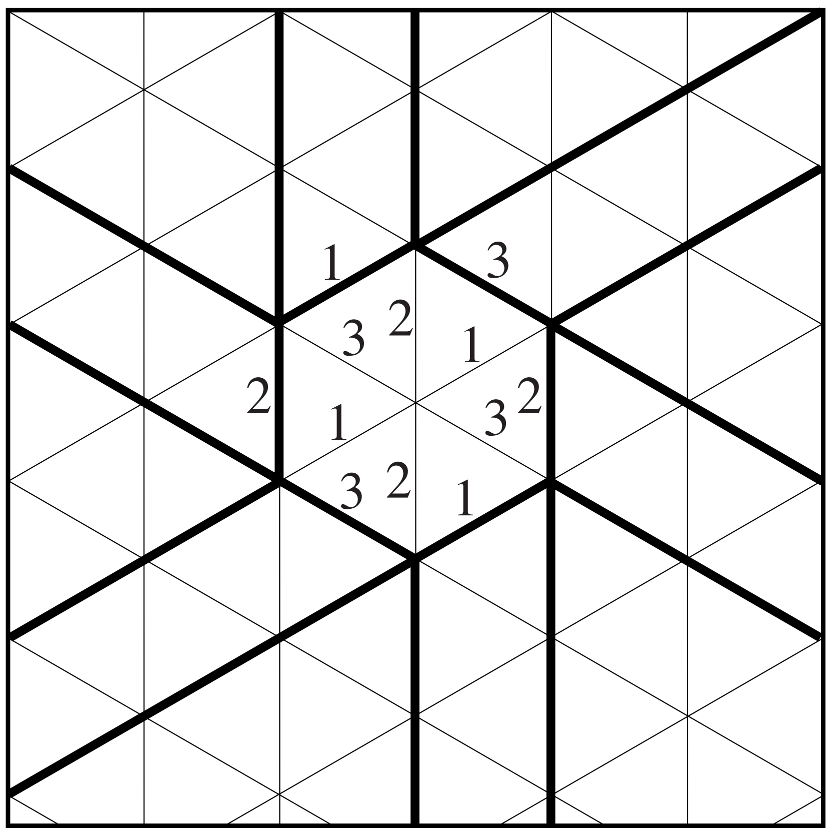
How? Take a flat-folded crease pattern of the triangle lattice.
Overlay the canonical Grünbaum coloring on the **folded** lattice.
Then **unfold** it, and let the colors follow the creases.

When unfolding, triangles are preserved, so it'll still be a valid Grünbaum.

Another counting question

- But what about the other direction? (Do Grünbaum colorings determine a unique flat-foldable crease pattern of the triangle lattice???)

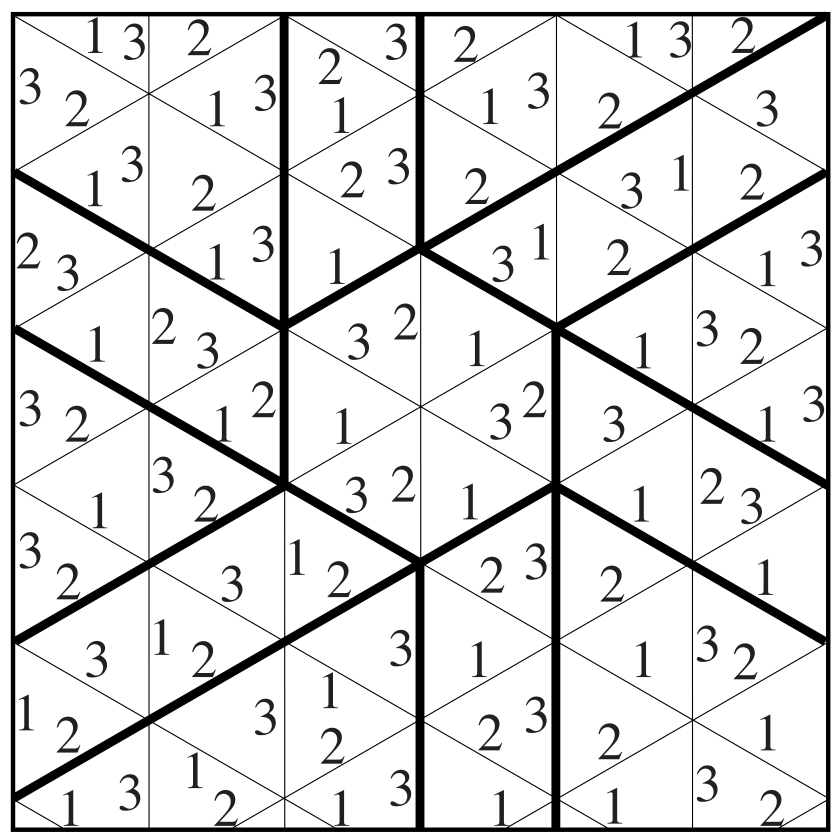
- **Activity!** Find the Grünbaum coloring of the triangle lattice that generates the hexagon twist you made. Start with the following canonical coloring in the center hexagon:



Another counting question

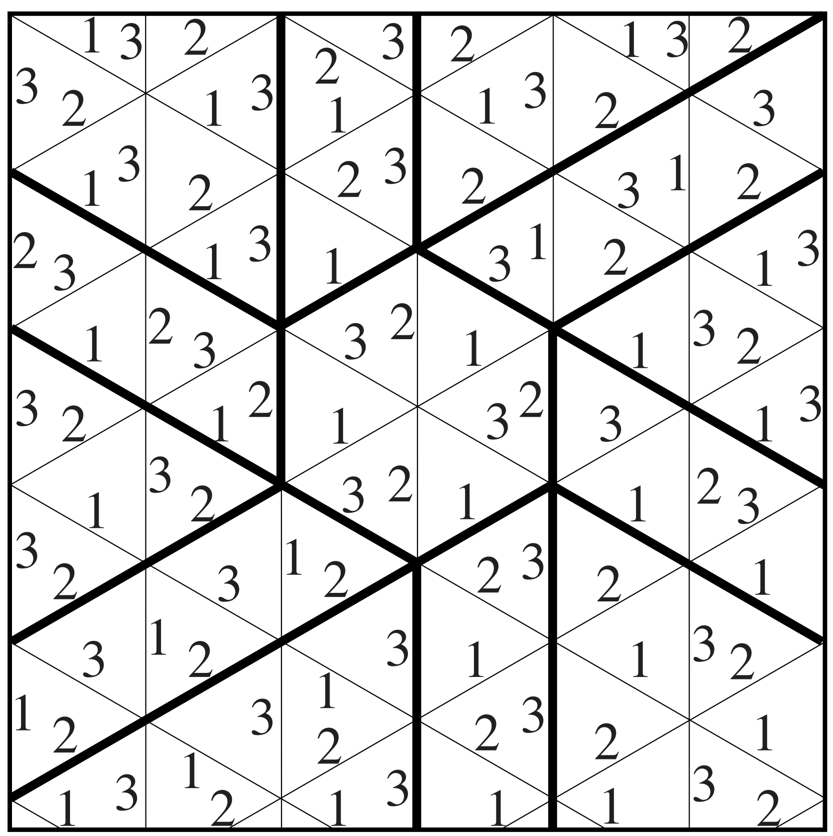
- But what about the other direction? (Do Grünbaum colorings determine a unique flat-foldable crease pattern of the triangle lattice???)

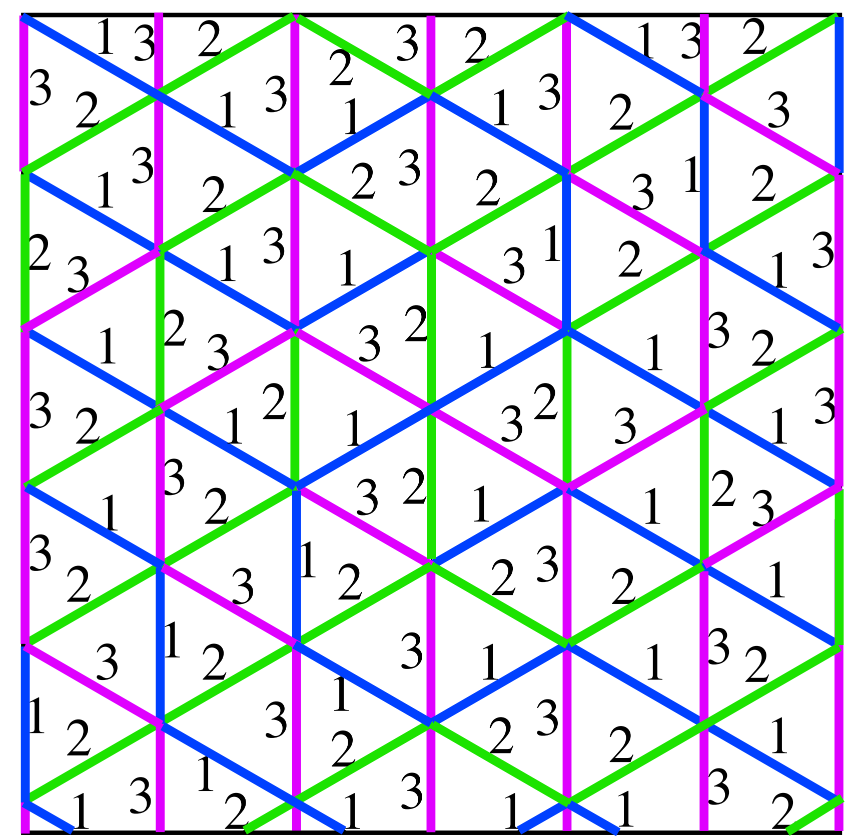
- **Activity!** Find the Grünbaum coloring of the triangle lattice that generates the hexagon twist you made. Start with the following canonical coloring in the center hexagon:



Another counting question

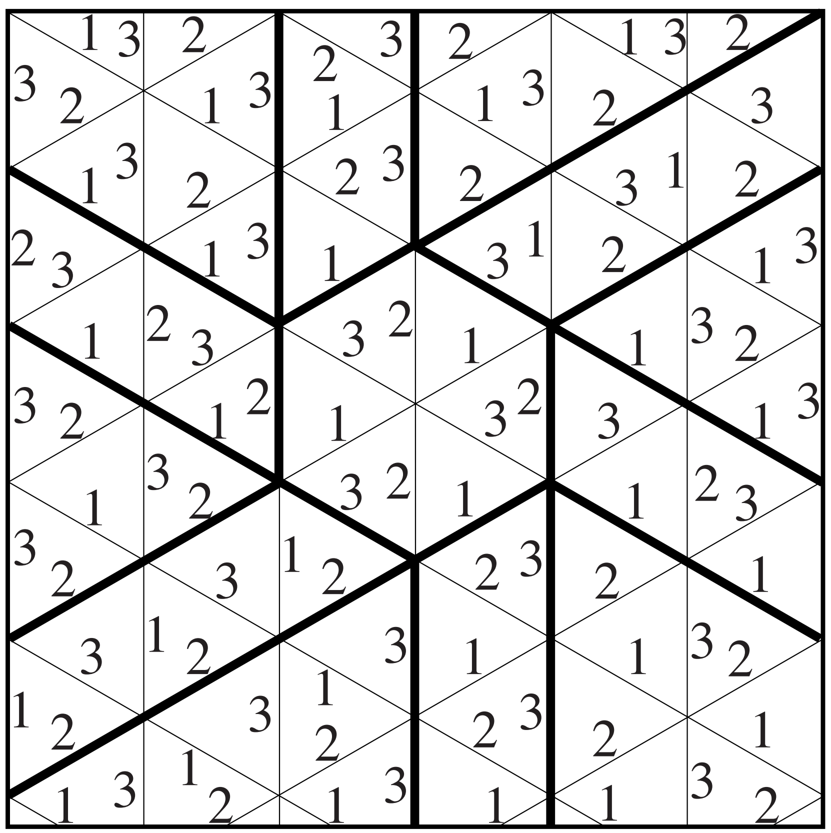
- With color?

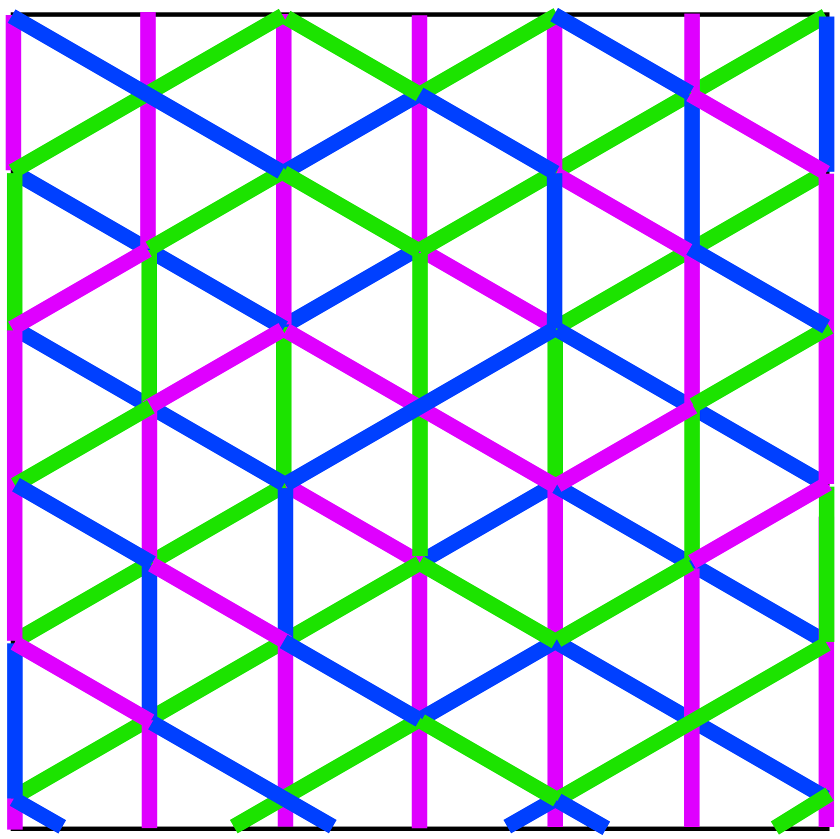




Another counting question

- With color?





Another counting question

- So what?

- Grünbaum colorings of the triangle lattice = 3-edge colorings of the hexagonal lattice. (by taking the dual)

- Physicists already proved that if a hexagonal lattice has L vertices (for L LARGE), then the number of proper 3-edge colorings of the lattice is
$$\approx (1.20872...)^L$$
where
$$1.20872... = \sqrt{\frac{2^2}{1 \cdot 3}\frac{5^2}{4 \cdot 6}\frac{8^2}{7 \cdot 9}\cdots}$$

Thus the number of ways one can fold a big triangle lattice with L triangles is
$$\approx (1.20872...)^L$$