

# Mathematical Methods in Physics – 231A

*Monday - Wednesday 12-2pm in PAB-4-330*  
*Office hours Monday 2-3pm and 5-6 pm, or by appointment, in PAB 4-929*

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The purpose of the course 231A is to present standard and widely used mathematical methods in Physics, including functions of a complex variable, linear algebra, differential equations, and special functions associated with eigenvalue problems of ordinary and partial differential operators.

# Bibliography

## Standard texts

1. *Complex Variables with an introduction to Conformal Mapping and applications* by Murray Spiegel, Schaum's outline series, McGraw Hill (1964);
2. *Mathematical Methods of Physics*, (based in part on Feynman lecture notes) by J. Mathews and R.L. Walker, Benjamin/Cummings (1979);
3. *Mathematical Methods for Physics and Engineering*, third edition, by K.F. Riley, M.P. Hobson and S.J. Bence, Cambridge University Press (2018);
4. *Mathematical Methods for Physicists*, fifth edition, by G.B. Arfken, H.J. Weber, Harcourt, Academic Press (2001);
5. *Advanced Mathematical Methods for Scientists and Engineers*, by C.M. Bender and S.A. Orszag, Springer (1999).

## Classics

6. *A course in Modern Analysis* by E.T. Whittaker and G.N. Watson, Cambridge University Press (1969);
7. *Methods of Mathematical Physics Volume I*, by R. Courant and D. Hilbert, John Wiley & sons (1989) ;
8. *Methods of Theoretical Physics. Vols I, II*, P.M. Morse and H. Feshbach, McGraw -Hill (my copy dates from 1953)

## More specialized

9. *Geometry, Topology and Physics*, by M. Nakahara, Institute of Physics Publishing (2005);
10. *The Geometry of Physics*, by T. Frenkel, Cambridge University Press (2006).

## An indispensable reference with lots of formulas

11. *Bateman Manuscript project, Higher transcendental Functions*, Vols I, II, III, by A. Erdelyi editor, R.E. Krieger Publishing (1981).

# 1 Complex Analytic functions

Gerolamo Cardano, in the 16-th century, is credited for introducing complex numbers. The set of complex numbers  $\mathbb{C}$  forms a quadratic extension of the real numbers  $\mathbb{R}$  by the square root of  $-1$ , which is denoted by  $i$  so that we have,

$$i^2 = -1 \tag{1.1}$$

Every complex number  $z \in \mathbb{C}$ , and its complex conjugate  $\bar{z} \in \mathbb{C}$ , may be uniquely decomposed into a pair of real numbers  $x, y \in \mathbb{R}$  which are respectively referred to as the real and imaginary part of  $z$ ,

$$\begin{aligned} z &= x + iy & x &= \operatorname{Re}(z) = \frac{1}{2}(z + \bar{z}) \\ \bar{z} &= x - iy & y &= \operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z}) \end{aligned} \tag{1.2}$$

Thus, the set of complex numbers is isomorphic to the plane  $\mathbb{C} \approx \mathbb{R}^2$  and we often refer to  $\mathbb{C}$  as the *complex plane*, represented schematically in Figure 1. Addition and

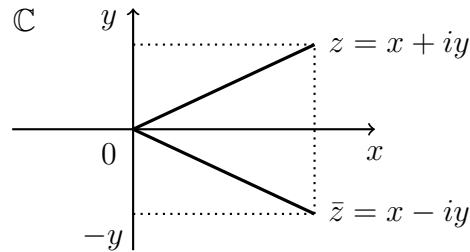


Figure 1: The isomorphism between  $\mathbb{C}$  and the plane  $\mathbb{R}^2$ .

multiplication of complex numbers  $z = x + iy$  and  $z' = x' + iy'$  proceeds as follows,

$$\begin{aligned} z + z' &= (x + x') + i(y + y') \\ z \cdot z' &= (xx' - yy') + i(xy' + x'y) \end{aligned} \tag{1.3}$$

These operations promote  $\mathbb{C}$  into a *commutative field*. The *modulus*  $|z|$  of  $z$  is defined by  $|z| = (z\bar{z})^{\frac{1}{2}}$  taking the positive branch of the square root  $|z| \geq 0$ . The modulus is positive definite, and satisfies the triangle inequality  $|z + z'| \leq |z| + |z'|$  for all  $z, z' \in \mathbb{C}$ . The modulus thus provides a norm and a distance function which induce the metric topology on  $\mathbb{C}$  whose open sets are open discs of arbitrary radius centered at arbitrary points in  $\mathbb{C}$  (see Section 2 for an introduction to topological spaces).

## 1.1 Holomorphic (or complex analytic) Functions

A function of a complex variable is a map from  $\mathbb{C}$  to  $\mathbb{C}$ . Equivalently it is a map from  $\mathbb{R}^2$  to  $\mathbb{R}^2$ , namely from  $(x, y) \in \mathbb{R}^2$  to  $(u, v) \in \mathbb{R}^2$ , given by  $u = u(x, y)$  and  $v = v(x, y)$ . Treating  $z$  and  $\bar{z}$  as *independent variables*, just as  $x$  and  $y$  are independent variables and performing a change of variables from  $(x, y)$  to  $(z, \bar{z})$  using (1.2) we collect the pair of real functions  $(u, v)$  into a single complex function  $f$  defined by,

$$f(z, \bar{z}) = u(x, y) + iv(x, y) \quad (1.4)$$

The function  $f$  is a general complex-valued function of the complex variables  $z, \bar{z}$ .

The change of variables (1.2) may be applied to the partial derivatives with respect to the variables  $x, y$  and  $z, \bar{z}$  and gives the following relations,

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_x - i\partial_y) & \partial_x &= \partial_z + \partial_{\bar{z}} \\ \partial_{\bar{z}} &= \frac{1}{2}(\partial_x + i\partial_y) & \partial_y &= i\partial_z - i\partial_{\bar{z}} \end{aligned} \quad (1.5)$$

where we systematically use the modern notation  $\partial_x = \partial/\partial x$ ,  $\partial_y = \partial/\partial y$  etc.

A complex-valued function  $f(z, \bar{z})$  is *holomorphic* or *complex analytic* in a region  $\mathcal{R}$  of the complex plane provided it satisfies the differential equation,

$$\partial_{\bar{z}}f(z, \bar{z}) = 0 \quad (1.6)$$

and the derivative  $\partial_z f(z, \bar{z})$  is continuous everywhere in  $\mathcal{R}$ . The condition (1.6) means that the function  $f$  is independent of the variable  $\bar{z}$  and only depends on  $z$  in the region  $\mathcal{R}$ . Translating this condition into differential equations on the real functions  $u, v$  using the change of variables (1.5), we obtain the *Cauchy-Riemann equations*,

$$\partial_x u - \partial_y v = 0 \quad \partial_x v + \partial_y u = 0 \quad (1.7)$$

Holomorphic functions are usually denoted by  $f(z)$  in which the variable  $\bar{z}$  is omitted since the function does not depend on it. The rules for differentiation in  $z$  are identical to the ones for differentiation of a function of a single real variable.

### 1.1.1 Single-valued functions

Examples of holomorphic functions may be obtained by replacing the real independent variable in some of the familiar functions by the complex variable  $z$ . It will be convenient to classify holomorphic functions according to the nature of their domain of holomorphicity, and the singularities they exhibit outside the domain of analyticity. In this subsection, we carry out this classification for *single-valued functions*  $f(z)$  which assign a unique value  $f(z)$  to every  $z \in \mathbb{C}$ . The possible behaviors of a function  $f(z)$  at a point  $z_0$  are as follows.

- A function  $f(z)$  is holomorphic at a *regular point*  $z_0$  if  $\partial_{\bar{z}}f = 0$  and  $\partial_z f(z)$  exists in an open neighborhood around the point  $z_0$ .
- A function  $f(z)$  has an *isolated singularity at a point*  $z_0$  when there exists a  $\delta > 0$  such that  $f$  is holomorphic inside the open disc of radius  $\delta$  centered at  $z_0$  minus the point  $z_0$  (*the disc with puncture*  $z_0$ ).
- The point  $z_0$  is a *pole of order*  $n \geq 1$  of  $f(z)$  if the limit  $\lim_{z \rightarrow z_0} (z - z_0)^n f(z)$  exists and is non-zero, so that  $f(z)$  is singular at  $z_0$ .
- A singular point  $z_0$  of a function  $f(z)$  which is not a pole is an *essential singularity*, sometimes also referred to as a pole of infinite order.

The main two classes of single-valued holomorphic functions that arise in view of the above classification of singularities are as follows.

- **Entire functions** are holomorphic throughout the complex plane  $\mathbb{C}$ .  
Examples are polynomials  $P(z)$  of arbitrary degree; the exponential function  $e^z$ ; the trigonometric functions  $\cos z$  and  $\sin z$ ; the hyperbolic functions  $\operatorname{ch}z$  and  $\operatorname{sh}z$ ; and compositions of all of the above, such as  $e^{\sin z}$  etc. Non-constant entire functions all blow up at  $\infty$  (as a corollary of Liouville's theorem, to be discussed in Section 3). We note the Euler relations,

$$e^{iz} = \cos z + i \sin z \qquad \operatorname{ch}(iz) = \cos z \qquad \operatorname{sh}(iz) = i \sin z \qquad (1.8)$$

- **Meromorphic functions** are holomorphic throughout the complex plane  $\mathbb{C}$ , except for *isolated poles of arbitrary but finite orders*.  
Meromorphic functions may be expressed as the ratio of two holomorphic functions. Examples are rational functions  $P(z)/Q(z)$  where  $P(z)$  and  $Q(z)$  are polynomials of arbitrary degree; trigonometric functions  $1/\cos z$ ,  $1/\sin z$ ,  $\tan z$ ; hyperbolic functions  $1/\operatorname{ch}z$ ,  $1/\operatorname{sh}z$ . The space of meromorphic functions forms a function field over  $\mathbb{C}$ . Compositions of meromorphic functions are not, however, necessarily meromorphic. This may be seen by considering the function  $1/\sin(\pi/z)$  which has poles at  $1/n$  for all  $n \in \mathbb{Z}$ , and these poles are not isolated. Another example is  $e^{1/z}$  which has an essential singularity at  $z = 0$ .

### 1.1.2 Multiple-valued functions and Riemann surfaces

The inverse function of a single-valued function is not necessarily a single-valued function and is generally *multiple-valued*. For example, consider the polynomial function  $f(z) = z^2$  which is holomorphic in  $\mathbb{C}$  and invariant under  $z \rightarrow -z$ . Its inverse function  $f^{-1}(z) = z^{\frac{1}{2}}$  is *double-valued* since at every non-zero value of  $f$  there correspond two different pre-images. More generally, the function  $f(z) = z^n$  with  $n$  integer and  $n > 2$

is invariant under  $z \rightarrow z e^{2\pi i/n}$ . Its inverse function  $f^{-1}(z) = z^{\frac{1}{n}}$  is  $n$ -valued since to every non-zero value of  $f$  there correspond  $n$  different pre-images. The inverse functions of general polynomials of degree  $n \geq 2$  and rational functions are likewise multiple-valued function, though these inverse functions are generally harder to write down explicitly. Exponential, trigonometric, and hyperbolic functions are periodic functions. For example, the single-valued function  $e^z$  is invariant under the infinite discrete group of shifts  $z \rightarrow z + 2\pi i k$  for any  $k \in \mathbb{Z}$  and therefore its inverse function the logarithm  $\ln(z)$  is infinitely multiple-valued.

There are two ways of handling multiple-valued functions.

- By introducing a *branch cut* in the complex plane and restricting the definition of the function to the *cut plane* in which the function is now holomorphic. For example, for the function  $f(z) = z^{\frac{1}{2}}$  we may cut the complex plane  $\mathbb{C}$  by removing the positive real axis, as shown in Figure 2. The branch cut is an arbitrary continuous curve starting at the point 0 and ending at  $\infty$ , which are both referred to as *branch points*.

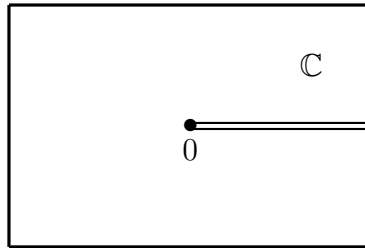


Figure 2: The branch cut for  $f(z) = z^{\frac{1}{2}}$  chosen along the positive real axis.

- By defining the function  $f$  on the union of two copies of the cut complex plane, referred to as *sheets*, such that the first sheet corresponds to one sign of the square root while the second sheet corresponds to the opposite sign. The sheets are *glued together* at the branch cut by smoothly moving from the first to the second sheet as one crosses the real positive axis, or vice-versa, as depicted in Figure 3. The resulting surface is a simple example of a *Riemann surface*. An arbitrary point on the surface may be labelled by the pair  $(z, w)$  with  $w^2 = z$ , where  $z$  is the standard coordinate on a single copy of the complex plan, and the inclusion of  $w$  in the data specifies the sheet.

More generally, the multiple-valued function  $f(z) = z^{\frac{1}{n}}$  for arbitrary  $n \geq 2$  may defined with the same branch cut along the positive real axis, but the corresponding Riemann surface now consists of  $n$  sheets, glued together at the branch cut.

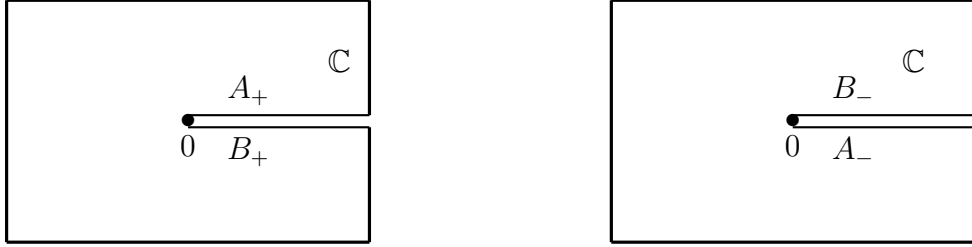


Figure 3: The double cover image of the function  $f(z) = z^{\frac{1}{2}}$  is obtained by sewing together two copies of the plane with a branch cut, and identifying sides  $A_{\pm}$  with one another, and sides  $B_{\pm}$  with one another.

For  $f(z) = \ln z$ , the Riemann surface consists of an infinite number of sheets glued together in an infinite winding staircase. Much more will be discussed about multiple-valued functions when we consider integration in Section 3.

## 1.2 Orthogonal families of curves - conformal mapping

Let  $f(z)$  be a holomorphic function with  $z = x + iy$  and  $f(z) = u(x, y) + iv(x, y)$ .

1. The functions  $u, v$  are harmonic,

$$(\partial_x^2 + \partial_y^2)u = 0 \qquad (\partial_x^2 + \partial_y^2)v = 0 \qquad (1.9)$$

2. The curves  $u(x, y) = \alpha$  and  $v(x, y) = \beta$  are orthogonal for each  $\alpha, \beta \in \mathbb{R}$  at all points  $z$  where  $f'(z) \neq 0$ .
3. The transformation (or map) from the variable  $z$  to the variable  $w = f(z)$  is *conformal* at all points  $z$  where  $f'(z) \neq 0$ .

Harmonicity follows from the Cauchy-Riemann equations obeyed by  $u, v$  along with the identity  $\partial_x \partial_y = \partial_y \partial_x$ . The curves are orthogonal provided their respective tangent vectors are orthogonal to one another at an arbitrary point. The tangent vectors  $\mathbf{t}^\alpha = (t_x^\alpha, t_y^\alpha)$  and  $\mathbf{t}^\beta = (t_x^\beta, t_y^\beta)$  obey the following equations,

$$\begin{aligned} u(x + \varepsilon t_x^\alpha, y + \varepsilon t_y^\alpha) &= u(x, y) + \mathcal{O}(\varepsilon^2) \\ v(x + \varepsilon t_x^\beta, y + \varepsilon t_y^\beta) &= v(x, y) + \mathcal{O}(\varepsilon^2) \end{aligned} \qquad (1.10)$$

Expanding to first order in  $\varepsilon$ , and using the Cauchy-Riemann equations to eliminate the  $v$ -derivatives in terms of the  $u$ -derivatives,

$$\begin{aligned} t_x^\alpha \partial_x u + t_y^\alpha \partial_y u &= 0 \\ -t_x^\beta \partial_y u + t_y^\beta \partial_x u &= 0 \end{aligned} \qquad (1.11)$$

Since the vectors  $(\partial_x u, \partial_y u)$  and  $(-\partial_y u, \partial_x u)$  are non-vanishing in view of the assumption  $f'(z) \neq 0$ , and are manifestly orthogonal to one another, it follows that  $\mathbf{t}^\alpha$  and  $\mathbf{t}^\beta$  must be orthogonal to one another. Hence the curves are orthogonal to one another.

To show item 3. we recall that, in any dimension  $n$ , a map is conformal at a point  $x$  provided it preserves the angles between arbitrary vectors at  $x$ . To define angles it suffices to have a metric by which one can measure distances and from distances deduce angles. The flat space Euclidean metric on  $\mathbb{R}^n$  is given by  $ds_1^2 = \sum_{i=1}^n dx^i dx^i$  (on a curved space we use a general Riemannian metric). Scaling the metric by an overall scalar function is referred to as a *Weyl transformation* resulting in a metric  $ds_\rho^2 = \rho^2(x) \sum_{i=1}^n dx^i dx^i$ . We shall now show that a Weyl transformation is conformal. To do so, we consider the inner product of two arbitrary vectors  $U(x) = (U^1, \dots, U^n)$  and  $V(x) = (V^1, \dots, V^n)$  at the point  $x$  is given by,

$$U(x) \cdot V(x) = \rho^2(x) \sum_{i=1}^n U^i V^i \quad (1.12)$$

The angle  $\theta$  between the vectors  $U(x)$  and  $V(x)$  is given by,

$$\cos \theta = \frac{U(x) \cdot V(x)}{(U \cdot U)^{\frac{1}{2}} (V \cdot V)^{\frac{1}{2}}} \quad (1.13)$$

which is independent of the Weyl factor  $\rho^2$ . Returning now to the case at hand of the flat two-dimensional plane, the flat metric is  $ds_1^2 = dx^2 + dy^2$  and its Weyl-rescaled form is  $ds_\rho^2 = \rho(x, y)^2(dx^2 + dy^2)$ . Expressing the metrics in complex coordinates we have  $ds_1^2 = |dz|^2$  and  $ds_\rho^2 = \rho(z, \bar{z})^2 |dz|^2$ . Now carry out the transformation from the variable  $z$  to the variables  $w = f(z)$  with  $f(z)$  holomorphic and  $f'(z) \neq 0$ . The transformation on the differential is  $dw = f'(z)dz$  and on the metric is as follows,

$$|dw|^2 = |f'(z)|^2 |dz|^2 \quad (1.14)$$

This transformation is conformal at any point where  $f'(z) \neq 0$ , thus proving item 3.

### 1.2.1 Examples of conformal mappings

Conformal mappings allow one to relate holomorphic and harmonic functions on different domains in the complex plane. *Global conformal transformations* constitute a special set of conformal transformations which are given by a Möbius transformation,

$$f(z) = \frac{az + b}{cz + d} \quad f'(z) = \frac{1}{(cz + d)^2} \quad ad - bc = 1 \quad (1.15)$$



Note that  $f'(z)$  is non-zero throughout the complex plane and thus globally conformal. Representing the data in the form of a matrix,

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det F = 1 \quad (1.16)$$

the composition of two Möbius transformations corresponds to the multiplication of the corresponding matrices,  $(f_1 \circ f_2)(z)$  corresponds to the matrix  $F_1 F_2$ . The set of such matrices of unit determinant forms a group under multiplication, denoted by  $SL(2, \mathbb{C})$  when  $a, b, c, d \in \mathbb{C}$ ,  $SL(2, \mathbb{R})$  when  $a, b, c, d \in \mathbb{R}$ , and  $SL(2, \mathbb{Z})$  when  $a, b, c, d \in \mathbb{Z}$ . There are further an infinity of other possibilities, which are all subgroups of  $SL(2, \mathbb{Z})$  and referred to as arithmetic groups.

An key global conformal map is from the upper half plane  $H = \{z \in \mathbb{C}, \text{Im}(z) > 0\}$  to the unit disc centered at zero  $D = \{w \in \mathbb{C}, |w| < 1\}$ ,

$$w(z) = \frac{i - z}{i + z} \quad z = i \frac{1 - w}{1 + w} \quad (1.17)$$

Note that this transformation maps the real line into the unit circle, the point  $z = i$  to the center of the disc  $w = 0$  and  $z = \infty$  to the point  $w = 1$ . Examples of global conformal transformations other than Möbius transformations are as follows.

- The map  $w = e^z$  from the cylinder  $\{z \in \mathbb{C}, z \sim z + 2\pi, \ell < \text{Im}(z) < L\}$  to the annulus  $\{w \in \mathbb{C}, e^\ell < |w| < e^L\}$ , where  $\sim$  stands for periodic identification;
- The map  $w = e^z$  from the infinite strip  $\{z \in \mathbb{C}, 0 < \text{Im}(z) < \pi\}$  to the upper half plane  $w$ .

Some transformations are conformal except at isolated points. The example from which most others are constructed is given by the map  $w = z^\mu$  for  $\mu \in \mathbb{R}$  and  $\mu \geq \frac{1}{2}$  from a wedge  $\{z \in \mathbb{C}, z = \rho e^{i\theta}, 0 < \theta < \pi/\mu\}$  to the upper half plane. The transformation fails to be conformal at  $z = 0$  which allows the vertex of the wedge of opening angle  $\alpha = \pi/\mu \leq 2\pi$  to be mapped onto a point on the real axis where the opening angle is  $\pi$ . We may express the inverse transformation as follows,

$$z = w^{\alpha/\pi} \quad dz = \frac{\alpha}{\pi} w^{\alpha/\pi-1} dw \quad (1.18)$$

Consider a general planar polygon, with  $n$  vertices  $z_1, \dots, z_n \in \mathbb{C}$  with opening angles  $\alpha_1, \dots, \alpha_n$  ordered along the boundary of the polygon, as depicted for the case  $n = 5$  in Figure 4. The *Schwarz-Christoffel transformation* maps this polygon onto the upper half  $w$ -plane, with the vertices  $z_i$  mapped to an ordered set of points  $x_i$  on the real line, and  $w$  and  $z$  related by the differential relation,

$$dz = A \prod_{i=1}^n (w - x_i)^{\alpha_i/\pi-1} dw \quad (1.19)$$

where  $A$  is a constant independent of  $z, w$ . To prove this formula, we make use of the argument function of a complex number, defined by,

$$\text{Arg}(z) = \theta \quad z = |z| e^{i\theta} \quad (1.20)$$

The Arg-functions acts like a logarithm, so that  $\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$  and  $\text{Arg}(z^\mu) = \mu \text{Arg}(z)$ . Applying Arg to bot sides of (1.18), we have,

$$\text{Arg}(dz) = \text{Arg}(dw) + \text{Arg}(A) + \sum_{i=1}^n \left( \frac{\alpha_i}{\pi} - 1 \right) \text{Arg}(w - x_i) \quad (1.21)$$

Moving  $w$  along the real line from right to left starting from  $+\infty$ , the function  $\text{Arg}(w - x_i)$  vanishes for  $w > x_i$  and equals  $\pi$  for  $w < x_i$ . Thus, as we move between two consecutive points  $x_i$  and  $x_{i+1}$ , the value of  $\text{Arg}(dz)$  remains constant, i.e. the slope of the line followed by  $z$  is constant. Crossing a point  $x_i$  increases  $\text{Arg}(dz)$  by  $\alpha_i - \pi$ , just as in the case of a single wedge. The fact that the sum of the angles  $\alpha_i$  is  $2\pi$  guarantees that the polygon closes.

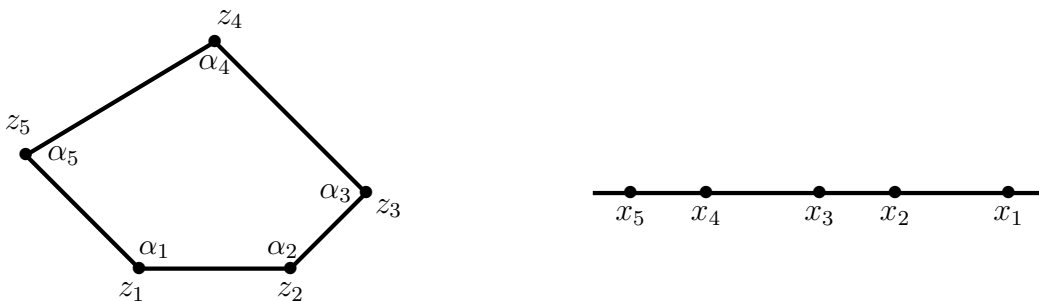


Figure 4: Vertices  $z_i$ , edges  $[z_i, z_{i+1}]$  and angles  $\alpha_i$  between consecutive edges for a planar pentagon in the case  $n = 5$  are mapped onto the real line with marked points  $x_i$  so that the interior of the polygon is mapped into the upper half plane.

For example, for the case of a rectangle, we have  $n = 4$  and  $\alpha_i = \frac{\pi}{2}$  for  $i = 1, \dots, 4$  so that the transformation is given by,

$$dz = \frac{A dw}{\sqrt{(w - x_1)(w - x_2)(w - x_3)(w - x_4)}} \quad (1.22)$$

Its solution  $z = f^{-1}(w)$  is given by an elliptic integral whose inverse  $w = f(z)$  by an elliptic function, to be discussed later.

### 1.3 Two-dimensional electrostatics and fluid flows

The mathematical problems of solving for two-dimensional electrostatics and stationary fluid flows are identical and reduce to obtaining harmonic functions in the presence of sources. Concentrating on electrostatics we seek the electro-static potential  $\Phi(x, y)$  which is a real function satisfying the Poisson equation,

$$(\partial_x^2 + \partial_y^2)\Phi(x, y) = -2\pi\rho(x, y) \quad (1.23)$$

where  $\rho(x, y)$  is the density of electric charges, and  $(-\partial_x\Phi, -\partial_y\Phi)$  is the associated electric field. Away from electric charges,  $\Phi$  is harmonic. The potential for an electric charge of strength  $q$  located at the point  $(x_0, y_0)$  is given by,

$$\Phi_0(x, y) = -q \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} \quad (1.24)$$

Complex analysis is particularly useful to solve electrostatics problems when the charge distribution is supported by isolated point-charges, or by charges distributed on a curve or line interval. In those cases, the electric potential is harmonic in the bulk of the plane. We now introduce a complex potential  $\Omega$ , defined by,

$$\Omega(x, y) = \Phi(x, y) + i\Psi(x, y) \quad (1.25)$$

Requiring  $\Omega$  to be holomorphic guarantees that  $\Phi$  and  $\Psi$  are conjugate harmonic functions satisfying the Cauchy-Riemann equations. The lines of constant  $\Phi$  are electric equipotentials, while the lines of constant  $\Psi$  are electric field lines. For a distribution of  $n$  point charges with strengths  $q_i$  and positions  $z_i$  in the complex plane for  $i = 1, \dots, n$ , the complex potential is given by,

$$\Omega(z) = - \sum_{i=1}^n q_i \ln(z - z_i) \quad (1.26)$$

To obtain a charge distribution along a line segment, one may take the limit as  $n \rightarrow \infty$  and transform the above sum into a line integral.

#### 1.3.1 Example of conformal mapping in electrostatics

As an example, we want to obtain the electrostatic potential for an array of  $n$  charges with strength  $q_i$  located at points  $w_i$  in the interior of the unit disc  $|w| < 1$  whose boundary unit circle is grounded at zero potential. The problem is linear, so it suffices to solve it for a single charge and then take the linear superposition of  $n$  charges. It is easy to solve this problem for the upper half plane with complex coordinate  $w$ . By introducing an opposite image charge we ground the real axis to zero potential,

$$\begin{aligned} \Omega_i^H(z) &= -q_i \ln(z - z_i) + q_i \ln(z - \bar{z}_i) \\ \Phi_i^H(z, \bar{z}) &= -q_i \ln |z - z_i|^2 + q_i \ln |z - \bar{z}_i|^2 \end{aligned} \quad (1.27)$$

where it is clear from the second line that  $\Phi = 0$  for  $z \in \mathbb{R}$ . Now make the conformal transformation (1.17) from the upper half  $z$ -plane to the unit  $w$ -disc by setting  $\Phi_i^D(w, \bar{w}) = \Phi_i^H(z, \bar{z})$ , and we find,

$$\Phi_i^D(w, \bar{w}) = -q_i \ln \left| \frac{1-w}{1+w} - \frac{1-w_i}{1+w_i} \right|^2 + q_i \ln \left| \frac{1-w}{1+w} + \frac{1-\bar{w}_i}{1+\bar{w}_i} \right|^2 \quad (1.28)$$

After some evident simplifications, we obtain the following equivalent form,

$$\Phi_i^D(w, \bar{w}) = q_i \ln \left| \frac{1-w\bar{w}_i}{w-w_i} \right|^2 \quad (1.29)$$

a result one would probably not have guessed so easily. One verifies that  $\Phi_i^D$  indeed vanishes for  $|w| = 1$ . The electric potential  $\Phi^D$  on the grounded disc with  $n$  charges is the linear superposition of the contributions for one charge,

$$\Phi^D(w, \bar{w}) = \sum_{i=1}^n \Phi_i^D(w, \bar{w}) \quad (1.30)$$

Thus we have solve a problem on the unit disc by mapping it to a problem on the upper half where the solution may be obtained much more simply than on the disc by using the symmetrical disposition of the image charges, as illustrated in Figure 5.

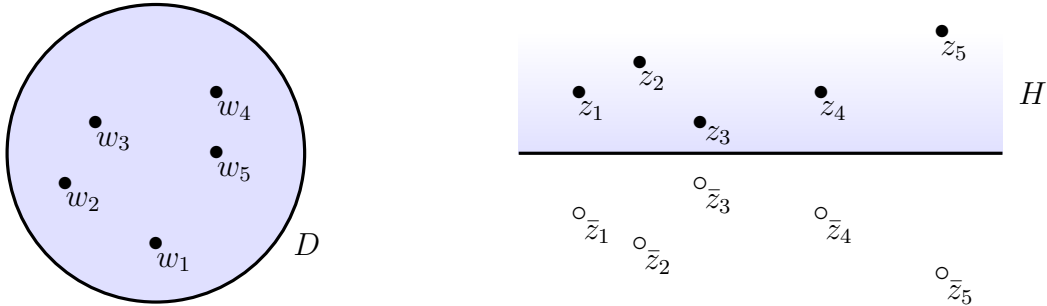


Figure 5: The unit disc  $D$  is on the left with vanishing potential on the unit circle and point charges at the points  $w_i$ . The upper half plane  $H$  is on the right with vanishing potential on the real line, charges at  $z_i$  in the upper half plane indicated bold dots and opposite image charges at  $\bar{z}_i$  in the lower half plane indicated in circles.

## 2 Topological Interlude

Topology is the modern language in terms of which analysis, including the local properties of functions such as continuity, differentiability, and integrability are formulated. It is also the language in which shapes of sets can be grouped into equivalence classes under the equivalence of being related to one another by a continuous deformation. We shall introduce here some of the most basic definitions and results in topology.

### 2.1 Basic definitions

We begin with some very basic definitions.

**Topology:** A class of subsets of a non-empty set  $X$  is a *topology*  $\mathcal{T}$  if and only if the following axioms are satisfied,

1.  $X$  itself and the empty set  $\emptyset$  belong to  $\mathcal{T}$ ;
2. The union of any number of sets in  $\mathcal{T}$  belongs to  $\mathcal{T}$ ;
3. the intersection of any *finite* number of sets in  $\mathcal{T}$  belongs to  $\mathcal{T}$ .

When these axioms are satisfied, the members of  $\mathcal{T}$  are referred to as *open sets of  $\mathcal{T}$*  and the pair  $(X, \mathcal{T})$  is referred to as a *topological space*. Note that a given space  $X$  may be endowed with different topologies. One example is the *trivial topology* which consists of  $\mathcal{T} = \{X, \emptyset\}$  for example, but this topology is essentially useless. If  $\mathcal{O}$  is an open set, which is not equal to  $X$  or  $\emptyset$ , then the complement  $X \setminus \mathcal{O}$  is a *closed set* and vice-versa.

The definition given above is incredibly general, which gives it great flexible power. In practice, given a space  $X$ , we need some concrete prescription for describing its open sets which will be useful for doing analysis. Almost always in physics, one has some *metric* or *distance function* available.

**Metric topology:** The general definition of a metric is a real-valued function  $d(x, y)$  for  $x, y \in X$  which satisfies the following axioms,

1. symmetry:  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
2. positivity:  $d(x, y) \geq 0$  for all  $x, y \in X$ ;
3. definiteness:  $d(x, y) = 0$  if and only if  $x = y$ ;
4. triangle inequality:  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

An important special case is when we are dealing with a vector space which is endowed with a norm. In this case we can take the distance function to be simply the norm of the difference between the two vectors. Thus, for example in  $\mathbb{R}$  we have the

norm=distance function  $|x - y|$  for  $x, y \in \mathbb{R}$ ; in  $\mathbb{C}$  the modulus is given by the modulus  $z - w$  with  $z, w \in \mathbb{C}$ ; and more generally in  $\mathbb{R}^n$  and  $\mathbb{C}^n$  we have the norms,

$$\|(x_1, \dots, x_n)\|^2 = \sum_{i=1}^n x_i^2 \qquad \|(z_1, \dots, z_n)\|^2 = \sum_{i=1}^n |z_i|^2 \qquad (2.1)$$

In fact one may extend these norms to infinite-dimensional spaces and we then enter the subject of Hilbert spaces which we shall discuss in section 5. A set equipped with a metric is referred to as a *metric space*.

Given a metric  $d(x, y)$  on a set  $X$ , we may define open sets to be *open balls* of arbitrary radius  $\varepsilon > 0$  centered at arbitrary points  $x_0$ ,

$$D_\varepsilon(x_0) = \{x \in X, d(x, x_0) < \varepsilon\} \qquad (2.2)$$

Note the crucial strict inequality in this definition. The *closed ball* corresponds to replacing the strict inequality by  $d(x, x_0) \leq \varepsilon$ . The class of open balls does not by itself define a topology because for example the intersection and union of two open balls is not necessarily an open ball. But the open balls form a *basis for a topology*: the other open sets may be obtained by applying the rules for union and intersection, given in the definition of a topology, to the open balls. By this process, one constructs the *metric topology*  $\mathcal{T}$  associated with the metric  $d(x, y)$  on  $X$ .

For example, the open sets of the metric topology of  $\mathbb{R}$  are the open intervals  $]a, b[$  with  $a < b \in \mathbb{R}$  and all possible unions thereof. The open sets of the metric topology of  $\mathbb{C}$  are generated by the open discs  $D_\varepsilon(z_0)$  for arbitrary radius  $\varepsilon > 0$  and center  $z_0$ .

Note that there is a very good reason for the condition of taking the intersection of only a finite number of open sets in the third axiom of a topology. Consider for example the open sets  $D_{\frac{2}{n}}(0) = ]-\frac{1}{n}, +\frac{1}{n}[$  for  $n \in \mathbb{N}$ . For each value of  $n$ , the point 0 is contained in  $D_{\frac{2}{n}}(0)$ , so that the infinite intersection  $\bigcap_{n=1}^{\infty} ]-\frac{1}{n}, +\frac{1}{n}[ = \{0\}$  but this set is not open; in fact it is closed !

**Accumulation point:** A point  $x$  is an *accumulation point* of a subset  $A \in X$  if every open set  $\mathcal{O}$  which contains  $x$  contains at least one point of  $A$  different from  $x$ . For example, the point 0 is an accumulation point of the set of points  $\{\frac{1}{n}\}_{n \in \mathbb{N}}$  on the real line. Also, the points  $a$  and  $b$  are accumulation points of the open interval  $]a, b[$  for  $a < b \in \mathbb{R}$ .

**Closure of a set:** Given a subset  $A$  of a topological space  $X$ , its closure  $\bar{A}$  is the union of  $A$  and of all its accumulation points. For example, the closure of the open interval  $]a, b[$  for  $a < b \in \mathbb{R}$  is the interval with its two accumulation point included, which makes it into the interval  $[a, b]$  which is closed, since its complement is open.

**Interior, exterior, boundary:** An point  $x$  which belongs to a subset  $A$  of a topological space  $X$  is an interior point of  $A$  if  $x$  belongs to an open set  $\mathcal{O} \subset A$ . The

set of all interior points of  $A$  is the interior of  $\text{int}(A)$  of  $A$ . The exterior  $\text{rnext}(A)$  is the interior of the complement of  $A$ , and the boundary is the set of point which are neither interior nor exterior to  $A$ , given by,  $\partial A = X \setminus (\text{int}(A) \cup \text{ext}(A))$ . For example, the interior, exterior, and boundary of the subset  $A = ]a, b]$  are respectively  $\text{int}(A) = ]a, b[$ ,  $\text{ext}(A) = ] - \infty, a[ \cup ] b, \infty[$  and  $\partial A = \{a, b\}$ .

## 2.2 Sequences

Consider a topological space  $X$ . An infinite sequence in  $X$  is a set of points  $x_n \in X$  indexed and ordered by the integers  $n \in \mathbb{N}$ , or an infinite subset thereof. This set of point may have zero, one, or several accumulation points. More specifically, the sequence may or may not converge. There are two convergence criteria which are both important and widely used.

- **Convergence to a point in a topological space  $X$ :** A sequence of points  $\{x_n\}_{n \in \mathbb{N}}$  with  $x_n \in X$  converges to a point  $x$  if and only if for every open set  $\mathcal{O}$  containing  $x$  there exists an  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $x_n \in \mathcal{O}$ .
- **Cauchy sequences in a metric space  $X$ :** A sequence  $\{x_n\}_{n \in \mathbb{N}}$  with  $x_n \in X$  is a Cauchy sequence if for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $m, n > N$  we have  $d(x_m, x_n) < \varepsilon$ . A fundamental result is that every convergent sequence is a Cauchy sequence. Thus, the notion of Cauchy sequence is more general than that of a convergent sequence. A metric space  $X$  is *complete* if every Cauchy sequence in  $X$  converges to a point in  $X$ . The spaces  $\mathbb{R}$  and  $\mathbb{C}$  and more generally  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are all complete metric spaces.

## 2.3 Continuous functions

Let  $X$  and  $Y$  be two topological spaces (their respective topologies will be understood throughout), and let  $f : X \rightarrow Y$  be a function from  $X$  to  $Y$ . The function  $f$  is *continuous* if the image under the inverse function  $f^{-1}$  of every open set of the topology of  $Y$  is an open set of the topology of  $X$ . In fact, it suffices to require that  $f^{-1}$  of every open set of a basis of open sets of  $Y$  is an open set of  $X$ .

For example, in the case of a real-valued function  $f(x)$  on a subset  $A$  of the real line, we can make this condition completely explicit. The open sets are generated by the open intervals  $]a, b[$  contained by  $A$ . The criterion is that  $f^{-1}(]a, b[)$  is an open interval for ever  $a < b \in \mathbb{R}$ . Let us characterize the interval instead by a center point  $x_0$  and a radius  $\varepsilon > 0$ , so that now the criterion is that for all  $\varepsilon > 0$  and for all corresponding  $x_0$  with  $D_\varepsilon(x_0) \subset A$  and such that  $|f(x) - f(x_0)| < \varepsilon$  the point  $x$  is in an open set around the point  $x_0$  which means that there exists some  $\delta > 0$  such that

$|x - x_0| < \delta$ . We have just recovered the well-known Weierstrass characterization of a real continuous function of a real variable. But the characterization by open sets is much more general and useful.

## 2.4 Connectedness

A subset  $A$  of a topological space  $X$  is *connected* if any two points  $x, y \in A$  can be joined by a continuous function  $f : [0, 1] \rightarrow A$  given by  $f(s)$  for  $s \in [0, 1]$ , such that  $f(0) = x$  and  $f(1) = y$ . Note the criteria of continuity and that the image of  $[0, 1]$  must be entirely contained in  $A$ . A set  $A$  which is not connected is *disconnected*. The set of all disconnected components of  $A$  is denoted by  $\pi_0(A)$  and referred to as the *zero-th homotopy of  $A$* .

A discrete set  $A$  of points  $x_n \in X$  with  $n \in \mathbb{N}$  is disconnected, and  $\pi_0(A) = A$ . An interval  $[a, b]$  with  $a < b$  is connected. The union of two intervals  $[a, b] \cup [c, d]$  with  $a < b$  and  $c < d$  is connected when  $c \leq b$  but disconnected when  $c > b$ . A disc  $D_\varepsilon(x_0)$  in  $\mathbb{R}^n$  of radius  $R > 0$  centered at an arbitrary point  $x_0$  is connected.

## 2.5 Simply-connectedness

This notion will be very important in complex analysis. Let  $A$  be a connected subset of a topological space  $X$ . A *closed curve*  $\mathcal{C} \subset A$  is given by a continuous function  $\mathcal{C} : [0, 1] \rightarrow A$  such that  $\mathcal{C}(0) = \mathcal{C}(1)$ . A subset  $A$  is *simply-connected* if every closed curve in  $A$  can be continuously deformed, while remaining in  $A$ , to a point. To make this more explicit, a closed curve  $\mathcal{C}$  can be deformed to a point  $p$  provided there exists a continuous function  $\hat{\mathcal{C}}$  from  $[0, 1] \times [0, 1] \rightarrow A$  such that,

$$\hat{\mathcal{C}}(s, 1) = \mathcal{C}(s) \qquad \hat{\mathcal{C}}(s, 0) = p \qquad s \in [0, 1] \qquad (2.3)$$

Subsets which are not simply-connected are also very important, and one can provide a measure by how much they fail to be simply connected. The key ingredient is the notion of *homotopy*. Two curves  $\mathcal{C}_0$  and  $\mathcal{C}_1$  in  $A$  which have the same end points  $p, q$  are homotopic to one another provided there exists a continuous function  $\hat{\mathcal{C}} : [0, 1] \times [0, 1] \rightarrow A$  such that,

$$\begin{aligned} \hat{\mathcal{C}}(s, 1) &= \mathcal{C}_1(s) & \hat{\mathcal{C}}(1, t) &= \mathcal{C}_1(1) = \mathcal{C}_0(1) = p \\ \hat{\mathcal{C}}(s, 0) &= \mathcal{C}_0(s) & \hat{\mathcal{C}}(0, t) &= \mathcal{C}_1(0) = \mathcal{C}_0(0) = q \end{aligned} \qquad (2.4)$$

Homotopy induces an equivalence relation between curves, either with the same end-points or between closed curves in  $A$ . Thus we may state the condition of simply-connectedness as the fact that all closed curves in  $A$  are homotopic to points.



The equivalence classes of closed curves in  $A$  under the homotopy equivalence relation are referred to as the elements of the *first homotopy group*  $\pi_1(A)$ . The classes form a group under composition of curves by choosing two representatives  $\mathcal{C}_1$  and  $\mathcal{C}_2$  which have a point  $p$  in common and then composing the curves at the point  $p$ .

For example, the real line  $\mathbb{R}$ , the complex plane  $\mathbb{C}$ , as well as  $\mathbb{R}^n$  and  $\mathbb{C}^n$  are all simply-connected. The sphere  $S^n$  with  $n \geq 2$  are all simply connected, but the circle  $S^1$ , the annulus, and the  $n$ -dimensional torus  $T^n$  are not simply-connected. Their first homotopy groups are respectively  $\pi_1(S^1) = \mathbb{Z}$ ,  $\pi_1(A) = \mathbb{Z}$  and  $\pi_1(T^n) = \mathbb{Z}^n$ .

## 2.6 Compactness

We begin by defining a *cover* of a subset  $A$  of a topological space  $X$  as a class of open set  $\mathcal{O}_n$  with  $n \in \mathbb{N}$  such that  $A \subset \bigcup \mathcal{O}_n$ .

**Compactness:** A subspace  $A$  of a topological space is *compact* if for every cover  $\{\mathcal{O}_n\}_{n \in \mathbb{N}}$  one can extract a finite sub-cover.

**Heine-Borel theorem:** A subset of Euclidean  $\mathbb{R}^n$  is compact if and only if it is closed and bounded in the sense of the Euclidean metric.

The Heine-Borel theorem does not extend to infinite dimensions. The closed unit ball in finite dimension is compact by the Heine-Borel theorem, but in infinite dimension (say a Hilbert space), the closed unit ball is not compact.

One of the many key properties of a compact space is that a continuous function maps a compact set to a compact set. In particular this means that a continuous real function on a compact space  $A$  attains its minimum and its maximum in  $A$ .

## 2.7 Manifolds

**Definition of a topological manifold:** A topological space  $M$  is a topological manifold provided every point  $x \in M$  has an open neighborhood which is homeomorphic to an open set in  $\mathbb{R}^n$ .

A map is a homeomorphism provided it is bijective, continuous, and its inverse map is also continuous. By continuity, the value of  $n$  must be constant throughout the manifold, and  $n$  is referred to as the *dimension of  $M$* .

Some important aspects of manifolds are as follows.

1. The homeomorphism  $\varphi_\alpha$  from an open neighborhood  $\mathcal{U}_\alpha$  of a point  $x$  into  $\mathbb{R}^n$  provides “local coordinates” for the neighborhood  $\mathcal{U}_\alpha \subset M$ .
2. The pair  $(\mathcal{U}_\alpha, \varphi_\alpha)$  is called a *chart* (think of a geographical map).

3. An *atlas*  $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha \in S}$  of charts is a set of charts such that

$$\bigcup_{\alpha \in S} \mathcal{U}_\alpha = M \quad (2.5)$$

namely all points in  $M$  are in at least one chart, and may thus be described by at least one local coordinate system.

4. Since the  $\mathcal{U}_\alpha$  are open sets, the only way they can cover  $M$  is by having non-trivial intersection. The only way two open sets  $\mathcal{U}_\alpha$  and  $\mathcal{U}_\beta$  can intersect non-trivially is in an open set  $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ . But in this open set we now have two different homeomorphisms (i.e. coordinate sets) for the same point  $x$ , namely  $\varphi_\alpha(x)$  and  $\varphi_\beta(x)$ . Hence we can form the composition,

$$\psi_{\alpha,\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (2.6)$$

Since the functions  $\varphi_\alpha$  and  $\varphi_\beta$  and their inverses are continuous, so must the composite map  $\psi_{\alpha,\beta}$  then also be automatically be continuous. The functions  $\psi_{\alpha,\beta}$  are referred to as *transition functions*.

While it follows from the definition of a topological manifold that the transition functions  $\psi_{\alpha,\beta}$  are continuous, we may impose conditions on  $\psi_{\alpha,\beta}$  which are stronger than mere continuity. Here are some of the most frequently used extra conditions.

- *Differentiable manifold* or more precisely  $C^k$  differentiable manifold provided the order  $k$  derivatives of the transition functions exist and are continuous;
- *Real analytic manifold* provided the transition functions are real analytic, i.e. given by their Taylor series;
- *Complex manifold* exists provided  $n = 2m$  is even, and the transition functions are holomorphic functions of several complex variables.
- *Riemann surfaces* are complex manifold of two real, or one complex dimension with holomorphic transition functions. When we speak of *gluing* different sheets together, we mean that the transition functions used to do so are holomorphic.

### 3 Complex Analysis

We now return to complex analysis.

#### 3.1 Line integrals, Green's theorem

In this subsection, we shall define the integral of a holomorphic function  $f(z)$  in the variable  $z$  along an oriented curve  $\mathcal{C}$  which is the boundary of a region  $\mathcal{R} \subset \mathbb{C}$ , as depicted for example in Figure 6.

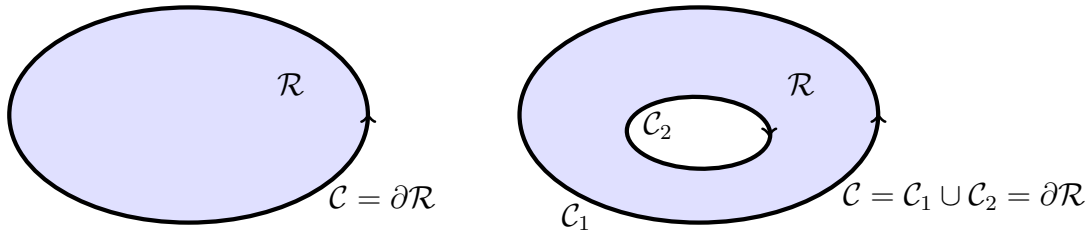


Figure 6: Examples of regions  $\mathcal{R}$  and their boundaries. The orientations of the inner and outer boundary curves are respectively clockwise and counter-clockwise.

**Green's theorem:** Let  $p(x, y)$  and  $q(x, y)$  be continuous functions of  $x, y$  with continuous first order partial derivatives in a region  $\mathcal{R}$  and on its boundary  $\mathcal{C} = \partial\mathcal{R}$ . The line integral of the differential one-form  $p(x, y)dx + q(x, y)dy$  along  $\mathcal{C}$  satisfies,

$$\oint_{\mathcal{C}} (p dx + q dy) = \int_{\mathcal{R}} dx dy (\partial_x q - \partial_y p) \quad (3.1)$$

where the orientation of the boundary curve  $\mathcal{C} = \partial\mathcal{R}$  is such that traveling along the boundary with positive orientation leaves the interior on  $\mathcal{R}$  to the left. The theorem is valid for regions  $\mathcal{R}$  that may be connected or have several disconnected components, and that are simply-connected or not simply connected.

To prove Green's theorem, we partition  $\mathcal{R}$  into rectangular coordinate regions, given by  $\mathcal{R} = \{(x, y), x_1 \leq x \leq x_2 \text{ and } y_1 \leq y \leq y_2\}$ . For each rectangular region, the theorem is proven by integrating the right side of (3.1),

$$\begin{aligned} \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy (\partial_x q - \partial_y p) &= \int_{y_1}^{y_2} dy (q(x_2, y) - q(x_1, y)) \\ &\quad + \int_{x_1}^{x_2} dx (q(x, y_1) - q(x, y_2)) \end{aligned} \quad (3.2)$$

which produces the line integral of  $p(x, y)dx + q(x, y)dy$  along the closed curve  $\mathcal{C}$ . The process of partitioning  $\mathcal{R}$  into a union of coordinate rectangles generally involves a

limit of small rectangles near the boundary. To avoid this limit, one may equivalently proceed to including also triangles with two coordinate sides and the third side being a monotonic boundary curve which is differentiable.

Equivalently, Green's theorem may be formulated in terms of a general complex function  $f(z, \bar{z})$  which is continuous with continuous first order partial derivatives in  $\mathcal{R}$  and on its boundary  $\mathcal{C} = \partial\mathcal{R}$ ,

$$\oint_{\mathcal{C}} dz f(z, \bar{z}) = 2i \int_{\mathcal{R}} dx dy \partial_{\bar{z}} f(z, \bar{z}) \quad (3.3)$$

To prove (3.3), we recast the left in terms of real variables using (1.2) and (1.4),

$$\oint_{\mathcal{C}} dz f(z, \bar{z}) = \oint_{\mathcal{C}} (dx u - dy v) + i \oint_{\mathcal{C}} (dx v + dy u) \quad (3.4)$$

and then use (3.1) on the real and imaginary parts of the result.

## 3.2 Complex integration

Line integrals of holomorphic functions behave as ordinary integrals, just as their derivatives behave as the derivatives of functions of a single real variable. To make this correspondence precise we begin by discussing Cauchy's theorem.

**Cauchy's Theorem:** *A function  $f(z)$  which is holomorphic in a region  $\mathcal{R}$  and on its boundary curve  $\mathcal{C}$  satisfies,*

$$\oint_{\mathcal{C}} dz f(z) = 0 \quad (3.5)$$

The theorem follows from Green's theorem by taking  $f$  holomorphic.

An immediate consequence of Cauchy's theorem may be deduced for open line integrals. We now consider a more restricted setting where the region  $\mathcal{R}$  is connected and simply connected. Recall that a region is simply connected provided every closed curve in  $\mathcal{R}$  can be continuously shrunk to a point through  $\mathcal{R}$ , as illustrated for the planar case in Figure 7.

Let  $f(z)$  be a holomorphic function in a connected and simply-connected region  $\mathcal{R}$  and let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two oriented curves in  $\mathcal{R}$  both of which begin at the point  $z_1 \in \mathcal{R}$  and end at the point  $z_2 \in \mathcal{R}$ . The integrals of  $f$  along the curves  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are then equal to one another and the integrals depend only on the end-points  $z_1, z_2$ ,

$$\int_{\mathcal{C}_1} dz f(z) = \int_{\mathcal{C}_2} dz f(z) = \int_{z_1}^{z_2} dz f(z) \quad (3.6)$$

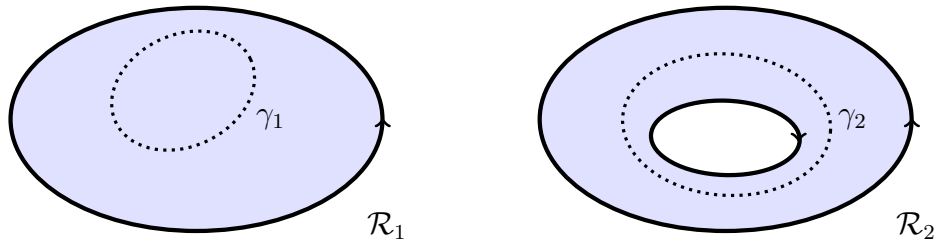


Figure 7: The shaded region  $\mathcal{R}_1$  is simply connected as any closed curve  $\gamma_1$  can be continuously shrunk to a point through  $\mathcal{R}$ , while the shaded region  $\mathcal{R}_2$  is non-simply-connected since the curve  $\gamma_2$  cannot be continuously shrunk to a point through  $\mathcal{R}$ .

The result is readily proven by noting that the difference of the integrals over  $\mathcal{C}_1$  and  $\mathcal{C}_2$  is an integral over the closed oriented curve  $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ , which is the boundary of a region in  $\mathcal{R} \subset \mathbb{C}$ , so which vanishes by (3.5). Note that it is essential for the region  $\mathcal{R}$  to be simply-connected since otherwise  $\mathcal{C}_1 - \mathcal{C}_2$  may not be the boundary of any region in  $\mathcal{R}$ , as illustrated in Figure 8.

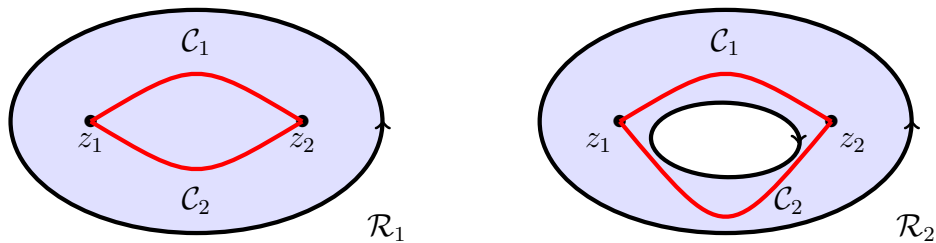


Figure 8: Line integrals of holomorphic functions depend only on the endpoints in a simply connected region  $\mathcal{R}_1$  but not in a non-simply-connected region  $\mathcal{R}_2$ .

The result (3.6) fails to hold when either  $f$  is not holomorphic in  $\mathcal{R}$  and/or when  $\mathcal{R}$  is not simply-connected. For example, the function  $f(z) = 1/z$  is holomorphic in the annulus  $\mathcal{R} = \{z \in \mathbb{C}, 1/2 \leq |z| \leq 2\}$ , which is not simply-connected and its integral over the unit circle  $|z| = 1$  may be evaluated by changing variables  $z = \rho e^{i\theta}$ , with  $dz = (d\rho + i\rho d\theta)e^{i\theta}$  with  $0 \leq \theta \leq 2\pi$ , and we find,

$$\oint_{|z|=1} \frac{dz}{z} = \int_0^{2\pi} i d\theta = 2\pi i \quad (3.7)$$

which does not vanish. Viewing the unit circle as the boundary of the unit disc, which is simply-connected, the same integral now fails to vanish because  $f(z) = 1/z$  fails to be holomorphic in the unit disc, since it has a pole at  $z = 0$ .

Actually, a very useful result that follows using the same methods is as follows,

$$\oint_{|z|=R} \frac{dz}{z^n} = 2\pi i \delta_{n,1} \quad (3.8)$$

for any radius  $R > 0$  and any integer  $n \in \mathbb{Z}$ . To prove it, we use the change of variables  $z = R e^{i\theta}$  with  $0 \leq \theta \leq 2\pi$  and that the integral over  $\theta$  vanishes for  $n \neq 1$ .

### 3.3 Cauchy's integral formulas

**Cauchy's integral formula:** Let  $f(z)$  be analytic inside a simply connected region  $\mathcal{R}$  and on its boundary  $\mathcal{C} = \partial\mathcal{R}$ . Then for every point  $w \in \mathcal{R}$  we have,

$$f(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \frac{f(z)}{z-w} \quad (3.9)$$

To prove this result, we use the fact that the function  $f(z)/(z-w)$  is holomorphic in  $\mathcal{R}$  except at the point  $w$ . To make use of Cauchy's theorem, we choose a contour  $\mathcal{C} \cup \gamma$  of integration which excludes the point  $w$ , as depicted in Figure 9.

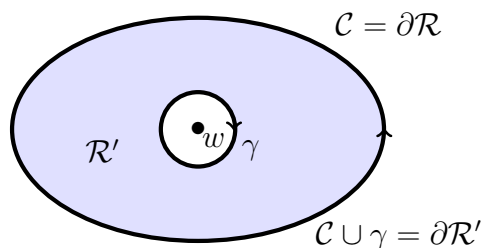


Figure 9: Oriented contour of integration  $\mathcal{C} \cup \gamma$  in Cauchy's integral formulas.

Cauchy's theorem applies to the integration over the contour  $\mathcal{C} \cup \gamma$ , and decomposing the integral over  $\mathcal{C} \cup \gamma$  into separate integrals over  $\mathcal{C}$  and  $\gamma$ , we find,

$$0 = \oint_{\mathcal{C} \cup \gamma} dz \frac{f(z)}{z-w} = \oint_{\mathcal{C}} dz \frac{f(z)}{z-w} + \oint_{\gamma} dz \frac{f(z)}{z-w} \quad (3.10)$$

The first integral on the right side is the one we want to obtain. To evaluate the second integral we choose  $\gamma$  to be a circle of radius  $\varepsilon > 0$  centered at  $w$ , and parametrize  $z \in \gamma$  by setting  $z = w + \varepsilon e^{-i\theta}$ , with  $dz = -i\varepsilon e^{-i\theta} d\theta$  with the angle  $\theta$  running from 0 to  $2\pi$  to account for the clock-wise orientation of  $\gamma$ , and we find,

$$\oint_{\gamma} dz \frac{f(z)}{z-w} = \int_0^{2\pi} (-i) d\theta f(w + \varepsilon e^{-i\theta}) = -2\pi i f(w) \quad (3.11)$$

Combining these results proves Cauchy's integral formula.

**Analyticity Theorem:** *If  $f(z)$  is holomorphic inside a simply-connected region  $\mathcal{R}$ , then all its derivatives are holomorphic in  $\mathcal{R}$ .*

This is an powerful result without analogue in the world of functions of a real variable. It means that as soon as a function is differentiable once in the complex sense with continuous first derivative, then it is differentiable any number of times. The proof follows directly from Cauchy's integral formulas, since by differentiating both sides of (3.9) we have,

$$f^{(n)}(w) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} dz \frac{f(z)}{(z-w)^{n+1}} \quad (3.12)$$

which is well-defined as long as  $f(z)$  is analytic.

### 3.4 Liouville's theorem, fundamental theorem of algebra

**Liouville's Theorem:** *A function  $f(z)$  which is holomorphic and bounded throughout the entire complex plane  $\mathbb{C}$  is constant.*

The proof proceeds from the first derivative of Cauchy's integral formula,

$$f'(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \frac{f(z)}{(z-w)^2} \quad (3.13)$$

We choose  $\mathcal{C}$  to be the circle of arbitrary radius  $R > 0$  centered at  $w$ , and assume  $|f(z)| < M$  for some positive constant  $M$  and all  $z \in \mathbb{C}$ . Taking the absolute value of both sides in (3.13) and bounding the integral using Schwarz's identity, we have,

$$|f'(w)| \leq \frac{1}{2\pi} \oint_{\mathcal{C}} |dz| \frac{|f(z)|}{|z-w|^2} \leq \frac{M}{R} \quad (3.14)$$

But  $R$  is arbitrary and may be taken to be arbitrarily large, so that  $|f'(w)|$  is bounded from above by an arbitrarily small number. Therefore  $f'(z) = 0$  and  $f(z)$  is constant.

**Fundamental Theorem of Algebra:** *A polynomial of degree  $n$  which is given by  $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$  and  $a_n, \dots, a_0 \in \mathbb{C}$  with  $a_n \neq 0$  has  $n$  zeros.*

The proof of the Theorem proceeds by contradiction using Liouville's theorem. We begin by proving that a polynomial of degree  $n \geq 1$  has at least one zero. Suppose the contrary, namely  $P(z)$  is a polynomial of degree  $n \geq 1$  and has no zeros. Then by continuity  $P(z)$  must be bounded from below and its inverse  $f(z) = 1/P(z)$  is bounded from above and holomorphic throughout  $\mathbb{C}$ . By Liouville's theorem,  $P(z)$  is then constant which is in contradiction to the assumption that  $P(z)$  has degree  $n \geq 1$ . Therefore, the initial assumption that  $P(z)$  has no zeros must be false and  $P(z)$  has

at least one zero. Having shown the existence of one zero, we proceed to prove the Theorem by induction on the degree. For degree 1, we are done. For degree 2, we have proven that there is at least one zero, say  $z_1$ . Thus  $P(z)/(z - z_1)$  is a polynomial of degree 1 for which we repeat the process and so on.

### 3.5 Taylor series

One of the most useful tools for analytical as well as numerical calculations is the representation of functions by Taylor series or Laurent series. We provide here the precise statements of these results.

**Taylor's Theorem:** *Let  $f(z)$  be a holomorphic function inside and on a circle  $\mathcal{C}$  of radius  $R$  centered at the point  $a$ , then for any  $z$  inside  $\mathcal{C}$  the Taylor expansion is,*

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a) \quad (3.15)$$

and this series has radius of convergence  $R$  or greater.

To prove the theorem we use the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dw \frac{f(w)}{w-z} \quad (3.16)$$

where  $\mathcal{C} = \{w \in \mathbb{C}, |w-a| = R\}$  as depicted in the left panel of Figure 10. Now expand the denominator as follows,

$$\frac{1}{w-z} = \frac{1}{(w-a) - (z-a)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} \quad (3.17)$$



Figure 10: The curves  $\mathcal{C}$  and  $\mathcal{C}'$  are circles centered at the point  $a$  of respective radii  $R$  and  $r < R$ . The shaded areas are the regions where the function  $f$  is holomorphic respectively for the Taylor and Laurent expansions.



This series is absolutely convergent since we have  $|w - a| > |z - a|$  for  $z$  inside  $\mathcal{C}$ . Substituting the result into Cauchy's integral formula, we find,

$$f(z) = \sum_{n=0}^{\infty} (z - a)^n \frac{1}{2\pi i} \oint_{\mathcal{C}} dw \frac{f(w)}{(w - a)^{n+1}} \quad (3.18)$$

To complete the proof, we use Cauchy's integral formula (3.12) for the  $n$ -th derivative of  $f$  and express the integrals on the right side in terms of  $f^{(n)}(a)$ .

The radius of convergence of the Taylor series in (3.15) is greater than or equal to  $R$ . As a result, the radius of convergence of an entire function  $f(z)$  is infinite, since  $f(z)$  is holomorphic inside a circle of arbitrarily large radius. On the other hand, the radius of convergence for a meromorphic function  $f(z)$  is given by the distance of the point  $a$  to the nearest pole or other type of singularity of  $f(z)$ . For example, the following Taylor series at the point  $a = 0$ ,

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \quad -\ln(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (3.19)$$

both have unit radius of convergence, namely  $|z| < 1$  which corresponds to the distance to the singularity at  $z = 1$ , which is a pole for the first function and a branch point for the second function.

### 3.6 Laurent series

**Laurent's Theorem:** *Let  $f(z)$  be a holomorphic function inside an annulus of inner radius  $r$  and outer radius  $R$  centered at the point  $a$ , then for any  $z$  inside the annulus we have the Laurent expansion,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - a)^n} \quad (3.20)$$

where the Laurent coefficients are given by,

$$a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} dw \frac{f(w)}{(w - a)^{n+1}} \quad a_{-n} = \frac{1}{2\pi i} \oint_{\mathcal{C}'} dw \frac{f(w)}{(w - a)^{-n+1}} \quad (3.21)$$

with the contours  $\mathcal{C}$  and  $\mathcal{C}'$  as depicted in the right drawing of Figure 10.

To prove the theorem, we again use Cauchy's integral formula in the form,

$$f(z) = \oint_{\mathcal{C}} dw \frac{f(w)}{w - z} + \oint_{\mathcal{C}'} dw \frac{f(w)}{w - z} \quad (3.22)$$

For the integral along  $\mathcal{C}$ , we use the expansion of (3.17), while for the integral along  $\mathcal{C}'$ , we use the expansion

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = -\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} \quad (3.23)$$

This series is absolutely convergent since we have  $|w-a| < |z-a|$  for  $w \in \mathcal{C}'$  and  $z$  inside the annulus. Substituting both expansion into (3.22) gives (3.20).

The contributions from  $a_{-n}$  with  $n \geq 1$  are referred to as the *pole part of the Laurent series*. The pole part vanishes for an entire function; it truncates to a finite sum for a meromorphic function; and contains an infinite number of terms for an essential singularity.

### 3.7 The residue theorem

**Residue Theorem:** *Let  $\mathcal{R}$  be a simply connected region with boundary  $\mathcal{C}$  (oriented clock-wise) and  $f(z)$  a single-valued function which is holomorphic in  $\mathcal{R}$  and on its boundary, except at isolated singularities at points  $z_k$  in the interior of  $\mathcal{R}$  for  $k = 1, \dots, N$ . Then we have the following formula,*

$$\oint_{\mathcal{C}} dz f(z) = 2\pi i \sum_{k=1}^N r_k \quad (3.24)$$

where  $r_k$  is the residue of  $f(z)$  at the point  $z_k$  defined to be the coefficient  $a_{-1}(r_k)$  of the Laurent series of  $f(z)$  at the point  $z_k$ .

The proof of the theorem is analogous to the proof of Cauchy's integral formula. We draw a disc  $D_k = \{z \in \mathbb{C}, |z - z_k| < \varepsilon\}$  of radius  $\varepsilon > 0$  centered at the pole  $z_k$  and choose  $\varepsilon$  sufficiently small such that none of the circles overlap with one another. The fact that the poles are isolated guarantees that we can find an  $\varepsilon > 0$  such that this can be done. The integral of  $f(z)$  over the region  $\mathcal{R}$  minus the union of the discs vanishes by Cauchy's theorem, so that,

$$\oint_{\mathcal{C}} dz f(z) = \sum_{k=1}^N \oint_{\partial D_k} dz f(z) \quad (3.25)$$

where the orientation of  $\partial D_k$  is chosen to be clockwise. Using now the Laurent expansion of  $f(z)$  at each point  $z_k$ , and the fact that,

$$\oint_{\partial D_k} \frac{dz}{(z-z_k)^n} = 2\pi i \delta_{n,1} \quad \oint_{\partial D_k} dz f(z) = 2\pi i a_{-1}(r_k) \quad (3.26)$$

so that the residue theorem follows. This formula is valid both whether  $z_k$  is a pole or an essential singularity.

To compute the residues, one does not always need to evaluate the entire Laurent series, and this constitutes a great advantage of the residue method of calculation. When the singularity  $z_k$  is a pole of order  $n$ , the residue is given by,

$$r_k = \lim_{z \rightarrow z_k} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left( (z - z_k)^n f(z) \right) \quad (3.27)$$

This formula is not applicable, of course, for essential singularities since then  $n = \infty$ .

### 3.8 Examples of calculations using residues

In this subsection, we apply Cauchy's theorem and the residue formulas to a set of examples of increasing difficulty and physical significance.

#### 3.8.1 A standard integral

A standard integral, which can be easily computed using residues is the following integral over the real line,

$$I_n(a) = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^n} \quad (3.28)$$

where we take  $a \in \mathbb{R}$ ,  $a > 0$  and  $n \in \mathbb{N}$ . The integral is then absolutely convergent.

We begin by transforming this integral over  $\mathbb{R}$  into an integral over a closed contour in the complex plane. A guiding principle when using this method is to choose a contour which includes the real line (giving the integral we want to evaluate in the first place) union a contour along which the integral either vanishes or is easy to evaluate. In practice, we compute the integral over the line segment  $[-R, R]$  and let  $R \rightarrow \infty$  at the end of the calculation. The contour we choose is given in Figure 11.

The integral along the semi-circle of radius  $R$  tends to zero in the limit  $R \rightarrow \infty$  and thus will not contribute in the limit. But now we see that there is a single pole inside  $\mathcal{C}$  and so the integral may be computed by evaluating the residue  $r_a$  at the point  $ia$ ,

$$I_n(a) = \oint_{\mathcal{C}} \frac{dz}{(z^2 + a^2)^n} = 2\pi i r_a \quad (3.29)$$

Using the formula (3.27) to evaluate  $r_a$ , we have,

$$r_a = \lim_{z \rightarrow ia} \left( \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \frac{(z - ia)^n}{(z^2 + a^2)^n} \right) = (-1)^{n-1} \frac{\Gamma(2n-1)}{\Gamma(n)^2 (2ia)^{2n-1}} \quad (3.30)$$

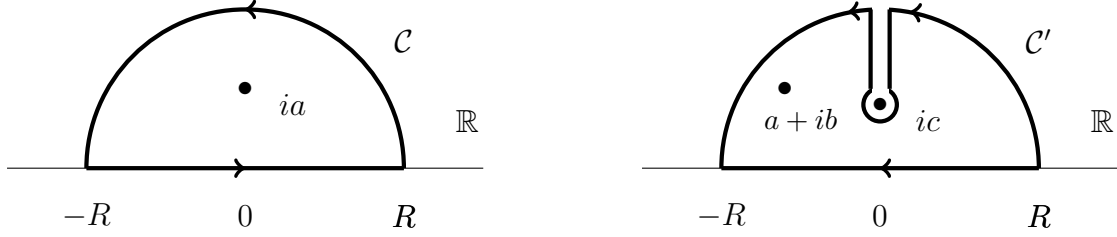


Figure 11: The curves  $\mathcal{C}$  and  $\mathcal{C}'$  used to compute the integrals  $I_n(a)$  and  $J(a, b, c)$ .

Taking the limit  $R \rightarrow \infty$ , we have,

$$I_n(a) = 2\pi \frac{\Gamma(2n-1)}{\Gamma(n)^2 (2a)^{2n-1}} \quad (3.31)$$

Note that this expression is positive, scales with  $a$  as  $a^{-2n+1}$ , and evaluates to  $\pi/a$  for  $n = 1$ , as expected from its definition. We also note that  $I_n(a)$  could have been computed using the recursion relation  $\partial I_n(a)/\partial a^2 = -nI_{n+1}(a)$ .

### 3.8.2 An integral involving a logarithm

A second example is a somewhat less standard integral,

$$J(a, b, c) = \int_{-\infty}^{\infty} dx \frac{\ln(x^2 + c^2)}{(x-a)^2 + b^2} \quad (3.32)$$

for  $a, b, c \in \mathbb{R}$  and we may assume  $b, c > 0$  without loss of generality. Using the contour of Figure 11, the fact that the function  $\ln(z+ic)$  is holomorphic in the upper half plane provided we choose its branch cut to lie in the lower half plane, and the fact that the integrand is holomorphic inside the contour of  $\mathcal{C}$  except for a simple pole at  $a+ib$ , we evaluate the following integral by computing its residue at the pole  $a+ib$ , and we find,

$$\oint_{\mathcal{C}} dz \frac{\ln(z+ic)}{(z-a)^2 + b^2} = \frac{\pi}{b} \ln(a+ib+ic) \quad (3.33)$$

Since the contribution from the semi-circle tends to zero as  $R \rightarrow \infty$ , the integral  $J(a, b, c)$  equals the above integral plus its complex conjugate and we find,

$$J(a, b, c) = \frac{\pi}{b} \ln(a^2 + (b+c)^2) \quad (3.34)$$

Just to make sure that we understand things properly, let us verify that a direct evaluation of the integral involving  $\ln(z-ic)$  yields the same result. We cannot

integrate  $\ln(z - ic)$  over the contour  $\mathcal{C}$  because  $\ln(z - ic)$  has a branch cut in the upper half plane. So, we take a more complicated contour  $\mathcal{C}'$  also given in Figure 8. The cut semi-circle at  $\infty$  again tends to zero in the limit  $R \rightarrow \infty$ , and so does the contribution from the small circle around  $ic$  as its radius tends to zero. The remaining contributions are from the residue at  $a + ib$  and from the two vertical parts of the contour which we parametrize by  $z = iy$  with  $y \in [c, R]$ , and we find,

$$\oint_{\mathcal{C}} dz \frac{\ln(z - ic)}{(z - a)^2 + b^2} = \frac{\pi}{b} \ln(a + ib - ic) + i \int_c^\infty dy \frac{\text{disc } \ln(iy - ic)}{(iy - a)^2 + b^2} \quad (3.35)$$

where the discontinuity of the log evaluates to,

$$\text{disc } \ln(iy - ic) = \lim_{\varepsilon \rightarrow 0} \left( \ln(iy - ic + \varepsilon) - \ln(iy - ic - \varepsilon) \right) = 2\pi i \quad (3.36)$$

Evaluating the integral over  $y$  by partial fraction decomposition of the integrand gives,

$$-2\pi \int_c^\infty dy \frac{1}{(iy - a)^2 + b^2} = -\frac{\pi}{b} \ln \frac{a + ib - ic}{a - ib - ic} \quad (3.37)$$

Combining the two terms in (3.35) confirms that we indeed obtain the complex conjugate of (3.33).

### 3.8.3 An integral involving a pole on the real axis

A type of integral which is often encountered in physics is given by,

$$\theta_a(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi i} \frac{e^{ikx}}{k - a - i\varepsilon} \quad (3.38)$$

where  $a, x \in \mathbb{R}$  and  $\varepsilon > 0$ . For  $\varepsilon = 0$ , there is a pole on the real axis and the integral is ill-defined. The addition of the imaginary part  $\varepsilon > 0$  moves the pole into the complex plane. The integral is now well-defined for  $x \neq 0$  despite the fact that it is not absolutely convergent on  $\mathbb{R}$  because the exponential provides sufficient damping for large  $k$ . In the physics literature, one is instructed to take the limit  $\varepsilon \rightarrow 0$ .

To evaluate the integral we proceed as follows. If  $x > 0$ , then the exponential  $e^{ixz}$  rapidly decays to zero for  $\text{Im}(z) \gg 1$ . Thus, we may close the contour in the upper half plane, just as we did in the left figure of Figure 8. Since  $\varepsilon > 0$ , there a single pole at  $k = a + i\varepsilon$  in the upper half plane and thus we find by evaluating its residue,

$$\begin{aligned} \theta_a(x) &= e^{ixa} & x > 0 \\ &= 0 & x < 0 \end{aligned} \quad (3.39)$$

For  $x < 0$ , the exponential  $e^{ixz}$  rapidly decays to zero for  $\text{Im}(z) \ll -1$ . But in the lower half plane there are no poles, the integrand is holomorphic, and we have  $\theta_a(x) = 0$  for  $x < 0$ . Note that for  $a = 0$ , the function  $\theta_0(x)$  is just the step function. The value at  $x = 0$  is undetermined.

### 3.8.4 Calculation of the Feynman propagator

In relativistic quantum field theory, one encounters a slight generalization of the above case. The Green function  $G$  for a scalar field of mass  $m$  in  $d$ -dimensional Minkowski space-time is defined to satisfy the differential equation,

$$(\partial_t^2 - \Delta + m^2)G(t - t', \vec{x} - \vec{x}') = i\delta(t - t')\delta^{(d-1)}(\vec{x} - \vec{x}') \quad (3.40)$$

where we have used units in which the speed of light is set to 1. By translation invariance in time and space, we may set  $t' = \vec{x}' = 0$  and solve the equation by Fourier transform. We adopt here the Feynman  $i\varepsilon$  prescription so that,

$$G(t, \vec{x}) = \int \frac{dk_0 d^{d-1}\vec{k}}{(2\pi)^d} \frac{i e^{i(k_0 t + \vec{k} \cdot \vec{x})}}{k_0^2 - \vec{k}^2 - m^2 + i\varepsilon} \quad (3.41)$$

where  $(k_0, \vec{k})$  is the momentum vector, and  $\varepsilon > 0$  is to be taken to zero. The denominator has zeros at  $k_0 = \pm(\omega(\vec{k}) - i\varepsilon)$  where  $\omega(\vec{k})^2 = \vec{k}^2 + m^2$  and  $\omega(\vec{k}) > 0$ . Thus, one pole is in the upper half plane while the other is in the lower half plane. For  $t > 0$ , we can close the contour in the upper half plane, while for  $t < 0$ , we can close in the lower half plane and pick up the corresponding poles. Hence we have,

$$G(t, \vec{x}) = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} \frac{e^{i\vec{k} \cdot \vec{x}}}{2\omega(\vec{k})} \left( \theta(t) e^{-i\omega(\vec{k})t} + \theta(-t) e^{+i\omega(\vec{k})t} \right) \quad (3.42)$$

Let us double check that this solution indeed satisfies the differential equation by computing the first time derivative,

$$\partial_t G(t, \vec{x}) = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}} e^{i\vec{k} \cdot \vec{x}} \left( -\frac{i}{2}\theta(t) e^{-i\omega(\vec{k})t} + \frac{i}{2}\theta(-t) e^{+i\omega(\vec{k})t} \right) \quad (3.43)$$

The contributions from the  $t$ -derivatives of  $\theta(\pm t)$  cancel since their exponential factors are equal to one another at  $t = 0$ . The second time derivative produces two types of terms. Applied to the exponentials, it brings down a factor of  $\omega(\vec{k})$  which combines with the spatial Laplacian and the mass term, while applied to  $\theta(\pm t)$  and performing the  $\vec{k}$  integral we obtain  $\delta(t)\delta^{(d-1)}(\vec{x})$ . Thus,  $G$  satisfies the differential equation.

## 3.9 Convergence of infinite series and integrals

In some of the preceding sections, we have shown how to obtain Taylor and Laurent series of analytic functions. However, functions are often defined by infinite series or integrals of holomorphic functions, and here we shall give some results on the

analyticity of the resulting sums and integrals. Our definitions are given for infinite series, and is similar for functions defined by integrals.

Let  $\{f_n(z)\}_{n \in \mathbb{N}}$  be a sequence of functions. The sequence converges to a function  $f(z)$  in a region  $\mathcal{R}$  provided for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n > N$  we have  $|f_n(z) - f(z)| < \varepsilon$ . Note that in this definition the number  $N$  may depend on the point  $z$ , and is therefore referred to as *point-wise convergent*. If there exists a number  $N$  which is *independent* of  $z \in \mathcal{R}$  then the sequence is *uniformly convergent in the region  $\mathcal{R}$* .

An important special case of a sequence of functions is given by series,

$$f_n(z) = \sum_{m=1}^n \varphi_m(z) \qquad f(z) = \sum_{n=1}^{\infty} \varphi_n(z) \qquad (3.44)$$

One example is a Taylor series where  $\varphi_n(z) = c_n z^n$  for some sequence of numbers  $c_n$ ; another example is a Fourier series with  $\varphi_n(z) = c_n \sin(nz)$ . The question is what the holomorphicity properties are of  $f(z)$ .

The series is said to be *absolutely convergent* in a region  $\mathcal{R}$  if the series  $\sum_{n=1}^{\infty} |\varphi_n(z)|$  converges for every point  $z \in \mathcal{R}$ . When the series defining  $F(z)$  converges, but  $\sum_{n=1}^{\infty} |\varphi_n(z)|$  diverges, then the series is said to be *conditionally convergent*. Every absolutely convergent series is convergent. Uniform convergence for infinite series is defined just as it is for sequences. We now have the following results.

1. The terms in an absolutely convergent series may be ordered arbitrarily.
2. Different orderings of the terms in a conditionally convergent series produce different sums.

Furthermore, we have the following important results for an infinite series (3.44) which is uniformly convergent in a region  $\mathcal{R}$ .

- If each function  $f_n(z)$  is continuous in  $\mathcal{R}$ , then the limit is continuous in  $\mathcal{R}$ .
- If each function  $f_n(z)$  is holomorphic in  $\mathcal{R}$ , then the limit is holomorphic in  $\mathcal{R}$ .
- If  $f_n(z)$  is continuous, and  $f'_n(z)$  exists for all  $n$ , and the series  $\sum_{n=1}^{\infty} f'_n(z)$  converges uniformly while the series  $\sum_{n=1}^{\infty} f_n(z)$  converges point-wise, then the infinite sum symbol may be interchanged with the differentiation and we have,

$$f'(z) = \sum_{n=1}^{\infty} f'_n(z) \qquad (3.45)$$

- If  $f_n(z)$  is continuous and the series is uniformly convergent in  $\mathcal{R}$  then we may interchange the infinite sum symbol with the integration over a curve  $\mathcal{C} \subset \mathcal{R}$ ,

$$\int_{\mathcal{C}} dz f(z) = \sum_{n=1}^{\infty} \int_{\mathcal{C}} dz f_n(z) \quad (3.46)$$

The above results typically fail for series which are point-wise convergent in a region  $\mathcal{R}$  but fail to be uniformly convergent in  $\mathcal{R}$ .

### 3.10 Analytic continuation

The domain  $\mathcal{R}$  of uniform convergence of a sequence, infinite series, or integral of a family of holomorphic functions may sometimes be extended to a larger region  $\mathcal{R} \subset \mathcal{R}'$  by the method of *analytic continuation*. This method is used very frequently in physics. The fundamental ingredient is the following theorem.

#### Theorem

(a) Let  $f(z)$  be holomorphic in a region  $\mathcal{R}$  and let  $\mathcal{C}$  be an arc completely contained in  $\mathcal{R}$ . If  $f(z) = 0$  for all  $z \in \mathcal{C}$  then  $f(z) = 0$  for all  $z \in \mathcal{R}$ .

(b) As a corollary, if  $f_1(z)$  is holomorphic in region  $\mathcal{R}_1$  and  $f_2(z)$  is holomorphic in region  $\mathcal{R}_2$  and  $f_1(z) = f_2(z)$  for every point of an arc  $\mathcal{C} \subset (\mathcal{R}_1 \cap \mathcal{R}_2)$  then  $f_1(z) = f_2(z)$  for every point  $z \in \mathcal{R}_1 \cap \mathcal{R}_2$  and one may define a function  $f(z) = f_1(z)$  for  $z \in \mathcal{R}_1$  and  $f(z) = f_2(z)$  for  $z \in \mathcal{R}_2$  which is analytic in the region  $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ .

It suffices to prove (a) as (b) immediately follows from (a). To prove (a), we choose a point  $z_0 \in \mathcal{C} \subset \mathcal{R}$  and expand  $f(z)$  in Taylor series at  $z_0$ ,

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!} \quad (3.47)$$

Its radius of convergence is set by the distance of  $z_0$  to the nearest singularity of  $f(z)$ , which is larger or equal to the distance of  $z_0$  to the boundary of  $\mathcal{R}$  and non-zero since  $f(z)$  is holomorphic in  $\mathcal{R}$ . But since  $z_0 \in \mathcal{C}$ , we can now parametrize  $z$  along  $\mathcal{C}$  where  $f(z) = 0$  for every point  $z \in \mathcal{C}$ . Thus  $f^{(n)}(z_0) = 0$  for all  $n$ , and hence  $f(z) = 0$  in  $\mathcal{R}$ .

#### 3.10.1 Analytic continuation of the Euler Gamma-function

Euler's  $\Gamma$ -function is defined by its integral representation,

$$\Gamma(z) = \int_0^{\infty} dt t^{z-1} e^{-t} \quad (3.48)$$



The integral is absolutely convergent for  $\operatorname{Re}(z) > 0$  and therefore defines a holomorphic function for  $\operatorname{Re}(z) > 0$ . The value at  $z = 1$  is given by  $\Gamma(1) = 1$  and integration by parts shows that  $\Gamma$  satisfies a functional relation for  $\operatorname{Re}(z) > 0$ ,

$$\Gamma(z + 1) = z \Gamma(z) \quad (3.49)$$

so that  $\Gamma(n + 1) = n!$  for any positive integer. Analytic continuation allows us to extend  $\Gamma(z)$  to a meromorphic function in  $\mathbb{C}$  with simple poles at  $z = 0, -1, -2, \dots$ . To see this, we partition the integration range of the integral as follows,

$$\Gamma(z) = \int_1^\infty dt t^{z-1} e^{-t} + \int_0^1 dt t^{z-1} e^{-t} \quad (3.50)$$

The first integral defined an entire function in  $z$ . Expanding the exponential in the second integral in a Taylor series and evaluating the  $t$ -integrals, we find,

$$\int_0^1 dt t^{z-1} e^{-t} = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int_0^1 dt t^{z-1+n} = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{1}{z+n} \quad (3.51)$$

While initially defined only for  $\operatorname{Re}(z) > 0$ , each term in the last series may be trivially analytically continued to the entire complex plane at the cost of simple poles at zero and the negative integers. The same result may be derived by analytic continuation directly of the functional relation, and gives the same values of the residues.

### 3.10.2 Analytic continuation of the Euler Beta-function

To obtain the Euler Beta function, we consider the product of two  $\Gamma$ -functions,

$$\Gamma(x)\Gamma(y) = \int_0^\infty ds \int_0^\infty dt s^{x-1} t^{y-1} e^{-s-t} \quad (3.52)$$

Changing integration variables from  $s, t$  to  $u = s + t$  and  $s = uv$  decouples the integrals in  $u$  and  $v$ . Performing the integral over  $u$  in terms of the  $\Gamma$ -function gives the expression for the Euler Beta-function,

$$\int_0^1 dv v^{x-1} (1-v)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (3.53)$$

Since we already know the analytic continuation of the  $\Gamma$ -function, the analytic continuation of the Beta-function integral is now also available.

As an application we obtain another functional relation for the  $\Gamma$ -function,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (3.54)$$

To prove this formula, we set  $x = z$  and  $y = 1 - z$  in the formula for the Euler beta function, evaluate the remaining  $z$ -integral for  $0 < \text{Re}(z) < 1$ ,

$$\Gamma(z)\Gamma(1-z) = \int_0^1 dv v^{z-1}(1-v)^{-z} \quad (3.55)$$

and then establish the result for general  $z \in \mathbb{C}$  by analytic continuation. To evaluate the integral, we change variables by setting  $v = t/(1+t)$ , in terms of which we have,

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty dt \frac{t^{z-1}}{t+1} \quad (3.56)$$

This integral may be evaluated in turn by contour integration around a contour  $\mathcal{C}$  consisting of concentric circles with origin 0 and radii  $\varepsilon > 0$  and  $R > \varepsilon$  in the limit of  $\varepsilon \rightarrow 0$  and  $R \rightarrow \infty$ , cut along the real axis to avoid the branch cut of  $t^{z-1}$  along the positive real axis. Picking up the residue at  $t = -1 = e^{i\pi}$ , which lies inside the contour, we have,

$$\oint_{\mathcal{C}} dw \frac{w^{z-1}}{w+1} = 2\pi i e^{i\pi(z-1)} \quad (3.57)$$

The contributions from the circles vanish in these limit, and the contributions from the positive real axis gives,

$$\lim_{\varepsilon \rightarrow 0} \lim_{R \rightarrow \infty} \oint_{\mathcal{C}} dw \frac{w^{z-1}}{w+1} = \int_0^\infty dt \frac{t^{z-1}}{t+1} - e^{2\pi i(z-1)} \int_0^\infty dt \frac{t^{z-1}}{t+1} \quad (3.58)$$

from which we recover the claimed formula for  $0 < \text{Re}(z) < 1$ . Both sides of the equation are holomorphic functions away from the poles at integers, so the formula extend throughout  $z \in \mathbb{C}$ . The formula guarantees that the analytic continuation of  $\Gamma(z)$  from  $\text{Re}(z) > 0$  to  $\text{Re}(z) \leq 0$  is manifest, and the poles are readily exposed. It is also manifest from this relation that  $\Gamma(z)$  nowhere vanishes in  $\mathbb{C}$ .

### 3.11 Asymptotic series

Very often in physics, we use a perturbative expansion in one of the parameters of the problem, but the resulting series does not converge. An example is to take an ordinary integral which mimics the general problem, including of perturbation theory in quantum mechanics of quantum field theory. Thus, we consider,

$$Z(m, \lambda) = \int_{-\infty}^{+\infty} d\phi \exp\left\{-m^2\phi^2 - \lambda^2\phi^4\right\} \quad (3.59)$$

The integral is absolutely convergent for  $\text{Re}(\lambda^2) > 0$ , regardless of  $m^2 \in \mathbb{C}$ . There are now two possible expansions; the first in power of  $m$ , the second in powers of  $\lambda$ . The expansion in powers of  $m$  yields,

$$\begin{aligned} Z(m, \lambda) &= \sum_{n=0}^{\infty} \frac{(-m^2)^n}{\Gamma(n+1)} \int_{-\infty}^{+\infty} d\phi \phi^{2n} \exp\left\{-\lambda^2 \phi^4\right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{2\sqrt{\lambda}} \frac{\Gamma(\frac{n}{2} + \frac{1}{4})}{\Gamma(n+1)} \left(-\frac{m^2}{\lambda}\right)^n \end{aligned} \quad (3.60)$$

while the expansion in powers of  $\lambda$  yields,

$$\begin{aligned} Z(m, \lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} (-\lambda^2)^n \int_{-\infty}^{+\infty} d\phi \phi^{4n} \exp\left\{-m^2 \phi^2\right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{m} \frac{\Gamma(2n + \frac{1}{2})}{\Gamma(n+1)} \left(-\frac{\lambda^2}{m^4}\right)^n \end{aligned} \quad (3.61)$$

Clearly, from analyzing the  $n$ -dependence of the ratios of  $\Gamma$  functions, the expansion in  $m$  is convergent (with  $\infty$  radius of convergence) while the expansion in power of  $\lambda$  has zero radius of convergence. This comes as no surprise. If we had flipped the sign of  $m^2$ , the integral itself would still be fine and convergent. On the other hand, if we had flipped the sign of  $\lambda^2$ , the integral would badly diverge. Therefore, the series in powers of  $\lambda$  cannot have a finite radius of convergence around 0. The moral of the story is that perturbation theory around a quadratic exponent in a higher degree term is always given by an asymptotic series which has zero radius of convergence. Nonetheless, these expansions have a well-defined meaning.

Generally an infinite series given by,

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^n} \quad (3.62)$$

is *asymptotic at  $\infty$*  provided for any given integer  $N > 1$ , we have,

$$\lim_{z \rightarrow \infty} z^N \left( f(z) - \sum_{n=1}^N \frac{a_n}{z^n} \right) = 0 \quad (3.63)$$

Asymptotic series may be added, multiplied, or integrated term by term. However, they may generally not be differentiated term by term. If an asymptotic series exists it is unique.

### 3.12 The method of steepest descent

A problem often encountered in physics is to find the asymptotic behavior, and the asymptotic series expansion of an integral of the following type,

$$f(z) = \int_{\mathcal{C}} dt e^{zg(t)} \quad (3.64)$$

for some path  $\mathcal{C}$  in the complex plane, and  $z$  real. We assume that the integral converges for all values of  $z > 0$ . For large values of  $z$ , the integral will be dominated by contributions from the points  $t$  where  $g(t)$  attains its largest value. We deform the contour of integration such that the new  $\mathcal{C}$  passes through  $t_0$ . Let's assume that there is only one such point. In fact, this point does not always have to be a maximum, but may be just a saddle, whence the name *saddle-point approximation*. Assuming the function  $g(t)$  to be continuous and sufficiently differentiable, a saddle point is reached for a point  $t_0$  where  $g'(t_0) = 0$ . Again, we assume that this point is unique. Expanding  $g(t)$  around  $t = t_0$ , we have,

$$g(t) = g(t_0) + \frac{1}{2}(t - t_0)^2 g''(t_0) + \mathcal{O}(t - t_0)^3 \quad (3.65)$$

If  $z$  is large enough, the function may be approximated as follows,

$$f(z) = e^{zg(t_0)} \int_{\mathcal{C}} dt e^{\frac{1}{2}g''(t_0)(t-t_0)^2} \quad (3.66)$$

Extending the integration to the full real axis, and performing the Gaussian integral, we obtain,

$$f(z) = \frac{e^{zg(t_0)}}{\sqrt{-2/\pi g''(t_0)}} \quad (3.67)$$

Higher order corrections may be obtained by carrying out a systematic perturbation theory which results in an asymptotic series as illustrated by the first example above.

A simple but useful example is given by the asymptotic behavior of the  $\Gamma$ -function,

$$\Gamma(z + 1) = \int_0^{\infty} dt t^z e^{-t} = \int_0^{\infty} dt e^{-t + z \ln t} \quad (3.68)$$

The argument of the exponential is the function,

$$g_z(t) = -t + z \ln t \quad (3.69)$$

We find a unique saddle point  $t_0 = z$  for  $\text{Re}(z) > 0$ . The second derivative at the saddle point is  $g''(t_0) = -1/z$ , so that we get the following large  $z$  approximation to  $\Gamma(z + 1)$  referred to as *Sterling's formula*,

$$\Gamma(z + 1) = \sqrt{2\pi z} e^{-z + z \ln z} (1 + \mathcal{O}(1/z)) \quad (3.70)$$

## 4 Periodic and elliptic functions

A beautiful application of the complex analysis studied in the preceding section is to periodic functions, elliptic functions and their inverse functions. We shall start with a brief review of periodic functions of a real variable, and then give two different points of view on meromorphic periodic functions and their inverse functions. These new points of view will be the ones in terms of which it is natural to introduce elliptic functions and their inverse functions.

Period functions associated with periodic motion are pervasive in physics. Taking an example from classical mechanics, we can ask whether the motion of a single degree of freedom  $x(t)$  in a potential  $V(x)$  is periodic and if yes, why so. In Figure 12 we present the cases of a harmonic potential in the left figure, and a more general potential in the right figure. In both cases is the range of  $x$  constrained by the energy conservation relation  $E = \dot{x}^2 + V(x)$ , and is the motion periodic in  $t$ . At the turning points  $a, b$  the potential energy is  $E$  and the kinetic energy vanishes, which allows the particle to turn around and start the cycle all over. For every potential, the mechanical motion defines a periodic function  $x(t)$ . Trigonometric functions are associated with the harmonic oscillator, elliptic functions with a potential which is a quartic polynomial in  $x$  while polynomial potentials of arbitrary even degree will produce hyper-elliptic functions.

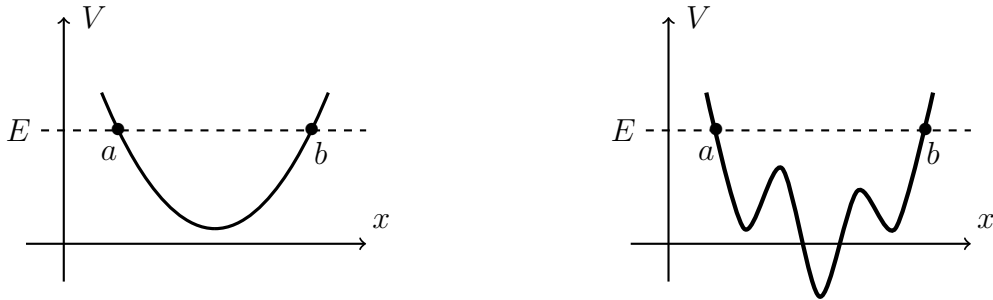


Figure 12: For systems with one degree of freedom  $x(t)$ , and bounded range of  $x$  at given energy  $E$ , the energy conservation equation forces the motion to be periodic.

### 4.1 Periodic functions of a real variable

We shall now give some more precise definitions. A function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is periodic with period  $a \in \mathbb{R}$  if for all  $x \in \mathbb{R}$  it satisfies,

$$f(x + a) = f(x) \tag{4.1}$$

It is straightforward to generalize this construction to the case of functions  $\mathbb{R} \rightarrow \mathbb{C}^n$ . There are various methods for constructing periodic functions out of non-periodic functions. One is by starting with a function  $g : \mathbb{R} \rightarrow \mathbb{C}$  which decays sufficiently rapidly at  $\infty$  and constructing  $f$  by the *method of images*,

$$f(x) = \sum_{n \in \mathbb{Z}} g(x + na) \quad (4.2)$$

The functions  $\psi_m(x) = e^{2\pi imx}$  for  $m \in \mathbb{Z}$  are all periodic with period 1, and in fact form a basis for the square-integrable periodic functions  $L^2(S^1)$ . One way to think about this is that they are eigenfunctions of the free Schrödinger operator on the unit interval with periodic boundary conditions, which we can write as  $\mathbb{R}/\mathbb{Z}$ ,

$$H = -\frac{d^2}{dx^2} \quad H\psi_m(x) = 4\pi^2 m^2 \psi_m(x) \quad (4.3)$$

for all  $m \in \mathbb{Z}$ . Since the operator  $H$  is self-adjoint (to be defined more generally in section 6) its eigenvalues are real and its eigenfunctions are mutually orthogonal,

$$\int_0^1 dx e^{2\pi imx} e^{-2\pi im'x} = \delta_{m,m'} \quad (4.4)$$

Finally, the eigenfunctions  $\psi_m(x)$  form a basis for  $L^2(\mathbb{R}/\mathbb{Z})$ , as expressed by the completeness relations (in the sense of distributions),

$$\sum_{m \in \mathbb{Z}} e^{2\pi imx} e^{-2\pi imy} = \sum_{n \in \mathbb{Z}} \delta(x - y + n) \quad (4.5)$$

Thus, any periodic function  $f$  with period 1 may be decomposed in a Fourier series,

$$f(x) = \sum_{m \in \mathbb{Z}} f_m e^{2\pi imx} \quad f_m = \int_0^1 dx f(x) e^{-2\pi imx} \quad (4.6)$$

We note that in the mathematics literature, one often uses the notation  $e(x) = e^{2\pi ix}$ .

## 4.2 Unfolding trick and Poisson summation formula

An incredibly simple yet extremely useful tool is the *unfolding trick*. If  $g : \mathbb{R} \rightarrow \mathbb{C}$  decays sufficiently rapidly as its argument  $x \rightarrow \infty$ , then we have,

$$\sum_{n \in \mathbb{Z}} \int_0^1 dx g(x + n) = \int_{-\infty}^{\infty} dx g(x) \quad (4.7)$$

We can now apply this to a combination of the method of images and Fourier decomposition. We construct a periodic function  $f$  from a non-periodic function  $g$  using the method of images in (4.2), and then calculate the Fourier coefficients  $f_m$  of (4.6) using the unfolding trick in (4.7),

$$f_m = \sum_{n \in \mathbb{Z}} \int_0^1 dx g(x+n) e^{-2\pi i m x} = \int_{-\infty}^{\infty} dx g(x) e^{-2\pi i m x} \quad (4.8)$$

The Fourier transform on the full real line will be denoted by a hat,

$$\hat{g}(y) = \int_{-\infty}^{\infty} dx g(x) e^{-2\pi i x y} \quad (4.9)$$

so that  $f_m = \hat{g}(m)$ . Therefore, the function  $f$  may be expressed in two different ways,

$$f(x) = \sum_{n \in \mathbb{Z}} g(x+n) = \sum_{m \in \mathbb{Z}} \hat{g}(m) e^{2\pi i m x} \quad (4.10)$$

Setting  $x = 0$  (or two any integer) gives the *Poisson summation formula*,

$$\sum_{n \in \mathbb{Z}} g(n) = \sum_{m \in \mathbb{Z}} \hat{g}(m) \quad (4.11)$$

An immediate application is to the case where  $g$  is a Gaussian. It will be convenient to normalize Gaussians as follows,

$$g(x) = e^{-\pi t x^2} \quad \hat{g}(y) = \frac{1}{\sqrt{t}} e^{-\pi y^2/t} \quad (4.12)$$

The formulas are valid as long as  $\text{Re}(t) > 0$ , and may be continued to  $\text{Re}(t) = 0$ . The corresponding Poisson summation formula then reads,

$$\sum_{n \in \mathbb{Z}} e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{m \in \mathbb{Z}} e^{-\pi m^2/t} \quad (4.13)$$

We shall see later on that this relation admits an important generalization to Jacobi  $\vartheta$ -functions, and corresponds to a special case of modular transformations.

### 4.3 Periodic functions of a complex variable

Periodic holomorphic or meromorphic functions have a very rich structure, and there are various complementary viewpoints one may adopt to describe them.

### 4.3.1 First viewpoint

The first viewpoint is obtained by constructing holomorphic periodic functions by the method of images. For example, starting with a function  $g$  which has a single simple pole  $g(z) = 1/z$  the method of images produces a periodic meromorphic function,

$$\sum_{n \in \mathbb{Z}} \frac{1}{z+n} \quad (4.14)$$

The sum is conditionally convergent so that its value depends on the way we order (or in physics parlance *regularize*) the series. A natural way to do this is by requiring  $f(-z) = -f(z)$  and taking a symmetric limit of a finite sum with this property, or equivalently by grouping opposite  $n$  terms together under the summation,

$$f(z) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z+n} + \frac{1}{z-n} \right) \quad (4.15)$$

The meromorphic nature of  $f$  allows us to evaluate the sum by finding a known function with identical poles, namely a simple pole with unit residue at every integer, and identical symmetry property  $f(-z) = -f(z)$ , referred to as the Lifschytz formula,

$$f(z) = \pi \frac{\cos \pi z}{\sin \pi z} = -i\pi \frac{1 + e^{2\pi iz}}{1 - e^{2\pi iz}} \quad (4.16)$$

Since the poles and residues of these meromorphic functions match in (4.15) and (4.16), their difference must be constant, but by their oddness of the function  $f$  we have, for example,  $f(1/2) + f(-1/2) = 0$  and hence this constant must vanish. Integrating both expressions for  $f$  and determining the integration constant by matching the behavior of  $z = 0$  gives Euler's product formula for the sin-function,

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \quad (4.17)$$

We could also proceed differently by deriving a differential equation for  $f$  directly from its series representation. For  $|z| < 1$ , we may expand the second representation in (4.15) in a convergent power series in  $z$ ,

$$f(z) = \frac{1}{z} - 2 \sum_{m=1}^{\infty} z^{2m-1} \zeta(2m) \quad (4.18)$$

where  $\zeta(s)$  is the Riemann zeta-function, defined by,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (4.19)$$



The combination  $f' + f^2$  is free of poles, bounded in  $\mathbb{C}$ , and thus constant by Liouville's theorem. The constant is computed by evaluating the series near  $z = 0$ , and we find,

$$f' + f^2 = -6\zeta(2) = -\pi^2 \quad (4.20)$$

which is a well-known differential equation for trigonometric functions.

### 4.3.2 Second viewpoint

The second viewpoint is by considering the inverse functions of periodic functions, which by construction must be multiple-valued. To do so, let's start from a single-valued meromorphic function, such as  $1/w$ . Now the line integral of  $1/w$  is well-known to be the logarithm of a complex variable,

$$\int_1^z \frac{dw}{w} = \ln(z) \quad (4.21)$$

But the integral on the left side actually depends on the path chosen to go from the point 1 to the point  $z$ , and more specifically on how it goes around the singularity  $w = 0$  of the integrand. The values around different paths differ by shifts by integer multiples of  $2\pi i$ . This “explains” why the logarithm function that it defines on the right side is a multiple-valued function by shifts by  $2\pi i\mathbb{Z}$ . The same argument can be made for an integral involving a square root, such as,

$$\int_1^z \frac{dw}{\sqrt{w^2 - 1}} = \cosh^{-1}(z) \quad (4.22)$$

## 4.4 Definition of elliptic functions

Periodic functions may be generalized to several variables by considering the function to be periodic in each variable separately. An *elliptic function* is a doubly periodic function in two real dimensions, which is also a meromorphic function. The combination of double-periodicity and meromorphicity imposes extremely strong conditions on the structure of the space of such functions. Just as for singly-periodic functions, elliptic functions may be approached by two complementary points of view: by the method of images, and by integrals over rational functions for their inverses.

We shall now define elliptic functions and discuss them from these two viewpoints. A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is elliptic with periods  $\omega_1, \omega_2 \in \mathbb{C}$  satisfying  $\omega_2/\omega_1 \notin \mathbb{R}$  if,

- $f$  is meromorphic in  $\mathbb{C}$ ;
- $f$  is periodic with periods  $\omega_1, \omega_2$ , namely  $f(z + \omega_1) = f(z + \omega_2) = f(z)$ .

It is advantageous to introduce the infinite lattice of periods depicted in Figure 13,

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \quad (4.23)$$

The second condition above is then simply  $f(z+\omega) = f(z)$  for all  $\omega \in \Lambda$ . Elliptic functions may alternatively be viewed as single-valued functions on the two-dimensional torus  $\mathbb{C}/\Lambda \rightarrow \mathbb{C}$ . It is often in this role that they appear in physics, namely in the function theory of the torus  $\mathbb{C}/\Lambda$ .

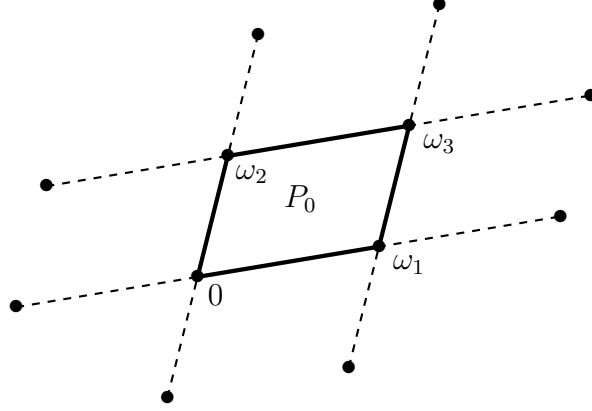


Figure 13: Schematic representation of the periods  $\omega_1, \omega_2$  spanning the infinite lattice  $\Lambda$ , and the fundamental parallelogram  $P_0$  where  $\omega_3 = \omega_1 + \omega_2$ . The fundamental parallelogram  $P_\mu$  is obtained from  $P_0$  by translation by  $\mu \in \mathbb{C}$ .

Sums, products, and inverses of elliptic functions for a given lattice  $\Lambda$  are elliptic functions for the same lattice  $\Lambda$ , and the constant function with value 1 is the identity under multiplication, so that elliptic functions for a given  $\Lambda$  form a *function field*.

We define  $\text{res}_f(w)$  to be the residue of  $f$  at the point  $w$ , and  $\text{ord}_f(w) = n$  to be the order of  $f$  at the point  $w$  by the integer  $n$  such that  $f(u)(u-w)^{-n}$  is a non-zero constant as  $u \rightarrow w$ . When  $\text{ord}_f(w) > 0$ , we have a zero of that order  $\text{ord}_f(w)$ , while when  $n < 0$  we have a pole of order  $-\text{ord}_f(w) > 0$ . In terms of these definitions we have the following results,

$$\begin{aligned} \sum_{w \in P_\mu} \text{res}_f(w) &= 0 \\ \sum_{w \in P_\mu} \text{ord}_f(w) &= 0 \end{aligned} \quad (4.24)$$

obtained by integrating in turn the functions  $f$  and  $f'/f$  over the boundary  $\partial P_\mu$ . We obtain the conditions that are familiar from electro-static charge distributions on a

compact surface: the sum of the residues vanishes and the number of poles equals the number of zeros (both counted with multiplicities).

## 4.5 The Weierstrass elliptic function $\wp$

All elliptic functions may be built up from the Weierstrass elliptic function  $\wp$ , which has one double pole. The Weierstrass elliptic function  $\wp(z) = \wp(z; \Lambda)$  for the lattice  $\Lambda$  may be constructed with the method of images. It would be natural to consider,

$$\sum_{\omega \in \Lambda} \frac{1}{(z + \omega)^2} \quad (4.25)$$

This series diverges, so we shall define  $\wp$  by the manifestly convergent series,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda'} \left( \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right) \quad (4.26)$$

where  $\Lambda' = \Lambda \setminus \{0\}$ .  $\wp$  is even in  $z$  since the lattice is invariant under  $\Lambda \rightarrow -\Lambda$ .

We start by describing the field of elliptic functions in terms of  $\wp(z)$  for even functions. As a function of  $z$  for fixed  $w$ , the elliptic function  $\wp(z) - \wp(w)$  has a double pole at  $z = 0$  and no other poles, and therefore must have two zeros. Since  $z = \pm w$  are manifestly two distinct zeros, they must be the only zeros. If  $w = -w \pmod{\Lambda}$ , then the zeros are double. There are exactly four points in  $P_0$  with  $w \equiv -w \pmod{\Lambda}$ , namely the half periods  $0, \omega_1/2, \omega_2/2$ , and  $\omega_3/2 \pmod{\Lambda}$ . At the three non-zero half periods  $w$ , the function  $\wp(z) - \wp(w)$  has a double zero in  $z$ .

The first key result is that every even elliptic function  $f$  for the lattice  $\Lambda$  is a rational function of  $\wp(z)$ , given by,

$$f(z) = \prod_{w \in P_0} \left( \wp(z) - \wp(w) \right)^{\text{ord}_f(w)} \quad (4.27)$$

which may be easily established by matching poles and zeros.

To incorporate elliptic functions which are odd under  $z \rightarrow -z$  and functions without definite parity we differentiate  $\wp$ ,

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z + \omega)^3} \quad (4.28)$$

This series is absolutely convergent and we have  $\wp'(-z) = -\wp'(z)$ . The function  $\wp'(z)^2$  is even and has a single pole of order 6 at  $z = 0$ . Thus, it must be expressible as a polynomial of degree 3 in  $\wp(z)$ . To determine this polynomial, we proceed by

analogy with the trigonometric case. We expand both  $\wp$  and  $\wp'$  in powers of  $z$  near  $z = 0$  in a convergent series,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{m=1}^{\infty} (m+1)G_{m+2} z^m \\ \wp'(z) &= -\frac{2}{z^3} + \sum_{m=1}^{\infty} m(m+1)G_{m+2} z^{m-1}\end{aligned}\quad (4.29)$$

The coefficients  $G_m$  depend only on  $\Lambda$  and are given by the convergent series,

$$G_m = \sum_{\omega \in \Lambda'} \frac{1}{\omega^m} \quad m \geq 3 \quad (4.30)$$

The reflection symmetry  $-\Lambda = \Lambda$  implies that  $G_{2m+1} = 0$  for all  $m \in \mathbb{N}$  so that,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \mathcal{O}(z^6) \\ \wp'(z) &= -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \mathcal{O}(z^5)\end{aligned}\quad (4.31)$$

Therefore, the pole of order six is cancelled in the combination  $(\wp')^2 - 4\wp^3$ , which in fact has a pole only of order two in  $z$ . By matching all terms of positive or zero power in  $z$  and then using Liouville's theorem, we find the relation,

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6 \quad (4.32)$$

The roots of the cubic polynomial must all produce double zeros as a function of  $z$ , since the left side is a perfect square. But we had seen earlier that double zeros of  $\wp(z) - \wp(w)$  can occur only if  $w \equiv -w \pmod{\Lambda}$ , namely at the three non-zero half-periods. Introducing the values of  $\wp$  at the half periods,

$$e_i = \wp\left(\frac{\omega_i}{2}\right) \quad i = 1, 2, 3 \quad (4.33)$$

we obtain the following factorized form of the cubic polynomial,

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \quad (4.34)$$

where the three symmetric functions of  $e_i$  may be identified as,

$$\begin{aligned}e_1 + e_2 + e_3 &= 0 \\ e_1 e_2 + e_2 e_3 + e_3 e_1 &= -15G_4 \\ e_1 e_2 e_3 &= 35G_6\end{aligned}\quad (4.35)$$

Every elliptic function in  $z$  may be expressed as a rational function of  $\wp(z)$  and  $\wp'(z)$ .

### 4.5.1 Addition Theorem

An immediate application is the existence of an addition theorem,

$$\wp(z+w) + \wp(z) + \wp(w) - \frac{1}{4} \left( \frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} \right)^2 = 0 \quad (4.36)$$

To prove it, one begins by showing that the left side has no poles. For fixed generic  $w$ , the function  $\wp(z+w)$  is an elliptic function in  $z$  with one double pole at  $z = -w \pmod{\Lambda}$ , and thus a rational function of  $\wp(z)$  and  $\wp'(z)$ . Now  $\wp(z) - \wp(w)$  has a simple zero at  $z = -w$ , so it should occur to the power  $-2$ , but it also has a simple zero at  $w = z$ , which is cancelled by multiplying by the square of  $\wp'(z) - \wp'(w)$ . The combination has a double pole in  $z$  at  $z = 0$  which is cancelled by the addition of  $\wp(z)$  and, by symmetry under the interchange of  $z$  and  $w$ , also  $\wp(w)$ . This shows that the left side has no poles and is therefore constant by Liouville's theorem. The constant vanishes by evaluation at a single point, for example  $z = \omega_1/2$ .

## 4.6 Abelian differentials and elliptic integrals

In the previous subsection, we took the generalization of the first point of view we used for trigonometric functions, namely by constructing them by infinite sums. Here we shall now adopt the second point of view and construct inverse functions of elliptic functions by integrating algebraic differentials.

The model used for the inverse trigonometric functions was the following integral,

$$\int_{w_1}^z \frac{dw}{\sqrt{(w-w_1)(w-w_2)}} = \cosh^{-1} \left( \frac{2z - w_1 - w_2}{w_1 - w_2} \right) \quad (4.37)$$

The reason this function must be multiple-valued is that we may take  $|z - w_i| \gg 1$  and complete a closed circle. In this approximation the integrand is just  $dw/w$  so we get back the multiple-valuedness of the log.

The inverse function of an elliptic function is multiple-valued with two independent periods, which we referred to as  $\omega_1$  and  $\omega_2$ . How could we get a second period? Consider the following 1-form,

$$\kappa = \frac{dw}{\sqrt{(w-w_1)(w-w_2)(w-w_3)(w-w_4)}} \quad (4.38)$$

for four distinct points  $w_i \in \mathbb{C}$  with  $i = 1, 2, 3, 4$ . The square root on this fourth degree polynomial may be well-defined by using two different branch cuts in the plane and constructing a double cover on which the differential  $\kappa$  is well-defined, and in fact  $\kappa$

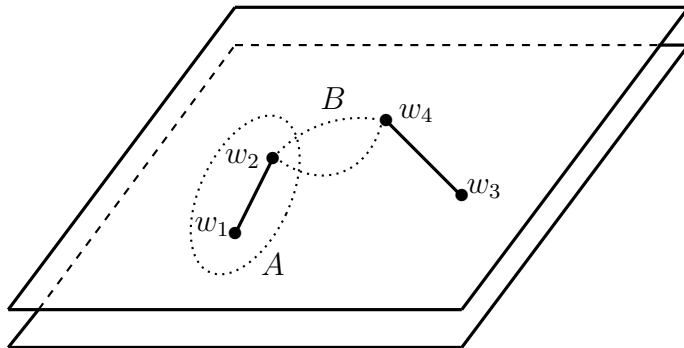


Figure 14: The double cover of  $\mathbb{C}$  used to define the holomorphic differential  $\kappa$ , and the inequivalent cycles  $A$  and  $B$ .

is a *holomorphic differential or Abelian differential of the first kind*. One choice of branch cuts is illustrated in Figure 14.

By construction, the *Abelian integral*,

$$\int_{z_0}^z \kappa = \int_{z_0}^z \frac{dw}{\sqrt{(w-w_1)(w-w_2)(w-w_3)(w-w_4)}} \quad (4.39)$$

is multiple-valued when  $z$  is taken around one of the branch cuts, say  $[w_1, w_2]$ , represented by  $A$  in Figure 14. When  $z$  is taken simultaneously around both branch cuts the integral of  $\kappa$  is single-valued since for large  $w$  the integrand behaves as  $dw/w^2$  whose integral is single-valued. Thus, the two independent periods cannot be generated simply by contours around the two branch cuts since their sum is single-valued. But there is in fact a second cycle that has opened up between the two cuts, a choice of which has been represented by cycle  $B$  in Figure 14. Thus we should expect an identification between the periods  $\omega_1$  and  $\omega_2$  of the lattice  $\Lambda$  of the following form,

$$\oint_A \kappa \approx \omega_1 \qquad \oint_B \kappa \approx \omega_2 \quad (4.40)$$

We shall now relate this approach to the Weierstrass  $\wp$ -function to obtain a precise correspondence. Under a Möbius transformation of  $SL(2, \mathbb{C})$ , the compactified complex plane (namely  $\mathbb{C} \cup \{\infty\}$ ) maps to itself bijectively, and may be used to map  $\kappa$  into a standard form. Consider an arbitrary such transformation by letting,

$$w = \frac{ax+b}{cx+d} \qquad w_i = \frac{ax_i+b}{cx_i+d} \quad (4.41)$$

for  $a, b, c, d \in \mathbb{C}$  and  $ad - bc = 1$ , and points  $x_i$  which map to  $w_i$  for  $i = 1, 2, 3, 4$ .

Under this transformation, we have,

$$dw = \frac{dx}{(cx + d)^2} \quad w - w_i = \frac{x - x_i}{(cx + d)(cx_i + d)} \quad (4.42)$$

Therefore, the differential in terms of the coordinate  $x$  and the points  $x_i$  becomes,

$$\kappa = \frac{(cx_1 + d)(cx_2 + d)(cx_3 + d)(cx_4 + d)}{\sqrt{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}} dx \quad (4.43)$$

Hence, up to a multiplicative factor which does not depend on  $x$ , the differential  $\kappa$  transforms into a form of the same type. Given  $w_1, w_2, w_3, w_4$ , we shall now choose the Möbius transformation such that  $x_4 = \infty$  by setting  $a = w_4c$ , as well as  $x_1 + x_2 + x_3 = 0$  by further choices of  $b$  and  $d$ , and normalize the prefactor so that it is of the form,

$$\kappa = \frac{dx}{\sqrt{4x^3 - g_2x - g_3}} \quad (4.44)$$

Using the differential relation satisfied by  $\wp$  in (4.32), setting  $g_2 = 60G_4$ ,  $g_3 = 140G_6$ , and  $x = \wp(z; \Lambda)$  for a lattice  $\Lambda$  generated by the period  $\omega_1$  and  $\omega_2$ , we find that,

$$\kappa = dz \quad (4.45)$$

But now we are done because we know that the periods of  $\wp(z; \Lambda)$  are  $\omega_1$  and  $\omega_2$ , so that the integrals around closed cycles of  $\kappa = dz$  must all lie in the lattice  $\Lambda$ .

## 4.7 Jacobi $\vartheta$ -functions

In the Weierstrass approach to elliptic functions, the basic building block is the meromorphic  $\wp$ -function and its derivative  $\wp'$ , in terms of which every elliptic function is a rational function. Instead, the Jacobi  $\vartheta$ -function approach produces elliptic functions in terms of Jacobi  $\vartheta$ -functions, which are holomorphic, at the cost of being multiple-valued on  $\mathbb{C}/\Lambda$ . We shall scale the lattice  $\Lambda$  so that  $\omega_1 = 1$  and  $\omega_2 = \tau$  with  $\text{Im}(\tau) > 0$ . The Jacobi  $\vartheta$ -function is then defined by,

$$\vartheta(z|\tau) = \sum_{n \in \mathbb{Z}} e^{i\pi\tau n^2 + 2\pi inz} \quad (4.46)$$

and is often denoted simply by  $\vartheta(z)$  when the  $\tau$ -dependence is clear. The series is absolutely convergent for  $\text{Im}(\tau) > 0$  and defines a holomorphic function in  $z \in \mathbb{C}$ . The function  $\vartheta(z|\tau)$  is manifestly even in  $z \rightarrow -z$ , and transforms as follows under shifts in the lattice  $\Lambda$ ,

$$\begin{aligned} \vartheta(z + 1|\tau) &= \vartheta(z|\tau) \\ \vartheta(z + \tau|\tau) &= \vartheta(z|\tau) e^{-i\pi\tau - 2\pi iz} \end{aligned} \quad (4.47)$$

Thus,  $\vartheta(z|\tau)$  is not an elliptic function in  $z$ . Indeed, it could never be as a doubly periodic holomorphic must be constant. In fact,  $\vartheta$  can be naturally viewed as a holomorphic section of a holomorphic line bundle on  $\mathbb{C}/\Lambda$ .

To find the number of zeros of  $\vartheta(z|\tau)$  as a function of  $z$ , we integrate its logarithmic derivative along the closed boundary of the fundamental parallelogram  $P_0$ , and decompose the integration as a sum of the line integrations along the four edges,

$$\oint_{\partial P_0} d \ln \vartheta(z|\tau) = \left( \int_0^1 + \int_1^{1+\tau} + \int_{1+\tau}^\tau + \int_\tau^0 \right) d \ln \vartheta(z|\tau) \quad (4.48)$$

By periodicity of  $\vartheta(z|\tau)$  under  $z \rightarrow z + 1$ , the contributions from the second and fourth integrals cancel. The contribution of the third integral is just a translate and opposite of the first integral. Using the second relation in (4.47), this integral is readily evaluated, and we find  $2\pi i$  which implies that  $\vartheta(z|\tau)$  has exactly one zero in  $P_0$ . Shifting  $z$  by  $\frac{1}{2} + \frac{\tau}{2}$  produces an equivalent  $\vartheta$ -function, which we shall define more precisely by,

$$\vartheta_1(z|\tau) = \vartheta \left( z + \frac{1}{2} + \frac{\tau}{2} | \tau \right) e^{i\pi z + i\pi \frac{\tau}{4}} = \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} e^{i\pi \tau \nu^2 + 2\pi i \nu (z + \frac{1}{2})} \quad (4.49)$$

Since the function  $\vartheta_1(z|\tau)$  is odd in  $z \rightarrow -z$ , it vanishes at the origin, which is its only zero up to translates by  $\Lambda$ .

It is now a simple result to prove that any elliptic function  $f(z)$  with periods 1 and  $\tau$ , zeros  $a_i$  and poles  $b_i$  for  $i = 1, \dots, N$ , may be expressed as a ratio of  $\vartheta_1(z|\tau)$ -functions by matching the zeros and poles of  $f(z)$ ,

$$f(z) = \prod_{i=1}^N \frac{\vartheta_1(z - a_i | \tau)}{\vartheta_1(z - b_i | \tau)} \quad (4.50)$$

for  $a_i, b_i \in \mathbb{C}$ , not necessarily distinct. The function  $f$  is manifestly periodic under  $z \rightarrow z + 1$ , but generally has a non-trivial transformation property under  $z \rightarrow z + \tau$ . Double periodicity is assured by the condition,

$$\sum_{i=1}^N (a_i - b_i) \in \mathbb{Z} \quad (4.51)$$

Note that the dependence on  $z$  in the exponential of the transformation law cancels out automatically since the number of zeros and poles coincide.

Another construction is by taking a single logarithmic derivative,

$$\varphi_{ab}(z) = \partial_z \ln \frac{\vartheta_1(z - a | \tau)}{\vartheta_1(z - b | \tau)} \quad (4.52)$$



for  $a, b \in \mathbb{C}$ . This combination is manifestly invariant under  $z \rightarrow z + 1$ , while under  $z \rightarrow z + \tau$  the logarithm of the ratio is shifted by a constant so that  $\varphi_{ab}$  is an elliptic function with simple poles at  $a, b$  with residues  $\pm 1$  at the points  $a, b$ . It is in fact more proper to think of  $\varphi_{ab}$  as a meromorphic differential 1-form  $\varphi_{ab}(z)dz$ , referred to as an *Abelian differential of the third kind*.

The third construction is by taking a double logarithmic derivative,

$$\varphi_a(z) = \partial_a \partial_z \ln \vartheta_1(z - a | \tau) = \partial_a \varphi_{ab}(z) \quad (4.53)$$

Again, it is proper to view  $\varphi_a(z)$  as a meromorphic differential 1-form, referred to as an *Abelian differential of the second kind*. It has a double pole at  $z = a$ . Translating  $a$  to zero,  $\varphi_0(z)$  has a double pole at 0 and differs from  $\wp(z)$  by a constant. The constant may be evaluated by expanding near  $z = 0$ , and we find the relation,

$$\wp(z) = \varphi_0(z) - \frac{2}{3} \frac{\vartheta_1'''(0 | \tau)}{\vartheta_1'(0 | \tau)} \quad (4.54)$$

where the primes on  $\vartheta_1$  denote taking the derivative with respect to its first argument. Integrating this formula twice in  $z$  and matching the integration constants gives the product formula for the  $\vartheta_1$ -function, with  $q = e^{2\pi i \tau}$ ,

$$\vartheta_1(z | \tau) = 2q^{\frac{1}{8}} \sin \pi z \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i z})(1 - q^n e^{-2\pi i z})(1 - q^n) \quad (4.55)$$

We shall leave its derivation as a problem set.

## 4.8 Scalar Green function on the torus

The scalar Green function  $G(z, w | \Lambda)$  on  $\mathbb{C}/\Lambda$ , for the flat metric  $ds^2 = |dz|^2$  on  $\mathbb{C}/\Lambda$  for  $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ , is defined to be a real-valued symmetric function which is the inverse of the Laplace operator  $\Delta = -4\partial_z \partial_{\bar{z}}$  on the space of functions transverse to constants. A standard normalization of the scalar Green function is obtained by assuming translation invariance on  $\mathbb{C}/\Lambda$  so that  $G(z, w | \Lambda) = G(z - w | \tau)$ ,

$$\Delta_z G(z - w | \tau) = 4\pi \delta(z - w) - \frac{4\pi}{\text{Im}(\tau)} \quad (4.56)$$

The integration measure and Dirac  $\delta$ -function are normalized as follows,

$$d^2 z = \frac{i}{2} dz \wedge d\bar{z} \quad \int_{\mathbb{C}/\Lambda} d^2 z \delta(z - w) f(z) = f(w) \quad (4.57)$$

so that the area of  $\mathbb{C}/\Lambda$  is  $\text{Im}(\tau)$ . The solution for  $G$  is easy to construct. For small  $|z - w|$  we have  $G(z - w|\tau) \sim -\ln|z - w|^2$ . The function  $\vartheta_1(z - w)/\vartheta_1'(0)$  behaves as  $z - w$  for small  $|z - w|$ , but it is not doubly periodic, so we must add a correction,

$$G(z - w|\tau) = -\ln \left| \frac{\vartheta_1(z - w|\tau)}{\vartheta_1'(0|\tau)} \right|^2 - \frac{\pi}{2\text{Im}(\tau)}(z - w - \bar{z} + \bar{w})^2 + G_0(\tau) \quad (4.58)$$

where  $G_0(\tau)$  is independent of  $z$ , and may be fixed by requiring  $\int d^2z G(z - w|\tau) = 0$ .

## 4.9 Modular transformations

We specified the lattice of periods  $\Lambda$  by giving two periods  $\omega_1$  and  $\omega_2$  such that  $\omega_1/\omega_2 \notin \mathbb{R}$ . But this specification is not unique. Given two other points  $\omega'_1, \omega'_2$  with  $\omega'_1/\omega'_2$  be in the lattice, we can ask when  $\Lambda$  is equivalently generated by these new periods (i.e. we are seeking the *automorphisms of the lattice*  $\Lambda$ ). Since  $\omega'_1, \omega'_2 \in \Lambda$  they are linear combinations with integer coefficients of  $\omega_1, \omega_2$ ,

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (4.59)$$

with  $a, b, c, d$ . To have  $\omega'_1, \omega'_2$  generate the same lattice we need this transformation to be invertible, which requires  $ad - bc = \pm 1$ . To preserve the orientation of the lattice (the ordering of  $\omega_1$  and  $\omega_2$ , we must take  $ad - bc = 1$ , so that  $\det M = 1$ . These transformations form the *modular group*  $SL(2, \mathbb{Z})$ . Elliptic functions have interesting transformation laws under  $SL(2, \mathbb{Z})$ , but this topic goes beyond the scope of these lectures.

## 5 Fourier analysis and linear differential operators

In this section, we begin the study of linear differential operators, their symmetries, spectrum, eigenfunctions, Green function, and resolvent. A natural setting for the function spaces on which linear operators act is Hilbert space, familiar from quantum mechanics. A good test case for the objects we study and the kind of questions we ask is provided by Fourier series and the Fourier transform.

### 5.1 The Fourier transform

It is natural to associate the Fourier series and transform with a linear operator, such as the Schrödinger operator of the *free particle on an interval with periodic boundary conditions or on the real line*,<sup>1</sup>

$$H = -\frac{d^2}{dx^2} \quad (5.1)$$

This operator is self-adjoint (to be defined in detail later) so that its eigenvalues are real and its eigenvectors are mutually orthogonal. The operator  $H$  is invariant under translations, whose infinitesimal generator is the momentum operator,

$$P = -i\frac{d}{dx} \quad (5.2)$$

The operator  $P$  is self-adjoint so that its eigenvalues are real. Since  $H$  and  $P$  commute they may be diagonalized in the same basis. In fact, we may therefore generalize our considerations to any operator  $\mathfrak{H}(P)$  which is a degree  $n$  polynomial of  $P$ ,

$$\mathfrak{H}(P) = a_n P^n + a_{n-1} P^{n-1} + \cdots + a_1 P + a_0 I \quad (5.3)$$

with real coefficients  $a_n \in \mathbb{R}$ . Any such  $\mathfrak{H}(P)$  is a linear differential operator with constant real coefficients and may be diagonalized in the same basis as  $P$ . (Further generalizations to an arbitrary real-analytic function  $\mathfrak{H}(P)$  will lead us to linear operators which are not necessarily local.) The operator  $P$  may be easily diagonalized, and we shall denote its eigenvalues by  $k$  and associated eigenfunctions by  $\psi_k(x)$ ,

$$\psi_k(x) = e^{ikx} \quad (5.4)$$

It is then immediate that  $\psi_k(x)$  are also eigenfunctions of  $H$  with eigenvalue  $k^2$  and of  $\mathfrak{H}(P)$  with eigenvalue  $\mathfrak{H}(k)$ . For  $x \in [-\pi, \pi]$ , the periodicity condition requires

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<sup>1</sup>Throughout, we shall strip physical quantities of mathematically irrelevant parameters such as  $\hbar, c$ , masses and coupling parameters. Their dependence may be easily restored using rescaling.

$k \in \mathbb{Z}$ , and the spectra of  $P, H, \mathfrak{H}(P)$  are all *discrete*. For  $x \in \mathbb{R}$ , the range of  $k$  is  $\mathbb{R}$ , and the spectra are *continuous*. Orthogonality for each case is expressed by,

$$\begin{aligned} \int_{-\pi}^{\pi} dx \overline{\psi_k(x)} \psi_\ell(x) &= 2\pi \delta_{k,\ell} & k, \ell \in \mathbb{Z} \\ \int_{\mathbb{R}} dx \overline{\psi_k(x)} \psi_\ell(x) &= 2\pi \delta(k - \ell) & k, \ell \in \mathbb{R} \end{aligned} \quad (5.5)$$

From quantum mechanics, we know that the space of eigenfunctions of a self-adjoint operator is complete, so that we also have the following relations,

$$\begin{aligned} \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \overline{\psi_k(x)} \psi_k(y) &= \delta(x - y) & x, y \in [-\pi, \pi] \\ \int_{\mathbb{R}} \frac{dk}{2\pi} \overline{\psi_k(x)} \psi_k(y) &= \delta(x - y) & x, y \in \mathbb{R} \end{aligned} \quad (5.6)$$

Note that on the interval  $[-\pi, \pi]$ , the eigenfunctions are normalizable and belong to  $L^2([-\pi, \pi])$  while on the full line  $\mathbb{R}$ , the eigenfunctions are not normalizable, and do not belong to  $L^2(\mathbb{R})$ . But allowing for the Dirac  $\delta$ -function distribution, one may nonetheless impose the above *continuum normalization*. With the help of the above orthogonality and completeness relations, any  $L^2$ -function  $f(x)$  may be decomposed into a Fourier series or Fourier transform,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ikx} & \hat{f}_k &= \int_{-\pi}^{\pi} dx e^{-ikx} f(x) \\ f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{f}(k) e^{ikx} & \hat{f}(k) &= \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \end{aligned} \quad (5.7)$$

Functional inner products of two functions and their Fourier series or transforms are related to one another by the *Plancherel Theorem*,

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{dx}{2\pi} \overline{f(x)} g(x) &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \overline{\hat{f}_k} \hat{g}_k \\ \int_{-\infty}^{\infty} dx \overline{f(x)} g(x) &= \int_{\mathbb{R}} \frac{dk}{2\pi} \overline{\hat{f}(k)} \hat{g}(k) \end{aligned} \quad (5.8)$$

The convolution of two functions  $f * g$  is (inverse) Fourier transform of the the product of the Fourier transforms  $\hat{f} \hat{g}$ , as given by the following formula (given here for  $\mathbb{R}$ ),

$$f * g(x) = \int_{\mathbb{R}} dy f(x - y) g(y) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{f}(k) \hat{g}(k) e^{ikx} \quad (5.9)$$

## 5.2 The Green function on a finite interval

The Green function  $G$  is the inverse of  $H$ , when the inverse exists,

$$HG = I \quad (5.10)$$

For a self-adjoint operator, the existence of zero eigenvalues will present a difficulty in defining the inverse. In fact, considering the operator  $H$  on the interval  $[-\pi, \pi]$  with periodic boundary conditions, it is clear that the constant function is an eigenfunction with zero eigenvalue and thus, strictly speaking, the operator does not have an inverse. We have here a problem of linear algebra that we shall address in all generality later on. However, we can modify the problem by considering the operator  $H$  acting only on functions that are orthogonal to the zero mode. This new operator is invertible, as it has no zero eigenvalues. The corresponding differential equation is as follows,

$$-\frac{d^2}{dx^2}G_H(x, y) = \delta(x - y) - \frac{1}{2\pi} \quad \int_{-\pi}^{\pi} dx G_H(x, y) = 0 \quad (5.11)$$

The right side has been modified so that it is orthogonal to the constant mode, i.e. it integrates to zero. Since the operator is self-adjoint and real, the Green function is naturally symmetric and real  $G_H(x, y) = G_H(y, x)$  and by translation invariance we have  $G_H(x, y) = G(x - y)$ . Thus, the equation determining  $G(x)$  are,

$$-\frac{d^2}{dx^2}G(x) = \delta(x) - \frac{1}{2\pi} \quad \int_{-\pi}^{\pi} dx G(x) = 0 \quad (5.12)$$

Decomposing  $G(x)$  in Fourier modes, we have,

$$G(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^0} \hat{G}_k e^{ikx} \quad k^2 \hat{G}_k = 1 \quad k \in \mathbb{Z}^0 \quad (5.13)$$

where  $\mathbb{Z}^0 = \mathbb{Z} \setminus \{0\}$ . We may also solve the differential equation for  $G(x)$  directly. Away from  $x = 0$  the solution is given by,

$$G(x) = -\frac{x^2}{4\pi} + a_{\pm}x + b_{\pm} \quad (5.14)$$

where  $\pm$  corresponds to  $x > 0$  and  $x < 0$ . Continuity of  $G(x)$  at  $x = 0$  requires  $b_- = b_+$ ; integrating the equation over a small interval  $[-\varepsilon, \varepsilon]$  for  $\varepsilon > 0$  gives  $G'(\varepsilon) - G'(-\varepsilon) = a_+ - a_- = -1$ ; and symmetry  $G(-x) = G(x)$  gives  $a_- = -a_+$ , so that,

$$G(x) = \frac{x^2}{4\pi} - \frac{1}{2}|x| + \frac{\pi}{6} \quad (5.15)$$

The constant term was fixed by requiring that the integral of  $G(x)$  vanish. Note that the function  $G(x)$  is automatically continuous at  $x = \pm\pi$ . This resulting expression may be recast in terms of a Bernoulli polynomial  $\pi B_2(|x|/2\pi)$ . If we had taken an interval of general length  $[-\pi L, \pi L]$ , the corresponding Green function would be,

$$G(x) = \frac{x^2}{4\pi L} - \frac{1}{2}|x| + \frac{\pi L}{6} \quad (5.16)$$

We see that in the limit  $L \rightarrow \infty$  at fixed  $x$ , the first term tends to zero, the second term is finite, and the third term blows up. To understand this better, we may look directly at the case of the full real line.

### 5.3 The Green function on $\mathbb{R}$

On the real line  $\mathbb{R}$ , we may attempt to define the Green function by,

$$HG_H = I \quad \Leftrightarrow \quad -\frac{d^2}{dx^2}G_H(x, y) = \delta(x - y) \quad (5.17)$$

Since the operator  $H$  is translation invariant, the Green function satisfies  $G_H(x, y) = G(x - y)$ . Since the operator  $H$  is self-adjoint but real, it is in fact *symmetric*, and it is natural to require its inverse  $G_H$  to be symmetric as well so that  $G_H(y, x) = G_H(x, y)$  and thus  $G(-x) = G(x)$ . Thus, it simply remains to solve the following equation,

$$-\frac{d^2}{dx^2}G(x) = \delta(x) \quad G(-x) = G(x) \quad (5.18)$$

There are several illuminating ways of solving this equation. The first, using the fact that we have a differential equation with constant coefficients, is to take the Fourier transform of both sides,

$$k^2\hat{G}(k) = 1 \quad G(x) = \int_{\mathbb{R}} \frac{dk}{2\pi} \hat{G}(k) e^{ikx} \quad (5.19)$$

We shall not attempt to evaluate the  $k$ -integral here since it diverges at  $k = 0$ . The second is to solve the differential equation as is familiar from undergraduate quantum mechanics. Since away from  $x = 0$  the right side of the equation vanishes,  $G(x)$  must be linear in  $x$ , though both sides of  $x = 0$  will have different coefficients,

$$G(x) = \begin{cases} a_+x + b_+ & x > 0 \\ a_-x + b_- & x < 0 \end{cases} \quad (5.20)$$

Symmetry requires  $a_- = -a_+$  and  $b_- = b_+$ , which automatically guarantees continuity of  $G(x)$  across  $x = 0$ . Integrating the differential equation over the interval  $[-\varepsilon, \varepsilon]$  for  $\varepsilon > 0$  gives  $G'(\varepsilon) - G'(-\varepsilon) = a_+ - a_- = -1$  and thus,

$$G(x) = -\frac{1}{2}|x| + c \quad (5.21)$$

Here  $c$  may be determined by boundary conditions. Looking back at the Fourier transform solution, we see that the  $k$ -integral diverges at  $k = 0$ , that  $G(x)$  is not an  $L^2$  function, and does not admit a Fourier transform. The different ways of *regularizing* this divergence lead to different values for the constant  $c$ .

The source of all of our problems is again that the range of the operator  $H$  is functions which integrate to zero. We may modify the definition of the Green function so that this property is manifest by introducing an opposite charge,

$$-\frac{d^2}{dx^2}G_H(x; y, z) = \delta(x - y) - \delta(x - z) \quad (5.22)$$

Now taking the Fourier transform in  $x$ , we find the solution,

$$G_H(x; y, z) = \int_{\mathbb{R}} \frac{dk}{2\pi} \frac{e^{ik(x-y)} - e^{ik(x-z)}}{k^2} \quad (5.23)$$

This integral is convergent, and may be evaluated by solving the differential equation,

$$G_H(x; y, z) = -\frac{1}{2}|x - y| + \frac{1}{2}|x - z| + x\lambda(y, z) + \mu(y, z) \quad (5.24)$$

where the functions  $\lambda(y, z)$  and  $\mu(y, z)$  must be determined by boundary conditions, or by differential equations on  $y$  and  $z$ .

## 5.4 The resolvent Green function

For a general self-adjoint operator, the resolvent Green function is a generalization of the ordinary Green function which is made to depend on an extra parameter  $E \in \mathbb{C}$ ,

$$(H - E)G_E = I \quad (5.25)$$

Taking  $\text{Im}(E) \neq 0$ , we see that  $H - E$  genuinely has an inverse and the combined operator has no zero modes. For simplicity let us assume that the spectrum of  $H$  is discrete, with eigenvalues  $\lambda_n \in \mathbb{R}$  and eigenfunctions  $\psi_n(x)$  so that,

$$H\psi_n(x) = \lambda_n\psi_n(x) \quad (5.26)$$

and we have the completeness relation,

$$\sum_n \psi_n(x)\overline{\psi_n(y)} = \delta(x, y) \quad (5.27)$$

Then the resolvent Green function may be solved for in terms of these data,

$$G_E(x, y) = \sum_n \frac{\psi_n(x)\overline{\psi_n(y)}}{\lambda_n - E} \quad (5.28)$$

The function  $G_E(x, y)$  is analytic in  $E \in \mathbb{C}$  except for poles on the real axis at the eigenvalues  $\lambda_n$ . By evaluating the Green function just above and just below the real axis, we can actually pick up these poles, using the distributional equation,

$$\lim_{\varepsilon \rightarrow 0} \left( \frac{1}{\lambda - i\varepsilon} - \frac{1}{\lambda + i\varepsilon} \right) = 2\pi i \delta(\lambda) \quad (5.29)$$

we find,

$$G_{\lambda - i\varepsilon}(x, y) - G_{\lambda + i\varepsilon}(x, y) = 2\pi i \sum_n \delta(\lambda - \lambda_n) \psi_n(x) \overline{\psi_n(y)} \quad (5.30)$$

Thus we see that the the entire spectrum of the operator may be read off from the resolvent Green function, including the normalized eigenfunctions.

However, the resolvent Green function may also be obtained by solving differential equations. Thus, the solutions of the differential equation will allow us to simply deduce the spectrum. To illustrate this method, we begin with the operator  $H$  on the interval  $[-\pi, \pi]$ , for which the resolvent Green function satisfies,

$$\left( -\frac{d^2}{dx^2} - \omega^2 \right) G_E(x, y) = 2\pi i \delta(x - y) \quad (5.31)$$

Taking  $E \in \mathbb{C} \setminus \mathbb{R}$ , the operator  $H - E$  has no zero modes, and is genuinely invertible. To obtain  $G_E(x, y)$ , we use translation invariance to set  $y = 0$ , set  $E = \omega^2$ , choose  $\text{Im}(\omega) > 0$ , and solve the differential equation with periodic boundary conditions on the interval  $[-\pi, \pi]$ . Away from  $x = 0$ , the solution is given by a linear combination of the exponentials  $e^{i\omega x}$  and  $e^{-i\omega x}$ . The conditions  $G_E(x, 0) = G_E(-x, 0)$  together with the discontinuity of the derivative at  $x = 0$  give for  $x > 0$ ,

$$G_E(x, 0) = \left( A + \frac{i}{4\omega} \right) e^{i\omega x} + \left( A - \frac{i}{4\omega} \right) e^{-i\omega x} \quad (5.32)$$

Continuity at  $x = \pm\pi$  is automatic by the symmetry  $G_E(x, 0) = G_E(-x, 0)$ , while continuity of the first derivative requires  $1/A = -4\omega \tan(\pi\omega)$ , so that for  $x > 0$ ,

$$G_E(x, 0) = \frac{i}{2\omega} \left( \frac{e^{i\omega x}}{1 - e^{-2i\pi\omega}} + \frac{e^{-i\omega x}}{e^{2i\pi\omega} - 1} \right) \quad (5.33)$$

Clearly, we have simple poles at integer values of  $\omega$ , and the residues are precisely the normalized eigenfunctions.



## 5.5 The resolvent and the zeta function

Closely related to the resolvent Green function is the resolvent, defined by,

$$R(E) = \text{Tr} \left( \frac{1}{H - E} \right) = \sum_n \frac{1}{\lambda_n - E} \quad (5.34)$$

whenever it exists. For an operator  $H$  with positive spectrum, one also defines the zeta-function,

$$\zeta_H(s) = \text{Tr} \left( \frac{1}{H^s} \right) = \sum_n \frac{1}{\lambda_n^s} \quad (5.35)$$

If there are zero modes, the summation is over the non-zero eigenvalues only, an operation usually denoted by  $\text{Tr}'$  instead of  $\text{Tr}$ . Introducing the spectral density,

$$\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n) \quad (5.36)$$

these quantities may be recast in integral form,

$$R(E) = \int_{\mathbb{R}} d\lambda \frac{\rho(\lambda)}{\lambda - E} \quad \zeta_H(s) = \int_0^{\infty} d\lambda \lambda^{-s} \rho(\lambda) \quad (5.37)$$

## 6 Linear differential operators

In this section, we shall give the definition of a group, a field, a vector space, and a Hilbert space, and linear operators acting on the Hilbert space.

### 6.1 Groups and Fields

A set  $G$  equipped with an operation  $\star$  is a *group*  $G_\star$  provided  $G$  and  $\star$  satisfy,

1. *Closure*: For any pair of elements  $g_1, g_2 \in G$  the operation obeys  $g_1 \star g_2 \in G$ ;
2. *Associativity*: For any triplet of elements  $g_1, g_2, g_3 \in G$  the operation obeys  $(g_1 \star g_2) \star g_3 = g_1 \star (g_2 \star g_3) = g_1 \star g_2 \star g_3$ ;
3. *Identity*: There exists an element  $e \in G$  such that  $e \star g = g \star e = g$  for all  $g \in G$ ;
4. *Inverse*: For every  $g \in G$ , there exists a  $g^{-1} \in G$  such that  $g \star g^{-1} = g^{-1} \star g = e$ .

It follows from the definition of a group that the identity element  $e$  is unique, and that for every element  $g$ , the inverse  $g^{-1}$  is unique. A subset  $H$  of the group  $G_\star$  is a *subgroup*  $H_\star$  of  $G_\star$  provided  $h_1 \star h_2 \in H$  for all  $h_1, h_2 \in H$  and  $h^{-1} \in H$  for all  $h \in H$ . A group  $G_\star$  is *commutative* or *Abelian* if the product satisfies  $g_1 \star g_2 = g_2 \star g_1$  for all  $g_1, g_2 \in G$ . Otherwise, the group is said to be *non-commutative* or *non-Abelian*.

Basic examples of groups under the operations of addition and multiplication are,

integers	$\mathbb{Z}_+$	
rational	$\mathbb{Q}_+$	$\mathbb{Q}_\times^0$
real	$\mathbb{R}_+$	$\mathbb{R}_\times^0$
complex	$\mathbb{C}_+$	$\mathbb{C}_\times^0$

(6.1)

with the subgroup inclusion relations  $\mathbb{Z}_+ \subset \mathbb{Q}_+ \subset \mathbb{R}_+ \subset \mathbb{C}_+$  and  $\mathbb{Q}_\times^0 \subset \mathbb{R}_\times^0 \subset \mathbb{C}_\times^0$  where the subscript 0 stands for the removal of 0, the identity element of addition.

Another example is the *group of permutations*  $\mathfrak{S}_n$  of a set  $A_n$  with  $n$  elements. For  $n \leq 4$  this group is Abelian, but for  $n \geq 5$  it is non-Abelian. More generally, a group  $G$  acting on a set  $A$  by permuting its elements is a *transformation group* even when it is not the full permutation group. When the set  $A$  is a vector space, and  $G$  acts by a linear transformation on  $A$ , we have a *representation of  $G$  on  $A$* .

A set  $F$  equipped with two operations, addition  $+$ , and multiplication  $\times$ , is a *field* provided  $F_+$  is an Abelian group with identity element 0,  $F_\times^0$  is an Abelian group, and the operations are related by the property of *distributivity* for all  $a, b, c \in F$ ,

$$a \times (b + c) = a \times b + a \times c \tag{6.2}$$

The most frequently used examples of fields are  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$ , which allow us to do all operations needed in physics.

## 6.2 Vector spaces and Hilbert spaces

A vector space  $V$  over a field  $F$  is a set  $V$ , equipped with two operation,

1. *addition of vectors in  $V$*  under which  $V$  is an Abelian group;
2. *scalar multiplication* under which  $F \times V \rightarrow V$  with the operation denoted as multiplication,  $(\alpha, \mathbf{u}) \in F \times V \rightarrow \alpha\mathbf{u} \in V$  satisfying the following axioms of generalized distributivity for all  $\alpha, \beta \in F$  and all  $\mathbf{u}, \mathbf{v} \in V$ ,

$$\begin{aligned}(\alpha + \beta)\mathbf{u} &= \alpha\mathbf{u} + \beta\mathbf{u} \\ \alpha(\mathbf{u} + \mathbf{v}) &= \alpha\mathbf{u} + \alpha\mathbf{v} \\ (\alpha\beta)\mathbf{u} &= \alpha(\beta\mathbf{u})\end{aligned}\tag{6.3}$$

and with the zero  $0$  of  $F$  related to the zero  $\mathbf{0}$  of  $V$  by  $0\mathbf{u} = \mathbf{0}$ , and the unit  $1$  of  $F$  acting trivially on  $V$  by  $1\mathbf{u} = \mathbf{u}$ .

The fields of interest in physics will be  $\mathbb{R}$  or  $\mathbb{C}$  giving respectively a real or a complex vector space. The vector space  $V$  will be either isomorphic to  $\mathbb{R}^n$ ,  $\mathbb{C}^n$  or be an infinite-dimensional generalization, such as a complex function space.

Consider a complex vector space  $V$  and denote its elements by  $\mathbf{u}$  or  $|u\rangle$  in Dirac notation. A *Hermitian inner product* is a map  $V \times V \rightarrow \mathbb{C}$ , denoted by  $(\mathbf{u}, \mathbf{v})$  or  $\langle u|v\rangle$  for  $\mathbf{u}, \mathbf{v} \in V$  such that for all  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and all  $\alpha, \beta \in \mathbb{C}$  we have,

$$\begin{aligned}(\mathbf{v}, \mathbf{u}) &= (\mathbf{u}, \mathbf{v})^* \\ (\mathbf{u}, \alpha\mathbf{v} + \beta\mathbf{w}) &= \alpha(\mathbf{u}, \mathbf{v}) + \beta(\mathbf{u}, \mathbf{w}) \\ (\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}) &= \alpha^*(\mathbf{u}, \mathbf{w}) + \beta^*(\mathbf{v}, \mathbf{w})\end{aligned}\tag{6.4}$$

From the Hermitian inner product, one defines  $\|\mathbf{u}\|^2 = (\mathbf{u}, \mathbf{u})$  which is a norm on  $V$  provided it is (1) positive  $\|\mathbf{u}\|^2 \geq 0$  for all  $\mathbf{u} \in V$ ; (2) definite so that  $\|\mathbf{u}\| = 0$  implies  $\mathbf{u} = \mathbf{0}$ ; (3) satisfies the triangle inequality  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$  for all  $\mathbf{u}, \mathbf{v} \in V$ . A vector space with a norm is referred to as a *normed vector space*, which is automatically a metric topological space, with the topology induced by the norm.

A Hilbert space  $\mathcal{H}$  is a complex vector space endowed with a Hermitian inner product and associated norm, which is *metrically complete*, i.e. every Cauchy sequence in  $\mathcal{H}$  converges in  $\mathcal{H}$ . The completeness property is automatically satisfied if the vector space has finite dimension, but gives a non-trivial condition when the dimension is infinite. The Hilbert spaces in quantum mechanics are required to be *separable*, which means that they admit a countable orthonormal basis. If the number of orthonormal basis vectors of a separable Hilbert space  $\mathcal{H}$  is  $N < \infty$ , then  $\mathcal{H}$  is isomorphic to  $\mathbf{C}^N$ . On the other hand, all separable Hilbert spaces of infinite dimension are isomorphic to one another.

A *linear form* on  $\mathcal{H}$  is a continuous map from  $\mathcal{H} \rightarrow \mathbb{C}$  and every linear form may be expressed as a map  $v^\dagger = (\mathbf{v}, \cdot) \rightarrow \mathbb{C}$  for some  $\mathbf{v} \in \mathcal{H}$ , or  $\langle v |$  in Dirac notation. The space of all continuous linear forms on  $\mathcal{H}$  is by definition the Hilbert space dual to  $\mathcal{H}$  which for a Hilbert space coincides with  $\mathcal{H}$ .

### 6.3 Linear operators on finite-dimensional Hilbert spaces

In a Hilbert space  $\mathcal{H}$  of finite dimension  $N$ , we may choose an orthonormal basis of vectors  $\mathbf{u}_1, \dots, \mathbf{u}_N \in \mathcal{H}$  which satisfy  $(\mathbf{u}_m, \mathbf{u}_n) = \delta_{m,n}$  for all  $1 \leq m, n \leq N$ . It follows that an arbitrary vector  $\mathbf{v} \in \mathcal{H}$  may be decomposed into a linear combination of  $\mathbf{u}_n$ ,

$$\mathbf{v} = \sum_{n=1}^N v_n \mathbf{u}_n \quad v_n = (\mathbf{u}_n, \mathbf{v}) \quad (6.5)$$

The inner product of two vectors  $\mathbf{v}, \mathbf{w}$  is then given by,

$$(\mathbf{v}, \mathbf{w}) = \sum_{n=1}^N \bar{v}_n w_n \quad v_n = (\mathbf{u}_n, \mathbf{v}), w_n = (\mathbf{u}_n, \mathbf{w}) \quad (6.6)$$

The arbitrary vector  $\mathbf{v} \in \mathcal{H}$  and the orthonormal basis vectors  $\mathbf{u}_1, \dots, \mathbf{u}_N \in \mathcal{H}$  may be represented by column matrices, as follows,

$$\mathbf{v} \rightarrow V = \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_N \end{pmatrix} \quad U_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \dots \quad U_N = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \quad (6.7)$$

The linear forms may similarly be represented by row vectors,

$$\begin{aligned} (\mathbf{v}, \cdot) \rightarrow V^\dagger &= (v_1^* \ v_2^* \ \dots \ v_N^*) \\ U_1^\dagger &= (1 \ 0 \ \dots \ 0) \\ U_2^\dagger &= (0 \ 1 \ \dots \ 0) \\ &\dots \\ U_N^\dagger &= (0 \ 0 \ \dots \ 1) \end{aligned} \quad (6.8)$$

The inner product of two vectors is then given by matrix contraction,

$$(\mathbf{v}, \mathbf{w}) = V^\dagger W = \sum_{n=1}^N \bar{v}_n w_n \quad (6.9)$$

The associated  $\|\mathbf{v}\|^2$  satisfies all the axioms for a positive definite norm.

A *linear operator* (or simply operator when linearity is clear from the context) is a linear map from  $\mathcal{H} \rightarrow \mathcal{H}$ . Every linear operator  $A$  on an  $N$ -dimensional Hilbert space  $\mathcal{H}$  is equivalent to an  $N \times N$  matrix, whose entries  $A_{mn}$  are the *matrix elements* in a given basis, such as the orthonormal basis  $\{\mathbf{u}_n\}_{n=1,\dots,N}$  constructed earlier,

$$A_{mn} = (\mathbf{u}_m, A\mathbf{u}_n) \quad (\mathbf{u}_m, \mathbf{u}_n) = \delta_{mn} \quad (6.10)$$

The identity operator may be represented by the *completeness relation*,

$$I_{\mathcal{H}} = \sum_{n=1}^N U_n U_n^\dagger = \sum_{n=1}^N |u_n\rangle\langle u_n| = \sum_{n=1}^N \mathbf{u}_n(\mathbf{u}_n, \cdot) \quad (6.11)$$

The product of operators maps to matrix multiplication. If  $A_{mn}$  and  $B_{mn}$  are the matrix elements of  $A$  and  $B$  in a common orthonormal basis  $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$  of  $\mathcal{H}$ , then the matrix elements  $(AB)_{mn}$  of the product  $AB$  are analogously defined by  $(AB)_{mn} = (\mathbf{u}_m, AB\mathbf{u}_n)$ . Upon inserting the completeness relation between  $A$  and  $B$ , we find,

$$(AB)_{mn} = \sum_{p=1}^N A_{mp} B_{pn} \quad (6.12)$$

The product is associative and generally non-commutative. The identity operator is the identity matrix  $I_{\mathcal{H}}$  and the inverse of an operator  $A$  is the inverse matrix  $A^{-1}$  when the inverse exists.

### 6.3.1 Hermitian operators and unitary operators

A Hermitian operator is represented by a Hermitian matrix  $A^\dagger = A$ , whose matrix elements, for all  $m, n = 1, \dots, N$ , satisfy,

$$A_{mn} = A_{nm}^* \quad (6.13)$$

Hermitian operators correspond to *physical observables* in quantum mechanics and therefore play a central role. Their spectrum obeys a set of fundamental properties.

1. The eigenvalues of a Hermitian matrix are real;
2. Eigenvectors corresponding to distinct eigenvalues are mutually orthogonal;
3. A Hermitian matrix may be written as a direct sum of mutually orthogonal projection operators, weighted by the distinct eigenvalues.

Let's prove the assertions using the Dirac notation. Let  $A$  be a Hermitian matrix with eigenvalue  $a$  and associated eigenvector  $|\varphi\rangle \neq 0$ ,

$$A|\varphi\rangle = a|\varphi\rangle \quad (6.14)$$

Taking the  $\dagger$  of this equation gives  $\langle\varphi|A^\dagger = a^*\langle\varphi|$ , and using the fact that  $A^\dagger = A$ , simplifies this equation to  $\langle\varphi|A = a^*\langle\varphi|$ . Taking the inner product of this equation with  $|\varphi\rangle$  and of the eigenvalue equation with  $\langle\varphi|$ , we obtain,

$$\langle\varphi|A|\varphi\rangle = a\langle\varphi|\varphi\rangle = a^*\langle\varphi|\varphi\rangle \quad (6.15)$$

Since  $|\varphi\rangle \neq 0$ , we have  $\langle\varphi|\varphi\rangle \neq 0$ , and hence  $a^* = a$ , which proves the first assertion.

To prove the second assertion, let  $a' \neq a$  be two distinct eigenvalues (which are both real by item 1.), and let  $|\varphi\rangle$  and  $|\varphi'\rangle$  be their associated eigenvectors,

$$\begin{aligned} A|\varphi\rangle &= a|\varphi\rangle \\ A|\varphi'\rangle &= a'|\varphi'\rangle \end{aligned} \quad (6.16)$$

Taking the inner product of the first line with  $\langle\varphi'|$  and of the second line by  $\langle\varphi|$ , and using  $\langle\varphi|\varphi'\rangle = \langle\varphi'|\varphi\rangle^*$ , and  $\langle\varphi|A|\varphi'\rangle = \langle\varphi'|A^\dagger|\varphi\rangle^* = \langle\varphi'|A|\varphi\rangle$ , we find that

$$\langle\varphi'|A|\varphi\rangle = a\langle\varphi|\varphi'\rangle = a'\langle\varphi|\varphi'\rangle \quad (6.17)$$

Since  $a' \neq a$ , we must have  $\langle\varphi'|\varphi\rangle = 0$  which proves item 2..

Finally, to prove item 3. we note that a given eigenvalue  $a_i$  may correspond to one or several linearly independent eigenvectors, which span the entire eigenspace  $\mathcal{E}_i$  associated with  $a_i$ . By the result of item 2. the eigenspaces associated with distinct eigenvalues are also mutually orthogonal, so that,

$$A = \sum_i a_i P_i \quad a_i \neq a_j \quad \text{when} \quad i \neq j \quad (6.18)$$

where  $P_i$  represents the projection operator on eigenspace  $\mathcal{E}_i$ . The dimension  $\dim P_i$  is referred to as the *degeneracy (or multiplicity) of the eigenvalue  $a_i$* . This number clearly coincides with the degeneracy of the root  $a_i$  in the characteristic equation. This produces a block-diagonal representation of  $A$ , which proves item 3.,

$$A = \begin{pmatrix} a_1 I_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 I_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 I_3 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & a_m I_m \end{pmatrix} \quad (6.19)$$

Constructing eigenvalues and eigenvectors in general is difficult, even in finite dimension. If  $A$  is an  $N \times N$  matrix, the eigenvalues obey the *characteristic equation*,

$$\det(aI - A) = a^N + c_1 a^{N-1} + c_2 a^{N-2} + \cdots + c_N = 0 \quad (6.20)$$

where  $c_1 = -\text{tr}A$  and  $c_N = (-)^N \det A$ . Clearly, for Hermitian  $A$ , all coefficients  $c_n$  are real, and all roots are real. Assuming that, given  $A$ , the roots  $a_1, a_2, \dots, a_N$  of this algebraic equation have been found (possibly numerically), then finding the associated eigenvectors reduces to a linear problem,

$$(a_n I - A) |\varphi\rangle = 0 \quad (6.21)$$

which can be solved by standard methods of matrix algebra.

An matrix  $U$  is unitary provided  $U^\dagger U = I$ . These operators are also playing a key role in quantum mechanics, and in particular include the time-evolution operator. Unitary operators satisfy properties analogous to Hermitian operators,

1. The eigenvalues of a unitary matrix take the form  $e^{i\phi_i}$  for  $\phi_i \in \mathbb{R}$  ;
2. Eigenvectors corresponding to distinct eigenvalues are mutually orthogonal;
3. A unitary matrix may be written as a direct sum of mutually orthogonal projection operators, weighted by the distinct eigenvalues.

The proof of these properties proceeds analogously to the case of Hermitian operators.

## 6.4 Infinite-dimensional Hilbert spaces

A infinite-dimensional Hilbert space  $\mathcal{H}$  in quantum physics will be separable and have a countable orthonormal basis  $\{|n\rangle\}$  where  $n$  may run over  $\mathbb{N}$  or  $\mathbb{Z}$  depending on which one is more convenient (both are countable !). All separable Hilbert spaces are isomorphic to one another, but they may arise in different guises. An arbitrary vector  $|\varphi\rangle \in \mathcal{H}$  may be represented by the expansion,

$$|\varphi\rangle = \sum_n c_n |n\rangle \quad \|\varphi\|^2 = \sum_n |c_n|^2 < \infty \quad (6.22)$$

where again  $n$  may run over  $\mathbb{N}$  or  $\mathbb{Z}$ . The simplest example of an infinite-dimensional separable Hilbert space is given by,

$$L^2 \equiv \{c = (c_1, c_2, c_3, \dots); c_n \in \mathbf{C}\} \quad (c, d) \equiv \sum_{n \in \mathbf{N}} c_n^* d_n \quad (6.23)$$

A more complicated example is provided by spaces of square integrable complex functions on some interval  $S$  (or all of) the real line  $\mathbf{R}$ , defined by

$$L^2(S) \equiv \{f : S \rightarrow \mathbf{C}; (f, f) < \infty\} \quad (f, g) \equiv \int_S dx f(x)^* g(x) \quad (6.24)$$

These function spaces are ubiquitous in quantum mechanics. The Fourier transform gives a convenient description of  $L^2(S)$ . For example, on an interval  $S = [-\pi\ell, +\pi\ell]$  with periodic boundary conditions (or equivalently a circle of radius  $\ell$ ), we have

$$f(x) = \sum_{m \in \mathbf{Z}} f_m \frac{e^{imx/\ell}}{\sqrt{2\pi\ell}} \quad (f, g) = \sum_{m \in \mathbf{Z}} f_m^* g_m \quad (6.25)$$

where  $g_m$  are the Fourier components of  $g$ . This shows that  $L^2$  and  $L^2(S)$  are isomorphic. The basis for  $L^2(S)$  used here corresponds to

$$|m\rangle \sim \frac{e^{imx/\ell}}{\sqrt{2\pi\ell}} \quad (6.26)$$

and is orthonormal in view of the relation,

$$\int_{-\pi\ell}^{+\pi\ell} dx (e^{imx/\ell})^* e^{inx/\ell} = 2\pi\ell \delta_{m,n} \quad (6.27)$$

which is a standard relation of Fourier analysis.

## 6.5 Linear operators in infinite-dimensional Hilbert spaces

Whereas on a finite-dimensional Hilbert space  $\mathcal{H}$  every linear operator maps all of  $\mathcal{H}$  to  $\mathcal{H}$ , this may or may not be true in an infinite-dimensional Hilbert space. We give an example in each case by considering the action of the following two operators on  $L^2$  with basis vector  $|n\rangle$  for  $n \in \mathbb{N}$ ,

$$B|n\rangle = \frac{1}{n^2}|n\rangle \quad H|n\rangle = n^2|n\rangle \quad (6.28)$$

Both operators are well-defined on every basis vector of  $\mathcal{H}$ . Applying the operators to a general vector  $|\varphi\rangle \in L^2$ , as given in (6.22), and computing the norm squared of these states we find,

$$\|B|\varphi\rangle\|^2 = \sum_{n \in \mathbb{N}} \frac{|c_n|^2}{n^4} \quad \|H|\varphi\rangle\|^2 = \sum_{n \in \mathbb{N}} n^4 |c_n|^2 \quad (6.29)$$

Since we have  $\|B|\varphi\rangle\|^2 \leq \| |\varphi\rangle \|^2$  we see that whenever  $|\varphi\rangle \in L^2$  then we have  $B|\varphi\rangle \in L^2$ .



### 6.5.1 Bounded operators

More generally, an operator  $B$  is said to be a *bounded operator* if there exists a  $C_B > 0$  such that for all  $|\varphi\rangle \in L^2$  we have,

$$\|B|\varphi\rangle\|^2 \leq C_B \|\varphi\|^2 \quad (6.30)$$

Note that  $C_B$  depends only on the operator  $B$  and not on  $|\varphi\rangle$ . As an example, all unitary operators are bounded and  $C_B = 1$  and, for reference, any operator in a finite-dimensional Hilbert space is bounded. We see that in all generality, any bounded operator is well-defined on every vector  $|\varphi\rangle \in L^2$ . Since all separable Hilbert spaces of infinite dimension are isomorphic to one another, this property extends to an arbitrary separable Hilbert space.

### 6.5.2 Unbounded operators

By contrast, let us now consider the properties of the operator  $H$ . As long as only a finite number of  $c_n$  are non-zero, the norm  $\|H|\varphi\rangle\|^2$  is finite. But taking the state  $|\varphi\rangle$  to be given by an infinite superposition with  $c_n \sim 1/n$  as  $n \rightarrow \infty$ , we see that the norm  $\|H|\varphi\rangle\|^2$  is infinite, so that  $H|\varphi\rangle \notin L^2$ . Clearly,  $H$  is an *unbounded operator*, and it is not true that  $H$  can be defined on every vector in  $L^2$ . Instead of working in  $L^2$ , we may also see this phenomenon on differential operators acting on  $L^2(S)$ . The kinetic operator  $-\frac{d^2}{dx^2}$  does not map an arbitrary element of  $L^2(S)$  to a square-integrable function. To see this, consider the function  $f(x) = |x|$  on the interval  $[-\pi\ell, +\pi\ell]$ , and note that its second derivative is proportional to Dirac  $\delta$ -functions at  $x = 0, \pi\ell$ , which are certainly not square integrable. In fact, the operator  $H$  is nothing but the kinetic operator in Fourier modes, while one may think of the operator  $B$  as corresponding to the discrete spectrum of the Hydrogen atom.

To proceed, we define the *domain*  $\mathcal{D}(A)$  of an operator  $H$  as a *dense subset* of  $\mathcal{H}$  such that  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ . A dense subset  $\mathcal{D}$  of  $\mathcal{H}$  is such that every point in  $\mathcal{H}$  is the limit of a Cauchy sequence  $\{|\varphi_n\rangle\}_{n \in \mathbb{N}}$  where  $|\varphi_n\rangle \in \mathcal{D}$  for all  $n \in \mathbb{N}$ . For an operator in a finite-dimensional Hilbert space, or for a bounded operator in an infinite dimensional Hilbert space, the domain is always the entire Hilbert space  $\mathcal{H}$ . But for an unbounded operator  $A$  in an infinite-dimensional Hilbert space  $\mathcal{D}(A)$  is strictly included in  $\mathcal{H}$ . For example, in a Hilbert space of  $L^2(S)$  functions,  $\mathcal{D}$  may consist of the subset of functions of  $L^2(S)$  which are infinitely differentiable, or even analytic. More explicitly, we may define the domain as follows,

$$\mathcal{D}(A) = \{|\varphi\rangle \in \mathcal{H} \text{ such that } \|A|\varphi\rangle\| < \infty\} \quad (6.31)$$

Such spaces are referred to as *Sobolev spaces*, but we shall not need this concept here.

## 6.6 Hermitian, Adjoint, Self-adjoint, and Unitary Operators

- An operator  $A$  is *Hermitian* provided for all  $|\varphi\rangle, |\psi\rangle \in \mathcal{D}(A)$  we have,

$$(|\psi\rangle, A|\varphi\rangle) = (A|\psi\rangle, |\varphi\rangle) \quad (6.32)$$

For a finite-dimensional Hilbert space, or for a bounded operator in an infinite-dimensional Hilbert space, we have  $\mathcal{D}(A) = \mathcal{H}$ , and in the first case reduces to our definition of Hermitian matrices.

- The *adjoint* of an operator  $A$  with domain  $\mathcal{D}(A)$  is denoted by  $A^\dagger$  and its domain by  $\mathcal{D}(A^\dagger)$  and is defined to be such that for all  $|\varphi\rangle \in \mathcal{D}(A)$  we have,

$$(|\psi\rangle, A|\varphi\rangle) = (A^\dagger|\psi\rangle, |\varphi\rangle) \quad (6.33)$$

For a bounded operator  $A$  with  $\mathcal{D}(A) = \mathcal{H}$ , this relation is to hold for all  $|\psi\rangle \in \mathcal{H}$  so that  $\mathcal{D}(A^\dagger) = \mathcal{H}$ , as in the finite-dimensional case. For unbounded operator,  $\mathcal{D}(A^\dagger)$  is defined as a dense subset of  $\mathcal{H}$  such that (6.33) holds.

- An operator  $A$  is a *self-adjoint operator* provided  $A^\dagger = A$  and  $\mathcal{D}(A^\dagger) = \mathcal{D}(A)$ , or equivalently provided it is Hermitian and the domains coincide. For a finite-dimensional Hilbert space, a self-adjoint operator is simply a Hermitian operator satisfying  $A^\dagger = A$ . For infinite dimensional Hilbert spaces and bounded operators, the notions of Hermitian and self-adjoint coincide. However, for unbounded operators, in an infinite-dimensional Hilbert space, self-adjointness is a stronger requirement than Hermitian as the domains have to coincide.<sup>2</sup>

The rationale for the stronger condition is that the spectrum of a self-adjoint operator  $A$  shares some of the important properties of finite-dimensional Hermitian matrices discussed earlier,

1. The eigenvalues of a self-adjoint operator are real;
2. Eigenvectors corresponding to distinct eigenvalues of a self-adjoint operator are mutually orthogonal;
3. A self-adjoint operator may be written as a direct sum of mutually orthogonal projection operators, weighted by the distinct eigenvalues.

Note, however, that the eigenfunctions do not need to be  $L^2$ , as is familiar from the Fourier transform on the real line, and that the spectrum may be continuous.

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<sup>2</sup>The terminology of Hermitian versus self-adjointness is standard, see for example Reed and Simon's *Functional Analysis*, Wiley 1980. Amazingly, in Arfken and Weber's *Mathematical Methods for Physicists* the terminology is inexplicably reversed.

- An operator  $U$  is a *unitary operator* provided that for all  $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$ , we have

$$(U|\psi\rangle, U|\varphi\rangle) = (|\psi\rangle, |\varphi\rangle) \quad (6.34)$$

for all  $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$ . Clearly, a unitary operator is a bounded operator and is invertible. The inverse  $U^{-1}$  may be defined by setting  $U|\psi\rangle = |\varphi\rangle$ , so that  $(|u\rangle, U|\varphi\rangle) = (U^{-1}|u\rangle, |\varphi\rangle)$  for all  $|\varphi\rangle, |u\rangle \in \mathcal{H}$ . Using now the definition of the adjoint of  $U$ , we see that for a unitary operator, we have

$$U^{-1} = U^\dagger \quad U^\dagger U = I_{\mathcal{H}} \quad (6.35)$$

Unitary operators are key ingredients in quantum mechanics because they preserve probability amplitudes, and represent symmetries. Their spectral properties are analogous to those of finite-dimensional unitary matrices, though, as for self-adjoint operators, their eigenfunctions may not be  $L^2$  and that their spectrum may be continuous.

- Finally, to every self-adjoint operator  $H$  there corresponds a one-parameter family of unitary operators  $U(t)$  given by (the Schrödinger equation),

$$U(t) = e^{itH} \quad t \in \mathbb{R} \quad (6.36)$$

Under some conditions of differentiability, a one-parameter family of unitary operators  $U(t)$  also defines a self-adjoint operator by  $-i\dot{U}(0)$ . Note that the correspondence fails for a Hermitian operator which is NOT self-adjoint. A Hamiltonian which is Hermitian but not self-adjoint will lead to violations of the conservation of probability.

## 6.7 The example of a free quantum particle on an interval

To illustrate the significance of the domain of an unbounded operator, the adjoint, the meaning of self-adjointness and its failure is easily illustrated by an example of a free quantum particle with Hamiltonian,

$$H = -\frac{d^2}{dx^2} \quad (6.37)$$

on an interval, which we may choose to be  $x \in [0, 1]$ .

### 6.7.1 Periodic boundary conditions

We are already familiar with the case where the boundary conditions are periodic, so that the Hilbert space is the space of square integrable functions  $\mathcal{H} = L^2([0, 1])$  with periodic boundary conditions. In this case, we know the spectrum,

$$\psi_n(x) = e^{2\pi i n x} \quad H\psi_n(x) = 4\pi^2 n^2 \psi_n(x) \quad n \in \mathbb{Z} \quad (6.38)$$

so that  $H$  is manifestly unbounded. The domain may be taken to be periodic functions in  $L^2([0, 1])$  which are infinitely differentiable, as the eigenfunctions are. Given a function  $\varphi(x) \in L^2([0, 1])$ , we have its Fourier series representation,

$$\varphi(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x} \quad \sum_{n \in \mathbb{Z}} |c_n|^2 < \infty \quad (6.39)$$

We now define the sequence  $\varphi_N(x)$  of infinitely differentiable functions by,

$$\varphi_N(x) = \sum_{n=-N}^N c_n \psi_n(x) \quad (6.40)$$

This sequence converges to  $\varphi(x)$  and hence is a Cauchy sequence (recall that every convergent sequence in a metric space is a Cauchy sequence). As a result, the linear space generated by taking finitely many superpositions of the functions  $\psi_n(x)$  is dense in  $L^2([0, 1])$ , and we may take the domain  $\mathcal{D}(H)$  to be this space. Clearly,  $H$  is Hermitian, and in fact it is self-adjoint, which is confirmed by the fact that its eigenvalues are real and its eigenfunctions for different eigenvalues are mutually orthogonal.

### 6.7.2 Dirichlet, Neumann, and mixed boundary conditions

Now let's consider  $H$  on the interval  $[0, 1]$  but not with periodic boundary conditions. To get a self-adjoint operator, it must first of all be Hermitian, which requires,

$$(H|\varphi\rangle, |\psi\rangle) = (|\varphi\rangle, H|\psi\rangle) \quad (6.41)$$

or in terms of its  $x$ -space representation,

$$\int_0^1 dx (\varphi''\psi - \varphi\psi'') = 0 \quad (6.42)$$

The functions  $\varphi, \psi$  in the domain  $\mathcal{D}(H)$  must certainly be differentiable and we can take them to be infinitely differentiable. Integrating by parts, we find that the above condition is equivalent to,

$$(\varphi'\psi - \varphi\psi')(1) - (\varphi'\psi - \varphi\psi')(0) = 0 \quad (6.43)$$

Physically, the condition states that the probability current  $(\varphi'\psi - \varphi\psi')(x)$  must be the same at both ends so that total probability is conserved. The condition was clearly satisfied when  $\varphi$  and  $\psi$  were periodic. But now let's not take the functions to be periodic. One way to satisfy the conditions is to take  $\varphi, \psi$  to obey Dirichlet boundary conditions at both ends,  $\varphi(1) = \varphi(0) = \psi(1) = \psi(0) = 0$ , and this defines

a suitable domain for  $H$  making  $H$  Hermitian. But is  $H$  self-adjoint? This would require in addition to Hermiticity that  $\mathcal{D}(H^\dagger) = \mathcal{D}(H)$ . To determine  $\mathcal{D}(H^\dagger)$ , we use the definition of  $H^\dagger$ ,

$$(H^\dagger|\varphi\rangle, |\psi\rangle) = (|\varphi\rangle, H|\psi\rangle) \quad (6.44)$$

For all  $\psi \in \mathcal{D}(H)$ , the space of  $\varphi$  such that this equation holds will give the domain  $\mathcal{D}(H^\dagger)$ . The probability current at each end may be made to vanish by setting,<sup>3</sup>

$$\begin{aligned} \psi'(1) + \xi_1 \psi(1) &= 0 & \psi'(0) + \xi_0 \psi(0) &= 0 \\ \varphi'(1) + \xi_1 \varphi(1) &= 0 & \varphi'(0) + \xi_0 \varphi(0) &= 0 \end{aligned} \quad (6.45)$$

for real parameters  $\xi_1, \xi_0$ . In each case do we have a self-adjoint  $H$  since the domains of  $H$  and  $H^\dagger$  are equal to one another. The spectrum depends on the boundary conditions. For example, Dirichlet (D) corresponds to  $\xi = \infty$  while Neumann (N) corresponds to  $\xi = 0$ . The spectra are as follows,

$$\begin{array}{llll} (D, D) & E_n = \pi^2 n^2 & \psi_n(x) = \sin(\pi n x) & n \in \mathbb{N} \\ (N, N) & E_n = \pi^2 n^2 & \psi_n(x) = \cos(\pi n x) & n \in \mathbb{N} \cup \{0\} \\ (D, N) & E_n = \pi^2 \nu^2 & \psi_\nu(x) = \sin(\pi \nu x) & \nu \in \mathbb{N} - 1/2 \end{array} \quad (6.46)$$

For other values of  $\xi_1, \xi_0$ , the spectrum interpolates continuously between these cases.

### 6.7.3 The momentum operator

Now consider the momentum operator,

$$P = -i \frac{d}{dx} \quad (6.47)$$

on the interval  $[0, 1]$ . For periodic boundary conditions, the operator is self-adjoint and the eigenvalues are real. But now consider non-periodic boundary conditions. The condition for self-adjointness is then,

$$(P\varphi, \psi) - (\varphi, P\psi) = i\varphi^*(1)\psi(1) - i\varphi^*(0)\psi(0) = 0 \quad (6.48)$$

For periodic boundary conditions, this equation always holds, and  $P$  is self-adjoint.

But if we choose the domain  $\mathcal{D}(H)$  to consist of differentiable functions which obey Dirichlet boundary conditions  $\psi(1) = \psi(0) = 0$  then the equation (6.48) holds for  $\varphi$

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<sup>3</sup>Mathematically, the operator is actually referred to as *essentially self-adjoint*: its Von Neumann indices are non-zero but equal so that it admits an adjoint extension.

with *any boundary conditions*. Thus the adjoint operator  $H^\dagger$  has a domain which is different from  $\mathcal{D}(H)$ , and the operator is *not self-adjoint*. In fact, it is straightforward to verify that the eigenvalue equation  $P|\psi\rangle = E|\psi\rangle$  has no solutions, as the solution in the interior  $\psi(x) = e^{iEx}$  cannot satisfy the boundary conditions for any value of  $E$ . So the operator  $P$  with this domain has no spectrum at all.

If we choose the domain  $\mathcal{D}(H)$  to consist of functions that obey Dirichet boundary conditions on at one end  $\psi(1) = 0$ , then we must have  $\varphi(0) = 0$  with no conditions on  $\varphi(1)$ . Again one can see that the domains  $\mathcal{D}(H)$  and  $\mathcal{D}(H^\dagger)$  are different, and that the operator  $P$  has no spectrum. There is in fact no way to choose the domains so that  $\mathcal{D}(H) = \mathcal{D}(H^\dagger)$  and the operator is not self-adjoint.<sup>4</sup> However on a two-component wave-function the operator  $P$  is self-adjoint. Physically, a manifestation of this effect is that in a closed cavity, such as a sphere or bag, you cannot make the chiral Dirac operator self-adjoint.

## 6.8 Sturm-Liouville theory

Sturm-Liouville theory deals with self-adjoint second order differential operators in one variable on a subset  $S$  of  $\mathbb{R}$ . Consider a real Hilbert space  $\mathcal{H}$  with inner product,

$$(\phi, \psi) = \int_S dx w(x) \phi(x) \psi(x) \quad (6.49)$$

where  $w(x) > 0$  is a measure on  $S$ . Now consider the most general second order differential operator on one variable,

$$L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x) \quad (6.50)$$

We begin by requiring that the operator is Hermitian, namely  $(L\phi, \psi) = (\phi, L\psi)$  for all  $\phi, \psi \in D(L)$ , or explicitly,

$$\int_S dx w(a\phi'' + b\phi' + c\phi)\psi - \int_S dx w\phi(a\psi'' + b\psi' + c\psi) = 0 \quad (6.51)$$

Integrating the double derivatives by parts once, and regrouping terms gives,

$$\int_{\partial S} wa(\phi\psi' - \phi'\psi) + \int_S dx(\phi'\psi - \phi\psi')((wa)' - wb) = 0 \quad (6.52)$$

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<sup>4</sup>Mathematically, the operator has non-zero Von Neumann indices but they are now unequal and there exists no self-adjoint extension.

where  $\partial S$  stands for the boundary contributions if any. Setting  $w(x)a(x) = p(x)$  and  $w(x)c(x) = -q(x)$  we have  $w(x)b(x) = p'(x)$  so that the operator becomes,

$$L = \frac{1}{w(x)} \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) - \frac{q(x)}{w(x)} \quad (6.53)$$

which is the Sturm-Liouville operator. It remains to ensure the absence of the boundary terms which may be achieved by requiring  $p(x)$  to vanish at the boundary. Assuming that the Hermitian operator extends to a self-adjoint operator, the spectrum of  $L$  will enjoy the following properties,

1. The eigenvalues  $\lambda$  of  $L$  are real;
2. The corresponding eigenfunctions are mutually orthogonal;
3. The properly normalized eigenfunctions satisfy the completeness relation.

The equation satisfied by an eigenfunction  $\psi_\lambda(x)$  for eigenvalue  $\lambda$  is given by,

$$p\psi_\lambda'' + p'\psi_\lambda' - q\psi_\lambda = \lambda w\psi_\lambda \quad (6.54)$$

Two linearly independent eigenfunctions  $\phi_\lambda, \psi_\lambda$  associated with the same eigenvalue  $\lambda$  may be used to define the Wronskian,

$$W = \phi_\lambda' \psi_\lambda - \phi_\lambda \psi_\lambda' \quad W = \frac{W_0}{p} \quad (6.55)$$

where  $W_0$  is a constant.

### 6.8.1 Discrete spectrum

In the special case where the spectrum is discrete, we may write down some explicit formulas. We denote the eigenfunctions by  $\psi_n$ ,  $n \in \mathbb{N}$ , with real eigenvalue  $\lambda_n$ , where the eigenvalue may be double degenerate. The eigenfunction equation is then,

$$p\psi_n'' + p'\psi_n' - q\psi_n = \lambda_n w\psi_n \quad (6.56)$$

The orthogonality relation, with properly normalized eigenfunctions, reads,

$$\int_S dx w(x) \psi_m(x) \psi_n(x) = \delta_{m,n} \quad (6.57)$$

while the completeness relation for these orthonormal eigenfunctions is given by,

$$\sum_{n \in \mathbb{N}} \psi_n(x) \psi_n(y) = \frac{1}{w(x)} \delta(x - y) \quad (6.58)$$

for  $x, y \in S$ . Note the proper normalization for the Dirac  $\delta$ -function in the presence of a non-trivial measure factor. The Green function and the resolvent Green function, familiar from the case of the Fourier transform discussion, are defined as follows,

$$LG(x, y) = (L - E)G_E(x, y) = \frac{1}{w(x)}\delta(x - y) \quad (6.59)$$

where  $L$  acts on the coordinate  $x$ . The resolvent Green function and the Green function itself are solved for as follows,

$$G_E(x, y) = \sum_{n \in \mathbb{N}} \frac{\psi_n(x)\psi_n(y)}{\lambda_n - E} \quad G(x, y) = \lim_{E \rightarrow 0} G_E(x, y) \quad (6.60)$$

With the help of the completeness relation, we see that any function  $f(x)$  for  $x \in S$  may be expanded in the basis of eigenfunctions  $\phi_n(x)$ ,

$$f(x) = \sum_{n \in \mathbb{N}} f_n \psi_n(x) \quad f_n = \int_S dx w(x) f(x) \psi_n(x) \quad (6.61)$$

providing examples of the *integral transforms* for the case of a discrete spectrum and thereby generalizing the expansion of periodic functions in a Fourier series.

A simple example is provided by the harmonic oscillator for which  $S = \mathbb{R}$  and where  $w(x) = p(x) = 1$  and  $q(x) = \omega^2 x^2$ , so that,

$$L = \frac{d^2}{dx^2} - \omega^2 x^2 \quad (6.62)$$

The eigenfunctions are given by Hermite polynomials,

$$\psi_n(x) = N_n H_{n-1}(x) e^{-\omega x^2/2} \quad (6.63)$$

where  $N_n$  is a normalization constant which guarantees that (6.57) holds. One then automatically has the completeness relation,

$$\sum_{n \in \mathbb{N}} N_n^2 H_{n-1}(x) H_{n-1}(y) e^{-\omega(x^2+y^2)/2} = \delta(x - y) \quad (6.64)$$

Hermite polynomials will be discussed in more detail in section 8 on special functions.

### 6.8.2 Continuous spectrum

Let us also give a simple example for the continuous spectrum. Consider the  $1/r^2$  potential on the half line  $r \in \mathbb{R}^+$ , given by the Hamiltonian,

$$H = -\frac{d^2}{dr^2} + \frac{\nu^2 - \frac{1}{4}}{r^2} \quad (6.65)$$



This Hamiltonian is homogeneous in  $r$  and thus covariant under scaling transformations; it has deep group-theoretic significance as will be shown in 231B. But here we simply want to view it as a Hamiltonian with a continuous spectrum, as is dictated by its scaling symmetry. The eigenvalue equation  $H\psi_E = E\psi_E$  may be put in standard form by rescaling  $\psi_E$  as follows,

$$\psi_E(r) = \sqrt{r} \phi_E(r) \quad (6.66)$$

so that  $\phi_E(r)$  obeys the equation,

$$r^2 \phi_E'' + r \phi_E' + (Er^2 - \nu^2) \phi_E = 0 \quad (6.67)$$

Now there are two regimes, either  $E > 0$  corresponding to the scattering regime, or  $E < 0$  corresponding to the bound state regime. In the scattering regime we set  $E = k^2$  for  $k \in \mathbb{R}$  and redefine  $x = kr$ , while in the bound state regime we set  $E = -\kappa^2$  for  $\kappa \in \mathbb{R}$  and redefine  $x = \kappa r$ . We then obtain the equations,

$$\begin{aligned} E = +k^2 & \quad \phi_E(r) = Z_\nu(kr) & \quad x^2 Z_\nu''(x) + x Z_\nu'(x) - (\nu^2 - x^2) Z_\nu(x) = 0 \\ E = -\kappa^2 & \quad \phi_E(r) = M_\nu(\kappa r) & \quad x^2 M_\nu''(x) + x M_\nu'(x) - (\nu^2 + x^2) M_\nu(x) = 0 \end{aligned} \quad (6.68)$$

The solutions  $Z_\nu$  are linear combinations of the Bessel functions with standard normalization  $J_\nu$  and  $J_{-\nu}$  (or equivalently  $Y_\nu$ ), while the solutions  $M_\nu$  are linear combinations of modified Bessel functions with standard normalization  $I_\nu$  and  $K_\nu$  (the normalizations will be spelled out in section 8 on special functions). The asymptotic behavior of the Bessel functions  $J_\nu$ ,  $I_\nu$  and  $K_\nu$  is given by,

$$\begin{aligned} x \rightarrow 0 & \quad J_\nu(z) \approx z^\nu & \quad I_\nu(z) \approx z^\nu & \quad K_\nu(z) \approx z^{-\nu} \\ x \rightarrow \infty & \quad J_\nu(z) \approx \sqrt{z} e^{iz} & \quad I_\nu(z) \approx \sqrt{z} e^z & \quad K_\nu(z) \approx \sqrt{z} e^{-z} \end{aligned} \quad (6.69)$$

Clearly,  $I_\nu$  is not normalizable at  $\infty$ , so this function is excluded.

For  $\nu \geq 1/2$ , the Hamiltonian is positive, so there are no solutions with  $E < 0$ . In this case, the spectrum consists of all  $k \in \mathbb{R}^+$  and we should have the orthogonality relation between  $\psi_{k^2}(r) = \sqrt{r} J_\nu(kr)$  functions,

$$\int_0^\infty dr r J_\nu(kr) J_\nu(k'r) = f(k) \delta(k - k') \quad (6.70)$$

We shall verify this equation in section 8 on special functions, where we will also determine the normalization factor to be  $f(k) = 1/k$ . As a result, the completeness relation must then read,

$$\int_0^\infty dk k J_\nu(kr) J_\nu(kr') = \frac{1}{r} \delta(r - r') \quad (6.71)$$

which implies,

$$\int_0^\infty dk k \phi_{k^2}(r) \phi_{k^2}(r') = \delta(r - r') \quad (6.72)$$

precisely with the correct coefficient 1 for the  $\delta$ -function on the right side.

The Green function and the resolvent Green function may then be computed as well and we find,

$$G(r, r') = \int_0^\infty dk k \frac{\phi_{k^2}(r) \phi_{k^2}(r')}{k^2} = \sqrt{rr'} \int_0^\infty \frac{dk}{k} J_\nu(kr) J_\nu(kr') \quad (6.73)$$

Computing the Green function directly from the differential equation,

$$\left( -\frac{\partial^2}{\partial r^2} + \frac{\nu^2 - \frac{1}{4}}{r^2} \right) G(r, r') = \delta(r - r') \quad (6.74)$$

we find the solutions  $r^\alpha$  away from  $r = r'$  with  $\alpha = \frac{1}{2} \pm \nu$ . For  $r < r'$  we take the solution  $r^{\frac{1}{2}+\nu}$  that decays as  $r \rightarrow 0$  while for  $r > r'$  we take the solution  $r^{\frac{1}{2}-\nu}$  that decays as  $r \rightarrow \infty$ . The dependence on  $r'$  is fixed by requiring symmetry  $G(r', r) = G(r, r')$  and the overall normalization is fixed by the discontinuity of the first derivative at  $r = r'$ , and we find,

$$G(r, r') = \frac{\sqrt{rr'}}{2\nu} \left( \left( \frac{r}{r'} \right)^\nu \theta(r' - r) + \left( \frac{r'}{r} \right)^\nu \theta(r - r') \right) \quad (6.75)$$

This result may be deduced from the integral representation of the Green function.

The completeness relation may be used to decompose any function on  $\mathbb{R}^+$  in the basis of Bessel functions, which is referred to as the Bessel transform,

$$f(r) = \int_0^\infty dk k \tilde{f}(k) \sqrt{r} J_\nu(kr) \quad \tilde{f}(k) = \int_0^\infty dr f(r) \sqrt{r} J_\nu(kr) \quad (6.76)$$

in complete analogy with the Fourier transform.

## 7 Spectrum of the Laplace operator

The Laplace operator in  $\mathbb{R}^d$  with the flat Euclidean metric is familiar for dimensions  $d = 2, 3$ . Introducing global Cartesian coordinates  $\xi^i$ ,  $i = 1, \dots, d$  the Euclidean metric  $ds^2$  and the Laplace operator are given by,

$$ds^2 = \sum_{i=1}^d (d\xi^i)^2 \qquad \Delta = \sum_{i=1}^d \frac{\partial^2}{(\partial\xi^i)^2} \qquad (7.1)$$

It is often convenient to parametrize flat space by coordinates  $x^i$ ,  $i = 1, \dots, d$  other than Cartesian coordinates, such as spherical, hyperbolic, or parabolic coordinates. This may be because the new coordinates are better adapted to either the boundary conditions or to the inhomogeneous source term of the Laplace equation. Cartesian coordinates are then functions  $\xi^i(x)$  which are given in terms of  $x$ . In the general coordinates  $x^i$ , the flat metric then takes the form,

$$ds^2 = \sum_{i,j,k} \frac{\partial\xi^k}{\partial x^i} \frac{\partial\xi^k}{\partial x^j} dx^i dx^j \qquad (7.2)$$

and one may carry out the associated change of variables on the Laplacian. This operation can be considerably simplified and generalized as discussed in the next subsection.

### 7.1 The Laplace-Beltrami operator

Actually, one may make yet a further fundamental generalization and consider  $\mathbb{R}^d$  not with the flat Euclidean metric, but with a curved metric. For example, by stereographic projection onto  $\mathbb{R}^d$ , the metric on the round sphere may be represented this way. Thus, in an arbitrary system of local coordinates  $x^i$ ,  $i = 1, \dots, d$ , an arbitrary metric takes the following form,

$$ds^2 = \sum_{i,j=1}^d g_{ij}(x) dx^i dx^j \qquad (7.3)$$

where the functions  $g_{ij} = g_{ji}$  are real at every point  $x$ . Since the matrix  $g_{ij}$  is real symmetric it has real eigenvalues which may be positive, zero, or negative. We shall assume that no zero eigenvalues occur in which case the metric is said to be *non-degenerate*. The  $d$  eigenvalues may then be partitioned into  $p$  positive and  $q$  negative eigenvalues with  $p + q = d$ , and one says that the metric has *signature*  $(p, q)$ . Clearly an overall sign in the metric is just a matter of convention so that signatures  $(p, q)$  and  $(q, p)$  are equivalent and we may choose  $p \geq q$ , the *mostly positive convention*.

- *Euclidean signature* corresponds to  $(d, 0)$ ;
- *Minkowski signature* corresponds to  $(d - 1, 1)$ ;
- *Conformal signature* corresponds to  $(d - 2, 2)$ .

Finally, we may also replace  $\mathbb{R}^d$  by a more general manifold  $M^d$  of dimension  $d$ . A differentiable manifold  $M^d$  equipped with an metric of Euclidean signature is a *Riemannian manifold*, while if the metric has signature  $(p, q)$  with  $pq \neq 0$  it is said to be *pseudo-Riemannian*.

The Laplace-Beltrami operator  $\Delta_g$  on a Riemannian manifold  $M^d$  with local coordinates  $x^i$ ,  $i = 1, \dots, d$  and metric  $ds^2$  is given by,

$$ds^2 = \sum_{i,j=1}^d g_{ij}(x) dx^i dx^j \quad \Delta_g = \frac{1}{\sqrt{g}} \sum_{i,j=1}^d \partial_i (\sqrt{g} g^{ij} \partial_j) \quad (7.4)$$

where  $g = \det(g_{ij})$  and  $g^{ij}$  are the components of the inverse matrix of  $g_{ij}$  so that,

$$\sum_{k=1}^d g_{ik}(x) g^{kj}(x) = \delta_i^j \quad (7.5)$$

The advantage of these formulas for the metric and the Laplace-Beltrami operator is that they are completely *intrinsic*, namely independent of the local coordinates used to parametrize  $M^d$ . We start with the metric, and change coordinates from  $x^i$  to  $x'^i(x)$ , requiring that the metric transform as follows,

$$ds^2 = \sum_{i,j=1}^d g_{ij}(x) dx^i dx^j = \sum_{k,\ell=1}^d g'_{k,\ell}(x') dx'^k dx'^\ell \quad (7.6)$$

Using chain rule  $dx'^k = \sum_i \frac{\partial x'^k}{\partial x^i} dx^i$  and identifying components of  $dx^i dx^j$  we find,

$$g_{ij}(x) = \sum_{k,\ell=1}^d g'_{k,\ell}(x') \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^\ell}{\partial x^j} \quad (7.7)$$

relating the old metric  $g_{ij}(x)$  in the old coordinates to the new metric  $g'_{k,\ell}(x')$  in the new coordinates  $x'$ . Taking the determinant of this equation to get  $g$ , we see that,

$$g(x) = g'(x') \left( \det \left\{ \frac{\partial x'^k}{\partial x^i} \right\} \right)^2 \quad (7.8)$$

which implies that the volume element  $\sqrt{g(x)} d^d x$  is invariant under coordinate changes.

## 7.2 The Laplace operator in spherical coordinates

Spherical coordinates for flat Euclidean space are defined as follows,

$$x^i = r \hat{x}^i \quad i = 1, \dots, d \quad r^2 = \sum_{i=1}^d (x^i)^2 \quad (7.9)$$

As a result,  $\hat{x}^i$  are the coordinates of a unit vector or equivalently coordinates on the unit sphere  $S^{d-1}$ . Using the fact that the tangent vector  $d\hat{x}^i$  is orthogonal to the radial direction  $\hat{x}^i$ , the flat Euclidean metric becomes,

$$ds^2 = dr^2 + r^2 ds_{S^{d-1}}^2 \quad ds_{S^{d-1}}^2 = \sum_{i=1}^d (d\hat{x}^i)^2 \quad \sum_{i=1}^d (\hat{x}^i)^2 = 1 \quad (7.10)$$

Now  $\hat{x}^i$  are constrained coordinates on the sphere with one more  $\hat{x}^i$  than dimensions of  $S^{d-1}$ . Thus, we parametrize  $S^{d-1}$  by unconstrained local coordinates  $y^\alpha$ ,  $\alpha = 1, \dots, d-1$ , so that,

$$ds_{S^{d-1}}^2 = \sum_{\alpha=1}^{d-1} h_{\alpha\beta}(y) dy^\alpha dy^\beta \quad (7.11)$$

To compute the Laplace operator using (7.4) we evaluate  $\sqrt{g} d^d x = r^{d-1} dr \sqrt{h} d^{d-1} y$  where  $h = \det(h_{\alpha\beta})$ . Since neither the metric  $g_{ij}$  nor the inverse metric  $g^{ij}$  has cross-terms between  $dr$  and  $dy^\alpha$ , the Laplace operator becomes,

$$\Delta = \frac{1}{r^{d-1} \sqrt{h}} \partial_r (r^{d-1} \sqrt{h} \partial_r) + \sum_{\alpha, \beta=1}^{d-1} \frac{1}{r^{d-1} \sqrt{h}} \partial_\alpha (r^{d-3} \sqrt{h} h^{\alpha\beta} \partial_\beta) \quad (7.12)$$

In the first term the factor  $\sqrt{h}$  is independent of  $r$  and cancels, while in the second term  $r^{d-1}$  is independent of  $y^\alpha$  and cancels, so that we are left with,

$$\Delta = \frac{1}{r^{d-1}} \partial_r (r^{d-1} \partial_r) + \frac{1}{r^2} \Delta_{S^{d-1}} \quad \Delta_{S^{d-1}} = \sum_{\alpha, \beta=1}^{d-1} \frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta) \quad (7.13)$$

where we have recognized  $\Delta_{S^{d-1}}$  as the Laplace-Beltrami operator on the unit sphere. It is important to notice that the radial and angular coordinates have been decoupled, namely  $\Delta_{S^{d-1}}$  is independent of  $r$ . In quantum mechanics, for  $d = 3$ , this operator is simply related to the total angular momentum square operator,  $\Delta_{S^{d-1}} = -\mathbf{L}^2$ .

### 7.3 Separating the eigenvalue equation

Next, we concentrate on the eigenvalue equation  $-\Delta\Psi_E = E\Psi_E$  and express the Laplacian in spherical coordinates. Since the dependence on the radial coordinate  $r$  and the angular coordinates  $y^\alpha$  is separated, the eigenfunction may be expanded in a product basis,  $\Psi_E(r, y^\alpha) = \psi(r)\chi(y^\alpha)$ . The angular factor obeys the Laplace-eigenvalue equation on the sphere,

$$-\Delta_{S^{d-1}}\chi = \ell(\ell + d - 2)\chi \quad (7.14)$$

where we have used the above parametrization of the eigenvalues by a real number  $\ell$  in anticipation of the result to be derived below that  $\ell$  will be an integer. The radial part obeys the equation,

$$-\frac{1}{r^{d-1}}\partial_r(r^{d-1}\partial_r\psi) + \frac{\ell(\ell + d - 2)}{r^2}\psi = E\psi \quad (7.15)$$

which may be put it in *standard form* by setting  $\nu = \ell + \frac{d-2}{2}$  and rescaling  $\psi$ ,

$$\psi(r) = r^{1-\frac{d}{2}}\phi(r) \quad r^2\phi'' + r\phi' + (Er^2 - \nu^2)\phi = 0 \quad (7.16)$$

This is the differential equation for the Bessel functions, with solutions,

- $\phi(r) = J_{\pm\nu}(kr)$  for  $E = k^2 > 0$  corresponding to the scattering states;
- $\phi(r) = K_\nu(\kappa r)$  for  $E = -\kappa^2 < 0$  corresponding to the bound states.  
(There is also a linearly independent exponentially growing solution  $I_\nu(\kappa r)$ .)

Bessel functions will be studied in the next section.

### 7.4 Spectrum of the Laplace operator on the sphere

Spheres  $S^{d-1}$  may be parametrized by induction on the dimensions  $d-1$ . Slicing the sphere by a family of parallel planes parametrized by the angle  $\theta \in [0, \pi]$  produces spheres  $S^{d-2}$  with radius  $\sin\theta$ , as shown in Figure 15. The metric  $ds_{S^{d-1}}^2$  on the sphere of unit radius  $S^{d-1}$  may thus be parametrized in terms of  $\theta$  and the metric  $ds_{S^{d-2}}^2$  on the sphere of unit radius  $S^{d-2}$  by,

$$ds_{S^{d-1}}^2 = d\theta^2 + \sin^2\theta ds_{S^{d-2}}^2 \quad (7.17)$$

Using again the general metric expressions for the corresponding Laplace operators we find that they are related by,

$$\Delta_{S^{d-1}} = \frac{1}{\sin^{d-2}\theta} \partial_\theta (\sin^{d-2}\theta \partial_\theta) + \frac{1}{\sin^2\theta} \Delta_{S^{d-2}} \quad (7.18)$$

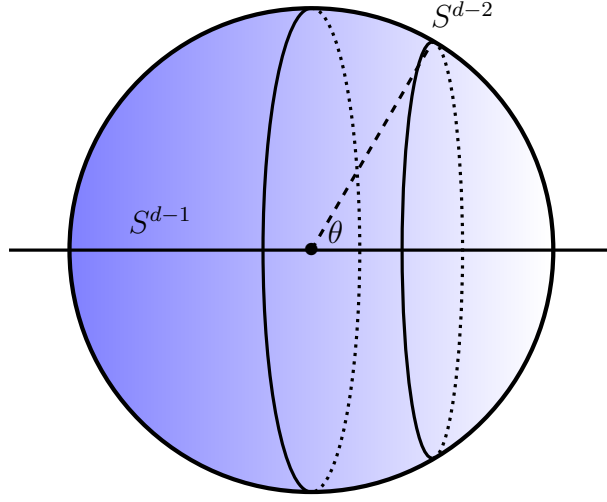


Figure 15: Slicing the sphere  $S^{d-1}$  by vertical planes into spheres  $S^{d-2}$ .

Since  $\Delta_{S^{d-2}}$  is independent of  $\theta$ , we have  $[\Delta_{S^{d-2}}, \Delta_{S^{d-1}}] = 0$ , and the operators may be diagonalized simultaneously. Assuming an eigenvalue  $\lambda_{d-2}$  of  $-\Delta_{S^{d-2}}$ , the remaining equation in  $\theta$  reduces to the following eigenvalue equation,

$$\frac{1}{\sin^{d-2} \theta} \partial_\theta \left( \sin^{d-2} \theta \partial_\theta \psi(\theta) \right) + \frac{\lambda_{d-2} \psi(\theta)}{\sin^2 \theta} + \lambda_{d-1} \psi(\theta) = 0 \quad (7.19)$$

Extracting a power of  $\sin \theta$  to make the function regular at  $\theta = 0, \pi$ , we set,

$$\psi(\theta) = \sin^\alpha \theta f(\cos \theta) \quad (7.20)$$

where  $\alpha$  remains to be determined by regularity conditions. In terms of  $f$ , the equation now becomes,

$$(1 - x^2) f'' - (2\alpha + d - 1) x f' + \frac{p f}{1 - x^2} + (\lambda_{d-1} - \alpha(\alpha + d - 2)) f = 0 \quad (7.21)$$

where  $p = \alpha(\alpha + d - 3) - \lambda_{d-2}$ . Regularity at  $x = \pm 1$  requires us to choose  $\alpha$  so that  $p = 0$  and thus

$$\lambda_{d-2} = \alpha(\alpha + d - 3) \quad (7.22)$$

so that the equation reduces to,

$$(1 - x^2) f'' - (2\alpha + d - 1) x f' + (\lambda_{d-1} - \alpha(\alpha + d - 2)) f = 0 \quad (7.23)$$

This equation is of the type of Jacobi functions or Gegenbauer functions, and has solutions which are polynomial in  $x$  of degree  $m$  provided,

$$\lambda_{d-1} = (m + \alpha)(m + \alpha + d - 2) \quad (7.24)$$

in which case the solutions are Jacobi polynomials  $P_n^{(\nu, \nu)}(x)$  with  $\nu = \alpha + \frac{d-3}{2}$ , often referred to as Gegenbauer or ultra-spherical polynomials for this special assignment of super-indices.

We solve for the eigenvalues recursively. For  $d = 3$ , we know the eigenvalues of  $\Delta_{S^1}$ , since they are given by  $\lambda_1 = m_1^2$  for  $m_1 \in \mathbb{Z}$ . Solving for  $\alpha$ , we find  $\alpha = m_1$ , and hence  $\lambda_2 = (m_2 + m_1)(m_2 + m_1 + 1)$ . Setting  $\ell_2 = m_2 + m_1$  we recover the familiar spectrum  $\ell_2(\ell_2 + 1)$  of orbital angular momentum states in three dimensions. Having  $\lambda_2$ , we may now seek the spectrum of  $\Delta_{S^3}$ , namely the eigenvalues  $\lambda_3$ . To do so, we solve for the corresponding  $\alpha = m_2 + m_1$ , and obtain  $\lambda_3 = (m_3 + m_2 + m_1)(m_3 + m_2 + m_1 + 2)$ . Setting  $\ell_3 = m_3 + m_2 + m_1$  we obtain the well-known spectrum of  $\Delta_{S^3}$  of the form  $\lambda_3 = \ell_3(\ell_3 + 2)$ . One obtains the general spectrum by induction on  $d$ ,

$$\lambda_{d-1} = \ell_d(\ell_d + d - 2) \quad (7.25)$$

In 231B we shall develop the group theoretic understanding of this construction. We have not been very careful here about the ranges of the different integers which enter, but we shall do so when we study the problem group-theoretically.



## 8 Special functions

Special functions arise in a wealth of physical and mathematical problems that are formulated in terms of second order differential equations in one or several variables, and with constant or non-constant coefficients. I will concentrate here on problems that arise often in quantum mechanics and electro-magnetism, but the same types of equations arise in fluid dynamics or even in general relativity.

### 8.1 Orthogonal polynomials

There is a whole collection of families of *orthogonal polynomials* giving the definite impression that every French mathematician of the 19-th century felt obliged to attach his name to a polynomial. To put some order into this collection, we present here a systematic viewpoint on these families of polynomials in one variable (there exist generalizations to several variables which we shall not discuss but which may be similarly constructed).

#### 8.1.1 Definition and construction

A set of polynomials  $\{f_n(x) \mid n \in \mathbb{Z}, n \geq 0\}$ , where  $n$  is the degree of  $f_n(x)$ , is said to be a *family of orthogonal polynomials with respect to the measure*  $w(x) \geq 0$  on a set  $S \subset \mathbb{R}$  provided  $w(x)$  and  $f_n(x)$  satisfy,

$$\int_S dx w(x) f_m(x) f_n(x) = h_m \delta_{m,n} \quad (8.1)$$

for all  $m, n \in \mathbb{Z}$  and  $m, n \geq 0$  and for  $h_m \in \mathbb{R}$  and  $h_n > 0$ . Setting  $h_m = 1$  for all  $m$  one obtains *orthonormal polynomials*. In practice, the set  $S$  may be taken to be a single interval  $[a, b]$  for  $a, b \in \mathbb{R}$ , the half line  $[a, \infty[$ , the half line  $] - \infty, b]$  or the entire real line  $\mathbb{R}$ . On a finite interval, continuity of polynomials guarantees that the integrals of (8.1) make sense for any continuous function  $w(x)$ , while when the set  $S$  is infinite the measure  $w(x)$  must decay faster than any inverse power of  $|x|$ .

Given a set  $S$  and a measure  $w(x)$ , the polynomials  $f_m(x)$  may be constructed recursively from the *moments*  $\mu_m$  of the measure  $w(x)$ , defined by,

$$\mu_m = \int_S dx w(x) x^m \quad m \geq 0 \quad (8.2)$$

Note that  $\mu_m$  may vanish for odd  $m$ , but it can never vanish for  $m$  even in view of the positivity of  $w(x)$ . We choose the polynomials  $f_m(x)$  to be *monic* which means that their highest degree term has unit coefficient,

$$f_m(x) = x^m + \sum_{k=0}^{m-1} \varphi_{m,k} x^k \quad (8.3)$$

and require  $f_m(x)$  to be orthogonal to any polynomial of degree  $0 \leq n < m$ . For this purpose we may use the basis of polynomials  $x^n$ , and we obtain,

$$\int_S dx w(x) f_m(x) x^n = \mu_{m+n} + \sum_{k=0}^{m-1} \varphi_{m,k} \mu_{k+n} = 0 \quad (8.4)$$

for all  $0 \leq n < m$ . Choosing  $n = m - 1$ , we see that the coefficient of  $\varphi_{m,m-1}$  is  $\mu_{2m-2}$  which is non-zero, and hence gives  $\varphi_{m,m-1}$  in terms of  $\varphi_{m,k}$  with  $k < m - 1$ , by

$$-\mu_{2m-2} \varphi_{m,m-1} = \mu_{2m-1} + \sum_{k=0}^{m-2} \varphi_{m,k} \mu_{k+n} \quad (8.5)$$

Eliminating  $\varphi_{m,m-1}$  from (8.4) and setting  $n$  to successively lower values starting at  $n = m - 2$  determines all coefficients recursively. The normalization  $h_m$  is then obtained by integrating the square of the monic polynomial and cannot vanish.

### 8.1.2 Recursion relation

All families of orthogonal polynomials satisfy three-step recursion relations. For monic polynomials they take the simple form,

$$f_{n+1}(x) = (x + a_n) f_n(x) + b_n f_{n-1}(x) \quad (8.6)$$

where  $a_n$  and  $b_n$  are constants which are independent of  $x$  but of course functionally depend on the measure  $w(x)$ . To prove this, we note that  $f_{n+1}(x) - x f_n(x)$  is a polynomial of degree at most  $n$ , and may thus be represented as a linear combination,

$$f_{n+1}(x) - x f_n(x) = \sum_{k=0}^n \lambda_k f_k(x) \quad (8.7)$$

Now integrate both sides against  $w(x) f_m(x)$  for any  $0 \leq m \leq n - 2$ ,

$$\int_S dx w(x) f_m(x) (f_{n+1}(x) - x f_n(x)) = h_m \lambda_m \quad (8.8)$$

But the left side vanishes, since the term in  $f_{n+1}$  integrates to zero since  $m < n + 1$  and the term in  $f_n$  integrates to zero against  $x f_m$  which is of degree at most  $n - 1$ . Hence  $\lambda_m = 0$  for all  $0 \leq m \leq n - 2$ , from which the recursion relation (8.6) readily follows with  $a_n = \lambda_n$  and  $b_n = \lambda_{n-1}$ . The coefficients  $a_n, b_n$  are obtained by setting  $m = n - 1$  and  $n = n$  in (8.8), and we find,

$$\begin{aligned} h_n a_n &= - \int_S dx w(x) x f_n(x)^2 \\ h_{n-1} b_n &= - \int_S dx w(x) x f_n(x) f_{n-1}(x) \end{aligned} \quad (8.9)$$

The recursion relations hold for arbitrary  $w(x)$ .

### 8.1.3 The classic orthogonal polynomials

The classic orthogonal polynomials satisfy three further, but equivalent, conditions. Conversely, any set of orthogonal polynomials which satisfies any one of these conditions is a classical orthogonal polynomial.

1. They satisfy a second order differential equation of the form,

$$A(x)f_n''(x) + B(x)f_n'(x) + C_n f_n(x) = 0 \quad (8.10)$$

where  $A(x)$  and  $B(x)$  are independent of  $n$  and  $C_n$  is independent of  $x$ ;

2. The family of polynomials  $\{f_n'(x)\}$  is orthogonal;
3. They satisfy a Rodrigues formula,

$$f_n(x) = \frac{k_n}{w(x)} \frac{d^n}{dx^n} (F(x)^n w(x)) \quad (8.11)$$

where  $k_n$  is constant and  $F(x)$  is a polynomial in  $x$  which is independent of  $n$ .

The classical orthogonal polynomials are given in Table 1.

Name	notation	$S$	$w(x)$	$F(x)$
Hermite	$H_n(x)$	$\mathbb{R}$	$e^{-x^2}$	1
Laguerre	$L_n(x)$	$[0, \infty]$	$e^{-x}$	$x$
Generalized Laguerre	$L_n^{(\alpha)}(x)$	$[0, \infty]$	$x^\alpha e^{-x}$	$x$
Legendre	$P_n(x)$	$[-1, +1]$	1	$1 - x^2$
Gegenbauer (ultra-spherical)	$P_n^{(\alpha)}(x)$	$[-1, +1]$	$(1 - x^2)^\alpha$	$1 - x^2$
Jacobi	$P_n^{(\alpha, \beta)}(x)$	$[-1, +1]$	$(1 - x)^\alpha (1 + x)^\beta$	$1 - x^2$

Table 1: The classical families of orthogonal polynomials.

As the notations suggest, Laguerre polynomials are special cases of the generalized Laguerre polynomials with  $L_n(x) = L_n^{(0)}(x)$ . Similarly, Legendre polynomials and Gegenbauer polynomials are special cases of Jacobi polynomials respectively with  $P_n(x) = P_n^{(0,0)}(x)$  and  $P_n^{(\alpha)}(x) = P_n^{(\alpha, \alpha)}(x)$ . Given these identifications, one may concentrate on Hermite, generalized Laguerre, and Jacobi polynomials. They satisfy the following differential equations,

$$\begin{aligned} f_n(x) = H_n(x) & \quad 0 = f_n'' - 2xf_n' + 2nf_n \\ f_n(x) = L_n^{(\alpha)}(x) & \quad 0 = xf_n'' + (\alpha + 1 - x)f_n' + nf_n \\ f_n(x) = P_n^{(\alpha, \beta)}(x) & \quad 0 = (1 - x^2)f_n'' + (\beta - \alpha - (\gamma + 1)x)f_n' + n(n + \gamma)f_n \end{aligned} \quad (8.12)$$

where  $\gamma = \alpha + \beta + 1$  on the last line. Note that these equations are linear and thus independent of the normalizations of the polynomials. It is straightforward to see that each differential equation admits a degree  $n$  polynomial solution.

**Hermite polynomials** solve the quantum harmonic oscillator with Hamiltonian (we have scaled out all mathematically irrelevant parameters),

$$H = -\frac{d^2}{dx^2} + x^2 \quad H\psi_n = (2n + 1)\psi_n \quad \psi_n(x) = H_n(x) e^{-x^2/2} \quad (8.13)$$

Recasting  $H$  in terms of raising and lowering operators,

$$H = a^\dagger a + 1 \quad a = \frac{d}{dx} + x \quad a^\dagger = -\frac{d}{dx} + x \quad (8.14)$$

the ground state satisfies  $a\psi_0 = 0$ , while excited states are related by  $\psi_{n+1} = a^\dagger\psi_n$ , which gives the following differential recursion relation for Hermite polynomials (with the standard normalization in which  $k_n = 1$  in the Rodrigues formula),

$$H'_n(x) = 2nH_{n-1}(x) \quad (8.15)$$

**Laguerre polynomials** arise when solving the Coulomb problem (see any textbook on quantum mechanics under the section of the Hydrogen atom).

**Jacobi polynomials** arise in many areas of physics. We consider here a simple electro-statics problem in the plane where they provide a solution. Place  $N + 2$  charges on the real line in the following arrangement. A charge  $p > 0$  is held fixed at  $z = 1$  and a charge  $q > 0$  is held fixed at  $z = -1$ , while  $N$  unit charges reside at points  $x_i$  with  $i = 1, \dots, N$  which are constrained to lie on the real line in the interval  $-1 < x_i < 1$ . Find the (unstable !) equilibrium position of the charges  $x_i$ . The electro-static energy of the configuration is given by,

$$E(x_i) = -\sum_{i=1}^N (p \ln(1 - x_i) + q \ln(1 + x_i)) - \sum_{1 \leq i < j \leq N} \ln|x_i - x_j| \quad (8.16)$$

Extremizing  $E$  with respect to  $x_i$  gives the electro-static equilibrium equations,

$$\frac{p}{x_i - 1} + \frac{q}{x_i + 1} + \sum_{j \neq i} \frac{1}{x_i - x_j} = 0 \quad (8.17)$$

It is these equations that we need to solve. By the way, very similar looking equations arise in the CHY formulation of scattering amplitudes to tree-level, and they also arise in string theory. To solve them, we introduce a polynomial  $f(x)$  of degree  $N$  whose zeros are the  $x_k$ ,

$$f(x) = \prod_{k=1}^N (x - x_k) \quad (8.18)$$

Evaluating its first and second derivatives at a point  $x_i$  we find,

$$f'(x_i) = \prod_{k \neq i} (x_i - x_k) \qquad f''(x_i) = \prod_{j \neq i} (x_i - x_j) \sum_{j \neq i} \frac{1}{x_i - x_j} \quad (8.19)$$

so that the electro-static equation becomes,

$$\frac{p}{x_i - 1} + \frac{q}{x_i + 1} + \frac{f''(x_i)}{f'(x_i)} = 0 \quad (8.20)$$

Let's now look for a polynomial of degree  $N$  which satisfies the equation for all  $x$ ,

$$\frac{p}{x - 1} f'(x) + \frac{q}{x + 1} f'(x) + f''(x) + \alpha(x) f(x) = 0 \quad (8.21)$$

where  $\alpha(x)$  is an undetermined function which is regular at all the points  $x_i$ . Multiplying through  $(1 - x^2)$  we have,

$$(1 - x^2) f''(x) + (q(1 - x) - p(1 + x)) f'(x) + \alpha(x)(1 - x^2) f(x) = 0 \quad (8.22)$$

Upon setting  $\alpha = p + 1$ ,  $\beta = q + 1$ , and  $(1 - x^2)\alpha(x) = N(N + \alpha + \beta + 1)$  we recover exactly the differential equation for the Jacobi polynomials  $P_N^{(p+1, q+1)}(x)$ , and the points  $x_i$  are the zeros of this Jacobi polynomial.

## 8.2 Bessel functions

We had encountered the differential equation for the Bessel functions  $Z_\nu(z)$ ,

$$z^2 Z_\nu''(z) + x Z_\nu'(z) - (\nu^2 - z^2) Z_\nu(z) = 0 \quad (8.23)$$

while the equation appeared in the physical settings where we encountered them for real  $z$  and real  $\nu$  we have here relax these restrictions and generally consider  $z$  and  $\nu$  complex. Thus,  $Z_\nu(z)$  will be a complex analytic function of  $z$  and  $\nu$ , which may of course have poles and branch cuts.

The behavior as  $z \rightarrow 0$  is dominated by the solutions to the homogeneous equation,

$$z^2 Z_\nu''(z) + x Z_\nu'(z) - \nu^2 Z_\nu(z) \approx 0 \quad (8.24)$$

plus corrections which are quadratic in  $z$  times  $Z_\nu(z)$ . The homogeneous equation is solved by  $Z_\nu(z) \approx z^\alpha$  with  $\alpha^2 = \nu^2$ , so that  $\alpha = \pm\nu$ . This makes sense: the sign of  $\nu$  is immaterial in the equation, but the solutions exist for both signs. We shall define the function  $J - \nu(z)$  to behave as  $z^\nu$  near  $z = 0$ , and we expect the following expansion near  $z = 0$ ,

$$J_\nu(z) = z^\nu f(z) \qquad f(z) = \sum_{k=0}^{\infty} f_k z^{2k} \quad (8.25)$$

The expansion will be in powers of  $z^2$  rather than  $z$  since the terms we have neglected to get the approximate behavior was in  $z^2$ . To determine  $f_k$ , we first obtain the differential equation for  $f$ ,

$$zf''(z) + (2\nu + 1)f'(z) + zf(z) = 0 \quad (8.26)$$

Substituting the power series into this equation gives,

$$\sum_{k=0}^{\infty} f_k \left( 2k(2k + 2\nu)z^{2k-1} + z^{2k+1} \right) = 0 \quad (8.27)$$

so that  $f_{k-1} = -4k(k + \nu)f_k$ , which is solved in terms of the first coefficient  $f_0$  by,

$$f_k = \frac{f_0 \Gamma(\nu + 1)}{(-4)^k \Gamma(k + \nu + 1) \Gamma(k + 1)} \quad (8.28)$$

The normalization is chosen so that  $f_0 = 2^{-\nu}/\Gamma(\nu + 1)$  so that the Bessel function with standard normalization is given by,

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{\Gamma(k + \nu + 1) \Gamma(k + 1)} \quad (8.29)$$

For any  $\nu \in \mathbb{C}$ , the series is absolutely convergent throughout  $z \in \mathbb{C}$  and thus  $z^{-\nu} J_\nu(z)$  is an entire function. However,  $J_\nu(z)$  itself is entire only when  $\nu \in \mathbb{N} \cup \{0\}$ , and has a branch point at  $z = 0$  for  $\nu \notin \mathbb{Z}$ . The functions  $J_\nu(z)$  thus defined are sometimes referred to as Bessel functions of the first kind.

The Bessel functions satisfy a simple recursion relation,

$$J_{\nu+1}(z) + J_{\nu-1}(z) - \frac{2\nu}{z} J_\nu(z) = 0 \quad (8.30)$$

To prove it, we use the Taylor series to compute,

$$J_{\nu+1}(z) + J_{\nu-1}(z) = \left(\frac{z}{2}\right)^{\nu-1} \sum_{k=0}^{\infty} \left( \frac{-(-z^2/4)^{k+1}}{\Gamma(k + \nu + 2) \Gamma(k + 1)} + \frac{(-z^2/4)^k}{\Gamma(k + \nu) \Gamma(k + 1)} \right) \quad (8.31)$$

Changing summation variables in the first sum  $k \rightarrow k - 1$  and combining the  $\Gamma$ -functions, we readily derive the recursion relation.

### 8.2.1 Bessel functions of the second kind

For  $\nu = -n$  and  $n \in \mathbb{N}$ , the terms with  $k < n$  in the series vanish and by shifting  $k \rightarrow k + n$  one readily establishes the relation,

$$J_{-n}(z) = (-)^n J_n(z) \quad (8.32)$$

As a result, the functions  $J_\nu(z)$  and  $J_{-\nu}(z)$  are linearly dependent when  $\nu \in \mathbb{Z}$ . Instead of  $J_{-\nu}(z)$  one may therefore use the following linear combinations,

$$Y_\nu(z) = \frac{\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z)}{\sin \pi\nu} \quad (8.33)$$

For  $\nu \in \mathbb{C} \setminus \mathbb{Z}$  this is just a linear combination, but for  $\nu \in \mathbb{Z}$ , the limit produces the second linearly independent solution to the Bessel equation. Linear independence can be investigated using the Wronskian of the differential equation. Two solutions  $Z_\nu, \tilde{Z}_\nu$  define the Wronskian,

$$W[Z_\nu, \tilde{Z}_\nu](z) = Z'_\nu(z)\tilde{Z}_\nu(z) - Z_\nu(z)\tilde{Z}'_\nu(z) \quad (8.34)$$

which satisfies  $W'(z) = -W(z)/z$  and is solved by  $W(z) = W_0/z$  for a constant  $W_0$  which may be obtained from the behavior of the functions as  $z \rightarrow 0$ ,

$$W[J_\nu, J_{-\nu}](z) = \frac{2\pi}{z} \sin \pi\nu \quad W[J_\nu, Y_\nu](z) = -\frac{2\pi}{z} \quad (8.35)$$

and thus we see that  $J_\nu$  and  $Y_\nu$  are linearly independent solutions for all values of  $\nu \in \mathbb{C}$ . The functions  $Y_\nu(z)$  are referred to as Bessel functions of the second kind.

### 8.2.2 Integral representation

A convenient integral representations of the Bessel function is given by,

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-1}^1 dt (1-t^2)^{\nu-\frac{1}{2}} e^{itz} \quad (8.36)$$

The integral is absolutely convergent for  $\text{Re}(\nu) > -\frac{1}{2}$  and may be analytically continued to  $\nu \in \mathbb{C}$ . To prove the validity of this integral representation, we expand the right side in powers of  $z$  and show that this expansion coincides with the one obtained in (8.29). Only even powers will contribute in the expansion, so we have,

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+1)} \int_{-1}^1 dt (1-t^2)^{\nu-\frac{1}{2}} t^{2k} \quad (8.37)$$

Using the invariance under  $t \rightarrow -t$  to restrict the integration region to the interval  $[0, 1]$  upon including a factor of 2, and changing variables  $t^2 = x$ , we obtain,

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \sum_{k=0}^{\infty} \frac{(-z^2)^k}{\Gamma(2k+1)} \int_0^1 dx (1-x)^{\nu-\frac{1}{2}} x^{k-\frac{1}{2}} \quad (8.38)$$

Evaluating the integral via  $\Gamma$ -functions and using the Gauss duplication formula,

$$\sqrt{\pi} \Gamma(2u+1) = 2^{2u} \Gamma(u+1) \Gamma(u + \frac{1}{2}) \quad (8.39)$$

we verify that indeed we recover (8.29).

### 8.2.3 Modified Bessel functions

We had encountered the differential equation for the modified Bessel functions  $M_\nu(z)$ ,

$$z^2 M_\nu''(z) + x M_\nu'(z) - (\nu^2 + z^2) M_\nu(z) = 0 \quad (8.40)$$

which corresponds to letting  $z \rightarrow iz$  in the differential equation for Bessel functions. The precise correspondence is as follows,

$$\begin{aligned} I_\nu(z) &= e^{-i\pi\nu/2} J_\nu(z e^{i\pi/2}) & -\pi < \arg(z) < \frac{\pi}{2} \\ &= e^{3i\pi\nu/2} J_\nu(z e^{-3i\pi/2}) & \frac{\pi}{2} < \arg(z) < \pi \end{aligned} \quad (8.41)$$

The prefactors have been chosen such that the Taylor expansion is as follows,

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{\Gamma(k+\nu+1)\Gamma(k+1)} \quad (8.42)$$

The analytic continuation of the integral representation for  $J_\nu(z)$  is given by,

$$I_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-1}^1 dt (1-t^2)^{\nu-\frac{1}{2}} e^{tz} \quad (8.43)$$

For integer  $\nu = n \in \mathbb{Z}$ , we have  $I_{-n}(z) = I_n(z)$  so that  $I_\nu$  and  $I_{-\nu}$  become linearly dependent for  $\nu \in \mathbb{Z}$ . Thus, in analogy with  $Y_\nu$ , we introduce  $K_\nu(z)$  as follows,

$$K_\nu(z) = \pi \frac{I_{-\nu}(z) - I_\nu(z)}{2 \sin \pi\nu} \quad (8.44)$$

### 8.2.4 Asymptotic behavior

The asymptotic behavior as  $z \rightarrow 0$  may be read off from the Taylor series expansions. As  $z \rightarrow \infty$ , we have,

$$\begin{aligned} J_\nu(z) &= \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{1}{2}\nu\pi\right) + \mathcal{O}(|z|^{-1} e^{|\operatorname{Im} z|}) \\ Y_\nu(z) &= \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{1}{2}\nu\pi\right) + \mathcal{O}(|z|^{-1} e^{|\operatorname{Im} z|}) \\ I_\nu(z) &= \frac{e^z}{\sqrt{2\pi z}} (1 + \mathcal{O}(|z|^{-1})) \\ K_\nu(z) &= \sqrt{\frac{\pi}{2z}} e^{-z} (1 + \mathcal{O}(|z|^{-1})) \end{aligned} \quad (8.45)$$

Given the asymptotic behaviors of  $J_\nu$  and  $Y_\nu$ , it is clearly natural to introduce linear combinations which are just plane waves. These are the Hankel functions,

$$\begin{aligned} H_\nu^+(z) &= J_\nu(z) + iY_\nu(z) \\ H_\nu^-(z) &= J_\nu(z) - iY_\nu(z) \end{aligned} \quad (8.46)$$



### 8.2.5 Spherical Bessel functions

Spherical Bessel functions are proportional to Bessel functions with  $\nu = \ell + \frac{1}{2}$  with  $\ell \in \mathbb{Z}$ , and one divides out by a power of  $z$  that makes them entire functions,

$$\begin{aligned} j_\ell(z) &= \sqrt{\frac{\pi}{2z}} J_{\ell+\frac{1}{2}}(z) \\ y_\ell(z) &= \sqrt{\frac{\pi}{2z}} Y_{\ell+\frac{1}{2}}(z) \end{aligned} \quad (8.47)$$

They arise in spherical decompositions in odd dimensions. Spherical Bessel functions may be expressed in terms of powers and trigonometric functions. For example,

$$j_0(z) = \frac{\sin z}{z} \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \quad (8.48)$$

The general formulas are as follows,

$$\begin{aligned} j_\ell(z) &= (-z)^\ell \left( \frac{1}{z} \frac{d}{dz} \right)^\ell \left( \frac{\sin z}{z} \right) \\ y_\ell(z) &= (-z)^\ell \left( \frac{1}{z} \frac{d}{dz} \right)^\ell \left( -\frac{\cos z}{z} \right) \end{aligned} \quad (8.49)$$

Analogous formulas hold for  $I_{n+\frac{1}{2}}$  and  $K_{n+\frac{1}{2}}$ .

### 8.2.6 Bessel functions of integer order

Bessel functions  $J_n(z)$  and  $I_n(z)$  of integer order  $n = 0$  or  $n \in \mathbb{N}$  are entire functions, and enjoy some extra properties. They arise in spherical decompositions in even dimensions. They are given by a simple generating function,

$$F(z, t) = \sum_{n=-\infty}^{\infty} t^n J_n(z) \quad F(z, t) = \exp \left\{ \frac{z}{2} \left( t - \frac{1}{t} \right) \right\} \quad (8.50)$$

To prove this formula we start from the recursion relation (8.30), and deduce the following relation on  $F$ ,

$$\left( t + \frac{1}{t} \right) F(z, t) - \frac{2t}{z} \frac{\partial F(z, t)}{\partial t} = 0 \quad (8.51)$$

Integrating this first order linear differential equation, we obtain the partial solution,

$$F(z, t) = e^{G(z)} \exp \left\{ \frac{z}{2} \left( t - \frac{1}{t} \right) \right\} \quad (8.52)$$

where  $G(z)$  is independent of  $t$ . Then we use the fact that  $J_n$  satisfies the Bessel differential equation,

$$(z^2 \partial_z^2 + z \partial_z + z^2 - t^2 \partial_t^2 - t \partial_t) F(z, t) = 0 \quad (8.53)$$

Substituting the partial solution into this equation, we obtain an equation for  $F$ —,

$$(t - t^{-1})G' + G'' + (G')^2 + z^{-1}G' = 0 \quad (8.54)$$

Since this equation has to hold for all  $t$ ,  $G$  must be constant. Using the asymptotic expansion as  $z \rightarrow 0$ , we see that the constant must be 1. From the generating function, we immediately deduce an integral representation,

$$J_n(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dt}{t^{n+1}} \exp \left\{ \frac{z}{2} \left( t - \frac{1}{t} \right) \right\} \quad (8.55)$$

or parametrizing  $t = e^{i\theta}$ , we have equivalently,

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \exp \{ i(z \sin \theta - n\theta) \} \quad (8.56)$$

### 8.2.7 Free particles inside a spherical cavity

The bag model of Hadrons assumes that these composite particles may be modeled by freely propagating quarks inside a spherical cavity of radius  $R$ , namely the bag. For simplicity here we shall assume the particles in the bag to be non-relativistic and their wave-function to be confined to the interior of the bag by requiring the potential energy inside the bag to be zero, and outside to be infinite. Thus, we take the Hamiltonian to be  $H = -\Delta$ , with vanishing Dirichlet boundary conditions on the bag. The eigenvalue equation is then  $H\Psi_{n,\ell,m} = E_{n,\ell}\Psi_{n,\ell,m}$ . In spherical coordinates,  $r, \theta, \phi$ , the wave-functions decompose as,

$$\Psi_{n,\ell,m}(r, \theta, \phi) = \frac{1}{\sqrt{r}} \psi_{n,\ell}(r) Y_\ell^m(\theta, \phi) \quad (8.57)$$

where  $Y_\ell^m(\theta, \phi)$  are the spherical harmonics wave-functions. The radial factor  $\psi_{n,\ell}(r)$  satisfies the Bessel equation for  $\nu = \ell + \frac{1}{2}$ ,

$$r^2 \psi'' + r \psi' - \nu^2 \psi + k^2 r^2 \psi = 0 \quad k^2 = E_{n,\ell} \quad (8.58)$$

with solutions which are normalizable at  $r = 0$  given by  $\psi_{n,\ell}(r) = J_\nu(kr)$ . Enforcing the Dirichlet boundary conditions at  $r = R$  requires  $kR$  to be a zero  $\mathfrak{z}_{n,\ell}$  of the Bessel function, labelled by  $n \in \mathbb{N}$ ,

$$E_{n,\ell} = \frac{\mathfrak{z}_{n,\ell}^2}{R^2} \quad J_{\ell+\frac{1}{2}}(\mathfrak{z}_{n,\ell}) = 0 \quad (8.59)$$

For example,  $\mathfrak{z}_{n,0} = n\pi$ ,  $\mathfrak{z}_{1,1} = 4.493$ ,  $\mathfrak{z}_{2,1} = 7.725$ ,  $\mathfrak{z}_{3,1} = 10.904$ ,  $\mathfrak{z}_{1,2} = 5.673$ , etc.

### 8.3 Hypergeometric functions

Recall that Jacobi polynomials  $P_n^{(\alpha,\beta)}(x)$  satisfy the differential equation,

$$(1-x^2)f'' + (\beta - \alpha - (\alpha + \beta + 2)x)f' + n(n + \alpha + \beta + 1)f = 0 \quad (8.60)$$

The existence of a polynomial solution requires  $n$  to be a positive integer. When  $n$  is not an integer the equation becomes the hypergeometric differential equation. It is customary to move the singular points from  $\{-1, 1, \infty\}$  to  $\{0, 1, \infty\}$  by translating and scaling to the new variable  $z = -1 + 2x$ , which we henceforth consider complex. The canonical form of the hypergeometric differential equation is then given by,

$$z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0 \quad (8.61)$$

where  $a, b, c$  are constants, which we will consider to be complex. All points  $z \in \mathbb{C}$  are regular points, except for  $z = 0, 1, \infty$  which are *regular singular points* (where the coefficients in the differential equation have zeros of integer multiplicity). Near each one of these points, we may obtain solutions whose leading behavior scales with some exponent  $\nu$  whose expression may be found by solving the differential equation near the point,

$$\begin{aligned} F(z) &= z^{\nu_0} \left( 1 + f_0^{(1)}z + f_0^{(2)}z^2 + \dots \right) \\ F(z) &= (z-1)^{\nu_1} \left( 1 + f_1^{(1)}(z-1) + f_1^{(2)}(z-1)^2 + \dots \right) \\ F(z) &= \frac{1}{z^{\nu_\infty}} \left( 1 + \frac{f_\infty^{(1)}}{z} + \frac{f_\infty^{(2)}}{z^2} + \dots \right) \end{aligned} \quad (8.62)$$

At each regular singular points, there are two possible exponents, given as follows,

$$\nu_0 = 0, 1 - c \quad \nu_1 = 0, c - a - b \quad \nu_\infty = a, b \quad (8.63)$$

The local behavior near each point provides a basis for the two linearly independent solutions to the differential equation.

#### 8.3.1 Gauss's hypergeometric function

The hypergeometric function  ${}_2F_1$ , which is often written just as  $F$  when no confusion is expected to arise, is defined to be the solution to the differential equation which admits a Taylor series at  $z = 0$  with the following normalization,

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{z^n}{n!} \quad (8.64)$$

To investigate convergence of the series, we use the asymptotic behavior of the following ratio of  $\Gamma$ -functions which is readily deduced from Sterling's formula,

$$\frac{\Gamma(a+n)}{\Gamma(b+n)} = n^{a-b}(1 + \mathcal{O}(1/n)) \quad (8.65)$$

Hence the coefficients of the Taylor series have the asymptotic behavior  $n^{a+b-c-1}$  for large  $n$ , so that the domain of absolute convergence of the Taylor series is  $|z| < 1$ . This is in accord with the fact that the differential equation has a singular point at  $z = 1$ . Now the analysis of the exponents at  $z = 0$  tells us that there must also be a solution which behaves as  $z^{1-c}$  near  $z = 0$ . Thus, the two solutions are given by,

$$\begin{aligned} u_0^+ &= F(a, b; c; z) \\ u_0^- &= z^{1-c} F(a+1-c, b+1-c; 2-c; z) \end{aligned} \quad (8.66)$$

Their domain of convergence is again  $|z| < 1$ . Some complications arise due to the degeneracies of the solutions when one or several of the parameters  $a, b, c$  or their differences  $a-b, b-c, c-a$  are integers, and for simplicity we shall assume that this is not the case in the sequel.

Euler gave the following integral representation,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \quad (8.67)$$

A very useful result, due to Gauss, is the value of  $F$  at  $z = 1$ ,

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (8.68)$$

It may be proven directly from the integral representation by setting  $z = 1$  and using the formula for the Euler Beta function.

### 8.3.2 Analytic continuation

But we could have carried out the same analysis around the point  $z = 1$  as well. We already know the allowed exponents, and we find two solutions with Taylor expansions around  $z = 1$ , as follows,

$$\begin{aligned} u_1^+ &= F(a, b; a+b+1-c; 1-z) \\ u_1^- &= (1-z)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-z) \end{aligned} \quad (8.69)$$

Finally, carrying out the same analysis around the point  $z = \infty$ , we find,

$$\begin{aligned} u_\infty^+ &= (-z)^{-a} F(a, a+1-c; a-b+1; z^{-1}) \\ u_\infty^- &= (-z)^{-b} F(b-c+1, b; b-a+1; z^{-1}) \end{aligned} \quad (8.70)$$

Since the differential equation is second order, it has two linearly independent solutions. As a result, the three pairs of solutions must be linearly related by invertible matrices  $M, N$  with constant coefficients,

$$\begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix} = M \begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} \qquad \begin{pmatrix} u_\infty^+ \\ u_\infty^- \end{pmatrix} = N \begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} \quad (8.71)$$

and thus also,

$$\begin{pmatrix} u_\infty^+ \\ u_\infty^- \end{pmatrix} = NM^{-1} \begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix} \quad (8.72)$$

These formulas provide us with the analytic continuation of the function  $F(a, b; c; z)$  from its domain of absolute convergence of the Taylor series  $|z| < 1$  to the full complex plane, which will require branch cuts. By inspection we see that the domains of absolute convergence of the solutions are as follows,

$$\begin{array}{ll} u_0^\pm(z) & |z| < 1 \\ u_1^\pm(z) & |z - 1| < 1 \\ u_\infty^\pm(z) & |z| > 1 \end{array} \quad (8.73)$$

While the domains of convergence of  $u_0^\pm(z)$  and  $u_\infty^\pm(z)$  have zero intersection, they both intersect non-trivially with  $u_1^\pm(z)$ . Hence they may be related to one another in these overlapping domain, so that the function  $F(a, b; c; z)$  may indeed be analytically continued to the entire plane.

### 8.3.3 Explicit transition formulas

The entries of  $M, N$ , respectively denoted by  $M_{ij}, N_{ij}$  with  $i, j = \pm$  are obtained as follows. Take the example of expressing  $u_1^+$  as a linear combination of  $u_0^\pm$ ,

$$u_1^+(z) = M_{++}u_0^+(z) + M_{+-}u_0^-(z) \quad (8.74)$$

Evaluating both sides at  $z = 0$  and using Gauss's formula (8.68), we obtain,

$$M_{++} = \frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \quad (8.75)$$

while setting  $z = 1$ , we obtain,

$$1 = M_{++} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + M_{+-} \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \quad (8.76)$$

Solving for  $M_{+-}$  by eliminating  $M_{++}$  and making use of the identity established earlier  $\Gamma(z)\Gamma(1-z)\sin\pi z = \pi$ , we find  $M_{+-}$ . The remaining entries may be determined in a similar manner, and we find,

$$\begin{aligned} M_{++} &= \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} & M_{+-} &= \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} \\ M_{-+} &= \frac{\Gamma(c+1-a-b)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} & M_{--} &= \frac{\Gamma(c+1-a-b)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)} \end{aligned} \quad (8.77)$$

Similarly, we may determined the entries of the matrix  $N$ , and we find,

$$\begin{aligned} N_{++} &= \frac{\Gamma(a+1-b)\Gamma(1-c)}{\Gamma(1-b)\Gamma(a+1-c)} & N_{+-} &= \frac{\Gamma(a+1-b)\Gamma(c-1)}{\Gamma(a)\Gamma(c-b)} e^{i\pi(c-1)} \\ N_{-+} &= \frac{\Gamma(b+1-a)\Gamma(1-c)}{\Gamma(1-a)\Gamma(b+1-c)} & N_{--} &= \frac{\Gamma(b+1-a)\Gamma(c-1)}{\Gamma(b)\Gamma(c-a)} e^{i\pi(c-1)} \end{aligned} \quad (8.78)$$

### 8.3.4 Monodromies

The function  $u_0^+(z)$  is holomorphic inside  $|z| < 1$ , but the function  $u_0^-(z)$  has a branch point at  $z = 0$ . Similarly,  $u_1^+(z)$  is holomorphic in the interior of  $|z-1| < 1$  but  $u_1^-(z)$  has a branch point at  $z = 1$ , while both  $u_\infty^\pm$  have branch points at  $z = \infty$ . We shall now define the operation of circling around each one of the branch point,

$$C_0(z) = e^{2\pi i} z \quad C_1(1-z) = e^{2\pi i}(1-z) \quad C_\infty(z^{-1}) = e^{2\pi i} z^{-1} \quad (8.79)$$

The effect of these transformations on the functions  $u_0^\pm, u_1^\pm, u_\infty^\pm$  may be represented by matrices, for  $i = 0, 1, \infty$ ,

$$\begin{pmatrix} u_i^+ \\ u_i^- \end{pmatrix} (C_i(z)) = \mathcal{C}_i \begin{pmatrix} u_i^+ \\ u_i^- \end{pmatrix} (z) \quad (8.80)$$

and the matrices  $\mathcal{C}_i$  may be read off from the definitions of the functions and we find,

$$\mathcal{C}_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi ic} \end{pmatrix} \quad \mathcal{C}_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i(c-a-b)} \end{pmatrix} \quad \mathcal{C}_\infty = \begin{pmatrix} e^{2\pi ia} & 0 \\ 0 & e^{2\pi ib} \end{pmatrix} \quad (8.81)$$

Using these results and the interrelations between the functions  $u_i^\pm$  then give formulas for all the monodromies,

$$\begin{pmatrix} u_i^+ \\ u_i^- \end{pmatrix} (C_j(z)) = \mathcal{C}_{ij} \begin{pmatrix} u_i^+ \\ u_i^- \end{pmatrix} (z) \quad (8.82)$$

where  $\mathcal{C}_{ii} = \mathcal{C}_i$  and the off-diagonal matrices  $\mathcal{C}_{ij}$  are given as follows,

$$\begin{aligned} \mathcal{C}_{01} &= M^{-1}\mathcal{C}_1M & \mathcal{C}_{10} &= M\mathcal{C}_0M^{-1} & \mathcal{C}_{\infty 0} &= N^{-1}\mathcal{C}_0N \\ \mathcal{C}_{0\infty} &= N^{-1}\mathcal{C}_\infty N & \mathcal{C}_{1\infty} &= MN^{-1}\mathcal{C}_\infty NM^{-1} & \mathcal{C}_{\infty 1} &= NM^{-1}\mathcal{C}_1MN^{-1} \end{aligned} \quad (8.83)$$

The Wronskian of the hypergeometric equation is given by,

$$W[f_1, f_2] = f_1' f_2 - f_1 f_2' = W_0 z^{-c} (1-z)^{c-a-b-a} \quad (8.84)$$

for a constant  $W_0$  which depends upon the normalizations, and is given as follows,

$$\begin{aligned} W[u_0^+, u_0^-] &= (c-1) z^{-c} (1-z)^{c-a-b-a} \\ W[u_1^+, u_1^-] &= (a+b-c) z^{-c} (1-z)^{c-a-b-a} \\ W[u_\infty^+, u_\infty^-] &= (a-b) e^{i\pi c} z^{-c} (1-z)^{c-a-b-a} \end{aligned} \quad (8.85)$$

The determinants of  $M$  and  $N$  are given by,

$$\det M = \frac{c-a-b}{c-1} \quad \det N = \frac{a-b}{c-1} e^{i\pi c} \quad (8.86)$$

One verifies that the Wronskians indeed satisfy the correct equations,

$$\begin{aligned} W[u_1^+, u_1^-] &= (\det M) W[u_0^+, u_0^-] \\ W[u_\infty^+, u_\infty^-] &= (\det N) W[u_0^+, u_0^-] \end{aligned} \quad (8.87)$$

### 8.3.5 Physical systems governed by hypergeometric functions