V. I. Arnold

Mathematical Methods of Classical Mechanics

Second Edition

Translated by K. Vogtmann
and A. Weinstein

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Preface

Many different mathematical methods and concepts are used in classical mechanics: differential equations and phase flows, smooth mappings and manifolds, Lie groups and Lie algebras, symplectic geometry and ergodic theory. Many modern mathematical theories arose from problems in mechanics and only later acquired that axiomatic-abstract form which makes them so hard to study.

In this book we construct the mathematical apparatus of classical mechanics from the very beginning; thus, the reader is not assumed to have any previous knowledge beyond standard courses in analysis (differential and integral calculus, differential equations), geometry (vector spaces, vectors) and linear algebra (linear operators, quadratic forms).

With the help of this apparatus, we examine all the basic problems in dynamics, including the theory of oscillations, the theory of rigid body motion, and the hamiltonian formalism. The author has tried to show the geometric, qualitative aspect of phenomena. In this respect the book is closer to courses in theoretical mechanics for theoretical physicists than to traditional courses in theoretical mechanics as taught by mathematicians.

A considerable part of the book is devoted to variational principles and analytical dynamics. Characterizing analytical dynamics in his "Lectures on the development of mathematics in the nineteenth century," F. Klein wrote that ". . . a physicist, for his problems, can extract from these theories only very little, and an engineer nothing." The development of the sciences in the following years decisively disproved this remark. Hamiltonian formalism lay at the basis of quantum mechanics and has become one of the most often used tools in the mathematical arsenal of physics. After the significance of symplectic structures and Huygens' principle for all sorts of optimization problems was realized, Hamilton's equations began to be used constantly in engineering calculations. On the other hand, the contemporary development of celestial mechanics, connected with the requirements of space exploration, created new interest in the methods and problems of analytical dynamics.

The connections between classical mechanics and other areas of mathematics and physics are many and varied. The appendices to this book are devoted to a few of these connections. The apparatus of classical mechanics is applied to: the foundations of riemannian geometry, the dynamics of an ideal fluid, Kolmogorov's theory of perturbations of conditionally periodic motion, short-wave asymptotics for equations of mathematical physics, and the classification of caustics in geometrical optics.

These appendices are intended for the interested reader and are not part of the required general course. Some of them could constitute the basis of special courses (for example, on asymptotic methods in the theory of nonlinear oscillations or on quasi-classical asymptotics). The appendices also contain some information of a reference nature (for example, a list of normal forms of quadratic hamiltonians). While in the basic chapters of the book the author has tried to develop all the proofs as explicitly as possible, avoiding references to other sources, the appendices consist on the whole of summaries of results, the proofs of which are to be found in the cited literature.

The basis for the book was a year-and-a-half-long required course in classical mechanics, taught by the author to third- and fourth-year mathematics students at the mathematics-mechanics faculty of Moscow State University in 1966–1968.

The author is grateful to I. G. Petrovsky, who insisted that these lectures be delivered, written up, and published. In preparing these lectures for publication, the author found very helpful the lecture notes of L. A. Bunimovich, L. D. Vaingortin, V. L. Novikov, and especially, the mimeographed edition (Moscow State University, 1968) organized by N. N. Kolesnikov. The author thanks them, and also all the students and colleagues who communicated their remarks on the mimeographed text; many of these remarks were used in the preparation of the present edition. The author is grateful to M. A. Leontovich, for suggesting the treatment of connections by means of a limit process, and also to I. I. Vorovich and V. I. Yudovich for their detailed review of the manuscript.

V. ARNOLD

The translators would like to thank Dr. R. Barrar for his help in reading the proofs. We would also like to thank many readers, especially Ted Courant, for spotting errors in the first two printings.

Berkeley, 1981

K. VOGTMANN
A. WEINSTEIN

Preface to the second edition

The main part of this book was written twenty years ago. The ideas and methods of symplectic geometry, developed in this book, have now found many applications in mathematical physics and in other domains of applied mathematics, as well as in pure mathematics itself. Especially, the short-wave asymptotical expansions theory has reached a very sophisticated level, with many important applications to optics, wave theory, acoustics, spectroscopy, and even chemistry; this development was parallel to the development of the theories of Lagrange and Legendre singularities, that is, of singularities of caustics and of wave fronts, of their topology and their perestroikas (in Russian metamorphoses were always called "perestroikas," as in "Morse perestroika" for the English "Morse surgery"; now that the word perestroika has become international, we may preserve the Russian term in translation and are not obliged to substitute "metamorphoses" for "perestroikas" when speaking of wave fronts, caustics, and so on).

Integrable hamiltonian systems have been discovered unexpectedly in many classical problems of mathematical physics, and their study has led to new results in both physics and mathematics, for instance, in algebraic geometry.

Symplectic topology has become one of the most promising and active branches of "global analysis." An important generalization of the Poincaré "geometric theorem" (see Appendix 9) was proved by C. Conley and E. Zehnder in 1983. A sequence of works (by M. Chaperon, A. Weinstein, J.-C. Sikorav, M. Gromov, Ja. M. Eliashberg, Ju. Tchekanov, A. Floer, C. Viterbo, H. Hofer, and others) marks important progress in this very living domain. One may hope that this progress will lead to the proof of many known conjectures in symplectic and contact topology, and to the discovery of new results in this new domain of mathematics, emerging from the problems of mechanics and optics.

The present edition includes three new appendices. They represent the modern development of the theory of ray systems (the theory of singularity and of perestroikas of caustics and of wave fronts, related to the theory of Coxeter reflection groups), the theory of integrable systems (the geometric theory of elliptic coordinates, adapted to the infinite-dimensional Hilbert space generalization), and the theory of Poisson structures (which is a generalization of the theory of symplectic structures, including degenerate Poisson brackets).

A more detailed account of the present state of perturbation theory may be found in the book, *Mathematical Aspects of Classical and Celestial Mechanics* by V. I. Arnold, V. V. Kozlov, and A. I. Neistadt, Encyclopaedia of Math. Sci., Vol. 3 (Springer, 1986); Volume 4 of this series (1988) contains a survey "Symplectic geometry" by V. I. Arnold and A. B. Givental', an article by A. A. Kirillov on geometric quantization, and a survey of the modern theory of integrable systems by S. P. Novikov, I. M. Krichever, and B. A. Dubrovin.

For more details on the geometry of ray systems, see the book *Singularities of Differentiable Mappings* by V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko (Vol. 1, Birkhäuser 1985; vol. 2, Birkhäuser, 1988). *Catastrophe Theory* by V. I. Arnold (Springer, 1986) (second edition) contains a long annotated bibliography.

Surveys on symplectic and contact geometry and on their applications may be found in the Bourbaki seminar (D. Bennequin, "Caustiques mystiques", February, 1986) and in a series of articles (V. I. Arnold, First steps of symplectic topology, Russian Math. Surveys, 41 (1986); Singularities of ray systems, Russian Math. Surveys, 38 (1983); Singularities in variational calculus, Modern Problems of Math., VINITI, 22 (1983) (translated in J. Soviet Math.); and O. P. Shcherbak, Wave fronts and reflection groups, Russian Math. Surveys, 43 (1988)).

Volumes 22 (1983) and 33 (1988) of the VINITI series, "Sovremennye problemy mathematiki. Noveishie dostijenia," contain a dozen articles on the applications of symplectic and contact geometry and singularity theory to mathematics and physics.

Bifurcation theory (both for hamiltonian and for more general systems) is discussed in the textbook *Geometrical Methods of the Theory of Ordinary Differential Equations* (Springer, 1988) (this new edition is more complete than the preceding one). The survey "Bifurcation theory and its applications in mathematics and mechanics" (XVIIth International Congress of Theoretical and Applied Mechanics in Grenoble, August, 1988) also contains new information, as does Volume 5 of the Encyclopaedia of Math. Sci. (Springer, 1989), containing the survey "Bifurcation theory" by V. I. Arnold, V. S. Afraimovich, Ju. S. Iljashenko, and L. P. Shilnikov. Volume 2 of this series, edited by D. V. Anosov and Ja. G. Sinai, is devoted to the ergodic theory of dynamical systems including those of mechanics.

The new discoveries in all these theories have potentially extremely wide applications, but since these results were discovered rather recently, they are discussed only in the specialized editions, and applications are impeded by the difficulty of the mathematical exposition for nonmathematicians. I hope that the present book will help to master these new theories not only to mathematicians, but also to all those readers who use the theory of dynamical systems, symplectic geometry, and the calculus of variations—in physics, mechanics, control theory, and so on. The author would like to thank Dr. T. Tokieda for his help in correcting errors in previous printings and for reading the proofs.

December 1988 V. I. Arnold

Translator's preface to the second edition

This edition contains three new appendices, originally written for inclusion in a German edition. They describe work by the author and his co-workers on Poisson structures, elliptic coordinates with applications to integrable systems, and singularities of ray systems. In addition, numerous corrections to errors found by the author, the translators, and readers have been incorporated into the text.

Contents

PART I
NEWTONIAN MECHANICS

Newtonian mechanics studies the motion of a system of point masses in three-dimensional euclidean space. The basic ideas and theorems of newtonian mechanics (even when formulated in terms of three-dimensional cartesian coordinates) are invariant with respect to the six-dimensional[1] group of euclidean motions of this space.

A newtonian potential mechanical system is specified by the masses of the points and by the potential energy. The motions of space which leave the potential energy invariant correspond to laws of conservation.

Newton's equations allow one to solve completely a series of important problems in mechanics, including the problem of motion in a central force field.

[1] And also with respect to the larger group of galilean transformations of space-time.

Experimental facts 1

In this chapter we write down the basic experimental facts which lie at the foundation of mechanics: Galileo's principle of relativity and Newton's differential equation. We examine constraints on the equation of motion imposed by the relativity principle, and we mention some simple examples.

1 The principles of relativity and determinacy

In this paragraph we introduce and discuss the notion of an inertial coordinate system. The mathematical statements of this paragraph are formulated exactly in the next paragraph.

A series of experimental facts is at the basis of classical mechanics.[2] We list some of them.

A *Space and time*

Our space is three-dimensional and euclidean, and time is one-dimensional.

B *Galileo's principle of relativity*

There exist coordinate systems (called inertial) possessing the following two properties:

1. All the laws of nature at all moments of time are the same in all inertial coordinate systems.
2. All coordinate systems in uniform rectilinear motion with respect to an inertial one are themselves inertial.

[2] All these "experimental facts" are only approximately true and can be refuted by more exact experiments. In order to avoid cumbersome expressions, we will not specify this from now on and we will speak of our mathematical models as if they exactly described physical phenomena.

In other words, if a coordinate system attached to the earth is inertial, then an experimenter on a train which is moving uniformly in a straight line with respect to the earth cannot detect the motion of the train by experiments conducted entirely inside his car.

In reality, the coordinate system associated with the earth is only approximately inertial. Coordinate systems associated with the sun, the stars, etc. are more nearly inertial.

C *Newton's principle of determinacy*

The initial state of a mechanical system (the totality of positions and velocities of its points at some moment of time) uniquely determines all of its motion.

It is hard to doubt this fact, since we learn it very early. One can imagine a world in which to determine the future of a system one must also know the acceleration at the initial moment, but experience shows us that our world is not like this.

2 The galilean group and Newton's equations

In this paragraph we define and investigate the galilean group of space-time transformations. Then we consider Newton's equation and the simplest constraints imposed on its right-hand side by the property of invariance with respect to galilean transformations.[3]

A *Notation*

We denote the set of all real numbers by $\mathbb{R}$. We denote by $\mathbb{R}^n$ an n-dimensional real vector space.

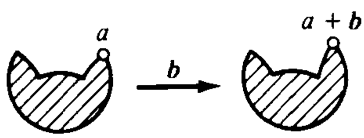

Figure 1 Parallel displacement

Affine n-dimensional space A^n is distinguished from $\mathbb{R}^n$ in that there is "no fixed origin." The group $\mathbb{R}^n$ acts on A^n as the *group of parallel displacements* (Figure 1):

$$a \to a + \mathbf{b}, \qquad a \in A^n, \mathbf{b} \in \mathbb{R}^n, a + \mathbf{b} \in A^n.$$

[Thus the sum of two points of A^n is not defined, but their difference is defined and is a vector in $\mathbb{R}^n$.]

[3] The reader who has no need for the mathematical formulation of the assertions of Section 1 can omit this section.

A *euclidean structure* on the vector space $\mathbb{R}^n$ is a positive definite symmetric bilinear form called a *scalar product*. The scalar product enables one to define the *distance*

$$\rho(x, y) = \|x - y\| = \sqrt{(x - y, x - y)}$$

between points of the corresponding *affine* space A^n. An affine space with this distance function is called a *euclidean space* and is denoted by E^n.

B *Galilean structure*

The galilean space-time structure consists of the following three elements:

1. The universe—a four-dimensional affine[4] space A^4. The points of A^4 are called *world points* or *events*. The parallel displacements of the universe A^4 constitute a vector space $\mathbb{R}^4$.
2. Time—a linear mapping $t: \mathbb{R}^4 \to \mathbb{R}$ from the vector space of parallel displacements of the universe to the real "time axis." The *time interval* from event $a \in A^4$ to event $b \in A^4$ is the number $t(b - a)$ (Figure 2). If $t(b - a) = 0$, then the events a and b are called *simultaneous*.

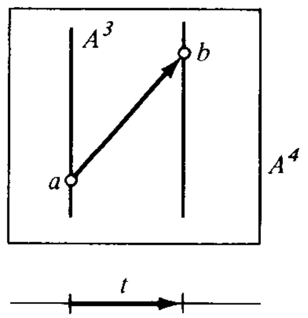

Figure 2 Interval of time t

The set of events simultaneous with a given event forms a three-dimensional affine subspace in A^4. It is called a *space of simultaneous events* A^3.

The kernel of the mapping t consists of those parallel displacements of A^4 which take some (and therefore every) event into an event simultaneous with it. This kernel is a three-dimensional linear subspace $\mathbb{R}^3$ of the vector space $\mathbb{R}^4$.

The galilean structure includes one further element.

3. The *distance between simultaneous events*

$$\rho(a, b) = \|a - b\| = \sqrt{(a - b, a - b)} \qquad a, b \in A^3$$

is given by a scalar product on the space $\mathbb{R}^3$. This distance makes every space of simultaneous events into a three-dimensional euclidean space E^3.

[4] Formerly, the universe was provided not with an affine, but with a linear structure (the geocentric system of the universe).

A space A^4, equipped with a galilean space-time structure, is called a *galilean space.*

One can speak of two events occurring simultaneously in different places, but the expression "two non-simultaneous events $a, b \in A^4$ occurring at *one and the same place in three-dimensional space*" has no meaning as long as we have not chosen a coordinate system.

The *galilean group* is the group of all transformations of a galilean space which preserve its structure. The elements of this group are called *galilean transformations*. Thus, galilean transformations are affine transformations of A^4 which preserve intervals of time and the distance between simultaneous events.

EXAMPLE. Consider the direct product[5] $\mathbb{R} \times \mathbb{R}^3$ of the t axis with a three-dimensional vector space $\mathbb{R}^3$; suppose $\mathbb{R}^3$ has a fixed euclidean structure. Such a space has a natural galilean structure. We will call this space *galilean coordinate space.*

We mention three examples of galilean transformations of this space. First, *uniform motion with velocity* $\mathbf{v}$:

$$g_1(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{v}t) \qquad \forall t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3.$$

Next, *translation of the origin*:

$$g_2(t, \mathbf{x}) = (t + s, \mathbf{x} + \mathbf{s}) \qquad \forall t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3.$$

Finally, *rotation of the coordinate axes*:

$$g_3(t, \mathbf{x}) = (t, G\mathbf{x}), \qquad \forall t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3,$$

where $G: \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation.

PROBLEM. Show that every galilean transformation of the space $\mathbb{R} \times \mathbb{R}^3$ can be written in a unique way as the composition of a rotation, a translation, and a uniform motion ($g = g_1 \circ g_2 \circ g_3$) (thus the dimension of the galilean group is equal to $3 + 4 + 3 = 10$).

PROBLEM. Show that all galilean spaces are isomorphic to each other[6] and, in particular, isomorphic to the coordinate space $\mathbb{R} \times \mathbb{R}^3$.

Let M be a set. A one-to-one correspondence $\varphi_1: M \to \mathbb{R} \times \mathbb{R}^3$ is called a *galilean coordinate system on the set M*. A coordinate system φ_2 *moves uniformly* with respect to φ_1 if $\varphi_1 \circ \varphi_2^{-1}: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ is a galilean transformation. The galilean coordinate systems φ_1 and φ_2 give M the same galilean structure.

[5] Recall that the direct product of two sets A and B is the set of ordered pairs (a, b), where $a \in A$ and $b \in B$. The direct product of two spaces (vector, affine, euclidean) has the structure of a space of the same type.

[6] That is, there is a one-to-one mapping of one to the other preserving the galilean structure.

C *Motion, velocity, acceleration*

A motion in $\mathbb{R}^N$ is a differentiable mapping $\mathbf{x}: I \to \mathbb{R}^N$, where I is an interval on the real axis.

The derivative

$$\dot{\mathbf{x}}(t_0) = \left.\frac{d\mathbf{x}}{dt}\right|_{t=t_0} = \lim_{h\to 0} \frac{\mathbf{x}(t_0 + h) - \mathbf{x}(t_0)}{h} \in \mathbb{R}^N$$

is called the *velocity vector* at the point $t_0 \in I$.

The second derivative

$$\ddot{\mathbf{x}}(t_0) = \left.\frac{d^2\mathbf{x}}{dt^2}\right|_{t=t_0}$$

is called the *acceleration vector* at the point t_0.

We will assume that the functions we encounter are continuously differentiable as many times as necessary. In the future, unless otherwise stated, mappings, functions, etc. are understood to be differentiable mappings, functions, etc. The image of a mapping $\mathbf{x}: I \to \mathbb{R}^N$ is called a *trajectory* or *curve* in $\mathbb{R}^N$.

PROBLEM. Is it possible for the trajectory of a differentiable motion on the plane to have the shape drawn in Figure 3? Is it possible for the acceleration vector to have the value shown?

ANSWER. Yes. No.

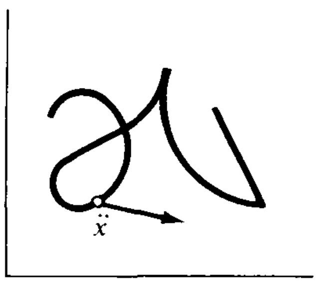

Figure 3 Trajectory of motion of a point

We now define a *mechanical system of n points moving in three-dimensional euclidean space.*

Let $\mathbf{x}: \mathbb{R} \to \mathbb{R}^3$ be a motion in $\mathbb{R}^3$. The graph[7] of this mapping is a curve in $\mathbb{R} \times \mathbb{R}^3$.

A curve in galilean space which appears in some (and therefore every) galilean coordinate system as the graph of a motion, is called a *world line* (Figure 4).

[7] The graph of a mapping $f: A \to B$ is the subset of the direct product $A \times B$ consisting of all pairs $(a, f(a))$ with $a \in A$.

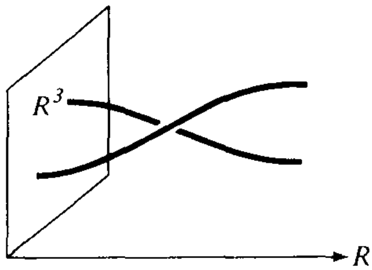

Figure 4 World lines

A motion of a system of n points gives, in galilean space, n world lines. In a galilean coordinate system they are described by n mappings $\mathbf{x}_i: \mathbb{R} \to \mathbb{R}^3$, $i = 1, \ldots, n$.

The direct product of n copies of $\mathbb{R}^3$ is called the *configuration space* of the system of n points. Our n mappings $\mathbf{x}_i: \mathbb{R} \to \mathbb{R}^3$ define one mapping

$$\mathbf{x}: \mathbb{R} \to \mathbb{R}^N \qquad N = 3n$$

of the time axis into the configuration space. Such a mapping is also called a *motion of a system of n points* in the galilean coordinate system on $\mathbb{R} \times \mathbb{R}^3$.

D *Newton's equations*

According to Newton's principle of determinacy (Section 1C) all motions of a system are uniquely determined by their initial positions ($\mathbf{x}(t_0) \in \mathbb{R}^N$) and initial velocities ($\dot{\mathbf{x}}(t_0) \in \mathbb{R}^N$).

In particular, the initial positions and velocities determine the acceleration. In other words, there is a function $\mathbf{F}: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ such that

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t). \tag{1}$$

Newton used Equation (1) as the basis of mechanics. It is called *Newton's equation.*

By the theorem of existence and uniqueness of solutions to ordinary differential equations, the function $\mathbf{F}$ and the initial conditions $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ uniquely determine a motion.[8]

For each specific mechanical system the form of the function $\mathbf{F}$ is determined experimentally. From the mathematical point of view the form of $\mathbf{F}$ for each system constitutes the *definition* of that system.

E *Constraints imposed by the principle of relativity*

Galileo's principle of relativity states that in physical space-time there is a selected galilean structure ("the class of inertial coordinate systems") having the following property.

[8] Under certain smoothness conditions, which we assume to be fulfilled. In general, a motion is determined by Equation (1) only on some *interval* of the time axis. For simplicity we will assume that this interval is the whole time axis, as is the case in most problems in mechanics.

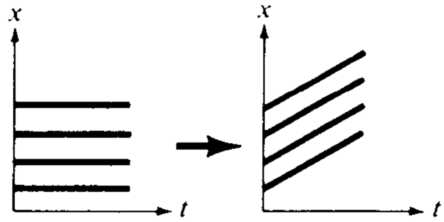

Figure 5 Galileo's principle of relativity

If we subject the world lines of all the points of any mechanical system[9] to one and the same galilean transformation, we obtain world lines of the same system (with new initial conditions) (Figure 5).

This imposes a series of conditions on the form of the right-hand side of Newton's equation written in an inertial coordinate system: Equation (1) must be invariant with respect to the group of galilean transformations.

EXAMPLE 1. Among the galilean transformations are the time translations. Invariance with respect to time translations means that "the laws of nature remain constant," i.e., if $\mathbf{x} = \boldsymbol{\varphi}(t)$ is a solution to Equation (1), then for any $s \in \mathbb{R}$, $\mathbf{x} = \boldsymbol{\varphi}(t + s)$ is also a solution.

From this it follows that the right-hand side of Equation (1) in an inertial coordinate system does not depend on the time:

$$\ddot{\mathbf{x}} = \boldsymbol{\Phi}(\mathbf{x}, \dot{\mathbf{x}}).$$

Remark. Differential equations in which the right-hand side does depend on time arise in the following situation.

Suppose that we are studying part I of the mechanical system I + II. Then the influence of part II on part I can sometimes be replaced by a time variation of parameters in the system of equations describing the motion of part I. For example, the influence of the moon on the earth can be ignored in investigating the majority of phenomena on the earth. However, in the study of the tides this influence must be taken into account; one can achieve this by introducing, instead of the attraction of the moon, periodic changes in the strength of gravity on earth.

[9] In formulating the principle of relativity we must keep in mind that it is relevant only to *closed* physical (in particular, mechanical) systems, i.e., that we must include in the system all bodies whose interactions play a role in the study of the given phenomena. Strictly speaking, we should include in the system all bodies in the universe. But we know from experience that one can disregard the effect of many of them: for example, in studying the motion of planets around the sun we can disregard the attractions among the stars, etc.

On the other hand, in the study of a body in the vicinity of earth, the system is not closed if the earth is not included; in the study of the motion of an airplane the system is not closed if it does not include the air surrounding the airplane, etc. In the future, the term "mechanical system" will mean a closed system in most cases, and when there is a non-closed system in question this will be explicitly stated (cf., for example, Section 3).

Equations with variable coefficients can appear also as the result of formal operations in the solution of problems.

EXAMPLE 2. Translations in three-dimensional space are galilean transformations. Invariance with respect to such translations means that *space is homogeneous*, or "has the same properties at all of its points." That is, if $\mathbf{x}_i = \boldsymbol{\varphi}_i(t)(i = 1, \ldots, n)$ is a motion of a system of n points satisfying (1), then for any $\mathbf{r} \in \mathbb{R}^3$ the motion $\boldsymbol{\varphi}_i(t) + \mathbf{r}\,(i = 1, \ldots, n)$ also satisfies Equation (1).

From this it follows that the right-hand side of Equation (1) in the inertial coordinate system can depend only on the "relative coordinates" $\mathbf{x}_j - \mathbf{x}_k$.

From invariance under passage to a uniformly moving coordinate system (which does not change $\ddot{\mathbf{x}}_i$ or $\mathbf{x}_j - \mathbf{x}_k$, but adds to each $\dot{\mathbf{x}}_j$ a fixed vector $\mathbf{v}$) it follows that the right-hand side of Equation (1) in an inertial system of coordinates can depend only on the *relative* velocities

$$\ddot{\mathbf{x}}_i = \mathbf{f}_i(\{\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k\}), \qquad i, j, k = 1, \ldots, n.$$

EXAMPLE 3. Among the galilean transformations are the rotations in three-dimensional space. Invariance with respect to these rotations means that *space is isotropic*; there are no preferred directions.

Thus, if $\boldsymbol{\varphi}_i : \mathbb{R} \to \mathbb{R}^3 (i = 1, \ldots, n)$ is a motion of a system of points satisfying (1), and $G: \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation, then the motion $G\boldsymbol{\varphi}_i : \mathbb{R} \to \mathbb{R}^3 (i, \ldots, n)$ also satisfies (1). In other words.

$$\mathbf{F}(G\mathbf{x}, G\,\dot{\mathbf{x}}) = G\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}),$$

where $G\mathbf{x}$ denotes $(G\mathbf{x}_1, \ldots, G\mathbf{x}_n)$, $\mathbf{x}_i \in \mathbb{R}^3$.

PROBLEM. Show that if a mechanical system consists of only one point, then its acceleration in an inertial coordinate system is equal to zero ("Newton's first law").

Hint. By Examples 1 and 2 the acceleration vector does not depend on $\mathbf{x}$, $\dot{\mathbf{x}}$, or t, and by Example 3 the vector $\mathbf{F}$ is invariant with respect to rotation.

PROBLEM. A mechanical system consists of two points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points will stay on the line which connected them at the initial moment.

PROBLEM. A mechanical system consists of three points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points always remain in the plane which contained them at the initial moment.

PROBLEM. A mechanical system consists of two points. Show that for any initial conditions there exists an inertial coordinate system in which the two points remain in a fixed plane.

Problem. Show that mechanics "through the looking glass" is identical to ours.

Hint. In the galilean group there is a reflection transformation, changing the orientation of $\mathbb{R}^3$.

Problem. Is the class of inertial systems unique?

Answer. No. Other classes can be obtained if one changes the units of length and time or the direction of time.

3 Examples of mechanical systems

We have already remarked that the form of the function $\mathbf{F}$ in Newton's equation (1) is determined experimentally for each mechanical system. Here are several examples.

In examining concrete systems it is reasonable not to include all the objects of the universe in a system. For example, in studying the majority of phenomena taking place on the earth we can ignore the influence of the moon. Furthermore, it is usually possible to disregard the effect of the processes we are studying on the motion of the earth itself; we may even consider a coordinate system attached to the earth as "fixed." It is clear that the principle of relativity no longer imposes the constraints found in Section 2 for equations of motion written in such a coordinate system. For example, near the earth there is a distinguished direction, the vertical.

A *Example 1: A stone falling to the earth*

Experiments show that

$$\ddot{x} = -g, \quad \text{where } g \approx 9.8 \text{ m/s}^2 \text{ (Galileo)*} \tag{2}$$

where x is the height of a stone above the surface of the earth.

If we introduce the "potential energy" $U = gx$, then Equation (2) can be written in the form

$$\ddot{x} = -\frac{dU}{dx}.$$

If $U: E^N \to \mathbb{R}$ is a differentiable function on euclidean space, then we will denote by $\partial U/\partial \mathbf{x}$ the gradient of the function U. If $E^N = E^{n_1} \times \cdots \times E^{n_k}$ is a direct product of euclidean spaces, then we will denote a point $\mathbf{x} \in E^N$ by $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$, and the vector $\partial U/\partial \mathbf{x}$ by $(\partial U/\partial \mathbf{x}_1, \ldots, \partial U/\partial \mathbf{x}_k)$. In particular, if $x_1, \ldots, x_N$ are cartesian coordinates in E^N, then the components of the vector $\partial U/\partial \mathbf{x}$ are the partial derivatives $\partial U/\partial x_1, \ldots, \partial U/\partial x_N$.

Experiments show that the radius vector of the stone with respect to some point 0 on the earth satisfies the equation

$$\ddot{\mathbf{x}} = -\frac{\partial U}{\partial \mathbf{x}}, \quad \text{where } U = -(\mathbf{g}, \mathbf{x}) \tag{3}$$

* In this and other sections, the mass of a particle is taken to be 1.

The vector in the right-hand side is directed towards the earth. It is called the *gravitational acceleration vector* **g**. (Figure 6.)

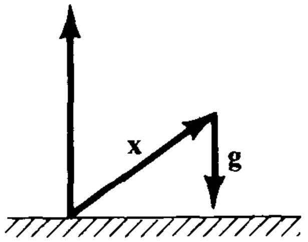

Figure 6 A stone falling to the earth

B *Example 2: Falling from great height*

Like all experimental facts, the law of motion (2) has a restricted domain of application. According to a more precise law of falling bodies, discovered by Newton, acceleration is inversely proportional to the square of the distance from the center of the earth:

$$\ddot{x} = -g\frac{r_0^2}{r^2},$$

where $r = r_0 + x$ (Figure 7).

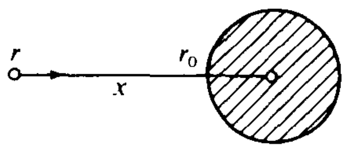

Figure 7 The earth's gravitational field

This equation can also be written in the form (3), if we introduce the potential energy

$$U = -\frac{k}{r} \qquad k = gr_0^2,$$

inversely proportional to the distance to the center of the earth.

Problem. Determine with what velocity a stone must be thrown in order that it fly infinitely far from the surface of the earth.[10]

Answer. ≥ 11.2 km/sec.

[10] This is the so-called second cosmic velocity v_2. Our equation does not take into account the attraction of the sun. The attraction of the sun will not let the stone escape from the solar system if the velocity of the stone with respect to the earth is less than 16.6 km/sec.

C *Example 3: Motion of a weight along a line under the action of a spring*

Experiments show that under small extensions of the spring the equation of motion of the weight will be (Figure 8)

$$\ddot{x} = -\alpha^2 x.$$

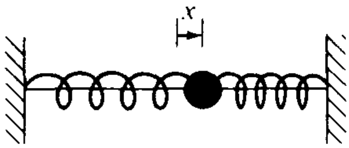

Figure 8 Weight on a spring

This equation can also be written in the form (3) if we introduce the potential energy

$$U = \frac{\alpha^2 x^2}{2}.$$

If we replace our one weight by two weights, then it turns out that, under the same extension of the spring, the acceleration is half as large.

It is experimentally established that for any two bodies the ratio of the accelerations $\ddot{x}_1/\ddot{x}_2$ under the same extension of a spring is fixed (does not depend on the extent of extension of the spring or on its characteristics, but only on the bodies themselves). The value inverse to this ratio is by definition the ratio of masses:

$$\frac{\ddot{x}_1}{\ddot{x}_2} = \frac{m_2}{m_1}.$$

For a unit of mass we take the mass of some fixed body, e.g., one liter of water. We know by experience that the masses of all bodies are positive. The product of mass times acceleration $m\ddot{x}$ does not depend on the body, and is a characteristic of the extension of the spring. This value is called the *force* of the spring acting on the body.

As a unit of force, we take the "newton." If one liter of water is suspended on a spring at the surface of the earth, the spring acts with a force of 9.8 newtons (= 1 kg).

D *Example 4: Conservative systems*

Let $E^{3n} = E^3 \times \cdots \times E^3$ be the configuration space of a system of n points in the euclidean space E^3. Let $U: E^{3n} \to \mathbb{R}$ be a differentiable function and let $m_1, \ldots, m_n$ be positive numbers.

Definition. The motion of n points, of masses $m_1, \ldots, m_n$, in the potential field with potential energy U is given by the system of differential equations

$$m_i \ddot{\mathbf{x}}_i = -\frac{\partial U}{\partial \mathbf{x}_i} \qquad i = 1, \ldots, n. \tag{4}$$

The equations of motion in Examples 1 to 3 have this form. The equations of motion of many other mechanical systems can be written in the same form. For example, the three-body problem of celestial mechanics is problem (4) in which

$$U = -\frac{m_1 m_2}{\|\mathbf{x}_1 - \mathbf{x}_2\|} - \frac{m_2 m_3}{\|\mathbf{x}_2 - \mathbf{x}_3\|} - \frac{m_3 m_1}{\|\mathbf{x}_3 - \mathbf{x}_1\|}.$$

Many different equations of entirely different origin can be reduced to form (4), for example the equations of electrical oscillations. In the following chapter we will study mainly systems of differential equations in the form (4).

Investigation of the equations of motion

2

In most cases (for example, in the three-body problem) we can neither solve the system of differential equations nor completely describe the behavior of the solutions. In this chapter we consider a few simple but important problems for which Newton's equations can be solved.

4 Systems with one degree of freedom

In this paragraph we study the phase flow of the differential equation (1). A look at the graph of the potential energy is enough for a qualitative analysis of such an equation. In addition, Equation (1) is integrated by quadratures.

A *Definitions*

A *system with one degree of freedom* is a system described by one differential equation

(1) $$\ddot{x} = f(x) \qquad x \in \mathbb{R}.$$

The *kinetic energy* is the quadratic form*

$$T = \tfrac{1}{2}\dot{x}^2.$$

The *potential energy* is the function

$$U(x) = -\int_{x_0}^{x} f(\xi)d\xi.$$

The sign in this formula is taken so that the potential energy of a stone is larger if the stone is higher off the ground.

Notice that the potential energy determines f. Therefore, to specify a system of the form (1) it is enough to give the potential energy. Adding a constant to the potential energy does not change the equation of motion (1).

* see footnote on p. 11.

The *total energy* is the sum

$$E = T + U.$$

In general, the total energy is a function, $E(x, \dot{x})$, of x and $\dot{x}$.

Theorem (The law of conservation of energy). *The total energy of points moving according to the equation* (1) *is conserved*: $E(x(t), \dot{x}(t))$ *is independent of* t.

PROOF.

$$\frac{d}{dt}(T + U) = \dot{x}\ddot{x} + \frac{dU}{dx}\dot{x} = \dot{x}(\ddot{x} - f(x)) = 0. \qquad \square$$

B *Phase flow*

Equation (1) is equivalent to the system of two equations:

$$(2) \qquad \dot{x} = y \qquad \dot{y} = f(x).$$

We consider the plane with coordinates x and y, which we call the *phase plane* of Equation (1). The points of the phase plane are called *phase points*. The right-hand side of (2) determines a vector field on the phase plane, called the *phase velocity vector field*.

A solution of (2) is a motion $\boldsymbol{\varphi}: \mathbb{R} \to \mathbb{R}^2$ of a phase point in the phase plane, such that the velocity of the moving point at each moment of time is equal to the phase velocity vector at the location of the phase point at that moment.[11]

The image of $\boldsymbol{\varphi}$ is called the *phase curve*. Thus the phase curve is given by the parametric equations

$$x = \varphi(t) \qquad y = \dot{\varphi}(t).$$

PROBLEM. Show that through every phase point there is one and only one phase curve.

Hint. Refer to a textbook on ordinary differential equations.

We notice that a phase curve could consist of only one point. Such a point is called an *equilibrium position*. The vector of phase velocity at an equilibrium position is zero.

The law of conservation of energy allows one to find the phase curves easily. On each phase curve the value of the total energy is constant. Therefore, each phase curve lies entirely in one energy level set $E(x, y) = h$.

C *Examples*

EXAMPLE 1. The basic equation of the theory of oscillations is

$$\ddot{x} = -x.$$

[11] Here we assume for simplicity that the solution $\boldsymbol{\varphi}$ is defined on the whole time axis $\mathbb{R}$.

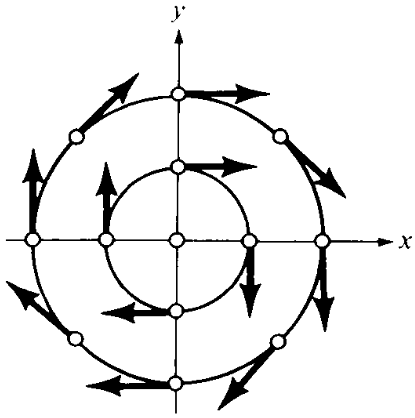

Figure 9 Phase plane of the equation $\ddot{x} = -x$

In this case (Figure 9) we have:

$$T = \frac{\dot{x}^2}{2} \qquad U = \frac{x^2}{2} \qquad E = \frac{\dot{x}^2}{2} + \frac{x^2}{2}.$$

The energy level sets are the concentric circles and the origin. The phase velocity vector at the phase point (x, y) has components $(y, -x)$. It is perpendicular to the radius vector and equal to it in magnitude. Therefore, the motion of the phase point in the phase plane is a uniform motion around 0: $x = r_0 \cos(\varphi_0 - t)$, $y = r_0 \sin(\varphi_0 - t)$. Each energy level set is a phase curve.

Example 2. Suppose that a potential energy is given by the graph in Figure 10. We will draw the energy level sets $\frac{1}{2}y^2 + U(x) = E$. For this, the following facts are helpful.

1. Any equilibrium position of (2) must lie on the x axis of the phase plane. The point $x = \xi$, $y = 0$ is an equilibrium position if ξ is a critical point of the potential energy, i.e., if $(\partial U/\partial x)|_{x=\xi} = 0$.
2. Each level set is a *smooth* curve in a neighborhood of each of its points which is not an equilibrium position (this follows from the implicit function theorem). In particular, if the number E is not a critical value of the potential energy (i.e., is not the value of the potential energy at one of its critical points), then the level set on which the energy is equal to E is a smooth curve.

It follows that in order to study the energy level curve, we should turn our attention to the critical and near-critical values of E. It is convenient here to imagine a little ball rolling in the potential well U.

For example, consider the following argument: "Kinetic energy is nonnegative. This means that potential energy is less than or equal to the total energy. The smaller the potential energy, the greater the velocity." This translates to: "The ball cannot jump out of the potential well, rising

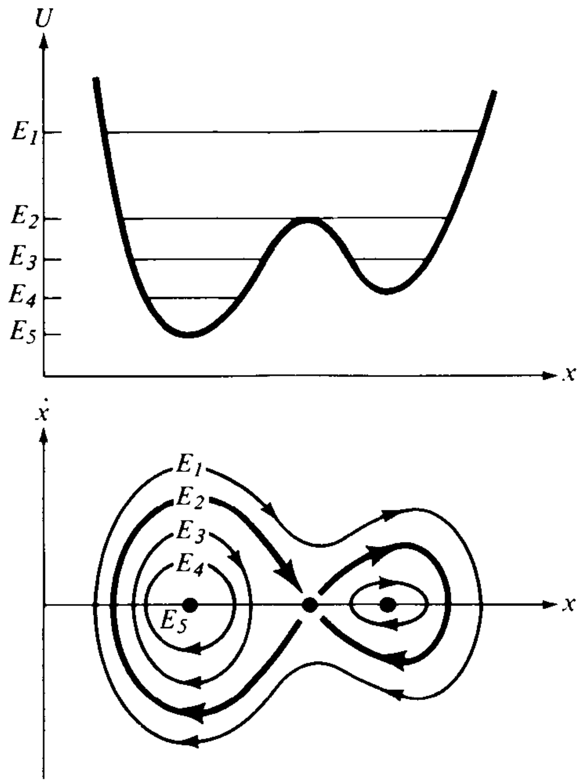

Figure 10 Potential energy and phase curves

higher than the level determined by its initial energy. As it falls into the well, the ball gains velocity." We also notice that the local maximum points of the potential energy are unstable, but the minimum points are stable equilibrium positions.

Problem. Prove this.

Problem. How many phase curves make up the separatrix (figure eight) curve, corresponding to the level E_2?

Answer. Three.

Problem. Determine the duration of motion along the separatrix.

Answer. It follows from the uniqueness theorem that the time is infinite.

Problem. Show that the time it takes to go from x_1 to x_2 (in one direction) is equal to

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}}.$$

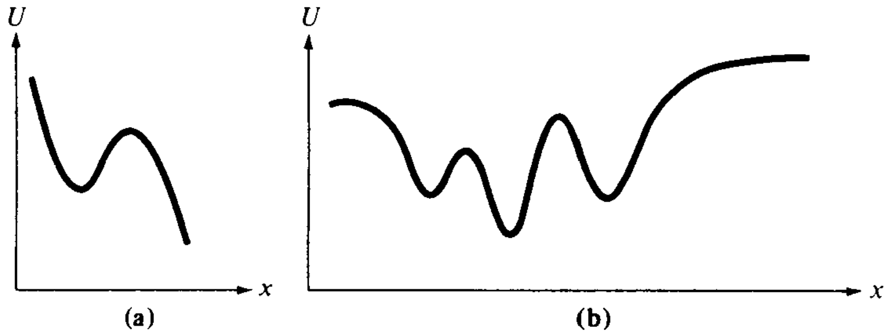

Figure 11 Potential energy

Problem. Draw the phase curves, given the potential energy graphs in Figure 11.

Answer. Figure 12.

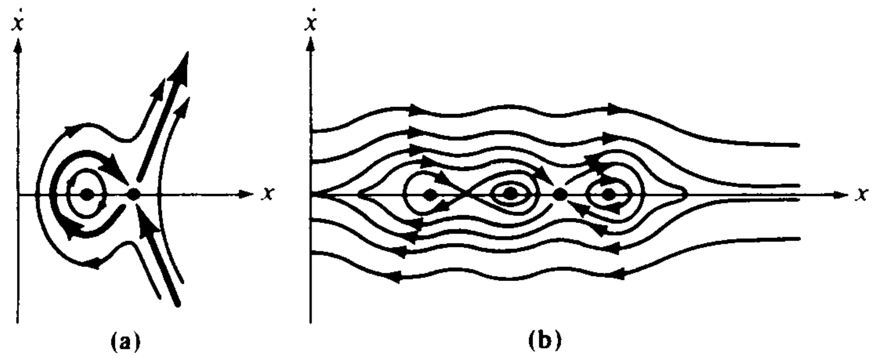

Figure 12 Phase curves

Problem. Draw the phase curves for the "equation of an ideal planar pendulum": $\ddot{x} = -\sin x$.

Problem. Draw the phase curves for the "equation of a pendulum on a rotating axis": $\ddot{x} = -\sin x + M$.

Remark. In these two problems x denotes the angle of displacement of the pendulum. The phase points whose coordinates differ by 2π correspond to the same position of the pendulum. Therefore, in addition to the phase plane, it is natural to look at the phase cylinder $\{x(\text{mod } 2\pi), y\}$.

Problem. Find the tangent lines to the branches of the critical level corresponding to maximal potential energy $E = U(\xi)$ (Figure 13).

Answer. $y = \pm\sqrt{-U''(\xi)}(x - \xi)$.

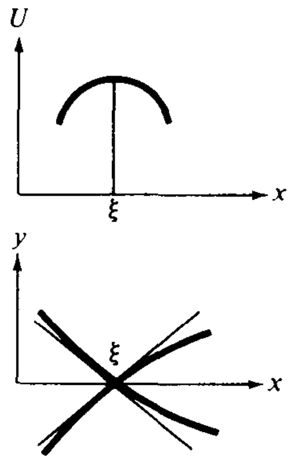

Figure 13 Critical energy level lines

PROBLEM. Let $S(E)$ be the area enclosed by the closed phase curve corresponding to the energy level E. Show that the period of motion along this curve is equal to

$$T = \frac{dS}{dE}.$$

PROBLEM. Let E_0 be the value of the potential function at a minimum point ξ. Find the period $T_0 = \lim_{E \to E_0} T(E)$ of small oscillations in a neighborhood of the point ξ.

ANSWER. $2\pi/\sqrt{U''(\xi)}$.

PROBLEM. Consider a periodic motion along the closed phase curve corresponding to the energy level E. Is it stable in the sense of Liapunov?[12]

ANSWER. No.[13]

D *Phase flow*

Let M be a point in the phase plane. We look at the solution to system (2) whose initial conditions at $t = 0$ are represented by the point M. We assume that any solution of the system can be extended to the whole time axis. The value of our solution at any value of t depends on M. We denote the resulting phase point (Figure 14) by

$$M(t) = g^t M.$$

In this way we have defined a mapping of the phase plane to itself, $g^t: \mathbb{R}^2 \to \mathbb{R}^2$. By theorems in the theory of ordinary differential equations,

[12] For a definition, see, e.g., p. 155 of *Ordinary Differential Equations* by V. I. Arnold, MIT Press, 1973.

[13] The only exception is the case when the period does not depend on the energy.

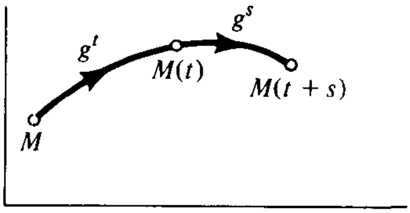

Figure 14 Phase flow

the mapping g^t is a diffeomorphism (a one-to-one differentiable mapping with a differentiable inverse). The diffeomorphisms g^t, $t \in \mathbb{R}$, form a group: $g^{t+s} = g^t \circ g^s$. The mapping g^0 is the identity ($g^0 M = M$), and g^{-t} is the inverse of g^t. The mapping $g\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $g(t, M) = g^t M$ is differentiable. All these properties together are expressed by saying that the transformations g^t form a *one-parameter group of diffeomorphisms* of the phase plane. This group is also called the *phase flow*, given by system (2) (or Equation (1)).

EXAMPLE. The phase flow given by the equation $\ddot{x} = -x$ is the group g^t of rotations of the phase plane through angle t around the origin.

PROBLEM. Show that the system with potential energy $U = -x^4$ does not define a phase flow.

PROBLEM. Show that if the potential energy is positive, then there is a phase flow.

Hint. Use the law of conservation of energy to show that a solution can be extended without bound.

PROBLEM. Draw the image of the circle $x^2 + (y - 1)^2 < \frac{1}{4}$ under the action of a transformation of the phase flow for the equations (a) of the "inverse pendulum," $\ddot{x} = x$ and (b) of the "nonlinear pendulum," $\ddot{x} = -\sin x$.

ANSWER. Figure 15.

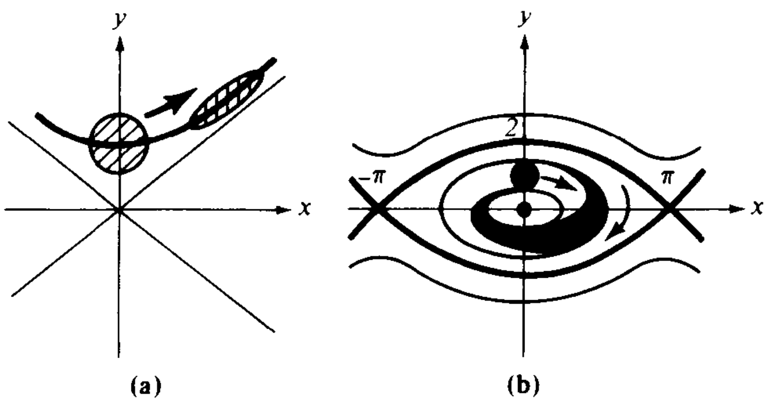

Figure 15 Action of the phase flow on a circle

5 Systems with two degrees of freedom

Analyzing a general potential system with two degrees of freedom is beyond the capability of modern science. In this paragraph we look at the simplest examples.

A *Definitions*

By a system with two degrees of freedom we will mean a system defined by the differential equations

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in E^2, \tag{1}$$

where $\mathbf{f}$ is a vector field on the plane.

A system is said to be *conservative* if there exists a function $U\colon E^2 \to \mathbb{R}$ such that $\mathbf{f} = -\partial U/\partial \mathbf{x}$. The equation of motion of a conservative system then has the form[14] $\ddot{\mathbf{x}} = -\partial U/\partial \mathbf{x}$.

B *The law of conservation of energy*

Theorem. *The total energy of a conservative system is conserved, i.e.,*

$$\frac{dE}{dt} = 0, \quad \text{where } E = \tfrac{1}{2}\dot{\mathbf{x}}^2 + U(\mathbf{x}),\ \dot{\mathbf{x}}^2 = (\dot{\mathbf{x}}, \dot{\mathbf{x}}).$$

PROOF. $dE/dt = (\dot{\mathbf{x}}, \ddot{\mathbf{x}}) + (\partial U/\partial \mathbf{x}, \dot{\mathbf{x}}) = (\ddot{\mathbf{x}} + (\partial U/\partial \mathbf{x}), \dot{\mathbf{x}}) = 0$ by the equation of motion. □

Corollary. *If at the initial moment the total energy is equal to E, then all trajectories lie in the region where $U(\mathbf{x}) \le E$, i.e., a point remains inside the potential well $U(x_1, x_2) \le E$ for all time.*

Remark. In a system with one degree of freedom it is always possible to introduce the potential energy

$$U(x) = -\int_{x_0}^{x} f(\xi)\,d\xi.$$

For a system with two degrees of freedom this is not so.

PROBLEM. Find an example of a system of the form $\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in E^2$, which is not conservative.

C *Phase space*

The equation of motion (1) can be written as the system:

$$\begin{aligned} \dot{x}_1 &= y_1 & \dot{x}_2 &= y_2 \\ \dot{y}_1 &= -\frac{\partial U}{\partial x_1} & \dot{y}_2 &= -\frac{\partial U}{\partial x_2} \end{aligned} \tag{2}$$

[14] In cartesian coordinates on the plane E^2, $\ddot{x}_1 = -\partial U/\partial x_1$ and $\ddot{x}_2 = -\partial U/\partial x_2$.

The *phase space* of a system with two degrees of freedom is the four-dimensional space with coordinates x_1, x_2, y_1, and y_2.

The system (2) defines the phase velocity vector field in four space as well as[15] the phase flow of the system (a one-parameter group of diffeomorphisms of four-dimensional phase space). The phase curves of (2) are subsets of four-dimensional phase space. All of phase space is partitioned into phase curves. Projecting the phase curves from four space to the x_1, x_2 plane gives the trajectories of our moving point in the x_1, x_2 plane. These trajectories are also called orbits. Orbits can have points of intersection even when the phase curves do not intersect one another. The equation of the law of conservation of energy

$$E = \frac{\dot{\mathbf{x}}^2}{2} + U(\mathbf{x}) = \frac{y_1^2 + y_2^2}{2} + U(x_1, x_2)$$

defines a three-dimensional hypersurface in four space: $E(x_1, x_2, y_1, y_2) = E_0$; this surface, π_{E_0}, remains invariant under the phase flow: $g^t\pi_{E_0} = \pi_{E_0}$. One could say that the phase flow flows along the energy level hypersurfaces. The phase velocity vector field is tangent at every point to π_{E_0}. Therefore, π_{E_0} is entirely composed of phase curves (Figure 16).

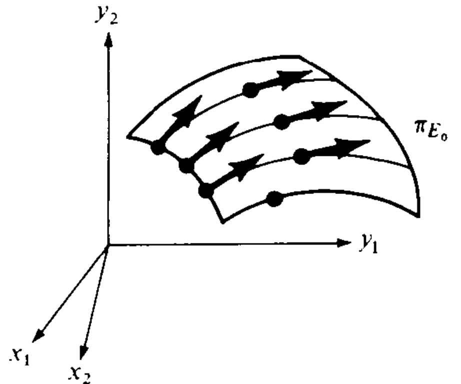

Figure 16 Energy level surface and phase curves

EXAMPLE 1 ("*small oscillations of a spherical pendulum*"). Let $U = \frac{1}{2}(x_1^2 + x_2^2)$. The level sets of the potential energy in the x_1, x_2 plane will be concentric circles (Figure 17).

The equations of motion, $\ddot{x}_1 = -x_1$, $\ddot{x}_2 = -x_2$, are equivalent to the system

$$\dot{x}_1 = y_1 \qquad \dot{x}_2 = y_2$$
$$\dot{y}_1 = -x_1 \qquad \dot{y}_2 = -x_2.$$

This system decomposes into two independent ones; in other words, each of the coordinates x_1 and x_2 changes with time in the same way as in a system with one degree of freedom.

[15] With the usual limitations.

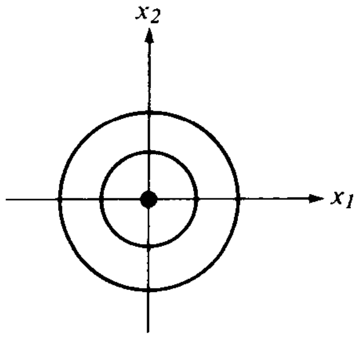

Figure 17 Potential energy level curves for a spherical pendulum

A solution has the form

$$x_1 = c_1 \cos t + c_2 \sin t \qquad x_2 = c_3 \cos t + c_4 \sin t$$
$$y_1 = -c_1 \sin t + c_2 \cos t \qquad y_2 = -c_3 \sin t + c_4 \cos t.$$

It follows from the law of conservation of energy that

$$E = \tfrac{1}{2}(y_1^2 + y_2^2) + \tfrac{1}{2}(x_1^2 + x_2^2) = \text{const},$$

i.e., the level surface π_{E_0} is a sphere in four space.

PROBLEM. Show that the phase curves are great circles of this sphere. (A great circle is the intersection of a sphere with a two-dimensional plane passing through its center.)

PROBLEM. Show that the set of phase curves on the surface π_{E_0} forms a two-dimensional sphere. The formula $w = (x_1 + iy_1)/(x_2 + iy_2)$ gives the "Hopf map" from the three sphere π_{E_0} to the two sphere (the complex w-plane completed by the point at infinity). Our phase curves are the pre-images of points under the Hopf map.

PROBLEM. Find the projection of the phase curves on the x_1, x_2 plane (i.e., draw the orbits of the motion of a point).

EXAMPLE 2 ("*Lissajous figures*"). We look at one more example of a planar motion ("small oscillations with two degrees of freedom"):

$$\ddot{x}_1 = -x_1 \qquad \ddot{x}_2 = -\omega^2 x_2.$$

The potential energy is

$$U = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}\omega^2 x_2^2.$$

From the law of conservation of energy it follows that, if at the initial moment of time the total energy is

$$\tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + U(x_1, x_2) = E,$$

then all motions will take place inside the ellipse $U(x_1, x_2) \le E$.

Our system consists of two independent one-dimensional systems. Therefore, the law of conservation of energy is satisfied for each of them separately, i.e., the following quantities are preserved

$$E_1 = \tfrac{1}{2}\dot{x}_1^2 + \tfrac{1}{2}x_1^2 \qquad E_2 = \tfrac{1}{2}\dot{x}_2^2 + \tfrac{1}{2}\omega^2 x_2^2 \qquad (E = E_1 + E_2).$$

Consequently, the variable x_1 is bounded by the region $|x_1| \leq A_1$, $A_1 = \sqrt{2E_1(0)}$, and x_2 oscillates within the region $|x_2| \leq A_2$. The intersection of these two regions defines a rectangle which contains the orbits (Figure 18).

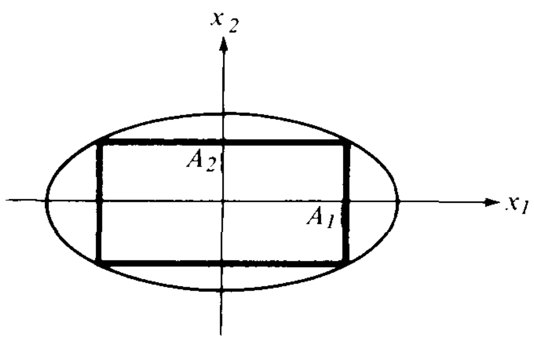

Figure 18 The regions $U \leq E$, $U_1 \leq E$ and $U_2 \leq E$

PROBLEM. Show that this rectangle is inscribed in the ellipse $U \leq E$.

The general solution of our equations is $x_1 = A_1 \sin(t + \varphi_1)$, $x_2 = A_2 \sin(\omega t + \varphi_2)$; a moving point independently performs an oscillation with frequency 1 and amplitude A_1 along the horizontal and an oscillation with frequency ω and amplitude A_2 along the vertical.

Consider the following method of describing an orbit in the x_1, x_2 plane. We look at a cylinder with base $2A_1$ and a band of width $2A_2$. We draw on the band a sine wave with period $2\pi A_1/\omega$ and amplitude A_2 and wind the band onto the cylinder (Figure 19). The orthogonal projection of the sinusoid

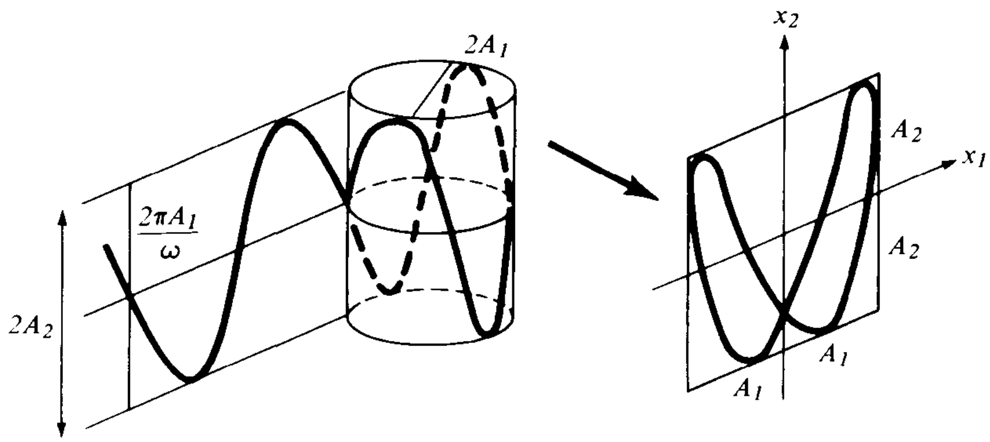

Figure 19 Construction of a Lissajous figure

wound around the cylinder onto the x_1, x_2 plane gives the desired orbit, called a *Lissajous figure.*

Lissajous figures can conveniently be seen on an oscilloscope which displays independent harmonic oscillations on the horizontal and vertical axes.

The form of a Lissajous figure very strongly depends on the frequency ω. If $\omega = 1$ (the spherical pendulum of Example 1), then the curve on the cylinder is an ellipse. The projection of this ellipse onto the x_1, x_2 plane depends on the difference $\varphi_2 - \varphi_1$ between the phases. For $\varphi_1 = \varphi_2$ we get a segment of the diagonal of the rectangle; for small $\varphi_2 - \varphi_1$ we get an ellipse close to the diagonal and inscribed in the rectangle. For $\varphi_2 - \varphi_1 = \pi/2$ we get an ellipse with major axes x_1, x_2; as $\varphi_2 - \varphi_1$ increases from $\pi/2$ to π the ellipse collapses onto the second diagonal; as $\varphi_2 - \varphi_1$ increases further the whole process is repeated from the beginning (Figure 20).

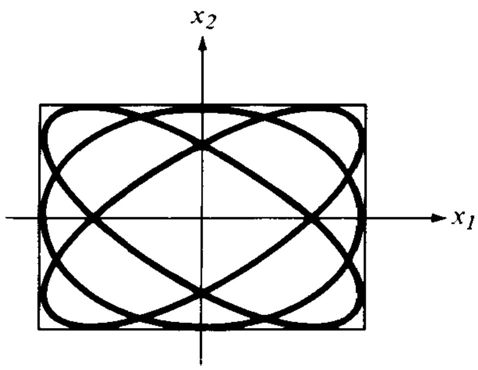

Figure 20 Series of Lissajous figures with $\omega = 1$

Now let the frequencies be only approximately equal: $\omega \approx 1$. The segment of the curve corresponding to $0 \leq t \leq 2\pi$ is very close to an ellipse. The next loop also reminds one of an ellipse, but here the phase shift $\varphi_2 - \varphi_1$ is greater than in the original by $2\pi(\omega - 1)$. Therefore, the Lissajous curve with $\omega \approx 1$ is a distorted ellipse, slowly progressing through all phases from collapsed onto one diagonal to collapsed onto the other (Figure 21).

If one of the frequencies is twice the other ($\omega = 2$), then for some particular phase shift the Lissajous figure becomes a doubly traversed arc (Figure 22).

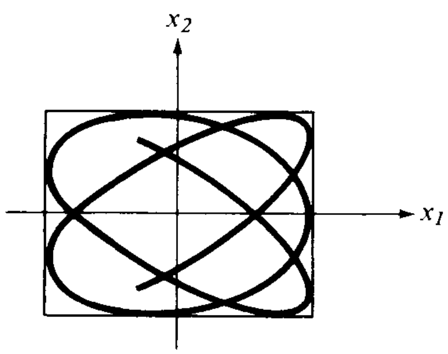

Figure 21 Lissajous figure with $\omega \approx 1$

PROBLEM. Show that this curve is a parabola. By increasing the phase shift $\varphi_2 - \varphi_1$ we get in turn the curves in Fig. 23.

In general, if one of the frequencies is n times bigger than the other ($\omega = n$), then among the graphs of the corresponding Lissajous figures there is the graph of a polynomial of degree n (Figure 24); this polynomial is called a *Chebyshev polynomial.*

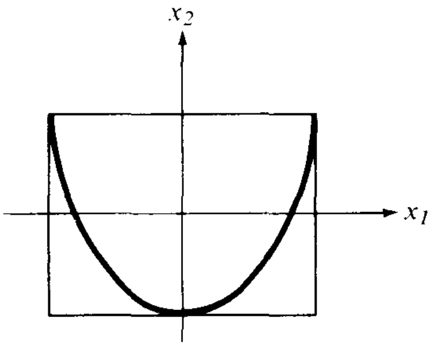

Figure 22 Lissajous figure with $\omega = 2$

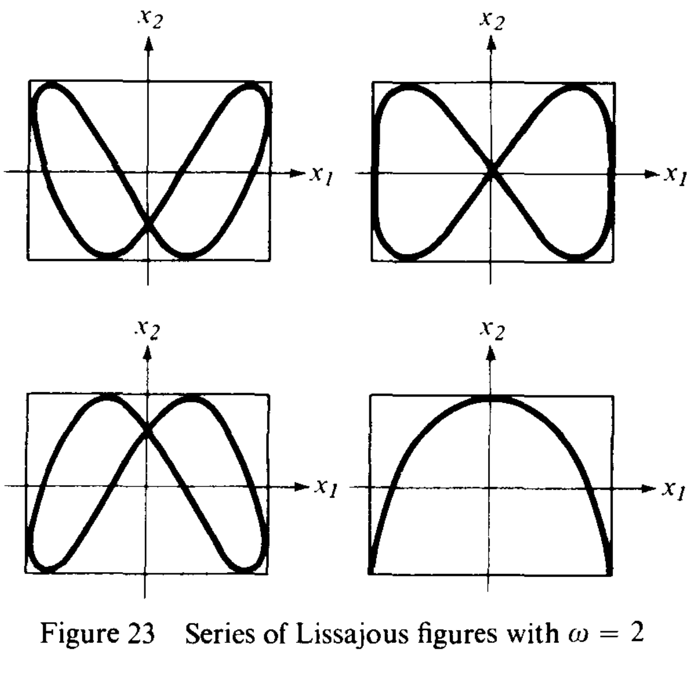

Figure 23 Series of Lissajous figures with $\omega = 2$

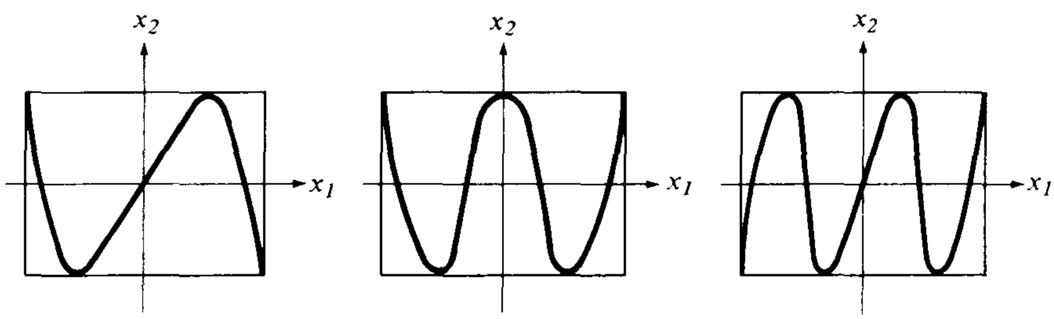

Figure 24 Chebyshev polynomials

PROBLEM. Show that if $\omega = m/n$, then the Lissajous figure is a closed algebraic curve; but if ω is irrational, then the Lissajous figure fills the rectangle everywhere densely. What does the corresponding phase trajectory fill out?

6 Conservative force fields

In this section we study the connection between work and potential energy.

A *Work of a force field along a path*

Recall the definition of the work by a force **F** on a path **S**. The work of the constant force **F** (for example, the force with which we lift up a load) on the

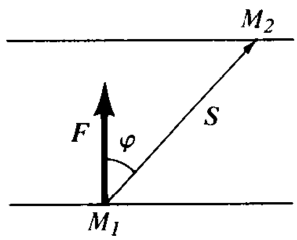

Figure 25 Work of the constant force **F** along the straight path **S**

path $\mathbf{S} = \overrightarrow{M_1M_2}$ is, by definition, the scalar product (Figure 25)

$$A = (\mathbf{F}, \mathbf{S}) = |\mathbf{F}||\mathbf{S}| \cdot \cos \varphi.$$

Suppose we are given a vector field **F** and a curve l of finite length. We approximate the curve l by a polygonal line with components $\Delta\mathbf{S}_i$ and denote by $\mathbf{F}_i$ the value of the force at some particular point of $\Delta\mathbf{S}_i$; then the work of the field **F** on the path l is by definition (Figure 26)

$$A = \lim_{|\Delta\mathbf{S}_i| \to 0} \sum (\mathbf{F}_i, \Delta\mathbf{S}_i).$$

In analysis courses it is proved that if the field is continuous and the path rectifiable, then the limit exists. It is denoted by $\int_l (\mathbf{F}, d\mathbf{S})$.

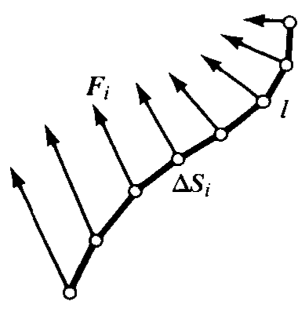

Figure 26 Work of the force field **F** along the path l

B *Conditions for a field to be conservative*

Theorem. *A vector field* $\mathbf{F}$ *is conservative if and only if its work along any path* M_1M_2 *depends only on the endpoints of the path, and not on the shape of the path.*

Proof. Suppose that the work of a field $\mathbf{F}$ does not depend on the path. Then

$$U(M) = -\int_{M_0}^{M} (\mathbf{F}, d\mathbf{S})$$

is well defined as a function of the point M. It is easy to verify that

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{x}},$$

i.e., the field is conservative and U is its potential energy. Of course, the potential energy is defined only up to the additive constant $U(M_0)$, which can be chosen arbitrarily.

Conversely, suppose that the field F is conservative and that U is its potential energy. Then it is easily verified that

$$\int_{M_0}^{M} (\mathbf{F}, d\mathbf{S}) = -U(M) + U(M_0),$$

i.e., the work does not depend on the shape of the path. □

Problem. Show that the vector field $F_1 = x_2, F_2 = -x_1$ is not conservative (Figure 27).

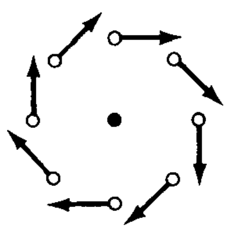

Figure 27 A non-potential field

Problem. Is the field in the plane minus the origin given by $F_1 = x_2/(x_1^2 + x_2^2)$, $F_2 = -x_1/(x_1^2 + x_2^2)$ conservative? Show that a field is conservative if and only if its work along any closed contour is equal to zero.

C *Central fields*

Definition. A vector field in the plane E^2 is called *central* with center at 0, if it is invariant with respect to the group of motions[16] of the plane which fix 0.

[16] Including reflections.

PROBLEM. Show that all vectors of a central field lie on rays through 0, and that the magnitude of the vector field at a point depends only on the distance from the point to the center of the field.

It is also useful to look at central fields which are not defined at the point 0.

EXAMPLE. The newtonian field $\mathbf{F} = -k(\mathbf{r}/|r|^3)$ is central, but the field in the problem in Section 6B is not.

Theorem. *Every central field is conservative, and its potential energy depends only on the distance to the center of the field, $U = U(r)$.*

PROOF. According to the previous problem, we may set $\mathbf{F}(\mathbf{r}) = \Phi(r)\mathbf{e}_r$, where $\mathbf{r}$ is the radius vector with respect to 0, r is its length and the unit vector $\mathbf{e}_r = \mathbf{r}/|r|$ its direction. Then

$$\int_{M_1}^{M_2} (\mathbf{F}, d\mathbf{S}) = \int_{r(M_1)}^{r(M_2)} \Phi(r)dr,$$

and this integral is obviously independent of the path. □

PROBLEM. Compute the potential energy of the newtonian field.

Remark. The definitions and theorems of this paragraph can be directly carried over to a euclidean space E^n of any dimension.

7 Angular momentum

We will see later that the invariance of an equation of a mechanical problem with respect to some group of transformations always implies a conservation law. A central field is invariant with respect to the group of rotations. The corresponding first integral is called the angular momentum.

Definition. The *motion of a material point (with unit mass) in a central field on a plane* is defined by the equation

$$\ddot{\mathbf{r}} = \Phi(r)\mathbf{e}_r,$$

where $\mathbf{r}$ is the radius vector beginning at the center of the field 0, r is its length, and $\mathbf{e}_r$ its direction. We will think of our plane as lying in three-dimensional oriented euclidean space.

Definition. The *angular momentum* of a material point of unit mass relative to the point 0 is the vector product

$$\mathbf{M} = [\mathbf{r}, \dot{\mathbf{r}}].$$

The vector $\mathbf{M}$ is perpendicular to our plane and is given by one number: $\mathbf{M} = M\mathbf{n}$, where $\mathbf{n} = [\mathbf{e}_1, \mathbf{e}_2]$ is the normal vector, $\mathbf{e}_1$ and $\mathbf{e}_2$ being an oriented frame in the plane (Figure 28).

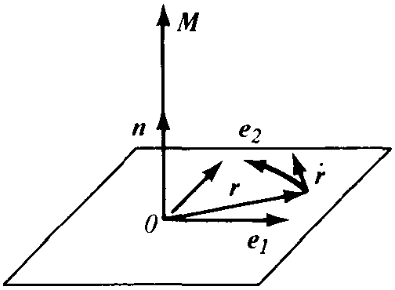

Figure 28 Angular momentum

Remark. In general, the moment of a vector $\mathbf{a}$ "applied at the point $\mathbf{r}$" relative to the point 0 is $[\mathbf{r}, \mathbf{a}]$; for example, in a school statics course one studies the moment of force. [The literal translation of the Russian term for angular momentum is "kinetic moment." (Trans. note)]

A *The law of conservation of angular momentum*

Lemma. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two vectors changing with time in the oriented euclidean space* $\mathbb{R}^3$. *Then*

$$\frac{d}{dt}[\mathbf{a}, \mathbf{b}] = [\dot{\mathbf{a}}, \mathbf{b}] + [\mathbf{a}, \dot{\mathbf{b}}].$$

PROOF. This follows from the definition of derivative. □

Theorem (The law of conservation of angular momentum). *Under motions in a central field, the angular momentum* $\mathbf{M}$ *relative to the center of the field* 0 *does not change with time.*

PROOF. By definition $\mathbf{M} = [\mathbf{r}, \dot{\mathbf{r}}]$. By the lemma, $\dot{\mathbf{M}} = [\dot{\mathbf{r}}, \dot{\mathbf{r}}] + [\mathbf{r}, \ddot{\mathbf{r}}]$. Since the field is central it is apparent from the equations of motion that the vectors $\ddot{\mathbf{r}}$ and $\mathbf{r}$ are collinear. Therefore $\dot{\mathbf{M}} = 0$. □

B *Kepler's law*

The law of conservation of angular momentum was first discovered by Kepler through observation of the motion of Mars. Kepler formulated this law in a slightly different way.

We introduce polar coordinates r, φ on our plane with pole at the center of the field 0. We consider, at the point $\mathbf{r}$ with coordinates $(|\mathbf{r}| = r, \varphi)$, two unit vectors: $\mathbf{e}_r$, directed along the radius vector so that

$$\mathbf{r} = r\mathbf{e}_r,$$

and $\mathbf{e}_\varphi$, perpendicular to it in the direction of increasing φ. We express the velocity vector $\dot{\mathbf{r}}$ in terms of the basis $\mathbf{e}_r$, $\mathbf{e}_\varphi$ (Figure 29).

Lemma. *We have the relation*

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi.$$

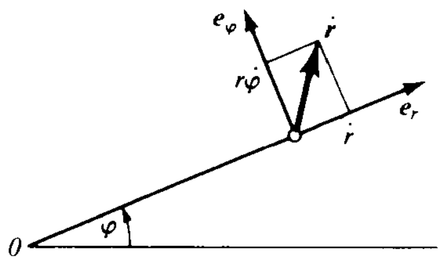

Figure 29 Decomposition of the vector $\dot{\mathbf{r}}$ in terms of the basis $\mathbf{e}_r$, $\mathbf{e}_\varphi$

PROOF. Clearly, the vectors $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ rotate with angular velocity $\dot{\varphi}$, i.e.,

$$\dot{\mathbf{e}}_r = \dot{\varphi}\mathbf{e}_\varphi \qquad \dot{\mathbf{e}}_\varphi = -\dot{\varphi}\mathbf{e}_r.$$

Differentiating the equality $\mathbf{r} = r\mathbf{e}_r$ gives us

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi. \qquad \square$$

Consequently, the angular momentum is

$$\mathbf{M} = [\mathbf{r}, \dot{\mathbf{r}}] = [\mathbf{r}, \dot{r}\mathbf{e}_r] + [\mathbf{r}, r\dot{\varphi}\mathbf{e}_\varphi] = r\dot{\varphi}[\mathbf{r}, \mathbf{e}_\varphi] = r^2\dot{\varphi}[\mathbf{e}_r, \mathbf{e}_\varphi].$$

Thus, the quantity $M = r^2\dot{\varphi}$ is preserved. This quantity has a simple geometric meaning.

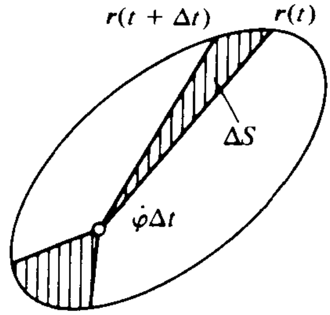

Figure 30 Sectorial velocity

Kepler called the rate of change of the area $S(t)$ swept out by the radius vector the *sectorial velocity* C (Figure 30):

$$C = \frac{dS}{dt}.$$

The law discovered by Kepler through observation of the motion of the planets says: in equal times the radius vector sweeps out equal areas, so that the sectorial velocity is constant, $dS/dt = \text{const}$. This is one formulation of the law of conservation of angular momentum. Since

$$\Delta S = S(t + \Delta t) - S(t) = \tfrac{1}{2}r^2\dot{\varphi}\Delta t + o(\Delta t),$$

this means that the sectorial velocity

$$C = \frac{dS}{dt} = \tfrac{1}{2}r^2\dot{\varphi} = \tfrac{1}{2}M$$

is half the angular momentum of our point of mass 1, and therefore constant.

EXAMPLE. Some satellites have very elongated orbits. By Kepler's law such a satellite spends most of its time in the distant part of its orbit, where the magnitude of $\dot{\varphi}$ is small.

8 Investigation of motion in a central field

The law of conservation of angular momentum lets us reduce problems about motion in a central field to problems with one degree of freedom. Thanks to this, motion in a central field can be completely determined.

A *Reduction to a one-dimensional problem*

We look at the motion of a point (of mass 1) in a central field on the plane:

$$\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}}, \qquad U = U(r).$$

It is natural to use polar coordinates r, φ.

By the law of conservation of angular momentum the quantity $M = \dot{\varphi}(t)r^2(t)$ is constant (independent of t).

Theorem. *For the motion of a material point of unit mass in a central field the distance from the center of the field varies in the same way as r varies in the one-dimensional problem with potential energy*

$$V(r) = U(r) + \frac{M^2}{2r^2}.$$

PROOF. Differentiating the relation shown in Section 7 ($\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi$), we find

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\varphi}^2)\mathbf{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\mathbf{e}_\varphi.$$

Since the field is central,

$$\frac{\partial U}{\partial \mathbf{r}} = \frac{\partial U}{\partial r}\mathbf{e}_r.$$

Therefore the equation of motion in polar coordinates takes the form

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\partial U}{\partial r} \qquad 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0.$$

But, by the law of conservation of angular momentum,

$$\dot{\varphi} = \frac{M}{r^2},$$

where M is a constant independent of t, determined by the initial conditions. Therefore,

$$\ddot{r} = -\frac{\partial U}{\partial r} + r\frac{M^2}{r^4} \quad \text{or} \quad \ddot{r} = -\frac{\partial V}{\partial r}, \quad \text{where } V = U + \frac{M^2}{2r^2}.$$

The quantity $V(r)$ is called the *effective potential energy*. □

Remark. The total energy in the derived one-dimensional problem

$$E_1 = \frac{\dot{r}^2}{2} + V(r)$$

is the same as the total energy in the original problem

$$E = \frac{\dot{\mathbf{r}}^2}{2} + U(\mathbf{r}),$$

since

$$\frac{\dot{\mathbf{r}}^2}{2} = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\varphi}^2}{2} = \frac{\dot{r}^2}{2} + \frac{M^2}{2r^2}.$$

B *Integration of the equation of motion*

The total energy in the derived one-dimensional problem is conserved. Consequently, the dependence of r on t is defined by the quadrature

$$\dot{r} = \sqrt{2(E - V(r))} \qquad \int dt = \int \frac{dr}{\sqrt{2(E - V(r))}}.$$

Since $\dot{\varphi} = M/r^2$, $d\varphi/dr = (M/r^2)/\sqrt{2(E - V(r))}$, and the equation of the orbit in polar coordinates is found by quadrature,

$$\varphi = \int \frac{M/r^2\, dr}{\sqrt{2(E - V(r))}}.$$

C *Investigation of the orbit*

We fix the value of the angular momentum at M. The variation of r with time is easy to visualize, if one draws the graph of the effective potential energy $V(r)$ (Figure 31).

Let E be the value of the total energy. All orbits corresponding to the given E and M lie in the region $V(r) \leq E$. On the boundary of this region, $V = E$,

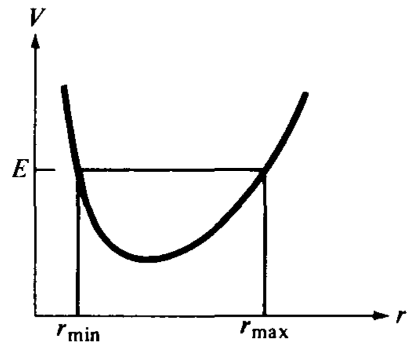

Figure 31 Graph of the effective potential energy

i.e., $\dot{r} = 0$. Therefore, the velocity of the moving point, in general, is not equal to zero since $\dot{\varphi} \neq 0$ for $M \neq 0$.

The inequality $V(r) \leq E$ gives one or several annular regions in the plane:

$$0 \leq r_{min} \leq r \leq r_{max} \leq \infty.$$

If $0 \leq r_{min} < r_{max} < \infty$, then the motion is bounded and takes place inside the ring between the circles of radius r_{min} and r_{max}.

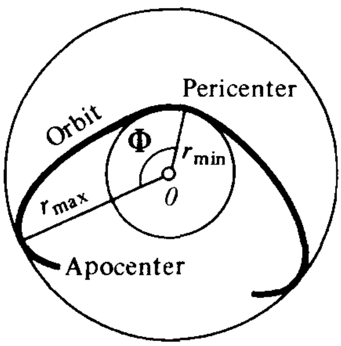

Figure 32 Orbit of a point in a central field

The shape of an orbit is shown in Figure 32. The angle φ varies monotonically while r oscillates periodically between r_{min} and r_{max}. The points where $r = r_{min}$ are called *pericentral*, and where $r = r_{max}$, *apocentral* (if the center is the earth—perigee and apogee; if it is the sun—perihelion and aphelion; if it is the moon—perilune and apolune).

Each of the rays leading from the center to the apocenter or to the pericenter is an axis of symmetry of the orbit.

In general, the orbit is not closed: the angle between the successive pericenters and apocenters is given by the integral

$$\Phi = \int_{r_{min}}^{r_{max}} \frac{M/r^2\, dr}{\sqrt{2(E - V(r))}}.$$

The angle between two successive pericenters is twice as big.

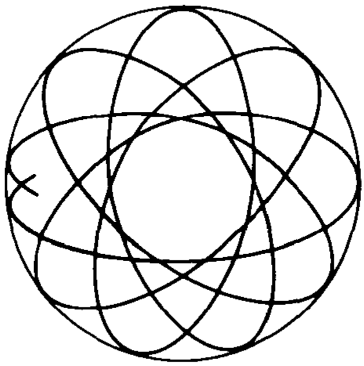

Figure 33 Orbit dense in an annulus

The orbit is closed if the angle Φ is commensurable with 2π, i.e., if $\Phi = 2\pi(m/n)$, where m and n are integers.

It can be shown that if the angle Φ is not commensurable with 2π, then the orbit is everywhere dense in the annulus (Figure 33).

If $r_{\min} = r_{\max}$, i.e., E is the value of V at a minimum point, then the annulus degenerates to a circle, which is also the orbit.

PROBLEM. For which values of α is motion along a circular orbit in the field with potential energy $U = r^\alpha$, $-2 \leq \alpha < \infty$, Liapunov stable?

ANSWER. Only for $\alpha = 2$.

For values of E a little larger than the minimum of V the annulus $r_{\min} \leq r \leq r_{\max}$ will be very narrow, and the orbit will be close to a circle. In the corresponding one-dimensional problem, r will perform small oscillations close to the minimum point of V.

PROBLEM. Find the angle Φ for an orbit close to the circle of radius r.

Hint. Cf. Section D below.

We now look at the case $r_{\max} = \infty$. If $\lim_{r\to\infty} U(r) = \lim_{r\to\infty} V(r) = U_\infty < \infty$, then it is possible for orbits to go off to infinity. If the initial energy E is larger than U, then the point goes to infinity with finite velocity $\dot{r}_\infty = \sqrt{2(E - U_\infty)}$. We notice that if $U(r)$ approaches its limit slower than r^{-2}, then the effective potential V will be attracting at infinity (here we assume that the potential U is attracting at infinity).

If, as $r \to 0$, $|U(r)|$ does not grow faster than $M^2/2r^2$, then $r_{\min} > 0$ and the orbit never approaches the center. If, however, $U(r) + (M^2/2r^2) \to -\infty$ as $r \to 0$, then it is possible to "fall into the center of the field." Falling into the center of the field is possible even in finite time (for example, in the field $U(r) = -1/r^3$).

PROBLEM. Examine the shape of an orbit in the case when the total energy is equal to the value of the effective energy V at a local maximum point.

D Central fields in which all bounded orbits are closed

It follows from the following sequence of problems that there are only two cases in which all the bounded orbits in a central field are closed, namely,

$$U = ar^2, \qquad a \geq 0$$

and

$$U = -\frac{k}{r}, \qquad k \geq 0.$$

PROBLEM 1. Show that the angle Φ between the pericenter and apocenter is equal to the semiperiod of an oscillation in the one-dimensional system with potential energy $W(x) = U(M/x) + (x^2/2)$.

Hint. The substitution $x = M/r$ gives

$$\Phi = \int_{x_{\min}}^{x_{\max}} \frac{dx}{\sqrt{2(E - W)}}.$$

PROBLEM 2. Find the angle Φ for an orbit close to the circle of radius r.

ANSWER. $\Phi \approx \Phi_{\text{cir}} = \pi(M/r^2\sqrt{V''(r)}) = \pi\sqrt{U'/(3U' + rU'')}$.

PROBLEM 3. For which values of U is the magnitude of Φ_{cir} independent of the radius r?

ANSWER. $U(r) = ar^\alpha$ ($\alpha \geq -2, \alpha \neq 0$) and $U(r) = b \log r$.

It follows that $\Phi_{\text{cir}} = \pi/\sqrt{\alpha + 2}$ (the logarithmic case corresponds to $\alpha = 0$). For example, for $\alpha = 2$ we have $\Phi_{\text{cir}} = \pi/2$, and for $\alpha = -1$ we have $\Phi_{\text{cir}} = \pi$.

PROBLEM 4. Let in the situation of problem 3 $U(r) \to \infty$ as $r \to \infty$. Find $\lim_{E\to\infty} \Phi(E, M)$.

ANSWER. $\pi/2$.

Hint. The substitution $x = yx_{\max}$ reduces Φ to the form

$$\Phi = \int_{y_{\min}}^{1} \frac{dy}{\sqrt{2(W^*(1) - W^*(y))}}, \qquad W^*(y) = \frac{y^2}{2} + \frac{1}{x_{\max}^2} U\left(\frac{M}{yx_{\max}}\right).$$

As $E \to \infty$ we have $x_{\max} \to \infty$ and $y_{\min} \to 0$, and the second term in W^* can be discarded.

PROBLEM 5. Let $U(r) = -kr^{-\beta}$, $0 < \beta < 2$. Find $\Phi_0 = \lim_{E\to -0}\Phi$.

ANSWER. $\Phi_0 = \int_0^1 dx/\sqrt{x^\beta - x^2} = \pi/(2-\beta)$. Note that Φ_0 does not depend on M. (Για το ολοκλ: W=x^2/2-kx^b/M^b, x_m^(2-b)=2k/M^b, -2W=xm^(2-b) x^b - x^2. Τελος, t^2=x^(2-b).)

PROBLEM 6. Find all central fields in which bounded orbits exist and are all closed.

(Αν Φ/(2π) ρητός, όπως απαιτείται για να κλείνει η τροχιά, τοτε

ANSWER. $U = ar^2$ or $U = -k/r$. επειδή Φ συνεχής ως προς E, M, η Φ θα πρεπει να είναι η σταθερή συνάρτηση και να μην εξαρταται απο E και M)

Solution. If all bounded orbits are closed, then, in particular, $\Phi_{\text{cir}} = 2\pi(m/n) = \text{const}$. According to Problem 3, $U = ar^\alpha(\alpha \geq -2)$, or $U = b \ln r$ $(\alpha = 0)$. In both cases $\Phi_{\text{cir}} = \pi/\sqrt{\alpha + 2}$. If $\alpha > 0$, then according to Problem 4, $\lim_{E\to\infty}\Phi(E, M) = \pi/2$. Therefore, $\Phi_{\text{cir}} = \pi/2$, $\alpha = 2$. If $\alpha < 0$, then according to Problem 5, $\lim_{E\to -\infty}\Phi(E, M) = \pi/(2+\alpha)$. Therefore, $\pi/(2+\alpha) = \pi/\sqrt{2+\alpha}$, $\alpha = -1$. In the case $\alpha = 0$ we find $\Phi_{\text{cir}} = \pi/\sqrt{2}$, which is not commensurable with 2π. Therefore, all bounded orbits can be closed only in fields where $U = ar^2$ or $U = -k/r$. In the field $U = ar^2$, $a > 0$, all the orbits are closed (these are ellipses with center at 0, cf. Example 1, Section 5). In the field $U = -k/r$ all bounded orbits are also closed and also elliptical, as we will now show.

(επειδή πρεπει και Φκ=Φο)

E *Kepler's problem*

This problem concerns motion in a central field with potential $U = -k/r$ and therefore $V(r) = -(k/r) + (M^2/2r^2)$ (Figure 34).

By the general formula

$$\varphi = \int \frac{M/r^2\, dr}{\sqrt{2(E - V(r))}}.$$

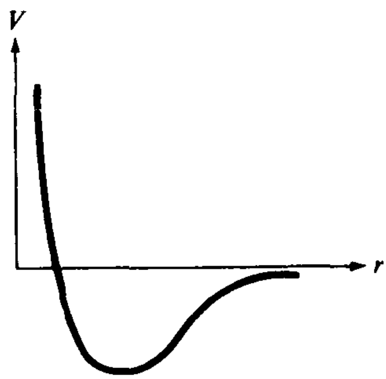

Figure 34 Effective potential of the Kepler problem

Integrating, we get

$$\varphi = \arc\cos \frac{\dfrac{M}{r} - \dfrac{k}{M}}{\sqrt{2E + \dfrac{k^2}{M^2}}}.$$

To this expression we should have added an arbitrary constant. We will assume it equal to zero; this is equivalent to the choice of an origin of reference for the angle φ at the pericenter. We introduce the following notation:

$$\frac{M^2}{k} = p \qquad \sqrt{1 + \frac{2EM^2}{k^2}} = e.$$

Now we get $\varphi = \arc\cos((p/r) - 1)/e$, i.e.,

$$r = \frac{p}{1 + e\cos\varphi}.$$

This is the so-called *focal equation* of a conic section. The motion is bounded (Figure 35) for $E < 0$. Then $e < 1$, i.e., the conic section is an ellipse. The number p is called the *parameter* of the ellipse, and e the *eccentricity*. Kepler's first law, which he discovered by observing the motion of Mars, consists of the fact that the planets describe ellipses, with the sun at one focus.

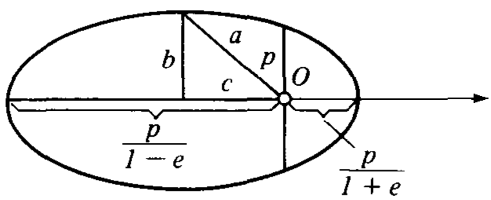

Figure 35 Keplerian ellipse

If we assume that the planets move in a central field of gravity, then Kepler's first law implies Newton's law of gravity: $U = -(k/r)$ (cf. Section 2D above).

The parameter and eccentricity are related with the semi-axes by the formulas

$$2a = \frac{p}{1-e} + \frac{p}{1+e} = \frac{2p}{1-e^2},$$

i.e.,

$$a = \frac{p}{1-e^2},$$

$e = c/a = \sqrt{a^2 - b^2}/a$, where $c = ae$ is the distance from the center to the focus (cf. Figure 35).

Remark. An ellipse with small eccentricity is very close to a circle.[17] If the distance from the focus to the center is small of first order, then the difference between the semi-axes is of second order: $b = a\sqrt{1 - e^2} \approx a(1 - \frac{1}{2}e^2)$. For example, in the ellipse with major semi-axes of 10 cm and eccentricity 0.1, the difference of the semi-axes is 0.5 mm, and the distance between the focus and the center is 1 cm.

The eccentricities of planets' orbits are very small. Therefore, Kepler originally formulated his first law as follows: the planets move around the sun in circles, but the sun is not at the center.

Kepler's second law, that the sectorial velocity is constant, is true in any central field.

Kepler's third law says that the period of revolution around an elliptical orbit depends only on the size of the major semi-axes.

The squares of the revolution periods of two planets on different elliptical orbits have the same ratio as the cubes of their major semi-axes.[18]

Proof. We denote by T the period of revolution and by S the area swept out by the radius vector in time T. $2S = MT$, since $M/2$ is the sectorial velocity. But the area of the ellipse, S, is equal to πab, so $T = 2\pi ab/M$. Since

$$a = \frac{M^2/k}{2|E|\dfrac{M^2}{k^2}} = \frac{k}{2|E|}$$

(from $a = p/(1 - e^2)$), and

$$b = \frac{M^2}{k} \cdot \frac{1}{\sqrt{2|E|}\dfrac{M}{k}} = \frac{M}{\sqrt{2|E|}},$$

then $T = 2\pi(k/(\sqrt{2|E|})^3)$; but $2|E| = k/a$, so $T = 2\pi a^{3/2}k^{-1/2}$. □

We note that the total energy E depends only on the major semi-axis a of the orbit and is the same for the whole set of elliptical orbits, from a circle of radius a to a line segment of length $2a$.

Problem. At the entry of a satellite into a circular orbit at a distance 300 km from the earth the direction of its velocity deviates from the intended direction by 1° towards the earth. How is the perigee changed?

Answer. The height of the perigee is less by approximately 110 km.

[17] Let a drop of tea fall into a glass of tea close to the center. The waves collect at the symmetric point. The reason is that, by the focal definition of an ellipse, waves radiating from one focus of the ellipse collect at the other.

[18] By planets we mean here points in a central field.

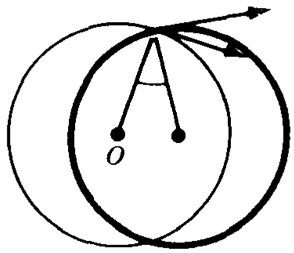

Figure 36 An orbit which is close to circular

Hint. The orbit differs from a circle only to second order, and we can disregard this difference. The radius has the intended value since the initial energy has the intended value. Therefore, we get the true orbit (Figure 36) by twisting the intended orbit through 1°.

PROBLEM. How does the height of the perigee change if the actual velocity is 1 m/sec less than intended?

PROBLEM. The *first cosmic velocity* is the velocity of motion on a circular orbit of radius close to the radius of the earth. Find the magnitude of the first cosmic velocity v_1 and show that $v_2 = \sqrt{2}v_1$ (cf. Section 3B).

ANSWER. 8.1 km/sec.

PROBLEM.[19] During his walk in outer space, the cosmonaut A. Leonov threw the lens cap of his movie camera towards the earth. Describe the motion of the lens cap with respect to the spaceship, taking the velocity of the throw as 10 m/sec.

ANSWER. The lens cap will move relative to the cosmonaut approximately in an ellipse with major axis about 32 km and minor axis about 16 km. The center of the ellipse will be situated 16 km in front of the cosmonaut in his orbit, and the period of circulation around the ellipse will be equal to the period of motion around the orbit.

Hint. We take as our unit of length the radius of the space ship's circular orbit, and we choose a unit of time so that the period of revolution around this orbit is 2π. We must study solutions to Newton's equation

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3},$$

close to the circular solution with $r_0 = 1$, $\varphi_0 = t$. We seek those solutions in the form

$$r = r_0 + r_1 \qquad \varphi = \varphi_0 + \varphi_1 \qquad r_1 \ll 1, \varphi_1 \ll 1.$$

[19] This problem is taken from V. V. Beletskii's delightful book, "Notes on the Motion of Celestial Bodies," Nauka, 1972.

By the theorem on the differentiability of a solution with respect to its initial conditions, the functions $r_1(t)$ and $\varphi_1(t)$ satisfy a system of linear differential equations (equations of variation) up to small amounts which are of higher than first order in the initial deviation.

By substituting the expressions for r and φ in Newton's equation, we get, after simple computation, the variational equations in the form

$$\ddot{r}_1 = 3r_1 + 2\dot{\varphi}_1 \qquad \ddot{\varphi}_1 = -2\dot{r}_1.$$

After solving these equations for the given initial conditions ($r_1(0) = \varphi_1(0) = \dot{\varphi}_1(0) = 0, \dot{r}_1(0) = -(1/800)$), we get the answer given above.

Disregarding the small quantities of second order gives an effect of under 1/800 of the one obtained (i.e., on the order of 10 meters on one loop). Thus the lens cap describes a 30 km ellipse in an hour-and-a-half, returns to the space ship on the side opposite the earth, and goes past at the distance of a few tens of meters.

Of course, in this calculation we have disregarded the deviation of the orbit from a circle, the effect of forces other than gravity, etc.

9 The motion of a point in three-space

In this paragraph we define the angular momentum relative to an axis and we show that, for motion in an axially symmetric field, it is conserved.

All the results obtained for motion in a plane can be easily carried over to motions in space.

A *Conservative fields*

We consider a motion in the conservative field

$$\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}},$$

where $U = U(\mathbf{r})$, $\mathbf{r} \in E^3$.

The *law of conservation of energy* holds:

$$\frac{dE}{dt} = 0, \quad \text{where } E = \tfrac{1}{2}\dot{\mathbf{r}}^2 + U(\mathbf{r}).$$

B *Central fields*

For motion in a central field the vector $\mathbf{M} = [\mathbf{r}, \dot{\mathbf{r}}]$ does not change: $d\mathbf{M}/dt = 0$.

Every central field is conservative (this is proved as in the two-dimensional case), and

$$\frac{d\mathbf{M}}{dt} = [\dot{\mathbf{r}}, \dot{\mathbf{r}}] + [\mathbf{r}, \ddot{\mathbf{r}}] = 0,$$

since $\ddot{\mathbf{r}} = -(\partial U/\partial \mathbf{r})$, and the vector $\partial U/\partial \mathbf{r}$ is collinear with $\mathbf{r}$ since the field is central.

Corollary. *For motion in a central field, every orbit is planar.*

PROOF. $(\mathbf{M}, \mathbf{r}) = ([\mathbf{r}, \dot{\mathbf{r}}], \mathbf{r}) = 0$; therefore $\mathbf{r}(t) \perp \mathbf{M}$, and since $\mathbf{M} = \text{const.}$, all orbits lie in the plane perpendicular to $\mathbf{M}$.[20] □

Thus the study of orbits in a central field in space reduces to the planar problem examined in the previous paragraph.

PROBLEM. Investigate motion in a central field in n-dimensional euclidean space.

C *Axially symmetric fields*

Definition. A vector field in E^3 has *axial symmetry* if it is invariant with respect to the group of rotations of space which fix every point of some axis.

PROBLEM. Show that if a field is axially symmetric and conservative, then its potential energy has the form $U = U(r, z)$, where r, φ, and z are cylindrical coordinates.

In particular, it follows from this that the vectors of the field lie in planes through the z axis.

As an example of such a field we can take the gravitational field created by a solid of revolution.

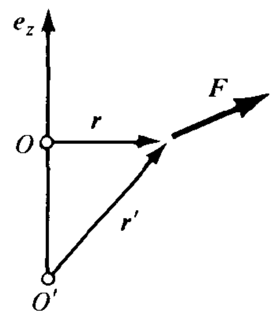

Figure 37 Moment of the vector **F** with respect to an axis

Let z be the axis, oriented by the vector $\mathbf{e}_z$ in three-dimensional euclidean space E^3; **F** a vector in the euclidean linear space $\mathbb{R}^3$; 0 a point on the z axis; $\mathbf{r} = x - 0 \in \mathbb{R}^3$ the radius vector of the point $x \in E^3$ relative to 0 (Figure 37).

Definition. The moment M_z relative to the z axis of the vector **F** applied at the point **r** is the projection onto the z axis of the moment of the vector **F** relative to some point on this axis:

$$M_z = (\mathbf{e}_z, [\mathbf{r}, \mathbf{F}]).$$

[20] The case $\mathbf{M} = 0$ is left to the reader.

The number M_z does not depend on the choice of the point 0 on the z axis. In fact, if we look at a point $0'$ on the axis, then by properties of the triple product, $M'_z = (\mathbf{e}_z, [\mathbf{r}', \mathbf{F}]) = ([\mathbf{e}_z, \mathbf{r}'], \mathbf{F}) = ([\mathbf{e}_z, \mathbf{r}], \mathbf{F}) = M_z$.

Remark. M_z depends on the choice of the *direction* of the z axis: if we change $\mathbf{e}_z$ to $-\mathbf{e}_z$, then M_z changes sign.

Theorem. *For a motion in a conservative field with axial symmetry around the z axis, the moment of velocity relative to the z axis is conserved.*

PROOF. $M_z = (\mathbf{e}_z, [\mathbf{r}, \dot{\mathbf{r}}])$. Since $\ddot{\mathbf{r}} = \mathbf{F}$, it follows that $\mathbf{r}$ and $\ddot{\mathbf{r}}$ lie in a plane passing through the z axis, and therefore $[\mathbf{r}, \ddot{\mathbf{r}}]$ is perpendicular to $\mathbf{e}_z$.

Therefore,

$$\dot{M}_z = (\mathbf{e}_z, [\dot{\mathbf{r}}, \dot{\mathbf{r}}]) + (\mathbf{e}_z, [\mathbf{r}, \ddot{\mathbf{r}}]) = 0. \qquad \square$$

Remark. This proof works for any force field in which the force vector $\mathbf{F}$ lies in the plane spanned by $\mathbf{r}$ and $\mathbf{e}_z$.

10 Motions of a system of n points

In this paragraph we prove the laws of conservation of energy, momentum, and angular momentum for systems of material points in E^3.

A *Internal and external forces*

Newton's equations for the motion of a system of n material points, with masses m_i and radius vectors $\mathbf{r}_i \in E^3$ are the equations

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \qquad i = 1, 2, \ldots, n.$$

The vector $\mathbf{F}_i$ is called the *force acting on the i-th point.*

The forces $\mathbf{F}_i$ are determined experimentally. We often observe in a system that for two points these forces are equal in magnitude and act in opposite directions along the straight line joining the points (Figure 38).

Figure 38 Forces of interaction

Such forces are called *forces of interaction* (example: the force of universal gravitation).

If all forces acting on a point of the system are forces of interaction, then the system is said to be *closed*. By definition, the force acting on the i-th point of a closed system is

$$\mathbf{F}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbf{F}_{ij}.$$

The vector $\mathbf{F}_{ij}$ is the force with which the j-th point acts on the i-th.

Since the forces $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$ are opposite ($\mathbf{F}_{ij} = -\mathbf{F}_{ji}$), we can write them in the form $\mathbf{F}_{ij} = f_{ij}\mathbf{e}_{ij}$, where $f_{ij} = f_{ji}$ is the magnitude of the force and $\mathbf{e}_{ij}$ is the unit vector in the direction from the i-th point to the j-th point.

If the system is not closed, then it is often possible to represent the forces acting on it in the form

$$\mathbf{F}_i = \sum \mathbf{F}_{ij} + \mathbf{F}'_i,$$

where $\mathbf{F}_{ij}$ are forces of interaction and $\mathbf{F}'_i(\mathbf{r}_i)$ is the so-called *external force*.

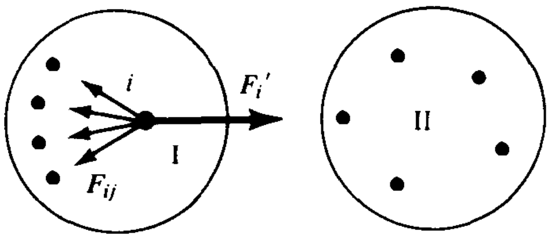

Figure 39 Internal and external forces

EXAMPLE. (Figure 39) We separate a closed system into two parts, I and II. The force $\mathbf{F}_i$ applied to the i-th point of system I is determined by forces of interaction inside system I and forces acting on the i-th point from points of system II, i.e.,

$$\mathbf{F}_i = \sum_{\substack{j \in I \\ j \neq i}} \mathbf{F}_{ij} + \mathbf{F}'_i.$$

$\mathbf{F}'_i$ is the external force with respect to system I.

B *The law of conservation of momentum*

Definition. The *momentum* of a system is the vector

$$\mathbf{P} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i.$$

Theorem. *The rate of change of momentum of a system is equal to the sum of all external forces acting on points of the system.*

PROOF. $d\mathbf{P}/dt = \sum_{i=1}^{n} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} \mathbf{F}_i = \sum_{i,j} \mathbf{F}_{ij} + \sum_i \mathbf{F}'_i = \sum_i \mathbf{F}'_i$; $\sum_{i,j} \mathbf{F}_{ij} = 0$, since for forces of interaction $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. □

Corollary 1. *The momentum of a closed system is conserved.*

Corollary 2. *If the sum of the exterior forces acting on a system is perpendicular to the x axis, then the projection P_x of the momentum onto the x axis is conserved:* $P_x = \text{const}$.

Definition. The *center of mass* of a system is the point

$$\mathbf{r} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}.$$

PROBLEM. Show that the center of mass is well defined, i.e., does not depend on the choice of the origin of reference for radius vectors.

The momentum of a system is equal to the momentum of a particle lying at the center of mass of the system and having mass $\sum m_i$.

In fact, $(\sum m_i)\mathbf{r} = \sum (m_i \mathbf{r}_i)$, from which it follows that $(\sum m_i)\dot{\mathbf{r}} = \sum m_i \dot{\mathbf{r}}_i$.

We can now formulate the theorem about momentum as a theorem about the motion of the center of mass.

Theorem. *The center of mass of a system moves as if all masses were concentrated at it and all forces were applied to it.*

PROOF. $(\sum m_i)\dot{\mathbf{r}} = \mathbf{P}$. Therefore, $(\sum m_i)\ddot{\mathbf{r}} = d\mathbf{P}/dt = \sum_i \mathbf{F}_i$. □

Corollary. *If a system is closed, then its center of mass moves uniformly and linearly.*

C *The law of conservation of angular momentum*

Definition. The *angular momentum of a material point of mass m relative to the point 0*, is the moment of the momentum vector relative to 0:

$$\mathbf{M} = [\mathbf{r}, m\dot{\mathbf{r}}].$$

The *angular momentum of a system relative to 0* is the sum of the angular momenta of all the points in the system:

$$\mathbf{M} = \sum_{i=1}^{n} [\mathbf{r}_i, m_i \dot{\mathbf{r}}_i].$$

Theorem. *The rate of change of the angular momentum of a system is equal to the sum of the moments of the external forces*[21] *acting on the points of the system.*

PROOF. $d\mathbf{M}/dt = \sum_{i=1}^{n} [\dot{\mathbf{r}}_i, m_i \dot{\mathbf{r}}_i] + \sum_{i=1}^{n} [\mathbf{r}_i, m_i \ddot{\mathbf{r}}_i]$. The first term is equal to zero, and the second is equal to

$$\sum_{i=1}^{n} [\mathbf{r}_i, \mathbf{F}_i] = \sum_{i=1}^{n} \left[\mathbf{r}_i, \left(\sum_{i \neq j} \mathbf{F}_{ij} + \mathbf{F}'_i\right)\right] = \sum_{i=1}^{n} [\mathbf{r}_i, \mathbf{F}'_i],$$

by Newton's equations.

[21] The moment of force is also called the *torque* [Trans. note].

The sum of the moments of two forces of interaction is equal to zero since

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}, \text{ so } [\mathbf{r}_i, \mathbf{F}_{ij}] + [\mathbf{r}_j, \mathbf{F}_{ji}] = [(\mathbf{r}_i - \mathbf{r}_j), \mathbf{F}_{ij}] = 0.$$

Therefore, the sum of the moments of all forces of interaction is equal to zero:

$$\sum_{i=1}^{n} \left[\mathbf{r}_i, \sum_{i \neq j} \mathbf{F}_{ij}\right] = 0.$$

Therefore, $d\mathbf{M}/dt = \sum_{i=1}^{n} [\mathbf{r}_i, \mathbf{F}'_i]$. □

Corollary 1 (The law of conservation of angular momentum). *If the system is closed, then* $\mathbf{M} = \mathbf{const}$.

We denote the sum of the moments of the external forces by $\mathbf{N} = \sum_{i=1}^{n} [\mathbf{r}_i, \mathbf{F}'_i]$.

Then, by the theorem above, $d\mathbf{M}/dt = \mathbf{N}$, from which we have

Corollary 2. *If the moment of the external forces relative to the z axis is equal to zero, then M_z is constant.*

D *The law of conservation of energy*

Definition. The *kinetic energy of a point of mass m* is

$$T = \frac{m\dot{\mathbf{r}}^2}{2}.$$

Definition. The *kinetic energy of a system of mass points* is the sum of the kinetic energies of the points:

$$T = \sum_{i=1}^{n} \frac{m_i \dot{\mathbf{r}}_i^2}{2},$$

where the m_i are the masses of the points and $\dot{\mathbf{r}}_i$ are their velocities.

Theorem. *The increase in the kinetic energy of a system is equal to the sum of the work of all forces acting on the points of the system.*

PROOF.

$$\frac{dT}{dt} = \sum_{i=1}^{n} m_i(\dot{\mathbf{r}}_i, \ddot{\mathbf{r}}_i) = \sum_{i=1}^{r} (\dot{\mathbf{r}}_i, m_i\ddot{\mathbf{r}}_i) = \sum_{i=1}^{n} (\dot{\mathbf{r}}_i, \mathbf{F}_i).$$

Therefore,

$$T(t) - T(t_0) = \int_{t_0}^{t} \frac{dT}{dt}\, dt = \sum_{i=1}^{n} \int_{t_0}^{t} (\dot{\mathbf{r}}_i, \mathbf{F}_i) dt = \sum_{i=1}^{n} A_i.$$ □

The configuration space of a system of n mass points in E^3 is the direct product of n euclidean spaces: $E^{3n} = E^3 \times \cdots \times E^3$. It has itself the structure of a euclidean space.

Let $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n)$ be the radius vector of a point in the configuration space, and $\mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_n)$ the force vector. We can write the theorem above in the form

$$T(t_1) - T(t_0) = \int_{r(t_0)}^{r(t_1)} (\mathbf{F}, d\mathbf{r}) = \int_{t_0}^{t_1} (\dot{\mathbf{r}}, \mathbf{F})dt.$$

In other words:

The increase in kinetic energy is equal to the work of the "force" $\mathbf{F}$ on the "path" $\mathbf{r}(t)$ in configuration space.

Definition. A system is called conservative if the forces depend only on the location of a point in the system ($\mathbf{F} = \mathbf{F}(\mathbf{r})$), and if the work of $\mathbf{F}$ along any path depends only on the initial and final points of the path:

$$\int_{M_1}^{M_2} (\mathbf{F}, d\mathbf{r}) = \Phi(M_1, M_2).$$

Theorem. *For a system to be conservative it is necessary and sufficient that there exist a potential energy, i.e., a function $U(\mathbf{r})$ such that*

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{r}}.$$

PROOF. Cf. Section 6B. □

Theorem. *The total energy of a conservative system ($E = T + U$) is preserved under the motion: $E(t_1) = E(t_0)$.*

PROOF. By what was shown earlier,

$$T(t_1) - T(t_0) = \int_{\mathbf{r}(t_0)}^{\mathbf{r}(t_1)} (\mathbf{F}, d\mathbf{r}) = U(\mathbf{r}(t_0)) - U(\mathbf{r}(t_1)).$$ □

Let all the forces acting on the points of a system be divided into forces of interaction and external forces:

$$\mathbf{F}_i = \sum_{i \neq j} \mathbf{F}_{ij} + \mathbf{F}'_i,$$

where $\mathbf{F}_{ij} = -\mathbf{F}_{ji} = f_{ij}\mathbf{e}_{ij}$.

Proposition. *If the forces of interaction depend only on distance, $f_{ij} = f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)$, then they are conservative.*

PROOF. If a system consists entirely of two points i and j, then, as is easily seen, the potential energy of the interaction is given by the formula

$$U_{ij}(\mathbf{r}) = \int_{r_0}^{r} f_{ij}(\rho)d\rho.$$

We then have

$$-\frac{\partial U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)}{\partial \mathbf{r}_i} = -f_{ij}\frac{\partial |\mathbf{r}_i - \mathbf{r}_j|}{\partial \mathbf{r}_i} = f_{ij}\mathbf{e}_{ij}.$$

Therefore, the potential energy of the interaction of all the points will be

$$U(\mathbf{r}) = \sum_{i>j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|). \qquad \square$$

If the external forces are also conservative, i.e., $\mathbf{F}'_i = -(\partial U'_i/\partial \mathbf{r}_i)$, then the system is conservative, and its total potential energy is

$$U(\mathbf{r}) = \sum_{i>j} U_{ij} + \sum_i U'_i.$$

For such a system the total mechanical energy

$$E = T + U = \sum_i \frac{\dot{\mathbf{r}}_i^2}{2} + \sum_{i>j} U_{ij} + \sum_i U'_i$$

is conserved.

If the system is not conservative, then the total mechanical energy is not generally conserved.

Definition. A decrease in the mechanical energy $E(t_0) - E(t_1)$ is called an increase in the *non-mechanical energy* E':

$$E'(t_1) - E'(t_0) = E(t_0) - E(t_1).$$

Theorem (The law of conservation of energy). *The total energy $H = E + E'$ is conserved.*

This theorem is an obvious corollary of the definition above. Its value lies in the fact that in concrete physical systems, expressions for the size of the non-mechanical energy can be found in terms of other physical quantities (temperature, etc.).

E *Example: The two-body problem*

Suppose that two points with masses m_1 and m_2 interact with potential U, so that the equations of motion have the form

$$m_1\ddot{\mathbf{r}}_1 = -\frac{\partial U}{\partial \mathbf{r}_1} \qquad m_2\ddot{\mathbf{r}}_2 = -\frac{\partial U}{\partial \mathbf{r}_2}, \qquad U = U(|\mathbf{r}_1 - \mathbf{r}_2|).$$

Theorem. *The time variation of* $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ *in the two-body problem is the same as that for the motion of a point of mass* $m = m_1 m_2/(m_1 + m_2)$ *in a field with potential* $U(|\mathbf{r}|)$.

We denote by $\mathbf{r}_0$ the radius vector of the center of mass: $\mathbf{r}_0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$. By the theorem on the conservation of momentum, the point $\mathbf{r}_0$ moves uniformly and linearly.

We now look at the vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Multiplying the first of the equations of motion by m_2, the second by m_1, and computing, we find that $m_1 m_2 \ddot{\mathbf{r}} = -(m_1 + m_2)(\partial U/\partial \mathbf{r})$, where $U = U(|\mathbf{r}_1 - \mathbf{r}_2|) = U(|\mathbf{r}|)$.

In particular, in the case of a Newtonian attraction, the points describe conic sections with foci at their common center of mass (Figure 40).

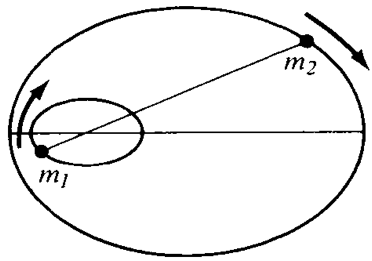

Figure 40 The two body problem

Problem. Determine the major semi-axis of the ellipse which the center of the earth describes around the common center of mass of the earth and the moon. Where is this center of mass, inside the earth or outside? (The mass of the moon is 1/81 times the mass of the earth.)

11 The method of similarity

In some cases it is possible to obtain important information from the form of the equations of motion without solving them, by using the methods of similarity and dimension. The main idea in these methods is to choose a change of scale (of time, length, mass, etc.) under which the equations of motion preserve their form.

A *Example*

Let $\mathbf{r}(t)$ satisfy the equation $m(d^2\mathbf{r}/dt^2) = -(\partial U/\partial \mathbf{r})$. We set $t_1 = \alpha t$ and $m_1 = \alpha^2 m$. Then $\mathbf{r}(t_1)$ satisfies the equation $m_1 \cdot (d^2\mathbf{r}/dt_1^2) = -(\partial U/\partial \mathbf{r})$. In other words:

If the mass of a point is decreased by a factor of 4, then the point can travel the same orbit in the same force field twice as fast.[22]

[22] Here we are assuming that U does not depend on m. In the field of gravity, the potential energy U is proportional to m, and therefore the acceleration does not depend on the mass m of the moving point.

B *A problem*

Suppose that the potential energy of a central field is a homogeneous function of degree ν:

$$U(\alpha r) = \alpha^{\nu} U(r) \quad \text{for any } \alpha > 0.$$

Show that if a curve γ is the orbit of a motion, then the homothetic curve $\alpha\gamma$ is also an orbit (under the appropriate initial conditions). Determine the ratio of the circulation times along these orbits. Deduce from this the isochronicity of the oscillation of a pendulum ($\nu = 2$) and Kepler's third law ($\nu = -1$).

PROBLEM. If the radius of a planet is α times the radius of the earth and its mass β times that of the earth, find the ratio of the acceleration of the force of gravity and the first and second cosmic velocities to the corresponding quantities for the earth.

ANSWER. $\gamma = \beta\alpha^{-2}$, $\delta = \sqrt{\beta/\alpha}$.

For the moon, for example, $\alpha = 1/3.7$ and $\beta = 1/81$. Therefore, the acceleration of gravity is about 1/6 that of the earth ($\gamma \approx 1/6$), and the cosmic velocities are about 1/5 those for the earth ($\delta \approx 1/4.7$).

PROBLEM.[23] A desert animal has to cover great distances between sources of water. How does the maximal time the animal can run depend on the size L of the animal?

ANSWER. It is directly proportional to L.

Solution. The store of water is proportional to the volume of the body, i.e., L^3; the evaporation is proportional to the surface area, i.e., L^2. Therefore, the maximal time of a run from one source to another is directly proportional to L.

We notice that the maximal distance an animal can run also grows proportionally to L (cf. the following problem).

PROBLEM.[24] How does the running velocity of an animal on level ground and uphill depend on the size L of the animal?

ANSWER. On level ground $\sim L^0$, uphill $\sim L^{-1}$.

[23] J. M. Smith, *Mathematical Ideas in Biology*, Cambridge University Press, 1968.

[24] *Ibid.*

Solution. The power developed by the animal is proportional to L^2 (the percentage used by muscle is constant at about 25%, the other 75% of the chemical energy is converted to heat; the heat output is proportional to the body surface, i.e., L^2, which means that the effective power is proportional to L^2).

The force of air resistance is directly proportional to the square of the velocity and the area of a cross-section; the power spent on overcoming it is therefore proportional to v^2L^2v. Therefore, $v^3L^2 \sim L^2$, so $v \sim L^0$. In fact, the running velocity on level ground, no smaller for a rabbit than for a horse, in practice does not specifically depend on the size.

The power necessary to run uphill is $mgv \sim L^3v$; since the generated power is $\sim L^2$, we find that $v \sim L^{-1}$. In fact, a dog easily runs up a hill, while a horse slows its pace.

PROBLEM.[24a] How does the height of an animal's jump depend on its size?

ANSWER. $\sim L^0$.

Solution. For a jump of height h one needs energy proportional to L^3h, and the work accomplished by muscular strength F is proportional to FL. The force F is proportional to L^2 (since the strength of bones is proportional to their section). Therefore, $L^3h \sim L^2L$, i.e., the height of a jump does not depend on the size of the animal. In fact, a jerboa and a kangaroo can jump to approximately the same height.

[24a] *Ibid.*