## Mathematical Model of Physical Systems

- Mechanical, electrical, thermal, hydraulic, economic, biological, etc, systems, may be characterized by differential equations.
- The response of dynamic system to an input may be obtained if these differential equations are solved.
- The differential equations can be obtained by utilizing physical laws governing a particular system, for example, Newtons laws for mechanical systems, Kirchhoffs laws for electrical systems, etc.

Mathematical models: The mathematical description of the dynamic characteristic of a system.
$\Rightarrow$ The first step in the analysis of dynamic system is to derive its model.
$\Rightarrow$ Models may assume different forms, depending on the particular system and the circumstances.
$\Rightarrow$ In obtaining a model, we must make a compromise between the simplicity of the model and the accuracy of results of the analysis.

Linear systems: Linear systems are one in which the equations of the model are linear.
A differential equation is linear, if the coefficients are constant or functions only of the independent variable (time).
(1) $\frac{d^{2} y}{d t^{2}}+3 \frac{d y}{d t}+6 y=e^{t} \quad$ is linear differential equation.
(2) $\frac{d^{3} y}{d t^{3}}+(6-t) \frac{d^{2} y}{d t^{2}}+t^{2} \frac{d y}{d t}+y=\sin t \quad$ is linear differential equation. or $\quad \dddot{y}+(6-t) \ddot{y}+t^{2} \dot{y}+y=\sin t$

* The most important properties of linear system is that the principle of superposition is applicable.
* In an experimental investigation of dynamic system, if cause and effect are proportional, thus implying that the principle of superposition holds, then the system can be considered linear.

Linear time invariant systems: the system which represented by differential equation whose coefficients are function of time for example $\dddot{y}+(t-6) \ddot{y}+2 t^{2} \dot{y}+4 y=e^{-2 t}$
An example of time varying control system is a spacecraft control system.

Non linear system: Non linear system are ones which are represented by non linear equations. Examples are $\mathrm{Y}=\sin (\mathrm{x}), \mathrm{y}=\mathrm{x}^{2}, \quad \mathrm{z}=\mathrm{x}^{2}+\mathrm{y}^{2}$

A differential equation is called non linear if it is not linear. For examples $\frac{d^{2} y}{d t^{2}}+\left(\frac{d y}{d t}\right)^{2}+y=A \sin w t$
$\frac{d^{2} y}{d t^{2}}+\left(y^{2}-1\right) \frac{d y}{d t}+y=0$
$\frac{d^{2} y}{d t^{2}}+\frac{d y}{d t}+y+y^{3}=0$


* The most important properties of non linear system is that the principle of superposition is not applicable.

Because of the mathematical difficulties attached to nonlinear systems, one often finds it necessary to introduce equivalent linear system which are valid for only a limited range of operation.

Transfer functions: The transfer function of a linear time-invariant system is define to be the ratio of the Laplace transform ( z transform for sampled data systems) of the output to the Laplace transform of the input (driving function), under the assumption that all initial conditions are zero.

Example: Consider the linear time-invariant system
$a_{0} y^{(n)}+a_{1} y^{(n-1)}+\ldots+a_{n-1} \dot{y}+a_{n} y=b_{0} x^{(m)}+b_{1} x^{(m-1)}+\ldots+b_{m-1} \dot{x}+b_{m} x \quad n \geq m$
take the Laplace transform $(\ell)$ of $\mathrm{y}(\mathrm{t})$
$\ell\left(a_{0} y^{(n)}\right)=a_{0} \ell\left(y^{(n)}\right)=a_{0}\left[s^{n} Y(s)-s^{n-1} y(0)-s^{(n-2)} y^{\prime}(0)-\ldots-y^{(n-1)}(0)\right.$
$\ell\left(a_{1} y^{(n-1)}\right)=a_{1} \ell\left(y^{(n-1)}\right)=a_{1}\left[s^{n-1} Y(s)-s^{n-2} y(0)-s^{(n-3)} y^{\prime}(o)-\ldots-y^{(n-2)}(0)\right.$
$\ell\left(\mathrm{a}_{\mathrm{n}} \mathrm{y}\right)=\mathrm{a}_{\mathrm{n}} \ell(\mathrm{y})=\mathrm{a}_{\mathrm{n}} \mathrm{Y}(\mathrm{s})$
same thing is applied to obtain the L.T. of $\mathrm{x}(\mathrm{t})$.by substitute all initial condition to zero. The transfer function of the system become.

Transfer Function $=G(s)=\frac{b_{0} s^{m}+b_{1} s^{m-1}+\ldots+b_{m-1} s+b_{m}}{a_{0} s^{n}+a_{1} s^{n-1}+\ldots+a_{n-1} s+a_{n}}$

- Transfer function is not provide any information concerning the physical structure of the system (the T.F. of many physically different system can be identical).
- The highest power of $s$ in the denominator of T. F. is equal to the order of the highest derivative term of the output. If the highest power of s is equal to n the system is called an $\mathrm{n}^{\text {th }}$ order system.


## How you can obtain the transfer function (T. F.)?

1- Write the differential equation of the system
2- Take the L. T. of the differential equation, assuming all initial condition to be zero.
3- Take the ratio of the output to the input. This ratio is the T. F. .

## Mechanical translation system

Consider the mass - spring - dashpot system
1- Mass


A force applied to the mass produces an acceleration of the mass. The reaction force $f_{m}$ is equal to the product of mass and acceleration and is opposite in direction to the applied force in term of displacement $y$, a velocity v , and acceleration a, the force equation is

$$
\mathrm{F}_{\mathrm{m}}=\mathrm{Ma}=\mathrm{MDv}=\mathrm{MD}^{2} \mathrm{y} \quad \text { where } \mathrm{D}=\mathrm{d} / \mathrm{dt}
$$

## 2- Spring



The elastance, or stiffness $k$ provides a restoring force as represented by a spring. The reaction force $f_{k}$ on each end of the spring is the same and is equal to the product of stiffness k and the amount of deformation of the spring.

End c has a position $\mathrm{y}_{\mathrm{c}}$ and end d has a position $\mathrm{y}_{\mathrm{d}}$ measured from the respective equilibrium positions. The force equation, in accordance with the Hoks law is

$$
\mathrm{f}_{\mathrm{k}}=\mathrm{k}\left(\mathrm{y}_{\mathrm{c}}-\mathrm{y}_{\mathrm{d}}\right)
$$

If the end $d$ is stationary, then $y_{d}=0$ and the above equation reduces to

$$
\mathrm{F}_{\mathrm{k}}=\mathrm{kyc}
$$

## 3- dashpot



The reaction damping force $f_{B}$ is approximated by the product of damping B and the relative velocity of the two ends of the dashpot. The direction of this force depend on the relative magnitude and direction of the velocity $\mathrm{Dy}_{\mathrm{e}}$ and $\mathrm{Dy}_{\mathrm{f}}$

$$
F_{B}=B\left(V_{e}-V_{f}\right)=B\left(D y_{e}-D y_{f}\right)
$$

Mechanical translation quantity and units

Quantity
Force
Distance
Velocity
Acceleration
Mass
Stiffness coefficient Damping coefficient
U. S. customary units
pounds
feet
feet/second
feet/second ${ }^{2}$
slugs=pound*second ${ }^{2} /$ foot
pounds/foot
pounds/(foot/second)
metric units
newtons
meters
meter/second meter/second ${ }^{2}$
kilogram newtons/meter newtons/(meter/second)

Example: find the T. F. of the figure below


The fundamental law governing mechanical system is Newtons law for translational system, the law state that

$$
\begin{aligned}
& M a=\sum F \\
& M \frac{d^{2} y}{d t^{2}}=-f \frac{d y}{d t}-k y+x
\end{aligned}
$$

Taking the Laplace transform of each term

$$
\begin{aligned}
& \ell\left[M \frac{d^{2} y}{d t^{2}}\right]=M\left(s^{2} Y(s)-s y(0)-y^{\prime}(0)\right) \\
& \ell\left[f \frac{d y}{d t}\right]=f[s Y(s)-y(0)] \\
& \ell[k y]=k Y(s) \\
& \ell(x)=X(s)
\end{aligned}
$$

Set the initial conditions to zero so that $y(0)=0, \dot{y}(0)=0$ the Laplace transform can be written
$\left(M s^{2}+f s+k\right) Y(s)=X(s)$
Taking the ratio of $\mathrm{Y}(\mathrm{s})$ to $\mathrm{X}(\mathrm{s})$, we find that the transfer function of the system is
T.F. $=G(s)=\frac{Y(s)}{X(s)}=\frac{1}{M s^{2}+f s+k}$

Example: find the T.F. of simple mass-spring-damper mechanical system


To draw the mechanical network, the points $\mathrm{x}_{\mathrm{a}}, \mathrm{x}_{\mathrm{b}}$ and the reference are located. The complete mechanical network is drawn in fig. below.


For nodes $\mathrm{a} \& \mathrm{~b}$

$$
\begin{align*}
& f=f_{k}=k\left(x_{a}-x_{b}\right)  \tag{1}\\
& f_{k}=f_{m}+f_{B}=M D^{2} x_{b}+B D x_{b}
\end{align*}
$$

(2) where $\mathrm{D}=\mathrm{d} / \mathrm{dt}$

$$
\text { or } \quad f_{k}=f_{m}+f_{B}=M \ddot{x}_{b}+B D \dot{x}_{b}
$$

It is possible to obtain one equation relating $x_{a}$ to $f, x_{b}$ to $x_{a}$, or $x_{b}$ to $f$ by combining equation (1) \& (2)
$k\left(M D^{2}+B D\right) x_{a}=\left(M D^{2}+B D+k\right) f$
(3) check

$$
\begin{align*}
& \left(M D^{2}+B D+k\right) x_{b}=k x_{a}  \tag{4}\\
& \left(M D^{2}+B D\right) x_{b}=f \tag{5}
\end{align*}
$$

Let $\quad G_{1}=\frac{x_{a}}{f}=\frac{M D^{2}+B D+k}{k\left(M D^{2}+B D\right)}$

$$
\begin{align*}
G_{2} & =\frac{x_{b}}{x_{a}}=\frac{k}{M D^{2}+B D+k}  \tag{7}\\
G & =\frac{x_{b}}{f}=\frac{1}{M D^{2}+B D} \tag{8}
\end{align*}
$$

Or by taking L.T. to equation $3,4,5$ with all initial condition equal to zero. The transfer functions are

$$
\begin{align*}
& G_{1}(s)=\frac{x_{a}(s)}{f(s)}=\frac{M s^{2}+B s+k}{k\left(M s^{2}+B s\right)}  \tag{9}\\
& G_{2}(s)=\frac{x_{b}(s)}{x_{a}(s)}=\frac{k}{M s^{2}+B s+k}  \tag{10}\\
& G=\frac{x_{b}(s)}{f(s)}=\frac{1}{M s^{2}+B s} \tag{11}
\end{align*}
$$

Note that the equations $8 \& 11$ are equal to the product of the (6) $*(7) \&(9) *(10)$ $\mathrm{G}=\mathrm{G}_{1} \mathrm{G}_{2} \quad \& \quad \mathrm{G}(\mathrm{s})=\mathrm{G}_{1}(\mathrm{~s}) \mathrm{G}_{2}(\mathrm{~s})$

The block diagram representing the mathematic operation


## Example:

Automobile suspension some system of one wheel
$\mathrm{M}_{1}=$ mass of the automobile
$B=$ the shock absorber
$\mathrm{k}_{1}=$ the spring
$\mathrm{k}_{2}=$ elestance of the tire
$\mathrm{M}_{2}=$ mass of the wheel

Two independent displacement exist, so we must write Two equations

$$
\begin{gathered}
M_{1} \frac{d^{2} x_{1}}{d t^{2}}=-B\left(\frac{d x_{1}}{d t}-\frac{d x_{2}}{d t}\right)-k_{1}\left(x_{1}-x_{2}\right) \\
M_{1} \frac{d^{2} x_{2}}{d t^{2}}=f(t)-B\left(\frac{d x_{2}}{d t}-\frac{d x_{1}}{d t}\right)-k_{1}\left(x_{2}-x_{1}\right)-k_{2} x_{2} \\
\mathrm{M}_{1} \mathrm{~s}^{2} \mathrm{X}_{1}(\mathrm{~s})+\mathrm{B}\left(\mathrm{sX}_{1}(\mathrm{~s})-\mathrm{sX}_{2}(\mathrm{~s})\right)+\mathrm{k}_{1}\left(\mathrm{X}_{1}(\mathrm{~s})-\mathrm{X}_{2}(\mathrm{~s})\right)=0 \\
\mathrm{M}_{2} \mathrm{~s}^{2} \mathrm{X}_{2}(\mathrm{~s})+\mathrm{B}\left(\mathrm{sX} 2(\mathrm{~s})-\mathrm{sX}_{1}(\mathrm{~s})\right)+\mathrm{k}_{1}\left(\mathrm{X}_{2}(\mathrm{~s})-\mathrm{X}_{1}(\mathrm{~s})\right)+\mathrm{k}_{2} \mathrm{X}_{2}(\mathrm{~s})=\mathrm{F}(\mathrm{~s}) \\
T(s)=\frac{X_{1}(s)}{F(s)}=\frac{B_{s}+k_{1}}{M_{1} M_{2} s^{4}+B\left(M_{1}+M_{2}\right) s^{3}+\left(K_{1} M_{2}+k_{2} M_{1}\right) s^{2}+k_{2} B s+k_{1} k_{2}}
\end{gathered}
$$

## Mechanical Rotational System

Consider the system shown below. The system consists of a load inertia and a viscous - friction damper.
$\mathrm{J}=$ moment of inertia of The load kg . $\mathrm{m}^{2}$

$\mathrm{f}=$ Viscous - Friction coefficient Newton $/(\mathrm{rad} / \mathrm{sec})$
$\omega=$ angular velocity, red $/ \mathrm{sec}$
T = torque applied to the system, Newton.m

For mechanical rotational systems Newtons low states that

$$
\begin{aligned}
& J \alpha=\sum T \text { Where } \alpha=\text { angular acceleration, } \mathrm{rad} / \mathrm{sec}^{2} \\
& \text { Applying Newtons law to the present system, We obtain } \\
& J \dot{\omega}+f \omega=T
\end{aligned}
$$

Then the transfer function of this system is found to be
$\frac{\Omega(s)}{T(s)}=\frac{1}{J s+f} \quad$ where $\Omega(s)=\ell\{\omega(t)\}, \quad T(s)=\ell\{T(t)\}$

## Electrical Systems

LRC circuit. Applying kirchhoffs voltage law to the system shown. We obtain the following equation;

$$
\begin{gathered}
L \frac{d i}{d t}+R i+\frac{1}{C} \int_{i d t}=e_{i} . \\
\frac{1}{C} \int_{i d t}=e_{o} \ldots \ldots . .
\end{gathered}
$$



Equation (1) \& (2) give a mathematical model of the circuit. Taking the L.T. of equations (1) \&(2), assuming zero initial conditions, we obtain $\operatorname{LsI}(s)+R I(s)+\frac{1}{C} \frac{1}{s} I(s)=E_{i}(s)$

$$
\frac{1}{C} \frac{1}{s} I(s)=E_{o}(s)
$$

the transfer function $\frac{E_{0}(s)}{E_{i}(s)}=\frac{1}{L C s^{2}+R C s+1}$

Complex impedances. Consider the circuit shown below then the T.F of this circuit is
$\frac{E_{o}(s)}{E_{i}(s)}=\frac{Z_{2}(s)}{Z_{1}(s)+Z_{2}(s)}$
Where

$$
\mathrm{Z}_{1}=\mathrm{Ls}+\mathrm{R}
$$

$$
Z_{2}=\frac{1}{C s}
$$


$\mathrm{Z}(\mathrm{s})=\mathrm{E}(\mathrm{s}) / \mathrm{I}(\mathrm{s})$
Hence the T.F. $\mathrm{E}_{\mathrm{o}}(\mathrm{s}) / \mathrm{E}_{\mathrm{i}}(\mathrm{s})$ can be found as follows;
$\frac{E_{o}(s)}{E_{i}(s)}=\frac{\frac{1}{C s}}{L s+R+\frac{1}{C s}}=\frac{1}{L C s^{2}+R C s+1}$

## Example

## Armature-Controlled dc motors

The dc motors have separately excited fields. They are either armature-controlled with fixed field or field-controlled with fixed armature current. For example, dc motors used in instruments employ a fixed permanent-magnet field, and the controlled signal is applied to the armature terminals.

Consider the armature-controlled dc motor shown in the following figure.

$\mathrm{R}_{\mathrm{a}}=$ armature-winding resistance, ohms
$\mathrm{L}_{\mathrm{a}}=$ armature-winding inductance, henrys
$\mathrm{i}_{\mathrm{a}}=$ armature-winding current, amperes
$\mathrm{i}_{\mathrm{f}}=$ field current, a-pares
$e_{a}=$ applied armature voltage, volt
$\mathrm{e}_{\mathrm{b}}=$ back emf, volts
$\theta=$ angular displacement of the motor shaft, radians
$\mathrm{T}=$ torque delivered by the motor, Newton*meter
$\mathrm{J}=$ equivalent moment of inertia of the motor and load referred to the motor shaft $\mathrm{kg} . \mathrm{m}^{2}$
$\mathrm{f}=$ equivalent viscous-friction coefficient of the motor and load referred to the motor shaft. Newton*m/rad/s
$\mathrm{T}=\mathrm{k}_{1} \mathrm{i}_{\mathrm{a}} \psi \quad$ where $\quad \psi$ is the air gap flux, $\psi=\mathrm{k}_{\mathrm{f}} \mathrm{i}_{\mathrm{f}}, \quad k_{1}$ is constant
$\mathrm{T}=\mathrm{k}_{\mathrm{f}} \mathrm{i}_{\mathrm{f}} \mathrm{k}_{1} \mathrm{i}_{\mathrm{a}}$
For a constant field current
$\mathrm{T}=\mathrm{k} \mathrm{i}_{\mathrm{a}} \quad$ where k is a motor-torque constant $\quad\left\{\left(\mathrm{e}_{\mathrm{b}}=\mathrm{k}_{2} \psi \omega\right)\right\}$
For constant flux

$$
\begin{equation*}
e_{b}=k_{b} \frac{d \vartheta}{d t} \quad \text { where } \mathrm{k}_{\mathrm{b}} \text { is a back emf constant. } \tag{1}
\end{equation*}
$$

The differential equation for the armature circuit is

$$
\begin{equation*}
L_{a} \frac{d i_{a}}{d t}+R_{a} i_{a}+e_{b}=e_{a} \tag{2}
\end{equation*}
$$

The armature current produces the torque which is applied to the inertia and friction; hence

$$
\begin{equation*}
\frac{J d^{2} \vartheta}{d t^{2}}+f \frac{d \vartheta}{d t}=T=K i_{a} . \tag{3}
\end{equation*}
$$

Assuming that all initial conditions are condition are zero/and taking the L.T. of equations (1),(2)\&(3), we obtain

$$
\begin{aligned}
& \mathrm{K}_{\mathrm{p}} \mathrm{~s} \theta(\mathrm{~s})=\mathrm{E}_{\mathrm{b}}(\mathrm{~s}) \\
& \left(\mathrm{L}_{\mathrm{a}} \mathrm{~s}+\mathrm{R}_{\mathrm{a}}\right) \mathrm{I}_{\mathrm{a}}(\mathrm{~s})+\mathrm{E}_{\mathrm{b}}(\mathrm{~s})=\mathrm{E}_{\mathrm{a}}(\mathrm{~s}) \\
& \left(\mathrm{Js}^{2}+\mathrm{fs}\right) \theta(\mathrm{s})=\mathrm{T}(\mathrm{~s})=\mathrm{KI}_{\mathrm{a}}(\mathrm{~s})
\end{aligned}
$$

The T.F can be obtained is
$\frac{\theta(s)}{E_{a}(s)}=\frac{K}{s\left(L_{a} J s^{2}+\left(L_{a} f+R_{a} J\right) s+R_{a} f+K K_{b}\right)} . \ldots . . . . . c h e k$ ?

## Example:

## Field-Controlled de motor

Find the T.F $\frac{\theta(s)}{E_{f}(s)}$ For the field-contnalled dc motor shown in figure below


The torque T developed by the motor is proportional to the product of the air gap flux $\psi$ and armature current $i_{a}$ so that
$\mathrm{T}=\mathrm{k}_{1} \psi \mathrm{i}_{\mathrm{a}} \quad$ Where $\mathrm{k}_{1} \quad$ is constant
$\mathrm{T}=\mathrm{k}_{2} \mathrm{i}_{\mathrm{f}} \quad$ Where $\mathrm{k}_{2}$ is constant
$L_{f} \frac{d i_{f}}{d t}+R_{f} i_{f}=e_{f}$
$J \frac{d^{2} \theta}{d t^{2}}+f \frac{d \theta}{d t}=T=k_{2} i_{f}$

By taking the L.T. of eqs. (1)\&(2) \& assuming zero initial conditions we get $\left(L_{f} \mathrm{~s}+\mathrm{R}_{\mathrm{f}}\right) \mathrm{I}_{\mathrm{f}}(\mathrm{s})=\mathrm{E}_{\mathrm{f}}(\mathrm{s})$
$\left(\mathrm{Js}^{2}+\mathrm{fs}\right) \quad \theta(\mathrm{s})=\mathrm{k}_{2} \mathrm{I}_{\mathrm{f}}(\mathrm{s})$
The T.F. of the this system is obtained as
$\frac{\theta(s)}{E_{f}(s)}=\frac{K_{2}}{S\left(L_{f} s+R_{f}\right)(J s+f)} \ldots \ldots \ldots . . . . \operatorname{chek}$ ?

## H.W



For the positional servomechanism obtain the closed-loop T.F. for the positional servomechanism shown below, Assume that the in-put and out put of the system are input shaft position and the output shaft
$r=$ reference input shaft, radian
$\mathrm{c}=$ out put shaft, radian
$\theta=$ motor shaft, radian
$\mathrm{k}_{1}=$ gain of potentiometer error detector $=24 / \pi \mathrm{volt} / \mathrm{rad}$
$\mathrm{k}_{\mathrm{p}}=$ amplifier gain $=10$
$\mathrm{k}_{\mathrm{b}}=$ back emf const. $=5.5 * 10^{-2}$ volts-sec $/ \mathrm{rad}$
$\mathrm{K}=$ motor torque constant $=6^{*} 10^{-5} \mathrm{Ib}-\mathrm{ft}-\mathrm{sec}^{2}$
$\mathrm{R}_{\mathrm{a}}=0.2 \Omega$
$\mathrm{L}_{\mathrm{a}}=$ negligible
$\mathrm{J}_{\mathrm{m}}=1 * 10^{-3} \quad \mathrm{Ib}-\mathrm{ft}-\mathrm{sec}^{2}$
$\mathrm{f}_{\mathrm{m}}=$ negligble
$\mathrm{J}_{1}=4.4^{*} 10^{-3} \mathrm{Ib}-\mathrm{ft}-\mathrm{sec}^{2}$
$\mathrm{f}_{\mathrm{L}}=4^{*} 10^{-2} \mathrm{Ib}-\mathrm{ft} / \mathrm{rad} / \mathrm{sec}$
$\mathrm{n}=$ gear ratio $\mathrm{N}_{1} / \mathrm{N}_{2}=1 / 10$
Hint $J=J_{m}+n^{2} J_{1} \quad, f=f_{m}+n^{2} f_{L}$
Answer $\frac{\theta(s)}{E_{a}(s)}=\frac{0.72}{s(0.13 s+1)}$

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Subject: Control Engineering Year: Third
time allowed 75 minute

Attempt All Questions

Q1: (A) Define four of the following statements, (1) state vector, (2) transducer, (3) open loop control system, (4) transfer function, (5) linear time invariant system.
(B) For the electrical circuit shown in Fig. (1)
(1) Derive the transfer function $\mathrm{Y}(\mathrm{s}) / \mathrm{U}(\mathrm{s})$.
(2) Find the state space equations and then draw the block diagram of state space equations.

Fig. (1)


Q2: For the block diagram of feedback control system shown in Fig.(2) the output is $\quad \mathrm{Y}(\mathrm{s})=\mathrm{M}(\mathrm{s}) \mathrm{R}(\mathrm{s})+\mathrm{M}_{\mathrm{w}}(\mathrm{s}) \mathrm{W}(\mathrm{s})$

Find the transfer functions $\mathrm{M}(\mathrm{s})$ and $\mathrm{M}_{\mathrm{w}}(\mathrm{s})$. [4 Marks]


Fig. (2)

