UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Monday, 8 June, 2020 3 hours

PAPER 1

Before you begin read these instructions carefully

Candidates are required to comply with the Code of Conduct for Part IB Online Examinations. This is a closed-book examination.

You must begin ***each answer*** *on a* ***separate sheet****.*

Answers must be ***handwritten*** *(unless you have an approved adjustment).*

You should ensure that your answers are ***legible****; otherwise you place yourself at a significant disadvantage. You are advised to write on one side of the paper only.*

Candidates have THREE HOURS to complete the examination

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may attempt ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

At the end of the examination:

Separate your answers to each question. Make sure that the question number, e.g. 7D, and your Blind Grade ID, e.g. 1234A, are written clearly on the first page of each answer.

Scan ***each*** *answer into a* ***separate*** *PDF file.*

Name each PDF file by the relevant question number, for example 7D.pdf *for question number 7D.*

Complete the PDF cover sheet to show all the questions you have attempted.

Sign the PDF declaration that you have complied with the Code of Conduct.

Upload the PDF for each answer, together with your PDF declaration and coversheet, to Moodle and submit.

If these administrative tasks take you more than 45 minutes then email your Tutor explaining why.

SECTION I

1F Linear Algebra

Define what it means for two $n \times n$ matrices A and B to be *similar*. Define the *Jordan normal form* of a matrix.

Determine whether the matrices

$$A = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$$

are similar, carefully stating any theorem you use.

2E Geometry

Define the *Gauss map* of a smooth embedded surface. Consider the surface of revolution S with points

$$\begin{pmatrix} (2+\cos v)\cos u \\ (2+\cos v)\sin u \\ \sin v \end{pmatrix} \in \mathbb{R}^3$$

for $u, v \in [0, 2\pi]$. Let f be the Gauss map of S. Describe f on the $\{y = 0\}$ cross-section of S, and use this to write down an explicit formula for f.

Let U be the upper hemisphere of the 2-sphere S^2, and K the Gauss curvature of S. Calculate $\int_{f^{-1}(U)} K \, dA$.

3G Complex Analysis or Complex Methods

Let D be the open disc with centre $e^{2\pi i/6}$ and radius 1, and let L be the open lower half plane. Starting with a suitable Möbius map, find a conformal equivalence (or conformal bijection) of $D \cap L$ onto the open unit disc.

4A Quantum Mechanics

Define what it means for an operator Q to be *hermitian* and briefly explain the significance of this definition in quantum mechanics.

Define the *uncertainty* $(\Delta Q)_\psi$ of Q in a state ψ. If P is also a hermitian operator, show by considering the state $(Q + i\lambda P)\psi$, where λ is a real number, that

$$\langle Q^2 \rangle_\psi \, \langle P^2 \rangle_\psi \;\geqslant\; \frac{1}{4} \, | \, \langle i[Q,P] \rangle_\psi \, |^2 \, .$$

Hence deduce that

$$(\Delta Q)_\psi \, (\Delta P)_\psi \;\geqslant\; \frac{1}{2} | \, \langle i[Q,P] \rangle_\psi \, | \, .$$

Give a physical interpretation of this result.

5C Numerical Analysis

(a) Find an LU factorisation of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 2 & 12 \\ 0 & 5 & 7 & 32 \\ 3 & -1 & -1 & -10 \end{bmatrix},$$

where the diagonal elements of L are $L_{11} = L_{44} = 1$, $L_{22} = L_{33} = 2$.

(b) Use this factorisation to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} -3 \\ -12 \\ -30 \\ 13 \end{bmatrix}.$$

6H Statistics

Suppose $X_1, \ldots, X_n$ are independent with distribution $N(\mu, 1)$. Suppose a prior $\mu \sim N(\theta, \tau^{-2})$ is placed on the unknown parameter μ for some given deterministic $\theta \in \mathbb{R}$ and $\tau > 0$. Derive the posterior mean.

Find an expression for the mean squared error of this posterior mean when $\theta = 0$.

7H Optimisation

Solve the following optimisation problem using the simplex algorithm:

$$\begin{array}{rl} \text{maximise} & x_1 + x_2 \\ \text{subject to} & |x_1 - 2x_2| \leqslant 2, \\ & 4x_1 + x_2 \leqslant 4, \qquad x_1, x_2 \geqslant 0. \end{array}$$

Suppose the constraints above are now replaced by $|x_1 - 2x_2| \leqslant 2 + \epsilon_1$ and $4x_1 + x_2 \leqslant 4 + \epsilon_2$. Give an expression for the maximum objective value that is valid for all sufficiently small non-zero ϵ_1 and ϵ_2.

SECTION II

8F Linear Algebra

Let $\mathcal{M}_n$ denote the vector space of $n \times n$ matrices over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. What is the *rank* $r(A)$ of a matrix $A \in \mathcal{M}_n$?

Show, stating accurately any preliminary results that you require, that $r(A) = n$ if and only if A is non-singular, i.e. $\det A \neq 0$.

Does $\mathcal{M}_n$ have a basis consisting of non-singular matrices? Justify your answer.

Suppose that an $n \times n$ matrix A is non-singular and every entry of A is either 0 or 1. Let c_n be the largest possible number of 1's in such an A. Show that $c_n \leqslant n^2 - n + 1$. Is this bound attained? Justify your answer.

[*Standard properties of the adjugate matrix can be assumed, if accurately stated.*]

9G Groups Rings and Modules

State the structure theorem for a finitely generated module M over a Euclidean domain R in terms of invariant factors.

Let V be a finite-dimensional vector space over a field F and let $\alpha \colon V \to V$ be a linear map. Let V_α denote the $F[X]$-module V with X acting as α. Apply the structure theorem to V_α to show the existence of a basis of V with respect to which α has the rational canonical form. Prove that the minimal polynomial and the characteristic polynomial of α can be expressed in terms of the invariant factors. [*Hint: For the characteristic polynomial apply suitable row operations.*] Deduce the Cayley–Hamilton theorem for α.

Now assume that α has matrix (a_{ij}) with respect to the basis $v_1, \ldots, v_n$ of V. Let M be the free $F[X]$-module of rank n with free basis $m_1, \ldots, m_n$ and let $\theta \colon M \to V_\alpha$ be the unique homomorphism with $\theta(m_i) = v_i$ for $1 \leqslant i \leqslant n$. Using the fact, which you need not prove, that $\ker\theta$ is generated by the elements $Xm_i - \sum_{j=1}^{n} a_{ji}\, m_j$, $1 \leqslant i \leqslant n$, find the invariant factors of V_α in the case that $V = \mathbb{R}^3$ and α is represented by the real matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

with respect to the standard basis.

10E Analysis and Topology

State what it means for a function $f : \mathbb{R}^m \to \mathbb{R}^r$ to be *differentiable* at a point $x \in \mathbb{R}^m$, and define its derivative $f'(x)$.

Let $\mathcal{M}_n$ be the vector space of $n \times n$ real-valued matrices, and let $p : \mathcal{M}_n \to \mathcal{M}_n$ be given by $p(A) = A^3 - 3A - I$. Show that p is differentiable at any $A \in \mathcal{M}_n$, and calculate its derivative.

State the inverse function theorem for a function f. In the case when $f(0) = 0$ and $f'(0) = I$, prove the existence of a continuous local inverse function in a neighbourhood of 0. [The rest of the proof of the inverse function theorem is not expected.]

Show that there exists a positive ϵ such that there is a continuously differentiable function $q : D_\epsilon(I) \to \mathcal{M}_n$ such that $p \circ q = \mathrm{id}|_{D_\epsilon(I)}$. Is it possible to find a continuously differentiable inverse to p on the whole of $\mathcal{M}_n$? Justify your answer.

11E Geometry

Let $\mathcal{C}$ be the curve in the (x,z)-plane defined by the equation

$$(x^2-1)^2+(z^2-1)^2=5\,.$$

Sketch $\mathcal{C}$, taking care with inflection points.

Let S be the surface of revolution in $\mathbb{R}^3$ given by spinning $\mathcal{C}$ about the z-axis. Write down an equation defining S. Stating any result you use, show that S is a smooth embedded surface.

Let r be the radial coordinate on the (x,y)-plane. Show that the Gauss curvature of S vanishes when $r=1$. Are these the only points at which the Gauss curvature of S vanishes? Briefly justify your answer.

12G Complex Analysis or Complex Methods

Let $\ell(z)$ be an analytic branch of $\log z$ on a domain $D \subset \mathbb{C}\setminus\{0\}$. Write down an analytic branch of $z^{1/2}$ on D. Show that if $\psi_1(z)$ and $\psi_2(z)$ are two analytic branches of $z^{1/2}$ on D, then either $\psi_1(z)=\psi_2(z)$ for all $z\in D$ or $\psi_1(z)=-\psi_2(z)$ for all $z\in D$.

Describe the principal value or branch $\sigma_1(z)$ of $z^{1/2}$ on $D_1=\mathbb{C}\setminus\{x\in\mathbb{R}: x\leqslant 0\}$. Describe a branch $\sigma_2(z)$ of $z^{1/2}$ on $D_2=\mathbb{C}\setminus\{x\in\mathbb{R}: x\geqslant 0\}$.

Construct an analytic branch $\varphi(z)$ of $\sqrt{1-z^2}$ on $\mathbb{C}\setminus\{x\in\mathbb{R}: -1\leqslant x\leqslant 1\}$ with $\varphi(2i)=\sqrt{5}$. [If you choose to use σ_1 and σ_2 in your construction, then you may assume without proof that they are analytic.]

Show that for $0<|z|<1$ we have $\varphi(1/z)=-i\sigma_1(1-z^2)/z$. Hence find the first three terms of the Laurent series of $\varphi(1/z)$ about 0.

Set $f(z)=\varphi(z)/(1+z^2)$ for $|z|>1$ and $g(z)=f(1/z)/z^2$ for $0<|z|<1$. Compute the residue of g at 0 and use it to compute the integral

$$\int_{|z|=2} f(z)\,dz\,.$$

13D Variational Principles

A motion sensor sits at the origin, in the middle of a field. The probability that you are detected as you sneak from one point to another along a path $\mathbf{x}(t) : 0 \leqslant t \leqslant T$ is

$$P[\mathbf{x}(t)] = \lambda \int_0^T \frac{v(t)}{r(t)}\, dt\,,$$

where λ is a positive constant, $r(t)$ is your distance to the sensor, and $v(t)$ is your speed. (If $P[\mathbf{x}(t)] \geqslant 1$ for some path then you are detected with certainty.)

You start at point $(x, y) = (A, 0)$, where $A > 0$. Your mission is to reach the point $(x, y) = (B\cos\alpha, B\sin\alpha)$, where $B > 0$. What path should you take to minimise the chance of detection? Should you tiptoe or should you run?

A new and improved sensor detects you with probability

$$\tilde{P}[\mathbf{x}(t)] = \lambda \int_0^T \frac{v(t)^2}{r(t)}\, dt\,.$$

Show that the optimal path now satisfies the equation

$$\left(\frac{dr}{dt}\right)^2 = Er - h^2$$

for some constants E and h that you should identify.

14B Methods

Consider the equation

$$\nabla^2\phi = \delta(x)g(y) \qquad (*)$$

on the two-dimensional strip $-\infty < x < \infty$, $0 \leqslant y \leqslant a$, where $\delta(x)$ is the delta function and $g(y)$ is a smooth function satisfying $g(0) = g(a) = 0$. $\phi(x, y)$ satisfies the boundary conditions $\phi(x, 0) = \phi(x, a) = 0$ and $\lim_{x\to\pm\infty} \phi(x, y) = 0$. By using solutions of Laplace's equation for $x < 0$ and $x > 0$, matched suitably at $x = 0$, find the solution of $(*)$ in terms of Fourier coefficients of $g(y)$.

Find the solution of $(*)$ in the limiting case $g(y) = \delta(y - c)$, where $0 < c < a$, and hence determine the Green's function $\phi(x, y)$ in the strip, satisfying

$$\nabla^2\phi = \delta(x - b)\delta(y - c)$$

and the same boundary conditions as before.

15A Quantum Mechanics

Consider a quantum system with Hamiltonian H and wavefunction Ψ obeying the time-dependent Schrödinger equation. Show that if Ψ is a *stationary state* then $\langle Q \rangle_\Psi$ is independent of time, if the observable Q is independent of time.

A particle of mass m is confined to the interval $0 \leqslant x \leqslant a$ by infinite potential barriers, but moves freely otherwise. Let $\Psi(x,t)$ be the normalised wavefunction for the particle at time t, with

$$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)$$

where

$$\psi_1(x) \; = \; \left(\frac{2}{a}\right)^{1/2} \sin\frac{\pi x}{a}\,, \qquad \psi_2(x) \; = \; \left(\frac{2}{a}\right)^{1/2} \sin\frac{2\pi x}{a}$$

and c_1, c_2 are complex constants. If the energy of the particle is measured at time t, what are the possible results, and what is the probability for each result to be obtained? Give brief justifications of your answers.

Calculate $\langle \hat{x} \rangle_\Psi$ at time t and show that the result oscillates with a frequency ω, to be determined. Show in addition that

$$\left| \langle \hat{x} \rangle_\Psi - \frac{a}{2} \right| \leqslant \frac{16a}{9\pi^2}\,.$$

16D Electromagnetism

Write down the electric potential due to a point charge Q at the origin.

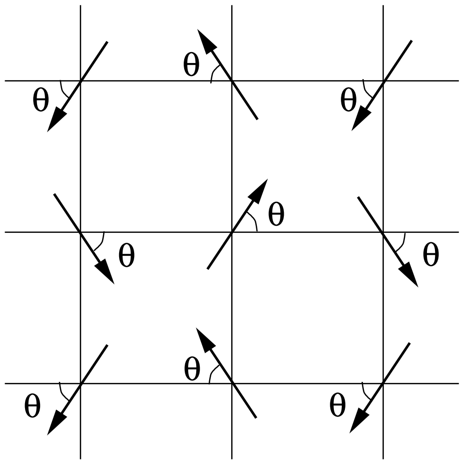
A dipole consists of a charge Q at the origin, and a charge $-Q$ at position $-\mathbf{d}$. Show that, at large distances, the electric potential due to such a dipole is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}\cdot\mathbf{x}}{|\mathbf{x}|^3}\,,$$

where $\mathbf{p} = Q\mathbf{d}$ is the dipole moment. Hence show that the potential energy between two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$, with separation $\mathbf{r}$, where $|\mathbf{r}| \gg |\mathbf{d}|$, is

$$U = \frac{1}{8\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})}{r^5}\right)\,.$$

Dipoles are arranged on an infinite chessboard so that they make an angle θ with the horizontal in an alternating pattern as shown in the figure. Compute the energy between a given dipole and its four nearest neighbours, and show that this is independent of θ.



17C Fluid Dynamics

Steady two-dimensional potential flow of an incompressible fluid is confined to the wedge $0 < \theta < \alpha$, where (r, θ) are polar coordinates centred on the vertex of the wedge and $0 < \alpha < \pi$.

(a) Show that a velocity potential ϕ of the form

$$\phi(r, \theta) = A r^{\gamma} \cos(\lambda \theta),$$

where A, γ and λ are positive constants, satisfies the condition of incompressible flow, provided that γ and λ satisfy a certain relation to be determined.

Assuming that u_θ, the θ-component of velocity, does not change sign within the wedge, determine the values of γ and λ by using the boundary conditions.

(b) Calculate the shape of the streamlines of this flow, labelling them by the distance $r_{\min}$ of closest approach to the vertex. Sketch the streamlines.

(c) Show that the speed $|\mathbf{u}|$ and pressure p are independent of θ. Assuming that at some radius $r = r_0$ the speed and pressure are u_0 and p_0, respectively, find the pressure difference in the flow between the vertex of the wedge and r_0.

[*Hint: In polar coordinates* (r, θ),

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta} \right) \quad \textit{and} \quad \boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r F_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$$

for a scalar f *and a vector* $\mathbf{F} = (F_r, F_\theta)$.]

18C Numerical Analysis

(a) Given a set of $n+1$ distinct real points $x_0, x_1, \ldots, x_n$ and real numbers $f_0, f_1, \ldots, f_n$, show that the interpolating polynomial $p_n \in \mathbb{P}_n[x]$, $p_n(x_i) = f_i$, can be written in the form

$$p_n(x) = \sum_{k=0}^{n} a_k \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}\,, \qquad x \in \mathbb{R}\,,$$

where the coefficients a_k are to be determined.

(b) Consider the approximation of the integral of a function $f \in C[a,b]$ by a finite sum,

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{s-1} w_k f(c_k)\,, \tag{1}$$

where the weights $w_0, \ldots, w_{s-1}$ and nodes $c_0, \ldots, c_{s-1} \in [a,b]$ are independent of f. Derive the expressions for the weights w_k that make the approximation (1) exact for f being any polynomial of degree $s-1$, i.e. $f \in \mathbb{P}_{s-1}[x]$.

Show that by choosing $c_0, \ldots, c_{s-1}$ to be zeros of the polynomial $q_s(x)$ of degree s, one of a sequence of orthogonal polynomials defined with respect to the scalar product

$$\langle u, v\rangle = \int_a^b u(x)v(x)dx\,, \tag{2}$$

the approximation (1) becomes exact for $f \in \mathbb{P}_{2s-1}[x]$ (i.e. for all polynomials of degree $2s-1$).

(c) On the interval $[a,b] = [-1,1]$ the scalar product (2) generates orthogonal polynomials given by

$$q_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n, \quad n = 0, 1, 2, \ldots\,.$$

Find the values of the nodes c_k for which the approximation (1) is exact for all polynomials of degree 7 (i.e. $f \in \mathbb{P}_7[x]$) but no higher.

19H Statistics

Let $X_1, \ldots, X_n$ be i.i.d. $U[0, 2\theta]$ random variables, where $\theta > 0$ is unknown.

(a) Derive the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) What is a *sufficient statistic*? What is a *minimal sufficient statistic*? Is $\hat{\theta}$ sufficient for θ? Is it minimal sufficient? Answer the same questions for the sample mean $\tilde{\theta} := \sum_{i=1}^{n} X_i/n$. Briefly justify your answers.

[You may use any result from the course provided it is stated clearly.]

(c) Show that the mean squared errors of $\hat{\theta}$ and $\tilde{\theta}$ are respectively

$$\frac{2\theta^2}{(n+1)(n+2)} \qquad \text{and} \qquad \frac{\theta^2}{3n}.$$

(d) Show that for each $t \in \mathbb{R}$, $\lim_{n\to\infty} \mathbb{P}(n(1 - \hat{\theta}/\theta) \geqslant t) = h(t)$ for a function h you should specify. Give, with justification, an approximate $1 - \alpha$ confidence interval for θ whose expected length is

$$\left(\frac{n\theta}{n+1}\right)\left(\frac{\log(1/\alpha)}{n - \log(1/\alpha)}\right).$$

[*Hint:* $\lim_{n\to\infty}(1 - \frac{t}{n})^n = e^{-t}$ *for all* $t \in \mathbb{R}$.]

20H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix P. What is a *stopping time* of $(X_n)_{n\geqslant 0}$? What is the *strong Markov property*?

A porter is trying to apprehend a student who is walking along a long narrow path at night. Being unaware of the porter, the student's location Y_n at time $n \geqslant 0$ evolves as a simple symmetric random walk on $\mathbb{Z}$. The porter's initial location Z_0 is $2m$ units to the right of the student, so $Z_0 - Y_0 = 2m$ where $m \geqslant 1$. The future locations Z_{n+1} of the porter evolve as follows: The porter moves to the left (so $Z_{n+1} = Z_n - 1$) with probability $q \in (\frac{1}{2}, 1)$, and to the right with probability $1 - q$ whenever $Z_n - Y_n > 2$. When $Z_n - Y_n = 2$, the porter's probability of moving left changes to $r \in (0, 1)$, and the probability of moving right is $1 - r$.

(a) By setting up an appropriate Markov chain, show that for $m \geqslant 2$, the expected time for the porter to be a distance $2(m-1)$ away from the student is $2/(2q-1)$.

(b) Show that the expected time for the porter to catch the student, i.e. for their locations to coincide, is

$$\frac{2}{r} + \left(m + \frac{1}{r} - 2\right)\frac{2}{2q-1}.$$

[*You may use without proof the fact that the time for the porter to catch the student is finite with probability* 1 *for any* $m \geqslant 1$.]

END OF PAPER