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Mathematicalising Behavioural Finance

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Abstract

This article presents an overview of the recent development on mathematical treatment of behavioural finance, primarily in the setting of continuous-time portfolio choice under the cumulative prospect theory. Financial motivations and mathematical challenges of the problem are highlighted. It is demonstrated that the solutions to the problem have in turn led to new financial and mathematical problems and machineries.

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1. Introduction

Finance ultimately deals with the interplay between market risks and human judgement. The history of finance theory over the last 60 years has been char-

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acterised by two revolutions. The first is neoclassical or maximising finance starting in the 1950s, including mean–variance portfolio selection and expected utility maximisation, the capital asset pricing model (CAPM), efficient market theory, and option pricing. The foundation of neoclassical finance is that the world and its participants are rational "wealth maximisers"; hence finance and economics, albeit primarily about human activities, can be made as logical and predictable as natural sciences. The other revolution is behavioural finance, starting in the 1980s. Its key components are (cumulative) prospect theory, security–potential/aspiration (SP/A) theory, regret and self-control, heuristics and biases. The behavioural theories posit that emotion and psychology do influence our decisions when faced with uncertainties, causing us to behave in unpredictable, inconsistent, incompetent, and most of all, irrational ways. Behavioural finance attempts to explain how and why emotions and cognitive errors influence investors and create stock market anomalies such as bubbles and crashes. It seeks to explore the consistency and predictability in human flaws so that such flaws can be understood, avoided or beneficially exploited.

Mathematical and quantitative approaches have played a pivotal rule in the development of neoclassical finance, and they have led to several ground-breaking, Nobel-prize-winning works. For instance, Markowitz's mean–variance portfolio selection model (Markowitz 1952), which uses probabilistic terms to quantify the risks as well as quadratic programming to derive the solutions, is widely regarded as the cornerstone of quantitative finance. Black–Scholes–Merton's option pricing theory (Black and Scholes 1973, Merton 1973), which employs the Itô calculus and partial differential equations as the underlying mathematical tools, is a fine example of "mathematicalising finance". On the other hand, while Daniel Kahneman won a Nobel prize in 2002 for his work on the prospect theory, behavioural finance is still a relatively new field in which research has so far been largely limited to be descriptive, experimental, and empirical. Rigorous mathematical treatment of behavioural finance, especially that for the continuous-time setting, is very rare in the literature. An important reason for this is that behavioural problems bring in highly unconventional and challenging features for which the known mathematical techniques and machineries almost all fall apart. Therefore, new mathematical theories and approaches, instead of mere extensions of the existing ones, are called for in formulating and solving behavioural models.

This article is to give an account of the recent development on mathematical behavioural finance theory, primarily in the realm of continuous-time behavioural portfolio selection. Study on continuous-time portfolio choice has so far predominantly centred around expected utility maximisation since the seminal papers of Merton (1969, 1971). Expected utility theory (EUT), developed by von Neumann and Morgenstern (1944) based on an axiomatic system, is premised upon the assumptions that decision makers are rational and risk averse when facing uncertainty. In the context of financial portfolio choice, its basic tenets are: investors evaluate wealth according to final asset positions; they are uniformly risk averse; and they are able to evaluate probabilities objectively. These, however, have long been challenged by many observed and repeatable empirical patterns as well as a number of famous paradoxes and puzzles such as Allais paradox (Allais 1953), Ellesberg paradox (Ellesberg 1961), Friedman and Savage puzzle (Friedman and Savage 1948), and the equity premium puzzle (Mehra and Prescott 1985).

Hence, many alternative preference measures to expected utility have been proposed, notably Yaari's *dual theory of choice* (Yaari 1987) which attempts to resolve a number of puzzles and paradoxes associated with the expected utility theory. To illustrate Yaari's theory, consider first the following expected utility

$$Eu(X) = \int_{-\infty}^{+\infty} u(x)dF_X(x) \tag{1}$$

where X is a random payoff with $F_X(\cdot)$ as its cumulative distribution function (CDF), and $u(\cdot)$ is a utility function. This expression shows that $u(\cdot)$ can be regarded as a *nonlinear* "distortion" on payment when evaluating the mean of X (if $u(x) \equiv x$, then the expression reduces to the mean). Yaari (1987) introduces the following criterion

$$V(X) = \int_{-\infty}^{+\infty} w(P(X > x))dx, \tag{2}$$

where $w(\cdot)$, called the *probability distortion* (or *weighting*) *function*, maps from $[0, 1]$ onto $[0, 1]$, with $w(0) = 0, w(1) = 1$. Mathematically, (2) involves the so-called *Choquet integral* with respect to the *capacity* $w \circ P$ (see Denneberg 1994 for a comprehensive account on Choquet integrals). This criterion can be rewritten, assuming $w(\cdot)$ is suitably differentiable, as

$$V(X) = \int_{-\infty}^{+\infty} xd[-w(1 - F_X(x))] = \int_{-\infty}^{+\infty} xw'(1 - F_X(x))dF_X(x). \tag{3}$$

The first identity in (3) suggests that the criterion involves a distortion on the CDF, in contrast to (1). The second identity reveals the role $w(\cdot)$ plays in this new risk preference measure. The term $w'(1 - F_X(x))$ puts a weight on the payment x. If $w(\cdot)$ is convex, the value of $w'(p)$ is greater around $p = 1$ than around $p = 0$; so $V(X)$ overweights payoffs close to the low end and underweights those close to the high end. In other words, the agent is risk averse. By the same token, the agent is risk seeking when $w(\cdot)$ is concave. Thus, in Yaari's theory risk attitude is captured by the nonlinear distortion of decumulative distribution rather than the utility of payoff.

Probability distortion has been observed in many experiments. Here we present two (rather simplified) examples. We write a random variable (*prospect*) $X = (x_i, p_i; i = 1, 2, \cdots, m)$ if $X = x_i$ with probability p_i. We write $X \succ Y$ if prospect X is preferred than propsect Y. Then it has been observed that $(£5000, 0.1; £0, 0.9) \succ (£5, 1)$ although the two prospects have the same mean.

One of the explanations is that people usually exaggerate the small probability associated with a big payoff (so people buy lotteries). On the other hand, it is usual that $(-\pounds 5, 1) \succ (-\pounds 5000, 0.1; \pounds 0, 0.9)$, indicating an inflation of the small probability in respect of a big loss (so people buy insurances).

Other theories developed along this line of involving probability distortions include Lopes' SP/A model (Lopes 1987) and, most significantly, Kahneman and Tversky's (cumulative) prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992), both in the paradigm of modern behavioural decision-making. Cumulative prospect theory (CPT) uses cognitive psychological techniques to incorporate anomalies in human behaviour into economic decision-making. In the context of financial asset allocation, the key elements of CPT are:

- People evaluate assets on *gains* and *losses* (which are defined with respect to a *reference point*), instead of on final wealth positions;
- People behave differently on gains and on losses; they are not uniformly risk averse, and are distinctively more sensitive to losses than to gains (the latter is a behaviour called *loss aversion*);
- People overweight small probabilities and underweight large probabilities.

The significance of the reference point and the presence of non-uniform risk preferences can be demonstrated by the following two experiments.

Experiment 1 You have been given £1000. Now choose between 1A) Win £1000 with 50% chance and £0 with 50% chance, and 1B) Win £500 with 100% chance.

Experiment 2 You have been given £2000. Now choose between 2A) Lose £1000 with 50% chance, and £0 with 50% chance, and 2B) Lose £500 with 100% chance.

It turns out that 1B) and 2A) were more popular in Experiments 1 and 2 respectively[1]. However, if one takes the initial amounts (£1000 and £2000 respectively) into consideration then it is easy to see that 1A) and 2A) are exactly the same *as random variables*, and so are 1B) and 2B). The different choices of references points (£1000 and £2000 in these experiments) have led to completely opposite decisions. On the other hand, the choice of 2A) in Experiment 2 indicates that in a *loss* situation, people favours risky prospects (namely they become risk-seeking), in sharp contrast to a *gain* situation in Experiment 1.

The loss aversion can be defined as $(x, 0.5; -x, 0.5) \succ (y, 0.5; -y; 0.5)$ when $y > x > 0$ are gains with respect to some reference point. So the marginal utility of gaining an additional $\pounds 1$ is lower than the marginal disutility of losing an additional $\pounds 1$.

[1] The outcomes of these experiments – or their variants – are well documented in the literature. I have myself conducted them in a good number of conference and seminar presentations, and the results have been very consistent.

The aforementioned CPT elements translate respectively into the following technical features when formulating a CPT portfolio choice model:

- A reference point in wealth that defines gains and losses;
- A value function or utility function, *concave for gains* and *convex for losses* (such a function is called *S-shaped*), and steeper for losses than for gains;
- A probability distortion (or weighting) that is a *nonlinear* transformation of the probability measure, which inflates a small probability and deflates a large probability.

There have been burgeoning research interests in incorporating behavioural theories into portfolio choice; nonetheless these have been hitherto overwhelmingly limited to the single-period setting; see for example Benartzi and Thaler (1995), Lopes and Oden (1999), Shefrin and Statman (2000), Bassett *et al.* (2004), Gomes (2005), and De Giorgi and Post (2008). Most of these works focus on empirical and numerical studies, and some of them solve the underlying optimisation problems simply by heuristics. Recently, Bernard and Ghossoub (2009) and He and Zhou (2009) have carried out analytical treatments on single-period CPT portfolio choice models and obtained closed-form solutions for a number of interesting cases.

There has been, however, little analytical treatment on *dynamic*, especially continuous-time, asset allocation featuring behavioural criteria. Such a lack of study on continuous-time behavioural portfolio choice is certainly not because the problem is uninteresting or unimportant; rather, we believe, it is because all the main mathematical approaches dealing with the conventional expected utility maximisation model fail completely. To elaborate, despite the existence of thousands of papers on the expected utility model, there are essentially only two approaches involved. One is the stochastic control or dynamic programming approach, initially proposed by Merton (1969), which transforms the problem into solving a partial differential equation, the Hamilton-Jacobi-Bellman (HJB) equation. The other one is the martingale approach. This approach, developed by Harrison and Kreps (1979) and Harrison and Pliska (1981), employs a martingale characterisation to turn the dynamic wealth equation into a static budget constraint and then identifies the optimal terminal wealth via solving a static optimisation problem. If the market is complete, an optimal strategy is then derived by replicating the established optimal terminal wealth, in the spirit of perfectly hedging a contingent claim. In an incomplete market with possible portfolio constraints, the martingale approach is further developed to include the so-called convex duality machinery; see, e.g., Cvitanić and Karatzas (1992).

Now, nonlinear probability distortions in behavioural finance abolish virtually all the nice properties associated with the standard additive probability and linear expectation. In particular, the time-consistency of the conditional expectation with respect to a filtration, which is the core of the dynamic programming principle, is absent due to the distorted probability. Moreover, in the

CPT framework, the utility function is non-convex and non-concave, while the global convexity/concavity is a necessity in traditional optimization. Worse still, the coupling of these two ill-behaved features greatly amplifies the difficulty of the problem.

Berkelaar, Kouwenberg and Post (2004) study a CPT mode with a specific two-piece power utility function. They employ a convexification technique to tackle the non-convexity of the problem. However, the probability distortion, which is one of the major ingredients of all the behavioural theories and which causes a main technical difficulty, is absent in that paper. Jin and Zhou (2008) develop a new theory in solving systematically continuous-time CPT models, featuring both S-shaped utility functions and probability distortions. The whole machinery is very involved; however its essential ideas are clear and intuitive. It constitutes several steps. First, to handle the S-shaped utility function one decomposes the problem, by parameterising some key variables, into a gain part problem and a loss part problem. The gain part problem is a Choquet maximisation problem involving a concave utility function and a probability distortion. The difficulty arising from the distortion is overcome by a so-called *quatile formulation* which changes the decision variable from the random variable X to its quantile function $G(\cdot)$. This quantile formulation has been used by several authors, such as Schied (2004, 2005), Dana (2005), Carlier and Dana (2005), in ad hoc ways to deal with problems with convex/concave probability distortions. It has recently been further developed by He and Zhou (2010) into a general paradigm of solving non-expected utility maximisation models. The loss part problem, on the other hand, is more subtle and difficult to handle even with the quantile formulation, because it is to *minimise* a *concave* functional (thanks, of course, to the original S-shaped utility function). Hence it is essentially a combinatorial optimisation in an infinite dimension. The problem is solved by noting that such a problem must have corner-point solutions, which are step functions in a function space. Once the gain and loss part problems are solved, their solutions are then appropriately pasted by optimising the parameters introduced in the first step.

The rest of this article is organised as follows. Section 2 presents the continuous-time CPT portfolio selection model and the approach to solve the model. Motivated by the gain part problem of the CPT model, Section 3 discusses about the quantile formulation that is a powerful tool in dealing with many non-expected utility models. Section 4 is concerned with the loss part problem and its solution procedure. Finally, Section 5 concludes.

2. The CPT Model

2.1. Model formulation. Consider a CPT agent with an investment planning horizon $[0, T]$ and an initial endowment $x_0 > 0$, both exogenously fixed throughout this paper, in an arbitrage-free economy[2]. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a standard filtered complete probability space representing the underlying uncertainty, and ρ be the *pricing kernel* (also known as the *stochastic discount factor* in the economics literature), which is an $\mathcal{F}_T$-measurable random variable, so that any $\mathcal{F}_T$-measurable and lower bounded contingent claim ξ has a unique price $E[\rho\xi]$ at $t = 0$ (provided that $E[\rho\xi] < +\infty$). The technical requirements on ρ throughout are that $0 < \rho < +\infty$ a.s., $0 < E\rho < +\infty$, and ρ admits no atom, i.e. $P(\rho = x) = 0$ for any $x \in \mathbb{R}^+$.

The key underlying assumption in such an economy is that "the price is linear". The general existence of a pricing kernel ρ can be derived, say, by Riesz's representation theorem under the price linearity in an appropriate Hilbert space. Hence, our setting is indeed very general. It certainly covers the continuous-time complete market considered in Jin and Zhou (2008) with general Itô processes for asset prices, in which case ρ is the usual pricing kernel having an explicit form involving the market price of risk. It also applies to a continuous-time *incomplete* market with a deterministic investment opportunity set, where ρ is the minimal pricing kernel; see, e.g., Föllmer and Kramkov (1997).

The agent risk preference is dictated by CPT. Specifically, she has a reference point B at the terminal time T, which is an $\mathcal{F}_T$-measurable random variable with $E[\rho B] < +\infty$. The reference point B determines whether a given terminal wealth position is a gain (excess over B) or a loss (shortfall from B). It could be interpreted as a liability the agent has to fulfil (e.g. a house downpayment), or an aspiration she strives to achieve (e.g. a target profit aspired by, or imposed on, a fund manager). The agent utility (value) function is S-shaped: $u(x) = u_+(x^+)\mathbf{1}_{x\geq 0}(x) - u_-(x^-)\mathbf{1}_{x<0}(x)$, where the superscripts $^{\pm}$ denote the positive and negative parts of a real number, u_+, u_- are *concave* functions on $\mathbb{R}^+$ with $u_{\pm}(0) = 0$, reflecting risk-aversion on gains and risk-seeking on losses. There are also probability distortions on both gains and losses, which are captured by two nonlinear functions w_+, w_- from $[0, 1]$ onto $[0, 1]$, with $w_{\pm}(0) = 0, w_{\pm}(1) = 1$ and $w_{\pm}(p) > p$ (respectively $w_{\pm}(p) < p$) when p is close to 0 (respectively 1).

The agent preference on a terminal wealth X (which is an $\mathcal{F}_T$-random variable) is measured by the prospective functional

$$V(X - B) := V_+((X - B)^+) - V_-((X - B)^-),$$

where $V_+(Y) := \int_0^{+\infty} w_+(P(u_+(Y) \geq y))dy,\ V_-(Y) := \int_0^{+\infty} w_-(P(u_-(Y) \geq$

[2]In our model the agent is a "small investor"; so her preference only affects her own utility function – and hence *her* portfolio choice – but not the overall economy. Therefore issues like "the limit of arbitrage" and "market equilibrium" are beyond the scope of this article.

$y))dy$. Thus, the CPT portfolio choice problem is to

$$\begin{array}{ll} \underset{X}{\text{Maximise}} & V(X-B) \\ \text{subject to} & \begin{cases} E[\rho X] = x_0 \\ X \text{ is } \mathcal{F}_T - \text{measurable and lower bounded.} \end{cases} \end{array} \tag{4}$$

Here the lower boundedness corresponds to the requirement that the admissible portfolios be "tame", i.e., each of the admissible portfolios generates a lower bounded wealth process, which is standard in the continuous-time portfolio choice literature (see, e.g., Karatzas and Shreve 1998 for a discussion).

We introduce some notation related to the pricing kernel ρ. Let $F_\rho(\cdot)$ be the cumulative distribution function (CDF) of ρ, and $\bar{\rho}$ and $\underline{\rho}$ be respectively the essential lower and upper bounds of ρ, namely,

$$\begin{aligned} \bar{\rho} &\equiv \text{esssup}\ \rho := \sup\left\{a \in \mathbb{R} : P\{\rho > a\} > 0\right\}, \\ \underline{\rho} &\equiv \text{essinf}\ \rho := \inf\left\{a \in \mathbb{R} : P\{\rho < a\} > 0\right\}. \end{aligned} \tag{5}$$

The following assumption is introduced in Jin and Zhou (2008) in solving (4).

Assumption 1. $u_+(\cdot)$ is strictly increasing, strictly concave and twice differentiable, with the Inada conditions $u_+'(0+) = +\infty$ and $u_+'(+\infty) = 0$, and $u_-(\cdot)$ is strictly increasing, and strictly concave at 0. Both $w_+(\cdot)$ and $w_-(\cdot)$ are non-decreasing and differentiable. Moreover, $F_\rho^{-1}(z)/w_+'(z)$ is non-decreasing in $z \in (0,1]$, $\liminf_{x\to+\infty}\left(\frac{-xu_+''(x)}{u_+'(x)}\right) > 0$, and $E\left[u_+\left((u_+')^{-1}(\frac{\rho}{w_+'(F_\rho(\rho))})\right)w_+'(F_\rho(\rho))\right] < +\infty$.

By and large, the monotonicity of the function $F_\rho^{-1}(z)/w_+'(z)$ can be interpreted economically as a requirement that the probability distortion $w_+(\cdot)$ on gains should not be too large in relation to the market (or, loosely speaking, the agent should not be over-optimistic about huge gains); see Jin and Zhou (2008), Section 6.2, for a detailed discussion. Other conditions in Assumption 1 are mild and economically motivated.

2.2. Ill-Posedness.

In general we say a maximisation problem is *well-posed* if its supremium is finite; otherwise it is *ill-posed*. Well-posedness is more a modelling issue; an ill-posed model sets incentives in such a way that the decision-maker could achieve an infinitely favourable value without having to consider trade-offs.

In classical portfolio selection literature (see, e.g., Karatzas and Shreve 1998) the utility function is typically assumed to be globally concave along with other nice properties; thus the problem is guaranteed to be well-posed in most cases[3].

[3] Even with a global concave utility function the underlying problem could still be ill-posed; see counter-examples and discussions in Jin, Xu and Zhou (2008).

However, for the CPT model (4) the well-posedness becomes a more significant issue, and that probability distortions in gains and losses play prominent, yet somewhat opposite, roles.

Theorem 1. *(Jin and Zhou 2008, Theorems 3.1 and 3.2) Problem (4) is ill-posed under either of the following two conditions:*

(i) *There exists a nonnegative $\mathcal{F}_T$-measurable random variable X such that $E[\rho X] < +\infty$ and $V_+(X) = +\infty$.*

(ii) $u_+(+\infty) = +\infty$*,* $\bar{\rho} = +\infty$*, and* $w_-(x) = x$*.*

Theorem 1-(i) says that the model is ill-posed if one can find a nonnegative claim having a finite price yet an infinite prospective value. In this case the agent can purchase such a claim initially (by taking out a loan if necessary) and reach the infinite value at the end. Here we reproduce Example 3.1 in Jin and Zhou (2008) for the existence of such a claim in a simple case with very "nice" parameter specifications. Let ρ be such that $F_\rho(\cdot)$ is continuous and strictly increasing, with $E\rho^3 < +\infty$ (e.g., when ρ is lognormal). Take $w_+(p) := p^{1/4}$ on $p \in [0, 1/2]$ and $u_+(x) := x^{1/2}$. Set $Z := F_\rho(\rho) \sim U(0, 1)$ and define $X := Z^{-1/2} - 1$. Then it is an easy exercise to show that $E[\rho X] < +\infty$ and $V_+(X) = +\infty$. Notice that the culprit of the ill-posedness in this case is the probability distortion $w_+(\cdot)$ which has very large curvatures around 0. In other words, the agent is excessively optimistic in the sense that she over-exaggerates the tiny probability of a huge gain, so much so that her resulting risk-seeking behaviour overrides the risk-averse part of the utility function in the gain domain. This in turn leads to a problem without trade offs (an ill-posed one, that is). So the agent is misled by her own "psychological illusion" (her preference set) to take the most risky exposures.

Theorem 1-(ii) shows that a probability distortion on *losses* is *necessary* for the well-posedness if the market upside potential is unlimited (as implied by $\bar{\rho} = +\infty$). In this case, the agent would borrow an enormous amount of money to purchase a claim with a huge payoff, and then bet the market be "good" leading to the realization of that payoff. If, for the lack of luck, the market turns out to be "bad", then the agent ends up with a loss; however due to the non-distortion on losses its damage to the prospective value is bounded[4]. In plain words, if the agent has no fear in the sense that she does not exaggerate the small probabilities of huge losses, and the market has an unlimited potential of going up, then she would be lured by her CPT criterion to take the infinite risky exposure (again an ill-posed model).

To exclude the ill-posed case identified by Theorem 1-(i), we introduce the following assumption.

[4]This argument is no longer valid if the wealth is *constrained* to be bounded from below. This is why in Berkelaar *et al.* (2004) the model is well-posed even though no probability distortion is considered, as the wealth process there is constrained to be non-negative.

Assumption 2. $V_+(X) < +\infty$ for any nonnegative, $\mathcal{F}_T$-measurable random variable X satisfying $E[\rho X] < +\infty$.

2.3. Solutions.

The original problem (4) is solved in two steps involving three sub-problems, which are described in what follows.

Step 1. In this step we consider two problems respectively:

- *The Gain Part Problem*: A problem with parameters (A, x_+):

$$\begin{array}{ll} \underset{X}{\text{Maximise}} & V_+(X) = \int_0^{+\infty} w_+(P(u_+(X) > y))dy \\ \text{subject to} & E[\rho X] = x_+, \;\; X \geq 0 \text{ a.s.}, \;\; X = 0 \text{ a.s. on } A^C, \end{array} \tag{6}$$

  where $x_+ \geq (x_0 - E[\rho B])^+$ (≥ 0) and $A \in \mathcal{F}_T$ are given. Thanks to Assumption 2, $V_+(X)$ is a finite number for any feasible X. We define the optimal value of Problem (6), denoted $v_+(A, x_+)$, in the following way. If $P(A) > 0$, in which case the feasible region of (6) is non-empty [$X = (x_+\mathbf{1}_A)/(\rho P(A))$ is a feasible solution], then $v_+(A, x_+)$ is defined to be the supremum of (6). If $P(A) = 0$ and $x_+ = 0$, then (6) has only one feasible solution $X = 0$ a.s. and $v_+(A, x_+) := 0$. If $P(A) = 0$ and $x_+ > 0$, then (6) has no feasible solution, where we define $v_+(A, x) := -\infty$.

- *The Loss Part Problem*: A problem with parameters (A, x_+):

$$\begin{array}{ll} \underset{X}{\text{Minimise}} & V_-(X) = \int_0^{+\infty} w_-(P(u_-(X) > y))dy \\ \text{subject to} & \left\{ \begin{array}{l} E[\rho X] = x_+ - x_0 + E[\rho B], \; X \geq 0 \text{ a.s.}, \; X = 0 \text{ a.s. on } A, \\ X \text{ is upper bounded a.s.}, \end{array} \right. \end{array} \tag{7}$$

  where $x_+ \geq (x_0 - E[\rho B])^+$ and $A \in \mathcal{F}_T$ are given. Similarly to the gain part problem we define the optimal value $v_-(A, x_+)$ of Problem (7) as follows. When $P(A) < 1$ in which case the feasible region of (7) is non-empty, $v_-(A, x_+)$ is the infimum of (7). If $P(A) = 1$ and $x_+ = x_0 - E[\rho B]$ where the only feasible solution is $X = 0$ a.s., then $v_-(A, x_+) := 0$. If $P(A) = 1$ and $x_+ \neq x_0 - E[\rho B]$, then there is no feasible solution, in which case we define $v_-(A, x_+) := +\infty$.

Step 2. In this step we solve

$$\begin{array}{ll} \underset{(A, x_+)}{\text{Maximise}} & v_+(A, x_+) - v_-(A, x_+) \\ \text{subject to} & \left\{ \begin{array}{l} A \in \mathcal{F}_T, \; x_+ \geq (x_0 - E[\rho B])^+, \\ x_+ = 0 \text{ when } P(A) = 0, \;\; x_+ = x_0 - E[\rho B] \text{ when } P(A) = 1. \end{array} \right. \end{array} \tag{8}$$

The interpretations of the gain and loss part problems, as well as the parameters (A, x_+), are intuitive. If X is any feasible solution to (4), then its deviation from the reference point B can be decomposed by $X - B = (X - B)^+ - (X - B)^-$.

Let $A := \{X \geq B\}$, the event of ending up with gains, and $x_+ := E[\rho(X-B)^+]$, the price of the gains, then $(X-B)^+$ and $(X-B)^-$ are respectively feasible solutions to (6) and (7) with the parameters (A, x_+).

If Step 1 above is to "divide" – to decompose the original problem into two sub-problems, then Step 2 is to "conquer" – to combine the solutions of the sub-problems in the best way so as to solve the original one. Problem (8) is to find the "best" split between good states and bad states of the world, as well as the corresponding price of the gains. Mathematically, it is an optimisation problem with the decision variables being a real number, x_+, and a random event, A, the latter being very hard to handle. However, Jin and Zhou (2008), Theorem 5.1, shows that one needs only to consider the type of events $A = \{\rho \leq c\}$, where c is a real number in certain range, when optimising (8). This important result in turn suggests that the event of having gains is completely characterised by the pricing kernel and a critical threshold.

With all these preliminaries at hand, we can now state the solution to (4) in terms of the following two-dimensional mathematical programme with the decision variables (c, x_+), which is intimately related to (but not the same as) Problem (8):

$$\begin{aligned} \underset{(c,x_+)}{\text{Maximise}} \quad & v(c,x_+) = E\left[u_+\left((u_+')^{-1}\left(\tfrac{\lambda(c,x_+)\rho}{w_+'(F_\rho(\rho))}\right)\right) w_+'(F_\rho(\rho))\mathbf{1}_{\rho\leq c}\right] \\ & \qquad - u_-\left(\tfrac{x_+-(x_0-E[\rho B])}{E[\rho\mathbf{1}_{\rho>c}]}\right) w_-(1-F_\rho(c)) \\ \text{subject to} \quad & \begin{cases} \underline{\rho} \leq c \leq \bar{\rho}, \quad x_+ \geq (x_0 - E[\rho B])^+, \\ x_+ = 0 \text{ when } c = \underline{\rho}, \quad x_+ = x_0 - E[\rho B] \text{ when } c = \bar{\rho}, \end{cases} \end{aligned} \tag{9}$$

where $\lambda(c,x_+)$ satisfies $E[(u_+')^{-1}(\frac{\lambda(c,x_+)\rho}{w_+'(F_\rho(\rho))})\rho\mathbf{1}_{\rho\leq c}] = x_+$, and we use the following convention:

$$u_-\left(\frac{x_+ - (x_0 - E[\rho B])}{E[\rho\mathbf{1}_{\rho>c}]}\right) w_-(1-F_\rho(c)) := 0 \quad \text{when } c = \bar{\rho} \text{ and } x_+ = x_0 - E[\rho B]. \tag{10}$$

Theorem 2. *(Jin and Zhou 2008, Theorem 4.1) We have the following conclusions:*

(i) *If X^* is optimal for Problem (4), then $c^* := F_\rho^{-1}(P\{X^* \geq B\})$, $x_+^* := E[\rho(X^*-B)^+]$, are optimal for Problem (9).*

(ii) *If (c^*, x_+^*) is optimal for Problem (9), then $\{X^* \geq B\}$ and $\{\rho \leq c^*\}$ are identical up to a zero probability event. In this case*

$$X^* = \left[(u_+')^{-1}\left(\frac{\lambda\rho}{w_+'(F_\rho(\rho))}\right) + B\right]\mathbf{1}_{\rho\leq c^*} - \left[\frac{x_+^* - (x_0 - E[\rho B])}{E[\rho\mathbf{1}_{\rho>c^*}]} - B\right]\mathbf{1}_{\rho>c^*}$$

is optimal for Problem (4).

The explicit form of the optimal terminal wealth profile, X^*, is sufficiently informative to reveal the key qualitative and quantitative features of the corresponding optimal portfolio[5]. The following summarise the economical interpretations and implications of Theorem 2, including those of c^* and x_+^*:

- The future world at $t = T$ is divided by two classes of states: "good" ones (having gains) or "bad" ones (having losses). Whether the agent ends up with a good state is *completely* determined by $\rho \le c^*$, which in statistical terms is a simple hypothesis test involving a constant c^*, *à la* Neyman–Pearson's lemma (see, e.g., Lehmann 1986).

- Optimal strategy is a *gambling* policy, betting on the good states while accepting a loss on the bad. Specifically, at $t = 0$ the agent needs to sell the "loss" lottery, $\left[\frac{x_+^* - (x_0 - E[\rho B])}{E[\rho \mathbf{1}_{\rho > c^*}]} - B\right] \mathbf{1}_{\rho > c^*}$, in order to raise fund to purchase the "gain" lottery, $\left[(u_+')^{-1}\left(\frac{\lambda\rho}{w_+'(F_\rho(\rho))}\right) + B\right] \mathbf{1}_{\rho \le c^*}$.

- The probability of finally reaching a good state is $P(\rho \le c^*) \equiv F_\rho(c^*)$, which in general depends on the reference point B, since c^* depends on B via (9). Equivalently, c^* is the quantile of the pricing kernel evaluated at the probability of good states.

- x_+^* is the price of the terminal gains.

- The magnitude of potential losses in the case of a bad state is a *constant* $\frac{x_+^* - (x_0 - E[\rho B])}{E[\rho \mathbf{1}_{\rho > c^*}]} \ge 0$, which is endogenously *dependent* of B.

- $x_+^* + E[\rho B \mathbf{1}_{\rho \le c^*}] \equiv E[\rho X^* \mathbf{1}_{\rho > c^*}]$ is the $t = 0$ price of the gain lottery. Hence, if B is set too high such that $x_0 < x_+^* + E[\rho B \mathbf{1}_{\rho \le c^*}]$, i.e., the initial wealth is not sufficient to purchase the gain lottery[6], then the optimal strategy *must* involve a leverage.

- If $x_0 < E[\rho B]$, then the optimal $c^* < \bar{\rho}$ (otherwise by the constraints of (9) it must hold that $x_+^* = x_0 - E[\rho B] < 0$ contradicting the non-negativeness of x_+^*); hence $P(\rho > c^*) > 0$. This shows that if the reference point is set too high compared with the initial endowment, then the odds are not zero that the agent ends up with a bad state.

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[5]The specific optimal trading *strategy* depends on the underlying economy, in particular the form of the asset prices. For instance, for a complete continuous-time market, the optimal strategy is the one that replicates X^* in a Black–Schole way. If the market is incomplete but with a deterministic investment opportunity set, then ρ involved is the minimal pricing kernel, and X^* in Theorem 2-(ii) is automatically a monotone functional of ρ and hence replicable. However, we do not actually need the form of the optimal strategy in our subsequent discussions.

[6]It is shown in Jin and Zhou (2009), Theorems 4 and 7, that $P(\rho \le c^*)$ converges to a constant when B goes to infinity, in the case when the utility function is two-piece CRRA and the pricing kernel is lognormal. So $x_+^* + E[\rho B \mathbf{1}_{\rho \le c^*}]$ will be sufficiently large when B is sufficiently large.

2.4. An example: Two-piece CRRA utility. We now illustrate the general results of Theorem 2 by a benchmark case where ρ is lognormal, i.e., $\log\rho \sim N(\mu,\sigma^2)$ with $\sigma > 0$, and the utility function is two-piece CRRA, i.e.,

$$u_+(x) = x^\alpha,\ \ u_-(x) = kx^\alpha,\ x \geq 0$$

where $k > 0$ (the loss aversion coefficient) and $0 < \alpha < 1$ are constants. In this case $\bar\rho = +\infty$ and $\underline{\rho} = 0$. This setting is general enough to cover, for example, a continuous-time economy with Itô processes for multiple asset prices (Karatzas and Shreve 1998, Jin and Zhou 2008) and Kahneman–Tversky's utility functions (Tversky and Kahneman 1992) with $\alpha = 0.88$.

In this case, the crucial mathematical programme (9) has the following more specific form (see Jin and Zhou 2008, eq. (9.3)):

$$\begin{array}{ll} \underset{(c,x_+)}{\text{Maximise}} & v(c,x_+) = \varphi(c)^{1-\alpha}x_+^\alpha - \frac{kw_-(1-F_\rho(c))}{(E[\rho\mathbf{1}_{\rho>c}])^\beta}(x_+-\tilde{x}_0)^\alpha, \\ \text{subject to} & \begin{cases} 0 \leq c \leq +\infty,\ \ x_+ \geq \tilde{x}_0^+, \\ x_+ = 0 \text{ when } c = 0,\ \ x_+ = \tilde{x}_0 \text{ when } c = +\infty, \end{cases} \end{array} \tag{11}$$

where $\tilde{x}_0 := x_0 - E[\rho B]$ and

$$\varphi(c) := E\left[\left(\frac{w_+'(F_\rho(\rho))}{\rho}\right)^{1/(1-\alpha)}\rho\mathbf{1}_{\rho\leq c}\right]\mathbf{1}_{c>0},\ \ 0 \leq c \leq +\infty.$$

It turns out that (11) can be solved explicitly.

Introduce the following function:

$$k(c) := \frac{kw_-(1-F_\rho(c))}{\varphi(c)^{1-\alpha}(E[\rho\mathbf{1}_{\rho>c}])^\alpha} > 0,\ \ c > 0.$$

We state the results for two different cases: one when the agent is initially in the gain domain and the other in the loss domain.

Theorem 3. *(Jin and Zhou 2008, Theorem 9.1) Assume that* $x_0 \geq E[\rho B]$*.*

(i) *If* $\inf_{c>0} k(c) \geq 1$*, then the optimal solution to (4) is*

$$X^* = \frac{x_0 - E[\rho B]}{\varphi(+\infty)}\left(\frac{w_+'(F_\rho(\rho))}{\rho}\right)^{1/(1-\alpha)} + B.$$

(ii) *If* $\inf_{c>0} k(c) < 1$*, then (4) is ill-posed.*

Theorem 4. *(Jin and Zhou 2008, Theorem 9.2) Assume that* $x_0 < E[\rho B]$*.*

(i) *If* $\inf_{c>0} k(c) > 1$*, then (4) is well-posed. Moreover, (4) admits an optimal solution if and only if the following optimisation problem attains an optimal solution*

$$\text{Min}_{0\leq c<+\infty} \quad \left[\left(\frac{kw_-(1-F_\rho(c))}{(E[\rho\mathbf{1}_{\rho>c}])^\alpha}\right)^{1/(1-\alpha)} - \varphi(c)\right]. \tag{12}$$

Furthermore, if an optimal solution c^* of (12) satisfies $c^* > 0$, then the optimal solution to (4) is

$$X^* = \frac{x_+^*}{\varphi(c^*)}\left(\frac{w_+'(F_\rho(\rho))}{\rho}\right)^{1/(1-\alpha)}\mathbf{1}_{\rho\le c^*} - \frac{x_+^* - (x_0 - E[\rho B])}{E[\rho\mathbf{1}_{\rho>c^*}]}\mathbf{1}_{\rho>c^*} + B, \tag{13}$$

where $x_+^* := \frac{-(x_0-E[\rho B])}{k(c^*)^{1/(1-\alpha)}-1}$. If $c^* = 0$ is the only minimiser in (12), then the unique optimal solution to (4) is $X^* = \frac{x_0-E[\rho B]}{E\rho} + B$.

(ii) *If $\inf_{c>0} k(c) = 1$, then the supremum value of (4) is 0, which is however not achievable.*

(iii) *If $\inf_{c>0} k(c) < 1$, then (4) is ill-posed.*

As seen from the preceding theorems the characterising condition for well-posedness in both cases is $\inf_{c>0} k(c) \ge 1$, which is equivalent to

$$k \ge \left(\inf_{c>0} \frac{w_-(1-F_\rho(c))}{\varphi(c)^{1-\alpha}(E[\rho\mathbf{1}_{\rho>c}])^\alpha}\right)^{-1} := k_0.$$

Recall that k is the loss aversion level of the agent ($k = 2.25$ in Tversky and Kahneman 1992). Thus the agent must be *sufficiently* loss averse in order to have a well-posed portfolio choice model.

Another interesting observation is that the optimal portfolios behave fundamentally different depending on whether the agent starts with a gain or loss situation (determined by the initial wealth in relation to the discounted reference point). If she starts in a gain territory, then the optimal strategy is simply to spend $x_0 - E[\rho B]$ buying a contingent claim that delivers a payoff in excess of X, reminiscent of a classical utility maximizing agent (although the allocation to stocks is “distorted” due to the probability distortion). If the initial situation is a loss, then the agent needs to get “out of the hole” soonest possible. As a result, the optimal strategy is a gambling policy which involves raising additional capital to purchase a claim that delivers a higher payoff in the case of a good state of the market and incurs a fixed loss in the case of a bad one. Finally, if $x_0 = E[\rho B]$, then the agent simply buy the claim B at price x_0. If in particular B is the risk-free payoff, then the optimal portfolio is *not* to invest in risky asset at all. Notice that this case underlines a natural psychological reference point – the risk-free return – for many people. This, nonetheless, does explain why most households do not invest in equities at all[7].

As described by Theorem 4-(i), the solution of (4) relies on some attainability condition of a minimisation problem (12), which is rather technical (or, shall we say, mathematical) without clear economical interpretation. The following

[7] A similar result is derived in Gomes (2005) for his portfolio selection model with loss averse investors, albeit in the single-period setting without probability distortions.

Theorem 5, however, gives a sufficient condition in terms of the probability distortion on *losses*.

Theorem 5. *(Jin and Zhou 2009, Theorem 3) Assume that* $x_0 < E[\rho B]$, *and* $\inf_{c>0} k(c) > 1$. *If there exists* $\gamma < \alpha$ *such that* $\liminf_{p\downarrow 0} \frac{w_-(p)}{p^\gamma} > 0$, *or equivalently (by l'Hôpital's rule),* $\liminf_{p\downarrow 0} \frac{w'_-(p)}{p^{\gamma-1}} > 0$, *then (12) must admit an optimal solution* $c^* > 0$ *and hence (13) solves (4).*

The conditions of Theorem 5 stipulate that the curvatures of the probability distortion on losses around 0 must be sufficiently significant in relation to her risk-seeking level (characterised by α). In other words, the agent must have a strong fear on the event of huge losses, in that she exaggerates its (usually) small probability, to the extent that it overrides her risk-seeking behavior in the loss domain.

If, on the other hand, the agent is not sufficiently fearful of big losses, then the risk-seeking part dominates and the problem is ill-posed, as stipulated in the following result.

Theorem 6. *(Jin and Zhou 2009, Proposition 1) Assume that* $x_0 < E[\rho B]$. *If there exists* $\gamma \geq \alpha$ *such that* $\limsup_{p\downarrow 0} \frac{w_-(p)}{p^\gamma} < +\infty$, *then* $\inf_{c\geq 0} k(c) = 0 < 1$, *and hence Problem (4) is ill-posed.*

3. Choquet Maximisation and Beyond: Quantile Formulation

3.1. The gain part problem. To solve the gain part problem (6), we may consider a more general maximisation problem involving the Choquet integral:

$$\begin{array}{ll} \underset{X}{\text{Maximise}} & C(X) := \int_0^{+\infty} w(P(u(X) > y))dy \\ \text{subject to} & E[\rho X] = a, \quad X \geq 0, \end{array} \tag{14}$$

where $a \geq 0$, $w(\cdot) : [0,1] \mapsto [0,1]$ is a non-decreasing, differentiable function with $w(0) = 0$, $w(1) = 1$, and $u(\cdot)$ is a strictly concave, strictly increasing, twice differentiable function with $u(0) = 0$, $u'(0) = +\infty$, $u'(+\infty) = 0$.

Although $u(\cdot)$ in this case is concave (instead of S-shaped), the preference functional $C(X)$ is still non-concave/non-convex in X, due to the probability distortion. The technique to overcome this difficulty is what we call the "quantile formulation", namely to change decision variable of Problem (14) from the random variable X to its quantile function $G(\cdot)$ (which is an appropriate inverse function of the CDF of X). This transformation will recover the concavity (in terms of $G(\cdot)$) for (14), as will be shown shortly.

The key properties of Problem (14) that make the quantile formulation work are the *law-invariance* of the preference functional $C(X)$ (namely $C(X) = C(Y)$

if $X \sim Y$) and the *monotonicity* of its supremium value with respect to the initial wealth a (as both $w(\cdot)$ and $u(\cdot)$ are increasing functions). The general logic of the quantile formulation goes like this: since $X \sim G_X(Z)$ for *any* $Z \sim U(0,1)$, where G_X is the quantile of X and $U(0,1)$ is the uniform distribution on $(0,1)$, we can replace X by $G_X(Z)$ without altering the value of $C(X)$. Now, since the value of Problem (14) is increasing in the initial price a, the optimal $G_X(Z)$ is necessarily the one that has the *cheapest* price, namely, one that makes $E[\rho G_X(Z)]$ the smallest. There is a beautiful result which states that $E[\rho G_X(Z)]$ achieves its minimum (over all possible $Z \sim U(0,1)$) at $Z_\rho := 1 - F_\rho(\rho)$. The precise statement of the result is as follows.

Lemma 1. *$E[\rho G_X(Z_\rho)] \leq E[\rho X]$ for any lower bounded random variable X whose quantile is $G_X(\cdot)$. Furthermore, if $E[\rho G_X(Z_\rho)] < \infty$, then the inequality becomes equality if and only if $X = G_X(Z_\rho)$, a.s..*

This lemma was originally due to Dybvig (1988) where a detailed proof for a finite discrete probability space was provided. The exact form of the lemma for general probability spaces needed for the present article was proved, with a different proof than Dybvig (1988), in Jin and Zhou (2008). The proof is based upon a lemma (Jin and Zhou 2008, Lemma B.1), which is closely related to the so-called Hardy–Littlewood's inequality (Hardy, Littlewood and Pòlya 1952, p. 278) in an integral form.

It follows from $Z_\rho := 1 - F_\rho(\rho)$ that $\rho = F_\rho^{-1}(1 - Z_\rho)$. Substituting this to (14) we can therefore consider the following problem

$$\begin{array}{ll} \underset{G_X(\cdot)}{\text{Maximise}} & C(G_X(Z_\rho)) \\ \text{subject to} & E[F_\rho^{-1}(1 - Z_\rho)G_X(Z_\rho)] = a,\ G(\cdot) \in \mathbb{G},\ G(0+) \geq 0, \end{array} \tag{15}$$

where $\mathbb{G}$ is the set of quantile functions of lower bounded random variables.

It may appear as if (15) were more complicated than (14), but it is actually not. Recall

$$\begin{aligned} C(X) &= \int_0^{+\infty} u(x)d[-w(1 - F_X(x))] \\ &= \int_0^{+\infty} u(x)w'(1 - F_X(x))dF_X(x) \\ &= \int_0^1 u(G_X(z))w'(1 - z)dz \\ &= E[u(G_X(Z_\rho))w'(1 - Z_\rho)], \end{aligned} \tag{16}$$

indicating that $C(X)$, while not concave in X, is indeed *concave* in $G_X(\cdot)$ and the presence of the distortion $w(\cdot)$ now becomes harmless.

We can then rewrite Problem (15) as follows

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Maximise}} & \tilde{C}(G(\cdot)) = \int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(1-z)G(z)dz = a,\ G(\cdot) \in \mathbb{G},\ G(0+) \geq 0. \end{array} \tag{17}$$

The above problem can be solved rather thoroughly via the Lagrange approach (see the next subsection). Finally, if $G^*(\cdot)$ solves (17), then we can recover the optimal terminal wealth X^* by the following formula

$$X^* = G^*(1 - F_\rho(\rho)). \tag{18}$$

3.2. General solution scheme for quantile formulation. Indeed, the law-invariance and monotonicity are inherent and common in many different continuous-time portfolio choice models, including expected utility maximisation, mean-variance, goal reaching, Yaari's dual model, Lopes' SP/A model, as well as those explicitly involving VaR and CVaR in preferences and/or constraints. Thus, like the gain part problem (6), these models all have quantile formulation and can be solved in a similar manner (although there may be technical subtleties with some of them); see He and Zhou (2010).

Let us consider the following general quantile formulation

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Maximise}} & U(G(\cdot)) = \int_0^1 u\left(G(z)\right)\psi(z)dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(1-z)G(z)dz = x_0,\ G(\cdot) \in \mathbb{G} \cap \mathbb{M}, \end{array} \tag{19}$$

where $\psi(z) \geq 0$ satisfies $\int_0^1 \psi(z)dz = 1$ and $\mathbb{M}$ specifies some other constraints on quantiles.

The solution scheme starts with removing the budget constraint in (19) via a Lagrange multiplier $\lambda \in \mathbb{R}$ and considering the following problem

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Maximise}} & U_\lambda(G(\cdot)) := \int_0^1 u\left(G(z)\right)\psi(z)dz - \lambda\left(\int_0^1 F_\rho^{-1}(1-z)G(z)dz - x_0\right) \\ \text{subject to} & G(\cdot) \in \mathbb{G} \cap \mathbb{M}. \end{array} \tag{20}$$

In solving the above problem one usually ignores the constraint, $G(\cdot) \in \mathbb{G} \cap \mathbb{M}$, in the first instance, since in many cases the optimal solution of the resulting unconstrained problem could be modified to satisfy this constraint under some reasonable assumptions. For some cases such a modification could be technically challenging; see for example the SP/A model tackled in He and Zhou (2008). In other cases the constraint may need to be dealt with separately, via techniques specific to each problem.

Once (20) is solved with an optimal solution $G^*_\lambda(\cdot)$, one then finds $\lambda^* \in \mathbb{R}$ that binds the original budget constraint, namely,

$$\int_0^1 F_\rho^{-1}(1-z)G^*_{\lambda^*}(z)dz = x_0.$$

The existence of such λ^* can usually be obtained by examining the monotonicity and continuity of $f(\lambda) := \int_0^1 F_\rho^{-1}(1-z)G_\lambda^*(z)dz$ in λ. Moreover, if the strict monotonicity can be established, then λ^* is unique.

Finally, $G^*(\cdot) := G^*_{\lambda^*}(\cdot)$ can be proved to be the optimal solution to (19). This is shown in the following way. Let $v(x_0)$ and $v_\lambda(x_0)$ be respectively the optimal value of (19) and (20). By their very definitions we have the following *weak duality*

$$v(x_0) \le \inf_{\lambda \in \mathbb{R}} v_\lambda(x_0) \quad \forall x_0 \in \mathbb{R}.$$

However,

$$v(x_0) \le \inf_{\lambda \in \mathbb{R}} v_\lambda(x_0) \le v_{\lambda^*}(x_0) = U_{\lambda^*}(G^*(\cdot)) = U(G^*(\cdot)) \le v(x_0).$$

This implies that $G^*(\cdot)$ is optimal to (19) (and, therefore, the *strong duality* $v(x_0) = \inf_{\lambda \in \mathbb{R}} v_\lambda(x_0)$ holds).

The uniqueness of the optimal solution can also be derived from that of (20). Indeed, suppose we have established the uniqueness of optimal solution to (20) for $\lambda = \lambda^*$, and λ^* is such that $G^*_{\lambda^*}(\cdot)$ binds the budget constraint. Then $G^*_{\lambda^*}(\cdot)$ is the unique optimal solution to (19). To see this, assume there exists another optimal solution $\tilde{G}^*(\cdot)$ to (19). Then

$$U_{\lambda^*}(\tilde{G}^*(\cdot)) \le U_{\lambda^*}(G^*_{\lambda^*}(\cdot)) = v(x_0) = U(\tilde{G}^*(\cdot)) = U_{\lambda^*}(\tilde{G}^*(\cdot)).$$

Hence, by the uniqueness of optimal solution to (20), we conclude $\tilde{G}^*(\cdot) = G^*_{\lambda^*}(\cdot)$.

Finally, once (19) has been solved with the optimal solution $G^*(\cdot)$, the corresponding optimal terminal cash flow can be recovered by

$$X^* = G^*(Z_\rho) \equiv G^*(1 - F_\rho(\rho)). \tag{21}$$

The general expression (21) shows that the optimal terminal wealth is *anti-comonotonic* with respect to the pricing kernel. One of its implications is that the mutual fund theorem holds in any market (complete or incomplete, with possible conic constraints on portfolios) having a deterministic opportunity set so long as all the agents follow the general model (19); see He and Zhou (2010), Theorem 5. Note that such a model covers a very diversified risk–return preferences including those of the classical utility maximisation, mean-variance and various behavioural models. Hence, the mutual fund theorem is somewhat inherent in financial portfolio selection, at least in markets with deterministic opportunity sets. As a consequence, the same risky portfolio is being held across neoclassical (rational) and behavioural (irrational) agents in the market. This, in turn, will shed light on the market equilibrium and capital asset pricing in markets where rational and irrational agents co-exist.

3.3. An example: Goal-reaching model. Let us demonstrate the preceding solution scheme by solving the following goal-reaching model:

$$\begin{array}{ll} \underset{X}{\text{Maximise}} & P(X \geq b) \\ \text{subject to} & E[\rho X] = x_0,\ X \geq 0,\ X \text{ is } \mathcal{F}_T\text{-measurable,} \end{array} \tag{22}$$

where $b > 0$ is the *goal* (level of wealth) intended to be reached by time T. This is called the *goal-reaching* problem, which was proposed by Kulldorff (1993), Heath (1993), and studied extensively (including various extensions) by Browne (1999, 2000).

First, if $x_0 \geq bE[\rho]$, then a trivial optimal solution is $X^* = b$ and the optimal value is 1. Therefore we confine us to the only interesting case $0 < x_0 < bE[\rho]$. Notice

$$P(X \geq b) = \int_0^{+\infty} \mathbf{1}_{\{x \geq b\}} dF_X(x) = \int_0^1 \mathbf{1}_{\{G(z) \geq b\}} dz,$$

and $X \geq 0$ is equivalent to $G(0+) \geq 0$. Hence problem (22) can be formulated in the following quantile version:

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Maximise}} & U(G(\cdot)) = \int_0^1 \mathbf{1}_{\{G(z) \geq b\}} dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(1-z) G(z) dz = x_0, \\ & G(\cdot) \in \mathbb{G},\ \ G(0+) \geq 0. \end{array} \tag{23}$$

This, certainly, specialises the general model (19) with a non-convex/concave "utility" function $u(x) = \mathbf{1}_{\{x \geq b\}}$ and $\psi(z) \equiv 1$.

Introducing the Lagrange multiplier $\lambda > 0$ (as will be evident from below in this case we need only to consider positive multipliers), we have the following family of problems

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Maximise}} & U_\lambda(G(\cdot)) := \int_0^1 \left[\mathbf{1}_{\{G(z) \geq b\}} - \lambda F_\rho^{-1}(1-z) G(z)\right] dz + \lambda x_0 \\ \text{Subject to} & G(\cdot) \in \mathbb{G},\ \ G(0+) \geq 0. \end{array} \tag{24}$$

Ignore the constraints for now, and consider the pointwise maximisation of the integrand above in the argument $x = G(z)$: $\max_{x \geq 0}[\mathbf{1}_{\{x \geq b\}} - \lambda F_\rho^{-1}(1-z)x]$. Its optimal value is $\max\{1 - \lambda F_\rho^{-1}(1-z)b, 0\}$ attained at $x^* = b\mathbf{1}_{\{1-\lambda F_\rho^{-1}(1-z)b \geq 0\}}$. Moreover, such an optimal solution is unique whenever $1 - \lambda F_\rho^{-1}(1-z)b > 0$. Thus, we define

$$G_\lambda^*(z) := b\mathbf{1}_{\{1-\lambda F_\rho^{-1}(1-z)b \geq 0\}}, \quad 0 < z < 1,$$

which is nondecreasing in z. Taking the left-continuous modification of $G_\lambda^*(\cdot)$ to be the optimal solution of (24), and the optimal solution is unique up to a null Lebesgue measure.

Now we are to find $\lambda^* > 0$ binding the budget constraint so as to conclude that $G^*_{\lambda^*}(\cdot)$ is the optimal solution to (23). To this end, let

$$\begin{aligned} f(\lambda) &:= \int_0^1 F_\rho^{-1}(1-z)G^*_\lambda(z)dz \\ &= b\int_0^1 F_\rho^{-1}(1-z)\mathbf{1}_{\{F_\rho^{-1}(1-z)\le 1/(\lambda b)\}}dz \\ &= b\int_0^{+\infty} x\mathbf{1}_{\{x\le 1/(\lambda b)\}}dF_\rho(x) \\ &= bE\left[\rho\mathbf{1}_{\{\rho\le 1/(\lambda b)\}}\right], \quad \lambda > 0. \end{aligned}$$

It is easy to see that $f(\cdot)$ is nonincreasing, continuous on $(0, +\infty)$, with $\lim_{\lambda\downarrow 0} f(\lambda) = bE[\rho]$ and $\lim_{\lambda\uparrow+\infty} f(\lambda) = 0$. Therefore, for any $0 < x_0 < bE[\rho]$, there exists $\lambda^* > 0$ such that $f(\lambda^*) = x_0$ or the budget constraint holds. As per discussed in the general solution scheme the corresponding $G^*_{\lambda^*}(\cdot)$ solves (23) and the terminal payment $X^* = G^*_{\lambda^*}(1 - F_\rho(\rho)) = b\mathbf{1}_{\{\rho\le c^*\}}$, where $c^* \equiv (\lambda^* b)^{-1}$ is such that the initial budget constraint binds, solves the original problem (22). Finally, the optimal solution is unique and the optimal value is $P(X^* \ge b) = P(\rho \le c^*) = F_\rho(c^*)$.

To summarise, we have

Theorem 7. *(He and Zhou 2010, Theorem 1) Assume that* $0 < x_0 < bE[\rho]$. *Then the unique solution to the goal-reaching problem (22) is* $X^* = b\mathbf{1}_{\{\rho\le c^*\}}$ *where* $c^* > 0$ *is the one such that* $E[\rho X^*] = x_0$. *The optimal value is* $F_\rho(c^*)$.

The solution above certainly reduces to that of Browne (1999) when the investment opportunity set is deterministic. However, the approach in Browne (1999) is rather ad hoc, in that a value function of the problem is *conjectured* and then verified to be the solution of the HJB equation. In contrast, the quantile approach *derives* the solution (without having to know its form a priori). Thus it can be easily adapted to more general settings. Indeed, the HJB equation fails to work with a stochastic investment opportunity set, which however can be treated by the quantile formulation at ease.

The quantile-based optimisation is proposed by Schied (2004, 2005) to solve a class of convex, robust portfolio selection problems, and employed by Dana (2005) and Carlier and Dana (2006) to study calculus of variations problems with law-invariant concave criteria. The results presented here are mainly taken from He and Zhou (2010) where the quantile approach is systematically developed into a general paradigm in solving non-expected, non-convex/concave utility maximization models, including both neoclassical and behavioural ones. The technique has been further applied to solve a continuous-time version of the SP/A model (He and Zhou 2008), a general risk-return model where the risk is quantified by a coherent risk measure (He, Jin and Zhou 2009) and an optimal stopping problem involving probability distortions (Xu and Zhou 2009).

4. Choquet Minimization: Combinatorial Optimisation in Function Spaces

The loss part problem (7) is a special case of the following general Choquet minimisation problem:

$$\begin{array}{ll} \underset{X}{\text{Minimise}} & C(X) := \int_0^{+\infty} w(P(u(X) > y))dy \\ \text{subject to} & E[\rho X] = a, \quad X \geq 0, \end{array} \tag{25}$$

where $a \geq 0$, $w(\cdot) : [0,1] \mapsto [0,1]$ is a non-decreasing, differentiable function with $w(0) = 0$, $w(1) = 1$, and $u(\cdot)$ is strictly increasing, concave, strictly concave at 0, with $u(0) = 0$.

A quantile formulation transforms (25) into

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Minimise}} & \tilde{C}(G(\cdot)) = \int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(z)G(z)dz = a, \ G(\cdot) \in \mathbb{G}, \ G(0+) \geq 0. \end{array} \tag{26}$$

Compared with (17), a critically different feature of (26) is that a *concave* functional is to be *minimised*. This, of course, originates from the S-shaped utility function in the CPT portfolio selection problem. The solution of (26) must have a very different structure compared with that of (17), which in turn requires a completely different technique (different from the Lagrange approach) to obtain. Specifically, the solution should be a "corner point solution"; in other words, the problem is essentially a combinatorial optimisation in an infinite dimensional space, which is a generally very challenging problem even in a finite dimension.

The question now is how to characterise a corner point solution in the present setting. A bit of reflection reveals that such a solution must be a step function, which is made precise in the following result.

Proposition 1. *(Jin and Zhou 2008, Propositions D.1 and D.2) The optimal solution to (26), if it exists, must be in the form* $G^*(z) = q(b)\mathbf{1}_{(b,1)}(z)$, $z \in [0,1)$, *with some* $b \in [0,1)$ *and* $q(b) := \frac{a}{E[\rho \mathbf{1}_{\{F_\rho(\rho)>b\}}]}$. *Moreover, in this case, the optimal solution to (25) is* $X^* = G^*(F_\rho(\rho))$.

Since $G(\cdot)$ in Proposition 1 is uniformly bounded in $z \in [0,1)$, it follows that any optimal solution X^* to (25) must be uniformly bounded from above.

Proposition 1 suggests that we only need to find an optimal *number* $b \in [0,1)$ so as to solve Problem (26), which motivates the introduction of the following problem

$$\begin{array}{ll} \underset{b}{\text{Minimise}} & f(b) := \int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} & G(\cdot) = \frac{a}{E[\rho \mathbf{1}_{\{F_\rho(\rho)>b\}}]}\mathbf{1}_{(b,1]}(\cdot), \quad 0 \leq b < 1. \end{array} \tag{27}$$

Problem (25) is then solved completely via the following result.

Theorem 8. *(Jin and Zhou 2008, Theorem D.1) Problem (25) admits an optimal solution if and only if the following problem*

$$\min_{0\le c<\bar{\rho}} u\left(\frac{a}{E[\rho \mathbf{1}_{\{\rho>c\}}]}\right) w(P(\rho>c))$$

admits an optimal solution c^*, *in which case the optimal solution to (25) is* $X^* = \frac{a}{E[\rho \mathbf{1}_{\{\rho>c^*\}}]}\mathbf{1}_{\rho>c^*}$.

5. Concluding Remarks

A referee who reviewed one of our mathematical behavioural finance papers questioned, 'There is a fundamental inconsistency underlying the problem being considered in this paper. The CPT is a theory that explains how investors are "irrational" - by over emphasising losses over gains, and by under emphasising very high and very low probabilities. In this paper the authors propose that the investor *rationally* account for their irrationalities (implicit in the CPT value function). How is this justified?'

A very good question indeed. Here is our response to the question.

'Although *irrationality* is the central theme in behavioural finance, irrational behaviours are by no means random or arbitrary. As pointed out by Dan Ariely, a behavioural economist, in his best-seller *Predictably Irrational* (Ariely 2008), "misguided behaviors ... are systamtic and predictable – making us predictably irrational". People working in behavioural finance have come up with various particular CPT values functions and probability weighting functions to examine and investigate the consistency, predictability, and rationality behind what appear as inconsistent, unpredictable and irrational human behaviours. These functions are dramatically different from those in a neoclassical model so as to systematically capture certain aspects of irrationalities such as risk-seeking, and hope and fear (reflected by the probability distortions). Tversky and Kahneman (1992) themselves state "a parametric specification for CPT is needed to provide a 'parsimonious' description of the the data". As in many other behavioural finance papers, here we use CPT and specific value functions as the carrier for exploring the "predictable irrationalities".'

To explore the consistent inconsistencies and the predictable unpredictabilities – it is the principal reason why one needs to research on "mathematicalising behavioural finance". The research is still in its infancy, but the potential is unlimited – or so we believe.

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