Bachelor Degree in Informatics Engineering

Facultat d'Informàtica de Barcelona

Mathematics 1

Part I: Graph Theory

Exercises and problems

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Departament de Matemàtiques

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The problems of this collection were initially gathered by Anna de Mier and Montserrat Maureso. Many of them were taken from the problem sets of several courses taught over the years by the members of the Departament de Matemàtica Aplicada 2. Other exercises came from the bibliography of the course or from other texts, and some of them were new. Since Mathematics 1 was first taught in 2010 several problems have been modified or rewritten by the professors involved in the teaching of the course.

We would like to acknowledge the assistance of the scholar Gabriel Bernardino in the writing of the solutions.

Translation by Anna de Mier and the scholar Bernat Coma.

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Contents

1
Graphs: basic concepts

1.1 Types of graphs. Subgraphs. Operations with graphs.

The following are some important families of graphs that we will use often. Let n be a positive integer and $V = \{x_1, x_2, \dots, x_n\}$.

The *null graph* of order n, denoted by N_n, is the graph of order n and size 0. The graph N_1 is called the *trivial graph*.

The *complete graph* of order n, denoted by K_n, is the graph of order n that has all possible edges. We observe that K_1 is a trivial graph too.

The *path graph* of order n, denoted by $P_n = (V, E)$, is the graph that has as a set of edges $E = \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\}$.

The *cycle graph* of order $n \geq 3$, denoted by $C_n = (V, E)$, is the graph that has as a set of edges $E = \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1\}$.

The *wheel graph* of order $n \geq 4$, denoted by $W_n = (V, E)$, is the graph that has as a set of edges $E = \{x_1x_2, x_2x_3, \dots, x_{n-1}x_1\} \cup \{x_nx_1, x_nx_2, \dots, x_nx_{n-1}\}$.

Let r and s be positive integers.

A graph is r*-regular* if all vertices have degree r.

A graph $G = (V, E)$ is *bipartite* if there are two non-empty subsets V_1 and V_2 such that $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and, for every edge $uv \in E$, we have $u \in V_1$ and $v \in V_2$, or vice versa. That is, there are no edges uv with $u, v \in V_1$ or $u, v \in V_2$. The sets V_1 and V_2 are called the *stable parts* of G. If every vertex from V_1 is adjacent to every vertex of V_2, we say that the graph is *complete bipartite* and we denote it by $K_{r,s}$, where $|V_1| = r$ and $|V_2| = s$. The graph $K_{1,s}$ is called a *star graph*.

Subgraphs

Let $G = (V, E)$ be a graph.

The graph $G' = (V', E')$ is a *subgraph of* G if $V' \subseteq V$ and $E' \subset E$. If $V' = V$, it is called *a spanning subgraph* of G.

Let $S \subseteq V$, $S \neq \emptyset$. The graph $G[S] = (S, E')$ with $E' = \{uv \in E : u, v \in S\}$ is called the *subgraph induced (or spanned) by the set of vertices* S .

Graphs derived from a graph

Consider a graph $G = (V, E)$.

The *complement* of G, denoted by G^c, is the graph with set of vertices V and set of edges $E^c = \{uv | \, uv \notin E\}$. A graph isomorphic to its complement is called *self-complementary.*

Let $S \subset V$. The graph obtained by *deleting the vertices* from S, denoted by $G - S$, is the graph having as vertices those of $V \setminus S$ and as edges those of G that are not incident to any vertex from S. In the case that $S = \{v\}$, we denote it $G - v$.

Let $S \subset E$. The graph obtained by *deleting the edges* from S, denoted by $G - S$, is the graph obtained from G by removing all the edges from S. That is, $G - S = (V, E \setminus S)$. If $S = \{e\}$, we write $G - e$.

Let u, v be vertices from G that are not adjacent. The graph obtained by *adding the edge* uv is the graph $G + uv = (V, E \cup \{uv\})$.

Operations with graphs

Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

The *union of* G_1 *and* G_2, denoted by $G_1 \cup G_2$, is the graph that has as set of vertices $V_1 \cup V_2$ and as set of edges $E_1 \cup E_2$.

The *product of* G_1 *and* G_2, denoted by $G_1 \times G_2$, is the graph that has as set of vertices $V_1 \times V_2$ and whose adjacencies are given by

$$(u_1, u_2) \sim (v_1, v_2) \Leftrightarrow (u_1v_1 \in E_1 \text{ and } u_2 = v_2) \text{ or } (u_1 = v_1 \text{ and } u_2v_2 \in E_2).$$

1.2 Exercises

1.1 For each of the graphs N_n, K_n, P_n, C_n and W_n, give:

1) a drawing for $n = 4$ and $n = 6$;
2) the adjacency matrix for $n = 5$;
3) the order, the size, the maximum degree and the minimum degree in terms of n.

1.2 For each of the following statements, find a graph with the required property, and give its adjacency list and a drawing.

1) A 3-regular graph of order at least 5.
2) A bipartite graph of order 6.
3) A complete bipartite graph of order 7.
4) A star graph of order 7.

1.3 Find out whether the complete graph, the path and the cycle of order $n \geq 1$ are bipartite and/or regular.

1.4 Give the size:

1) of an r-regular graph of order n;
2) of the complete bipartite graph $K_{r,s}$.

1.5 Let $V = \{a, b, c, d, e, f\}$, $E = \{ab, af, ad, be, de, ef\}$ and $G = (V, E)$. Determine all the subgraphs of G of order 4 and size 4.

1.6 The following five items refer to the graph G defined as follows. The set of vertices is $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, and two vertices u and v are adjacent if $|u - v| \in \{1, 4, 5, 8\}$. Determine the order and the size of the following subgraphs of G:

1) The subgraph induced by even vertices.
2) The subgraph induced by odd vertices.
3) The subgraph induced by the set $\{0, 1, 2, 3, 4\}$.
4) A spanning subgraph with as many edges as possible but without cycles.

1.7 Consider the graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{12, 13, 23, 24, 34, 45\}$. Give the set of edges, the incidence and adjacency matrices, and a drawing of the graphs G^c, $G - 4$, $G - 45$ and $G + 25$.

1.8 Consider a graph $G = (V, E)$ of order n and size m. Let v be a vertex and e an edge of G. Give the order and the size of G^c, $G - v$ and $G - e$.

1.9 Find out whether the complement of a regular graph is regular, and whether the complement of a bipartite graph is bipartite. If so, prove it; if not, give a counterexample.

1.10 Give the set of edges and a drawing of the graphs $K_3 \cup P_3$ and $K_3 \times P_3$, assuming that the sets of vertices of K_3 and P_3 are disjoint.

1.11 Consider the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Give the order, the degree of the vertices and the size of $G_1 \times G_2$ in terms of those of G_1 and G_2.

1.12 Prove or disprove the following statements:

1) If G_1 and G_2 are regular graphs, then $G_1 \times G_2$ is regular.
2) If G_1 and G_2 are bipartite graphs, then $G_1 \times G_2$ is bipartite.

1.13 Draw all the graphs that have $V = \{a, b, c\}$ as set of vertices.

1.14 Consider graphs whose set of vertices is $[7] = \{1, 2, 3, 4, 5, 6, 7\}$. Compute how many of them are there ...

1) ... with exactly 20 edges.
2) ... with exactly 16 edges.
3) ... in total.

1.15 For each of the following sequences, find out if there is any graph of order 5 such that the degrees of its vertices are given by that sequence. If so, give an example.

1) 3, 3, 2, 2, 2.
2) 4, 4, 3, 2, 1.
3) 4, 3, 3, 2, 2.
4) 3, 3, 3, 2, 2.
5) 3, 3, 3, 3, 2.
6) 5, 3, 2, 2, 2.

1.16 Prove that if a graph is regular of odd degree, then it has even order.

1.17 Let G be a bipartite graph of order n and regular of degree $d \geq 1$. Which is the size of G? Could it be that the order of G is odd?

1.18 Prove that the size of a bipartite graph of order n is at most $n^2/4$.

1.19 Let G be a graph with order 9 so that the degree of each vertex is either 5 or 6. Prove that there are either at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

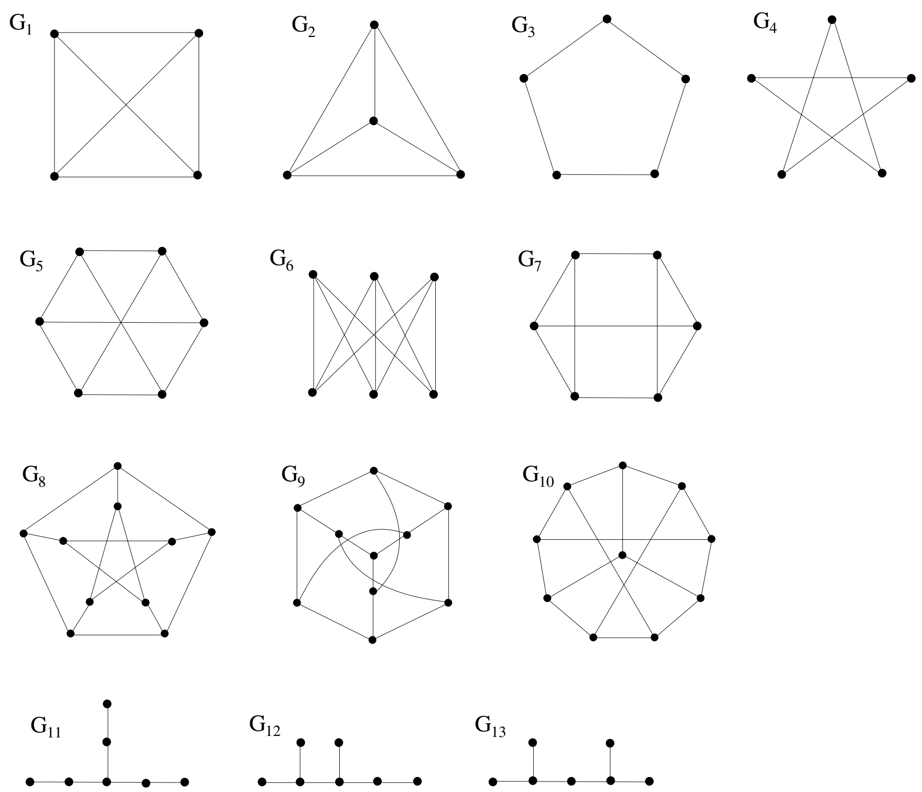
1.20 Alex and Leo are a couple, and they organize a party together with 4 other couples. There are a number of greetings but, naturally, nobody says hello to their own partner. At the end of the party Alex asks everyone how many people did they greet, receiving nine different answers. How many people did Alex greet and how many people did Leo greet?
Hint: Describe a graph that models the situation. Find out how many people did each member of a couple greet.

1.21 Determine, up to isomorphism, all the graphs of order four and size two.

1.22 Let $V = \{a, b, c, d\}$ and $E = \{ab, ac, ad, dc\}$. Determine, up to isomorphism, all the subgraphs of the graph $G = (V, E)$.

1.23 Classify by isomorphism type the graphs of Figure 1.1.

Figure 1.1:



1.24 Let $G = (V, E)$ and $H = (W, B)$ be two graphs. Prove that G and H are isomorphic if, and only if, G^c and H^c are isomorphic.

1.25 Determine up to isomorphism the number of graphs of order 20 and size 188.

1.26 A graph is *self-complementary* if it is isomorphic to its complement. Prove that there are no self-complementary graphs of order 3, but there are such graphs of order 4 and 5.

1.27 A graph is *self-complementary* if it is isomorphic to its complement.

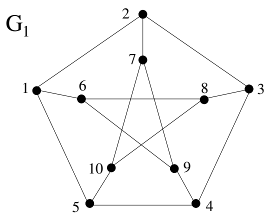
1) How many edges does a self-complementary graph of order n have?
2) Prove that if n is the order of a self-complementary graph, then n is congruent with 0 or with 1 modulo 4.
3) Check that for $n = 4k$ with $k \geq 1$, the following construction yields a self-complementary graph of order n: let us take $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where each V_i contains k vertices; the vertices of V_1 and V_2 induce complete graphs; also, we have all edges between V_1 and V_3, between V_3 and V_4, and between V_4 and V_2.
4) How could we modify the previous construction to build a self-complementary graph of order $4k + 1$?

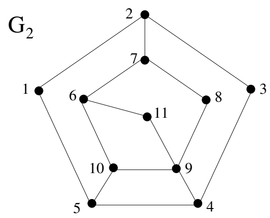
1.28 Let G be a graph of order $n \geq 6$.

1) Show that either G or G^c has a vertex v of degree at least 3.
2) Prove that G or G^c contains a cycle of length 3. (Consider the adjacencies between the neighbours of vertex v above.)
3) Prove that at a meeting of at least 6 people, there are always 3 that mutually know each other, or 3 that mutually do not know each other.

2
Walks, connectivity and distance

2.1 In each of the following graphs, find paths of length 9 and 11, and cycles of length 5, 6, 8 and 9, if possible.





2.2 Prove that if G is a graph of minimum degree d, then G contains a path of length d.

2.3 A graph has order 13 and 3 connected components. Prove that one of the components has at least 5 vertices.

2.4 Use the algorithm DFS to find out whether the following graphs, given by their adjacency lists, are connected, and otherwise determine their connected components. Consider that the set of vertices is alphabetically ordered.

1)

a	b	c	d	e	f	g	h	i	j
d	d	h	a	a	a	b	c	b	b
e	g		b	d	d	i		g	g
f	i		e			j			
	j		f						

2)

a	b	c	d	e	f	g	h	i	j	k	l	m
b	a	f	b	b	c	b	b	c	a	c		g
j	d	i	h	g		e	d	k	b	i		
	e	k				m						
	g											
	h											
	j											

2.5 Prove that if a graph has exactly two vertices of odd degree, then there is a path from one of them to the other.

2.6 Let G be a graph of order n that has exactly two connected components, both of them being complete graphs. Prove that the size of G is at least $(n^2 - 2n)/4$.

2.7 Let G be a graph of order n with exactly k connected components. Prove that the size of G is larger than or equal to $n - k$.

2.8 Let G be a graph of order n with exactly $k+1$ connected components. In this exercise we want to find an upper bound for the size of G. Toward this end, we define an auxiliary graph H of order n that has $k+1$ connected components: k components are isomorphic to K_1 and one component is isomorphic to K_{n-k}.

1) Compute the size of H.
2) Prove that the size of H is larger than or equal to the size of G.

2.9 Prove that a 3-regular graph has a cut vertex if, and only if, it has some bridge.

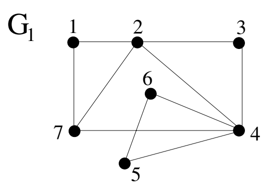
2.10 Find the smallest n for which there is a 3-regular graph of order n that has a bridge.

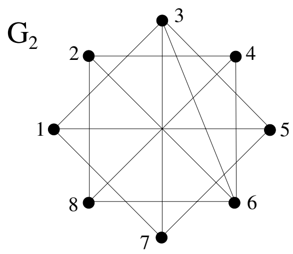
2.11 Let $G = (V, E)$ be a connected graph of order at least 2. Take $z \notin V$ and define $G + z$ as the graph that has $V \cup \{z\}$ as set of vertices and $E \cup \{zv : v \in V\}$ as set of edges. Prove that $G + z$ is 2-connected.

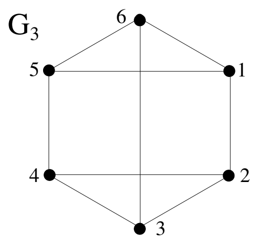
2.12 Let $G = (V, E)$ be a graph and v a vertex of G. Prove:

1) if G is not connected, then G^c is connected;
2) $(G - v)^c = G^c - v$;
3) if G is connected and v is a cut vertex of G, then v is not a cut vertex of G^c.

2.13 Find out whether any of the following graphs is 2-connected.







2.14 Let us consider the graphs from exercise 2.4. Using the algorithm BFS, find the distance from the vertices a and b to each of the other vertices of the connected component to which they belong.

2.15 Find the diameter of the following graphs.

1) K_n.
2) Graphs of exercise 2.1.
3) $K_{r,s}$.
4) C_n.
5) W_n.
6) P_n.

2.16 For each of the following statements, give a connected graph $G = (V, E)$ and a vertex $u \in V$ that satisfies it.

1) $D(G) = D(G - u)$.
2) $D(G) < D(G - u)$.
3) $D(G) > D(G - u)$.

Note: $D(G)$ is the diameter of G.

2.17 Let $G = (V, E)$ be a connected graph and $v \in V$. Let us introduce the following concepts:

- The *eccentricity of the vertex* v, $e(v)$, is the maximum of the distances from v to any other vertex of the graph, that is, $e(v) = \max\{d(v, x) : x \in V\}$.
- The *radius of* G, $r(G)$, is the minimum of the eccentricities of the vertices of G, that is, $r(G) = \min\{e(v) : v \in V\}$.
- A *central vertex of* G is a vertex u such that $e(u) = r(G)$.

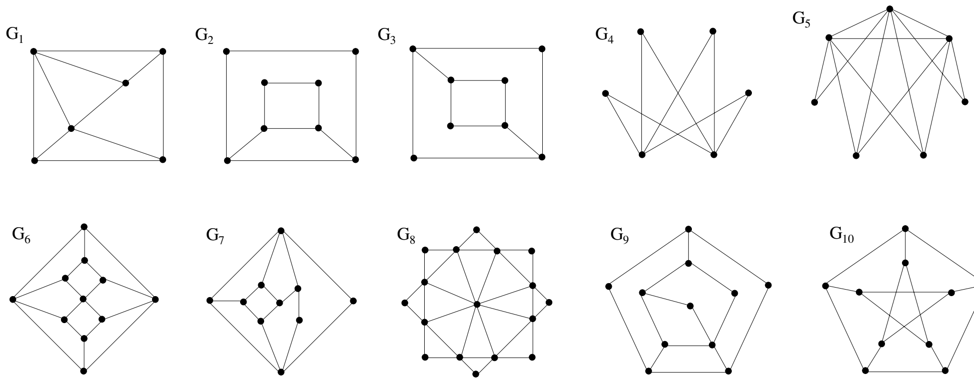
Answer the following questions.

1) Find the eccentricities, the radius and the central vertices of: a) the graphs from exercise 2.1; b) $G = ([8], \{12, 14, 15, 23, 34, 38, 46, 47, 56, 67, 78\})$.
2) Give an example of a graph with the same radius and diameter.
3) Give an example of a graph whose diameter is twice its radius.
4) Prove that, for each graph G, $r(G) \leq D(G) \leq 2r(G)$, where $D(G)$ is the diameter of G.

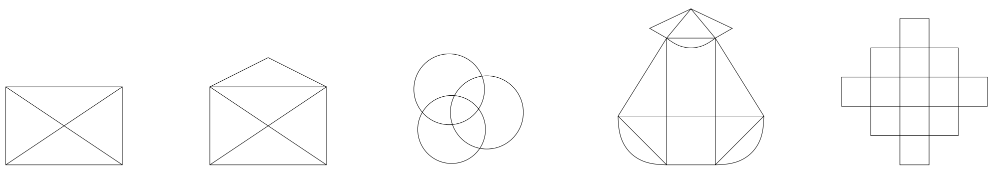
2.18 Let G be a graph of order 1001 so that each vertex has degree ≥ 500. Prove that G has diameter ≤ 2.

3
Eulerian and Hamiltonian graphs

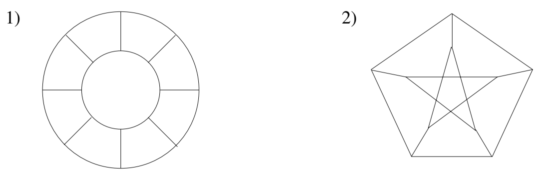
3.1 For each of the following graphs, either find an Eulerian circuit or prove that there is not one.



3.2 Find out if the following figures can be drawn without lifting the pencil from the paper and without repeating any line.



3.3 Determine the minimum number of times that one needs to lift the pencil from the paper to draw each of the figures below without repeating any line.



3.4 Find out for which values of r and s the complete bipartite graph $K_{r,s}$ is Eulerian.

3.5 Let G be a graph with exactly two connected components, both being Eulerian. Which is the minimum number of edges that need to be added to G to obtain an Eulerian graph?

3.6 Prove that a connected graph in which each vertex has even degree is bridgeless.

3.7 Find out if it is possible to put all the pieces of a domino set in a row so that the when two pieces are adjacent the values of the touching sides match, and moreover that the values at either end of the row also agree. If it is possible, give an explicit solution.

3.8 The *n-cube* is the graph Q_n with set of vertices $\{0,1\}^n$ and where two vertices $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ are adjacent if they differ exactly in one coordinate.

1) Draw Q_i for $1 \leq i \leq 4$.
2) Determine the order, the size and the degree sequence of Q_n.
3) Find for which values of n the graph Q_n is Eulerian.

3.9 For each of the graphs from exercise 3.1, either find a Hamiltonian cycle or prove that there is none.

3.10 Prove that if a bipartite graph is Hamiltonian, then the stable parts have the same cardinal.

3.11 Prove that a bipartite graph $K_{r,s}$ of order ≥ 3 is Hamiltonian if, and only if, $r = s$.

3.12 Let G be a graph that has exactly two connected components, both of them Hamiltonian graphs. Find the minimum number of edges that one needs to add to G to obtain a Hamiltonian graph.

3.13 Let G be a Hamiltonian graph that is not a cycle. Prove that G has at least 2 vertices of degree ≥ 3.

3.14 Cameron and Robin have rented an apartment together. They throw a dinner party where 10 other friends are invited. In the group of 12 people, each of them knows at least 6 other people. Prove that they can seat at a round table in such a way that everyone knows the two people sitting next to them.

At the last minute another person arrives, who also knows at least 6 of the people present. Can you ensure now that they can still sit at the table following the previous condition?

3.15 Let G be a d-regular graph of order $\geq 2d+2$, for $d \geq 1$. Prove that the complement of G is Hamiltonian.

3.16 Let G be a graph of order $n \geq 2$ such that each vertex has degree $\geq (n-1)/2$. Prove that G has a Hamiltonian path.

4
Trees

4.1 For each integer $n \geq 1$, let a_n be the number of non-isomorphic trees of order n. Check the values in the following table:

n	1	2	3	4	5	6	7
a_n	1	1	1	2	3	6	11

4.2 Prove that a tree of order $n \geq 2$ is a bipartite graph.

4.3 Let T_1 be a tree of order n and size 17, and let T_2 be a tree of order $2n$. Find n and the order and the size of T_2.

4.4 Find how many trees of order n there are with...

1) ...maximum degree $n-2$.
2) ...maximum degree $n-3$.

4.5 Let T be a tree of order 12 that has exactly 3 vertices of degree 3 and exactly one vertex of degree 2.

1) Find the degree sequence of T.
2) Find two non-isomorphic trees with this degree sequence.

4.6 Find a connected graph that is not a tree but in which every vertex of degree ≥ 2 is a cut vertex.

4.7

1) Let T be a tree of order $n \geq 2$. Prove that the number of leaves of T is

$$2 + \sum_{d(u) \geq 3} (d(u) - 2).$$

2) Let Δ be the maximum degree of T and let n_i be the number of vertices of degree i of T. Check that the previous formula can be written as

$$n_1 = 2 + \sum_{i=2}^{\Delta} (i-2)n_i.$$

3) Let G be now a connected graph of maximum degree Δ and with n_i vertices of degree i, for every i. Prove that the equality

$$n_1 = 2 + \sum_{i=2}^{\Delta} (i-2)n_i,$$

implies that G is a tree.

4.8 Let G be a connected graph where each vertex has degree either 1 or 4. Let k be the number of vertices of degree 4. Prove that G is a tree if, and only if, the number of vertices of degree 1 is $2k+2$.

4.9 Let T be a tree of order $n \geq 2$ and maximum degree Δ. Prove that T has at least Δ leaves.

4.10 Let T be a tree of order $n \geq 3$. Prove that the following statements are equivalent:

a) T is isomorphic to the star $K_{1,n-1}$.

b) T has exactly $n-1$ leaves.

c) T has maximum degree $n-1$.

d) T has diameter equal to 2.

4.11 Let G be a graph of order n and size m. Prove that the following statements are equivalent:

a) The graph G is connected and has only one cycle.

b) There is an edge e of G such that $G - e$ is a tree.

c) The graph G is connected and $n = m$.

4.12 Compute the number of spanning trees of the cycle graph C_n and of the complete bipartite graph $K_{2,r}$.

4.13 Give two non-isomorphic graphs of order $n \geq 4$ that have the following property: when we apply the algorithm BFS with initial vertex v, we obtain a star graph $K_{1,n-1}$ in which v is a leaf. Point out which is vertex v in each of the two graphs.

4.14 We apply the algorithm DFS to the complete graph $K_{r,r+3}$. How many non-isomorphic trees can we obtain, depending on the initial vertex?

4.15 Show that if T is a spanning tree of G, then the leaves of T are not cut vertices of G. Deduce that a connected graph of order ≥ 2 has at least two vertices that are not cut vertices.

4.16 Find the Prüfer sequences of the following trees:

$$\begin{aligned} T_1 &= ([6], \{12,\ 13,\ 14,\ 15,\ 56\}). \\ T_2 &= ([8], \{12,\ 13,\ 14,\ 18,\ 25,\ 26,\ 27\}). \\ T_3 &= ([11], \{12,\ 13,\ 24,\ 25,\ 36,\ 37,\ 48,\ 49,\ 5\ 10,\ 5\ 11\}). \end{aligned}$$

4.17 Find the trees that have the following Prüfer sequences:

1) (4,4,3,1,1), 2) (6,5,6,5,1), 3) (1,8,1,5,2,5), 4) (4,5,7,2,1,1,6,6,7).

4.18 Determine the trees whose Prüfer sequences have length 1.

4.19 Determine the trees whose Prüfer sequences are constant.

Review exercises

A.1 Find the adjacency matrix and the incidence matrix of the graph $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{ab, ac, bc, bd, cd, ce, de\}$.

A.2 Give the adjacency list and a drawing of the graph $G = ([5], E)$ whose adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

A.3 Prove that if the order of a graph is a multiple of 4 and the size is odd, then the graph is not regular.

A.4 Prove that if a graph G has minimum degree 1, maximum degree k and order $n > 2k$, then G has at least 3 vertices with the same degree.

A.5 Let G be a graph of order ≥ 7 such that each vertex has degree > 5. Prove that G has size ≥ 21.

A.6 Let $n \geq 3$ and $0 \leq k \leq n$ be integers, and consider the complete graph K_n with $[n]$ as set of vertices.

1) Compute the size of the subgraph induced by $[k]$.
2) Compute how many edges have an end in $[k]$ and the other in $[n] \setminus [k]$.
3) Compute the size of the subgraph induced by $[n] \setminus [k]$.
4) Using the previous results, prove that

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}.$$

A.7 Find, up to isomorphism, all 4-regular graphs of order 7.

A.8 Let G be a self-complementary graph of order n, for $n \equiv 1 \pmod 4$. Prove that there is an odd number of vertices of degree $(n-1)/2$ and, therefore, that G contains at least one vertex of degree $(n-1)/2$.

A.9 Consider the graph $G = (V, E)$ where $V = \{1, 2, \ldots, 15\}$ and two vertices i, j are adjacent if, and only if, their greatest common divisor is different than 1. Give the number of connected components of G and a path of maximum length.

A.10 Let G be a graph of order n and size m that does not have any cycle of length 3.

1) Prove that if u and v are adjacent vertices of G, then $d(u) + d(v) \leq n$.

2) Prove that if $n = 2k$, then $m \leq k^2$. *Hint:* Induction on $k \geq 1$.

A.11 Prove that in a connected graph two paths of maximum length have at least one vertex in common, but not necessarily an edge in common.
Hint: Assume that two paths of maximum length do not have any vertex in common and see if you can construct a path longer than the starting two.

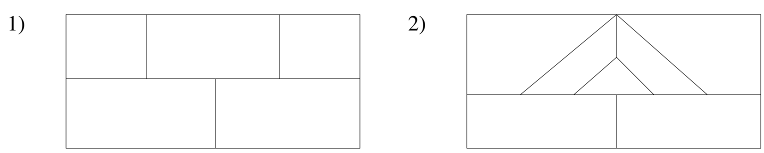
A.12 Let G be a connected bipartite graph of order $n \geq 3$ and d-regular. Prove that G is bridgeless.

A.13 Let G be a non-bipartite connected graph. Prove that between any two vertices of G there is a walk of odd length and a walk of even length.
Hint: the characterization of bipartite graphs can be useful.

A.14 Prove that if a graph is regular with even order and odd size, then it is not Eulerian.

A.15 Let G be a graph of odd order such that G and G^c are connected. Prove that G is Eulerian if, and only if, G^c is Eulerian.

A.16 In each of the following cases, find out if it is possible to draw a closed continuous line that crosses exactly once each interior segment of the rectangle.



A.17 Let G be a bipartite graph that has a Hamiltonian path and let V_1 and V_2 be the stable parts. Prove that $\big||V_1| - |V_2|\big| \leq 1$.

A.18 Prove that if $n \geq 1$ and $m = n + 1$, then the complete bipartite graph $K_{m,n}$ has a Hamiltonian path.

A.19 Seven people that assist to a conference want to have lunch together at a roundtable during the three days that the conference lasts. In order to get to know each other better, they decide to sit in such a way that two people are next to each other at most once. Can they achieve their goal? And what happens if the congress lasts 5 days?

A.20 Let G be a Hamiltonian graph that is not a cycle. Prove that if G has two non-adjacent vertices of degree 3, then it has at least another vertex of degree ≥ 3.

A.21 Prove that if G is a graph of order n and size $\geq \binom{n-1}{2} + 2$, then G is Hamiltonian.
Hint: use the Ore's theorem.

A.22 Find all graphs G such that G and G^c are trees.

A.23 Compute the number of edges that must be added to a forest of k connected components to obtain a tree.

A.24 Let T be a tree of order 7 with at least three vertices of degree 1 and at least two vertices of degree 3.

1) Find the degree sequence of T.

2) Find, up to isomorphism, all the trees that have this degree sequence.

A.25 Prove that if G is a graph of order ≥ 2 that has exactly one vertex of degree 1, then G has a cycle.

A.26 Prove that the following statements are equivalent for a tree T of order $n \geq 3$:

a) T is isomorphic to the path P_n.

b) T has maximum degree 2.

c) T has exactly 2 leaves.

d) T has diameter equal to $n - 1$.

A.27 Let G be a graph of order n and size $m = n - 1$ that is not a tree.

1) Prove that G has at least one connected component that is a tree and at least one that is not a tree.

2) Prove that if G has exactly two connected components, then the one that is not a tree has exactly one cycle.

A.28 Consider the wheel graph W_n of order $n \geq 4$. Give all the non-isomorphic trees that one can obtain by applying the algorithm BFS, depending on the initial vertex.

A.29 Give the Prüfer sequences that corresponds to each of the trees having [4] as set of vertices.

A.30 Determine the trees whose Prüfer sequences have all terms different.

1.31 We want to prove that a sequence of positive integers $d_1 \geq d_2 \geq \cdots \geq d_n \geq 1$ is the degree sequence of a tree of order $n \geq 2$ if, and only if, $d_1 + \cdots + d_n = 2(n-1)$. One of the implications is a direct consequence of the handshaking lemma (check it!). To prove the other implication, we will apply induction on n, according to the following steps:

1) Write down the implication that is not a consequence of the hanshaking lemma. Check the case $n = 2$. Write down the inductive hypothesis for $n-1$.
2) Let $n \geq 3$. Prove that if $d_1 + \cdots + d_n = 2(n-1)$ and $d_i \geq 1$ for each i, then $d_n = 1$ and $d_1 > 1$.
3) Apply the inductive hypothesis to $d_1 - 1, d_2, \ldots, d_{n-1}$ and deduce the desired result.

A.32 Let S be a set and $\mathcal{C}$ be a finite collection of subsets of S. The *intersection graph* $I(\mathcal{C})$ is the graph that has $\mathcal{C}$ as set of vertices and where two vertices $A, B \in \mathcal{C}$ are adjacent if $A \cap B \neq \emptyset$.

1) Let $S = [6]$ and $\mathcal{C} = \{\{1,2\},\{2,4\},\{1,2,3\},\{3,4,5\},\{5,6\}\}$. Draw the graph $I(\mathcal{C})$.
2) Consider the graph G that has [4] as set of vertices and edges 12, 23, 34 and 41. For each $i \in [4]$, consider the set S_i consisting of the vertex i and the two edges incident to i, that is: $S_1 = \{1, 12, 41\}, S_2 = \{2, 12, 23\}, S_3 = \{3, 23, 34\}, S_4 = \{4, 41, 34\}$. Let $S = S_1 \cup S_2 \cup S_3 \cup S_4$ and $\mathcal{C} = \{S_1, S_2, S_3, S_4\}$. Prove that $I(\mathcal{C})$ is isomorphic to G.
3) Prove that for any graph G, there exist a set S and a finite collection $\mathcal{C}$ of subsets of S such that G is isomorphic to the intersection graph $I(\mathcal{C})$.

A.33 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. Prove the following statements:

1) If G_1 and G_2 are connected, then $G_1 \times G_2$ is connected.
2) If G_1 and G_2 are Eulerian, then $G_1 \times G_2$ is Eulerian.
3) If $G_1 \times G_2$ is Eulerian, then G_1 and G_2 are either Eulerian or of even order.
4) If G is Hamiltonian, then $G \times K_2$ is Hamiltonian.

A.34 If G_1 is connected but G_2 is not, is the product $G_1 \times G_2$ connected?

A.35 Let $G = (V, E)$ be a graph. The *line graph* of G, LG, is the graph whose vertices are the edges of G and where two vertices of LG are adjacent if, as edges of G, they are incident.

1) Give the line graph of $K_{1,3}$, C_5 and $G = (\{1,2,3,4,5\},\{12,23,24,25,34,35,45\})$.

2) Give the order and the vertex degrees of LG in terms of the parameters of G.
3) Prove that if G is Eulerian, then LG is Hamiltonian.
4) Find a graph G such that LG is Hamiltonian but G is not Eulerian.
5) Prove that if G is Eulerian, then LG is Eulerian.
6) Find a graph G such that LG is Eulerian, but G is not.
7) Prove that if G is Hamiltonian, then LG is Hamiltonian.
8) Find a graph G such that LG is Hamiltonian, but G is not.