MATHEMATICS BOOK FOR MSQE

Further Reading Lists for MSQE & Other Entrances:

1. Mathematics for Economic Analysis by Sydsaeter Hammond
2. Mathematical Statistics by Gupta Kapoor
3. Basic Econometrics by Gujarati
4. Microeconomics by Satya Ranjan Chakraborty
5. Microeconomics by Hal Varian/Nicholson & Synder
6. Macroeconomics by H.L. Ahuja
7. Macroeconomics by Dornbusch & Fischer/Branson
8. Development Economics by Debraj Ray
9. Indian Economy by Sundaram &Dutt
10. An Introduction to Game Theory by Osborne

Arithmetic series

General (kth) term, $u_k = a + (k-1)d$
last (nth) term, $l = u_n = a + (n-1)d$
Sum to n terms, $S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a + (n-1)d]$

Geometric series

General (kth) term, $u_k = a\,r^{k-1}$

Sum to n terms, $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{a(r^n-1)}{r-1}$

Sum to infinity $S_\infty = \dfrac{a}{1-r}$, $-1 < r < 1$

Binomial expansions

When n is a positive integer

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots b^n, \; n \in \mathbb{N}$$

where

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!} \qquad \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

General case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots, \; |x| < 1, \; n \in \mathbb{R}$$

Logarithms and exponentials

$$e^{x\ln a} = a^x \qquad \log_a x = \frac{\log_b x}{\log_b a}$$

Numerical solution of equations

Newton-Raphson iterative formula for solving $f(x) = 0$, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Complex Numbers

$\{r(\cos\theta + j\sin\theta)\}^n = r^n(\cos n\theta + j\sin n\theta)$

$e^{j\theta} = \cos\theta + j\sin\theta$

The roots of $z^n = 1$ are given by $z = \exp(\frac{2\pi k}{n}j)$ for $k = 0, 1, 2, \ldots, n-1$

Finite series

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1) \qquad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Infinite series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r f^{(r)}(a)}{r!} + \ldots$$

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \ldots + \frac{x^r}{r!}f^{(r)}(a) + \ldots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots, \text{ all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots, \; -1 < x \leqslant 1$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r\frac{x^{2r+1}}{(2r+1)!} + \ldots, \text{ all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots, \text{ all } x$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r\frac{x^{2r+1}}{2r+1} + \ldots, \; -1 \leqslant x \leqslant 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots, \text{ all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots, \text{ all } x$$

$$\text{artanh}\, x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{(2r+1)} + \ldots, \; -1 < x < 1$$

Hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1$, $\sinh 2x = 2\sinh x\cosh x$, $\cosh 2x = \cosh^2 x + \sinh^2 x$

$\text{arsinh}\, x = \ln(x + \sqrt{x^2+1})$, $\text{arcosh}\, x = \ln(x + \sqrt{x^2-1})$, $x \geqslant 1$

$\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x| < 1$

Matrices

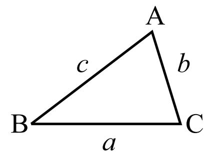
Anticlockwise rotation through angle θ, centre O: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Reflection in the line $y = x\tan\theta$: $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Cosine rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{(etc.)}$$

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{(etc.)}$$



Trigonometry

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}, \ [(\theta \pm \phi) \neq (k + \tfrac{1}{2})\pi]$$

For $t = \tan\frac{1}{2}\theta$: $\sin\theta = \dfrac{2t}{(1+t^2)}$, $\cos\theta = \dfrac{(1-t^2)}{(1+t^2)}$

$$\sin\theta + \sin\phi = 2\sin\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)$$

$$\sin\theta - \sin\phi = 2\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)$$

$$\cos\theta + \cos\phi = 2\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)$$

$$\cos\theta - \cos\phi = -2\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)$$

Vectors and 3-D coordinate geometry

(The position vectors of points A, B, C are **a**, **b**, **c**.)

The position vector of the point dividing AB in the ratio λ:μ is $\dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{(\lambda + \mu)}$

Line: Cartesian equation of line through A in direction **u** is

$$\frac{x - a_1}{u_1} = \frac{y - a_2}{u_2} = \frac{z - a_3}{u_3} \left(= t\right)$$

The resolved part of **a** in the direction **u** is $\dfrac{\mathbf{a}\,.\,\mathbf{u}}{|\mathbf{u}|}$

Plane: Cartesian equation of plane through A with normal **n** is

$$n_1 x + n_2 y + n_3 z + d = 0 \quad \text{where } d = -\mathbf{a}\,.\,\mathbf{n}$$

The plane through non-collinear points A, B and C has vector equation
$\mathbf{r} = \mathbf{a} + s(\mathbf{b} - \mathbf{a}) + t(\mathbf{c} - \mathbf{a}) = (1 - s - t)\,\mathbf{a} + s\mathbf{b} + t\mathbf{c}$
The plane through A parallel to **u** and **v** has equation
$\mathbf{r} = \mathbf{a} + s\mathbf{u} + t\mathbf{v}$

Perpendicular distance of a point from a line and a plane

Line: (x_1, y_1) from $ax + by + c = 0$: $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Plane: (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$: $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{(n_1^2 + n_2^2 + n_3^2)}}$

Vector product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$\mathbf{a}.\,(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \mathbf{b}.\,(\mathbf{c} \times \mathbf{a}) = \mathbf{c}.\,(\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c}\,.\,\mathbf{a})\,\mathbf{b} - (\mathbf{a}\,.\,\mathbf{b})\,\mathbf{c}$$

Conics

	Ellipse	**Parabola**	**Hyperbola**	**Rectangular hyperbola**
Standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$	$(ct, \frac{c}{t})$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm c\sqrt{2}, \pm c\sqrt{2})$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm c\sqrt{2}$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

Any of these conics can be expressed in polar coordinates (with the focus as the origin) as: $\dfrac{l}{r} = 1 + e\cos\theta$
where l is the length of the semi-latus rectum.

Mensuration

Sphere : Surface area $= 4\pi r^2$

Cone : Curved surface area $= \pi r \times$ slant height

Differentiation f(x)	f'(x)
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\text{cosec}^2 x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\arcsin x$	$\dfrac{1}{\sqrt{(1-x^2)}}$
$\arccos x$	$\dfrac{-1}{\sqrt{(1-x^2)}}$
$\arctan x$	$\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2 x$
$\text{arsinh}\, x$	$\dfrac{1}{\sqrt{(1+x^2)}}$
$\text{arcosh}\, x$	$\dfrac{1}{\sqrt{(x^2-1)}}$
$\text{artanh}\, x$	$\dfrac{1}{(1-x^2)}$

Integration f(x)	$\int$f(x) dx (+ a constant)
$\sec^2 kx$	$(1/k) \tan kx$
$\tan x$	$\ln \lvert\sec x\rvert$
$\cot x$	$\ln \lvert\sin x\rvert$
$\text{cosec}\, x$	$-\ln \lvert\text{cosec}\, x + \cot x\rvert = \ln \lvert\tan \frac{x}{2}\rvert$
$\sec x$	$\ln \lvert\sec x + \tan x\rvert = \ln \left\lvert\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\rvert$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a} \ln \left\lvert\dfrac{x-a}{x+a}\right\rvert$
$\dfrac{1}{\sqrt{(a^2-x^2)}}$	$\arcsin\left(\dfrac{x}{a}\right)$, $\lvert x\rvert < a$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a} \arctan\left(\dfrac{x}{a}\right)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a} \ln \left\lvert\dfrac{a+x}{a-x}\right\rvert = \dfrac{1}{a}\,\text{artanh}\left(\dfrac{x}{a}\right)$, $\lvert x\rvert < a$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\dfrac{1}{\sqrt{(a^2+x^2)}}$	$\text{arsinh}\left(\dfrac{x}{a}\right)$ or $\ln(x+\sqrt{x^2+a^2})$,
$\dfrac{1}{\sqrt{(x^2-a^2)}}$	$\text{arcosh}\left(\dfrac{x}{a}\right)$ or $\ln(x+\sqrt{x^2-a^2})$, $x > a$, $a > 0$

Quotient rule $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Trapezium rule $\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0+y_n) + 2(y_1+y_2+\ldots+y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

Integration by parts $\int u \dfrac{dv}{dx}\,dx = uv - \int v\dfrac{du}{dx}\,dx$

Area of a sector $A = \frac{1}{2}\int r^2\,d\theta$ (polar coordinates)

$A = \frac{1}{2}\int (x\dot{y} - y\dot{x})\,dt$ (parametric form)

Arc length $s = \int \sqrt{(\dot{x}^2+\dot{y}^2)}\,dt$ (parametric form)

$s = \int \sqrt{\left(1+\left[\dfrac{dy}{dx}\right]^2\right)}\,dx$ (cartesian coordinates)

$s = \int \sqrt{\left(r^2+\left[\dfrac{dr}{d\theta}\right]^2\right)}\,d\theta$ (polar coordinates)

Surface area of revolution $S_x = 2\pi\int y\,ds = 2\pi\int y\sqrt{(\dot{x}^2+\dot{y}^2)}\,dt$

$S_y = 2\pi\int x\,ds = 2\pi\int x\sqrt{(\dot{x}^2+\dot{y}^2)}\,dt$

Curvature

$$\kappa = \frac{d\psi}{ds} = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{\dfrac{d^2y}{dx^2}}{\left(1+\left[\dfrac{dy}{dx}\right]^2\right)^{3/2}}$$

Radius of curvature $\rho = \dfrac{1}{\kappa}$, Centre of curvature $\mathbf{c} = \mathbf{r} + \rho\,\hat{\mathbf{n}}$

L'Hôpital's rule

If f(a) = g(a) = 0 and g'(a) ≠ 0 then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$

Multi-variable calculus

grad $g = \begin{pmatrix} \partial g/\partial x \\ \partial g/\partial y \\ \partial g/\partial z \end{pmatrix}$ For $w = g(x, y, z)$, $\delta w = \dfrac{\partial w}{\partial x}\delta x + \dfrac{\partial w}{\partial y}\delta y + \dfrac{\partial w}{\partial z}\delta z$

Centre of mass (uniform bodies)

Triangular lamina:	$\frac{2}{3}$ along median from vertex
Solid hemisphere of radius r:	$\frac{3}{8}r$ from centre
Hemispherical shell of radius r:	$\frac{1}{2}r$ from centre
Solid cone or pyramid of height h:	$\frac{1}{4}h$ above the base on the line from centre of base to vertex
Sector of circle, radius r, angle 2θ:	$\dfrac{2r\sin\theta}{3\theta}$ from centre
Arc of circle, radius r, angle 2θ at centre:	$\dfrac{r\sin\theta}{\theta}$ from centre
Conical shell, height h:	$\frac{1}{3}h$ above the base on the line from the centre of base to the vertex

Motion in polar coordinates

Motion in a circle

Transverse velocity: $v = r\dot{\theta}$

Radial acceleration: $-r\dot{\theta}^2 = -\dfrac{v^2}{r}$

Transverse acceleration: $\dot{v} = r\ddot{\theta}$

General motion

Radial velocity: $\dot{r}$

Transverse velocity: $r\dot{\theta}$

Radial acceleration: $\ddot{r} - r\dot{\theta}^2$

Transverse acceleration: $r\ddot{\theta} + 2\dot{r}\dot{\theta} = \dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})$

Moments as vectors

The moment about O of $\mathbf{F}$ acting at $\mathbf{r}$ is $\mathbf{r} \times \mathbf{F}$

Moments of inertia (uniform bodies, mass M)

Thin rod, length $2l$, about perpendicular axis through centre:	$\frac{1}{3}Ml^2$
Rectangular lamina about axis in plane bisecting edges of length $2l$:	$\frac{1}{3}Ml^2$
Thin rod, length $2l$, about perpendicular axis through end:	$\frac{4}{3}Ml^2$
Rectangular lamina about edge perpendicular to edges of length $2l$:	$\frac{4}{3}Ml^2$
Rectangular lamina, sides $2a$ and $2b$, about perpendicular axis through centre:	$\frac{1}{3}M(a^2 + b^2)$
Hoop or cylindrical shell of radius r about perpendicular axis through centre:	Mr^2
Hoop of radius r about a diameter:	$\frac{1}{2}Mr^2$
Disc or solid cylinder of radius r about axis:	$\frac{1}{2}Mr^2$
Disc of radius r about a diameter:	$\frac{1}{4}Mr^2$
Solid sphere of radius r about a diameter:	$\frac{2}{5}Mr^2$
Spherical shell of radius r about a diameter:	$\frac{2}{3}Mr^2$

Parallel axes theorem: $I_A = I_G + M(AG)^2$

Perpendicular axes theorem: $I_z = I_x + I_y$ (for a lamina in the (x, y) plane)

Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)\,.\,P(B|A)$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

Bayes' Theorem: $P(A_j|B) = \dfrac{P(A_j)P(B|A_j)}{\Sigma P(A_i)P(B|A_i)}$

Populations

Discrete distributions

X is a random variable taking values x_i in a discrete distribution with $P(X = x_i) = p_i$

Expectation: $\mu = E(X) = \Sigma x_i p_i$

Variance: $\sigma^2 = Var(X) = \Sigma(x_i - \mu)^2 p_i = \Sigma x_i^2 p_i - \mu^2$

For a function g(X): $E[g(X)] = \Sigma g(x_i) p_i$

Continuous distributions

X is a continuous variable with probability density function (p.d.f.) f(x)

Expectation: $\mu = E(X) = \int x\, f(x)\,dx$

Variance: $\sigma^2 = Var(X) = \int (x - \mu)^2 f(x)\,dx = \int x^2 f(x)\,dx - \mu^2$

For a function g(X): $E[g(X)] = \int g(x) f(x)\,dx$

Cumulative distribution function $F(x) = P(X \leqslant x) = \int_{-\infty}^{x} f(t)\,dt$

Correlation and regression For a sample of n pairs of observations (x_i, y_i)

$$S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}, \quad S_{yy} = \Sigma(y_i - \bar{y})^2 = \Sigma y_i^2 - \frac{(\Sigma y_i)^2}{n},$$

$$S_{xy} = \Sigma(x_i - \bar{x})(y_i - \bar{y}) = \Sigma x_i y_i - \frac{(\Sigma x_i)(\Sigma y_i)}{n}$$

Covariance $\dfrac{S_{xy}}{n} = \dfrac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{n} = \dfrac{\Sigma x_i y_i}{n} - \bar{x}\,\bar{y}$

Product-moment correlation: Pearson's coefficient

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left[\Sigma(x_i - \bar{x})^2\,\Sigma(y_i - \bar{y})^2\right]}} = \frac{\dfrac{\Sigma x_i y_i}{n} - \bar{x}\,\bar{y}}{\sqrt{\left[\left(\dfrac{\Sigma x_i^2}{n} - \bar{x}^2\right)\left(\dfrac{\Sigma y_i^2}{n} - \bar{y}^2\right)\right]}}$$

Rank correlation: Spearman's coefficient

$$r_s = 1 - \frac{6\Sigma d_i^2}{n(n^2 - 1)}$$

Regression

Least squares regression line of y on x: $y - \bar{y} = b(x - \bar{x})$

$$b = \frac{S_{xy}}{S_{xx}} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2} = \frac{\dfrac{\Sigma x_i y_i}{n} - \bar{x}\,\bar{y}}{\dfrac{\Sigma x_i^2}{n} - \bar{x}^2}$$

Estimates

Unbiased estimates from a single sample

$\bar{X}$ for population mean μ; Var $\bar{X} = \dfrac{\sigma^2}{n}$

S^2 for population variance σ^2 where $S^2 = \dfrac{1}{n-1}\Sigma(x_i - \bar{x})^2 f_i$

Probability generating functions

For a discrete distribution

$G(t) = E(t^X)$

$E(X) = G'(1)$; $Var(X) = G''(1) + \mu - \mu^2$

$G_{X+Y}(t) = G_X(t)\,G_Y(t)$ for independent X, Y

Moment generating functions:

$M_X(\theta) = E(e^{\theta X})$

$E(X) = M'(0) = \mu$; $E(X^n) = M^{(n)}(0)$

$Var(X) = M''(0) - \{M'(0)\}^2$

$M_{X+Y}(\theta) = M_X(\theta)\,M_Y(\theta)$ for independent X, Y

Markov Chains

$\mathbf{p}_{n+1} = \mathbf{p}_n\mathbf{P}$

Long run proportion $\mathbf{p} = \mathbf{pP}$

Bivariate distributions

Covariance $\quad \mathrm{Cov}(X, Y) = \mathrm{E}[(X - \mu_X)(Y - \mu_Y)] = \mathrm{E}(XY) - \mu_X\mu_Y$

Product-moment correlation coefficient $\quad \rho = \dfrac{\mathrm{Cov}(X, Y)}{\sigma_X\,\sigma_Y}$

Sum and difference

$\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) \pm 2ab\ \mathrm{Cov}\ (X,Y)$

If X, Y are independent: $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

$\mathrm{E}(XY) = \mathrm{E}(X)\ \mathrm{E}(Y)$

Coding

$$\left.\begin{array}{l} X = aX' + b \\ Y = cY' + d \end{array}\right\} \Rightarrow \mathrm{Cov}(X, Y) = ac\ \mathrm{Cov}(X', Y')$$

Analysis of variance

One-factor model: $x_{ij} = \mu + \alpha_i + \varepsilon_{ij}$, where $\varepsilon_{ij} \sim \mathrm{N}(0,\sigma^2)$

$$SS_B = \sum_i n_i(\bar{x}_i - \bar{x})^2 = \sum_i \frac{T_i^2}{n_i} - \frac{T^2}{n}$$

$$SS_T = \sum_i\sum_j (x_{ij} - \bar{x})^2 = \sum_i\sum_j x_{ij}^2 - \frac{T^2}{n}$$

Regression

Y_i	RSS	No. of parameters, p
$\alpha + \beta x_i + \varepsilon_i$	$\Sigma(y_i - a - bx_i)^2$	2
$\alpha + \beta \mathrm{f}(x_i) + \varepsilon_i$	$\Sigma(y_i - a - b\mathrm{f}(x_i))^2$	2
$\alpha + \beta x_i + \gamma z_i + \varepsilon_i$	$\Sigma(y_i - a - bx_i - cz_i)^2$	3

$\varepsilon_i \sim \mathrm{N}(0, \sigma^2)$ $\quad a, b, c$ are estimates for α, β, γ. $\quad \hat{\sigma}^2 = \dfrac{\mathrm{RSS}}{n - p}$

For the model $Y_i = \alpha + \beta x_i + \varepsilon_i$,

$$b = \frac{S_{xy}}{S_{xx}}\ ,\quad b \sim \mathrm{N}\left(\beta, \frac{\sigma^2}{S_{xx}}\right)\ ,\quad \frac{b - \beta}{\sqrt{\hat{\sigma}^2 / S_{xx}}} \sim t_{n-2}$$

$$a = \bar{y} - b\bar{x}\ ,\ a \sim \mathrm{N}\left(\alpha, \frac{\sigma^2\Sigma x_i^2}{n\,S_{xx}}\right)$$

$$a + bx_0 \sim \mathrm{N}\left(\alpha + \beta x_0, \sigma^2\left\{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right\}\right)$$

$$\mathrm{RSS} = S_{yy} - \frac{(S_{xy})^2}{S_{xx}} = S_{yy}(1 - r^2)$$

Randomised response technique

$$\mathrm{E}(\hat{p}) = \frac{\frac{y}{n} - (1 - \theta)}{(2\theta - 1)} \qquad \mathrm{Var}(\hat{p}) = \frac{[(2\theta - 1)\,p + (1 - \theta)][\theta - (2\theta - 1)p]}{n(2\theta - 1)^2}$$

Factorial design

Interaction between 1st and 2nd of 3 treatments

$$(-)\left\{\frac{(Abc - abc) + (AbC - abC)}{2} - \frac{(ABc - aBc) + (ABC - aBC)}{2}\right\}$$

Exponential smoothing

$$\hat{y}_{n+1} = \alpha y_n + \alpha(1 - \alpha)y_{n-1} + \alpha(1 - \alpha)^2 y_{n-2} + \ldots + \alpha(1 - \alpha)^{n-1} y_1 + (1 - \alpha)^n y_0$$

$$\hat{y}_{n+1} = \hat{y}_n + \alpha(y_n - \hat{y}_n)$$

$$\hat{y}_{n+1} = \alpha y_n + (1 - \alpha)\,\hat{y}_n$$

Description	Test statistic	Distribution
Pearson's product moment correlation test	$r = \dfrac{\dfrac{\sum x_i y_i}{n} - \bar{x}\,\bar{y}}{\sqrt{\left[\left(\dfrac{\sum x_i^2}{n} - \bar{x}^2\right)\left(\dfrac{\sum y_i^2}{n} - \bar{y}^2\right)\right]}}$	
Spearman rank correlation test	$r_s = 1 - \dfrac{6\sum d_i^2}{n(n^2-1)}$	
Normal test for a mean	$\dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}}$	N(0, 1)
t-test for a mean	$\dfrac{\bar{x} - \mu}{s / \sqrt{n}}$	t_{n-1}
χ^2 test	$\sum \dfrac{(f_o - f_e)^2}{f_e}$	χ^2_{ν}
t-test for paired sample	$\dfrac{(\bar{x}_1 - \bar{x}_2) - \mu}{s/\sqrt{n}}$	*t* with $(n-1)$ degrees of freedom
Normal test for the difference in the means of 2 samples with different variances	$\dfrac{(\bar{x} - \bar{y}) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$	N(0, 1)

Description	Test statistic	Distribution
t-test for the difference in the means of 2 samples	$\dfrac{(\bar{x} - \bar{y}) - (\mu_1 - \mu_2)}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ where $s^2 = \dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2}$	$t_{n_1 + n_2 - 2}$
Wilcoxon single sample test	A statistic T is calculated from the ranked data.	See tables
Wilcoxon Rank-sum (or Mann-Whitney) 2-Sample test	Samples size m, n: $m \leqslant$ n Wilcoxon W = sum of ranks of sample size m Mann-Whitney $T = W - \frac{1}{2}m(m+1)$	See tables
Normal test on binomial proportion	$\dfrac{p - \theta}{\sqrt{\left(\dfrac{\theta(1-\theta)}{n}\right)}}$	N(0, 1)
χ^2 test for variance	$\dfrac{(n-1)s^2}{\sigma^2}$	χ^2_{n-1}
F-test on ratio of two variances	$\dfrac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}$, $s_1^2 > s_2^2$	$F_{n_1-1,\, n_2-1}$

Name	Function	Mean	Variance	p.g.f. G(t) (discrete) m.g.f. M(θ) (continuous)
Binomial B(n, p) *Discrete*	$P(X = r) = {}^nC_r\, q^{n-r} p^r$, for $r = 0, 1, \dots, n$, $0 < p < 1$, $q = 1 - p$	np	npq	$G(t) = (q + pt)^n$
Poisson (λ) *Discrete*	$P(X = r) = e^{-\lambda} \dfrac{\lambda^r}{r!}$, for $r = 0, 1, \dots$, $\lambda > 0$	λ	λ	$G(t) = e^{\lambda(t-1)}$
Normal N(μ, σ^2) *Continuous*	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right)$, $-\infty < x < \infty$	μ	σ^2	$M(\theta) = \exp(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)$
Uniform (Rectangular) on [a, b] *Continuous*	$f(x) = \dfrac{1}{b-a}$, $a \leqslant x \leqslant b$	$\dfrac{a+b}{2}$	$\tfrac{1}{12}(b-a)^2$	$M(\theta) = \dfrac{e^{b\theta} - e^{a\theta}}{(b-a)\theta}$
Exponential *Continuous*	$f(x) = \lambda e^{-\lambda x}$, $x \geqslant 0$, $\lambda > 0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$M(\theta) = \dfrac{\lambda}{\lambda - \theta}$
Geometric *Discrete*	$P(X = r) = q^{r-1}p$, $r = 1, 2, \dots$, $0 < p < 1$, $q = 1 - p$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$	$G(t) = \dfrac{pt}{1 - qt}$
Negative binomial *Discrete*	$P(X = r) = {}^{r-1}C_{n-1}\, q^{r-n} p^n$, $r = n, n+1, \dots$, $0 < p < 1$, $q = 1 - p$	$\dfrac{n}{p}$	$\dfrac{nq}{p^2}$	$G(t) = \left(\dfrac{pt}{1-qt}\right)^n$

Numerical Solution of Equations

The Newton-Raphson iteration for solving $f(x) = 0$: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Numerical integration

The trapezium rule

$$\int_a^b y\,dx \approx \tfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}, \text{ where } h = \frac{b-a}{n}$$

The mid-ordinate rule

$$\int_a^b y\,dx \approx h(y_{\frac{1}{2}} + y_{1\frac{1}{2}} + \dots + y_{n-1\frac{1}{2}} + y_{n-\frac{1}{2}}), \text{ where } h = \frac{b-a}{n}$$

Simpson's rule

for n even

$$\int_a^b y\,dx \approx \tfrac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\},$$

$$\text{where } h = \frac{b-a}{n}$$

The Gaussian 2-point integration rule

$$\int_{-h}^{h} f(x)\,dx \approx h\left[f\left(\frac{-h}{\sqrt{3}}\right) + f\left(\frac{h}{\sqrt{3}}\right)\right]$$

Interpolation/finite differences

Lagrange's polynomial : $P_n(x) = \sum L_r(x)f(x)$ where $L_r(x) = \prod\limits_{\substack{i=0\\ i\neq r}}^{n} \dfrac{x - x_i}{x_r - x_i}$

Newton's forward difference interpolation formula

$$f(x) = f(x_0) + \frac{(x - x_0)}{h}\Delta f(x_0) + \frac{(x - x_0)(x - x_1)}{2!h^2}\Delta^2 f(x_0) + \dots$$

Newton's divided difference interpolation formula

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \dots$$

Numerical differentiation

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Taylor polynomials

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \text{error}$$

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a + \xi),\ \ 0 < \xi < h$$

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \text{error}$$

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(\eta),\ a < \eta < x$$

Numerical solution of differential equations

For $\dfrac{dy}{dx} = f(x, y)$:

Euler's method : $y_{r+1} = y_r + hf(x_r, y_r)$; $x_{r+1} = x_r + h$

Runge-Kutta method (order 2) (modified Euler method)

$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$

where $k_1 = h\,f(x_r, y_r)$, $k_2 = h\,f(x_r + h, y_r + k_1)$

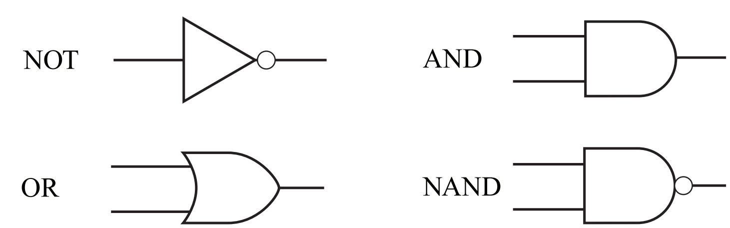
Runge-Kutta method, order 4:

$$y_{r+1} = y_r + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where $k_1 = hf(x_r, y_r)$ $k_2 = hf(x_r + \frac{1}{2}h, y_r + \frac{1}{2}k_1)$

$k_3 = hf(x_r + \frac{1}{2}h, y_r + \frac{1}{2}k_2)$ $k_4 = hf(x_r + h, y_r + k_3)$.

Logic gates



LIMIT FORMULAE

1. $\lim_{x\to a}\frac{x^a-a^n}{x-a}=na^{n-1}$.
2. $\lim_{x\to 0}\frac{sinx}{x}=\lim_{x\to 0}\frac{x}{sinx}=\lim_{x\to 0}\frac{sin^{-1}x}{x}=\lim_{x\to 0}\frac{x}{sin^{-1}x}=1$.
3. $\lim_{x\to\infty}\frac{sinx}{x}=0$.
4. $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$.
5. $\lim_{x\to 0}\frac{e^x-1}{x}=1$.
6. $\lim_{x\to 0}\frac{log_e(1+x)}{x}=1$.
7. $\lim_{x\to 0}\frac{a^x-1}{x}=log_e a\,,(a>0)$.
8. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$, $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$.
9. $\lim_{x\to\infty}\frac{logx}{x}=\lim_{x\to\infty}\frac{\log[x]}{x}=0$.
10. $\lim_{x\to\infty}\frac{x^n}{e^x}=0,(n>0)$, $\lim_{x\to\infty}\frac{e^x}{x^n}=\infty,(n>0)$, $\lim_{x\to\infty}\frac{x^n}{n!}=0$.
11. $\lim_{x\to 0}\frac{tanx}{x}=\lim_{x\to 0}\frac{tan^{-1}x}{x}=1$.
12. $\lim_{n\to\infty}\left(1+\frac{\mu}{n}\right)^n=e^{\mu}=\lim_{n\to\infty}(1+\mu n)^{\frac{1}{n}}$.
13. $\lim\limits_{x\to-\infty}e^x=0$, $\lim\limits_{x\to\infty}e^{-x}=0$.
14. $\lim_{x\to\infty}x^n=\begin{cases}0 & if\ \mid x\mid<1\\ \pm1 & \text{if } \mid x\mid=1\\ \pm\infty & if\ \mid x\mid>1\end{cases}$, $\lim_{n\to\infty}x^n=0\,,(-1<x<1)$.
15. $\lim_{x\to 0+}\frac{1}{x}=\infty$, $\lim_{x\to 0-}\frac{1}{x}=-\infty$, so $\lim_{x\to 0}\frac{1}{x}$ does not exist.
16. $\lim_{x\to 0}\frac{1}{x^2}=\infty$ but $\lim_{x\to\infty}\frac{1}{x^2}=0$.
17. $\lim_{x\to\infty}x^{\frac{1}{x}}=1$.
18. $\lim_{x\to a}\mid x\mid=\mid a\mid$.
19. $\lim\limits_{x\to\infty}xsin\frac{1}{x}=1$.
20. $\lim\limits_{x\to 0}sin\frac{1}{x}$ does not exist .

21. $\lim\limits_{x\to 0} x sin\frac{1}{x} = \lim\limits_{x\to 0} x^2 sin\frac{1}{x} = \lim\limits_{x\to 0} x sin\frac{1}{x^2} = \lim\limits_{x\to 0} x cos\frac{1}{x} = \lim\limits_{x\to 0} x cos\frac{1}{x^2} = \lim\limits_{x\to 0} \sqrt{x} sin\frac{1}{x} = 0$.
22. $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$
23. $\lim_{x\to a}(fog)(x) = f(\lim_{x\to a} g(x))$.
24. $\lim_{x\to a-}[x] = a-1$, $\lim_{x\to a+}[x] = a$, $a \in \mathbb{Z}$ (set of integers) .
25. $\lim_{x\to\infty} \frac{ae^x+be^{-x}}{ce^x+de^{-x}} = \frac{a}{c}$ $(c \neq 0)$, $\lim_{x\to -\infty} \frac{ae^x+be^{-x}}{ce^x+de^{-x}} = \frac{b}{d}$ $(d \neq 0)$.

SOME IMPORTANT FORMULAE TO REMEMBER

1. $(cos\theta + \mathrm{i}\sin\theta)^n = cosn\theta + \mathrm{i}\sin n\theta$.
2. $e^{i\theta} = cos\theta + \mathrm{i}\sin\theta$, $e^{-i\theta} = cos\theta - \mathrm{i}\sin\theta$.
3. If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist then $\lim_{x\to a}[f(x) \pm g(x)]$ does not exist .
4. If $\lim_{x\to a}[f(x)g(x)]$ *exists then* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ may not exist . [e.g. f(x) = x , g(x) = 1/x]
5. Product and Ratio of two odd function is even .

SOME USEFUL RESULTS ON CONTINUOUS FUNCTION

If f and g are continuous at x = a then

1. f ± g is continuous at x = a
2. f . g is continuous at x = a
3. f / g is continuous at x = a , where g(a) ≠ 0
4. f[g(x)] is continuous at x = a
5. Every Polynomial function is continuous at every point of the real line .
6. Every Rational function is continuous at every point where its denominator is different from zero .

SOME USEFUL RESULTS ON RELATION AND MAPPING

1. Let n(A) = m , n(B) = n ; then the total number of relations from A to B is 2^{mn} .
2. Let f : A→B be a function where n(A) = m , n(B) = n , then total number of functions is n^m .
3. If f : A→B be a function where n(A) = m , n(B) = n , then total number of injection is $= \begin{cases} nPm & , n \geq m \\ 0 \ , & n < m \end{cases}$
4. If f : A→B be a function where n(A) = n , n(B) = n , then total number of bijection is n! .
5. Let f : A→B be a function where n(A) = m , n(B) = n , then total number of onto functions is $= \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} r^m$; $1 \leq n \leq m$.

SOME USEFUL RESULTS ON APPLIED CALCULUS

1. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$, *where* $m > 0 , n > 0$ is called Beta Integral.
2. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$, *where* $n > 0$ is called Gamma Integral .
3. $\Gamma(n+1) = n\Gamma(n) = n!$, $\Gamma(n) = (n-1)!$, $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
4. B(m,n) = B(n,m) , where $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.
5. $\Gamma(m)\,\Gamma(1-m) = \pi \operatorname{cosec} m\pi$, $0 < m < 1$.
6. $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.
7. $\int_0^{\pi/2} \sin^p\theta \, . \cos^q\theta \, d\theta = \frac{1}{2} . B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$, *where* $p > -1, q > -1$.
8. $\int_0^\infty \frac{\sin bx}{x} dx = \pm \frac{\pi}{2}$ *according as* $b > 0 , b < 0$.
9. $\int_0^{\pi/2} \sin^p\theta \, d\theta = \int_0^{\pi/2} \cos^q\theta \, d\theta = \frac{\pi}{2} . \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p+2}{2})}$ (p > -1) .
10. $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$, $k > 0$.

SERIES FORMULAE

1. $(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$.
2. $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$
3. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \; -\infty < x < \infty$
4. $a^x = 1 + \frac{xloga}{1!} + \frac{(xloga)^2}{2!} + \frac{(xloga)^3}{3!} + \cdots$
5. $(1-t)^{-r} = \sum_{x=0}^{\infty} \binom{x+r-1}{x} t^x$ for $| t | < 1$.
6. tanx = $x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$ [Use these kinds of expansions only when the variable in the expansion tends to 0]
7. $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$
8. $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$
9. $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$
10. $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$
11. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
12. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
13. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
14. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
15. $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \cdots$
16. $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \cdots$

Permutations and Combinations

Fundamental principle of counting:

There are two fundamental counting principles i.e. *Multiplication principle* and *Addition principle*.

Multiplication principle: If an operation can be performed independently in 'm' different ways, and another operation can be performed in 'n' different ways, then both operations can be performed by m x n ways.

In other words, if a job has n parts and the job will be completed only when each part is completed, and the first part can be completed in a_1 ways, the second part can be completed in a_2 ways and so on… the n^{th} part can be completed in a_n ways then the total number of ways of doing the jobs is $a_1.a_2.a_3\cdots\cdots a_n$.

Ex: - A person can travel from Sambalpur to Bargarh in four routes and Bargarh to Bolangir in five routes then the number of routes that the person can travel is from Sambalpur to Bolangir via Bargarh is 4 x 5 = 20 routes.

Addition principle: If one operation can be performed independently in 'm' different ways, a second operation can be performed in 'n' different ways, then there are(m + n) possible ways when one of these operations be performed.

Ex: - A person has 4 shirts and 5 pants. The number of ways he wears a pant or shirt is 4 + 5 = 9 ways

Problems:

1. There are three letters and three envelopes. Find the total number of ways in which letters can be put in the envelopes so that each envelope has only one letter. [Ans:6]
2. Find the number of possible outcomes of tossing a coin twice.[Ans:4]
3. In a class there are 20 boys and 15 girls. In how many ways can the teacher select one boy and one girl from amongst the students of the class to represent the school in a quiz competition?[Ans:300]
4. A teacher has to select either a boy or a girl from the class of 12 boys and 15 girls for conducting a school function. In how many ways can she do it?[Ans:27]
5. There are 5 routes from A to B and 3 routes from place B to C. Find how many different routes are there from A to C?[Ans:15]
6. How many three lettered codes is possible using the first ten letters of the English alphabets if no letter can be repeated?[Ans:720]
7. If there are 20 buses plying between places A and B, in how many ways can a round trip from A be made if the return journey is made on
   i) same bus[Ans:20]   ii) a different bus[Ans:380]
8. A lady wants to choose one cotton saree and one polyester saree from 10 cotton and 12 polyester sarees in a textile shop. In how many ways she can choose?[Ans:120]
9. How many three digit numbers with distinct digits can be formed with out using the digits 0, 2, 3, 4, 5, 6.[Ans:24]
10. How many three digit numbers are there between 100 and 1000 such that every digit is either 2 or 9?[Ans:8]
11. In how many ways can three letters be posted in four letter boxes?[Ans:64]
12. How many different signals can be generated by arranging three flags of different colors vertically out of five flags?[Ans:60]

13. In how many ways can three people be seated in a row containing seven seats?[Ans:210]
14. There are five colleges in a city. In how many ways can a man send three of his children to a college if no two of the children are to read in the same college?[Ans:60]
15. How many even numbers consisting of 4 digits can be formed by using the digits 1, 2, 3, 5, 7?[Ans:24]
16. How many four digit numbers can be formed with the digits 4,3,2,0 digits not being repeated?[Ans:18]
17. How many different words with two letters can be formed by using the letters of the word JUNGLE, each containing one vowel and one consonant?[Ans:16]
18. How many numbers between 99 and 1000 can be formed with the digits 0, 1, 2, 3, 4 and 5?[Ans:180]
19. There are three multiple choice questions in an examination. How many sequences of answers are possible, if each question has two choices?[Ans:8]
20. There are four doors leading to the inside of a cinema hall. In how many ways can a person enter into it and come out?[Ans:16]
21. Find the number of possible outcomes if a die is thrown 3 times.[Ans:216]
22. How many three digit numbers can be formed from the digits 1,2,3,4, and 5, if the repetition of the digits is not allowed.[Ans:60]
23. How many numbers can be formed from the digits 1,2,3, and 9 , if the repetition of the digits is not allowed.[Ans:24]
24. How many four digit numbers greater than 2300 can be formed with the digits 0,1,2,3,4,5 and 6, no digit being repeated in any number.[Ans:560]
25. How many two digit even numbers can be formed from the digits 1,2,3,4,5 if the digits can be repeated?[Ans:10]
26. How many three digits numbers have exactly one of the digits as 5 if repetition is not allowed?[Ans:200]
27. How many 5 digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 59 and no digit appears more than once.[Ans:210]
28. In how many ways can four different balls be distributed among 5 boxes, when
    i) no box has more than one ball[Ans:120]
    ii) a box can have any number of balls[Ans:625]
29. Rajeev has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with?[Ans:6]
30. Ali has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can he carry these items choosing one each?[Ans:12]
31. How many three digit numbers with distinct digits are there whose all the digits are odd?[Ans:60]
32. A team consists of 7 boys and 3 girls plays singles matches against another team consisting of 5 boys and 5 girls. How many matches can be scheduled between the two teams if a boy plays against a boy and a girl plays against a girl.[Ans:50]
33. How many non- zero numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if repetition of the digits is not allowed? [Ans:600]
34. In how many ways can five people be seated in a car with two people in the front seat including driver and three in the rear, if two particular persons out of the five can not drive?[Ans:72]
35. How many A.P's with 10 terms are there whose first term belongs to the set{1,2,3} and common difference belongs to the set {1,2,3,4,5}[Ans:15]

Factorial: The product of first n natural numbers is generally written as n! or $\angle n$ and is read factorial n.

Thus, n! = 1. 2. 3.………..n.

Ex: $6! = 6\times5\times4\times3\times2\times1 = 720$

Note:

1) 0! =1
2) (−r)! = ∞

Problems:

1. Evaluate the following:

   i) 7! ii) 5! iii) 8! iv) 8!−5! v) 4!−3! vii) 7!−5! viii) $\frac{6!}{5!}$

   ix) $\frac{7!}{5!}$ x) $\frac{8!}{6!2!}$ xi) $\frac{12!}{10!\,2!}$ xii) (3!)(5!) xiii) $\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}$ xiv) $2!^{3!}$

2. Evaluate $\frac{n!}{r!(n-r)!}$, when

   i) n=7, r=3 ii) n=15, r=12 iii) n=5, r=2

3. Evaluate $\frac{n!}{(n-r)!}$, when

   i) n=9, r=5 ii) n=6, r=2

4. Convert the following into factorials:

   i) 1.3.5.7.9.11 ii) 2.4.6.8.10 iii) 5.6.7.8.9 iv) (n+1)(n+2)(n+3)…………2n

5. Find x if

   i) $\frac{1}{5!}+\frac{1}{6!}=\frac{x}{7!}$ ii) $\frac{1}{8!}+\frac{1}{9!}=\frac{x}{10!}$

6. Find the value of n if

   i) (n+1)!=12(n−1)! ii) (2n)!n!=(n+1)(n−1)!(2n−1)!

7. If $\frac{n!}{2!(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2:1 find the value of n.

8. Find the value of x if $\frac{(x+2)!}{(2x-1)!}.\frac{(2x+1)!}{(x+3)!}=\frac{72}{7}$ where $x\in N$

9. Show that n!(n+2)=n!+(n+1)!

10. Show that 27! Is divisible by 2^{12}. What is the largest natural number n such that 27! is divisible by 2^n.

11. Show that 24! + 1 is not divisible by any number between 2 to 24.

12. Prove that $(n!)^2 \le n^n\, n! < (2n)!$

13. Find the value of x if $\frac{(2x+3)!}{(x+1)!}.\frac{(x-1)!}{(2x+1)!}=7$

14. Prove that the product of k consecutive positive integers is divisible by k! for $k\ge2$

15. Show that 2.6.10……..to n factors $=\frac{(2n)!}{n!}$.

Permutation:- The different arrangements which can be made by taking some or all at a time from a number of objects are called permutations. In forming permutations we are concerned with the order of the things. For example the arrangements which can be made by taking the letters a, b, c two at a time are six numbers, namely,

ab , bc, ca, ba, cb, ac

Thus the permutations of 3 things taken two at a time are 6.

a) Without repetition:

i) If there are n distinct objects then the number of permutations of n objects taking r at a time with out repetition is denoted by ${}^{n}p_{r}$ or p (n ,r) and is defined as

$$ {}^{n}p_{r} = \frac{n!}{(n-r)!} \quad , \; 0 \le r \le n $$

Proof: Arrangements of n objects, taken r at a time, is same to filling r places with n things

1^{st} place can be filled up in n ways

2^{nd} place can be filled up in n-1 ways

3^{rd} place can be filled up in n-2 ways

…………………………………………….

…………………………………………….

r^{th} place can be filled up in n-(r-1) ways

$\therefore$ the number of arrangements

$$ {}^{n}p_{r} = n(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots(n-(r-1)) $$

$$ = \frac{n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-r+1)(n-r)!}{(n-r)!} $$

$$ {}^{n}p_{r} = \frac{n!}{(n-r)!}. $$

ii) Number of arrangements of n different things taken all at a time without repetition $= {}^{n}p_{n} = \frac{n!}{(n-n)!} = n!$

b) With repetition:

i) If there are n distinct objects then the number of permutations of n objects taking r at a time with repetition is n^r.

ii) Number of arrangements of n different things taken all at a time with repetition is n^n.

c) If p objects of one kind, q objects of second kind are there then the total number of permutations of all the

p + q objects are given by $\frac{(p+q)!}{p!q!}$.

In general If a_i objects of i^{th} kind, i= 1, 2, 3…..,r are there then the number of permutations of all the $a_1 + a_2 + \cdots\cdots\cdots + a_r$ objects is given by $\frac{(a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_r)!}{a_1!a_2!\ldots\ldots\ldots\ldots a_r!}$.

d) Circular arrangements:

i) The number of circular arrangements of n distinct objects taking all at a time is (n−1)!

ii) The number of circular arrangements of n distinct objects when clockwise and anti-clockwise circular permutations are considered as same is $\frac{(n-1)!}{2}$.

iii) The number of circular permutations of n different things taken r at a time is $\frac{{}^n p_r}{r}$ (if clockwise and anti-clockwise circular permutations are considered as different)

Ex: The number of which 29 persons be seated in a round table if there are 9 chairs is $\frac{{}^{29} p_9}{9}$

iv) The number of circular permutations of n different things taken r at a time is $\frac{{}^n p_r}{2r}$ (if clockwise and anti-clockwise circular permutations are considered as same).

Restricted permutations:

1) The number of permutations of n dissimilar things taken r at a time when one particular thing always occurs is r ${}^{n-1}P_{r-1}$
2) The number of permutations of n dissimilar things taken r at a time when one particular thing taken is ${}^{n-1}P_r$.
3) The number of permutations of n dissimilar things taken r at a time when p particular things always occurs $= {}^{n-p}C_{r-p}.r!$
4) The number of permutations of n dissimilar things taken r at a time when p particular things never occurs ${}^{n-p}C_r.r!$

Zero Factorial:

The value of Zero factorial is 1 i.e. 0! =1

Proof:

By the fundamental principle of counting we know that the number of permutations of n different objects taken all at a time with out repetition is $n(n-1)(n-2)..........3.2.1 = n!$ ⋯⋯⋯(1)

And we have seen ${}^n p_r = \frac{n!}{(n-r)!}$ ⋯⋯⋯⋯⋯ (2)

From (2) the number of permutations of n different objects taken all at a time with out repetition is

$$ {}^n p_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} \cdots\cdots\cdots\cdots\cdots\cdots (3)$$

from (1) and (3) $n! = \frac{n!}{0!}$

and this can be hold true if 0! is 1.

$$\therefore 0! = 1$$

Problems:

1. Find r if P(20,r) = 13. P(20,r−1)
2. Find n if P(n,4) = 12. P(n,2)
3. If P(n−1,3): P(n+1,3) = 5 : 12, find n
4. Find m and n if P(m+n,2)=56, P(m−n,2)=12
5. Show that P(n, n) = P(n, n−1) for all positive integers.
6. Show that P(m, 1)+ P(n ,1) = P(m+n, 1) for all positive integers
7. Prove that P(n,n) = 2 P(n, n−2)
8. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1:9$
9. Find r if $5\ {}^{4}P_r = 6\ {}^{5}P_{r-1}$
10. If ${}^{n}P_5 = 42\ {}^{n}P_3$, for n>4, then find the value of n.
11. If ${}^{n}P_4 = 360$, find n.
12. If ${}^{n}P_3 = 9240$, find n.
13. If ${}^{10}P_r = 720$, find r.
14. Find n if ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$
15. Prove that ${}^{1}P_1 + 2\ {}^{2}P_2 + 3\ {}^{3}P_3 + 4\ {}^{4}P_4 + \cdots\cdots\cdots + n\ {}^{n}P_n = {}^{n+1}P_{n+1} - 1$
16. In how many ways can five people be arranged in a row? [Ans: 5!]
17. In how many ways can three guests be seated if there are six chairs in your home?[Ans: ${}^{6}p_3$]
18. How many four digit numbers are there, with no digit repeated?[Ans: 9. ${}^{9}p_3$]
19. How many numbers of four digits can be formed with the digits 1, 2,4,5,7 if no digit being repeated?[Ans: ${}^{5}p_4$]
20. How many even numbers of three digits can be formed with the digits 1, 2, 3,4,5,7 if no digit being repeated?[Ans: $2.{}^{5}p_2$
21. How many numbers between 100 and 1000 can be formed with the digits 1,2,3,4,5,6,7 if no digit being repeated? [Ans: ${}^{7}p_3$]
22. How many different numbers greater than 5000 can be formed with the digits 0,1,5,9 if no digit being repeated? [Ans:12]
23. In how many ways can four persons sit in a row?[Ans:4!]
24. In how many ways can three men and four women be arranged in a row such that all the men sit together?[[Ans:5!3!]
25. In how many ways can three men and four women be arranged in a row such that all the men and all the women will sit together?[Ans:2!3!4!]
26. In how many ways can 8 Indians, 4 English men and 4 Americans be seated in a row so that all the persons of the same nationality sit together? [Ans:3!8!4!4!]
27. In how many ways can 10 question papers be arranged so that the best and the worst papers never come together?[Ans:10!−2!9!]
28. In how many ways can 5 boys and 3 girls be seated in a row so that all the three girls do not sit together?[Ans:8!−3!6!]
29. In how many ways can 5 boys and 4 girls be seated in a row so that no two girls sit together?[Ans: ${}^{7}p_4 5!$]
30. In how many ways the word MISSISSIPPI can be arranged?[Ans: $\frac{11!}{4!4!2!}$]
31. In how many ways the word MISSISSIPPI can be rearranged? [Ans: $\frac{8!}{4!4!2!} - 1$]
32. In how many ways the word GANESH can be arranged?[Ans:6!]

33.In how many ways can the word CIVILIZATION be arranged so that four I's come together?[Ans:9!]

34.In how many ways can 4 boys and 4 girls be seated in a row so that boys and girls occupy alternate seats?[2.4!.4!]

35.In a class there are 10 boys and 3 girls. In how many ways can they be arranged in a row so that no two girls come consecutive?[${}^{11}p_3 10!$]

36.How many different words can be formed with the letters of the word UNIVERSITY so that all the vowels are together?[Ans:7!$\frac{4!}{2!}$]

37.In how many ways can the letters of the word DIRECTOR be arranged so that the three vowels are never together?[Ans:$\frac{8!}{2!}-\frac{6!}{2!}3!$]

38.Find the number of rearrangements of the letters of the word BENEVOLENT. How many of them end with L.[Ans: $\frac{10!}{3!2!}, \frac{9!}{3!2!}$]

39.In how many ways the letters of the word ALZEBRA can be arranged in a row if

i) the two A's are together[Ans: $\frac{6!2!}{2!}$ ii) the two A's are not together[Ans:$\frac{7!}{2!}-\frac{6!2!}{2!}$]

40.How many words can be formed with the letters of the word PATALIPUTRA with out changing the relative order of the vowels and consonants?[$\frac{6!}{2!2!}.\frac{5!}{3!}$]

41.How many different can be formed if with the letters of the word PENCIL when vowels occupy even places.[${}^{3}p_2 4!$]

42.In how many ways can the letters of the word ARRANGE be arranged so that

i) the two R's are never together

ii) the two A's are together but not the two R's

iii) neither the two R's nor two A's are together

41.The letters of the word OUGHT are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word TOUGH in this dictionary.[Ans:89]

42.Find the number of words which can be made using all the letters of the word AGAIN. If these words are written in a dictionary, what will be the fiftieth word?[Ans:NAAIG]

43.In how many ways can 8 people sit in a round table?[Ans:7!]

44.In how many ways three men and three women sit in a round table so that no two men can occupy adjacent positions?[Ans:2!3!]

45.In how many ways a garland can be prepared if there are ten flowers of different colors?[Ans:$\frac{9!}{2}$]

46.In how many ways can four people be seated in a round table if six places are available?

[Ans: $\frac{{}^{6}p_4}{4}$]

Combination: - The different groups or selections which can be made by taking some or all at a time from a number of things are called combinations. Thus in combinations we are only concerned with the number of things each group contains irrespective of the order.

For examples the combinations which can be made by taking the letters a, b, c two at a time are 3 in number namely, ab, bc, ca

The number of combinations of n dissimilar things taken r at a time denoted by ^nC_r or C(n,r) and is given by $^nc_r = \dfrac{n!}{r!(n-r)!}$

Proof:

Let there are n objects and let us denote the number of combinations of n objects taking r at a time as ^nc_r. Therefore every combination contains r objects and these r objects can be arranged in r! ways, which gives us the total number of permutations of n objects taking r at a time.

Hence $^np_r = r!\,^nc_r$

$$\Rightarrow {}^nc_r = \frac{^np_r}{r!}$$

$$\Rightarrow {}^nc_r = \frac{n!}{r!(n-r)!}$$

Note: Relation between ^np_r and ^nc_r is $^np_r = r!\,^nc_r$

Restricted combinations

1) The number of combinations of n dissimilar thing taken r at a time when p particular things always occur = $^{n-p}C_{r-p}$

2) The number of combinations of n dissimilar things taken r at a time when p particular things never occur = $^{n-p}C_r$

Properties of ^nc_r :

1) $^nc_r = {}^nc_{n-r} = \dfrac{n}{r}\,^{n-1}c_{r-1}$

Proof:

$$^nc_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)!}{r(r-1)![(n-1)-(r-1)]!} = \frac{n}{r}\,^{n-1}c_{r-1}$$

3) If $^nc_x = {}^nc_y$ then either $x = y \text{ or } x + y = n$

Proof:

Case (i) given $^nc_x = {}^nc_y$

$$\Rightarrow x = y$$

Case (ii) given $^nc_x = {}^nc_y$

$$\Rightarrow {}^nc_x = {}^nc_{n-y} \Rightarrow x = n - y \Rightarrow x + y = n$$

4) $^nc_r + {}^nc_{r-1} = {}^{n+1}c_r$

Proof: we have

$$^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r.(r-1)!(n-1)!} + \frac{n!}{(r-1)!(n-r+1).(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!} \cdot \frac{n+1}{r(n-r+1)}$$

$$= \frac{(n+1)n!}{r!(n-r+1)!}$$

$$= {}^{n+1}C_r$$

$$Hence\, {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

5) $^nC_r = \frac{^np_r}{r!} = \frac{n(n-1)(n-2)........(n-r+1)}{r!}$

6) $^nC_n = \frac{n!}{n!(n-n!)} = \frac{n!}{n!0!} = 1$

7) $^nC_0 = \frac{n!}{0!(n-0!)} = \frac{n!}{n!0!} = 1$

8) $\sum_{r=1}^{n} c(n,r) = \sum_{r=1}^{n} \frac{p(n,r)}{r!} = 2^n - 1$

9) Number of divisors or factors of a given number n>1, which can be expressed as $p_1^{k_1}.p_2^{k_2}..........p_r^{k_r}$ where $p_1, p_2,.........., p_r$ are distinct primes and $k_1, k_2,.........., k_r$ are positive integers, are $(k_1+1)(k_2+1).........(k_r+1)$ (including 1and n).

10) Number of selections from n objects, taking at least one is $^nc_1 + {}^nc_2 + {}^nc_3 + \cdots\cdots\cdots. + {}^nc_n = 2^n - 1$

Ex: There are 15 bulbs in a room. Each one of them can operated independently. The number of ways in which the room can be lightened is $^{15}c_1 + {}^{15}c_2 + {}^{15}c_3 + \cdots\cdots\cdots. + {}^{15}c_{15} = 2^{15} - 1$

11) The number selection of r objects out of n identical objects is 1.

12) The number of selection of none or more objects from n identical objects is equal to n+ 1.

13) Number of ways of dividing m different things into 3 sets consisting a, b, c things such that a, b, c are distinct and a + b + c=m is $^mc_a \, {}^{m-a}c_b \, {}^{m-a-b}c_c = \frac{m!}{a!b!c!}$

14) Number of ways of distributing m different things among three persons such that each person gets a, b, c things is $\frac{m!}{a!b!c!}3!$

15) Number of ways dividing 3m different things into three groups having m things in each group is $\frac{m!}{(m!)^3 3!}$

16) Number of ways distributing 3m different things to three persons having m things is $\frac{m!}{(m!)^3}$

17) If there are n points in the plane then the number of line segments can be drawn is ${}^{n}c_2$

18) If there are n points out of which m are collinear then the number of line segments can be drawn is ${}^{n}c_2 - {}^{m}c_2 + 1 = \frac{1}{2}(n-m)(n+m-1)$

19) If there are n points in the plane then the number of triangles can be drawn is ${}^{n}c_3$

20) If there are n points out of which m are collinear then the number of triangles can be drawn is ${}^{n}c_3 - {}^{m}c_3$

21) Number of diagonals in a regular polygon having n sides is ${}^{n}c_2 - n$.

Ex: Number of diagonals in a regular decagon is ${}^{10}c_2 - 10$.

Problems:

1. Compute the following
   i) ${}^{12}c_3$ ii) ${}^{15}c_{12}$ iii) ${}^{9}c_4 + {}^{9}c_5$ iv) ${}^{7}c_3 + {}^{6}c_4 + {}^{6}c_3$
2. Prove that $\sum_{r=1}^{5} {}^{5}c_r = 31$
3. Evaluate ${}^{25}c_{22} - {}^{24}c_{21}$
4. If ${}^{5}c_{3r} = {}^{15}c_{r+3}$, find r
5. If ${}^{18}c_r = {}^{18}c_{r+2}$, find ${}^{r}c_5$
6. Determine n, if ${}^{2n}c_3 : {}^{n}c_3$=11:1.
7. If ${}^{n}c_8 = {}^{n}c_6$, determine n and hence find ${}^{n}c_2$
8. Determine n, if ${}^{n}c_6 : {}^{n-3}c_3$=33 : 4.
9. Prove that ${}^{n}c_r \times {}^{r}c_s = {}^{n}c_s \times {}^{n-s}c_{r-s}$
10. If ${}^{n-1}c_r : {}^{n}c_r : {}^{n+1}c_r = 6:9:13$, find n and r
11. Find the value of the expression ${}^{47}c_4 + \sum_{j=1}^{5} {}^{52-j}c_3$
12. How many diagonals does a polygon have?[${}^{n}c_2 - n$]
13. Find the number of sides of a polygon having 44 diagonals.[Ans:11]
14. In how many ways three balls can be selected from a bag containing 10 balls?[${}^{10}c_3$]
15. In how many ways two black and three white balls are selected from a bag containing 10black and 7 white balls? [${}^{10}c_2\ {}^{7}c_3$]
16. A delegation of 6 members is to be sent abroad out of 12 members. In how many ways can the selection be made so that i) a particular person always included [${}^{11}c_5$] ii) a particular person never included[${}^{11}c_6$]
17. A man has six friends. In how many ways can he invite two or more friends to a dinner party?[Ans:57]
18. In how many ways can a student choose 5 courses out of the courses $c_1, c_2, \ldots\ldots\ldots, c_9$ if c_1, c_2 are compulsory and c_6, c_8 can not be taken together?
19. In a class there are 20 students. How many Shake hands are available if they shake hand each other?[${}^{20}c_2$]

20.Find the number of triangles which can be formed with 20 points in which no two points are collinear?[${}^{20}c_3$]

21.There are 15 points in a plane, no three points are collinear. Find the number of triangles formed by joining them. [${}^{15}c_3$]

22.How many lines can be drawn through 21 points on a circle?[${}^{21}c_2$]

23.There are ten points on a plane, from which four are collinear. No three of remaining six points are collinear. How many different straight lines and triangles can be formed by joining these points?[Ans: ${}^{10}c_2 - {}^4c_2 + 1,\ {}^{10}c_3 - {}^4c_3$]

24.To fill 12 vacancies there are 25 candidates of which 5 are from S.C. If three of the vacancies are reserved for scheduled caste, find the number of ways in which the selections can be made. [Ans: ${}^{20}c_9\,{}^5c_3$]

25.On a New Year day every student of a class sends a card to every other student. If the post man delivers 600 cards. How many students are there in the class?[Ans:25]

26.There are n stations on a railway line. The number of kinds of tickets printed (no return tickets) is 105. Find the number of stations.[Ans:15]

27.In how many ways a cricket team containing 6 batsmen and 5 bowlers can be selected from 10 batsmen and 12 bowlers?[${}^{10}c_6\,{}^{12}c_5$]

28.How many words can be formed out of ten consonants and 4 vowels, such that each contains three consonants and two vowels?[${}^{10}c_3\,{}^4c_2\,5!$]

29.How many words each of three vowels and two consonants can be formed from the letters of the word INVOLUE? [${}^4c_3\,{}^3c_2\,5!$]

30.A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of i) exactly 3 girls[Ans: ${}^9c_4\,{}^4c_3$]

ii) at least three girls.[${}^9c_4\,{}^4c_3 + {}^9c_3\,{}^4c_4$]

31.A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has i) no girls ii) at least one boy iii) at least one boy and one girl iv) at least three girls.

32.In how many ways four cards selected from the pack of 52 cards? [${}^{52}c_4$]

33.How many factors do 210 have?[16(including 1) and 15(excluding 1)]

34.How many factors does 1155 have that are divisible by 3?[Ans:8]

35.Find the number of divisors of 21600.[71(excluding 1)]

36.In an examination minimum is to be scored in each of the five subjects for a pass. In how many ways can a student fail?[Ans:31]

37.In how many number of ways 4 things are distributed equally among two persons. [$\frac{4!}{(2!)^2}$]

38.In how many ways 12 different things can be divided in three sets each having four things? [Ans: $\frac{12!}{(4!)^3 3!}$]

39.In how many ways 12 different things can be distributed equally among three persons?[Ans: $\frac{12!}{(4!)^3}$]

40.How many different words of 4 letters can be made by using the letters of the word EXAMINATION?[Ans:2454]

41.How many different words of 4 letters can be made by using the letters of the word BOOKLET?[

42.How many different 5 lettered words can be made by using the letters of the word INDEPENDENT?[Ans:72]

43.From 5 apples, 4 oranges and 3 mangos how many selections of fruits can be made?[Ans:119]

44.Find the number of different sums that can be formed with one rupee, one half rupee and one quarter rupee coin.[Ans:7]

45.There are 5 questions in a question paper. In how many ways can boy solve one or more questions?[Ans:31]

Important formulas:

1. The number of arrangements taking not more than q objects from n objects, provided every object can be used any number of times is given by $\sum_{r=1}^{q} n^r$.
2. Number of integers from 1 to n which are divisible by k is $\left[\frac{n}{k}\right]$, where [ ] denotes the greatest integral function.
3. The total number of selections of taking at least one out of $p_1 + p_2 + \dots\dots\dots + p_n$ objects where p_1 are alike of one kind, p_2 are alike of another kind and so on ……..p_n are alike of another kind is equal to $[(p_1+1)(p_2+1)\dots\dots\dots(p_n+1)]-1$
4. The total number of selections taking of at least one out of $p_1 + p_2 + \dots\dots\dots + p_n + s$ objects where p_1 are alike of one kind, p_2 are alike of another kind and so on ……..p_n are alike of another kind and s are distinct are equal to $\{[(p_1+1)(p_2+1)\dots\dots\dots(p_n+1)]2^s\}-1$
5. The greatest value of ${}^nc_r$ is ${}^nc_k$ where

$$k = \frac{n}{2} \text{ if } n \in 2m, m \in N$$
$$= \frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ if } n \in 2m+1 \, \forall m \in N$$

6. Number of rectangles of any size in a square of size $n \times n = \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$
7. Number of squares of any size in a square of size $n \times n = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$
8. Number of squares of any size in a rectangle of size $m \times n = \sum_{r=1}^{n} (m-r+1)(n-r+1)$
9. If m points of one straight line are joined to n points on the another straight line, then the number of points of intersections of the line segment thus obtained $= {}^mc_2 \, {}^nc_2 = \frac{mn(m-1)(n-1)}{4}$.
10. Number of rectangles formed on a chess board is ${}^9c_2 \, {}^9c_2$.
11. Number of rectangles of any size in a rectangle of size $m \times n = (n \le m) = {}^{m+1}c_2 \, {}^{n+1}c_2 = \frac{mn}{4}(m+1)(n+1)$
12. The total number of ways of dividing n identical objects into r groups if blank groups are allowed is ${}^{n+r-1}c_{r-1}$.

13. The total number of ways of dividing n identical objects into r groups if blank groups are not allowed is ${}^{n-1}c_{r-1}$.
14. The exponent of k in n! is $E_k(n!) = \left[\frac{n}{k}\right] + \left[\frac{n}{k^2}\right] + \left[\frac{n}{k^3}\right] + \left[\frac{n}{k^4}\right] + \ldots\ldots \left[\frac{n}{k^p}\right]$, where $k^p < n$
15. The sum of the digits in unit's place of the numbers formed by n nonzero distinct digits is
(sum of the digits) (n−1)!
16. The sum of the numbers formed by n nonzero distinct digits is (sum of the digits) (n−1)!$\left(\frac{10^n - 1}{9}\right)$
17. **Derangements:** If n items are arranged in a row, then the number of ways in which they can be rearranged so that no one of them occupies the place assigned to it is $n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots\ldots\ldots + (-1)^n \frac{1}{n!}\right]$

Exercise:

1. In how many ways can 5 beads out 7 different beads be strung into a string?
2. A person has 12 friends, out of them 8 are his relatives. In how many ways can he invite his 7 friends so as to include his 5 relatives?
(a) ${}^8C_3 \times {}^4C_2$ (b) ${}^{12}C_7$ (c) ${}^{12}C_5 \times {}^4C_3$ (d) none of these
3. It is essential for a student to pass in 5 different subjects of an examination then the no. of method so that he may failure
(a) 31 (b) 32 (c) 10 (d) 15
4. The number of ways of dividing 20 persons into 10 couples is
(a) $\frac{20!}{2^{10}}$ (b) ${}^{20}C_{10}$ (c) $\frac{20!}{(2!)^9}$ (d) none of these
5. The number of words by taking 4 letters out of the letters of the word 'COURTESY', when T and S are always included are
(a) 120 (b) 720 (c) 360 (d) none of these
6. The number of ways to put five letters in five envelopes when one letter is kept in right envelope and four letters in wrong envelopes are−
(a) 40 (b) 45 (c) 30 (d) 70
7. ${}^{47}C_4 + \sum_{r=1}^{5} {}^{52-r}C_3$ is equal to
(a) ${}^{51}C_4$ (b) ${}^{52}C_4$ (c) ${}^{53}C_4$ (d) none of these
8. A candidate is required to answer 6 out of 10 questions which are divided into two groups each containing 5 questions and he is not permitted to attempt more than 4 from each group. The number of ways in which he can make up his choice is
(a) 100 (b) 200 (c) 300 (d) 400

9.Out of 10 white, 9 black and 7 red balls, the number of ways in which selection of one or more balls can be made, is

(a) 881 (b) 891 (c) 879 (d) 892

10.The number of diagonals in an octagon are

(a) 28 (b) 48 (c) 20 (d) none of these

Q26.Out of 10 given points 6 are in a straight line. The number of the triangles formed by joining any three of them is

(a) 100 (b) 150 (c) 120 (d) none of these

Q27.In how many ways the letters AAAAA, BBB, CCC, D, EE, F can be arranged in a row when the letter C occur at different places?

(a) $\frac{12!}{5!3!2!}$ x $^{13}C_3$ (b) $\frac{12!}{5!3!2!}$ x $^{13}P_3$ (c) $\frac{13!}{5!3!2!3!}$ (d) none of these

Q28.A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of chosen P and Q so that P Ç Q = f is

(a) $2^{2n} - {}^{2n}C_n$ (b) 2^n (c) $2^n - 1$ (d) 3^n

Q29.A parallelogram is cut by two sets of m lines parallel to the sides, the number of parallelograms thus formed is

(a) $\frac{m^2}{4}$ (b) $\frac{(m+1)^2}{4}$ (c) $\frac{(m+2)^2}{4}$ (d) $\frac{(m+2)^2(m+1)^2}{4}$

Q30.Along a railway line there are 20 stations. The number of different tickets required in order so that it may be possible to travel from every station to every station is

(a) 380 (b) 225 (c) 196 (d) 105

Q31.The number of ordered triplets of positive integers which are solutions of the equation x + y + z = 100 is

(a) 5081 (b) 6005 (c) 4851 (d) none of these

Q32.The number of numbers less than 1000 that can be formed out of the digits 0, 1, 2, 3, 4 and 5, no digit being repeated, is

(a) 130 (b) 131 (c) 156 (d) none of these

Q33.A variable name in certain computer language must be either a alphabet or alphabet followed by a decimal digit. Total number of different variable names that can exist in that language is equal to

(a) 280 (b) 290 (c) 286 (d) 296

Q34.The total number of ways of selecting 10 balls out of an unlimited number of identical white, red and blue balls is equal to

(a) $^{12}C_2$ (b) $^{12}C_3$ (c) $^{10}C_2$ (d) $^{10}C_3$

Q35.Total number of ways in which 15 identical blankets can be distributed among 4 persons so that each of them get atleast two blankets equal to

(a) $^{10}C_3$ (b) $^{9}C_3$ (c) $^{11}C_3$ (d) none of these

Q36.The number of ways in which three distinct numbers in AP can be selected from the set {1, 2, 3, ⋯, 24}, is equal to

(a) 66 (b) 132 (c) 198 (d) none of these

Q37.The number of ways of distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is:

(a) 5 (b) 21 (c) 3^8 (d) $^{8}C_3$

Q38.The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by:

(a) 6! x 5! (b) 30 (c) 5! x 4! (d) 7! x 5!

Q39.If $^{n}C_r$ denotes the number of combinations of n things taken r at a time, then the expression $^{n}C_{r+1} + {}^{n}C_{r-1} + 2 \times {}^{n}C_r$ equals:

(a) $^{n+2}C_r$ (b) $^{n+2}C_{r+1}$ (c) $^{n+1}C_r$ (d) $^{n+1}C_{r+1}$

Q40.If the letters of the word SACHIN are arranged in all possible ways and these are written out as in dictionary, then the word SACHIN appears at serial number

(a) 600 (b) 601 (c) 602 (d) 603

Q26.The number of numbers is there between 100 and 1000 in which all the digits are distinct is

(a)648 (b) 548 (c)448 (d) none of these

Q27.The number of arrangements of the letters of the word 'CALCUTTA' is

(a)5040 (b) 2550 (c)40320 (d) 10080

Q28. How many different words can be formed with the letters of the word "PATLIPUTRA" without changing the position of the vowels and consonants?
(a)2160 (b) 180 (c)720 (d) none of these

Q29. How many different words ending and beginning with a consonant can be formed with the letters of the word 'EQUATION'?
(a)720 (b) 4320 (c)1440 (d) none of these

Q30.The number of 4 digit numbers divisible by 5 which can be formed by using the digits 0, 2, 3, 4, 5 is
(a)36 (b) 42 (c) 48 (d) none of these

Q31.The number of ways in which 5 biscuits can be distributed among two children is
(a)32 (b) 31 (c) 30 (d) none of these

Q32.How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word "EQUATION" so that the two consonants occur together?
(a)1380 (b) 1420 (c)1440 (d) none

Q33.If the letters of the word 'RACHIT' are arranged in all possible ways and these words are written out as in a dictionary, then the rank of this word is
(a)365 (b) 702 (c)481 (d) none of these

Q34.On the occasion of Dipawali festival each student of a class sends greeting cards to the others. If there are 20 students in the class, then the total number of greeting cards exchanged by the students is
(a)$^{20}C_2$ (b) $2 \cdot {}^{20}C_2$ (c)$2 \cdot {}^{20}P_2$ (d) none of these

Q35.The sum of the digits in the unit place of all the numbers formed with the help of 3, 4, 5, 6 taken all at a time is
(a)18 (b) 108 (c)432 (d) 144

Q36.How many six digits numbers can be formed in decimal system in which every succeeding digit is greater than its preceding digit
(a) $^9P_6$ (b) $^{10}P_6$ (c)$^9P_3$ (d) none of these

Q37.How many ways are there to arrange the letters in the work GARDEN with the vowels in alphabetical order?

(a) 120 (b) 240 (c)360 (d) 480

Q38.A five-digit numbers divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 and 5, without repetition. The total number of ways this can be done is

(a)216 (b) 240 (c)600 (d) 3125

Q39.How many different nine digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions?

(a)16 (b) 36 (c) 60 (d) 180

Q40.The number of arrangements of the letters of the word BANANA in which the two N's do not appear adjacently is

(a)40 (b) 60 (c) 80 (d) 100

THE BINOMIAL THEOREM

Binomial expression:

An algebraic expression consisting of only two terms is called a binomial expression.

Ex: i) $x+y$ ii) $4x - 3y$ iii) x^2+y^2 iv) $x^2 - 1/a^2$

Binomial theorem:

The formula by which any power of a binomial expression can be expanded in the form of a series is known as binomial theorem. This theorem is given by Sir Issac Newton.

Binomial theorem for positive integral index:

If n is a positive integer

$$(x+y)^n = {}^nc_0 x^n y^0 + {}^nc_1 x^{n-1} y^1 + {}^nc_2 x^{n-2} y^2 + {}^nc_3 x^{n-3} y^3 + \cdots\cdots\cdots\cdots\cdots.. + {}^nc_n x^{n-n} y^n$$

Note:

1) Number of terms in the expansion of $(x+y)^n$ is $n+1$.
2) In the expansion of $(x+y)^n$, the sum of the powers of x and y is equal to n.
3) ${}^nc_0, {}^nc_1, {}^nc_2, \cdots\cdots\cdots, {}^nc_n$ are called coefficients of 1st, 2nd ,……..,(n+1)th terms respectively. These are called binomial coefficients.

Pascal's triangle:

The coefficients of the binomial expansion for different values of n are written in the form of triangle as shown below.

1 ———— n=0
1 1 ———— n=1
1 2 1 ———— n=2
1 3 3 1 ———— n=3
1 4 6 4 1 ———— n=4
1 5 10 10 5 1 ———— n=5
1 6 15 20 15 6 1 ———— n=6

This triangular array is called *Pascal's Triangle*.

Each row gives the binomial coefficients. That is, the row 1 2 1 are the coefficients of $(a+b)^2$. The next row, 1 3 3 1, are the coefficients of $(a+b)^3$; and so on.

To construct the triangle, write 1, and below it write 1 1. Begin and end each successive row with 1. To construct the intervening numbers, add the two numbers immediately above.

Thus to construct the third row, begin it with 1, and then add the two numbers immediately above: 1 + 1. Write 2. Finish the row with 1.

To construct the next row, begin it with 1, and add the two numbers immediately above: 1 + 2. Write 3. Again, add the two numbers immediately above: 2 + 1 = 3. Finish the row with 1.

Some special forms of Binomial expansion:

$$(x+y)^n = {}^nc_0x^ny^0 + {}^nc_1x^{n-1}y^1 + {}^nc_2x^{n-2}y^2 + {}^nc_3x^{n-3}y^3 + \cdots\cdots\cdots + {}^nc_nx^{n-n}y^n \cdots \quad (1)$$

$$= \sum_{r=0}^{n} {}^nc_rx^{n-r}y^r$$

Put −x in place of x, we get

$$(x-y)^n = {}^nc_0x^ny^0 - {}^nc_1x^{n-1}y^1 + {}^nc_2x^{n-2}y^2 - {}^nc_3x^{n-3}y^3 + \cdots\cdots + (-1)^n\,{}^nc_nx^{n-n}y^n \cdots (2)$$

$$= \sum_{r=0}^{n}(-1)^r\ {}^nc_rx^{n-r}y^r$$

Put x = 1 in (1)

$$(1+y)^n = {}^nc_01^ny^0 + {}^nc_11^{n-1}y^1 + {}^nc_21^{n-2}y^2 + {}^nc_31^{n-3}y^3 + \cdots\cdots\cdots + {}^nc_n1^{n-n}y^n$$

$$= 1 + {}^nc_1y + {}^nc_2y^2 + {}^nc_3y^3 + \cdots\cdots\cdots + y^n$$

$$= \sum_{r=0}^{n} {}^nc_ry^r$$

Put x = 1 in (2)

$$(1-y)^n = {}^nc_01^ny^0 - {}^nc_11^{n-1}y^1 + {}^nc_21^{n-2}y^2 - {}^nc_31^{n-3}y^3 + \cdots\cdots\cdots + (-1)^n\,{}^nc_n1^{n-n}y^n$$

$$= 1 - {}^nc_1y + {}^nc_2y^2 - {}^nc_3y^3 + \cdots\cdots\cdots + (-1)^ny^n$$

$$= \sum_{r=0}^{n}(-1)^r\ {}^nc_ry^r$$

Problems:

1) Expand $(x - 1)^6$.

Solution: According to Pascal's triangle, the coefficients are

1 6 15 20 15 6 1.

In the binomial, x is "x", and −1 is "y". The signs will alternate:

$$(x-1)^6 = x^6 - \underline{6}x^5\cdot 1 + \underline{15}x^4\cdot 1^2 - \underline{20}x^3\cdot 1^3 + \underline{15}x^2\cdot 1^4 - \underline{6}x\cdot 1^5 + 1^6$$

$$= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$

2) The term a^8b^4 occurs in the expansion of what binomial?

Answer. $(a + b)^{12}$. The sum of 8 + 4 is 12.

3). Use Pascal's triangle to expand the following.

a) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
b) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
c) $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
d) $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
e) $(x - 1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$
f) $(x + 2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
g) $(2x - 1)^3 = 8x^3 - 12x^2 + 6x - 1$

Exercise:

1) Expand i) $\left(x+\frac{1}{x}\right)^6$ ii) $\left(x-\frac{1}{y}\right)^4, y\neq 0$ iii) $(2x-3y)^4$ iv) $(x^2+2a)^5$ v) $(1+x+x^2)^3$ vi) $(1-x+x^2)^4$
2) Expand $(a+b)^6-(a-b)^6$ .hence find the value of $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$
3) Simplify $(x+\sqrt{x-1})^6-(x-\sqrt{x-1})^6$
4) If A be the sum of odd terms and B be the sum of even terms in the expansion of $(x+a)^n$, then prove that
   i) $A^2-B^2=(x^2-a^2)^n$ ii) $2(A^2+B^2)=(x+a)^{2n}+(x-a)^{2n}$
5) The first three terms in the expansion $(1+y)^n$ are 1, 10 and 40, find the expansion.
6) Using binomial theorem compute $(99)^5$
7) Find the exact value of $(1.01)^5$
8) Which is larger $(1.2)^{4000}$ or 800?
9) Which is greater $(1.1)^{10000}$ or 1000?
10) Show that $(101)^{50} > (100)^{50} + (99)^{50}$.
11) Prove that $\sum_{r=0}^{n} {}^nc_r 3^r = 4^n$.
12) Prove that ${}^nc_0 + {}^nc_1 + {}^nc_2 + {}^nc_3 + \cdots\cdots + {}^nc_n = 2^n$.
13) Prove that product of k consecutive numbers is divisible by k!.

General term in the expansion $(x+y)^n$:

$$(x+y)^n = {}^nc_0x^ny^0 + {}^nc_1x^{n-1}y^1 + {}^nc_2x^{n-2}y^2 + {}^nc_3x^{n-3}y^3 + \cdots\cdots + {}^nc_nx^{n-n}y^n$$

In the above expansion the (r+ 1)th term is given by

$$\boxed{T_{r+1} = {}^nc_rx^{n-r}y^r}$$

this is called the general term of the expansion.

Putting r=0,1,2,3,4…..,n we get 1^{st} ,2^{nd} ,……..,(n+ 1)th terms respectively.

Middle term in the expansion $(x+y)^n$:

Case- i) n is even

If n is even then the number of terms in the expansion is n+ 1 which is odd. Therefore the number of middle terms in the expansion is one and the term is $\frac{n}{2}+1$ th term.

Case- ii) n is odd

If n is odd then the number of terms in the expansion is n+1 which is even. Therefore the number middle terms in the expansion are two and the terms are $\frac{n+1}{2}$th and $\frac{n+3}{2}$th terms.

Greatest coefficient in the expansion (x + y)^n :

In any binomial expansion the middle term has the greatest coefficient. If there are two middle terms then their two coefficients are equal and greater.

Prob : If n be a positive integer, prove that the coefficients of the terms in the expansion of $(x+y)^n$ equidistant from the beginning and from the end are equal.

In the expansion of $(x+y)^n$

Co efficient of 1st term from beginning = ${}^nc_0$

Co efficient of 2nd term from beginning = ${}^nc_1$

Co efficient of 3rd term from beginning = ${}^nc_2$

……………………………………..

………………………………………

Co efficient of r th term from beginning = ${}^nc_{r-1}$

Now

Co efficient of 1st term from end = ${}^nc_n$

Co efficient of 2nd term from end = ${}^nc_{n-1}$

Co efficient of 3rd term from end = ${}^nc_{n-2}$

……………………………………..

………………………………………

Co efficient of r th term from end = ${}^nc_{n-(r-1)}$

Since ${}^nc_{r-1}$ = ${}^nc_{n-(r-1)}$ are equal. We can say in the expansion of $(x+y)^n$, the co efficient of r th term from beginning and end are equal.

Note: ***In the binomial expansion, the r th term from the end is equal to (n-r+2)th term from the beginning.***

Problems:

1) Find the 4 th term in the expansion of $(x-2y)^{12}$

2) Find the 13 th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0.$

3) Find the 5 th term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^9$.

4) Write the general term in the expansion of $(x^2-y)^6$.

5) If x >1 and the third term in the expansion of $\left(\frac{1}{x}+x^{\log_{10} x}\right)^5$ is1000, find the value of x.

6) If the 21^{st} and 22^{nd} terms in the expansion of $(1+x)^{44}$ are equal then find the value of x.

7) In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of 5^{th} and 6^{th} terms is zero, then find $\frac{a}{b}$.

8) Find the middle term in the expansion of $\left(\frac{x}{3}-9y\right)^{10}$.

9) Find the middle term in the expansion of $\left(x-\frac{1}{2x}\right)^{12}$.

10) Find the middle term in the expansion of $\left(2x^2-\frac{1}{x}\right)^7$.

11) Find the middle term in the expansion of $(1-2x+2x^2)^n$.

12) Prove that the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5.7......(2n-1)2^n}{n!}$

13) Show that the greatest coefficient in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5.7......(2n-1)2^n}{n!}$.

14) Show that the coefficient of the middle term in $(1+x)^{2n}$ is equal to the sum of the coefficients of two middle terms in $(1+x)^{2n-1}$.

15) Find the coefficient of $1/y^2$ in $\left(y-\frac{c^3}{y^2}\right)^{10}$.

16) Find the coefficient of x^9 in $(1+3x+3x^2+x^3)^{15}$.

17) Find the coefficient of x^{40} in $(1+2x+x^2)^{27}$.

18) Find the term independent of x in $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.

19) Given that the fourth term in the expansion of $\left(px+\frac{1}{x}\right)^n$.is 5/2, find n and p.

20) Find the value of k so that the term independent of x in $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$ is 405.

21) In the expansion of $(1+a)^{m+n}$, prove that the coefficient of a^m and a^n are equal.

22) Find a if the coefficient of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal.

23) If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1+a)^n$ are in A.P. prove that $n^2-n(4r+1)+4r^2-2=0$.

24) Find the coefficient of x^{-1} in $(1+3x^2+x^4)\left(1+\frac{1}{x}\right)^8$.

25) If n be a positive integer, then prove that $6^{2n}-35n-1$ is divisible by 1225.

26) Find the

a) 7^{th} term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

b) 9^{th} term in the expansion of $\left(\frac{x}{a}-\frac{3a}{x^2}\right)^{12}$

c) 5^{th} term in the expansion of $\left(\frac{a}{3}-3b\right)^7$ and $\left(2x^2-\frac{1}{3x^3}\right)^{10}$

27) Find a, if the 17^{th} and 18^{th} terms of the expansion $(2+a)^{50}$ are equal.

28) Find the r th term from the end in $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^9$

29) Write the general terms in the following expansions.

i) $(1-x^2)^{12}$. ii) $\left(x-\frac{3}{x^2}\right)^{10}$ iii) $\left(x^2-\frac{1}{x}\right)^{12}, x\neq 0$

30) Find the general term and middle term in the expansion of $\left(\frac{x}{y}+\frac{y}{x}\right)^{2n+1}$ n being positive integer.

31) If n is a positive integer, show that

i) 4^n-3n-1 is divisible by 9.

ii) $2^{5n}-31n-1$ is divisible by 961.

32) Using binomial theorem prove that 6^n-5n always leaves the remainder 1 when divided by 25 for all positive integers n.

33) Find the middle terms in the expansions

i) $\left(\frac{2x}{3}-\frac{3y}{2}\right)^{20}$ ii) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ iii) $\left(\frac{x}{y}-\frac{y}{x}\right)^7$ iv) $(1+x)^{2n}$ v) $(1-2x+x^2)^n$ vi) $\left(3-\frac{x^3}{6}\right)^7$

34) Find the coefficient of

i) x in the expansion of $\left(2x-\frac{3}{x}\right)^9$

ii) x^7 in the expansion of $\left(3x^2+\frac{1}{5x}\right)^{11}$

iii) x^9 in the expansion of $\left(2x^2-\frac{1}{x}\right)^{20}$

iv) x^{24} in the expansion of $\left(x^2-\frac{3a}{x}\right)^{15}$

v) x^9 in the expansion of $\left(x^2-\frac{1}{3x}\right)^9$

vi) x^{-7} in the expansion of $\left(2x-\frac{1}{3x^2}\right)^{11}$

vii) x^5 in the expansion of $(x+3)^8$

viii) x^5 in the expansion of $(x+3)^9$

ix) a^5b^7 in the expansion of $(a-2b)^{12}$

x) x^6y^3 in in the expansion of $(x+y)^9$

35) If the coefficients of x, x^2 and x^3 in the binomial expansion $(1+x)^{2n}$ are in A.P then prove that $2n^2-9n+7=0$.

36) Find the positive value of m for which the coefficient of x^2 in the expansion of $(1+x)^m$ is 6.

37) Find the term independent of x in the following binomial expansion($x\neq 0$).

i) $\left(x+\frac{1}{x}\right)^{2n}$ ii) $\left(x-\frac{1}{x}\right)^{14}$ iii) $\left(2x^2+\frac{1}{x}\right)^{13}$ iv) $\left(x^2+\frac{1}{x}\right)^{12}$ v) $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$

vi) $\left(2x^2-\frac{1}{x}\right)^{12}$ vii) $\left(2x^2-\frac{3}{x^3}\right)^{25}$ viii) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^{6}$ ix) $\left(x^3-\frac{3}{x^2}\right)^{15}$ x) $\left(x^2-\frac{3}{x^3}\right)^{10}$

xi) $\left(\frac{x^{1/3}}{2}+x^{-1/3}\right)^{8}$ xii) $\left(x-\frac{1}{x}\right)^{12}$ xiii) $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}$

38) If three consecutive coefficients in the expansion of $(1+x)^n$ be 56, 70 and 56., find n and the position of the coefficients.

39) If three successive coefficients in the expansion of $(1+x)^n$ be 220, 495 and 972., find n.

40) If coefficients of (r−1)th, rth and (r+1)th terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find n and r.

41) If the coefficients of 5^{th}, 6^{th} and 7^{th} terms in the expansion of $(1+x)^n$ are in A.P, Find n.

42) If the coefficients of 2^{nd} , 3^{rd} and 4^{th} terms in the expansion of $(1+x)^n$ are in A.P, show that $2n^2-9n+7=0$.

43) In the expansion of $(1+a)^{m+n}$, prove that the coefficient of a^m and a^n are equal.

44) Find a if the coefficient of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal.

45) If coefficients of a^{r-1}, a^r,a^{r+1} in the expansion of $(1+a)^n$ are in A.P. Prove that

$n^2-n(4r+1)+4r^2-2=0$.

46) Find the coefficient of x^4 in the expansion of $\left(1+3x+10x^2\right).\left(x+\frac{1}{x}\right)^{10}$

47) Find the coefficient of x^{-1} in the expansion of $\left(1+3x^2+x^4\right).\left(x+\frac{1}{x}\right)^{8}$

48) Find n if the if the coefficient of 4^{th} and 13^{th} terms in the expansion of $(a+b)^n$ are equal.

49) If in the expansion of $(1+x)^{43}$ the coefficient of (2r+1)th term is equal to the coefficient of (r+2) th term , find r.

50) If three consecutive coefficients in the expansion of $(1+x)^n$ be 165, 330 and 462., find n and the position of the coefficients.

51) If a_1,a_2,a_3 and a_4 be any four consecutive coefficients in the expansion of $(1+x)^n$, prove that $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$.

52) If 2nd, 3^{rd} and 4^{th} terms in the expansion of $(x+y)^n$ be 240,720 and 1080 respectively find x, y and n.

53) If the coefficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio 1:7:42. Find n .

54) if in the binomial expansion a,b,c and d be 6^{th} ,7^{th} ,8^{th} and 9^{th} terms respectively, prove that $\frac{b^2-ac}{c^2-bd}=\frac{4a}{3c}$.

Binomial expansion for fractional index:

$$(1+x)^{-n}=1-{}^{n}c_1x+{}^{n+1}c_2x^2-{}^{n+2}c_3x^3+\cdots+(-1)^r\,{}^{n+r-1}c_rx^r+\cdots\ldots\cdots,|x|<1,n\in Q$$

To determine numerically greatest term in the expansion of $(x+y)^n$ $(\forall n\in N)$:-

It is always better to consider $(1+x)^n$ in place of $(x+y)^n$. For this take one of x and y common preferably the greater one. For example $(5+7)^{10}=7^{10}\left(1+\frac{5}{7}\right)^{10}$, now one should find the greatest term of $\left(1+\frac{5}{7}\right)^{10}$ and multiply it by 7^{10}. So it is sufficient to consider the expansion of $(1+x)^n$, $|x|<1$.

Method to determine numerically greatest term in the expansion of $(1+x)^n$:

Steps:

1. Calculate $r=\left|\frac{x(n+1)}{x+1}\right|$
2. If r is an integer then T_r and T_{r+1} are equal and both are greatest terms.
3. If r is not an integer, there $T_{[r]+1}$ is the greatest term where[ ] denotes the greatest integer part.

Some important conclusions from the binomial theorem:

1) If n is odd then $(x+a)^n-(x-a)^n$ and $(x+a)^n+(x-a)^n$ both have equal no of terms and the number of terms are $\frac{n+1}{2}$.
2) If n is even then $(x+a)^n-(x-a)^n$ has $\frac{n}{2}$ terms and $(x+a)^n+(x-a)^n$ has $\frac{n}{2}+1$ terms.

Some important products:

1) $r^2=r(r-1)+r$
2) $r^3=r(r-1)(r-2)+3r(r-1)+r$
3) $r^4=r(r-1)(r-2)(r-3)+6r(r-1)(r-2)+7r(r-1)+r$
4) $(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$
5) $(x-a_1)(x-a_2)(x-a_3)(x-a_4)=x^4-\sum_{i=1}^{4}a_i\,x^3+\sum_{i\neq j=1}^{4}a_ia_j\,x^2-\sum_{i\neq j\neq k=1}^{4}a_ia_ja_k\,x-\prod_{i=1}^{4}a_i$

Some important short cuts:

1) If a, b, c are three consecutive coefficients in the expansion of $(1+x)^n$ then the values of n and r are given by

$$n = \frac{2ac + b(a+c)}{b^2 - ac} \quad \text{and} \quad r = \frac{a(b+c)}{b^2 - ac}$$

2) If the coefficient of x^r, x^{r+1} in the expansion $\left(a + \frac{x}{b}\right)^n$ are given then the value of n is

$$n = ab(r+1) + r$$

3) If the coefficients of T_r, T_{r+1}, T_{r+2} in the expansion of $(1+x)^n$ are in A.P then the value of r is given by

$$r = \frac{n \pm \sqrt{n+2}}{2}, \forall n \in N$$

4) If the coefficients of T_r, T_{r+1}, T_{r+2} in the expansion of $(1+x)^n, \forall n \in N$ are in the ratio a : b : c then the value of r is given by

$$r = \frac{a(b+c)}{b^2 - ac} \quad \text{and} \quad n = \frac{2ac + b(a+c)}{b^2 - ac}$$

5) If in the expansion of $(1+x)^n$, p^{th} term = q^{th} term then p + q = n + 2

Identities involving Binomial coefficients:

We know the binomial coefficients are ${}^nc_0$, ${}^nc_1$, ${}^nc_2$, ${}^nc_3$, …………, ${}^nc_n$. Through out this chapter we write these coefficients as $c_0, c_1, c_2, \ldots\ldots, c_n$ for convenience.

1. Prove that $c_0 + c_1 + c_2 + \ldots\ldots + c_n = 2^n$

Proof:

we have

$(1+y)^n = {}^nc_0 1^n y^0 + {}^nc_1 1^{n-1} y^1 + {}^nc_2 1^{n-2} y^2 + {}^nc_3 1^{n-3} y^3 + \cdots\cdots + {}^nc_n 1^{n-n} y^n$

Put y = 1 we get

$c_0 + c_1 + c_2 + \ldots\ldots + c_n = 2^n$ ……………. (1)

2. Prove that $c_0 - c_1 + c_2 - \ldots\ldots + (-1)^n c_n = 0$

Proof:

we have

$(1+y)^n = {}^nc_0 1^n y^0 + {}^nc_1 1^{n-1} y^1 + {}^nc_2 1^{n-2} y^2 + {}^nc_3 1^{n-3} y^3 + \cdots\cdots + {}^nc_n 1^{n-n} y^n$

Put y = −1 we get

$c_0 - c_1 + c_2 - \ldots\ldots + (-1)^n c_n = 0$ ……………. (2)

3. Prove that $c_0 + c_2 + c_4 + \ldots\ldots = 2^{n-1}$ and $c_1 + c_3 + c_5 + \ldots\ldots = 2^{n-1}$

Proof:

Adding (1) and (2) we get $c_0 + c_2 + c_4 + \ldots\ldots = 2^{n-1}$

Subtracting (1) and (2) we get $c_1 + c_3 + c_5 + \ldots\ldots = 2^{n-1}$

4. Prove that $\left(1+\frac{c_1}{c_0}\right)\left(1+\frac{c_2}{c_1}\right)\left(1+\frac{c_3}{c_2}\right)\ldots\ldots\ldots\ldots\left(1+\frac{c_n}{c_{n-1}}\right)=\frac{(n+1)^n}{n!}$

Proof:

Let us take $\frac{c_r}{c_{r-1}}=\frac{n!}{r!(n-r)!}\times\frac{(r-1)!(n-r+1)!}{n!}=\frac{n-r+1}{r}$

Now putting r=1,2,3⋯⋯,n we get

$$\frac{c_1}{c_0}=n,\ \frac{c_2}{c_1}=\frac{n-1}{2},\ \frac{c_3}{c_2}=\frac{n-2}{3}\ \cdots\cdots\cdots.,\ \frac{c_n}{c_{n-1}}=\frac{1}{n}$$

now

$$\left(1+\frac{c_1}{c_0}\right)\left(1+\frac{c_2}{c_1}\right)\left(1+\frac{c_3}{c_2}\right)\ldots\ldots\ldots\ldots\left(1+\frac{c_n}{c_{n-1}}\right)$$

$$=(1+n)\left(1+\frac{n-1}{2}\right)\left(1+\frac{n-2}{3}\right)\ldots\ldots\ldots\ldots\left(1+\frac{1}{n}\right)$$

$$=\frac{(1+n)(1+n)\ldots\ldots\ldots\ldots.(1+n)(n\,times)}{1.2.3\ldots\ldots\ldots n}=\frac{(n+1)^n}{n!}$$

5. If P be the sum of the odd terms and Q be the sum of the even terms in the expansion of $(a+x)^n$, then prove that $(a^2-x^2)^n=P^2-Q^2$

6. Find the sum of $1+\frac{1}{2}c_1+\frac{1}{3}c_2+\ldots\ldots\ldots.+\frac{1}{n+1}c_n$

Proof: 1st method

$$1+\frac{1}{2}c_1+\frac{1}{3}c_2+\ldots\ldots\ldots.+\frac{1}{n+1}c_n$$

$$=1+\frac{1}{2}n+\frac{1}{3}\frac{n(n-1)}{2!}+\ldots\ldots\ldots.+\frac{1}{n+1}$$

$$=\frac{1}{n+1}\left((n+1)+\frac{1}{2}n(n+1)+\frac{1}{3}\frac{(n+1)n(n-1)}{2!}+\ldots\ldots\ldots.+1\right)$$

$$=\frac{1}{n+1}\left({}^{n+1}c_1+{}^{n+1}c_2+{}^{n+1}c_3+\ldots\ldots\ldots+{}^{n+1}c_{n+1}\right)$$

$$=\frac{1}{n+1}\left(2^{n+1}-1\right)$$

2nd method

we have

$$(1+y)^n=1+c_1y+c_2y^2+c_3y^3+\cdots\cdots\cdots\cdots+c_ny^n$$

now integrating both sides w.r.to y under the limits 0 and 1 we get the answer

7. Find the sum of $\frac{c_1}{c_0}+2\frac{c_2}{c_1}+3\frac{c_3}{c_2}+\ldots\ldots\ldots\ldots.+n\frac{c_n}{c_{n-1}}$

Proof:

Let us take $\frac{c_r}{c_{r-1}}=\frac{n!}{r!(n-r)!}\times\frac{(r-1)!(n-r+1)!}{n!}=\frac{n-r+1}{r}$

Now putting r=1,2,3⋯⋯,n we get

$$\frac{c_1}{c_0}=n,\ \frac{c_2}{c_1}=\frac{n-1}{2},\ \frac{c_3}{c_2}=\frac{n-2}{3}\ \cdots\cdots\cdots.,\ \frac{c_n}{c_{n-1}}=\frac{1}{n}$$

now

$$\frac{c_1}{c_0}+2\frac{c_2}{c_1}+3\frac{c_3}{c_2}+.............+n\frac{c_n}{c_{n-1}}$$
$$=n+2\frac{n-1}{2}+3\frac{n-1}{3}+........+n\frac{1}{n}$$
$$=n+(n-1)+(n-2)+................+1$$
$$=\frac{n(n+1)}{2}$$

8. Show that i) $c_0^{\ 2}+c_1^{\ 2}+c_2^{\ 2}+..........+c_n^{\ 2}=\frac{(2n)!}{(n!)^2}$

ii) $c_0c_1+c_1c_2+c_2c_3+..........+c_{n-1}c_n=\frac{(2n)!}{(n-1)!(n+1)!}$

iii) $c_0c_2+c_1c_3+c_2c_4+..........+c_{n-2}c_n=\frac{(2n)!}{(n-2)!(n+2)!}$

Proof:

We have

$$(1+y)^n=c_0+c_1y+c_2y^2+c_3y^3+\cdots\cdots\cdots\cdots+c_ny^n\cdots\cdots\cdots\cdots\cdots.\ (1)$$

and $(y+1)^n=c_0y^n+c_1y^{n-1}+c_2y^{n-2}+c_3y^{n-3}+\cdots\cdots\cdots\cdots+c_n\cdots\cdots\cdots\cdots\cdots.(2)$

now multiplying (1) and (2) we get

$(1+y)^{2n}=(c_0+c_1y+c_2y^2+c_3y^3+\cdots\cdots+c_ny^n)(c_0y^n+c_1y^{n-1}+c_2y^{n-2}+\cdots..+c_n)\cdots\cdots..(3)$

from l.h.s

$(1+y)^{2n}={}^{2n}c_0+{}^{2n}c_1y+{}^{2n}c_2y^2+\cdots\cdots..+{}^{2n}c_{n-1}y^{n-1}+{}^{2n}c_ny^n+{}^{2n}c_{n+1}y^{n+1}+\cdots\cdots+{}^{2n}c_{2n}y^{2n}\cdots..(4)$

i) Equating the coefficients of y^n in the right hand side of (3) and (4) we get

$$c_0^{\ 2}+c_1^{\ 2}+c_2^{\ 2}+..........+c_n^{\ 2}={}^{2n}c_n$$
$$\Rightarrow c_0^{\ 2}+c_1^{\ 2}+c_2^{\ 2}+..........+c_n^{\ 2}=\frac{(2n)!}{(n!)^2}$$

ii) Equating the coefficients of y^{n-1} in the right hand side of (3) and (4) we get

$$c_0c_1+c_1c_2+c_2c_3+..........+c_{n-1}c_n={}^{2n}c_{n-1}$$
$$\Rightarrow c_0c_1+c_1c_2+c_2c_3+..........+c_{n-1}c_n=\frac{(2n)!}{(n-1)!(n+1)!}$$

iii) Equating the coefficients of y^{n-2} in the right hand side of (3) and (4) we get

$$c_0c_2+c_1c_3+c_2c_4+..........+c_{n-2}c_n={}^{2n}c_{n-2}$$
$$\Rightarrow c_0c_2+c_1c_3+c_2c_4+..........+c_{n-2}c_n=\frac{(2n)!}{(n-2)!(n+2)!}$$

9. Prove that $c_0 - 2^2 c_1 + 3^2 c_2 - \dots\dots\dots + (-1)^n (n+1)^2 c_n = 0$, $n > 2$

10. $\frac{c_0}{2} + \frac{c_1}{3} + \frac{c_2}{4} + \frac{c_3}{5} + \dots\dots\dots\dots + \frac{c_n}{n+2} = \frac{n2^{n+1}+1}{(n+1)(n+2)}$

11. Prove that

i) ${}^{2n}c_0 + {}^{2n}c_1 + {}^{2n}c_2 + \dots\dots + {}^{2n}c_{2n-1} + {}^{2n}c_{2n} = 2^{2n}$ [Hints: $c_0 + c_1 + c_2 + \dots\dots\dots + c_n = 2^n$]

ii) ${}^{2n}c_1 + {}^{2n}c_3 + {}^{2n}c_5 + \dots\dots + {}^{2n}c_{2n-1} = 2^{2n-1}$ [Hints: $c_1 + c_3 + c_5 + \dots\dots\dots = 2^{n-1}$]

iii) $c_1 + 2c_2 + 3c_3 + \dots\dots\dots + nc_n = n2^{n-1}$

[Hints: take $(1+x)^n$ then differentiate w.r.to x both sides then put x=1both sides]

iv) $c_0 + 3c_1 + 5c_2 + \dots\dots\dots + (2n+1)c_n = (n+1)2^n$

[Hints: write it as $(c_0 + c_1 + c_2 + \dots\dots\dots + c_n) + 2(c_1 + 2c_2 + 3c_3 + \dots\dots\dots + nc_n)$]

12. Find the sum of

i) $c_1 - 2c_2 + 3c_3 - \dots\dots\dots + (-1)^{n-1} nc_n$

[Hints: take $(1-x)^n$ then differentiate w.r.to x both sides then put x=1both sides]

ii) $1.2c_2 + 2.3c_3 + \dots\dots\dots + (n-1)nc_n$

[Hints: take $(1+x)^n$ then differentiate w.r.to x both sides then again differentiate both sides w.r.to x and then put x=1both sides]

iii) $c_1 + 2^2 c_2 + 3^2 c_3 + \dots\dots\dots + n^2 c_n$

[Hints: take $(1+x)^n$ then differentiate w.r.to x both sides then multiply x both sides then again differentiate both sides w.r.to x and then put x=1both sides]

iv) $c_0 + 2c_1 + 3c_2 + \dots\dots\dots + (n+1)c_n$

[Hints: take $(1+x)^n$ then multiply x both sides then differentiate w.r.to x both sides and then put x=1both sides]

v) $c_0 - 2c_1 + 3c_2 - \dots\dots\dots + (-1)^n (n+1)c_n$

[Hints: take $(1-x)^n$ then multiply x both sides then differentiate w.r.to x both sides and then put x=1both sides]

vi) $c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \dots\dots\dots + (-1)^n \frac{1}{n+1} c_n$

[Hints: take $(1-x)^n$ then integrate both sides w.r.to x under the limits 0 and 1]

13. Show that

i) $c_1^2 + 2c_2^2 + 3c_3^2 + \dots\dots\dots + nc_n^2 = \frac{(2n-1)!}{[(n-1)!]^2}$

[Hints: do like problem no.8]

ii) $c_2 + 2c_3 + 3c_4 + \dots\dots\dots + (n-1)c_n = 1 + (n-2)2^{n-1}$

Inequality

First we will discuss about AM (Arithmetic mean), GM (Geometric mean) and HM (Harmonic mean).

AM of any n *positive* real number is defined as = $(x_1 + x_2 + ...+ x_n)/n$.

GM of any n *positive* real number is defined as = $(x_1x_2....x_n)^{1/n}$

HM of any n *positive* real number is defined as = $n/\{(1/x_1) + (1/x_2) + ... + (1/x_n)\}$

Theorem: AM ≥ GM ≥ HM

Proof: Let, a and b are two positive real numbers.

Then, $(a - b)^2 \geq 0$
- ⇨ $(a + b)^2 - 4ab \geq 0$
- ⇨ $(a + b)^2 \geq 4ab$
- ⇨ $\{(a + b)/2\}^2 \geq ab$
- ⇨ $(a + b)/2 \geq \sqrt{(ab)}$
- ⇨ AM ≥ GM

This is proved for two positive real numbers. It can be extended to any number of positive real numbers.

Now, we have, $(a + b)/2 \geq \sqrt{(ab)}$
- ⇨ $1/\sqrt{(ab)} \geq 2/(a + b)$
- ⇨ $\sqrt{(ab)}/ab \geq 2/(a + b)$
- ⇨ $\sqrt{(ab)} \geq 2ab/(a + b)$
- ⇨ $\sqrt{(ab)} \geq 2/\{(1/a) + 1/b)\}$
- ⇨ GM ≥ HM

This is proved for two positive real numbers. It can be extended to any number of positive real numbers.

Equality holds when a = b.

Note that "positive" is written in *Italic* to emphasize on the word that whenever you will be using AM ≥ GM ≥ HM then all the real numbers must be positive.

Problem 1: Prove that $a^3 + b^3 + c^3 \geq 3abc$ where a, b, c are positive real numbers.

Solution 1: Applying AM ≥ GM on a^3, b^3 and c^3 we get,

$(a^3 + b^3 + c^3)/3 \geq (a^3b^3c^3)^{1/3}$

⇨ $(a^3 + b^3 + c^3)/3 \geq abc$

⇨ $a^3 + b^3 + c^3 \geq 3abc$.

Weighted AM, GM, HM: If $x_1, x_2, \ldots, x_n$ are n real numbers with weights $w_1, w_2, \ldots, w_n$ then weighted AM is defined as,

Weighted AM = $(x_1w_1 + x_2w_2 + \ldots + x_nw_n)/(w_1 + w_2 + \ldots + w_n)$

Weighted GM = $\{(x_1^{w_1})(x_2^{w_2})\ldots(x_n^{w_n})\}$^$\{1/(w_1 + w_2 + \ldots + w_n)\}$

Weighted HM = $(w_1 + w_2 + \ldots + w_n)/\{(w_1/x_1) + (w_2/x_2) + \ldots + (w_n/x_n)\}$

Also, Weighted AM ≥ Weighted GM ≥ Weighted HM but remember the word positive whenever applying this.

Combinatorics

Fundamental theorem: If a work can be done in m ways and another work can be done in n ways then the two works can be done simultaneously in mn ways.

Note: Emphasize on the word simultaneously because most of the students get confused where to apply multiplication and where to apply addition. When both the works need to be done to complete a set of work then apply multiplication and if the works are disjoint then apply addition.

Permutation: There are n things and we need to take r things at a time and we need to arrange it with respect to order then the total number of ways is ^nP_r where $^nP_r = n!/(n - r)!$.

For example there are 3 tuples (1, 2, 3)

6 permutations are possible = (1, 2, 3); (1, 3, 2); (2, 1, 3); (2, 3, 1); (3, 1, 2) and (3, 2, 1)

Now, we will check by the formula.

Here number of permutations = $^3P_3 = 3!/(3 - 3)! = 3\times2/1 = 6$.

Proof: We can take 1st thing in n ways, 2nd thing in (n − 1) ways, ..., r^{th} thing in {n − (r − 1)} = (n − r + 1) ways.

By fundamental theorem,
total number of ways = $n(n - 1)(n - 2)....(n - r + 1)$

$= n(n - 1)(n - 2)....(n - r + 1)(n - r)(n - r - 1)....2\times1/\{(n - r)(n - r - 1)..... \times2\times1\}$

(Multiplying numerator and denominator by $(n - r)(n - r - 1).... \times2\times1$)

$= n!/(n - r)!$

So, $^nP_r = n!/(n - r)!$

Proved.

Combination: If there are n things and we need to select r things at a time (order is not important) then total number of ways of doing this = ${}^{n}C_{r}$ = n!/(n − r)! ×r!.

For example, there are 5 numbers (1, 2, 3, 4, 5). We need to select 3 at a time.

Total number of ways = (1, 2, 3); (1, 2, 4); (1, 2, 5); (1, 3, 4); (1, 3, 5); (1, 4, 5); (2, 3, 4); (2, 3, 5); (2, 4, 5); (3, 4, 5) i.e. 10 number of ways.

Now, we will check by formula.

Here number of combinations = ${}^{5}C_{3}$ = 5!/(5 − 3)! ×2! = 5×4×3!/3! ×2 = 10. Note that here (1, 2, 3) is equivalent to (1, 3, 2) etc. as order is not important in combination but this is important in permutation.

Proof: Now, the order is not important.

Hence the number of ways r things can permutate among themselves is r!.

Therefore, $r! \times {}^{n}C_{r} = {}^{n}P_{r} = n!/(n-r)!$

⇨ ${}^{n}C_{r} = n!/\{(n-r)! \times r!\}$ Proved.

Number of non-negative solution:

Problem: $a_1 + a_2 + \dots + a_r = n$ where $a_1, a_2, \dots, a_n$, n are all integers. Prove that the number of non-negative solution of the equation is ${}^{n+r-1}C_{r-1}$ (or, $\binom{n+r-1}{r-1}$)

We will prove it by induction.

Clearly, this is true for r = 1.
Let, this is true for r = k i.e. number of non-negative solution when there are k variables in the LHS is ${}^{(n+k-1)}C_{(k-1)}$.

Now, number of non-negative solution for r = k + 1 i.e. when an extra variable gets added in LHS is $\sum_{i=0}^{n}\binom{n-i+k-1}{k-1}$

Now, we have to prove that, $\sum_{i=0}^{n}\binom{n-i+k-1}{k-1} = \binom{n+k}{k}$.
Now, we will prove this by another induction. For, n = 1,
LHS = ${}^{k}C_{(k-1)} + {}^{(k-1)}C_{(k-1)} = k + 1$
RHS = ${}^{(k+1)}C_{k} = k + 1$.

So, this is true for n = 1.

Let, this is true for n = p i.e. we have, $\sum_{i=0}^{p}\binom{p-i+k-1}{k-1} = \binom{p+k}{k}$.

For, n = p + 1, LHS = $\sum_{i=0}^{p+1}\binom{p+1-i+k-1}{k-1}$

= $^{(p+k)}C_{(k-1)} + \sum_{i=0}^{p}\binom{p-i+k-1}{k-1}$

= $^{(p+k)}C_{(k-1)} + {}^{(p+k)}C_k$

= (p + k)!/{(p + 1)!×(k − 1)!} + (p + k)!/{p!×k!}

= [(p + k)!/{(p + 1)!×k!}](k + p + 1)

= (p + k + 1)!/{(p + 1)!×k!}

= $\binom{p+1+k}{k}$

= RHS for n = p + 1.

Number of positive solutions:

Problem: $a_1 + a_2 + \ldots + a_r = n$ where $a_1, a_2, \ldots, a_r$, n are all positive integers. Prove that number of positive solutions of this equation is $^{n-1}C_{r-1}$.

Solution: We will prove this by induction.

Clearly this is true for r = 1.

Let this is true for r = k i.e. if there are k number of variables in LHS then number of solutions of the equation is $^{n-1}C_{k-1}$.

Now, for r = k + 1 i.e. if an extra variable gets added then number of solutions of the equation is $\sum_{i=1}^{n-k}\binom{n-i-1}{k-1}$

We have to prove that, $\sum_{i=1}^{n-k}\binom{n-i-1}{k-1} = {}^{n-1}C_k$. We will prove this by another induction.

Clearly this is true for n = 1.

Let this is true for n = p i.e. $\sum_{i=1}^{p-k}\binom{p-i-1}{k-1} = {}^{p-1}C_k$.

Now, for n = p + 1, LHS = $\sum_{i=1}^{n-k+1}\binom{p-i}{k-1} = {}^{p-1}C_{k-1} + {}^{p-1}C_k$ (from above)

= $^{p}C_k$ = RHS for n = p + 1.

Proved.

Problem: Find number of terms in the expansion of $(x + y + z + w)^n$.

Solution:

Now, there are 4 variables and any term consists of the 4 terms such that,

$x_1 + y_1 + z_1 + w_1 = n$ where x_1, y_1, z_1, w_1 are powers of x, y, z, w in any term.

Now, x_1, y_1, z_1, w_1 runs from 0 to n.

So, we need to find number of non-negative solution of this equation and we are done with number of terms of this equation.

From previous article we know, number of non-negative solution of this equation = ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$.

In general if there are r variables then number of terms = ${}^{n+r-1}C_{r-1}$.

Problem: Find number of terms which are independent of x in the expansion of $(x + y + z + w)^n$.

Solution:

Now, $x_1 = 0$. So, we need to find number of non-negative solution of the equation, $y_1 + z_1 + w_1 = n$ and then we need to subtract this from ${}^{n+3}C_3$ and we are done with the number of terms which are independent of x.

It is, ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$.

So, number of terms which are independent of x = ${}^{n+3}C_3 - {}^{n+2}C_2$.
In general if there are r variables then number of terms excluding one variable = ${}^{n+r-1}C_{r-1} - {}^{n+r-2}C_{r-2}$.
Similarly, we can find number of terms independent of x and y and so on.

Problem: Prove that number of ways of distributing n identical things among r members where every member gets at least 1 thing is ${}^{n-1}C_{r-1}$.

Solution: Let, first member gets x_1 things, second member gets x_2 things and so on i.e. t^{th} member gets x_t things.
So, $x_1 + x_2 + \dots + x_r = n$

Now, we need to find number of positive solutions of this equation because $x_1, x_2, \ldots, x_n > 0$.

From above it is ${}^{n-1}C_{r-1}$.

Problem: In an arrangement of m H's and n T's, an uninterrupted sequence of one kind of symbol is called a run. (For example, the arrangement HHHTHHTTTH of 6 H's and 4T's opens with an H-run of length 3, followed successively by a T-run of length 1, an H-run of length 2, a T-run of length 3 and, finally an H-run if length 1.)

Find the number of arrangements of m H's and n T's in which there are exactly k H-runs.

Solution: Now, m H's can be put in k places with k+1 holes (spaces) between them in ${}^{m-1}C_{k-1}$ ways.

Now, k – 1 spaces between the H's must be filled up by at least one T.

So, number of ways is ${}^{n-1}C_{k-2}$.

So, in this case number of ways = ${}^{m-1}C_{k-1} \times {}^{n-1}C_{k-2}$.

Now, if k spaces (i.e. one space from either side first or last) can be filled by n T's where in every space at least one T is there in ${}^{n-1}C_{k-1}2$ ways.

So, total number of ways in this case = $2 \times {}^{m-1}C_{k-1} \times {}^{n-1}C_{k-1}$.

Now, if k+1 spaces (i.e. including first and last space) can be filled up by n T's where in every space at least one T is there in ${}^{n-1}C_{k}$ ways.

So, in this case total number of ways = ${}^{m-1}C_{k-1} \times {}^{n-1}C_{k}$.

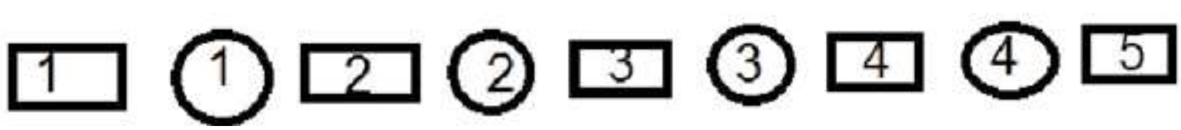
So, total number of ways = ${}^{m-1}C_{k-1}({}^{n-1}C_{k-2} + 2 \times {}^{n-1}C_{k-1} + {}^{n-1}C_{k})$

$= {}^{m-1}C_{k-1}({}^{n-1}C_{k-2} + {}^{n-1}C_{k-1} + {}^{n-1}C_{k-1} + {}^{n-1}C_{k})$

$= {}^{m-1}C_{k-1}({}^{n}C_{k-1} + {}^{n}C_{k}) = {}^{n-1}C_{k-1} * {}^{n+1}C_{k}$

Problem: Show that number of ways in which four distinct integers can be chosen from 1, 2, …, n (n ≥ 7) such that no two are consecutive is equal to ${}^{n-3}C_4$.

Solution:



We choose 4 integers as shown in figure by circle.

So, there are maximum 5 spaces between them shown in figure by boxes.

Now, let us say, 2, 3, 4 spaces i.e. boxes are to be filled by other n – 4 integers (4 integers already chosen for 4 circles).

Number of ways = ${}^{n-5}C_2$ (As number of ways is ${}^{n-1}C_{r-1}$ for at least one to be there)

Similarly, for 1, 2, 3, 4, boxes and 2, 3, 4, 5 boxes to be filled by other n– 4 integers number of ways = $2\times{}^{n-5}C_3$.

For 5 boxes to be filled by other n – 4 integers number of ways = ${}^{n-5}C_4$.
Total number of ways = ${}^{n-5}C_2 + 2*{}^{n-5}C_3 + {}^{n-5}C_4 = ({}^{n-5}C_2 + {}^{n-5}C_3) + ({}^{n-5}C_3 + {}^{n-5}C_4) = {}^{n-4}C_3 + {}^{n-4}C_4 = {}^{n-3}C_4$.

Problem: Prove that number of ways of distributing n identical things to r members (no condition) is ${}^{n+r-1}C_{r-1}$.

Solution: Let first member gets x_1 things, second member gets x_2 things and so on i.e. t^{th} member gets x_t things.
We have, $x_1 + x_2 + ... + x_r = n$

We need to find number of non-negative solutions of this equation.

From above it is ${}^{n+r-1}C_{r-1}$.

Proved.

Problem: Find the number of all possible ordered k-tuples of non-negative integers ($n_1, n_2, ..., n_k$) such that $\sum_{i=1}^{k} n_i = 100$.

Solution: Clearly, it needs number of non-negative solution of the equation, $n_1 + n_2 + ... + n_k = 100$.

It is ${}^{n+k-1}C_{k-1}$.

Problem: Show that the number of all possible ordered 4-tuples of non-negative integers (n1, n2, n3, n4) such that n1 + n2 + n3 + n4 ≤ 100 is $^{104}C_4$.

Solution: Clearly, required number = $^{3}C_3 + {}^{4}C_3 + {}^{5}C_3 + \ldots.. + {}^{103}C_3$
= $^{104}C_4$.

Problem: How many 6-letter words can be formed using the letters A, B and C so that each letter appears at least once in the word?

Solution: Let x_1 number of A, x_2 number of B and x_3 number of C are chosen where x_1, x_2, x_3 >0

Now, $x_1 + x_2 + x_3 = 6$.

Number of positive solution of this equation is $^{6-1}C_{3-1} = {}^{5}C_2 = 10$.

So, combinations are as follows,

4 A, 1 B, 1 C, number of words = 6!/4! = 30

3 A, 2 B, 1 C, number of words = 6!/(3!×2!) = 60

3 A, 1 B, 2 C, number of words = 6!/(3!×2!) = 60

2 A, 1 B, 3 C, number of words = 6!/(2!×3!) = 60

2 A, 2 B, 2 C, number of words = 6!/(2!×2!×2!) = 90

2 A, 3 B, 1 C, number of words = 6!/(2!×3!) = 60

1 A, 1 B, 4 C, number of words = 6!/4! = 30

1 A, 2 B, 3 C, number of words = 6!/(2!×3!) = 60

1 A, 3 B, 2 C, number of words = 6!/(3!×2!) = 60

1 A, 4 B, 1 C, number of words = 6!/4! = 30

So, total number of words = 30 + 60 + 60 + 60 + 90 + 60 + 30 + 60 + 60 + 30 = 540.

Problem: All the permutations of the letters a, b, c, d, e are written down and arranged in alphabetical order as in a dictionary. Thus the arrangement abcde is in the first position and abced is in the second position. What is the position of the arrangement debac?

Solution:

Now, first fix a at first place. Number of arrangements = 4!

Now, fix b at first place. Number of arrangements = 4!

Now, fix c at first place. Number of arrangements = 4!

Now comes d the first letter of the required arrangement.

Now fix d at first position & a at second position.
Number of arrangement = 3!

Fix b at second place. Number of arrangement = 3!

Fix c at second place. Number of arrangement = 3!

Now, comes e at second place and we have de.

Now, fix a at third place. Number of arrangement = 2!

Now comes b which is required and we have deb.

Then comes a and then c.

So, debac comes after (4! + 4! + 4! + 3! + 3! + 3! + 2!) = 92 arrangement.

So, it will take 92 + 1 = 93^{rd} position.

Problem: x red balls, y black balls and z white balls are to be arranged in a row. Suppose that any two balls of the same color are indistinguishable. Given that x + y + z = 30, show that the number of possible arrangements is the largest for x = y = z = 10.

Solution:
Clearly, number of possible arrangement is $(x + y + z)!/\{x!\times y!\times z!\}$
$= 30!/\{x!\times y!\times z!\}$

Now, it will be largest when $x!\times y!\times z!$ = minimum.

Let us say, x = 12 and y = 8

Now, $12!\times 8! = 12\times 11\times 10!\times 10!/(10\times 9) = (12*11/10*9)*(10!)^2$

⇨ $12!\times 8!/(10!)^2 = (12\times 11)/(10\times 9) > 1$

⇨ $(10!)^2 < 12\times 8!$

⇨ It will be least when $x = y = z = 10$.

Problem: Find number of arrangements of the letters of the word MISSISIPPI.

Solution:
Number of letters = 10

Number of I's = 4, number of S's = 3, number of P's = 2

Therefore, total number of words that can be formed from the letters of the word is $10!/(4!\times 3!\times 2!)$.

Problem: Find the number of words (meaningful or non-meaningful) that can be formed from the letters of the word MOTHER.

Solution:

Number of letters = 6.

All are distinct.

Hence total number of words = 6!

Problem: Show that the number of ways one can choose a set of distinct positive integers, each smaller than or equal to 50, such that their sum is odd, is 2^{49}.

Solution: The sum is odd.
We need to select odd number of integers.

Now, we can select 1 integer from 50 integers in ${}^{50}C_1$ ways.

We can select 3 integers from 50 integers in ${}^{50}C_3$ ways.

…

..

We can select 49 integers from 50 integers in ${}^{50}C_{49}$ ways.

So, number of ways = $^{50}C_1 + {}^{50}C_3 + \ldots.. + {}^{50}C_{49}$.

Now, $^{50}C_0 + {}^{50}C_1 + \ldots.. + {}^{50}C_{49} + {}^{50}C_{50} = 2^{50}$

Now, $^{50}C_0 - {}^{50}C_1 + \ldots.. - {}^{50}C_{49} + {}^{50}C_{50} = 0$

Subtracting the above two equations we get,

$2({}^{50}C_1 + {}^{50}C_3 + \ldots. + {}^{50}C_{49}) = 2^{50}$

⇨ $^{50}C_1 + {}^{50}C_3 + \ldots. + {}^{50}C_{49} = 2^{49}$.

Number of ways of distributing n distinct things to r persons (r < n) so that every person gets *at least* one thing.

Total number of ways = r^n

Now, let A_i denotes that the i th person doesn"t get a gift and B denotes that every person gets at least one gift.

Therefore, $r^n = |B| + |A_1 \cup A_2 \cup A_3 \cup \ldots.. \cup A_r|$

Now, $|A_1 \cup A_2 \cup \ldots. \cup A_r|$

$= \sum A_i - \sum A_i \cap A_j + \sum A_i \cap A_j \cap A_k - \ldots.. + (-1)^{r-1}[A_1 \cap A_2 \cap \ldots \cap A_r]$

Now, $|A_i| = (r-1)^n$, $|A_i \cap A_j| = (r-2)^n$,...., $A_1 \cap A_2 \cap \ldots. \cap A_r = (r-r)^n = 0^n$

So, $|A_1 \cup A_2 \cup \ldots \cup A_r| = {}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots. + (-1)^{r-}$

Therefore, $|B| = r^n - [{}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots. + (-1)^{r-1}{}^rC_r 0^n]$

$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \ldots. + (-1)^r {}^rC_r 0^n$

Polynomial

Let, P(x) be a polynomial of degree d, then P(x) can be written as,

$P(x) = a_1x^d + a_2x^{d-1} + a_3x^{d-2} + ... + a_{d-1}x^2 + a_dx + a_{d+1}$

Remainder theorem: Consider a polynomial of degree d > 1. P(x) gives the remainder P(a) if P(x) is divided by x – a.

Proof: Let P(x) = (x – a)Q(x) + R where R is constant as the divider is linear so at most degree of R is 0 i.e. free of x or constant.

Q(x) is the quotient and R is remainder.

Putting x = a in the above expression we get,

P(a) = (a – a)Q(a) + R

- R = P(a)

Problem 1: Find the remainder when $P(x) = x^2 + x + 1$ is divided by x + 1.

Solution 1: From above we have remainder = R = P(-1)
So, $R = P(-1) = (-1)^2 + (-1) + 1 = 1$.

Consider a polynomial P(x) of degree d > 1. We will now find the remainder if there is any repeated root in divider. So, we will find the remainder when P(x) is divided by $(x – a)^2$.

Let, $P(x) = (x – a)^2Q(x) + R(x)$ (Note that this time R(x) is not constant and have degree 1 as divider is quadratic)

Let, R(x) = Ax + B.

Q(x) is the quotient.

Now, putting the value of R(x) in the above equation the equation becomes,

$P(x) = (x – a)^2Q(x) + Ax + B$

Now, putting x = a in the above equation we get,

$P(a) = (a - a)^2Q(a) + Aa + B$

- $Aa + B = P(a)$

Now, differentiating the above equation w.r.t. x we get,

$P''(x) = 2(x - a)Q(x) + (x - a)^2Q''(x) + A$

Putting x = a in the above equation we get,

$P''(a) = 2(a - a)Q(a) + (a - a)^2Q''(a) + A$

$A = P''(a)$

$B = P(a) - Aa = P(a) - a\times P''(a)$

$R(x) = Ax + B = P''(a)x + P(a) - a*P''(a) = P''(a)(x - a) + P(a)$

Problem 1: Consider a polynomial P(x) of degree d > 2. Let R(x) be the remainder when P(x) is divided by $(x - 1)^2$. P'(1) = P(1) = 1. Find R(x).

Solution 1: From the above result we have,

$R(x) = P''(1)(x - 1) + P(1) = 1*(x - 1) + 1 = x.$

- Consider a polynomial P(x) of degree d > 1. Now we will find the remainder when the divider is quadratic and have two distinct roots. Let us find the remainder when P(x) is divided by (x – a)(x – b).

Let, $P(x) = (x - a)(x - b)Q(x) + R(x)$

Q(x) = quotient and R(x) = remainder = Ax + B.

Putting value of R(x) in the above equation we get,

$P(x) = (x - a)(x - b)Q(x) + Ax + B$

Putting x = a in the above equation we get,

$P(a) = (a - a)(a - b)Q(a) + Aa + B$

- $Aa + B = P(a)$ ....(i)

Now, putting x = b in the above equation we get,

$P(b) = (b - a)(b - b)Q(b) + Ab + B$

- $Ab + B = P(b)$ ......(ii)

Now, from (i) and (ii) we get, $Aa + B - Ab - B = P(a) - P(b)$

$A = \{P(a) - P(b)\}/(a - b)$

$B = \{aP(b) - bP(a)\}/(a - b)$

$R(x) = Ax + B = \{P(a) - P(b)\}x/(a - b) + \{aP(b) - bP(a)\}/(a - b)$

- Consider a polynomial of degree $d > 1$. Now, we will work with quotient. Let $Q(x)$ be the quotient when $P(x)$ is divided by $(x - a)$. Then we will have the relation $Q(a) = P''(a)$.

Proof: Let, $P(x) = (x - a)Q(x) + R$ where R is remainder and note that R is constant as divider is linear i.e. of degree 1.

Differentiating the above equation w.r.t. x we get,

$P''(x) = Q(x) + (x - a)Q''(x) + R''$

Note that $R'' = 0$ as R is constant.

So, we have, $P''(x) = Q(x) + (x - a)Q''(x)$

Putting $x = a$ in the above equation we get,

$P''(a) = Q(a) + (a - a)Q''(a)$

- $Q(a) = P''(a)$

Problem 1: Consider a polynomial $P(x) = x^3 + 3x^2 + 2x + 1$. Q(x) is the quotient when P(x) is divided by x - 1. Find the value of Q(1).

Solution 1: From the above we have the result, $Q(1) = P''(1)$

Given $P(x) = x^3 + 3x^2 + 3x + 1$

$P''(x) = 3x^2 + 6x + 3$

$P''(1) = 3\times1^2 + 6\times1 + 3$

$Q(1) = P''(1) = 12$.

Problem 2: Consider a polynomial P(x) of degree d > 1. Q(x) = 4x + 3 is the quotient when P(x) is divided by x - 7. Find the slope of P(x) at x = 7 or put in other way find P'(7).

Solution 2: From above we have, $P''(7) = Q(7) = 4*7 + 3 = 31$.

- Consider a polynomial P(x) of degree d > 1. Now, we will see relation between quotient i.e. Q(x) and P(x) when there is repeated root in the divider. Let, Q(x) is the quotient when P(x) is divided by $(x - a)^2$. Then we will have the relation Q(a) = P““(a)/2.

Proof: Let, $P(x) = (x - a)^2Q(x) + R(x)$ (Note that R(x) is linear here and so R““(x) = 0)

Differentiating w.r.t. x we get,

$P“(x) = 2(x - a)Q(x) + (x - a)^2Q“(x) + R“(x)$

Differentiating again w.r.t. x we get,

$P““(x) = 2Q(x) + 2(x - a)Q“(x) + 2(x - a)Q“(x) + (x - a)^2Q““(x) + R““(x)$

Putting x = a in the above expression we get,

$P““(a) = 2Q(a) + 2(a - a)Q“(a) + 2(a - a)Q“(a) + (a - a)^2Q““(a) + 0$

- Q(a) = P““(a)/2.

Problem 1: Let $Q(x) = 3x^2 + 2$ is the quotient when P(x) is divided by $(x - 1)^2$. Find the value of P““(1).

Solution 1: From the above result we have,

$P““(1) = 2Q(1) = 2(3*1^2 + 2) = 10.$

Problem 2: Let P(x) be a polynomial of degree d > 2. Q(x) is the quotient when P(x) is divided by $(x - 2)^2$. Q(2) = 4. Find P''(2).

Solution 2: From the above result we have,

P““(2) = 2Q(2) = 2*4 = 8.

Tips to solve problems:

1. The remainder of P(x) divided by x+ a can be found by putting x = -a i.e. P(-a) will give the remainder when P(x) is divided by x + a.
2. If there is a root between (a, b) then P(a) and P(b) will be of opposite sign.
3. If P(x) is strictly increasing or decreasing then P(x) have at most one real

root. P(x) can be proved increasing if P"(x) > 0 and decreasing if P"(x) < 0.

4. To find number of real roots in P(x) draw the graph of LHS and RHS and count the number of intersection points and that is the answer.
5. If there is any mention of sum of coefficients then think of P(1) and vice versa.
6. If there is any repeated root think of derivative.
7. If there is any involvement of quotient Q(x) then write the equation P(x) = D(x)Q(x) + R(x) where D(x) is the divider and think of derivative.
8. If there is any mention of becoming the polynomial prime think of P(0) i.e. the constant term.
9. R(x) has at most degree d - 1 where d is the degree of divider.
10. Complex roots come in pair. If a polynomial is of degree d which is odd then the polynomial must have at least one real root.

11. If there is any question/mention of multiplicity of a root then do derivative for m + 1 times where m is at most multiplicity of the root and show that $P^{(m+1)}(x)$ doesn't have the root.

Problem 1: Consider the polynomial $P(x) = 30x^7 - 35x^6 + 42x^5 + 210x^3 - 1470$. Prove that P(x) = 0 have only one real root and the root lies between (1, 2).

Solution 1: Now, $P(x) = 30x^7 - 35x^6 + 42x^5 + 210x^3 - 1470$

$P''(x) = 210x^6 - 210x^5 + 210x^4 + 630x^2$

$P''(x) = 210x^4(x^2 - x + 1) + 630x^2$

$P''(x) = 210x^4\{(x - \frac{1}{2})^2 + \frac{3}{4}\} + 630x^2$

Which is always greater than 0.

P(x) is increasing.

P(x) has at most one real root.

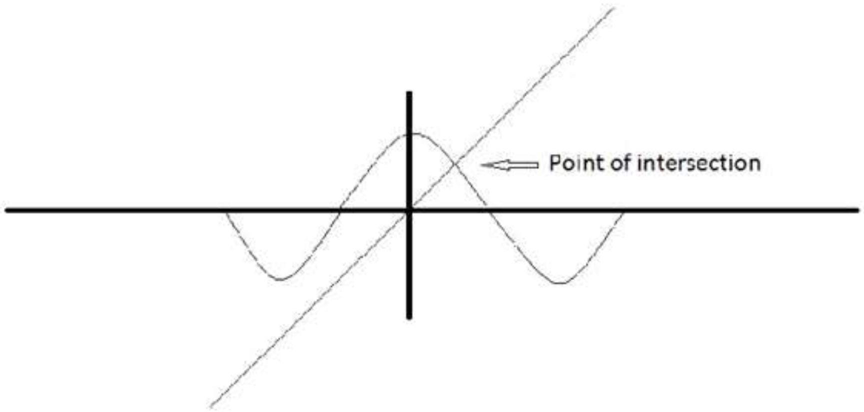
Now, $P(1) = 30\times1^7 - 35\times1^6 + 42\times1^5 + 210\times1^3 - 1470 < 0$

And, $P(2) = 30\times2^7 - 35\times2^6 + 42\times2^5 + 210\times2^3 - 1470 > 0$

There is a root between (1, 2) & this root is the only real root of P(x) = 0.

Problem 2: Prove that x = cosx has only one real root.

Solution 2: Now, drawing the graph of LHS and RHS i.e., y = x and y = cosx.



Clearly, there is one point of intersection. There is one real root.

Problem 3: P(x) and Q(x) are two polynomials such that the sum of the co-efficient is same for both. Prove that the remainders when P(x) and Q(x) are divided by x - 1 are same.

Solution 3: Let, $P(x) = a_1x^d + a_2x^{d-1} + ... + a_dx + a_{d+1}$

Putting x = 1 we get,

$P(1) = a_1 + a_2 + ... + a_d + a_{d+1}$ = sum of the coefficients.

Now, from remainder theorem (also tips number 1) we have the remainder P(x) gives when P(x) is divided by (x − 1) is P(1) = sum of coefficients.

Similar thing goes for Q(x) i.e., Q(1) = sum of coefficients.

Now, it is given that sum of coefficients of P(x) = sum of coefficients of Q(x)

P(1) = Q(1).

Problem 4: P(x) is a polynomial of degree d > 1 with integer coefficients. P(1) is divisible by 3. All the coefficients are placed side by side in any order to make a positive integer. For example if there is 2 coefficients 20 and 9 then the numbers formed are 209 or 920. Prove that the number thus generated is divisible by 3.

Solution 4: Now, in the previous example we have seen that P(1) = sum of coefficients of P(x).

Now, the sum is divided by 3.

As per the rule of divisibility by 3 says a number is divisible by 3 if sum of the digits is divisible by 3.

- ⇨ The sum of the digits of the coefficients of P(x) is divisible by 3.

Now, whatever be the order of placing the digits of the coefficients of P(x) the sum of their digits must be same.

- ⇨ The sum of the digits is divisible by 3.
- ⇨ Thus the number generated with whatever be the order of placing the digits is divisible by 3.

Problem 5: Show that the equation x(x - 1)(x - 2).....(x - 2009) = c has real roots of multiplicity at most 2.

Solution 7: (tips number 11)

As per the tips we need to show P““(x) doesn't have the root which P(x) and P“(x) has.

We have, x(x - 1)(x - 2).....(x - 2009) = c

Differentiating w.r.t. x we get,

$(x - 1)(x - 2)....(x - 2009) + x(x - 2)(x - 3)....(x - 2009) + x(x - 1)(x - 3)...(x - 2009) + ... + x(x - 1)(x - 2)....(x - 2008) = 0$

- ⇨ $c/x + c/(x - 1) + c/(x - 2) + ... + c/(x - 2009) = 0$

  (Putting value from the given equation)
- ⇨ $c\{1/x + 1/(x - 1) + 1/(x - 2) + ... + 1/(x - 2009)\} = 0$

Now, differentiating again w.r.t. x we get,

$-c \times \{1/x^2 + 1/(x - 1)^2 + 1/(x - 2)^2 + ... + 1/(x - 2009)^2\} = 0$

This equation cannot hold true as sum of squares of real numbers equal to 0 but they are always greater than 0.

The given polynomial cannot have real roots of multiplicity more than 2.

Problem 6: Prove that the polynomial $P(x) = x^3 + x - 2$ have at least one real root.

Solution 8: (Tips number 10)

Let all the roots of P(x) is complex.

Complex roots come in pair.

There needs to be 4 roots of P(x)

But P(x) have at most 3 roots as the degree of the polynomial is 3.

⇨ Our assumption was wrong. It may have at most 2 complex root.

There is at least one real root of P(x).

Solved examples:

1. **Consider a polynomial P(x) of degree d > 1. Q(x) is the quotient when P(x) is divided by x - a. Prove that P'(a) = Limiting value of Q(x) as x -> a.**

Solution:

We can write, P(x) = (x – a)Q(x) + R(x) where R(x) is remainder when P(x) is divided by (x – a).

⇨ R(x) is constant.

Now, P"(a) = {P(x) – P(a)}/(x – a) as x -> a

- ⇨ P"(a) = {(x – a)Q(x) + R(x) – R(a)}/(x – a) as x -> a (P(a) = R(a))
- ⇨ P"(a) = Q(x) as x -> a + {R(x) – R(a)}/(x – a) as x -> a
- ⇨ P"(a) = Q(x) as x -> a + R"(a)
- ⇨ P"(a) = Q(x) as x -> a (as R"(a) = 0 because R(x) is constant)

2. **Consider a polynomial P(x) of degree d > 2. Let R(x) be the remainder when P(x) is divided by $(x - 1)^2$. P'(1) = P(1) = 1. Find R(x).**

Solution: We can write, $P(x) = (x – 1)^2Q(x) + R(x)$ where Q(x) is the quotient when P(x) is divided by $(x – 1)^2$.

- ⇨ $P"(x) = 2(x – 1)Q(x) + (x – 1)^2Q"(x) + R"(x)$
- ⇨ P"(1) = R"(1) = 1

And, P(1) = R(1) = 1

Now, R(x) is remainder when P(x) is divided by $(x – 1)^2$

⇨ R(x) is linear.

Say, R(x) = ax + b

Now, R"(x) = a

R"(1) = a = 1

Now, R(1) = a + b = 1

⇨ $b = 0$. So,
$R(x) = x$.

3. Let P(x) be a polynomial of degree d > 2. Q(x) is the quotient when P(x) is divided by $(x - 2)^2$. Q(2) = 4. Find P''(2).

Solution: We can write, $P(x) = (x - 2)^2Q(x) + R(x)$ where R(x) is remainder when P(x) is divided by $(x - 2)^2$.
So, R(x) is linear.

Now, $P''(x) = 2(x - 2)Q(x) + (x - 2)^2Q''(x) + R''(x)$
$P''(x) = 2Q(x) + 2(x - 2)Q''(x) + 2(x - 2)Q''(x) + (x - 2)^2Q''(x) + R''(x)$
$P''(2) = 2Q(2) + R''''(2)$

Now, $R''(2) = 0$ as R(x) is linear.

$P''(2) = 2Q(2) = 2\times4 = 8$.

4. P(x) and Q(x) are two polynomials such that the sum of the coefficient is same for both. Prove that the remainders when P(x) and Q(x) are divided by x - 1 are same.

Solution: Sum of the coefficients of P(x) and Q(x) are same.

⇨ $P(1) = Q(1)$.
Now, as per Remainder theorem P(1) and Q(1) are remainders when P(x) and Q(x) are divided by $(x - 1)$ and they are clearly same.

5. P(x) is a polynomial of degree d > 1 with integer coefficients. P(1) is divisible by 3. All the coefficients are placed side by side in any order to make a positive integer. For example if there is 2 coefficients 20 and 9 then the numbers formed are 209 or 920. Prove that the number thus generated is divisible by 3.

Solution: P(1) is divisible by 3.
⇨ Sum of the coefficients is divisible by 3.

⇨ If we place the coefficients side by side then the number formed will be divisible by 3 as the sum of the digits is divisible by 3 as per P(1) is divisible by 3.

6. **Consider a polynomial P(x) of degree d > 2. Q(x) is the quotient when P(x) is divided by D(x). D(x) is quadratic and x = a is a root. P''(a) = b and Q(a) = b/2. Prove that D(x) has repeated root at x = a.**

Solution: Let, D(x) = (x − a)(x − d)

We can write, P(x) = D(x)Q(x) + R(x) where R(x) is remainder when P(x) is divided by D(x). R(x) is linear as D(x) is quadratic.

- ⇨ P(x) = (x − a)(x − d)Q(x) + R(x)
- ⇨ P"(x) = (x − d)Q(x) + (x − a)Q(x) + (x − a)(x − d)Q"(x) + R"(x)
- ⇨ P""(x) = Q(x) + (x − d)Q"(x) + Q(x) + (x − a)Q"(x) + (x − d)Q"(x) + (x − a)Q"(x) + (x − c)(x − d)Q"(x) + R"(x)
- ⇨ P"(a) = 2Q(a) + 2(a − d)Q"(a) + R"(a)
- ⇨ b = 2(b/2) + 2(a − d)Q"(a) (R""(a) = 0 as R(x) is linear)
- ⇨ (a − d)Q"(a) = 0
- ⇨ d = a
- ⇨ D(x) has repeated root at x = a.

7. **Consider two polynomials P(x) and Q(x) of degree d > 0 with integer coefficients. P(0) = Q(0). Prove that there exists an integer n which divides both P(n) and Q(n).**

Solution: P(0) = Q(0)

- ➢ The constant term of P(x) and Q(x) are equal. Let,
  $P(x) = a_1x^p + a_2x^{p-1} + \ldots.. + a_px + n$

Let, $Q(x) = b_1x^q + b_2x^{q-1} + \ldots.. + b_qx + n$

Clearly n divides both P(n) and Q(n).

8. **Let P(x) be a polynomial of degree 3d - 1 where d > 0.**
   Let $P^{(i)}(0) = 3\times(i!)$ where $P^{(i)}(x)$ is i-th derivative of P(x) w.r.t. x. Prove that P(1) is divisible by 9.

Solution:

$P(x) = a_1x^n + a_2x^{n-1} + \ldots. + a_{n-1}x^2 + a_nx + a_{n+1}$ P(0)

$= a_{n+1} = 3$

$P"(0) = a_n = 3$

$P"(0) = (2!)a_{n-1} = 3\times 2!$

⇨ $a_{n-1} = 3$.

Similarly, $a_{n-2} = a_{n-3} = \dots = a_1 = 3$.

Now, $P(1) = a_1 + a_2 + \dots + a_{n+1} = a_1 + a_2 + \dots + a_{3d}$ as $n = 3d - 1$.

⇨ $P(1) = 3 + 3 + \dots$ 3d times. $= 3\times 3d = 9d$

⇨ $P(1)$ is divisible by 9.

9. Consider a polynomial $P(x)$ of degree $d > 1$. Given $P(0) = 25$. All the roots of $P(x)$ are distinct positive integers. $P^{(d)}(0) = d!$. Find the value of $P^{(d-1)}(0)/(d-1)!$ where $P^{(m)}(x)$ is m-th derivative of $P(x)$ w.r.t. x.

Solution:

Let, $P(x) = a_1x^d + a_2x^{d-1} + \dots + a_{d-1}x^2 + a_dx + 25$ (As $P(0) = 25$)

Now, $P''(0) = a_d = 1$

$P''(0) = (2!)a_{d-1} = 2!$

⇨ $a_{d-1} = 1$

Similarly $a_{d-2} = a_{d-3} = a_{d-4} = \dots = a_2 = a_1 = 1$

Now, $P^{(d-1)}(0) = (d - 1)!\times a_2$

⇨ $P^{(d-1)}(0)/(d - 1)! = a_2 = 1$

10. Let $P(x)$ and $Q(x)$ be two polynomials of degree d_1 and d_2 respectively where d_1 and d_2 are both odd. Prove that the sum of the squares of the number of real roots of $P(x)$ and $Q(x)$ cannot be equal to a^n where a and n are positive integers, $n > 1$.

Solution: Now, if $P(x)$ have complex roots then they will come in pair (complex + conjugate)

So, number of real roots of $P(x)$ must be odd as degree $= d_1 =$ odd.

Similarly, number of real roots of $Q(x)$ must be odd.

Let, number of real roots of $P(x)$ and $Q(x)$ are u and v respectively.

Now, let, $u^2 + v^2 = a^n$

As u and v are both odd, a is even.

Now, dividing the equation by 4 we get,

$1 + 1 \equiv 0 \pmod 4$ as $n > 1$

⇨ $2 \equiv 0 \pmod 4$

Which is impossible

Sum of squares of the number of real roots of P(x) and Q(x) cannot be equal to a^n where a and n are positive integers, n > 1.

10. Let a, b, c be three distinct integers, and let P be a polynomial with integer coefficients. Show that in this case the conditions P(a) = b, P(b) = c, P(c) = a cannot be satisfied simultaneously.

Solution:

Suppose the conditions are satisfied. We derive a contradiction.

$P(x) - b = (x - a)P_1(x)$ ……………… (1)

$P(x) - c = (x - b)P_2(x)$ ……………… (2)

$P(x) - a = (x - c)P_3(x)$ ……………… (3)

Among the numbers a, b, c, we choose the pair with maximal absolute difference.

Suppose this is |a − c|. Then we have

$|a - b| < |a - c|$……………………. (4)

If we replace x by c in (1), then we get

$a - b = (c - a)P_1(c)$.

Since $P_1(c)$ is an integer, we have $|a - b| \geq |c - a|$, which contradicts (4).

12. Let f(x) be a monic polynomial with integral coefficients. If there are four different integers a, b, c, d, so that f(a) = f(b) = f(c) = f(d) = 5, then there is no integer k, so that f(k) = 8.

Solution: Monic polynomial means the highest degree coefficient is 1.

Now, $f(x) = (x - a)(x - b)(x - c)(x - d)Q(x) + 5$

Now, $f(k) = (k - a)(k - b)(k - c)(k - d)Q(k) + 5$

⇨ $(k - a)(k - b)(k - c)(k - d)Q(k) = 3$ (As f(k) = 8)

3 is factor of at least four distinct integers k − a, k − b, k − c, k − d as a, b, c, d are distinct.

But this is impossible as 3 is prime and may have maximum 3 factors viz. 3, 1, -1.

Problem: Check whether $n^{68} + n^{37} + 1$ is divisible by $n^2 + n + 1$.

Solution: As ω is cube root of unity so we have $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$.

Put ω in place of n and check whether the given expression is divisible by $\omega^2 + \omega + 1$ or not. This is possible because $n^3 \equiv 1 \pmod{n^2 + n + 1}$

So, $n^{68} + n^{37} + 1 = \omega^{68} + \omega^{37} + 1 = \omega^2 + \omega + 1 = 0$

Hence, $n^2 + n + 1$ divides $n^{68} + n^{37} + 1$.

How to solve the questions like "how many real solutions does this equation have?" and you have a four-degree equation.

1. First check if you can factorize using Vanishing method with 1, 2, - 1, -2, maximum verify by 3.
2. Then check the degree of the equation. If it is odd then it has at least one real solution. If it is even then it may have no real solution at all because complex roots come in pair.
3. Then use Descartes" sign rule to evaluate if there is any positive or negative real roots. Descartes" sign rule says: check number of sign changes of the coefficients from higher degree to lower degree of the polynomial and that says number of maximum possible positive roots of the equation. If it has 4 sign changes then it may have 4 or 2 or 0 number of positive roots i.e. it comes down by an even number 2. Check of negative roots is my same method but of the polynomial P(-x). So, put x = -x and then find number of negative roots of the equation. If there is no then all roots are complex, otherwise it may have real roots.
4. Check whether the polynomial is increasing or decreasing for some value of x. For example, it is a fourth degree equation and we have evaluated that it may have 4 positive roots. And you see the polynomial is increasing for $x > 0$. Implies the polynomial doesn't meet the x-axis after $x > 0$. Therefore, all the roots it has negative but from Descartes" sign rule we have zero negative roots. Implies all the roots of the equation are complex.
5. Take any complex root of the equation as a + ib, then a − ib is also a root of the equation. Now, do $P(a + ib) - P(a - ib) = 0$ and check whether $b = 0$ for sure. If it comes out to be $b = 0$, then it has no imaginary roots as the imaginary part of the root is zero.

Problem: The number of real roots of $x^5 + 2x^3 + x^2 + 2 = 0$

Solution: First check x = -1 is a solution. So, we will first factorize it by vanishing method.

Now, $x^5 + 2x^3 + x^2 + 2$

$= x^5 + x^4 - x^4 - x^3 + 3x^3 + 3x^2 - 2x^2 - 2x + 2x + 2$

$= x^4(x + 1) - x^3(x + 1) + 3x^2(x + 1) - 2x(x + 1) + 2(x + 1)$

$= (x + 1)(x^4 - x^3 + 3x^2 - 2x + 2)$

It is a fifth degree equation and we have evaluated one real root x = -1.

Now, we have a four degree equation, $x^4 - x^3 + 3x^2 - 2x + 2 = 0$

Number of sign change = 4. Therefore, it may have 4, 2 or 0 positive roots. And it has 0 negative roots.

$P(x) = x^4 - x^3 + 3x^2 - 2x + 2$

P(0) = 2, P(1) = 3, P(2) = 18 and we are seeing that it is increasing with (+)-ve value of x.

So, $P"(x) = 4x^3 - 3x^2 + 3x - 2$
$= x(4x^2 - 4x + 1) + (x^2 + 2x - 2)$
$= x(2x - 1)^2 + (x - 1)(x + 2) + x > 0$ for $x > 2$

- ⇨ P(x) is increasing for x > 2.
- ⇨ P(x) may have negative real roots but from Descartes" sign rule it has no negative roots.
- ⇨ All the roots of the four degree equation are complex.
- ⇨ The equation has only one real root and that is x = -1.

Elements of Combinatorial Probability

RULE –I : If there are two groups G_1 & G_2;

$G_1 = \{a_1, a_2, \dots.., a_n\}$ consisting of n elements and $G_2 = \{b_1, b_2, \dots.., b_m\}$ consisting of m elements then the no. of pairs (a_i, b_j) formed by taking one element $a_i\ from\ G_1\ and\ b_j\ from\ G_2\ is\ n \times m$.

If there are k groups $G_1, G_2, \dots\dots, G_k$, such that

$$G_1 = \{a_1, a_2, \dots\dots, a_{n_1}\}$$

$$G_2 = \{b_1, b_2, \dots\dots, b_{n_2}\}$$

$$\vdots$$

$$\vdots$$

$$G_k = \{t_1, t_2, \dots\dots, t_{n_k}\}$$

Then the number ordered k–tuples $(a_{i1}, b_{i2}, \dots\dots, t_{ik})$ formed by taking one element from each group is $= n_1 \times n_2 \times \dots \dots \times n_k$

Example: 'Placing balls into the cells' amounts to choose one cell for each ball. Let there are r balls and n cells. For the 1st ball, we can choose any one of the n cells. Similarly, for each of the balls, we have n choices, assuming the capacity of each cell is infinite or we can place more than one ball in each cell. Hence the r balls can be placed in the n cells in n^r ways.

Applications:

1. **A die is rolled r times. Find the probability that –**

i) No ace turns up.

ii) No ace turns up.

Solution:

i) The experiment of throwing a die r times has $6 \times 6 \times 6 \dots.r\ times = 6^r$ possible outcomes.

Assume that all possible cases are equally likely. The no. of cases favorable to the event (A), 'no ace turns up' is 5^r.

By Classical Definition, $P[A] = \frac{N(A)}{N} = \frac{5^r}{6^r}$.

ii) P[an ace turns up] $= 1 - P[no\ ace\ turns\ up] = 1 - \frac{5^r}{6^r}$.

Remark : The all possible outcomes of 'r' throw of a die correspond to the placing r balls into $n = 6$ cells.

RULE –II:

Ordered Samples: Consider a population of n elements $a_1, a_2, \dots . a_n$ any order arrangement $a_{j1}, a_{j2}, \dots . , a_{jr}$ of r elements is called an ordered sample of size r, drawn from the population. Two procedure are possible –

i) Sampling with replacement: Here an element is selected from the population and the selected element is returned to the population before the next selection is made. Each selection is made from the entire population, so that the same element can be drawn more than ones.

ii) Sampling without replacement : Here an element once chosen is removed from the population, so that the sample becomes an arrangement without repetition.

- For a population with n elements and a prescribed sample size r, there are n^r different ordered samples with replacement and n(n−1)…..(n−r+1) $= n_{p_r}$ or $(n)_r$ different ordered samples without replacement.

Remark:

1. $n_{p_r} = n(n-1) \dots . (n-r+1)$ is defined if $n \in N$ and r is a non−negative integers. But $(n)_r = n(n-1) \dots . (n-r+1)$ is defined if $n \in R$ and r is non−negative integer. In the same way if $n \in \mathbb{R}$ then

$$n_{c_R} = \binom{n}{r} = \frac{n(n-1) \dots \dots . (n-r+1)}{r!}$$

Example : 1) A random sample of size 'r' with replacement is taken from a population of n elements. Find the probability that in the sample no element appear twice.

Solution: There are n^r sample in all. As the samples are drawn randomly, all samples are equally likely. The no. of the samples in which in which no element appears twice is the no. of samples drawn without replacement.

Favorable sample is $= n(n-1) \dots . (n-r+1) = (n)_r$

Hence, the probability is $= \frac{(n)_r}{n^r}$

Example: 2) If n balls are randomly placed into n cells, what is the probability that each cell will be occupied.

Solution: $P(A) = \frac{n!}{n^n}$.

SOLVED EXAMPLES:

1. **Find the probability that among five randomly selected digits, all digits are different.**

Ans:

$$P(A) = \frac{(10)_5}{10^5}$$

2. **In a city seven accidents occur each week in a particular week there occurs one accidents per day. Is it surprising?**

Ans :

$$P(A) = \frac{7!}{7^7}$$

3. **An elevator (lift) stands with 7 passengers and stops at 10th floor. What is the probability that no two passengers leave at the same floor?**

Solution:

$$N = 10,10 \ldots..10 \ (7 \ times)$$

$$= 10^7$$

$$N(A) = 10.9.8.7.6.5.4$$

$$= (10)_7$$

$$P(A) = \frac{(10)_7}{10^7}$$

4. **What is the probability that r individuals have different birthdays? Also show that the probability is approximately equal to $e^{-r(r-1)}/730$. How many people are required to make the prob. of distinct birthdays less than ½ ?**

Solution:

$$p = \frac{(365)_r}{365^r} = \frac{365.364 \ldots \ldots.(365 - r + 1)}{365.365 \ldots \ldots \ldots \ldots..365}$$

$$= 1\left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \ldots \ldots \left(1 - \frac{r-1}{365}\right)$$

$$\therefore \ln p = \sum_{k=1}^{r-1} \ln\left(1 - \frac{k}{365}\right)$$

$$For, 0 < x < 1, \qquad \ln(1 - x) \cong -x.$$

$$\therefore \ln p \cong \sum_{k=1}^{r-1}\left(-\frac{k}{365}\right) = -\frac{1}{365}\left(\sum_{k=1}^{r-1} k\right)$$

$$= -\frac{r(r-1)}{2(365)} = -\frac{r(r-1)}{730}$$

$$\therefore p = e^{-r(r-1)/730}$$

$$For\ p = \frac{1}{2}, \ln p = -\ln 2 = -0.693,$$

$$\therefore \frac{r(r-1)}{730} = 0.693$$

$$\therefore r^2 - r - 506 = 0$$

$$\Rightarrow (r-23)(r+22) = 0$$

$$\Rightarrow r = 23$$

∴ More than 23 people are required.

5. **Six dice are thrown. What's the prob. that every possible number will appear.**

Hints: $p = \frac{6!}{6^6} = 0.0154$

6. **There are four children in a family. Find the prob. that**
   (a) **At least two of them have the same birthday?**
   (b) **Only the oldest and the youngest have the same birthday?**

Hints: (a)

$$p_1 = 1 - \left\{\frac{(365)_4}{365^4}\right\} = 1 - p\ \{them\ have\ different\ birthdays\}$$

(b)

$$p_2 = \frac{365 \times 365 \times 363}{365^4} = \frac{(365)_3}{365^4}$$

7. **The number 1, 2,….,n are arranging in a random order. Find the probability that digits (a) 1, 2 , (b) 1, 2, 3 appears as neighbours in the order named.**

Hints: consider (1, 2) as a single digit then there are (n−1) entities which can be arranged in (n−1)! ways.

(a) Required prob. is $= \frac{(n-1)!}{n!} = \frac{1}{n}$.
(b) Required prob. is $= \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$.

8. **(i) In sampling with replacement find the prob. that a fixed element be included at least once.**
   (i) In sampling without replacement find the prob. that a fixed element of a population of n elements to be included in a random sample of size r.

Hints:

(i) $P_1 = 1 - P$ [the fixed element is not included in the sample WOR]

$$= 1 - \frac{(n-1)^r}{n^r}$$

(ii) $P_2 = 1 - P$[a fixed element is not included in the sample WR]

$$= 1 - \frac{(n-1)_r}{(n)_r} \; 1 - \frac{n-r}{n} = \frac{r}{n}.$$

9. **There is 3 volume dictionary among 30 books is arranged in a shelf in random way. Find the prob. of 3 volume standing in an increasing order from left to right? (The vols. are not necessary side by side).**

Solution: The order of the 3 vols. doesn't depend on the arrangement of the remaining books. Here 3 vols. can be arranged in 3! ways of which only one case V_1, V_2, V_3 is favorable. Hence prob. is 1/3!.

10. **Two fair dice are thrown 10 times. Find the prob. that the first 3 throws result in a sum of 7 and the last 7 throws in a sum of 8.**

Solution: $\Omega_k = \{(i,j): i,j = 1\ (1)6\}, k = 1(1)10$, be the sample space of the kth throw of a pair of dice, the sample space of the experiment is

$$\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 \times \ldots . \times \Omega_{10}.$$

$$N = n(\Omega) = \text{n}(\,\Omega_1 \times \Omega_2 \times \Omega_3 \times \ldots . . \times \Omega_{10}) = 36^{10}$$

Let, $A = \{(i,j): i + j = 7, i,j = 1(1)6\}$, the event of getting a sum of 7 in a throw of a pair of dice.

And $B = \{(i,j): i + j = 8, i,j = 1(1)6\}$, the event of getting a sum of 8 in a throw of a pair of dice.

Our event is $= A \times A \times A \times B \times \ldots B$

Favorable cases are $= \{(3,4),(2,5),(1,6),(2,6),(3,5),(4,4)$

$$N(A) = \{n(A)\}^3\{n(B)\}^7 = 6^3 \times 5^7 \ldots\ldots\}$$

$$\therefore Required\ Probability = \frac{6^3 \times 5^7}{36^{10}}$$

11.

(i) **If n men, among whom A and B, stand in a row. What's the prob. that there will be exactly r men between A and B?**

(ii) **If they stand in a ring instead of in a row, show that the prob. is independent of 'r'.**

[In the circular arrangement, consider only that they are leading from A to B in the +ve direction.]

Solution:

(i) n persons can be arranged among themselves in n! ways. Since, the persons are randomly, all possible cases are equally likely. For the favorable cases if A occupies a position to the left of B, then A may choose any of the positions:

1st, 2nd,….. (n−r−1)th from the left, with r persons between A and B. The remaining (n−2) persons can stand in (n−2) places in (n−2)! Ways. Similar thing for B on the left of A.

Hence, no. of favorable cases, $N(A) = 2(n-r-1)(n-2)!$

Required probability

$$= \frac{2(n-r-1)(n-2)!}{n!} = \frac{2(n-r-1)}{n(n-1)}.$$

(ii) If they form a ring, then the no. of possible arrangement is (n−1)! which is obtained by keeping the place for any person fixed and arranging the remaining (n−1) persons.

For the favorable cases, we fixed the places for A and B, with r individuals between them and then remaining (n−2) persons can be arranged in (n−2)! ways.

$$Required\ probability = \frac{(n-2)!}{(n-1)!} = \frac{1}{n-1}, it\ is\ independent\ of\ r.$$

RULE−III:

Subpopulations and Groups: Consider a subpopulation of size 'r' from a given population of size 'n', let the no. of the groups of size r be x.

Now the r elements in a group can be arranged in r! ways. Hence x.r! ordered samples of size r.

$$\therefore x.r! = (n)_r$$

$$So, x = \binom{n}{r}$$

Application :

1. **Each of the 50 states has two senator. Find the prop. of the event that in a committee of 50 senators chosen randomly –**
(a) **A given state is represented.**
(b) **All states are represented.**

Solution: We can choose a group of 50 senators in $\binom{100}{50}$ ways & since 50 senators are chosen randomly 50 all possible outcomes are equally likely.

(a) There are 100 senators and 98 not from the given state.

Required probability $= P$[the given state is not represented] ^{c}

$$= 1 - \frac{\binom{98}{50}}{\binom{100}{50}}$$

(b) All states will be represented if one senators from each state is selected. A committee of 50 with one senator from 50 states can be selected in $\underbrace{2 \times 2 \times \dots \times 2}_{50\ times}$ ways.

$$Required\ probability\ = \frac{2^{50}}{\binom{100}{50}}$$

2. **If n balls are placed at random in n cells, find the probability that exactly one cell remains empty.**

Solution: $N = n^r$

Since k balls can be chosen in $\binom{r}{k}$ ways which are to be placed in the specified cells and the remaining (r−k) balls can be placed in the remaining (n−1) cells in $(n-1)^{r-k}$ ways.

$$Required\ prob. = \frac{\binom{r}{k}\,(n-1)^{r-k}}{(n)^r} = \binom{r}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{r-k}.$$

3. **If n balls are placed at a random order in n cells, find the prob. that exactly one cell remains empty.**

Solution: $N = n^n$

For the favorable cases, the empty cell can be chosen in n ways and the two balls to be kept in the same cell can be chosen in $\binom{n}{2}$ ways.

Consider the two balls as a single ball or entity, then (n−1) entities can be arranged in (n−1) cells in (n−1)! ways.

So, the required prob. $= \frac{n\binom{n}{2}(n-1)!}{n^n}$

4. **A closent contains n pairs of shoes. If 2r shoes chosen at random (2r < n). What is the prob. that there will be:**

(a) **No complete pair**
(b) **Exactly one complete pair**
(c) **Exactly two complete pair among them.**

Solution: (a)

$$required\ prob. = \frac{\binom{n}{2r}2^{2r}}{\binom{2n}{2r}}$$

(b)

$$Required\ prob. = \frac{\binom{n}{1}\binom{n-1}{2r-2}2^{2r-2}}{\binom{2n}{2r}}$$

(c)

$$Req.prob. = \frac{\binom{n}{2}\binom{n-2}{2r-4}.2^{2r-4}}{\binom{2n}{2r}}$$

5. **A car is parked among N cars in a row, not at either end. On the return the car owner finds that exactly r of the N places are still occupied. What's the prob. that both neighbouring places are empty?**

Solution: $Required\ Prob. = \frac{\binom{N-3}{r-1}}{\binom{N-1}{r-1}}$

RULE –IV:

The no. of ways in which a population of n elements can be divided into K−ordered parts of which 1st contains r_1, 2nd contains r_2 elements and so on is

$$\frac{n!}{r_1!\, r_2! \ldots \ldots r_k!}, where \sum_{i=1}^{k} r_i = n$$

Application:

1. **In a bridge table, calculate the prob. that**

(a) Each of the 4 players has an ace

(b) One of the player receives all 13 spades.

Solution:

(a) In a bridge table 52 cards are partitioned into four equal groups and the no. of different hands is

$$\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

For the favorable cases, 4 aces can be arranged in 4! ways and each arrangement represents one possibility of given one ace to each player and the remaining 48 cards can be distributed equally among the 4 players in

$$\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}\ ways$$

$$Required\ prob. = \frac{4!\ \binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}}{\binom{52}{12}\binom{39}{13}\binom{26}{13}\binom{13}{13}}$$

$$= \frac{4!\ \frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}}.$$

(b)

$$Required\ prob. = \frac{4!\frac{39!}{(13!)^3}}{\frac{52!}{(13!)^4}}$$

2. In a bridge hand of cards consists of 13 cards drawn at random WOR from a deck of 52 cards. Find the prob. that a hand of cards will contain

(a) γ_1 clubs, γ_2 spades, γ_3 diamonds

(b) γ aces

(c) γ_1 aces and γ_2 kings.

Solution: (a)

$$prob. = \frac{\binom{13}{\gamma_1}\binom{13}{\gamma_2}\binom{13}{\gamma_3}\binom{13}{13-\gamma_1-\gamma_2-\gamma_3}}{\binom{52}{13}}$$

(b)

$$Prob. = \frac{\binom{4}{\gamma}\binom{48}{13-\gamma}}{\binom{52}{13}}$$

(c)

$$Prob. = \frac{\binom{4}{\gamma_1}\binom{4}{\gamma_2}\binom{44}{4-\gamma_1-\gamma_2}}{\binom{52}{3}}$$

3. 4 cards are drawn at random from a full deck of 52 cards. What's the prob. that

(i) They are of different denominations?

(ii) They are of different suits?

(iii) Both?

Solution:

(i) In a deck of cards there are 13 denominations and 4 suits.

For favorable cases select a group of 4 denominations from 13 and then choose one card from each of the 4 denomination.

So, no. of favorable cases $= \binom{13}{4}\binom{4}{1}^4$.

$$Required\ prob. = \frac{\binom{13}{4}\binom{4}{1}^4}{\binom{52}{4}}$$

(ii)

$$required\ prob. = \frac{\binom{13}{1}^4}{\binom{52}{4}}$$

(iii) For favorable cases, selecting 4 denominations from 13 and then taking one card from the 1st denomination in 4 ways from the 4 suits. Then taking 2nd from the 2nd denomination in 3 ways & so on.

$$Required\ probability = \frac{\binom{13}{4} \times 4!}{\binom{52}{4}}$$

4. From a deck of 52 cards are drawn successively until an ace appears. What is the prob. that the 1st ace will appear

(a) At the nth draw,

(b) After the nth draw.

Solution:

(a) For the favourable cases, at the nth draw an ace can occur in 4 ways and the first (n−1) cards are to be taken from 48 non−ace cards which can be done in $(48)_{n-1}$ ways.

$$\therefore Required\ prob. = \frac{4 \times (48)_{n-1}}{(52)_n}$$

(b) For the favorable cases, 1st n cards contain no ace.

$$\therefore Req\ prob. = \frac{(48)_n}{(52)_n}$$

5. (Spread of Rumours) In a town of (n+1) inhabitants, a person tells a rumour to a second person, who in turn, repeats it to a third person, etc. At each step the receipt of the rumour is chosen at random from n people available.

(i) Find the prob. that the rumour will be told r times without

(a) Returning to the originator.

(b) Being repeated to any person.

(ii) Do the same problem when at each step the rumour is told by one person to a gathering of N randomly chosen individuals.

Solution:

(i) Since any person can tell the rumour to any one of the n available persons in n ways, total possible cases $= n^r$.

(a) The originator can tell the rumour to anyone of the remaining n persons in n ways & each of the (r−1) receipts of the rumour can tell to anyone of the remaining (n−1) persons without returning to the originator in (n−1) ways.

(b)

$$Req.prob. = \frac{n(n-1)^{r-1}}{n^r}$$

$$Req.prob. = \frac{(n)_r}{n^r}$$

(ii)

(a)

$$P_a = \frac{\binom{n}{N}\left\{\binom{n-1}{N}\right\}^{r-1}}{\left\{\binom{n}{N}\right\}^r}$$

$$= \left\{\frac{\binom{n-1}{N}}{\binom{n}{N}}\right\}^{r-1} = \left(1 - \frac{N}{n}\right)^{r-1}$$

(b)

$$P_b = \frac{\binom{n}{N}\binom{n-N}{N}\binom{n-2N}{N}\ldots\ldots\binom{n-\overline{r-1}N}{N}}{\left\{\binom{n}{N}\right\}^r}$$

$$= \frac{\frac{n!}{\{N!\}^r(n-rN)!}}{\left\{\binom{n}{N}\right\}^r}$$

$$= \frac{(n)_{rN}}{\{(n)_N\}^r}$$

6. 5 cards are taken at random from a full deck. Find the probability that

(a) They are different denominations?

(b) 2 are of same denominations?

(c) One pair if of one denomination & other pair of a different denomination and one odd?

(d) There are of one denomination & two scattered?

(e) 2 are of one denomination and 3 of another?

(f) 4 are of one denomination and 1 of another?

Solution : (a)

$$P(a) = \frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}}$$

(b)

$$P(b) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}}$$

(c)

$$P(c) = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

(d)

$$P(d) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

(e)

$$P(e) = \frac{\binom{13}{2}\binom{4}{2}\binom{13}{3}\binom{4}{3}}{\binom{52}{5}}$$

(f)

$$P(f) = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}$$

RULE−V:

- Occupancy Problem: In many situations it is necessary to treat the balls indistinguishable, e.g., in statistical studies of the distribution of accidents among week days, here one is interested only in the number of occurrences and not in the individual involved.

Such an example is completely described by its occupancy numbers $r_1, r_2, \dots, r_n$; where, r_k denotes the number of balls in the kth cell.

Here we are interested in number of possible distribution, i.e., the number of different n−tuples $(r_1, r_2, \dots\dots, r_n)$ such that $r_1 + r_2 + \cdots + r_n = r$ $(r_i \geq 0)$.

- Theorem 1: The number of different distributions of 'r' indistinguishable balls in n cells, i.e., the number of different solution of the above fact is

$$\binom{n+r-1}{n-1}.$$

- Theorem 2: The number of different distribution of 'r' indistinguishable balls in the n cells in which no cell remains empty is

$$\binom{r-1}{n-1}.$$

Ex: r distinguishable balls are distributed into n cells and all possible distributions are equally likely. Find the prob. that exactly m cells remain empty.

Solution: The m cells which are to be kept empty can be chosen from n cells in $\binom{n}{m}$ ways and r indistinguishable balls can be distributed in the remaining (n−m) cells so that no cell remain empty is in

$$\binom{r-1}{n-m-1} \ ways.$$

No. of favorable cases $= \binom{n}{m}\binom{r-1}{n-m-1}$

$$\therefore Required\ prob. = \frac{\binom{n}{m}\binom{r-1}{n-m-1}}{\binom{n+r-1}{r}}$$

- **Application:**

1. **Show that r indistinguishable balls can be distributed in n cells i.e., the no. of different solution $(\boldsymbol{r_1}, \boldsymbol{r_2}, \dots, \boldsymbol{r_n})$ such that $\{\boldsymbol{r_1} + \boldsymbol{r_2} + \cdots + \boldsymbol{r_n} = \boldsymbol{r}\}$ is $\binom{\boldsymbol{n}+\boldsymbol{r}-\mathbf{1}}{\boldsymbol{r}}$, where $\boldsymbol{r_i} \geq \mathbf{0}$.**

Solution : Denoting the choices of r_1, i.e., 0, 1, ….., r in the indices, we get the factors $(x^0 + x^1 + \cdots + x^r)$.

The no. of different solutions $(r_1, r_2, \dots, r_n)$ of

$$\sum_{i=1}^{n} r_i = r\,, where\ r_i \geq 0\ is$$

$=$ The coefficient of x^r in

$$\underbrace{(x^0 + x^1 + \cdots + x^r) \dots (x^0 + x^1 + \cdots + x^r)}_{n\ times}$$

$=$ The coefficient of x^r in

$$\left(\frac{1 - x^{r+1}}{1 - x}\right)^n$$

$=$ The coefficient of x^r in the expression $(1 - x^{r+1})^n\ (1 - x)^{-n}$

$=$ The coefficient of x^r in

$$\left\{1 - nx^{r+1} + \binom{n}{2}x^{2r+2} + \cdots\right\}\left\{1 + nx + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots\right\}$$

$$= \binom{n+r-1}{r}.$$

2. **Show that the no. of different distributions of r indistinguishable balls in n cells where no cell remains empty is $\binom{\boldsymbol{n}-\mathbf{1}}{\boldsymbol{r}-\mathbf{1}}$.**

Hints:

The coefficients of x^r in $(x + x^2 + \cdots + x^r)^n$

$=$ The coefficient of x^r in $x^n\ \left(\frac{1-x^r}{1-x}\right)^n$

$=$ The coefficient of $x^{r-n}\ in\ (1 - x^r)^n(1 - x)^{-n}$

$$= \binom{n+r-n-1}{r-n} = \binom{r-1}{n-1}.$$

STUDY MATERIALS ON BASIC ALGEBRA

TOPIC : COMPLEX NUMBERS (DE' MOIVRE'S THEOREM)

A complex number z is an ordered pair of real numbers (a,b): a is called Real part of z, denoted by Re z and b is called imaginary part of z, denoted by Im z. If Re z=0, then z is called purely imaginary; if Im z =0, then z is called real. On the set C of all complex numbers, the relation of equality and the operations of addition and multiplication are defined as follows:

(a,b)=(c,d) iff a=b and c=d, (a,b)+(c,d)=(a+c,b+d), (a,b).(c,d)= (ac-bd,ad+bc)

The set C of all complex numbers under the operations of addition and multiplication as defined above satisfies following properties:

- For $z_1,z_2,z_3 \in C$, (1) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$(associativity), (2)$z_1+(0,0)=z_1$, (3) for $z=(a,b)\in C$, there exists $-z=(-a,-b)\in C$ such that $(-z)+z=z+(-z)=(0,0)$, (4)$z_1+z_2=z_2+z_1$.
- For $z_1,z_2,z_3 \in C$, (1) $(z_1.\ z_2).z_3=z_1.(z_2.z_3)$(associativity), (2)$z_1.(1,0)=z_1$, (3) for $z=(a,b)\in C, z\neq(0,0)$, there exists $\frac{1}{z}\in C$ such that $z.\frac{1}{z}=\frac{1}{z}.z=1$, (4)$z_1.z_2=z_2.z_1$.
- For $z_1,z_2,z_3 \in C$, $z_1.(z_2+z_3)=(z_1.z_2)+(z_1.z_3)$.

Few Observations

(1) Denoting the complex number (0,1) by i and identifying a real complex number (a,0) with the real number a, we see z=(a,b)=(a,0)+(0,b)=(a,0)+(0,1)(b,0) can be written as z=a+ib.

(2) For two real numbers a,b , $a^2+b^2=0$ implies a=0=b; same conclusion need not follow for two complex numbers, for example, $1^2+i^2=0$ but $1\equiv(1,0)\neq(0,0)\equiv 0$ and $i=(0,1)\neq(0,0)$ (≡ denotes identification of a real complex number with the corresponding real number).

(3) For two complex numbers z_1,z_2, $z_1z_2=0$ implies $z_1=0$ or $z_2=0$.

(4) $i^2=(0,1)(0,1)=(-1,0)\equiv -1$.

(5) Just as real numbers are represented as points on a line, complex numbers can be represented as points on a plane: z=(a,b)↔P: (a,b). The line containing points representing the real complex numbers (a,0), a real, is called the real axis and the line containing points representing purely imaginary complex numbers (0,b) ≡ib is called the imaginary axis.

(6) The plane on which the representation is made is called Gaussian Plane or Argand Plane.

Definition 1.1 Let z=(a,b) ≡a+ib. The conjugate of z, denoted by $\bar{z}$, is (a,-b) ≡a-ib.

Geometrically, the point (representing) $\bar{z}$ is the reflection of the point (representing) z in the real axis. The conjugate operation satisfies the following properties:

(1) $\bar{\bar{z}}$=z , (2) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, (3) $\overline{z_1 z_2} = \overline{z_1} + \overline{z_2}$, (4)$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, (4) z+$\bar{z}$=2 Rez, z-$\bar{z}$=2i Im(z)

Definition 1.2 Let z=(a,b) ≡a+ib. The modulus of z, written as $|z|$, is defined as $\sqrt{a^2 + b^2}$.

Geometrically, $|z|$ represents the distance of the point representing z from the origin (representing complex number (0,0) ≡0+i0). More generally, $|z_1 - z_2|$ represents the distance between the points z_1 and z_2. The modulus operation satisfies the following properties:

(1) $|z_1 + z_2| \leq |z_1| + |z_2|$, (2) $|z_1 . z_2| = |z_1|.|z_2|$ (3) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(4) $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS: THE ARGAND PLANE

Let z=a+ib be a complex number. In the Argand plane, z is represented by the point whose Cartesian co-ordinates is (a,b) referred to two perpendicular lines as axes, the first co-ordinate axis is called the real axis and the second the imaginary axis. Taking the origin as the pole and the real axis as the initial line, let (r,θ) be the polar co-ordinates of the point (a,b). Then a=r cos θ, b=r sin θ. Also r=$\sqrt{a^2 + b^2}$=$|z|$. Thus z=a+ib=$|z|$(cos θ+isin θ): this is called modulus-amplitude form of z. For a given z≠0, there exist infinitely many values of θ differing from one another by an integral multiple of 2π: the collection of all such values of θ for a given z≠0 is denoted by Arg z or Amp z. The principal value of Arg z , denoted by arg z or amp z, is defined to be the angle θ from the collection Arg z that satisfies the condition $-\pi < \theta \leq \pi$. Thus Arg z={arg z+2nπ: n an integer}. arg z satisfies following properties: (1) arg($z_1 z_2$)=argz_1+argz_2+2kπ, where k is a suitable integer from the set{-1,0,1] such that $-\pi < \arg z_1+\arg z_2+2k\pi \leq \pi$, (2) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1-\arg z_2+2k\pi$, where k is a suitable integer from the set{-1,0,1] such that $-\pi < \arg z_1-\arg z_2+2k\pi \leq \pi$.

Note An argument of a complex number z=a+ib is to be determined from the relations $\cos\theta=a/|z|$, $\sin\theta= b/|z|$ simultaneously and not from the single relation $\tan\theta=b/a$.

Example1.1 Find arg z where $z=1+i\tan\frac{3\pi}{5}$.

»Let $1+i\tan\frac{3\pi}{5}=r(\cos\theta+i\sin\theta)$. Then $r^2= \sec^2\frac{3\pi}{5}$. Thus $r = -\sec\frac{3\pi}{5} > 0$. Thus $\cos\theta=-\cos\frac{3\pi}{5}$, $\sin\theta=-\sin\frac{3\pi}{5}$. Hence $\theta= \pi+\frac{3\pi}{5}$. Since $\theta> \pi$, arg $z= \theta-2\pi=-\frac{2\pi}{5}$.

Theorem 1.1 (De Moivre's Theorem) If n is an integer and θ is any real number, then $(\cos\theta+i\sin\theta)^n= \cos n\theta+i\sin n\theta$. If $n=\frac{p}{q}$, q natural, p integer, $|p|$and q are realtively prime, θ is any real number, then $(\cos\theta+i\sin\theta)^n$ has q number of values, one of which is $\cos n\theta+i\sin n\theta$.

Proof: <u>Case 1:</u> Let n be a positive integer.

Result holds for n=1: $(\cos\theta+i\sin\theta)^1= \cos 1\theta+i\sin 1\theta$. Assume result holds for some positive integer k: $(\cos\theta+i\sin\theta)^k= \cos k\theta+i\sin k\theta$.Then $(\cos\theta+i\sin\theta)^{k+1}=(\cos\theta+i\sin\theta)^k(\cos\theta+i\sin\theta)=(\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta)= \cos(k+1)\theta+i\sin(k+1)\theta$. Hence result holds by mathematical induction.

<u>Case 2</u>: Let n be a negative integer, say, n=-m, m natural.

$(\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)^{-m} = \frac{1}{(\cos\theta+i\sin\theta)^m} = \frac{1}{\cos m\theta+i\sin m\theta}$ (by case 1)

$= \cos m\theta-i\sin m\theta=\cos(-m)\theta+i\sin(-m)\theta= \cos n\theta+i\sin n\theta$.

<u>Case3</u>: n=0: proof obvious.

Case 4 Let $n=\frac{p}{q}$, q natural, p integer, $|p|$and q are realtively prime.

Let $(\cos\theta + i\sin\theta)^{\frac{p}{q}}= \cos\varphi+i\sin\varphi$. Then $(\cos\theta + i\sin\theta)^p= (\cos\varphi+i\sin\varphi)^q$. Thus $\cos p\theta+i\sin p\theta= \cos q\varphi+i\sin q\varphi$. Thus $q\varphi=2k\pi+p\theta$, that is, $\varphi=\frac{2k\pi+p\theta}{q}$. Hence $(\cos\theta + i\sin\theta)^{\frac{p}{q}}= \cos(\frac{2k\pi+p\theta}{q})+i\sin(\frac{2k\pi+p\theta}{q})$, where $k=0,1,\ldots,q-1$ are the distinct q values.

Some Applications of De' Moivre's Theorem

(1) Expansion of cos n θ, sin n θ and tan n θ where n is natural and θ is real. Cos n θ+i sin n θ=(cos θ+isin θ)^n=cos^n θ+${}^n_1C$ cos^{n-1} θisin θ+${}^n_2C$ cos^{n-2} θ $i^2sin^2\theta$+…+i^{n}sin^n θ= (cos^n θ-${}^n_2C$ cos^{n-2} θ $sin^2\theta$+…)+i(${}^n_1C$ cos^{n-1} θsin θ-${}^n_3C$ cos^{n-3} $\theta sin^3\theta$+…). Equating real and imaginary parts, cos n θ= cos^n θ-${}^n_2C$ cos^{n-2} θ $sin^2\theta$+… and sin n θ=${}^n_1C$ cos^{n-1} θsin θ-${}^n_3C$ cos^{n-3} $\theta sin^3\theta$+…

(2) Expansion of cos^n θ and sin^n θ in a series of multiples of θ where n is natural and θ is real.
Let x = cos $\theta + isin\theta$. Then x^n=cos n θ+isin n θ, x^{-n}= cos n θ-isin n θ. Thus (2 cos θ)^n=(x+$\frac{1}{x}$)^n

=(x^n+$\frac{1}{x^n}$) + ${}^n_1C$(x^{n-2}+$\frac{1}{x^{n-2}}$)+…=2 cos n θ+${}^n_1C$(2 cos(n-2) θ)+…

Similarly, expansion of sin^n θ in terms of multiple angle can be derived.

(3) Finding n th roots of unity
To find z satisfying z^n=1=cos(2kπ)+isin(2kπ), where k is an integer. Thus z=[cos(2kπ)+isin(2kπ)]$^{1/n}$=cos($\frac{2k\pi}{n}$) + $isin\left(\frac{2k\pi}{n}\right)$, k=0,1,…,n-1; replacing k by any integer gives rise to a complex number in the set A={ cos($\frac{2k\pi}{n}$) + $isin\left(\frac{2k\pi}{n}\right)$/ k=0,1,…,n-1}. Thus A is the set of all nth roots of unity.

Example1.2 Solve x^6+x^5+x^4+x^3+x^2+x+1=0

»We have the identity x^6+x^5+x^4+x^3+x^2+x+1=$\frac{x^7-1}{x-1}$. Roots of x^7-1=0 are cos$\frac{2k\pi}{7}$ + $isin\frac{2k\pi}{7}$, k=0,1,…,6. Putting k=0, we obtain root of x-1=0. Thus the roots of given equation are cos$\frac{2k\pi}{7}$ + $isin\frac{2k\pi}{7}$, k=1,…,6.

Example1.3 Prove that the sum of 99 th powers of all the roots of x^7-1=0 is zero.

»The roots of x^7-1=0 are $\{1, \alpha, \alpha^2, \ldots, \alpha^6\}$, where $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$. Thus sum of 99^{th} powers of the roots is $1+\alpha^{99}+(\alpha^2)^{99}+\cdots+(\alpha^6)^{99}=1+\alpha^{99}+(\alpha^{99})^2+\cdots+(\alpha^{99})^6=\frac{1-\alpha^{99.7}}{1-\alpha^{99}}=0$, since $\alpha^{99.7}=1$ and $\alpha^{99}\neq 1$.

Example1.4 If the amplitude of the complex number $\frac{z-i}{z+1}$ is $\frac{\pi}{4}$, show that z lies on a circle in the Argand plane.

»Let z=x+iy. Then $\frac{z-i}{z+1}=\frac{x^2+x+y^2-y}{(x+1)^2+y^2}+i\,\frac{y-x-1}{(x+1)^2+y^2}$. By given condition, $\frac{y-x-1}{x^2+x+y^2-y}=1$. On simplification, $(x+1)^2+(y-1)^2=1$. Hence z lies on the circle centred at (-1,1) and radius 1.

Example1.5 If A,B,C represent complex numbers z_1, z_2, z_3 in the Argand plane and $z_1+z_2+z_3=0$ and $|z_1|=|z_2|=|z_3|$, prove that ABC is an equilateral triangle.

»$z_1+z_2=-z_3$. Hence $|z_1+z_2|^2=|z_3|^2$, that is, $|z_1|^2+|z_2|^2+2z_1.z_2=|z_3|^2$. By given condition, $|z_1||z_2|\cos\theta=|z_1|^2$, where θ is the angle between z_1 and z_2.Thus $\cos\theta=-\frac{1}{2}$, that is, $\theta=120^0$. Hence the corresponding angle of the triangle ABC is 60^0. Similarly other angles are 60^0.

Example1.6 Let z and z_1 be two complex numbers satisfying $z=\frac{1+z_1}{1-z_1}$ and $|z_1|=1$. Prove that z lies on the imaginary axis.

» $z_1=\frac{z-1}{z+1}$. By given condition, $1=\left|\frac{z-1}{z+1}\right|=\frac{|z-1|}{|z+1|}$. If z=x+iy, x=0. Hence.

Example1.7 complex numbers z_1, z_2, z_3 satisfy the relation $z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_3z_1=0$ iff $|z_1-z_2|=|z_2-z_3|=|z_3-z_1|$.

» $0=z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_3z_1=(z_1+wz_2+w^2z_3)(z_1+w^2z_2+wz_3)$, where w stands for an imaginary cube roots of unity. If $z_1+wz_2+w^2z_3=0$, then $(z_1-z_2)=-w^2(z_3-z_2)$; hence $|z_1-z_2|=|w^2||z_2-z_3|=|z_2-z_3|$.similarly other part.

Conversely, if $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, then z_1,z_2,z_3 represent vertices of an equilateral triangle. Then $z_2-z_1=(z_3-z_1)(\cos 60^0+i\sin 60^0)$, $z_1-z_2=(z_3-z_2)(\cos 60^0+i\sin 60^0)$; by dividing respective sides, we get the result.

Example1.8 Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$, for two complex numbers z_1,z_2.

»$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$=$|z_1|^2 + |z_2|^2$+2 z_1z_2; similarly $|z_1 - z_2|^2$=$|z_1|^2 + |z_2|^2$-2 z_1z_2; Adding we get the result.

Example1.9 If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, then prove that (1) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$, (2) $\sum cos^2\alpha=\sum sin^2\alpha=3/2$.

»Let $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$. Then $x+y+z=0$. Thus $x^3+y^3+z^3=3xyz$. By De' Moivre's Theorem, $(\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$. Equating, we get result.

Let $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$. Then $x+y+z=0$. Also $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$; hence $xy+yz+zx=0$. Thus $x^2+y^2+z^2=0$. By De' Moivre's Theorem, $\cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$. Hence$\sum cos^2\alpha=3/2$. Using $\sin^2\alpha=1-\cos^2\alpha$, we get other part.

Example1.10 Find the roots of $z^n=(z+1)^n$, where n is a positive integer, and show that the points which represent them in the Argand plane are collinear.

Let $w=\frac{z+1}{z}. Then\ z=\frac{1}{w-1}$.Now $z^n=(z+1)^n$ implies $w^n=1$.Thus, $w=\cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n}$,k=0,…,n-1.

So $z=\frac{1}{\cos\frac{2k\pi}{n}+isin\frac{2k\pi}{n}}$, k=1,…,n-1

$=-\frac{1}{2}-\frac{i}{2}cot\frac{k\pi}{n}$. Thus all points z satisfying $z^n=(z+1)^n$ lie on the line $x=-\frac{1}{2}$.

TOPIC : THEORY OF EQUATIONS

An expression of the form $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, where $a_0,a_1,\ldots,a_n$ are real or complex constants, n is a nonnegative integer and x is a variable (over real or complex numbers) is a polynomial in x. If $a_0\neq0$, the polynomial is of degree n and a_0x^n is the leading term of the polynomial. A non-zero constant a_0 is a polynomial of degree 0 while a polynomial in which the coefficients of each term is zero is said to be a zero polynomial and no degree is assigned to a zero polynomial.

Equality: two polynomials $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ and $b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$ are equal iff $a_0=b_0,a_1=b_1,\ldots,a_n=b_n$.

Addition: Let $f(x)= a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, $g(x)= b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$. the sum of the polynomials f(x) and g(x) is given by

$$f(x)+g(x)= a_0x^n+\ldots+a_{n-m-1}x^{m+1}+(a_{n-m}+b_0)x^m+\ldots+(a_n+b_m),\text{ if } m<n$$

$$= (a_0+b_0)x^n+\ldots+(a_n+b_n),\text{ if } m=n$$

$$= b_0x^m+\ldots+b_{m-n-1}x^{n+1}+(b_{m-n}+a_0)x^n+\ldots+(b_m+a_n),\text{ if } m>n.$$

Multiplication: Let $f(x)= a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, $g(x)= b_0x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$. the product of the polynomials f(x) and g(x) is given by

$f(x)g(x)=c_0x^{m+n}+c_1x^{m+n-1}+\ldots+c_{m+n}$, where $c_i=a_0b_i+a_1b_{i-1}+\ldots+a_ib_0$. $c_0=a_0b_0\neq0$; hence degree of f(x)g(x) is m+n.

Division Algorithm: Let f(x) and g(x) be two polynomials of degree n and m respectively and $n\geq m$. Then there exist two uniquely determined polynomials q(x) and r(x) satisfying f(x)=g(x)q(x)+r(x), where the degree of q(x) is n-m and r(x) is either a zero polynomial or the degree of r(x) is less than m. In particular, if degree of g(x) is 1, then r(x) is a constant, identically zero or non-zero.

Theorem 1.1 (Remainder Theorem) If a polynomial f(x) is divided by x-a, then the remainder is f(a).

»Let q(x) be the quotient and r (constant)be the remainder when f is divided by x-a. then f(x)=(x-a)q(x)+r is an identity. Thus f(a)=r.

Theorem 1.2 (Factor Theorem) If f is a polynomial, then x-a is a factor of f iff f(a)=0.

» By Remainder theorem, f(a) is the remainder when f is divided by x-a; hence, if f(a)=0, then x-a is a factor of f. Conversely, if x-a is a factor of f, then f(x)=(x-a)g(x) and hence f(a)=0.

Example1.1 Find the remainder when $f(x)=4x^5+3x^3+6x^2+5$ is divided by 2x+1.

»The remainder on dividing f(x) by $x-(-\frac{1}{2})=x+\frac{1}{2}$ is $f(-\frac{1}{2})=6$. If q(x) be the quotient, then $f(x)=q(x)(x+\frac{1}{2})+6=\frac{q(x)}{2}(2x+1)+6$. Hence 6 is the remainder when f is divided by 2x+1.

Theorem 1.3 (Fundamental Theorem of Classical Algebra)

Every polynomial equation of degree≥1 has a root, real or complex.

Corollary A polynomial equation of degree n has exactly n roots, multiplicity of each root being taken into account.

Corollary If a polynomial f(x) of degree n vanishes for more than n distinct values of x, then f(x) =0 for all values of x.

Polynomial equations with Real Coefficients

Theorem 1.4 If a+ib is a root of multiplicity r of the polynomial equation f(x)=0 with real coefficients, then a-ib is a root of multiplicity r of f(x)=0.

Note: 1+i is a root of $x^2-(1+i)x=0$ but not so is 1- i.

Example1.2 Prove that the roots of $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}=\frac{1}{x}$ are all real.

» The given equation is $\frac{1}{x-1}+\frac{4}{x-2}+\frac{9}{x-3}=-5$ (*). Let a+ib be a root of the polynomial equation (*) with real coefficients. Then a-ib is also a root of (*).Thus $\frac{1}{(a-1)+ib}+\frac{4}{(a-2)+ib}+\frac{9}{(a-3)+ib}=-5$ and $\frac{1}{(a-1)-ib}+\frac{4}{(a-2)-ib}+\frac{9}{(a-3)-ib}=-5$. Subtracting, $-2ib[\frac{1}{(a-1)^2+b^2}+\frac{4}{(a-2)^2+b^2}+\frac{9}{(a-3)^2+b^2}]=0$ which gives b=0. Hence all roots of given equation must be real.

Example1.3 Solve the equation $f(x)=x^4+x^2-2x+6=0$, given that 1+i is a root.

» Since f(x)=0 is a polynomial equation with real coefficients, 1-i is also a root of f(x)=0. By factor theorem,$(x-1-i)(x-1+i)=x^2-2x+2$ is a factor of f(x). by division, $f(x)=(x^2-2x+2)(x^2+2x+3)$. Roots of $x^2+2x+3=0$ are $-1\pm\sqrt{2}i$. Hence the roots of f(x)=0 are $1\pm i$, $-1\pm\sqrt{2}i$.

Theorem 1.5 If $a+\sqrt{b}$ is a root of multiplicity r of the polynomial equation f(x)=0 with rational coefficients, then $a-\sqrt{b}$ is a root of multiplicity r of f(x)=0 where a,b are rational and b is not a perfect square of a rational number.

Since every polynomial with real coefficients is a continuous function from R to R, we have

Theorem 1.6 (Intermediate Value Property) Let f(x) be a polynomial with real coefficients and a,b are distinct real numbers such that f(a) and f(b) are of opposite signs. Then f(x)=0 has an odd number of roots between a and b. If f(a) and f(b) are of same sign, then there is an even number of roots of f(x)=0 between a and b.

Example1.4 Show that for all real values of a, the equation $(x+3)(x+1)(x-2)(x-4)+a(x+2)(x-1)(x-3)=0$ has all its roots real and simple.

»Let $f(x)= (x+3)(x+1)(x-2)(x-4)+a(x+2)(x-1)(x-3)$. Then $\lim_{x\to-\infty} f(x)=\infty$, $f(-2)<0$, $f(1)>0$, $f(3)<0$,$\lim_{x\to\infty} f(x)=\infty$. Thus each of the intervals $(-\infty,-2)$,$(-2,1)$,$(1,3)$,$(3, \infty)$ contains a real root of f(x)=0. Since the equation is of degree 4, all its roots are real and simple.

Theorem 1.7 (Rolle's Theorem) Let f(x) be a polynomial with real coefficients. Between two distinct real roots of f(x)=0 ,there is at least one real root of $f^{(1)}(x)=0$.

Note:

(1) Between two **consecutive** real roots of $f^{(1)}(x)=0$, there is at most one real root of f(x)=0.

(2) If all the roots of f(x)=0 be real and distinct, then all the roots of $f^{(1)}(x)=0$ are also real and distinct.

Example1.5 Show that the equation $f(x)=(x-a)^3+(x-b)^3+(x-c)^3+(x-d)^3=0$, where a,b,c,d are not all equal , has only one real root.

» Since f(x)=0 is a cubic polynomial equation with real coefficients, f(x)=0 has either one or three real roots. If α be a real multiple root of f(x)=0 with multiplicity 3, then α is also a real root of $f^{(1)}(x)=3[(x-a)^2+(x-b)^2+(x-c)^2+(x-d)^2]=0$, and hence α=a=b=c=d (since α,a,b,c,d are real), contradiction. If f(x)=0 has two distinct real roots, then in between should lie a real root of $f^{(1)}(x)=0$, contradiction since not all of a,b,c,d are equal. Hence f(x)=0 has only one real root.

Example1.6 Find the range of values of k for which the equation $f(x)=x^4+4x^3-2x^2-12x+k=0$ has four real and unequal roots.

» Roots of $f^{(1)}(x)=0$ are -3,-1,1. Since all the roots of f(x)=0 are to be real and distinct, they will be separated by the roots of $f^{(1)}(x)=0$. Now $\lim_{x\to-\infty} f(x)=\infty$, f(-3)=-9+k, f(-1)=7+k, f(1)=-9+k,$\lim_{x\to\infty} f(x)=\infty$. Since f(-3)<0, f(-1)>0 and f(1)<0, -7<k<9.

Example1.7 If $c_1,c_2,\ldots,c_n$ be the roots of $x^n+nax+b=0$, prove that $(c_1-c_2)(c_1-c_3)\ldots(c_1-c_n)=n(c_1^{n-1}+a)$.

» By factor theorem, $x^n+nax+b=(x-c_1)(x-c_2)\ldots(x-c_n)$.Differentiating w.r.t. x, $n(x^{n-1}+a)= (x-c_2)\ldots(x-c_n)+(x-c_1)(x-c_3)\ldots(x-c_n)+\ldots+(x-c_2)(x-c_3)\ldots(x-c_n)$.

Replacing x by c_1 in this identity, we obtain the result.

Example1.8 If a is a double root of $f(x)=x^n+p_1x^{n-1}+\ldots+p_n=0$, prove that a is also a root of $p_1x^{n-1}+2p_2x^{n-2}+\ldots+np_n=0$.

» Since a is a double root of f(x)=0, both f(a)=0 and $f^{(1)}(a)=0$ hold. Thus $a^n+p_1a^{n-1}+\ldots+p_n=0$ (1) and $na^{n-1}+(n-1)p_1a^{n-2}+\ldots+p_{n-1}=0$(2). Multiplying both side of (1) by n and both side of (2) by a and subtracting, we get $p_1a^{n-1}+2p_2a^{n-2}+\ldots+np_n=0$.Hence the result.

Example1.9 Prove that the equation $f(x)=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}=0$ cannot have a multiple root.

» If a is a multiple root of f(x)=0, then $1+a+\frac{a^2}{2!}+\ldots+\frac{a^n}{n!}=0$ and $1+a+\frac{a^2}{2!}+\ldots+\frac{a^{n-1}}{(n-1)!}=0$;it thus follows that $\frac{a^n}{n!}=0$, so that a=0; but 0 is not a root of given equation.

Hence no multiple root.

Descartes' Rule of signs

Theorem 1.8 The number of positive roots of an equation $f(x)=0$ with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of $f(x)$ and if less, it is less by an even number.

The number of negative roots of an equation $f(x)=0$ with real coefficients does not exceed the number of variations of signs in the sequence of the coefficients of $f(-x)$ and if less, it is less by an even number.

Example1.10 If $f(x)=2x^3+7x^2-2x-3$, express $f(x-1)$ as a polynomial in x. Apply Descartes' rule of signs to both the equations $f(x)=0$ and $f(-x)=0$ to determine the exact number of positive and negative roots of $f(x)=0$.

» Let $g(x)=f(x-1)=2x^3+x^2-10x+4$. By Descartes' Rule, $g(x)=0$ has exactly one negative root, say, c. Thus $g(c)=f(c-1)=0$; hence $c-1(<0)$ is a negative root of $f(x)=0$. Since there are 2 variations of signs in the sequence of coefficients of $f(-x)$and since c-1 is a negative root of $f(x)=0$, $f(x)=0$ has two negative roots. Also, $f(x)=0$ has exactly one positive root ,by Descartes' rule.

Relations between roots and coefficients

Let $c_1,\ldots,c_n$ be the roots of the equation $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$. By factor theorem,

$a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=a_0(x-c_1)(x-c_2)\ldots(x-c_n)$.

Equating coefficients of like powers of x,$a_1=a_0(-\sum c_1)$, $a_2=a_0\sum c_1\, c_2,\ldots,a_n=a_0\,(-1)^n c_1c_2\ldots c_n$. Hence

$\sum c_1=-\frac{a_1}{a_0},\sum c_1\, c_2=\frac{a_2}{a_0},\ldots,\ c_1c_2\ldots c_n=(-1)^n\frac{a_n}{a_0}$.

Example1.11 Solve the equation $2x^3-x^2-18x+9=0$ if two of the roots are equal in magnitude but opposite in signs.

» Let the roots be -a, a, b .Using relations between roots and coefficients, $b=(-a)+a+b=\frac{1}{2}$ and $-a^2b=-\frac{9}{2}$. Hence $a^2=9$, that is, $a=\pm 3$. Hence the roots are $3, -3, \frac{1}{2}$.

SETS AND FUNCTIONS

A set is a collection of objects having the property that given any abstract (the thought of getting 100%marks at the term-end examination) or concrete (student having a particular Roll No. of semester II mathematics general) object, we can say without any ambiguity whether that object belongs to the collection (collection of all thoughts that came to one's mind on a particular day or the collection of all students of this class) or not. For example, the collection of 'good' students of semester II will not be a set unless the criteria of 'goodness' is made explicit! The objects of which a set A is constituted of are called elements of the set A. If x is an element of a set A, we write $x\in A$; otherwise $x\notin A$. If every element of a set X is an element of set Y,X is a subset of Y, written as $X \subseteq Y$. X is a proper subset of Y if $X \subseteq Y$ and $Y \nsubseteq X$, written as $X \subsetneq Y$. For two sets X = Y iff (if and only if, bi-implication) $X \subseteq Y$ and $Y \subseteq X$. A set having no element is called null set, denoted by $\emptyset$.

Example1.1 $a\neq\{a\}$ (a letter inside envelope is different from a letter without envelope!), $\{a\}\in\{a,\{a\}\}$, $\{a\}\subsetneq\{a,\{a\}\}$, $\emptyset \subset A$(the premise $x\in \emptyset$ of the implication $x\in \emptyset \Rightarrow x\in A$ is false and so the implication holds vacuously), $A\subseteq A$, for every set A.

Set Operations: formation of new sets

Let X and Y be two sets. Union of X and Y, denoted by $X\cup Y$, is the set $\{a|\ a\in X$ or $a\in Y$ or both$\}$. Intersection of X and Y, denoted by $X\cap Y$, is the set $\{a|\ a\in X \text{ and } a\in Y\}$. The set difference of X and Y, denoted by X-Y, is the set $\{a|\ a\in X \text{ and } a\notin Y\}$. The set difference U-X is called complement of the set X, denoted by $X^{/}$, where U is the universal set. The symmetric set difference of X and Y, denoted by $X\Delta Y$, is the set (X-Y) U(Y-X). For any set X, the power set of X, P(X), is the set of all subsets of X. Two sets X and Y are disjoint iff $X\cap Y = \emptyset$. The Cartesian product of X and Y, denoted by X X Y, is defined as the set $\{(x,y)|\ x\in X,\ y\in Y\}$ [(x,y) is called an ordered pair. Two ordered pairs (x,y) and (u,v) are equal, written (x,y) = (u,v), iff x = u and y = v]. If we take X = {1,2} and Y = {3}, then X X Y = {(1,3),(2,3)} $\neq${(3,1),(3,2)} = Y X X. Thus Cartesian product between two distinct sets are not necessarily commutative (Is $\emptyset\, X\, \{1\} = \{1\} X\, \emptyset$?).

Laws governing set operations

For sets X, Y, Z,

- Idempotent laws: $X \cup X = X$, $X \cap X = X$
- Commutative laws: $X \cup Y = Y \cup X$, $X \cap Y = Y \cap X$
- Associative Laws: $(X \cup Y) \cup Z = X \cup (Y \cup Z)$, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$
- Distributive laws: $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
- Absorptive laws: $X \cap (X \cup Y) = X$, $X \cup (X \cap Y) = X$
- De' Morgan's laws: $X-(Y \cup Z) = (X-Y) \cap (X-Z)$, $X-(Y \cap Z) = (X-Y) \cup (X-Z)$

Example1.1 Let A, B, C be three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then prove B = C.

» $B = B \cup (A \cap B) = B \cup (A \cap C) = (B \cup A) \cap (B \cup C)$ (distributivity of $\cup$ over $\cap$) $= (C \cup A) \cap (B \cup C) = C \cup (A \cap B) = C \cup (A \cap C) = C$.

Example1.2 $A\Delta C = B\Delta C$ implies A = B: prove or disprove.

NOTE: Proving will involve consideration of arbitrary sets A,B,C satisfying the given condition, whereas disproving consists of giving counter-examples of three particular sets A,B,C that satisfies the hypothesis $A\Delta C = B\Delta C$ but for which the conclusion A = B is false.

NOTATION: N,Z,Q,R,C will denote set of all positive integers, integers, rational numbers ,real numbers and the complex numbers respectively.

A function from a set A to a set B, denoted by f: A→B, is a correspondence between elements of A and B having the properties:

- ✓ For every x∈A, the corresponding element $f(x) \in B$. f(x) is called the image of x under the correspondence f and x is called a pre-image of f(x). A is called domain and B is called the codomain of the correspondence. Note that we differentiate between f, the correspondence, and f(x), the image of x under f.
- ✓ For a fixed x∈A, $f(x) \in B$ is unique. For two different elements x and y of A, images f(x) and f(y) may be same or may be different.

In brief, a function is a correspondence under which

- ➢ both existence and uniqueness of image of all elements of the domain is guaranteed but
- ➢ neither the existence nor the uniqueness of pre-image of some element of codomain is guaranteed.

Example1.7 Prove that $f(A\cap B) \subseteq f(A) \cap f(B)$; give a counterexample to establish that the reverse inclusion may not hold.

» $y\in f(A\cap B) \Rightarrow y = f(x),\ x\in A\cap B \Rightarrow y = f(x),\ x\in A$ and $x\in B \Rightarrow y\in f(A)$ and $y\in f(B) \Rightarrow y\in f(A) \cap f(B)$. Hence $f(A\cap B) \subseteq f(A) \cap f(B)$. Consider the counter example: f: R→R, $f(x) = x^2$, A = {2}, B = {-2}.

<u>Example1.3</u> Let f: R→R, $f(x) = 3x^2-5$. f(x) = 70 implies x = ±5. Thus $f^{-1}\{70\} = \{-5, 5\}$.Hence $f[f^{-1}\{70\}] = \{f(-5), f(-5)\} = \{70\}$. Also, $f^{-1}(\{-11\}) = \emptyset$ $[x\in f^{-1}(\{-11\}) \Rightarrow 3x^2-5=-11 \Rightarrow x^2=-2]$.

A function under which uniqueness of pre-image is guaranteed is called an <u>injective function</u>. A function under which existence of pre-image is guaranteed is called a <u>surjective function</u>. Put in a different language, f: A→B is injective iff $a_1, a_2\in A$, $f(a_1) =$

$f(a_2)$ implies $a_1 = a_2$. f is <u>surjective</u> iff codomain and range coincide. A function which is both injective and surjective is called <u>bijective</u> .

<u>NOTE:</u> The injectivity, surjectivity and bijectivity depends very much on the domain and codomain sets and may well change with the variation of those sets even if expression of the function remains unaltered e.g. $f:Z\to Z$, $f(x) = x^2$ is not injective though $g:N\to Z$, $g(x) = x^2$ is injective.

<u>Example1.4</u> $f: R\to R$, $f(x) = x^2 - 3x+4$. $f(x_1) = f(x_2)$ implies $(x_1-x_2)(x_1+x_2-3) = 0$. Thus $f(1) = f(2)$ though $1\neq 2$; hence f is not injective [Note: for establishing non-injectivity, it is sufficient to consider particular values of x]. Let $y\in R$ and $x\in f^{-1}\{y\}$. Then $y = f(x) = x^2 - 3x+4$. We get a quadratic equation $x^2 - 3x+(4-y) = 0$ whose roots, considered as a quadratic in x, give pre-image(s) of y. But the quadratic will have real roots if the discriminant $4y-7\geq 0$, that is , only when $y\geq 7/4$. Thus, for example, $f^{-1}\{1\} =\emptyset$. Hence f is not surjective.

If $f:A\to B$ and $g:B\to C$, we can define a function $g_0f:A\to C$, called the composition of f and g, by $(g_0f)(a) = g(f(a))$, $a\in A$.

<u>Example1.5</u> $f:Z \to Z$ and $g: Z \to Z$ by $f(n) = (-1)^n$ and $g(n) = 2n$. Then $g_0f: Z \to Z$, $(g_0f)(n)=g((-1)^n) = 2(-1)^n$ and $(f_0g)(n) = (-1)^{2n}$. Thus $g_0f \neq f_0g$. Commutativity of composition of functions need not hold.

LINEAR ALGEBRA

MATRICES

Definition: A rectangular array of mn elements a_{ij} into m rows and n columns, where the elements a_{ij} belong to a field F, is called a matrix of order mxn over F. It is denoted by $[a_{ij}]_{mxn}$ or by $\begin{bmatrix} a_{11} a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} a_{m2} & \cdots & a_{mn} \end{bmatrix}$ or by $\begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{pmatrix}$. F is called field of scalars. In particular, if F be the field R of real numbers, a matrix over R is said to be a real matrix. The element a_{ij} appearing in the i th row and j th column of the matrix is said to be ij th element. If m=1, the matrix is said to be a row matrix and if n = 1, it is called a column matrix. If each element of a matrix is 0, it is called a null matrix and denoted by O_{mxn}. If m=n, matrix is called a square matrix. Two matrices $[a_{ij}]_{mxn}$ and $[b_{ij}]_{pxq}$ are equal iff m = p, n = q and $a_{ij} = b_{ij}$ for each i and j. A square matrix whose elements on the principal diagonal are all equal to 1 and all the elements off the main diagonal are 0 is called identity matrix and is denoted by I_n. If A = $[a_{ij}]_{mxn}$, then transpose of A, denoted by A^T, is defined as $A^T = [b_{ij}]_{nxm}$,where $b_{ij} = a_{ji}$, for each i and j. A square matrix is a diagonal matrix if all the elements not lying on the main diagonal are zero.

OPERATION ON MATRICES

Equality of matrices $[a_{ij}]_{mxn}=[b_{ij}]_{pxq}$ iff m=p, n=q and $a_{ij} = b_{ij}$, for each i,j.

Multiplication by a scalar for a scalar c, $c[a_{ij}]_{mxn} = [ca_{ij}]_{mxn}$

Addition two matrices $[a_{ij}]_{mxn}$, $[b_{ij}]_{pxq}$ are conformable for addition iff m = p and n = q and in that case

$$[a_{ij}]_{mxn}+[b_{ij}]_{mxn} = [a_{ij}+b_{ij}]_{mxn}$$

Multiplication two matrices $[a_{ij}]_{mxn}$, $[b_{ij}]_{pxq}$ are conformable for multiplication iff n = p. in that case

$$[a_{ij}]_{mxn}\,[b_{ij}]_{nxq} = [c_{ij}]_{mxq},\ c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}.$$

ALGEBRA OF MATRICES

1. Matrix addition is commutative and associative.
2. Matrix multiplication is NOT commutative.
3. Matrix multiplication is associative. Let A = $[a_{ij}]_{mxn}$, B = $[b_{ij}]_{nxp}$, C = $[c_{ij}]_{pxq}$. Then AB = $[d_{ij}]_{mxp,}$ where $d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Thus (AB)C = $[e_{ij}]_{mxq}$ where $e_{ij} = \sum_{s=1}^{p} d_{is} b_{sj} = \sum_{s=1}^{p} \sum_{k=1}^{n} a_{ik} b_{ks}\, c_{sj}$. Again, BC = $[f_{ij}]_{nxq}$, where $f_{ij} = \sum_{t=1}^{p} b_{it} c_{tj}$. Thus A(BC) = $[g_{ij}]_{mxq}$ where $g_{ij} = \sum_{u=1}^{n} a_{iu} f_{uj} = \sum_{u=1}^{n} \sum_{s=1}^{p} a_{iu} b_{ut} c_{tj}$ = e_{ij}, for all i,j. hence A(BC) = (AB)C.
4. Matrix multiplication is distributive over addition.
5. $(A^T)^T = A$
6. $(A+B)^T = A^T+B^T$.
7. $(AB)^T = B^TA^T$ (supposing A,B are conformable for product). Let A = $[a_{ij}]_{mxn}$, B = $[b_{ij}]_{nxp}$. AB =$[c_{ij}]_{mxp}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. So $(AB)^T$ =$[d_{ij}]_{pxm}$, $d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$. $B^T = [e_{ij}]_{pxn}$, $A^T = [f_{ij}]_{nxm}$ where $e_{ij} = b_{ji}, f_{ij} = a_{ji}$. Hence B^TA^T = $[g_{ij}]_{pxm}$, where $g_{ij} = \sum_{s=1}^{n} e_{is} f_{sj} = \sum_{s=1}^{n} a_{js} b_{si} = d_{ij}$.

Symmetric and skew-symmetric matrix

A square matrix A is symmetric iff $A = A^T$. A square matrix A is skew-symmetric iff $A = -A^T$.

Results involving symmetric and skew-symmetric matrices

(1) If A and B are symmetric matrices of the same order, then A+B is symmetric.

(2) If A and B are symmetric matrices of the same order , then AB is symmetric iff AB = BA.

» if AB is symmetric, then AB = $(AB)^T = B^T A^T$ = BA. If AB = BA, then AB = BA = $B^T A^T = (AB)^T$, so that AB is symmetric.

(3) AA^T and A^TA are both symmetric.

(4) A real or complex square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

» let A be a square matrix. then A can be expressed as A = $\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ where $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is skew-symmetric. Uniqueness can be proved.

DETERMINANT

Let M stand for the set of all square matrices over R. We define a function det: M→R inductively as follows:

Step 1 det X= $a_{11}=(-1)^{1+1}a_{11}$, if X= $[a_{11}]_{1x1}$. det X = $(-1)^{1+1}a_{11}det(a_{22}) + (-1)^{1+2}a_{21}det(a_{12})$, if X=$[a_{ij}]_{2x2}$= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Step 2 let us assume the definition is valid for a square matrix of order n: thus for X= $(a_{ij})_{nxn}$, det X=$\sum_{r=1}^{n}(-1)^{1+r}a_{1r}\det X_{1r}$, where X_{1r} is a matrix of order n-1, obtained from X by deleting the first row and r th column. We now consider X = $(a_{ij})_{(n+1)x(n+1)}$. We now define det X =$\sum_{r=1}^{n+1}(-1)^{1+r}a_{1r}\det \mathrm{X}_{1r}$, where X_{1r} is a matrix of order n, obtained from X by deleting the first row and r th column. By assumption, $\det \mathrm{X}_{1r}$ can be evaluated for all r. so the definition is valid for a square matrix of order (n+1).

Hence by induction, for a matrix X = $(a_{ij})_{nxn}$, we define det X $=\sum_{r=1}^{n}(-1)^{1+r}a_{1r}\det X_{1r}$, where X_{1r} is a matrix of order n-1, obtained from X by deleting the first row and r th column.

Following this definition , we can define det X, for any X ∈M.

Example 2.1 for the matrix X=$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, det X = $(-1)^{1+1}a_{11}X_{11}+(-1)^{1+2}a_{12}X_{12}+(-1)^{1+3}a_{13}X_{13}$, where X_{1r} is the determinant of the matrix obtained by deleting the first row and r th column of A. now $X_{11} = (-1)^{2+2}a_{22}\det(a_{33})+(-1)^{2+3}a_{23}\det(a_{32}) = a_{22}a_{33}-a_{23}a_{32}$. Similarly value for X_{12} and X_{13} can be found and finally value of det X can be calculated.

Properties of determinants

(1) det X = det X^T, where X =$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. By actual calculation, we can verify the result.

NOTE: By virtue of this property, a statement obtained from an established result by interchanging the words 'row' and 'column' thoroughly will be established.

(2) Let A be a matrix and B is obtained from A by interchanging any two rows (columns) of a matrix A, then detA = - det B.

(3) If A be a matrix containing two identical rows (or columns), then det A = 0. Result follows from (2).

(4) If elements of any row of a determinant is expressed as sum of two elements, then the determinant can be expressed as a sum of two determinants.

Example: det $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+p & a_{22}+q & a_{23}+r \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ = $det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ + $det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ p & q & r \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, which can be verified using definition and earlier properties.

(5) If elements of any row of a determinant is multiplied by a constant, then the determinant is multiplied by the same constant

Example: $det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$= c det $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, which can be verified using definition and earlier properties.

(6) In an nxn matrix A if a scalar multiple of one row(column) be added to another row(column), then detA remains unaltered.

» det $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+ca_{31} & a_{22}+ca_{32} & a_{23}+ca_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ = det $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ + det $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ca_{31} & ca_{32} & ca_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, [by property (4)]

=det$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$+cdet $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, [by property(5)]

= det$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ + c.0 , [by property(3)]

= det$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(7) In an nxn matrix A, if one row(column) be expressed as a linear combination of the remaining rows(columns), then detA = 0.

$$» \det\begin{bmatrix} c_1a_{21}+c_2a_{31} & c_1a_{22}+c_2a_{32} & c_1a_{23}+c_2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= c_1\det\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + c_2\det\begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0.$$

<u>Example:</u> Prove without expanding the determinant $\begin{vmatrix} 1 & a & b^2+c^2+bc \\ 1 & b & c^2+a^2+ca \\ 1 & c & a^2+b^2+ab \end{vmatrix} = 0$

$$» \begin{vmatrix} 1 & a & b^2+c^2+bc \\ 1 & b & c^2+a^2+ca \\ 1 & c & a^2+b^2+ab \end{vmatrix} = \begin{vmatrix} 1 & a & b^2+c^2 \\ 1 & b & c^2+a^2 \\ 1 & c & a^2+b^2 \end{vmatrix} + \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & -a^2 \\ 1 & b & -b^2 \\ 1 & c & -c^2 \end{vmatrix} + \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0.$$

<u>Cofactors and Minors :</u>

Let $A = [a_{ij}]_{mxn}$. M_{ij}, minor of the element a_{ij}, is the determinant of the matrix obtained by deleting i th row and j th column of the matrix A. A_{ij}, cofactor of the element a_{ij}, is defined as $(-1)^{i+j}M_{ij}$.

<u>Theorem:</u> For a matrix $A=[a_{ij}]_{3x3}$, $a_{i1}A_{k1}+ a_{i2}A_{k2}+ a_{i3}A_{k3}$ = det A, if i=k and = 0, if $i \neq k$.

» $a_{11}A_{11}+a_{12}A_{12}+A_{13} = a_{11}(-1)^{1+1}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{12}(-1)^{1+2}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}(-1)^{1+3}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ = detA, by definition. Also,

$a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}= \quad a_{11}(-1)^{2+1}\det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} +a_{12}(-1)^{2+2}\det \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} +a_{13}(-1)^{2+3}\det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$

$$=\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{(by definition of determinant)} = 0 \text{ (by property (3))}.$$

Similarly other statement can be proved.

<u>Multiplication of determinants</u>

<u>Theorem:</u> If A and B are two square matrices of the same order, then

$\det(AB) = \det A.\det B= \det A^T.\det B$

<u>Example:</u> prove $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}=(a^3+b^3+c^3-3abc)^2$

» we have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=-(a^3+b^3+c^3-3abc)$. Now $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.\begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}.$$

<u>Example:</u> Prove that $\begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-c)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix}=2(a-b)^2(b-c)^2(c-a)^2.$

» We have $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}= (a-b)(b-c)(c-a)$. Now $2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= -2\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$=\begin{vmatrix} a^2 & b^2 & c^2 \\ -2a & -2b & -2c \\ 1 & 1 & 1 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & (a-b)^2 & (a-c)^2 \\ (b-c)^2 & 0 & (b-c)^2 \\ (c-a)^2 & (c-b)^2 & 0 \end{vmatrix}.$$

<u>Definition:</u> If $A=(a_{ij})$ be a square matrix and A_{ij} be the cofactor of a_{ij} in det A, then $\det(A_{ij})$ is the adjoint of det A.

<u>Theorem:</u> If $A=(a_{ij})_{3x3}$, then $\det(A_{ij})=[\det(a_{ij})]^2$, if $\det(a_{ij})\neq 0$.

$$» \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$=\begin{vmatrix} a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13} & a_{11}A_{12}+a_{12}A_{22}+a_{13}A_{32} & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{vmatrix}$$

$= (\det A)^3$. Hence $\det(A_{ij})=[\det(a_{ij})]^2$, if $\det(a_{ij})\neq 0$.

<u>Example:</u> Prove $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}=(a^3+b^3+c^3-3abc)^2$

» we have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=-(a^3+b^3+c^3-3abc)$. Now $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$

$$= \operatorname{adj}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \text{RHS}$$

Adjoint of a square matrix

Definition Let A = (a_{ij}) be a square matrix. Let A_{ij} be the cofactor of a_{ij} in detA. The adjoint of A, denoted by Adj A, is defined as $(A_{ij})^T$.

Theorem: Let A be a square matrix of order n. then A. (Adj A) = (Adj A).A = $|A|I_n$.

» the i,j th element of A.(Adj A) is $a_{i1}A_{1j}+a_{i2}A_{I2}+\ldots+a_{in}A_{nj}$ which equals 0, if i≠j and equals $|A|$, if i=j. Hence A. (Adj A) = $\begin{bmatrix} |A| & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix}$ = $|A|I_n$. similarly other part.

Definition: A square matrix is singular if $|A|$=0 and is non-singular if $|A|$≠0.

Definition: A square matrix of order n is invertible if there exists a matrix B such that AB= BA = I_n. B is called an inverse to A. If C be an inverse to A also, then AC = CA = I_n. using associativity of product of matrices, it is easy to verify that B = C. So inverse of a square matrix , if it exists, is unique. Note also that since AB and BA both are to be defined, A must be a square matrix.

Theorem: An nxn matrix A is invertible iff it is non-singular.

» Necessity let A_{nxn} be invertible. Then there exists B_{nxn} such that AB = BA = I_n. then $|A||B|=|AB|=|I_n|=1$ so that $|A|\neq0$.

Sufficiency Let $|A| \neq 0$. we know that A.(Adj A) = (Adj A).A = $|A|$ I_n. hence A.$[\frac{1}{|A|}(Adj\ A)]$= $[\frac{1}{|A|}(Adj\ A)]$A = I_n, proving the result.

Theorem: If A,B be invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

» $|AB| = |A||B| \neq 0$, hence AB is invertible. Using associativity,

$(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$.

Theorem: If A be invertible, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.

» From $AA^{-1} = A^{-1}A = I_n$, it follows from the definition of inverse that $(A^{-1})^{-1} = A$.

Theorem: If A be an invertible matrix, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

» $A^T.(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n = (AA^{-1})^T = (A^{-1})^T.A^T$. Hence $(A^T)^{-1} = (A^{-1})^T$.

Orthogonal matrix

Definition: A square matrix A of order n is orthogonal iff $AA^T = I_n$.

Theorem: If A is orthogonal , then A is non-singular and $|A| = \pm 1$.

» $|A|^2 = |A||A^T| = |AA^T| = |I_n| = 1$ implies $|A| = \pm 1 \neq 0$.

Theorem: If A be an nxn orthogonal matrix, then $A^TA = I_n$.

» $AA^T = I_n \Rightarrow A^T(AA^T) = A^TI_n \Rightarrow (A^TA)A^T = A^T \Rightarrow (A^TA - I_n)A^T = O \Rightarrow (A^TA - I_n)[A^T(A^T)^{-1}] = O$ (A is orthogonal⇒A is non-singular⇒A^T is non-singular⇒A^T is invertible) $\Rightarrow A^TA = I_n$.

Theorem: If A and B are orthogonal matrices of the same order, then AB is orthogonal.

»$(AB)(AB)^T = (AB)(B^TA^T) = A(BB^T)A^T = (AI_n)A^T = AA^T = I_n$.

Theorem: If A is orthogonal, $A^{-1}$is orthogonal.

» $(A^{-1})(A^{-1})^T = (A^{-1})(A^T)^{-1} = (A^TA)^{-1} = I_n^{-1} = I_n$.

Note: If A be an orthogonal matrix, $A^T = A^{-1}$.

Rank of a matrix

Definition Let A be a non-zero matrix of order mxn. Rank of A is defined to be the greatest positive integer r such that the determinant of the matrix formed by elements of A lying at the intersection of some r rows and some r columns is nonzero. Rank of null matrix is defined to be zero.

Note: (1) $0<\text{rank } A \leq \min\{m,n\}$, for a non-zero matrix A.

(2) for a square matrix A of order n, rankA $< n$ or $=n$ according as A is singular or non-singular.

(3) Rank A = Rank A^T.

Elementary row operations

An elementary operation on a matrix A over a field F is an operation of the following three types:

- Interchange of two rows(columns) of A
- Multiplication of a row (or column) by a non-zero scalar $c \in F$
- Addition of a scalar multiple of one row (or column)to another row(or column)

When applied to rows, elementary operations are called elementary row operations.

Notation: interchange of i th and j th row will be denoted by R_{ij}. Multiplication of i th row by c will be denoted by cR_i. Addition of c times the j th row to the i th row is denoted by R_i+cR_j.

Definition: An mxn matrix B is row equivalent to a mxn matrix A over the same field F iff B can be obtained from A by a finite number of successive elementary row operations.

Note Since inverse of an elementary row operation is again an elementary row operation, if B is row equivalent to A, then A is also row equivalent to B.

Definition: An m x n matrix is row reduced iff

- The first non-zero element in a non-zero row is 1 and
- each column containing the leading 1of some row has all other elements zero.

Example: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 \cdots 0 & & 0 \end{bmatrix}$

Definition: An m x n matrix A is row reduced echelon matrix iff

- A is row reduced
- Each zero row comes below each non-zero row and
- If first r rows are non-zero rows of A and if the leading element of row I occurs in column k_i, then $k_1 < k_2 < \ldots < k_r$.

Example: $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Algorithm which row-reduces a matrix to echelon form

Step-1: suppose that j_1 column is the first column with a nonzero entry. Interchange the rows so that this nonzero entry appears in the first row, that is, so that $a_{1j_1} \neq 0$

Step-2: for each i>1, apply the operation $R_i \to -a_{1j_1}R_1 + a_{1j_1}R_i$.

Repeat steps 1 and 2 with the submatrix formed by all the rows excluding the first.

Theorem: For a given matrix A , a row-reduced echelon matrix B equivalent to A can be found by elementary row operations.

Example: $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & -2 \\ 2 & 6 & -2 \end{bmatrix} \xrightarrow[\frac{1}{2}R_1]{} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & -2 \\ 2 & 6 & -2 \end{bmatrix} \xrightarrow[R_2+2R_1]{} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 6 & -2 \end{bmatrix} \xrightarrow[R_3+2R_1]{} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 2 & 6 & 0 \end{bmatrix}$

$\xrightarrow[R_3-2R_2]{} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_{12}]{} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Theorem: If a matrix A is equivalent to a row-reduced enhelon matrix having r non-zero rows, then Rank A = r.

Example: Find the rank of the matrix A = $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

» First Method: rank A$\geq$ 1, since $det[1]$ = 1$\neq$0. Though det$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ = 0, det$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$\neq$0. Hence rank A$\geq$ 2. Since det A= 0, rank A is not equal to 3. hence rank A = 2.

Second Method: A $\xrightarrow[R_2-2R_1]{} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 6 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[R_1+\frac{1}{5}R_3]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow[R_3-\frac{5}{6}R_2]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow[\frac{1}{6}R_2]{} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence rank A = 2.

SYSTEM OF LINEAR EQUATIONS

A system of m linear equation in n unknowns $x_1, x_2, \ldots, x_n$ is of the form

$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n = b_1$

$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n = b_2$

…….................................

$a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n = b_m$, where a_{ij}'s and b_i's are given elements of a field.

An ordered n-tuple $(c_1,c_2,\ldots,c_n)$ is a solution of the system if each equation of the system is satisfied by $x_1=c_1,\ldots,x_n = c_n$. a system of equation is consistent if it has a solution; otherwise it is inconsistent.

Matrix representation Let $A = (a_{ij})_{mxn}$, $X = (x_j)_{nx1}$ and $B = (b_i)_{mx1}$.then the system can be written as AX = B. The matrix $\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} & b_1 \\ \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} \cdots a_{mn} & b_m \end{bmatrix}$ is called the augmented matrix, denoted by (A,B).

<u>Definition:</u> two systems AX=B and CX=D are equivalent systems if the augmented matrices (A,B) and (C,D) are row equivalent.

<u>Theorem:</u> if AX = B and CX = D are equivalent systems and if $(e_1,e_2,\ldots,e_n)$ be a solution of AX = B, then $(e_1,e_2,\ldots,e_n)$ is also a solution of CX = D. If one of two equivalent systems is inconsistent, then the other is also so.

<u>Ex :</u> Solve, if possible, the system $x_1+2x_2 - x_3 = 10$, $-x_1+x_2+2x_3 = 4$, $2x_1+x_2 - 3x_3 = 2$.

» $$\begin{bmatrix} 1 & 2 & -1 & 10 \\ -1 & 1 & 2 & 4 \\ 2 & 1 & -3 & 2 \end{bmatrix} \xrightarrow[R_2+R_1]{} \begin{bmatrix} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 14 \\ 2 & 1 & -3 & 2 \end{bmatrix} \xrightarrow[R_3-2R_1]{} \begin{bmatrix} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 14 \\ 0 & -3 & -1 & -18 \end{bmatrix}$$

$\xrightarrow[R_3+R_2]{} \begin{bmatrix} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 14 \\ 0 & 0 & 0 & -4 \end{bmatrix}$. Thus the given system is equivalent to $x_1+2x_2 - x_3 = 10$, $3x_2+x_3 = 14$, $0 = -4$, which is inconsistent. Thus the given system is inconsistent.

<u>Theorem:</u> a necessary and sufficient condition that a given system of linear equations AX=B is consistent is that rank A = rank(A,B).

Solution of a system of linear equations having same number of variables as that of equations in which coefficient matrix is nonsingular

METHOD 1: Cramer's rule

Let

$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n = b_1$

$a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n = b_2$

…………………………………..

$a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n = b_n$

be a system of n linear equations in n unknowns where det A= $\det(a_{ij})_{nxn}\neq 0$.Then there exists a unique solution of the system given by $x_1 = \frac{\det A_1}{\det A},\ldots,x_n = \frac{\det A_n}{detA}$, where A_i is the nxn matrix obtained from A by replacing its i th column by the column $[b_1\ \ b_2\ldots b_n]^T$, i = 1,2,…,n.

» $x_1$detA =det$\begin{bmatrix} x_1a_{11} & a_{12}\cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ x_1a_{n1} & a_{n2}\cdots & a_{nn} \end{bmatrix}$ = det$\begin{bmatrix} x_1a_{11}+x_2a_{12}+\cdots+x_na_{1n} & a_{12}\cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ x_1a_{n1}+x_2a_{n2}+\cdots+x_na_{nn} & a_{n2}\cdots & a_{nn} \end{bmatrix}$

=det$\begin{bmatrix} b_1 & a_{12}\cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ b_n & a_{n2}\cdots & a_{nn} \end{bmatrix}$=det A_1. Similarly others.

Example: let us consider the system x+2y-3z = 1, 2x – y+z = 4, x+3y = 5.

Determinant of the coefficient matrix = $\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 1 & 3 & 0 \end{vmatrix}$ = -22≠0. By Cramer's rule,

x = $\dfrac{\begin{vmatrix} 1 & 2 & -3 \\ 4 & -1 & 1 \\ 5 & 3 & 0 \end{vmatrix}}{-22}$=2, y=$\dfrac{\begin{vmatrix} 1 & 1 & -3 \\ 2 & 4 & 1 \\ 1 & 5 & 0 \end{vmatrix}}{-22}$=1, z=$\dfrac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 1 & 3 & 5 \end{vmatrix}}{-22}$=1.

METHOD 2: Matrix Inversion method

Let A = $(a_{ij})_{nxn}$, X = $[x_1,..,x_n]^T$, B=$[b_1,b_2,\ldots,b_n]^T$. Then the above system of linear equations can be written as AX=B,

where detA ≠0. Thus A^{-1} exists and X = A^{-1}B.

Example2.12 3x+y = 2, 2y+3z = 1, x+2z = 3.

Let A =$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, X=$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. detA≠0.

$A^{-1} = \frac{1}{\det A} AdjA$=$\begin{bmatrix} \frac{4}{15} & -\frac{2}{15} & \frac{3}{15} \\ \frac{3}{15} & \frac{6}{15} & -\frac{9}{15} \\ -\frac{2}{15} & \frac{1}{15} & \frac{6}{15} \end{bmatrix}$. X=$A^{-1}$B=$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Thus the solution is

x = 1,y = -1,z = 1.

EIGEN VECTOR AND EIGEN VALUE CORRESPONDING TO A SQUARE MATRIX

Let A be an nxn matrix over a field F. A non-zero vector $\underline{x} \in F^n$ is an eigen vector or a characteristic vector of A if there exists a scalar a∈F such that $A\underline{x} = c\underline{x}$ holds. Thus $(A-cI_n)\,\underline{x} = O$ holds. This is a homogeneous system of n equations in n unknowns. If $\det(A-cI_n) \neq 0$, then by Cramer's rule, $\underline{x} = \underline{0}$ will be the only solution .Since we are interested in non-zero solution, $\det(A-cI_n)=0$ equation is called the characteristic equation of A.

A root of the above equation equation in a, is called an eigen value of A.

<u>Example:</u> let A = $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. The characteristic equation is $\begin{vmatrix} 1-a & 3 \\ 4 & 5-a \end{vmatrix}$=0, or ,

a^2-6a-7 = 0.thus eigen values are -1,7. The eigenvector $\begin{bmatrix} x \\ y \end{bmatrix}$ corresponding to the eigenvalue -1 is given by $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=-1\begin{bmatrix} x \\ y \end{bmatrix}$. Thus 2x+3y = 0, 4x+6y = 0. Thus x =-3y/2.hence the eigenvector corresponding to -1 is $\begin{bmatrix} x \\ y \end{bmatrix}=k\begin{bmatrix} -3 \\ 2 \end{bmatrix}$, where k≠0. Similarly eigenvector corresponding to eigenvalue 7 can be found.

<u>Theorem:</u> The eigen values of a diagonal matrix are its diagonal elements.

<u>Theorem:</u>If c is an eigen value of a nonsingular matrix A,then c^{-1} is an eigen value of A^{-1}.

<u>Theorem:</u> If A and P be both nxn matrices and P be non-singular, then A and $P^{-1}AP$ have the same eigen values.

<u>Theorem:</u> To an eigen vector of A, there corresponds a unique eigen value of A.

» if possible, let there be two distinct eigen values c_1 and c_2 of A corresponding to an eigen vector $\underline{x}$. Thus $A\underline{x} = c_1\underline{x} = c_2\underline{x}$. hence $(c_1-c_2)\,\underline{x} = \underline{0}$; but this is a contradiction since $a_1 \neq a_2$ and $\underline{x}$ is non-zero vector.

Theorem: (**Cayley Hamilton theorem**)

Every square matrix satisfies its own characteristic equation.

Ex: let A=$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. Verify that A satisfies its characteristic equation. Hence find A^{-1}.

» The characteristic equation is $x^2 - 7x+7 = 0$. Now A^2-7A+7I =O can be verified by actual calculation. Hence Cayley Hamilton theorem is verified. Hence A $\left[-\frac{1}{7}(A - 7I_2)\right]$=$I_2$. Thus $A^{-1} = -\frac{1}{7}(A - 7I_2)$=$\frac{1}{7}\begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$.

REAL QUADRATIC FORM

An expression of the form $\sum_{i,j} a_{ij}x_i x_j$ (i,j = 1,2,…,n) where $a_{ij}$are real and $a_{ij} = a_{ji}$, is said to be a real quadratic form in n variables $x_1,x_2,\ldots,x_n$. the matrix notation for the quadratic form is $\underline{x}^T A\underline{x}$, where $\underline{x} = [x_1 \quad x_2 \ldots \quad x_n]^T$, A = $(a_{ij})_{nxn}$. A is a real symmetric matrix since $a_{ij} = a_{ji}$ for all i,j. A is called the matrix associated with the quadratic form.

Example2.27 x_1x_2-x_2x_3 is a real quadratic form in three variables x_1,x_2,x_3. The associated matrix is $\begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{bmatrix}$.

Definition: A real quadratic form $Q=\underline{x}^T A\underline{x}$ is

(1) Positive definite if $Q>0$ for all $\underline{x}\neq O$
(2) Positive semi definiteif $Q\geq 0$ for all $\underline{x}\neq O$
(3) Negative definite if $Q<0$ for all $\underline{x}\neq O$
(4) Negative semidefinite if $Q\leq 0$ for all $\underline{x}\neq O$
(5) Indefinite if $Q\geq 0$ for some $\underline{x}\neq O$ and $Q\leq 0$ for some other $\underline{x}\neq O$

Example: consider the quadratic form $Q(x_1,x_2,x_3) = x_1^2+2x_2^2+4x_3^2+2x_1x_2-4x_2x_3-2x_3x_1 = (x_1+x_2-x_3)^2+(x_2-x_3)^2+2x_3^2\geq 0$ and $Q = 0$ only when $x_1+x_2-x_3= x_2-x_3 = x_3 = 0$, that is, when $x_1=x_2=x_3 = 0$. Thus Q is positive definite.

For a real quadratic form $Q=X^TAX$ where A is real symmetric matrix of rank $r(\leq n)$, there exists a non-singular matrix P such that P^TAP becomes a diagonal matrix $\begin{bmatrix} I_m & & \\ & -I_{r-m} & \\ & & o \end{bmatrix}$ of rank r, where $0\leq m\leq r$. thus by a suitable transformation X=PY, where P is nonsingular, the real quadratic form Q transforms to $y_1^2+\ldots+y_m^2-y_{m+1}^2-\ldots-y_r^2$ where $0\leq m\leq r\leq n$. this is called normal form of Q.

Theorem: A real quadratic form of rank r and index m is

(1) Positive definite, if $r = n, m=r$
(2) Positive semidefinite, if $r<n, m= r$
(3) Negative definite, if $r = n, m = 0$
(4) Negative semidefinite, if $r<n, m= 0$
(5) Indefinite, if $r\leq n, 0<m<r$

Example: Reduce the quadratic form $5x^2+y^2+14z^2-4yz-10zx$ to its normal form and show that it is positive definite.

» The associated symmetric matrix is $A=\begin{bmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ -5 & -2 & 14 \end{bmatrix}$. Let us apply congruent operations on A to reduce it to the normal form.

$$A \xrightarrow{R_3+R_1} \begin{bmatrix} 5 & 0 & -5 \\ 0 & 1 & -2 \\ 0 & -2 & 9 \end{bmatrix} \xrightarrow{C_3+C_1} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 9 \end{bmatrix} \xrightarrow{R_3+2R_2} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{C_3+2C_2} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$\xrightarrow{\frac{1}{\sqrt{5}}R_1, \frac{1}{\sqrt{5}}R_3} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \xrightarrow{\frac{1}{\sqrt{5}}C_1, \frac{1}{\sqrt{5}}C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The normal form is $x^2+y^2+z^2$.the rank of the quadratic form is 3 and its signature is 3. Thus the quadratic form is positive definite.

1 Functions

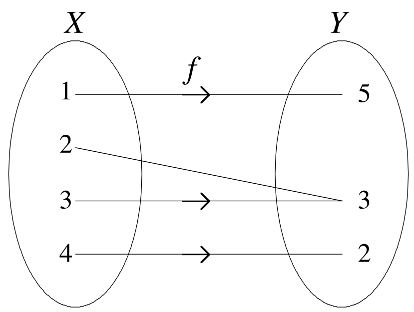
In this Chapter we will cover various aspects of functions. We will look at the definition of a function, the domain and range of a function, what we mean by specifying the domain of a function and absolute value function.

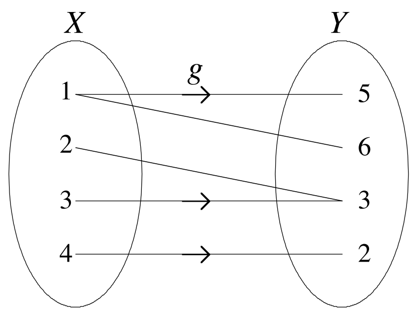
1.1 What is a function?

1.1.1 Definition of a function

> A function f from a set of elements X to a set of elements Y is a rule that assigns to each element x in X exactly one element y in Y.

One way to demonstrate the meaning of this definition is by using arrow diagrams.





$f : X \to Y$ is a function. Every element in X has associated with it exactly one element of Y.

$g : X \to Y$ is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y.

A function can also be described as a set of ordered pairs (x, y) such that for any x-value in the set, there is only one y-value. This means that there cannot be any repeated x-values with different y-values.

The examples above can be described by the following sets of ordered pairs.

F = {(1,5),(3,3),(2,3),(4,2)} is a function.

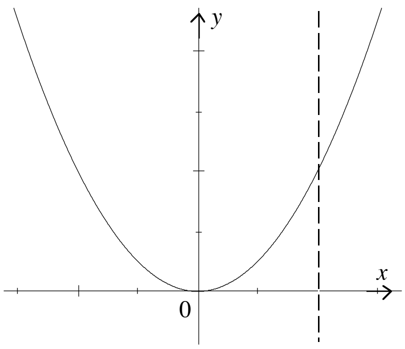
G = {(1,5),(4,2),(2,3),(3,3),(1,6)} is not a function.

The definition we have given is a general one. While in the examples we have used numbers as elements of X and Y, there is no reason why this must be so. However, in these notes we will only consider functions where X and Y are subsets of the real numbers.

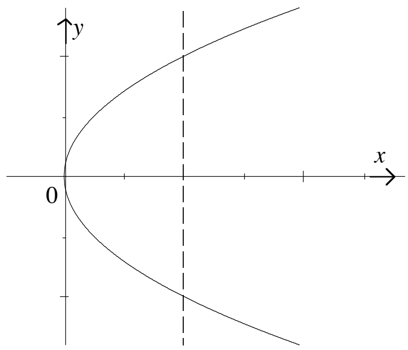
In this setting, we often describe a function using the rule, $y = f(x)$, and create a graph of that function by plotting the ordered pairs $(x, f(x))$ on the Cartesian Plane. This graphical representation allows us to use a test to decide whether or not we have the graph of a function: The Vertical Line Test.

1.1.2 The Vertical Line Test

The Vertical Line Test states that if it is *not possible* to draw a vertical line through a graph so that it cuts the graph in more than one point, then the graph *is* a function.



This is the graph of a function. All possible vertical lines will cut this graph only once.



This is not the graph of a function. The vertical line we have drawn cuts the graph twice.

1.1.3 Domain of a function

For a function $f : X \to Y$ the *domain* of f is the set X.

This also corresponds to the set of x-values when we describe a function as a set of ordered pairs (x, y).

If only the rule $y = f(x)$ is given, then the domain is taken to be the set of all real x for which the function is defined. For example, $y = \sqrt{x}$ has domain; all real $x \geq 0$. This is sometimes referred to as the *natural* domain of the function.

1.1.4 Range of a function

For a function $f : X \to Y$ the *range* of f is the set of y-values such that $y = f(x)$ for some x in X.

This corresponds to the set of y-values when we describe a function as a set of ordered pairs (x, y). The function $y = \sqrt{x}$ has range; all real $y \geq 0$.

Example

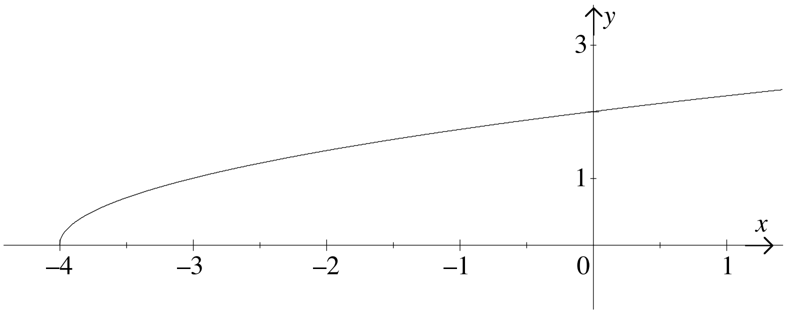
a. State the domain and range of $y = \sqrt{x + 4}$.

b. Sketch, showing significant features, the graph of $y = \sqrt{x + 4}$.

Solution

a. The domain of $y = \sqrt{x+4}$ is all real $x \geq -4$. We know that square root functions are only defined for positive numbers so we require that $x + 4 \geq 0$, ie $x \geq -4$. We also know that the square root functions are always positive so the range of $y = \sqrt{x+4}$ is all real $y \geq 0$.

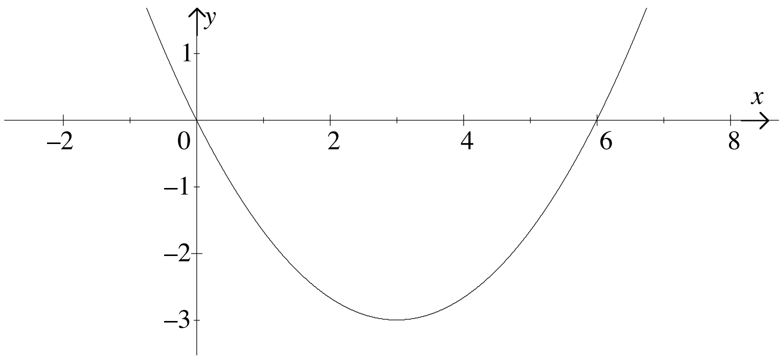
b.



The graph of $y = \sqrt{x+4}$.

Example

a. State the equation of the parabola sketched below, which has vertex $(3, -3)$.



b. Find the domain and range of this function.

Solution

a. The equation of the parabola is $y = \frac{x^2-6x}{3}$.

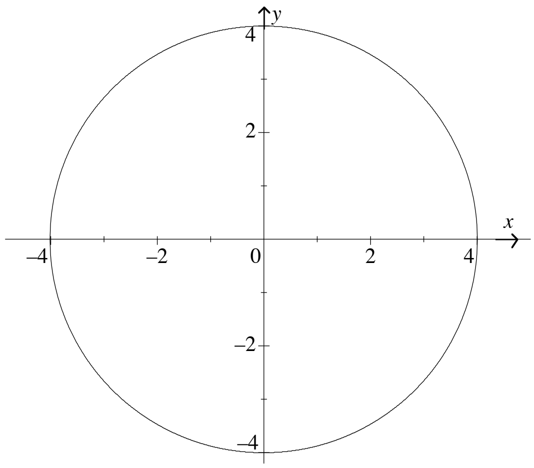
b. The domain of this parabola is all real x. The range is all real $y \geq -3$.

Example

Sketch $x^2 + y^2 = 16$ and explain why it is not the graph of a function.

Solution

$x^2 + y^2 = 16$ is not a function as it fails the vertical line test. For example, when $x = 0$ $y = \pm 4$.



The graph of $x^2 + y^2 = 16$.

Example

Sketch the graph of $f(x) = 3x - x^2$ and find

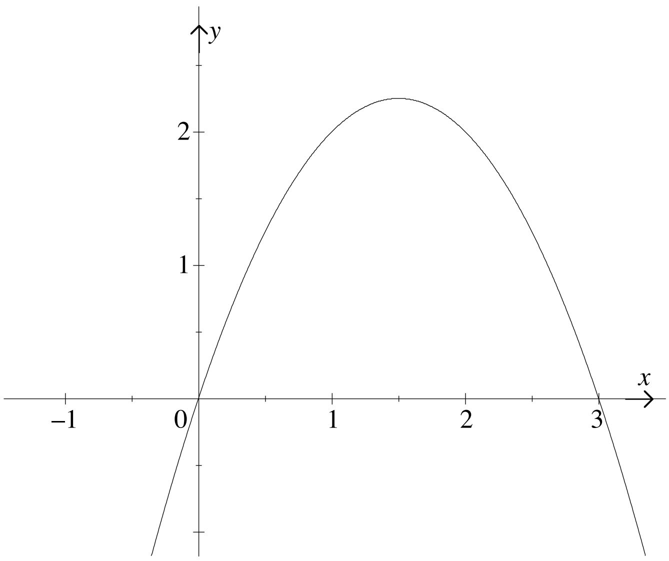
a. the domain and range

b. $f(q)$

c. $f(x^2)$

d. $\frac{f(2+h)-f(2)}{h}$, $h \neq 0$.

Solution



The graph of $f(x) = 3x - x^2$.

a. The domain is all real x. The range is all real y where $y \leq 2.25$.

b. $f(q) = 3q - q^2$

c. $f(x^2) = 3(x^2) - (x^2)^2 = 3x^2 - x^4$

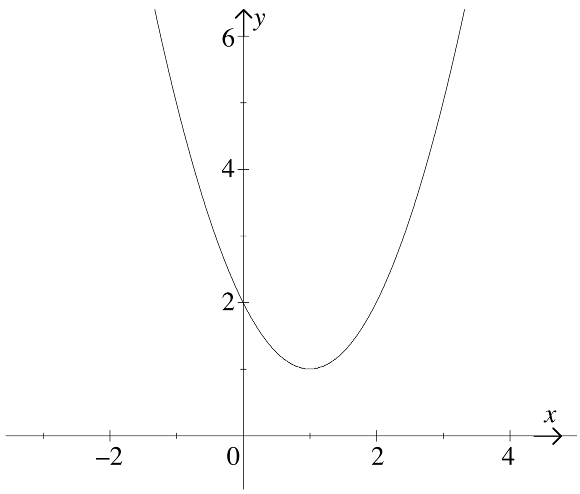
d.

$$\begin{aligned}
\frac{f(2+h)-f(2)}{h} &= \frac{(3(2+h)-(2+h)^2)-(3(2)-(2)^2)}{h} \\
&= \frac{6+3h-(h^2+4h+4)-2}{h} \\
&= \frac{-h^2-h}{h} \\
&= -h-1
\end{aligned}$$

Example

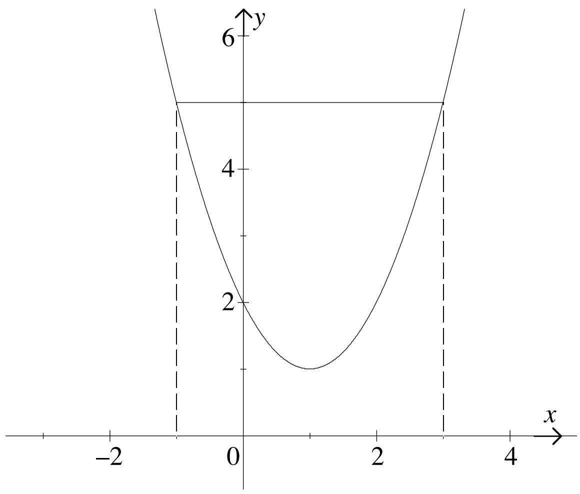
Sketch the graph of the function $f(x) = (x-1)^2 + 1$ and show that $f(p) = f(2-p)$. Illustrate this result on your graph by choosing one value of p.

Solution



The graph of $f(x) = (x-1)^2 + 1$.

$$\begin{aligned}
f(2-p) &= ((2-p)-1)^2+1 \\
&= (1-p)^2+1 \\
&= (p-1)^2+1 \\
&= f(p)
\end{aligned}$$



The sketch illustrates the relationship $f(p) = f(2-p)$ for $p = -1$. If $p = -1$ then $2 - p = 2 - (-1) = 3$, and $f(-1) = f(3)$.

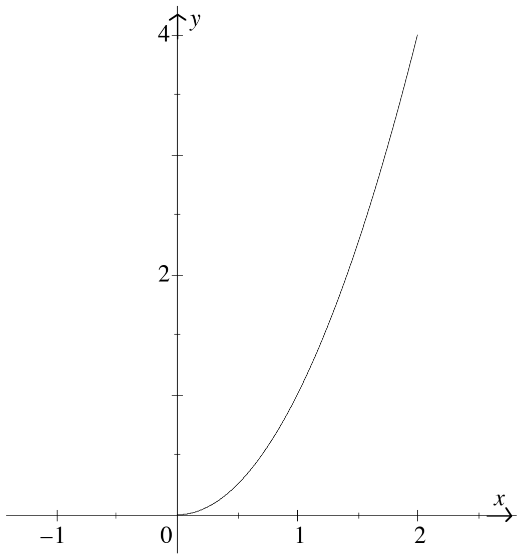
1.2 Specifying or restricting the domain of a function

We sometimes give the rule $y = f(x)$ along with the domain of definition. This domain may not necessarily be the natural domain. For example, if we have the function

$$y = x^2 \quad \text{for} \quad 0 \leq x \leq 2$$

then the domain is given as $0 \leq x \leq 2$. The natural domain has been restricted to the subinterval $0 \leq x \leq 2$.

Consequently, the range of this function is all real y where $0 \leq y \leq 4$. We can best illustrate this by sketching the graph.



The graph of $y = x^2$ for $0 \leq x \leq 2$.

1.3 The absolute value function

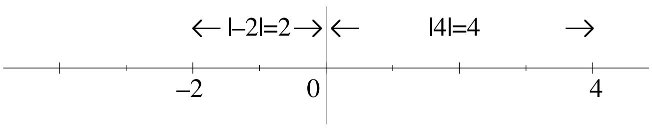
Before we define the absolute value function we will review the definition of the absolute value of a number.

The *Absolute value of a number* x is written $|x|$ and is defined as

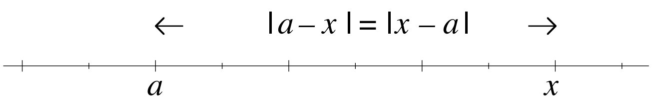
$$|x| = x \text{ if } x \geq 0 \qquad \text{or} \qquad |x| = -x \text{ if } x < 0.$$

That is, $|4| = 4$ since 4 is positive, but $|-2| = 2$ since -2 is negative.

We can also think of $|x|$ geometrically as the distance of x from 0 on the number line.



More generally, $|x - a|$ can be thought of as the distance of x from a on the numberline.



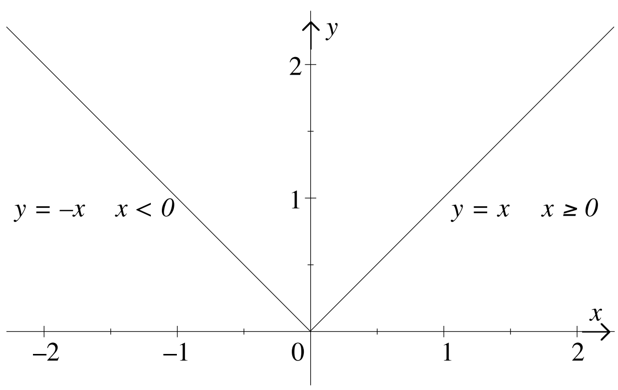
Note that $|a - x| = |x - a|$.

The absolute value *function* is written as $y = |x|$.

We define this function as

$$y = \begin{cases} +x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

From this definition we can graph the function by taking each part separately. The graph of $y = |x|$ is given below.



The graph of $y = |x|$.

Example

Sketch the graph of $y = |x-2|$.

Solution

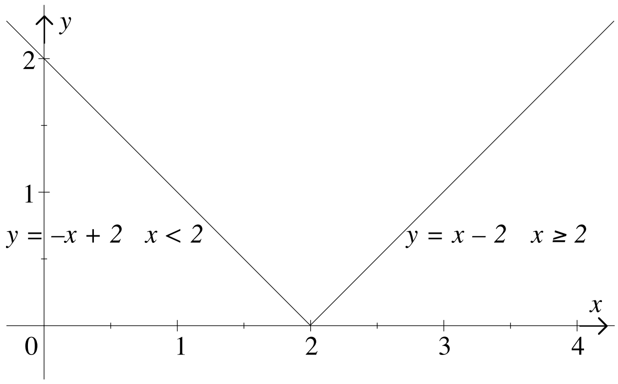
For $y = |x-2|$ we have

$$y = \begin{cases} +(x-2) & \text{when } x-2 \geq 0 \quad \text{or} \quad x \geq 2 \\ -(x-2) & \text{when } x-2 < 0 \quad \text{or} \quad x < 2 \end{cases}$$

That is,

$$y = \begin{cases} x-2 & \text{for } x \geq 2 \\ -x+2 & \text{for } x < 2 \end{cases}$$

Hence we can draw the graph in two parts.



The graph of $y = |x-2|$.

We could have sketched this graph by first of all sketching the graph of $y = x - 2$ and then reflecting the negative part in the x-axis. We will use this fact to sketch graphs of this type in Chapter 2.

1.4 Exercises

1. **a.** State the domain and range of $f(x) = \sqrt{9-x^2}$.

b. Sketch the graph of $y = \sqrt{9-x^2}$.

2. Given $\psi(x) = x^2 + 5$, find, in simplest form, $\dfrac{\psi(x+h) - \psi(x)}{h} \qquad h \neq 0.$

3. Sketch the following functions stating the domain and range of each:

a. $y = \sqrt{x-1}$

b. $y = |2x|$

c. $y = \frac{1}{x-4}$

d. $y = |2x| - 1$.

4. **a.** Find the perpendicular distance from $(0, 0)$ to the line $x + y + k = 0$

b. If the line $x + y + k = 0$ cuts the circle $x^2 + y^2 = 4$ in two distinct points, find the restrictions on k.

5. Sketch the following, showing their important features.

a. $y = \left(\frac{1}{2}\right)^x$

b. $y^2 = x^2$.

6. Explain the meanings of function, domain and range. Discuss whether or not $y^2 = x^3$ is a function.

7. Sketch the following relations, showing all intercepts and features. State which ones are functions giving their domain and range.

a. $y = -\sqrt{4 - x^2}$

b. $|x| - |y| = 0$

c. $y = x^3$

d. $y = \frac{x}{|x|}, x \neq 0$

e. $|y| = x$.

8. If $A(x) = x^2 + 2 + \frac{1}{x^2}$, $x \neq 0$, prove that $A(p) = A(\frac{1}{p})$ for all $p \neq 0$.

9. Write down the values of x which are not in the domain of the following functions:

a. $f(x) = \sqrt{x^2 - 4x}$

b. $g(x) = \frac{x}{x^2-1}$

10. If $\phi(x) = \log\left(\frac{x}{x-1}\right)$, find in simplest form:

a. $\phi(3) + \phi(4) + \phi(5)$

b. $\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n)$

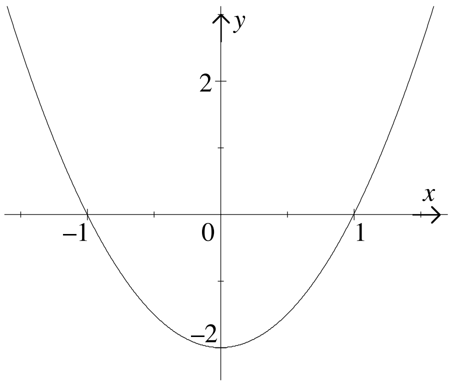
11. **a.** If $y = x^2 + 2x$ and $x = (z - 2)^2$, find y when $z = 3$.

b. Given $L(x) = 2x + 1$ and $M(x) = x^2 - x$, find

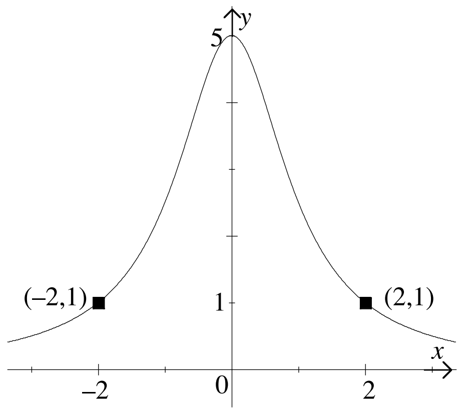
i $L(M(x))$

ii $M(L(x))$

12. Using the sketches, find the value(s) of the constants in the given equations:



$y = ax^2 - b$



$y = \frac{a}{bx^2+1}$

13. **a.** Define $|a|$, the absolute value of a, where a is real.

b. Sketch the relation $|x| + |y| = 1$.

14. Given that $S(n) = \frac{n}{2n+1}$, find an expression for $S(n-1)$.

Hence show that $S(n) - S(n-1) = \frac{1}{(2n-1)(2n+1)}$.

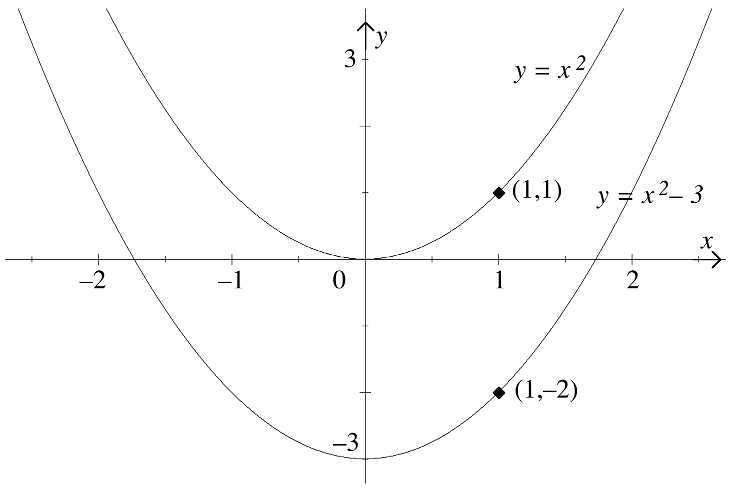
2 More about functions

In this Chapter we will look at the effects of stretching, shifting and reflecting the basic functions, $y = x^2$, $y = x^3$, $y = \frac{1}{x}$, $y = |x|$, $y = a^x$, $x^2 + y^2 = r^2$. We will introduce the concepts of even and odd functions, increasing and decreasing functions and will solve equations using graphs.

2.1 Modifying functions by shifting

2.1.1 Vertical shift

We can draw the graph of $y = f(x) + k$ from the graph of $y = f(x)$ as the addition of the constant k produces a *vertical shift*. That is, adding a constant to a function moves the graph up k units if $k > 0$ or down k units if $k < 0$. For example, we can sketch the function $y = x^2 - 3$ from our knowledge of $y = x^2$ by shifting the graph of $y = x^2$ *down* by 3 units. That is, if $f(x) = x^2$ then $f(x) - 3 = x^2 - 3$.

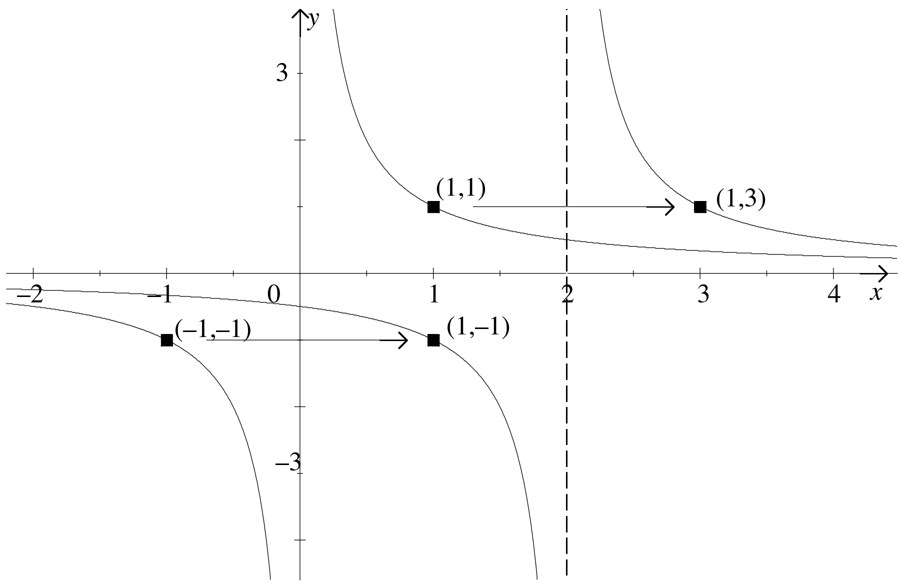


We can also write $y = f(x) - 3$ as $y + 3 = f(x)$, so replacing y by $y + 3$ in $y = f(x)$ also shifts the graph down by 3 units.

2.1.2 Horizontal shift

We can draw the graph of $y = f(x - a)$ if we know the graph of $y = f(x)$ as placing the constant a inside the brackets produces a *horizontal shift*. If we replace x by $x - a$ inside the function then the graph will shift to the left by a units if $a < 0$ and to the right by a units if $a > 0$.

For example we can sketch the graph of $y = \frac{1}{x-2}$ from our knowledge of $y = \frac{1}{x}$ by shifting this graph to the right by 2 units. That is, if $f(x) = \frac{1}{x}$ then $f(x-2) = \frac{1}{x-2}$.

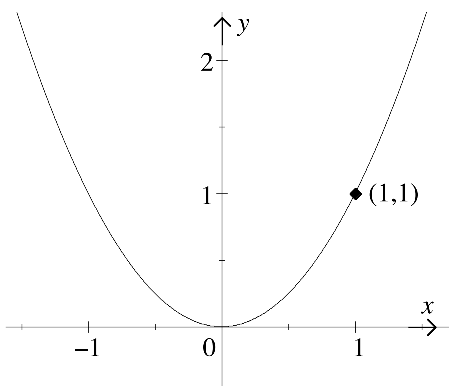


Note that the function $y = \frac{1}{x-2}$ is not defined at $x = 2$. The point $(1, 1)$ has been shifted to $(1, 3)$.

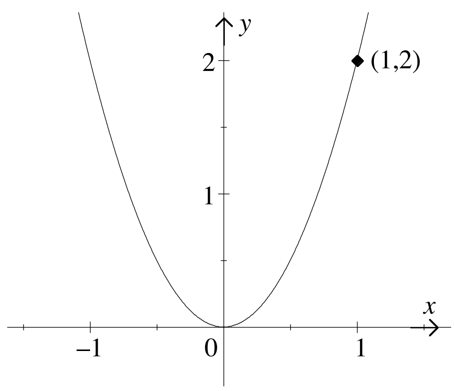
2.2 Modifying functions by stretching

We can sketch the graph of a function $y = bf(x)$ $(b > 0)$ if we know the graph of $y = f(x)$ as multiplying by the constant b will have the effect of stretching the graph in the y-direction by a factor of b. That is, multiplying $f(x)$ by b will change all of the y-values proportionally.

For example, we can sketch $y = 2x^2$ from our knowledge of $y = x^2$ as follows:

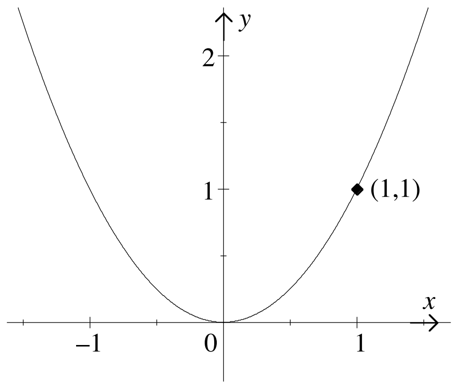


The graph of $y = x^2$.

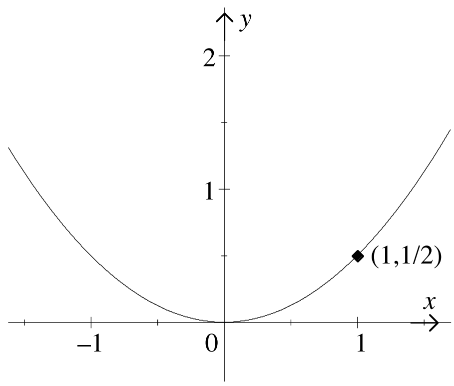


The graph of $y = 2x^2$. Note, all the y-values have been multiplied by 2, but the x-values are unchanged.

We can sketch the graph of $y = \frac{1}{2}x^2$ from our knowledge of $y = x^2$ as follows:



The graph of $y = x^2$.

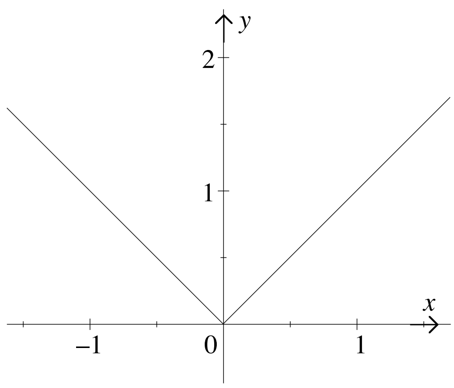


The graph of $y = \frac{1}{2}x^2$. Note, all the y-values have been multiplied by $\frac{1}{2}$, but the x-values are unchanged.

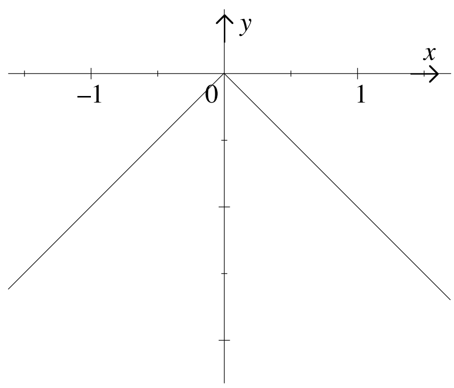
2.3 Modifying functions by reflections

2.3.1 Reflection in the x-axis

We can sketch the function $y = -f(x)$ if we know the graph of $y = f(x)$, as a minus sign in front of $f(x)$ has the effect of reflecting the whole graph in the x-axis. (Think of the x-axis as a mirror.) For example, we can sketch $y = -|x|$ from our knowledge of $y = |x|$.



The graph of $y = |x|$.

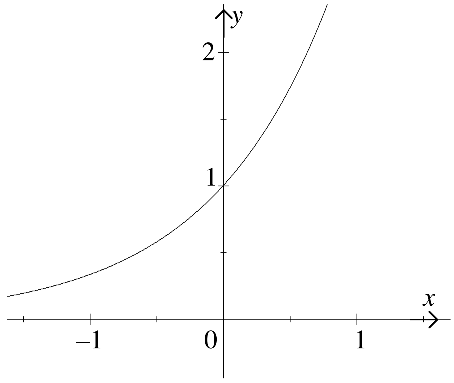


The graph of $y = -|x|$. It is the reflection of $y = |x|$ in the x-axis.

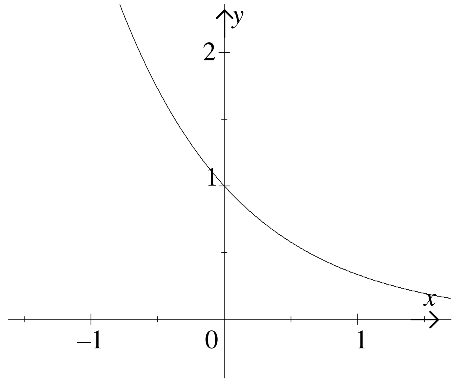
2.3.2 Reflection in the y-axis

We can sketch the graph of $y = f(-x)$ if we know the graph of $y = f(x)$ as the graph of $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.

For example, we can sketch $y = 3^{-x}$ from our knowledge of $y = 3^x$.



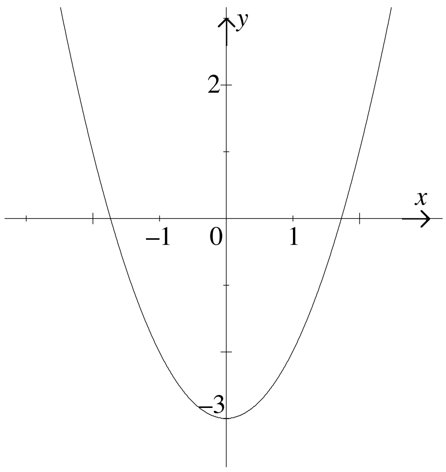
The graph of $y = 3^x$.



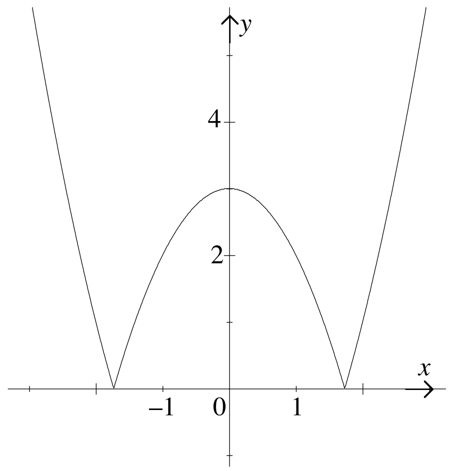
The graph of $y = 3^{-x}$. It is the reflection of $y = 3^x$ in the y-axis.

2.4 Other effects

We can sketch the graph of $y = |f(x)|$ if we know the graph of $y = f(x)$ as the effect of the absolute value is to reflect all of the *negative* values of $f(x)$ in the x-axis. For example, we can sketch the graph of $y = |x^2 - 3|$ from our knowledge of the graph of $y = x^2 - 3$.



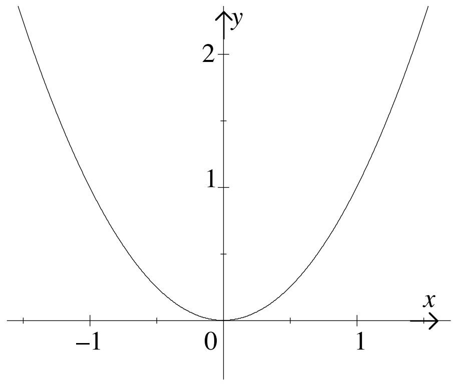
The graph of $y = x^2 - 3$.



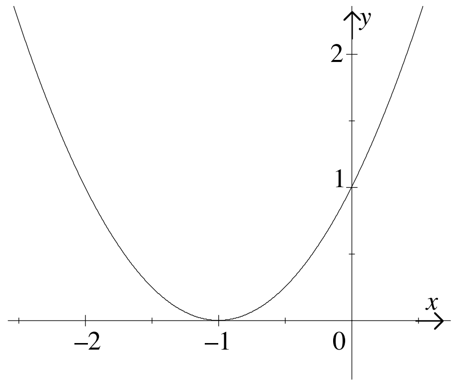
The graph of $y = |x^2 - 3|$. The negative values of $y = x^2 - 3$ have been reflected in the x-axis.

2.5 Combining effects

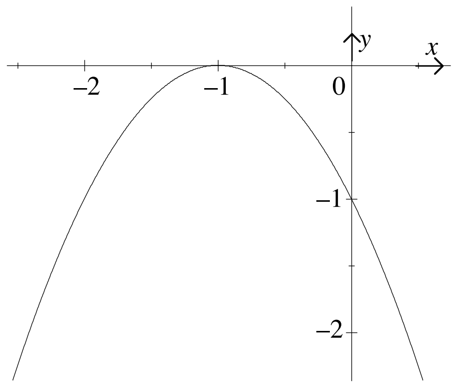
We can use all the above techniques to graph more complex functions. For example, we can sketch the graph of $y = 2 - (x + 1)^2$ from the graph of $y = x^2$ provided we can analyse the combined effects of the modifications. Replacing x by $x + 1$ (or $x - (-1)$) moves the graph to the left by 1 unit. The effect of the $-$ sign in front of the brackets turns the graph up side down. The effect of adding 2 moves the graph up 2 units. We can illustrate these effects in the following diagrams.



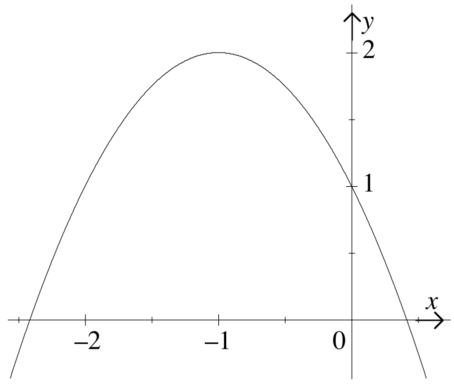
The graph of $y = x^2$.



The graph of $y = (x + 1)^2$. The graph of $y = x^2$ has been shifted 1 unit to the left.



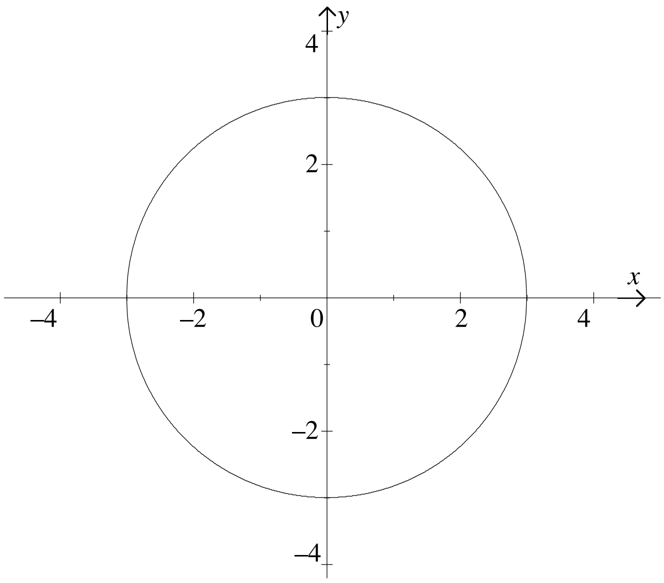
The graph of $y = -(x + 1)^2$. The graph of $y = (x + 1)^2$ has been reflected in the x-axis.



The graph of $y = 2-(x+1)^2$. The graph of $y = -(x + 1)^2$ has been shifted up by 2 units.

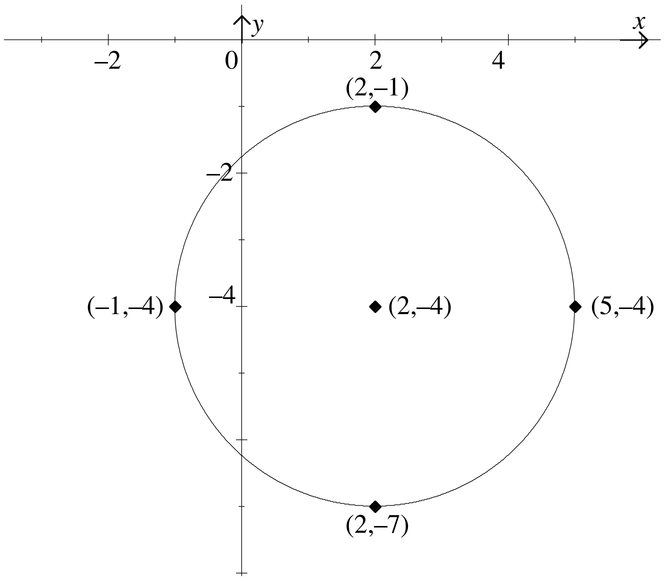
Similarly, we can sketch the graph of $(x-h)^2+(y-k)^2 = r^2$ from the graph of $x^2+y^2 = r^2$. Replacing x by $x - h$ shifts the graph sideways h units. Replacing y by $y - k$ shifts the graph up or down k units. (We remarked before that $y = f(x) + k$ could be written as $y - k = f(x)$.)

For example, we can use the graph of the circle of radius 3, $x^2 + y^2 = 9$, to sketch the graph of $(x - 2)^2 + (y + 4)^2 = 9$.



The graph of $x^2 + y^2 = 9$.

This is a circle centre $(0, 0)$, radius 3.



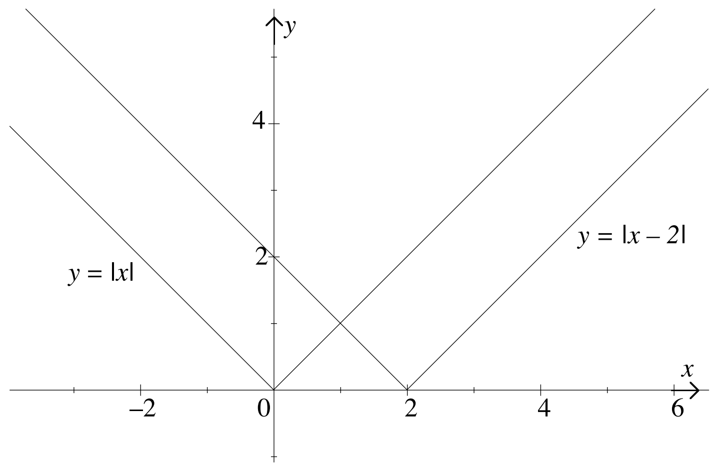
The graph of $(x - 2)^2 + (y + 4)^2 = 9$.

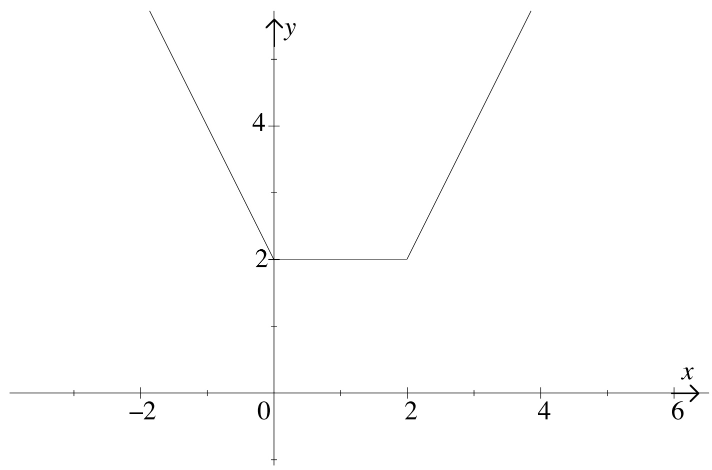
This is a circle centre $(2, -4)$, radius 3.

Replacing x by $x - 2$ has the effect of shifting the graph of $x^2 + y^2 = 9$ two units to the right. Replacing y by $y + 4$ shifts it down 4 units.

2.6 Graphing by addition of ordinates

We can sketch the graph of functions such as $y = |x| + |x - 2|$ by drawing the graphs of both $y = |x|$ and $y = |x - 2|$ on the same axes then adding the corresponding y-values.





The graph of $y = |x| + |x - 2|$.

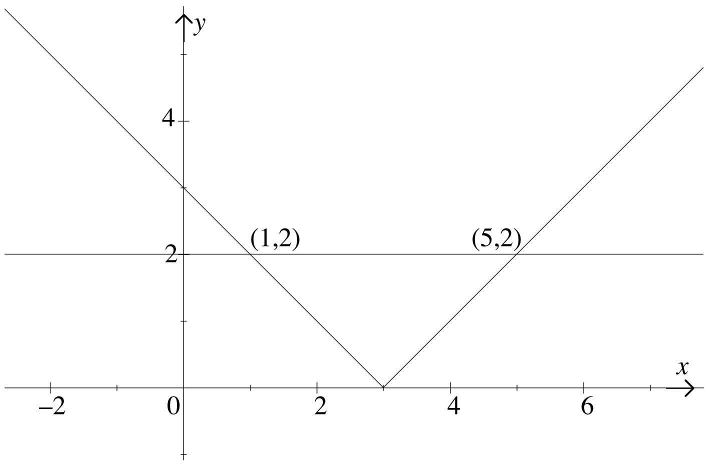
At each point of x the y-values of $y = |x|$ and $y = |x - 2|$ have been added. This allows us to sketch the graph of $y = |x| + |x - 2|$.

This technique for sketching graphs is very useful for sketching the graph of the sum of two trigonometric functions.

2.7 Using graphs to solve equations

We can solve equations of the form $f(x) = k$ by sketching $y = f(x)$ and the horizontal line $y = k$ on the same axes. The solution to the equation $f(x) = k$ is found by determining the x-values of any points of intersection of the two graphs.

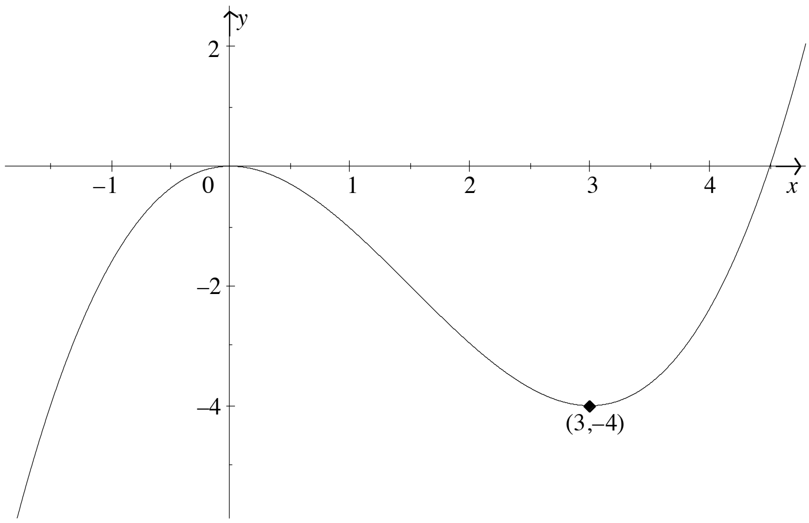
For example, to solve $|x - 3| = 2$ we sketch $y = |x - 3|$ and $y = 2$ on the same axes.



The x-values of the points of intersection are 1 and 5. Therefore $|x - 3| = 2$ when $x = 1$ or $x = 5$.

Example

The graph of $y = f(x)$ is sketched below.



For what values of k does the equation $f(x) = k$ have

1. 1 solution

2. 2 solutions

3. 3 solutions?

Solution

If we draw a horizontal line $y = k$ across the graph $y = f(x)$, it will intersect once when $k > 0$ or $k < -4$, twice when $k = 0$ or $k = -4$ and three times when $-4 < k < 0$. Therefore the equation $f(x) = k$ will have

1. 1 solution if $k > 0$ or $k < -4$

2. 2 solutions if $k = 0$ or $k = -4$

3. 3 solutions if $-4 < k < 0$.

2.8 Exercises

1. Sketch the following:

a. $y = x^2$ **b.** $y = \frac{1}{3}x^2$ **c.** $y = -x^2$ **d.** $y = (x+1)^2$

2. Sketch the following:

a. $y = \frac{1}{x}$ **b.** $y = \frac{1}{x-2}$ **c.** $y = \frac{-2}{x}$ **d.** $y = \frac{1}{x+1} + 2$

3. Sketch the following:

a. $y = x^3$ **b.** $y = |x^3 - 2|$ **c.** $y = 3 - (x-1)^3$

4. Sketch the following:

a. $y = |x|$ **b.** $y = 2|x-2|$ **c.** $y = 4 - |x|$

5. Sketch the following:

a. $x^2 + y^2 = 16$ **b.** $x^2 + (y+2)^2 = 16$ **c.** $(x-1)^2 + (y-3)^2 = 16$

6. Sketch the following:

a. $y = \sqrt{9 - x^2}$ **b.** $y = \sqrt{9 - (x-1)^2}$ **c.** $y = \sqrt{9 - x^2} - 3$

7. Show that $\dfrac{x-1}{x-2} = \dfrac{1}{x-2} + 1$.

Hence sketch the graph of $y = \dfrac{x-1}{x-2}$.

8. Sketch $y = \frac{x+1}{x-1}$.

9. Graph the following relations in the given interval:

a. $y = |x| + x + 1$ for $-2 \le x \le 2$ [Hint: Sketch by adding ordinates]

b. $y = |x| + |x-1|$ for $-2 \le x \le 3$

c. $y = 2^x + 2^{-x}$ for $-2 \le x \le 2$

d. $|x - y| = 1$ for $-1 \le x \le 3$.

10. Sketch the function $f(x) = |x^2 - 1| - 1$.

11. Given $y = f(x)$ as sketched below, sketch

a. $y = 2f(x)$

b. $y = -f(x)$

c. $y = f(-x)$

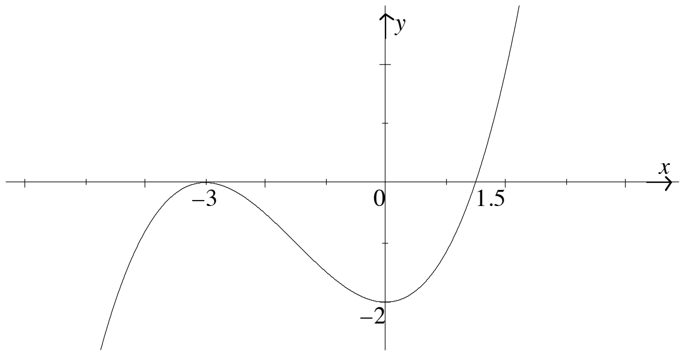
d. $y = f(x) + 4$

e. $y = f(x - 3)$

f. $y = f(x + 1) - 2$

g. $y = 3 - 2f(x - 3)$

h. $y = |f(x)|$



12. By sketching graphs solve the following equations:

a. $|2x| = 4$

b. $\frac{1}{x-2} = -1$

c. $x^3 = x^2$

d. $x^2 = \frac{1}{x}$

13. Solve $|x - 2| = 3$.

a. algebraically

b. geometrically.

14. The parabolas $y = (x - 1)^2$ and $y = (x - 3)^2$ intersect at a point P. Find the coordinates of P.

15. Sketch the circle $x^2 + y^2 - 2x - 14y + 25 = 0$. [Hint: Complete the squares.] Find the values of k, so that the line $y = k$ intersects the circle in two distinct points.

16. Solve $\frac{4}{5-x} = 1$, using a graph.

17. Find all real numbers x for which $|x - 2| = |x + 2|$.

18. Given that $Q(p) = p^2 - p$, find possible values of n if $Q(n) = 2$.

19. Solve $|x - 4| = 2x$.

a. algebraically

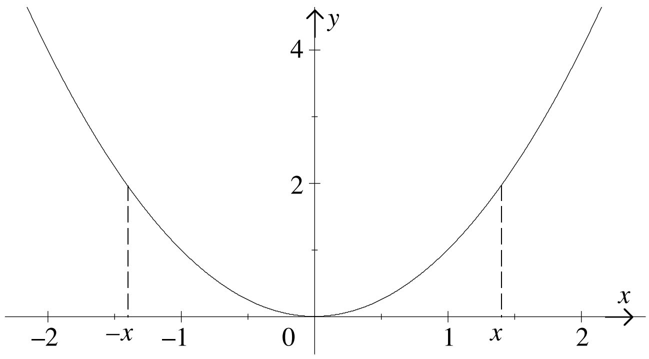
b. geometrically.

2.9 Even and odd functions

Definition:

> A function, $y = f(x)$, is *even* if $f(x) = f(-x)$ for all x in the domain of f.

Geometrically, an even function is symmetrical about the y-axis (it has line symmetry).
The function $f(x) = x^2$ is an even function as $f(-x) = (-x)^2 = x^2 = f(x)$ for all values of x. We illustrate this on the following graph.

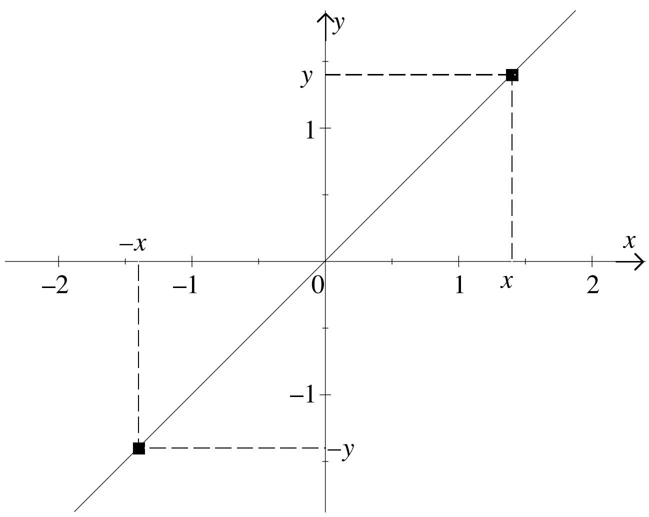


The graph of $y = x^2$.

Definition:

> A function, $y = f(x)$, is *odd* if $f(-x) = -f(x)$ for all x in the domain of f.

Geometrically, an odd function is symmetrical about the origin (it has rotational symmetry).
The function $f(x) = x$ is an odd function as $f(-x) = -x = -f(x)$ for all values of x. This is illustrated on the following graph.



The graph of $y = x$.

Example

Decide whether the following functions are even, odd or neither.

1. $f(x) = 3x^2 - 4$

2. $g(x) = \frac{1}{2x}$

3. $f(x) = x^3 - x^2$.

Solution

1.

$$f(-x) = 3(-x)^2 - 4 = 3x^2 - 4 = f(x)$$

The function $f(x) = 3x^2 - 4$ is even.

2.

$$g(-x) = \frac{1}{2(-x)} = \frac{1}{-2x} = -\frac{1}{2x} = -g(x)$$

Therefore, the function g is odd.

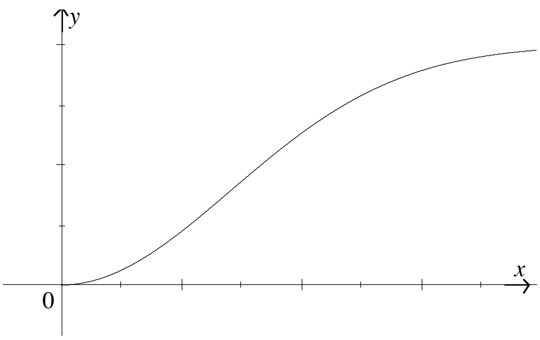
3.

$$f(-x) = (-x)^3 - (-x)^2 = -x^3 - x^2$$

This function is neither even (since $-x^3 - x^2 \neq x^3 - x^2$) nor odd (since $-x^3 - x^2 \neq -(x^3 - x^2)$).

Example

Sketched below is part of the graph of $y = f(x)$.

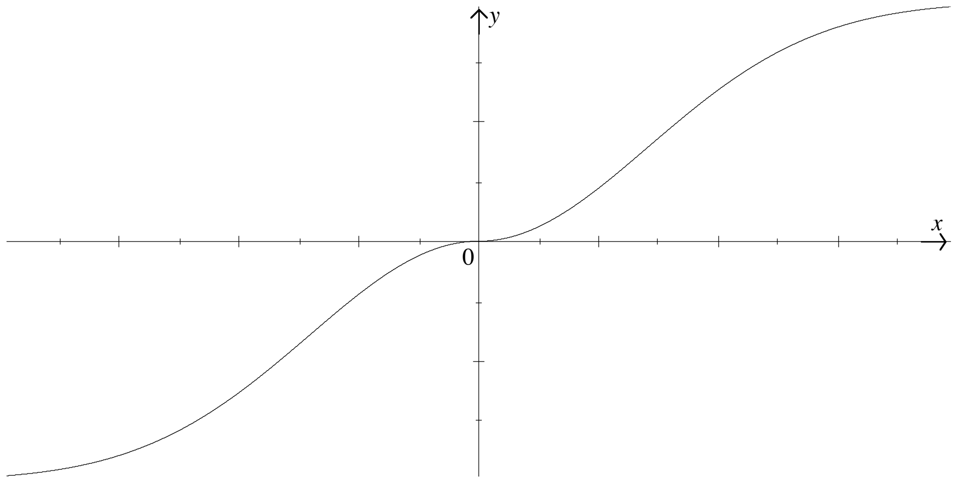


Complete the graph if $y = f(x)$ is

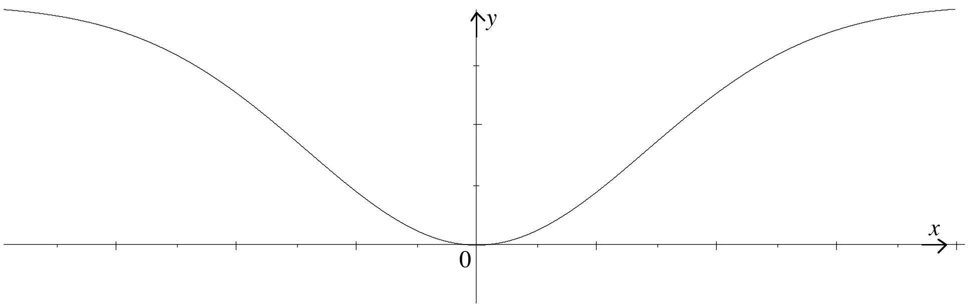
1. odd

2. even.

Solution



$y = f(x)$ is an odd function.



$y = f(x)$ is an even function.

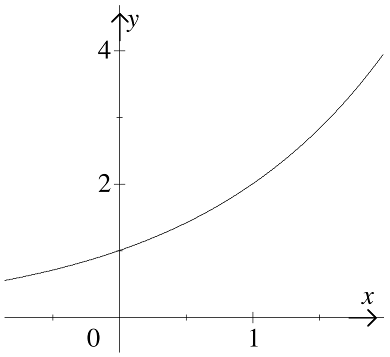
2.10 Increasing and decreasing functions

Here we will introduce the concepts of increasing and decreasing functions. In Chapter 5 we will relate these concepts to the derivative of a function.

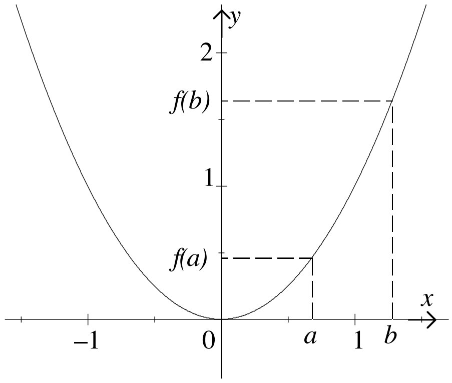
Definition:

> A function is *increasing* on an interval I, if for all a and b in I such that $a < b$, $f(a) < f(b)$.

The function $y = 2^x$ is an example of a function that is increasing over its domain. The function $y = x^2$ is increasing for all real $x > 0$.



The graph of $y = 2^x$. This function is increasing for all real x.



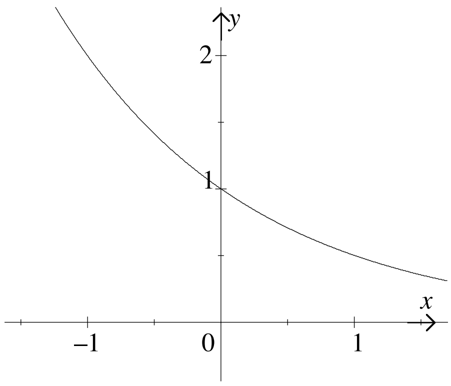
The graph of $y = x^2$. This function is increasing on the interval $x > 0$.

Notice that when a function is increasing it has a positive slope.

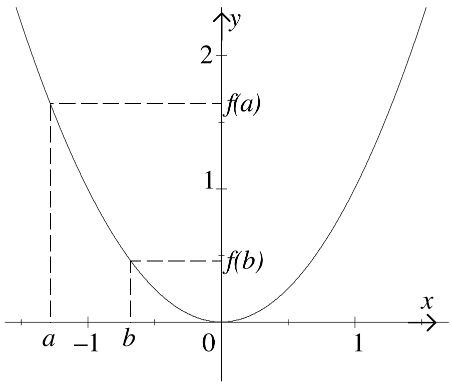
Definition:

> A graph is decreasing on an interval I, if for all a and b in I such that $a < b$, $f(a) > f(b)$.

The function $y = 2^{-x}$ is decreasing over its domain. The function $y = x^2$ is decreasing on the interval $x < 0$.



The graph of $y = 2^{-x}$. This function is decreasing for all real x.



The graph of $y = x^2$. This function is decreasing on the interval $x < 0$.

Notice that if a function is decreasing then it has negative slope.

2.11 Exercises

1. Given the graph below of $y = f(x)$:

a. State the domain and range.

b. Where is the graph

i increasing?

ii decreasing?

c. if k is a constant, find the values of k such that $f(x) = k$ has

i no solutions

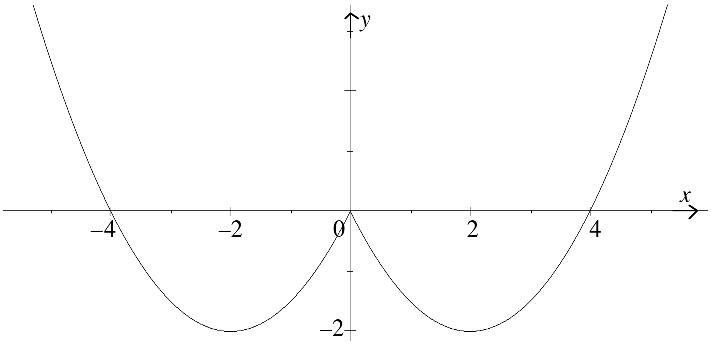
ii 1 solution

iii 2 solutions

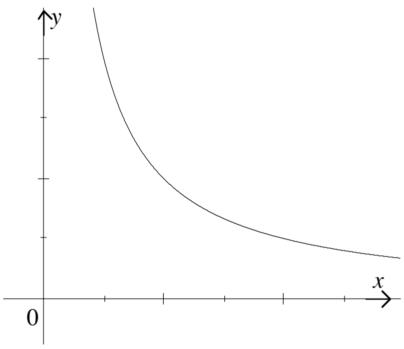
iv 3 solutions

v 4 solutions.

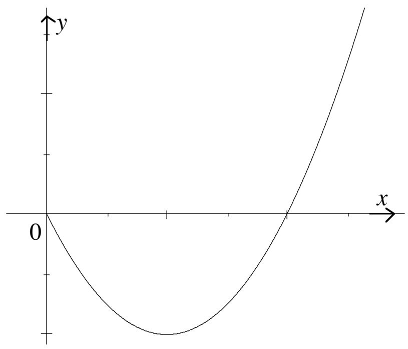
d. Is $y = f(x)$ even, odd or neither?



2. Complete the following functions if they are defined to be (a) even (b) odd.



$y = f(x)$



$y = g(x)$

3. Determine whether the following functions are odd, even or neither.

a. $y = x^4 + 2$ **b.** $y = \sqrt{4 - x^2}$ **c.** $y = 2^x$ **d.** $y = x^3 + 3x$

e. $y = \frac{x}{x^2}$ **f.** $y = \dfrac{1}{x^2 - 4}$ **g.** $y = \dfrac{1}{x^2 + 4}$ **h.** $y = \dfrac{x}{x^3 + 3}$

i. $y = 2^x + 2^{-x}$ **j.** $y = |x - 1| + |x + 1|$

4. Given $y = f(x)$ is even and $y = g(x)$ is odd, prove

a. if $h(x) = f(x) \cdot g(x)$ then $h(x)$ is odd

b. if $h(x) = (g(x))^2$ then $h(x)$ is even

c. if $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, then $h(x)$ is odd

d. if $h(x) = f(x) \cdot (g(x))^2$ then $h(x)$ is even.

5. Consider the set of all odd functions which are defined at $x = 0$. Can you prove that for every odd function in this set $f(0) = 0$? If not, give a counter-example.

3 Piecewise functions and solving inequalities

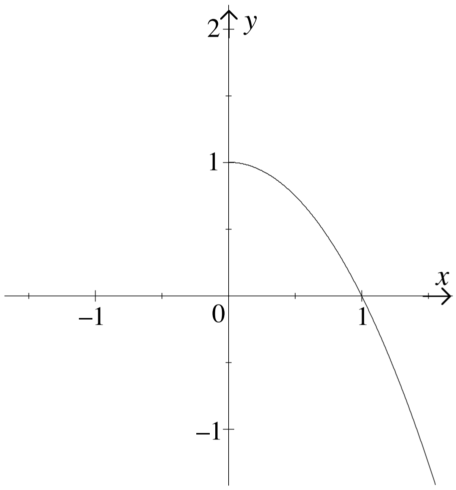
In this Chapter we will discuss functions that are defined piecewise (sometimes called piecemeal functions) and look at solving inequalities using both algebraic and graphical techniques.

3.1 Piecewise functions

3.1.1 Restricting the domain

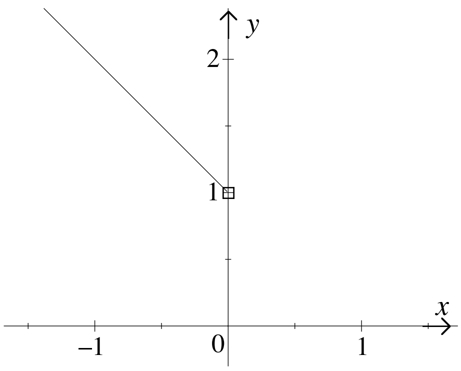
In Chapter 1 we saw how functions could be defined on a subinterval of their natural domain. This is frequently called *restricting* the domain of the function. In this Chapter we will extend this idea to define functions piecewise.

Sketch the graph of $y = 1 - x^2$ for $x \geq 0$.



The graph of $y = 1 - x^2$ for $x \geq 0$.

Sketch the graph of $y = 1 - x$ for $x < 0$.

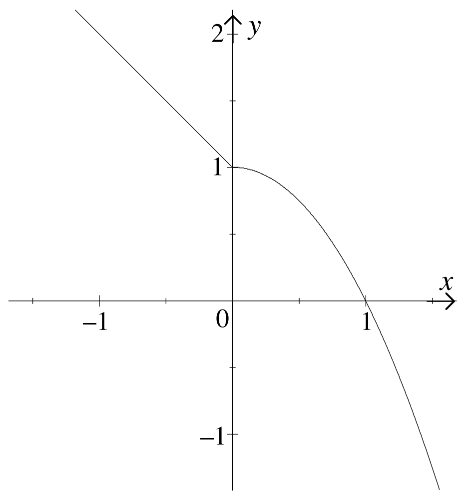


The graph of $y = 1 - x$ for $x < 0$.

We can now put these pieces together to define a function of the form

$$f(x) = \begin{cases} 1 - x^2 & \text{for } x \geq 0 \\ 1 - x & \text{for } x < 0 \end{cases}$$

We say that this function is defined *piecewise*. First note that it *is* a function; each value of x in the domain is assigned exactly one value of y. This is easy to see if we graph the function and use the vertical line test. We graph this function by graphing each piece of it in turn.



The graph shows that f defined in this way is a function. The two pieces of $y = f(x)$ meet so f is a continuous function.

The absolute value function

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

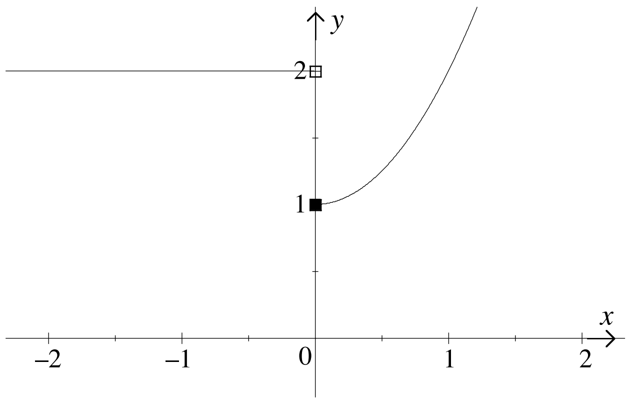
is another example of a piecewise function.

Example

Sketch the function

$$f(x) = \begin{cases} x^2 + 1 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$$

Solution



This function is not continuous at $x = 0$ as the two branches of the graph do not meet.

Notice that we have put an open square (or circle) around the point $(0, 2)$ and a solid square (or circle) around the point $(0, 1)$. This is to make it absolutely clear that $f(0) = 1$ and not 2. When defining a function piecewise, we must be extremely careful to assign to each x exactly one value of y.

3.2 Exercises

1. For the function

$$f(x) = \begin{cases} 1 - x^2 & \text{for } x \geq 0 \\ 1 - x & \text{for } x < 0 \end{cases}$$

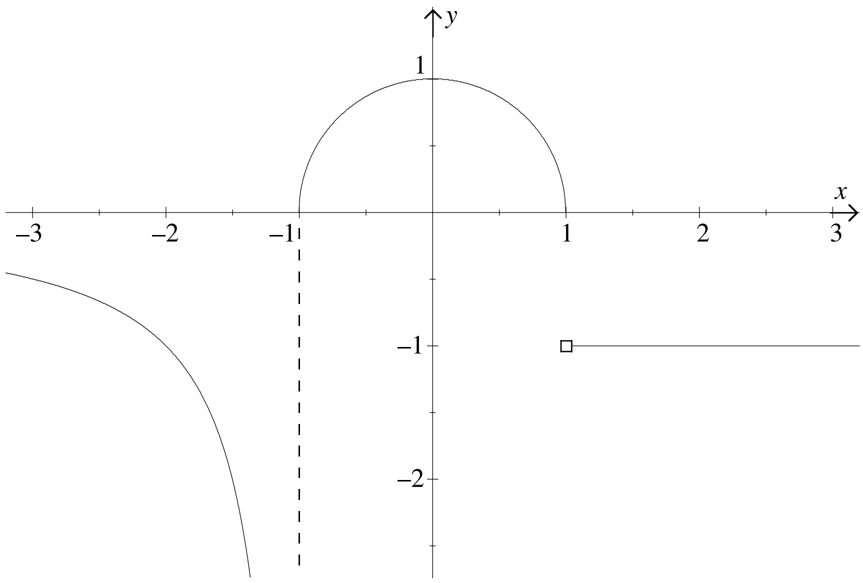
evaluate

a. $2f(-1) + f(2)$

b. $f(a^2)$

2. For the function given in **1**, solve $f(x) = 2$.

3. Below is the graph of $y = g(x)$. Write down the rules which define $g(x)$ given that its pieces are hyperbolic, circular and linear.



4. **a.** Sketch the graph of $y = f(x)$ if

$$f(x) = \begin{cases} -\sqrt{4-x^2} & \text{for } -2 \le x \le 0 \\ x^2 - 4 & \text{for } x > 0 \end{cases}$$

b. State the range of f.

c. Solve

i $f(x) = 0$

ii $f(x) = -3$.

d. Find k if $f(x) = k$ has

i 0

ii 1

iii 2 solutions.

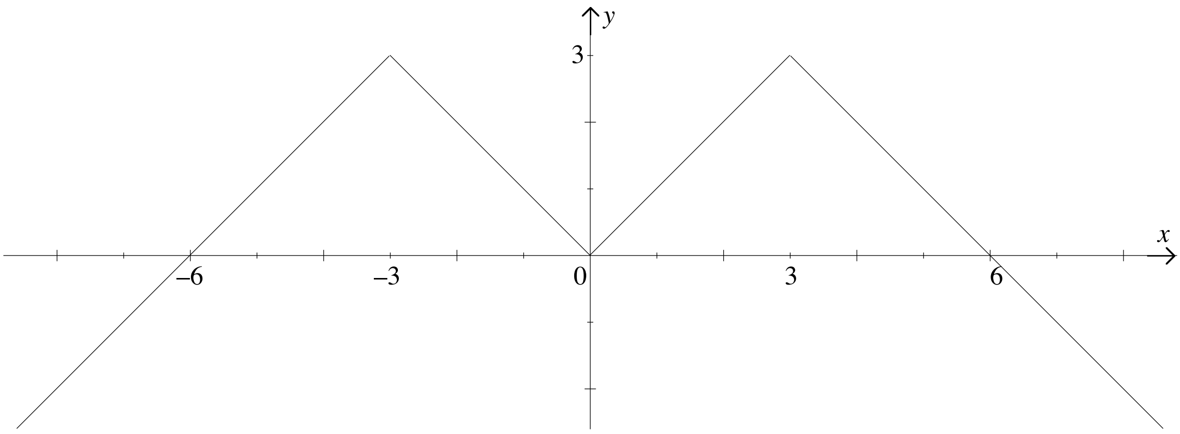
5. Sketch the graph of $y = f(x)$ if

$$f(x) = \begin{cases} 1 - |x-1| & \text{for } x \ge 0 \\ |x+1| & \text{for } x < 0 \end{cases}$$

6. Sketch the graph of $y = g(x)$ if

$$g(x) = \begin{cases} \frac{2}{x+2} & \text{for } x < -1 \\ 2 & \text{for } -1 \le x < 1 \\ 2^x & \text{for } x \ge 1 \end{cases}$$

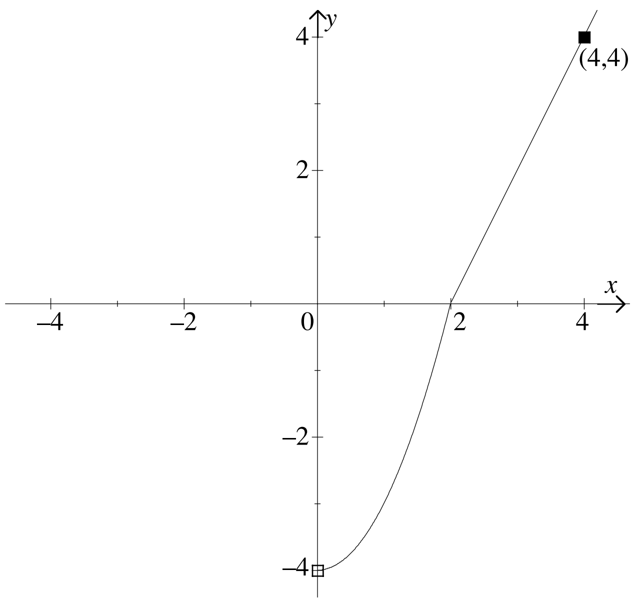
7. McMaths burgers are to modernise their logo as shown below.



Write down a piecewise function that represents this function using (a) 4 (b) 3 (c) 2 pieces (i.e. rules that define the function).

8. **a.** The following piecewise function is of the form

$$f(x) = \begin{cases} ax^2 + b & \text{for } 0 < x \leq 2 \\ cx + d & \text{for } x > 2 \end{cases}$$



Determine the values of a, b, c and d.

b. Complete the graph so that $f(x)$ is an odd function defined for all real x, $x \neq 0$.

c. Write down the equations that now define $f(x)$, $x \neq 0$.

3.3 Inequalities

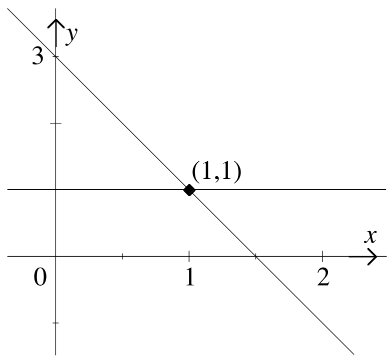
We can solve inequalities using both algebraic and graphical methods. Sometimes it is easier to use an algebraic method and sometimes a graphical one. For the following examples we will use both, as this allows us to make the connections between the algebra and the graphs.

Algebraic method

1. Solve $3 - 2x \geq 1$.
This is a (2 Unit) linear inequality. Remember to reverse the inequality sign when multiplying or dividing by a negative number.

$$\begin{aligned} 3 - 2x &\geq 1 \\ -2x &\geq -2 \\ x &\leq 1 \end{aligned}$$

Graphical method

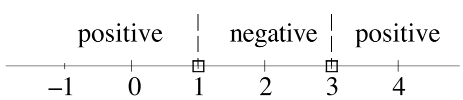


When is the line $y = 3 - 2x$ above or on the horizontal line $y = 1$? From the graph, we see that this is true for $x \leq 1$.

2. Solve $x^2 - 4x + 3 < 0$.
This is a (2 Unit) quadratic inequality. Factorise and use a number line.

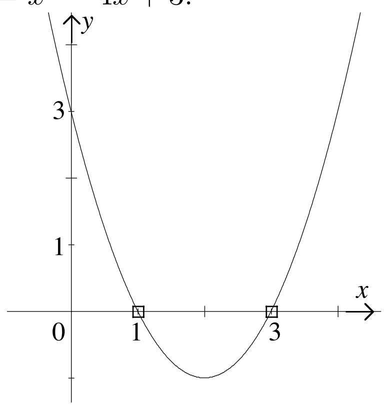
$$\begin{aligned} x^2 - 4x + 3 &< 0 \\ (x-3)(x-1) &< 0 \end{aligned}$$

The critical values are 1 and 3, which divide the number line into three intervals. We take points in each interval to determine the sign of the inequality; eg use $x = 0$, $x = 2$ and $x = 4$ as test values.



Thus, the solution is $1 < x < 3$.

Let $y = x^2 - 4x + 3$.

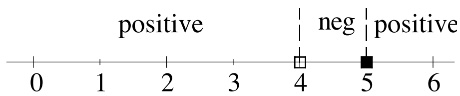


When does the parabola have negative y-values? OR When is the parabola under the x-axis? From the graph, we see that this happens when $1 < x < 3$.

3. Solve $\frac{1}{x-4} \le 1$.
This is a 3 Unit inequality. There is a variable in the denominator. Remember that a denominator can never be zero, so in this case $x \ne 4$. First multiply by the square of the denominator

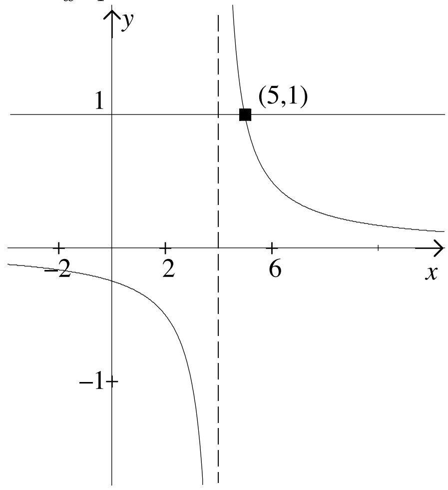
$$\begin{aligned} x-4 &\le (x-4)^2, x \ne 4 \\ x-4 &\le x^2-8x+16 \\ 0 &\le x^2-9x+20 \\ 0 &\le (x-4)(x-5) \end{aligned}$$

Mark the critical values on the number line and test $x = 0$, $x = 4.5$ and $x = 6$.



Therefore, $x < 4$ or $x \ge 5$.

Let $y = \frac{1}{x-4}$.



$y = \frac{1}{x-4}$ is not defined for $x = 4$. It is a hyperbola with vertical asymptote at $x = 4$. To solve our inequality we need to find the values of x for which the hyperbola lies on or under the line $y = 1$. $(5, 1)$ is the point of intersection. So, from the graph we see that $\frac{1}{x-4} \le 1$ when $x < 4$ or $x \ge 5$.

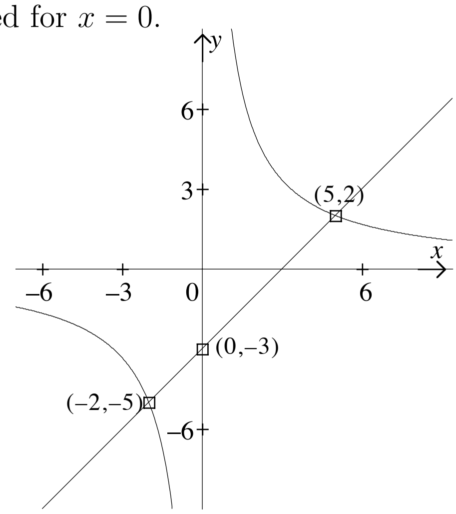
4. Solve $x - 3 < \frac{10}{x}$.
Consider $x - 3 = \frac{10}{x}$, $\quad x \ne 0$.
Multiply by x we get

$$\begin{aligned} x^2 - 3x &= 10 \\ x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

Therefore, the critical values are -2, 0 and 5 which divide the number line into four intervals. We can use $x = -3$, $x = -1$, $x = 1$ and $x = 6$ as test values in the inequality. The points $x = -3$ and $x = 1$ satisfy the inequality, so the solution is $x < -2$ or $0 < x < 5$.

(Notice that we had to include 0 as one of our critical values.)

Sketch $y = x - 3$ and then $y = \frac{10}{x}$. Note that second of these functions is not defined for $x = 0$.



For what values of x does the line lie under the hyperbola? From the graph, we see that this happens when $x < -2$ or $0 < x < 5$.

Example

Sketch the graph of $y = |2x - 6|$.

Hence, where possible,

a. Solve

i $|2x - 6| = 2x$

ii $|2x - 6| > 2x$

iii $|2x - 6| = x + 3$

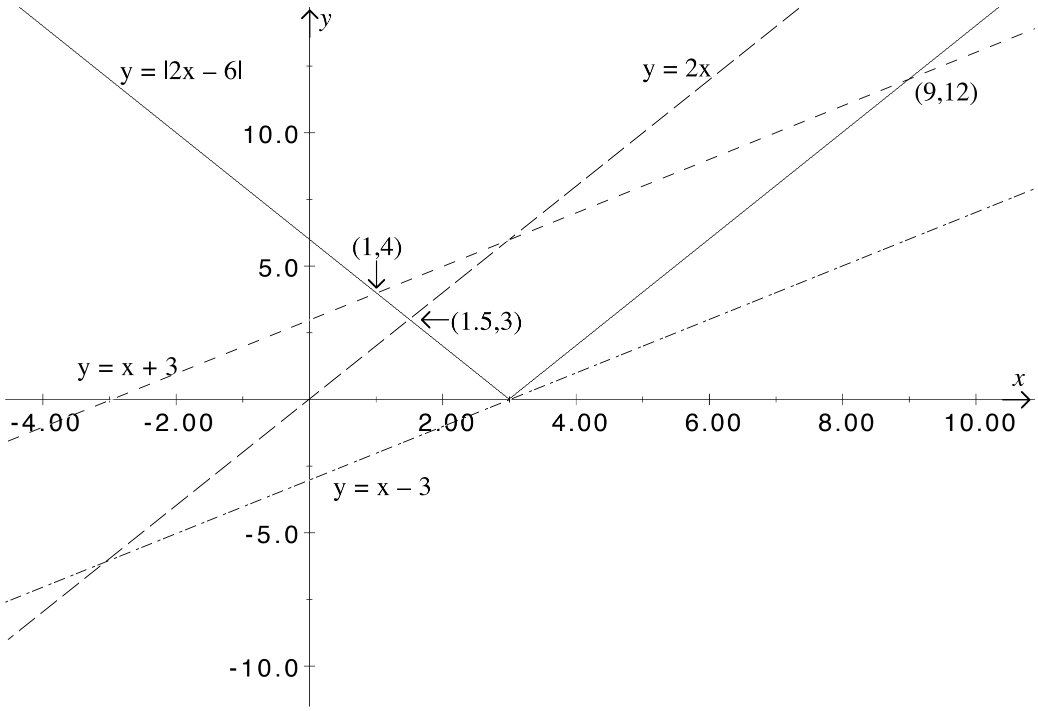
iv $|2x - 6| < x + 3$

v $|2x - 6| = x - 3$

b. Determine the values of k for which $|2x - 6| = x + k$ has exactly two solutions.

Solution

$$f(x) = |2x - 6| = \begin{cases} 2x - 6 & \text{for } x \geq 3 \\ -(2x - 6) & \text{for } x < 3 \end{cases}$$



a. i Mark in the graph of $y = 2x$. It is parallel to one arm of the absolute value graph. It has one point of intersection with $y = |2x - 6| = -2x + 6$ $(x < 3)$ at $x = 1.5$.

ii When is the absolute value graph above the line $y = 2x$? From the graph, when $x < 1.5$.

iii $y = x + 3$ intersects $y = |2x - 6|$ twice.

To solve $|2x - 6| = x + 3$, take $|2x - 6| = 2x - 6 = x + 3$ when $x \geq 3$. This gives us the solution $x = 9$. Then take $|2x - 6| = -2x + 6 = x + 3$ when $x < 3$ which gives us the solution $x = 1$.

iv When is the absolute value graph below the line $y = x + 3$?

From the graph, $1 < x < 9$.

v $y = x - 3$ intersects the absolute value graph at $x = 3$ only.

b. k represents the y-intercept of the line $y = x + k$. When $k = -3$, there is one point of intersection. (See (a) (v) above). For $k > -3$, lines of the form $y = x + k$ will have two points of intersection. Hence $|2x - 6| = x + k$ will have two solutions for $k > -3$.

3.4 Exercises

1. Solve

a. $x^2 \leq 4x$

b. $\frac{4p}{p+3} \leq 1$

c. $\frac{7}{9-x^2} > -1$

2. **a.** Sketch the graph of $y = 4x(x - 3)$.

b. Hence solve $4x(x - 3) \leq 0$.

3. **a.** Find the points of intersection of the graphs $y = 5 - x$ and $y = \frac{4}{x}$.

b. On the same set of axes, sketch the graphs of $y = 5 - x$ and $y = \frac{4}{x}$.

c. Using part (ii), or otherwise, write down all the values of x for which

$$5 - x > \frac{4}{x}$$

4. **a.** Sketch the graph of $y = 2^x$.

b. Solve $2^x < \frac{1}{2}$.

c. Suppose $0 < a < b$ and consider the points $\mathrm{A}(a, 2^a)$ and $\mathrm{B}(b, 2^b)$ on the graph of $y = 2^x$. Find the coordinates of the midpoint M of the segment AB.

Explain why

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$

5. **a.** Sketch the graphs of $y = x$ and $y = |x - 5|$ on the same diagram.

b. Solve $|x - 5| > x$.

c. For what values of m does $mx = |x - 5|$ have exactly

i two solutions

ii no solutions

6. Solve $5x^2 - 6x - 3 \leq |8x|$.

4 Polynomials

Many of the functions we have been using so far have been polynomials. In this Chapter we will study them in more detail.

Definition

A real *polynomial*, $P(x)$, of degree n is an expression of the form

$$P(x) = p_n x^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_2x^2 + p_1x + p_0$$

where $p_n \neq 0$, p_0, p_1, $\cdots$, p_n are real and n is an integer ≥ 0.

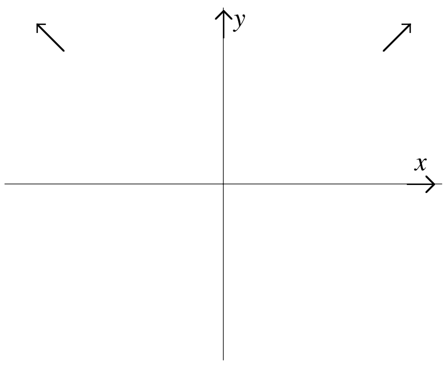
All polynomials are defined for all real x and are continuous functions.
We are familiar with the quadratic polynomial, $Q(x) = ax^2 + bx + c$ where $a \neq 0$. This polynomial has degree 2.
The function $f(x) = \sqrt{x} + x$ is not a polynomial as it has a power which is not an integer ≥ 0 and so does not satisfy the definition.

4.1 Graphs of polynomials and their zeros

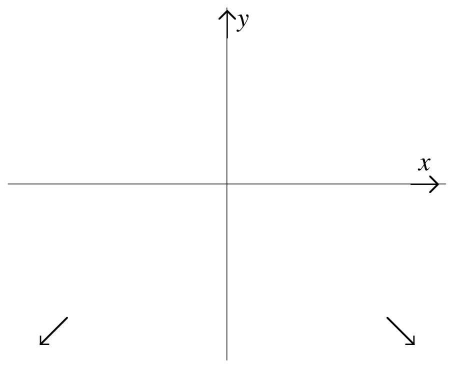
4.1.1 Behaviour of polynomials when $|x|$ is large

One piece of information that can be a great help when sketching a polynomial is the way it behaves for values of x when $|x|$ is large. That is, values of x which are large in magnitude.
The term of the polynomial with the highest power of x is called the *leading* or *dominant* term. For example, in the polynomial $P(x) = x^6 - 3x^4 - 1$, the term x^6 is the dominant term.
When $|x|$ is large, the dominant term determines how the graph behaves as it is so much larger in magnitude than all the other terms.

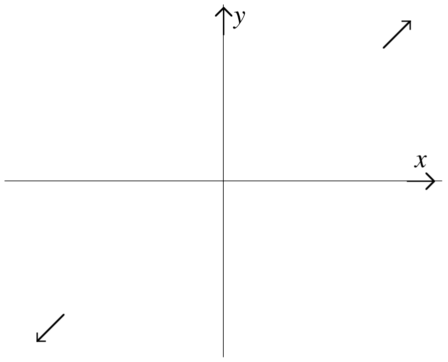
How the graph behaves for $|x|$ large depends on the *power* and *coefficient* of the dominant term.
There are four possibilities which we summarise in the following diagrams:



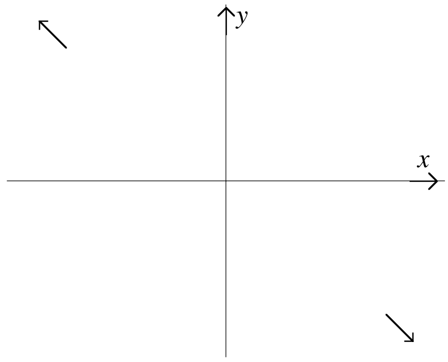
1. Dominant term with even power and positive coefficient, eg $y = x^2$.



2. Dominant term with even power and negative coefficient, eg $Q(x) = -x^2$.



3. Dominant term with odd power and positive coefficient, eg $y = x^3$.



4. Dominant term with odd power and negative coefficient, eg $Q(x) = -x^3$.

This gives us a good start to graphing polynomials. All we need do now is work out what happens in the middle. In Chapter 5 we will use calculus methods to do this. Here we will use our knowledge of the roots of polynomials to help complete the picture.

4.1.2 Polynomial equations and their roots

If, for a polynomial $P(x)$, $P(k) = 0$ then we can say

1. $x = k$ is a root of the equation $P(x) = 0$.

2. $x = k$ is a zero of $P(x)$.

3. k is an x-intercept of the graph of $P(x)$.

4.1.3 Zeros of the quadratic polynomial

The quadratic polynomial equation $Q(x) = ax^2 + bx + c = 0$ has two roots that may be:

1. real (rational or irrational) and distinct,

2. real (rational or irrational) and equal,

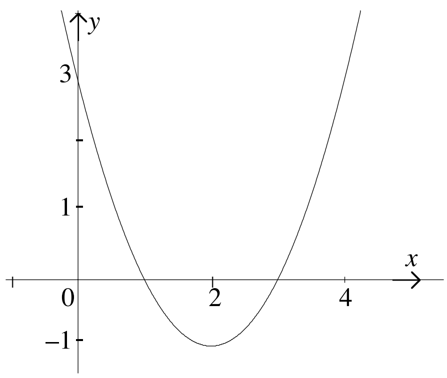
3. complex (not real).

We will illustrate all of these cases with examples, and will show the relationship between the nature and number of zeros of $Q(x)$ and the x-intercepts (if any) on the graph.

1. Let $Q(x) = x^2 - 4x + 3$.
We find the zeros of $Q(x)$ by solving the equation $Q(x) = 0$.

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ \text{Therefore } x &= 1 \text{ or } 3. \end{aligned}$$

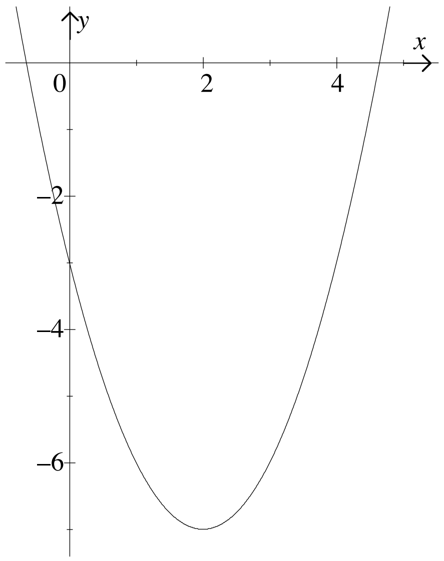
The roots are rational (hence real) and distinct.



2. Let $Q(x) = x^2 - 4x - 3$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x - 3 &= 0 \\ x &= \frac{4 \pm \sqrt{16 + 12}}{2} \\ \text{Therefore } x &= 2 \pm \sqrt{7}. \end{aligned}$$

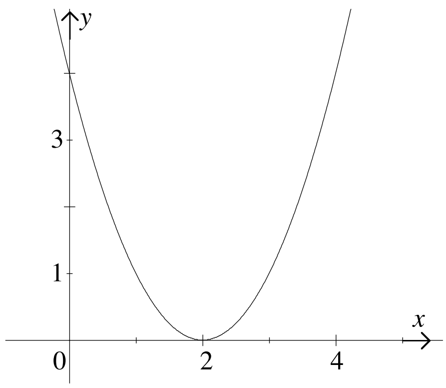
The roots are irrational (hence real) and distinct.



3. Let $Q(x) = x^2 - 4x + 4$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ (x - 2)^2 &= 0 \\ \text{Therefore } x &= 2. \end{aligned}$$

The roots are rational (hence real) and equal. $Q(x) = 0$ has a repeated or double root at $x = 2$.

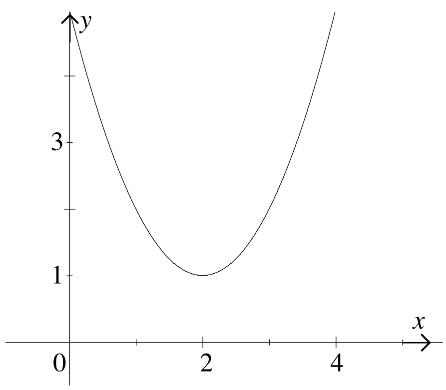


Notice that the graph turns at the double root $x = 2$.

4. Let $Q(x) = x^2 - 4x + 5$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^2 - 4x + 5 &= 0 \\ x &= \frac{4 \pm \sqrt{16 - 20}}{2} \\ \text{Therefore } x &= 2 \pm \sqrt{-4}. \end{aligned}$$

There are no real roots. In this case the roots are complex.



Notice that the graph does not intersect the x-axis. That is $Q(x) > 0$ for all real x. Therefore Q is positive definite.

We have given above four examples of quadratic polynomials to illustrate the relationship between the zeros of the polynomials and their graphs.

In particular we saw that:

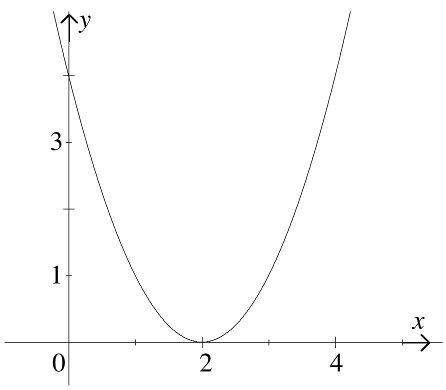
i if the quadratic polynomial has two real distinct zeros, then the graph of the polynomial cuts the x-axis at two distinct points;

ii if the quadratic polynomial has a real double (or repeated) zero, then the graph sits on the x-axis;

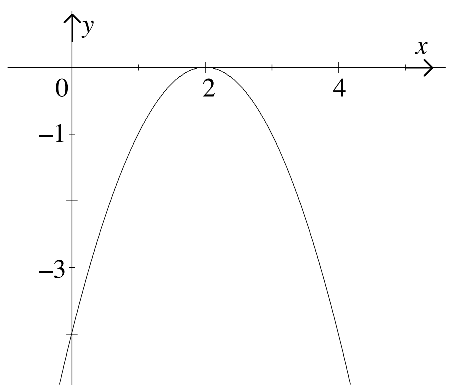
iii if the quadratic polynomial has no real zeros, then the graph does not intersect the x-axis at all.

So far, we have only considered quadratic polynomials where the coefficient of the x^2 term is positive which gives us a graph which is *concave up.* If we consider polynomials $Q(x) = ax^2 + bx + c$ where $a < 0$ then we will have a graph which is *concave down.*

For example, the graph of $Q(x) = -(x^2 - 4x + 4)$ is the reflection in the x-axis of the graph of $Q(x) = x^2 - 4x + 4$. (See Chapter 2.)



The graph of $Q(x) = x^2 - 4x + 4$.



The graph of $Q(x) = -(x^2 - 4x + 4)$.

4.1.4 Zeros of cubic polynomials

A real cubic polynomial has an equation of the form

$$P(x) = ax^3 + bx^2 + cx + d$$

where $a \neq 0$, a, b, c and d are real. It has 3 zeros which may be:

i 3 real distinct zeros;

ii 3 real zeros, all of which are equal (3 equal zeros);

iii 3 real zeros, 2 of which are equal;

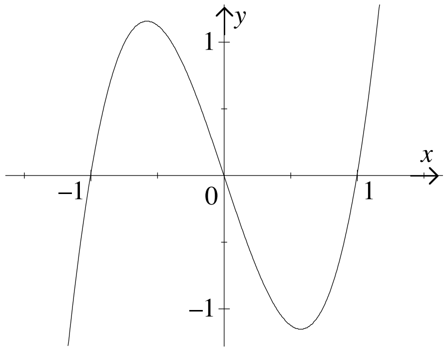
iv 1 real zero and 2 complex zeros.

We will illustrate these cases with the following examples:

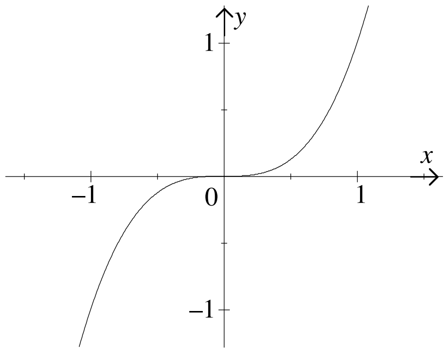
1. Let $Q(x) = 3x^3 - 3x$.
Solving the equation $Q(x) = 0$ we get:.

$$\begin{aligned} 3x^3 - 3x &= 0 \\ 3x(x-1)(x+1) &= 0 \\ \text{Therefore } x &= -1 \text{ or } 0 \text{ or } 1 \end{aligned}$$

The roots are real (in fact rational) and distinct.



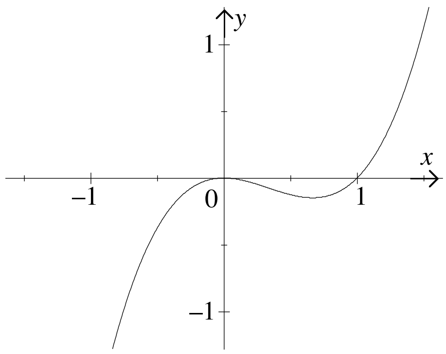
2. Let $Q(x) = x^3$.
Solving $Q(x) = 0$ we get that $x^3 = 0$.
We can write this as $(x - 0)^3 = 0$.
So, this equation has three equal real roots at $x = 0$.



3. Let $Q(x) = x^3 - x^2$.
Solving the equation $Q(x) = 0$ we get,

$$\begin{aligned} x^3 - x^2 &= 0 \\ x^2(x-1) &= 0 \\ \text{Therefore } x &= 0 \text{ or } 1. \end{aligned}$$

The roots are real with a double root at $x = 0$ and a single root at $x = 1$.

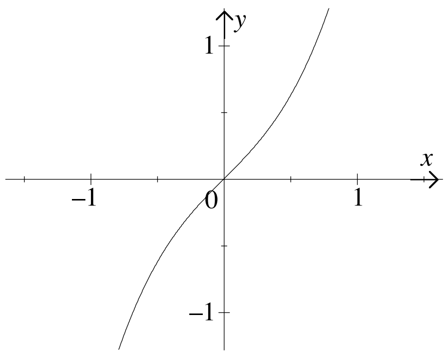


The graph turns at the double root.

4. Let $Q(x) = x^3 + x$.
Solving the equation $Q(x) = 0$ we get,

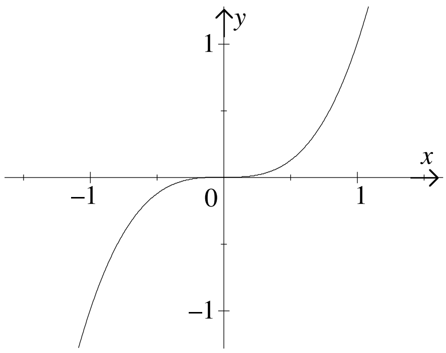
$$\begin{aligned} x^3 + x &= 0 \\ x(x^2+1) &= 0 \\ \text{Therefore } x &= 0. \end{aligned}$$

There is one real root at $x = 0$.
$x^2 + 1 = 0$ does not have any real solutions.

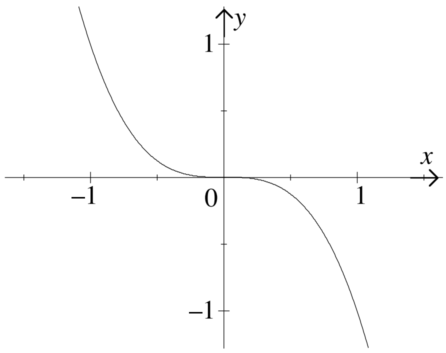


The graph intersects the x-axis once only.

Again, in the above examples we have looked only at cubic polynomials where the coefficient of the x^3 term is positive. If we consider the polynomial $P(x) = -x^3$ then the graph of this polynomial is the reflection of the graph of $P(x) = x^3$ in the x-axis.



The graph of $Q(x) = x^3$.



The graph of $Q(x) = -x^3$.

4.2 Polynomials of higher degree

We will write down a few rules that we can use when we have a polynomial of degree ≥ 3.

If $P(x)$ is a real polynomial of degree n then:

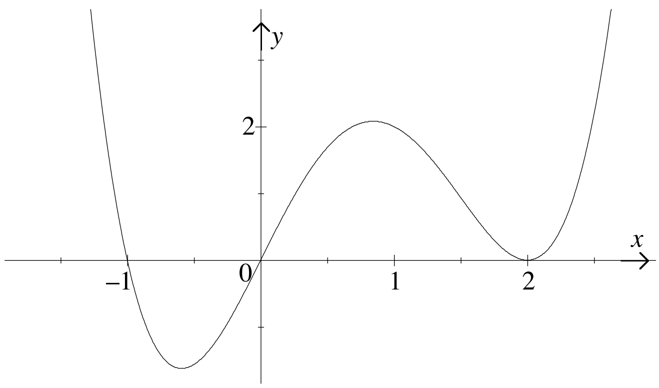
1. $P(x) = 0$ has at most n real roots;

2. if $P(x) = 0$ has a repeated root with an even power then the graph of $P(x)$ turns at this repeated root;

3. if $P(x) = 0$ has a repeated root with an odd power then the graph of $P(x)$ has a horizontal point of inflection at this repeated root.

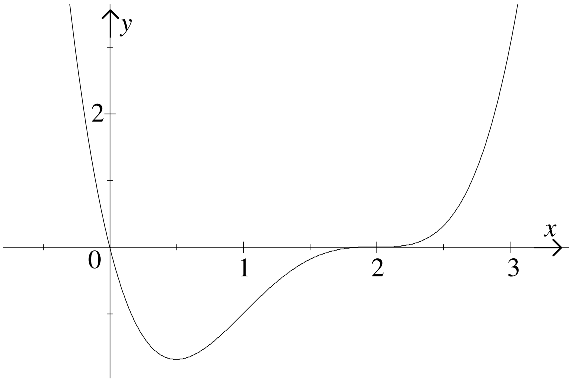
For example, **1.** tells us that if we have a quartic polynomial equation $f(x) = 0$. Then we know that $f(x) = 0$ has ≤ 4 real roots.

We can illustrate **2.** by the sketching $f(x) = x(x-2)^2(x+1)$. Notice how the graph sits on the x-axis at $x = 2$.



The graph of $f(x) = x(x+1)(x-2)^2$.

We illustrate **3.** by sketching the graph of $f(x) = x(x-2)^3$. Notice the horizontal point of inflection at $x = 2$.



The graph of $f(x) = x(x-2)^3$.

4.3 Exercises

1. Sketch the graphs of the following polynomials if $y = P(x)$ is:

a. $x(x+1)(x-3)$

b. $x(x+1)(3-x)$

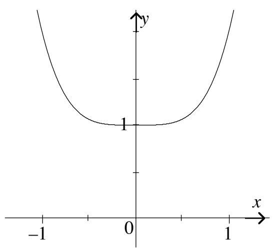
c. $(x+1)^2(x-3)$

d. $(x+1)(x^2-4x+5)$

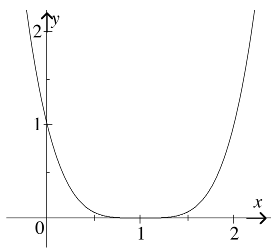
2. The graphs of the following quartic polynomials are sketched below. Match the graph with the polynomial.

a. $y = x^4$ **b.** $y = x^4 - 1$ **c.** $y = x^4 + 1$ **d.** $y = 1 - x^4$ **e.** $y = (x-1)^4$ **f.** $y = (x+1)^4$

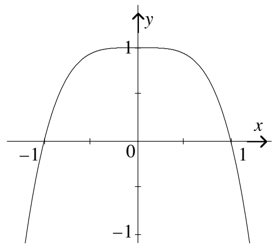
i



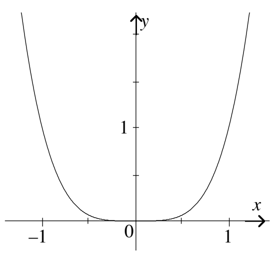
ii



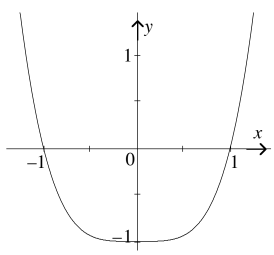
iii



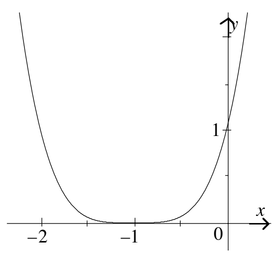
iv



v



vi



3. Sketch the graphs of the following quartic polynomials if $y = C(x)$ is:

a. $x(x-1)(x+2)(x+3)$

b. $x(x-1)(x+2)(3-x)$

c. $x^2(x-1)(x-3)$

d. $(x+1)^2(x-3)^2$

e. $(x+1)^3(x-3)$

f. $(x+1)^3(3-x)$

g. $x(x+1)(x^2-4x+5)$

h. $x^2(x^2-4x+5)$.

4. By sketching the appropriate polynomial, solve:

a. $x^2 - 4x - 12 < 0$

b. $(x+2)(x-3)(5-x) > 0$

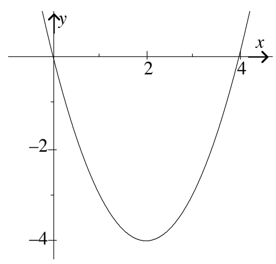
c. $(x+2)^2(5-x) > 0$

d. $(x+2)^3(5-x) \geq 0$.

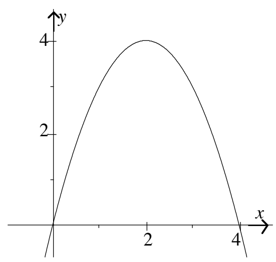
5. For what values of k will $P(x) \geq 0$ for all real x if $P(x) = x^2 - 4x - 12 + k$?

6. The diagrams show the graph of $y = P(x)$ where $P(x) = a(x-b)(x-c)^d$.
In each case determine possible values for a, b, c and d.

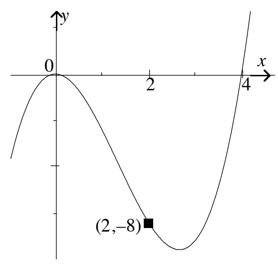
a.



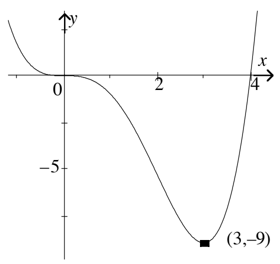
b.



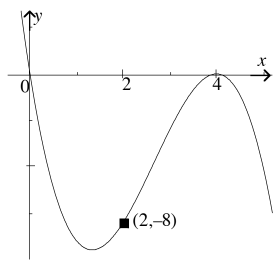
c.



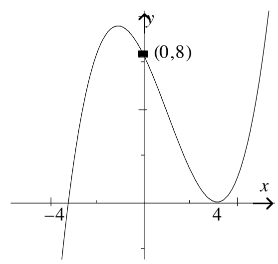
d.



e.



f.



7. The graph of the polynomial $y = f(x)$ is given below. It has a local maximum and minimum as marked. Use the graph to answer the following questions.

a. State the roots of $f(x) = 0$.

b. What is the value of the repeated root.

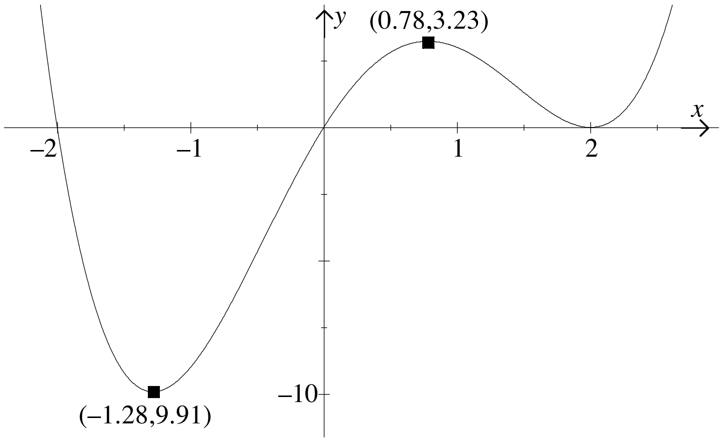
c. For what values of k does the equation $f(x) = k$ have exactly 3 solutions.

d. Solve the inequality $f(x) < 0$.

e. What is the *least* possible degree of $f(x)$?

f. State the value of the constant of $f(x)$.

g. For what values of k is $f(x) + k \geq 0$ for all real x.



The graph of the polynomial $y = f(x)$

4.4 Factorising polynomials

So far for the most part, we have looked at polynomials which were already factorised. In this section we will look at methods which will help us factorise polynomials with degree > 2.

4.4.1 Dividing polynomials

Suppose we have two polynomials $P(x)$ and $A(x)$, with the degree of $P(x) \geq$ the degree of $A(x)$, and $P(x)$ is divided by $A(x)$. Then

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)},$$

where $Q(x)$ is a polynomial called the *quotient* and $R(x)$ is a polynomial called the *remainder*, with the degree of $R(x) <$ degree of $A(x)$.

We can rewrite this as

$$P(x) = A(x) \cdot Q(x) + R(x).$$

For example: If $P(x) = 2x^3 + 4x + 3$ and $A(x) = x - 2$, then $P(x)$ can be divided by $A(x)$ as follows:

$$\begin{array}{r|rrrr}
 & 2x^2 + & 4x + & 12 & \\
\hline
x-2 & 2x^3 + 0x^2 + & 4x - & 3 & \\
 & \underline{2x^3 - 4x^2} & & & \\
 & 4x^2 + & 4x - & 3 & \\
 & \underline{4x^2 -} & \underline{8x} & & \\
 & & 12x - & 3 & \\
 & & \underline{12x -} & \underline{24} & \\
 & & & 21 &
\end{array}$$

The quotient is $2x^2 + 4x + 12$ and the remainder is 21. We have

$$\frac{2x^3 + 4x + 3}{x - 2} = 2x^2 + 4x + 12 + \frac{21}{x - 2}.$$

This can be written as

$$2x^3 + 4x - 3 = (x - 2)(2x^2 + 4x + 12) + 21.$$

Note that the degree of the "polynomial" 21 is 0.

4.4.2 The Remainder Theorem

If the polynomial $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

Proof:

Following the above, we can write

$$f(x) = A(x) \cdot Q(x) + R(x),$$

where $A(x) = (x - a)$. Since the degree of $A(x)$ is 1, the degree of $R(x)$ is zero. That is, $R(x) = r$ where r is a constant.

$$\begin{aligned} f(x) &= (x - a)Q(x) + r \quad \text{where } r \text{ is a constant.} \\ f(a) &= 0 \cdot Q(a) + r \\ &= r \end{aligned}$$

So, if $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

Example

Find the remainder when $P(x) = 3x^4 - x^3 + 30x - 1$ is divided by **a.** $x + 1$, **b.** $2x - 1$.

Solution

a. Using the Remainder Theorem:

$$\begin{aligned} \text{Remainder} &= P(-1) \\ &= 3 - (-1) - 30 - 1 \\ &= -27 \end{aligned}$$

b.

$$\begin{aligned} \text{Remainder} &= P(\frac{1}{2}) \\ &= 3(\frac{1}{2})^4 - (\frac{1}{2})^3 + 30(\frac{1}{2}) - 1 \\ &= \frac{3}{16} - \frac{1}{8} + 15 - 1 \\ &= 14\frac{1}{16} \end{aligned}$$

Example

When the polynomial $f(x)$ is divided by $x^2 - 4$, the remainder is $5x + 6$. What is the remainder when $f(x)$ is divided by $(x - 2)$?

Solution

Write $f(x) = (x^2 - 4) \cdot q(x) + (5x + 6)$. Then

$$\begin{aligned}\text{Remainder} &= f(2) \\ &= 0 \cdot q(2) + 16 \\ &= 16\end{aligned}$$

A consequence of the Remainder Theorem is the Factor Theorem which we state below.

4.4.3 The Factor Theorem

If $x = a$ is a zero of $f(x)$, that is $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$ and $f(x)$ may be written as

$$f(x) = (x - a)q(x)$$

for some polynomial $q(x)$.

Also, if $(x - a)$ and $(x - b)$ are factors of $f(x)$ then $(x - a)(x - b)$ is a factor of $f(x)$ and

$$f(x) = (x - a)(x - b) \cdot Q(x)$$

for some polynomial $Q(x)$.

Another useful fact about zeros of polynomials is given below for a polynomial of degree 3.

If a (real) polynomial

$$P(x) = ax^3 + bx^2 + cx + d,$$

where $a \neq 0$, a, b, c and d are real, has exactly 3 real zeros α, β and γ, then

$$P(x) = a(x - \alpha)(x - \beta)(x - \gamma) \qquad (1)$$

Furthermore, by expanding the right hand side of (1) and equating coefficients we get:

i
$$\alpha + \beta + \gamma = -\frac{b}{a};$$

ii
$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a};$$

iii
$$\alpha\beta\gamma = -\frac{d}{a}.$$

This result can be extended for polynomials of degree n. We will give the partial result for $n = 4$.

If

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

is a polynomial of degree 4 with real coefficents, and $P(x)$ has four real zeros α, β, γ and δ, then

$$P(x) = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

and expanding and equating as above gives

$$\alpha\beta\gamma\delta = \frac{e}{a}.$$

If $a = 1$ and the equation $P(x) = 0$ has a root which is an integer, then that integer must be a factor of the constant term. This gives us a place to start when looking for factors of a polynomial. That is, we look at all the factors of the constant term to see which ones (if any) are roots of the equation $P(x) = 0$.

Example

Let $f(x) = 4x^3 - 8x^2 - x + 2$

a. Factorise $f(x)$.

b. Sketch the graph of $y = f(x)$.

c. Solve $f(x) \geq 0$.

Solution

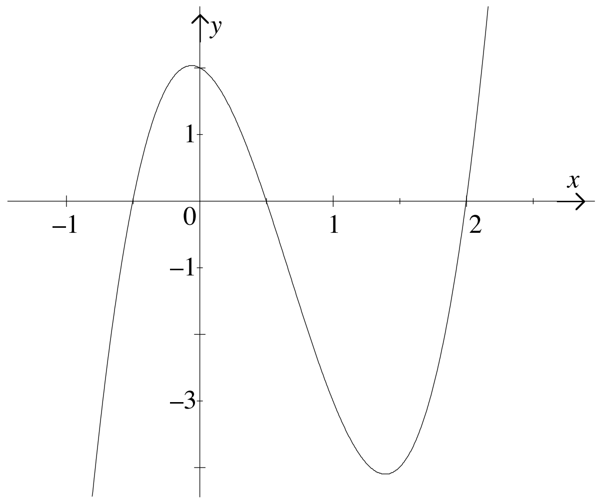
a. Consider the factors of the constant term, 2. We check to see if ± 1 and ± 2 are solutions of the equation $f(x) = 0$ by substitution. Since $f(2) = 0$, we know that $(x - 2)$ is a factor of $f(x)$. We use long division to determine the quotient.

$$\begin{array}{r|l} & 4x^2 \quad\;\; - 1 \\ \cline{2-2} x-2 & 4x^3 - 8x^2 - x + 2 \\ & \underline{4x^3 - 8x^2} \\ & \qquad\qquad - x + 2 \\ & \qquad\qquad \underline{- x + 2} \end{array}$$

So,

$$\begin{aligned} f(x) &= (x-2)(4x^2-1) \\ &= (x-2)(2x-1)(2x+1) \end{aligned}$$

b.



The graph of $f(x) = 4x^3 - 8x^2 - x + 2$.

c. $f(x) \geq 0$ when $-\frac{1}{2} \leq x \leq \frac{1}{2}$ or $x \geq 2$.

Example

Show that $(x - 2)$ and $(x - 3)$ are factors of $P(x) = x^3 - 19x + 30$, and hence solve $x^3 - 19x + 30 = 0$.

Solution

$P(2) = 8 - 38 + 30 = 0$ and $P(3) = 27 - 57 + 30 = 0$ so $(x - 2)$ and $(x - 3)$ are both factors of $P(x)$ and $(x - 2)(x - 3) = x^2 - 5x + 6$ is also a factor of $P(x)$. Long division of $P(x)$ by $x^2 - 5x + 6$ gives a quotient of $(x + 5)$.

So,

$$P(x) = x^3 - 19x + 30 = (x - 2)(x - 3)(x + 5).$$

Solving $P(x) = 0$ we get $(x - 2)(x - 3)(x + 5) = 0$.

That is, $x = 2$ or $x = 3$ or $x = -5$.

Instead of using long division we could have used the facts that

i the polynomial cannot have more than three real zeros;

ii the product of the zeros must be equal to -30.

Let α be the unknown root.

Then $2 \cdot 3 \cdot \alpha = -30$, so that $\alpha = -5$. Therefore the solution of $P(x) = x^3 - 19x + 30 = 0$ is $x = 2$ or $x = 3$ or $x = -5$.

4.5 Exercises

1. When the polynomial $P(x)$ is divided by $(x-a)(x-b)$ the quotient is $Q(x)$ and the remainder is $R(x)$.

a. Explain why $R(x)$ is of the form $mx+c$ where m and c are constants.

b. When a polynomial is divided by $(x-2)$ and $(x-3)$, the remainders are 4 and 9 respectively. Find the remainder when the polynomial is divided by x^2-5x+6.

c. When $P(x)$ is divided by $(x-a)$ the remainder is a^2. Also, $P(b)=b^2$. Find $R(x)$ when $P(x)$ is divided by $(x-a)(x-b)$.

2. a. Divide the polynomial $f(x)=2x^4+13x^3+18x^2+x-4$ by $g(x)=x^2+5x+2$. Hence write $f(x)=g(x)q(x)+r(x)$ where $q(x)$ and $r(x)$ are polynomials.

b. Show that $f(x)$ and $g(x)$ have no common zeros. (Hint: Assume that α is a common zero and show by contradiction that α does not exist.)

3. For the following polynomials,

i factorise

ii solve $P(x)=0$

iii sketch the graph of $y=P(x)$.

a. $P(x)=x^3-x^2-10x-8$

b. $P(x)=x^3-x^2-16x-20$

c. $P(x)=x^3+4x^2-8$

d. $P(x)=x^3-x^2+x-6$

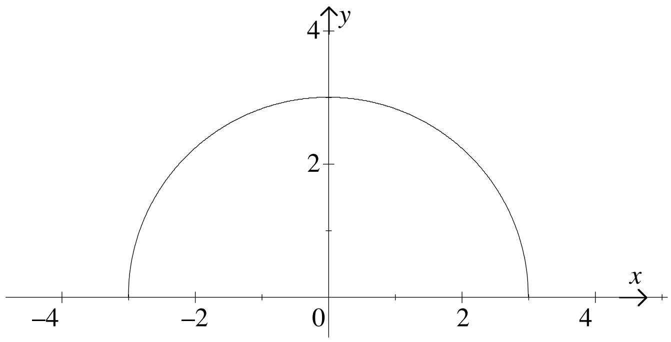
e. $P(x)=2x^3-3x^2-11x+6$

5 Solutions to exercises

1.4 Solutions

1. a. The domain of $f(x) = \sqrt{9 - x^2}$ is all real x where $-3 \le x \le 3$. The range is all real y such that $0 \le y \le 3$.

b.

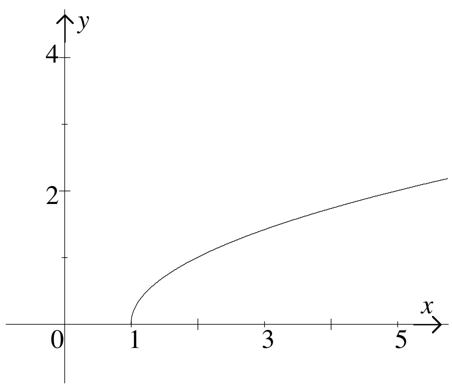


The graph of $f(x) = \sqrt{9 - x^2}$.

2.

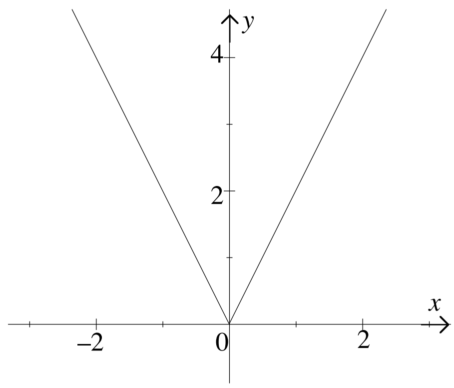
$$
\begin{aligned}
\frac{\psi(x+h) - \psi(x)}{h} &= \frac{(x+h)^2 + 5 - (x^2 + 5)}{h} \\
&= \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} \\
&= \frac{h^2 + 2xh}{h} \\
&= h + 2x
\end{aligned}
$$

3. a.



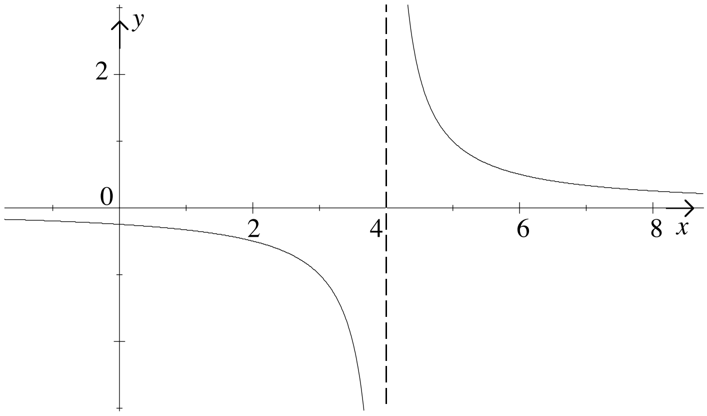
The graph of $y = \sqrt{x - 1}$. The domain is all real $x \ge 1$ and the range is all real $y \ge 0$.

b.



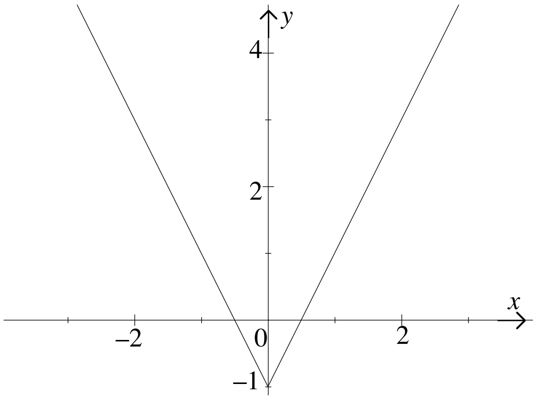
The graph of $y = |2x|$. Its domain is all real x and range all real $y \ge 0$.

c.



The graph of $y = \frac{1}{x-4}$. The domain is all real $x \neq 4$ and the range is all real $y \neq 0$.

d.

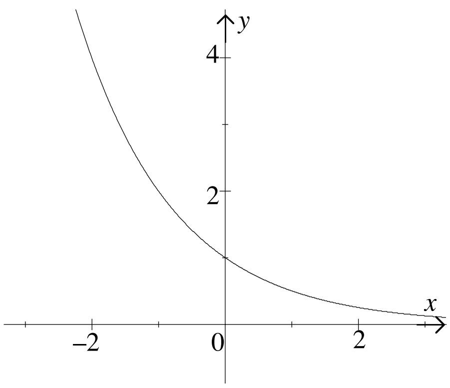


The graph of $y = |2x| - 1$. The domain is all real x, and the range is all real $y \geq -1$.

4. **a.** The perpendicular distance d from $(0, 0)$ to $x + y + k = 0$ is $d = |\frac{k}{\sqrt{2}}|$.

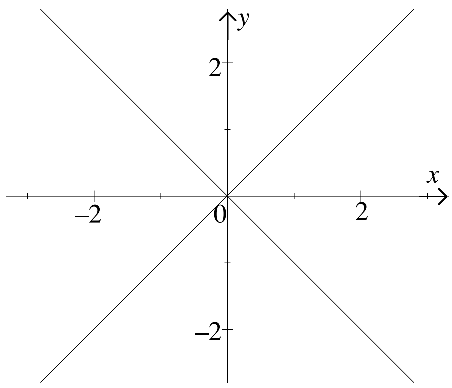
b. For the line $x+y+k = 0$ to cut the circle in two distinct points $d < 2$. ie $|k| < 2\sqrt{2}$ or $-2\sqrt{2} < k < 2\sqrt{2}$.

5. **a.**



The graph of $y = (\frac{1}{2})^x$.

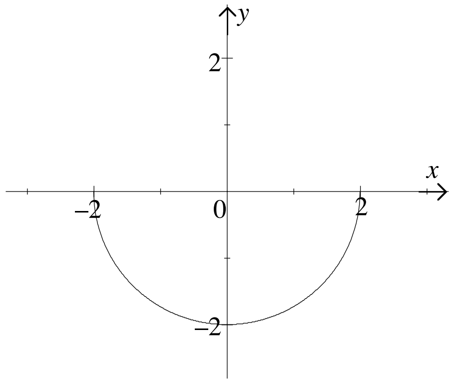
b.



The graph of $y^2 = x^2$.

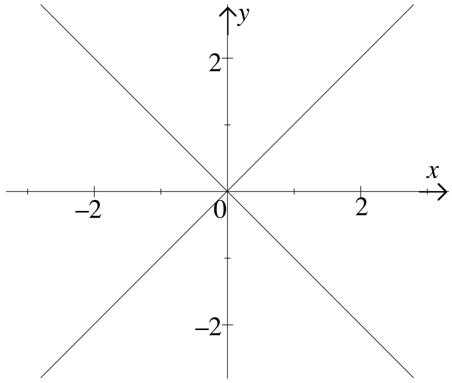
6. $y^2 = x^3$ is not a function.

7. a.



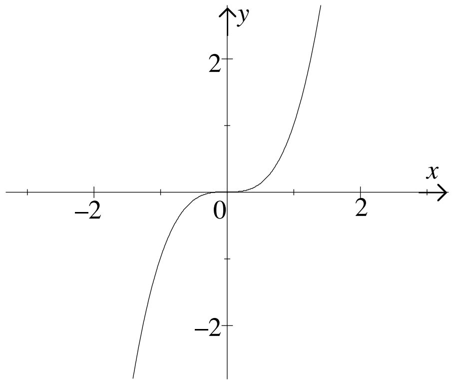
The graph of $y = -\sqrt{4-x^2}$. This is a function with the domain: all real x such that $-2 \leq x \leq 2$ and range: all real y such that $-2 \leq y \leq 0$.

b.



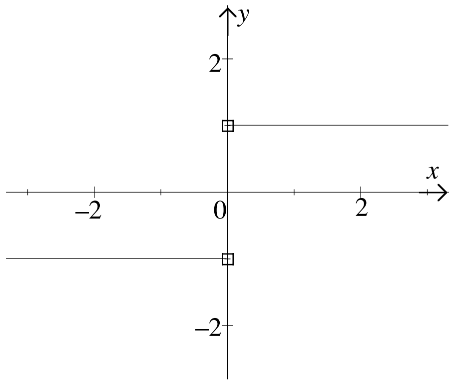
The graph of $|x| - |y| = 0$. This is not the graph of a function.

c.



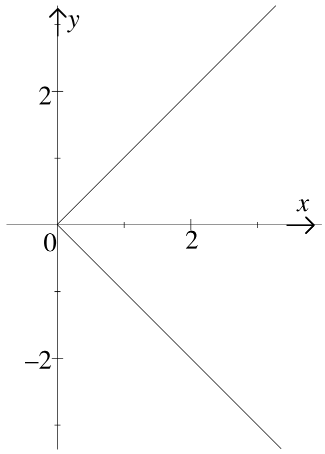
The graph of $y = x^3$. This is a function with the domain: all real x and range: all real y.

d.



The graph of $y = \frac{x}{|x|}$. This is the graph of a function which is not defined at $x = 0$. Its domain is all real $x \neq 0$, and range is $y = \pm 1$.

e.



The graph of $|y| = x$. This is not the graph of a function.

8.

$$
\begin{aligned}
A(\frac{1}{p}) &= (\frac{1}{p})^2 + 2 + \frac{1}{(\frac{1}{p})^2} \\
&= \frac{1}{p^2} + 2 + \frac{1}{\frac{1}{p^2}} \\
&= \frac{1}{p^2} + 2 + p^2 \\
&= A(p)
\end{aligned}
$$

9. a. The values of x in the interval $0 < x < 4$ are not in the domain of the function.

b. $x = 1$ and $x = -1$ are not in the domain of the function.

10. a. $\phi(3) + \phi(4) + \phi(5) = \log(2.5)$

b. $\phi(3) + \phi(4) + \phi(5) + \cdots + \phi(n) = \log(\frac{n}{2})$

11. a. $y = 3$ when $z = 3$.

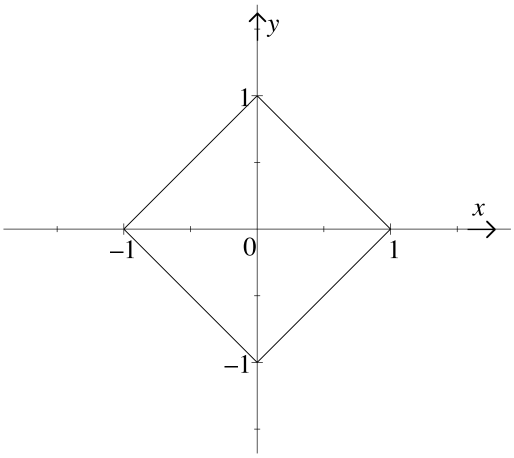
b. i $L(M(x)) = 2(x^2 - x) + 1$

ii $M(L(x)) = 4x^2 + 2x$

12. a. $a = 2$, $b = 2$ so the equations is $y = 2x^2 - 2$.

b. $a = 5$, $b = 1$ so the equation is $y = \frac{5}{x^2+1}$.

13. b.



The graph of $|x| + |y| = 1$.

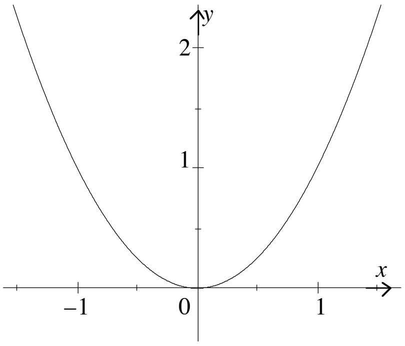
14. $S(n-1) = \frac{n-1}{2n-1}$

Hence

$$
\begin{aligned}
S(n) - S(n-1) &= \frac{n}{2n+1} - \frac{n-1}{2n-1} \\
&= \frac{n(2n-1) - (2n+1)(n-1)}{(2n-1)(2n+1)} \\
&= \frac{2n^2 - n - (2n^2 - n - 1)}{(2n-1)(2n+1)} \\
&= \frac{1}{(2n-1)(2n+1)}
\end{aligned}
$$

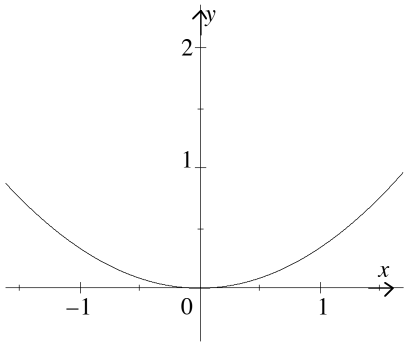
2.8 Solutions

1. a.



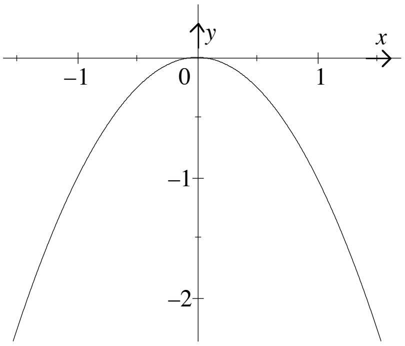
The graph of $y = x^2$.

b.



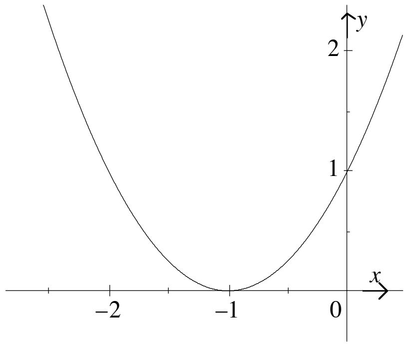
The graph of $y = \frac{x^2}{3}$.

c.



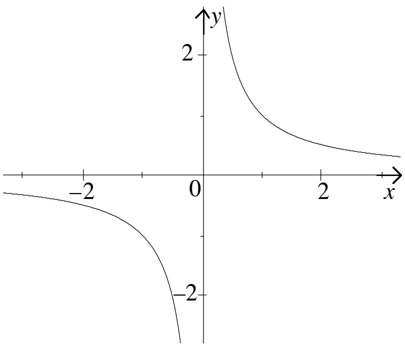
The graph of $y = -x^2$.

d.



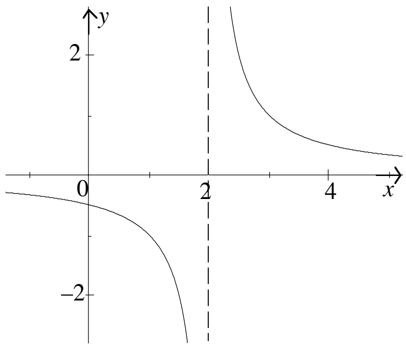
The graph of $y = (x + 1)^2$.

2. a.



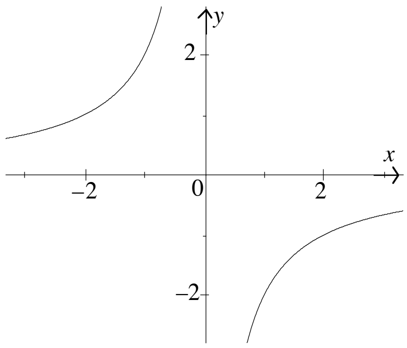
The graph of $y = \frac{1}{x}$.

b.



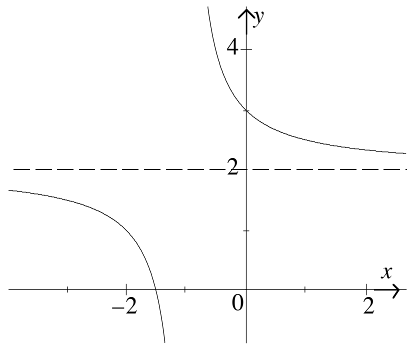
The graph of $y = \frac{1}{x-2}$.

c.



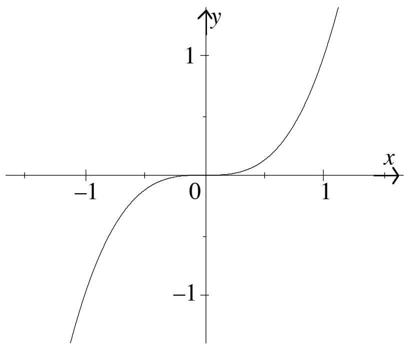
The graph of $y = \frac{-2}{x}$.

d.



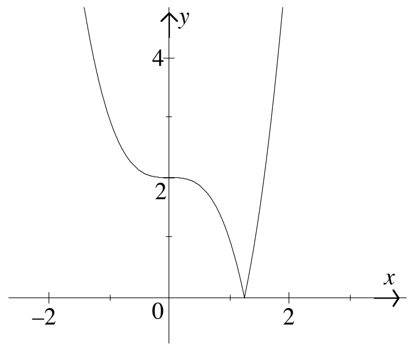
The graph of $y = \frac{1}{x+1} + 2$.

3. a.



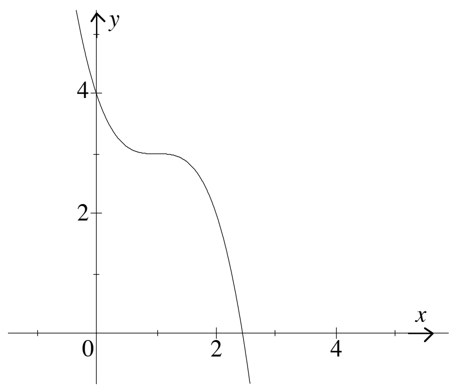
The graph of $y = x^3$.

b.



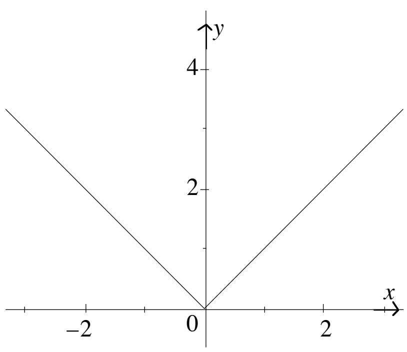
The graph of $y = |x^3 - 2|$.

c.



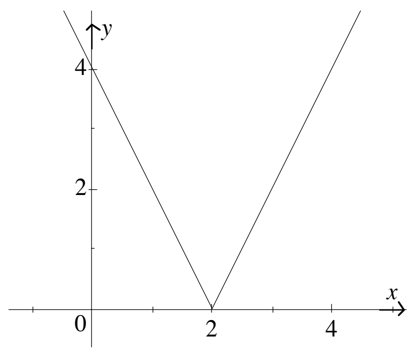
The graph of $y = 3 - (x - 1)^3$.

4. a.



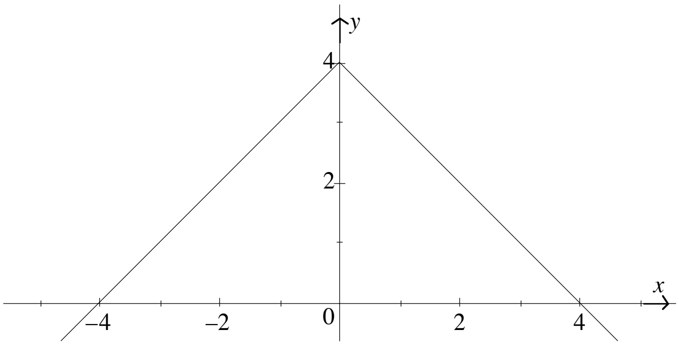
The graph of $y = |x|$.

b.



The graph of $y = 2|x - 2|$.

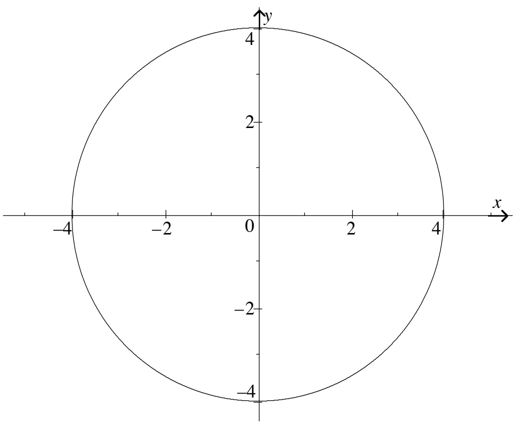
c.



The graph of $y = 4 - |x|$.

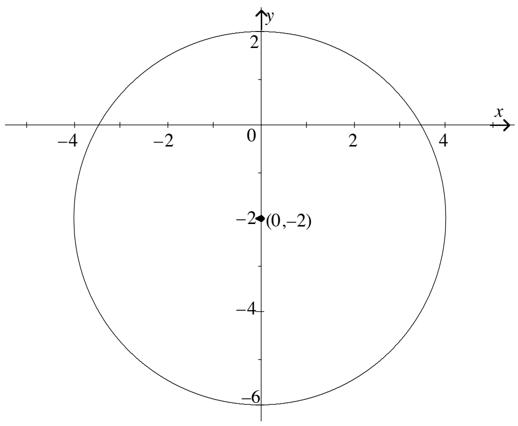
5.

a.



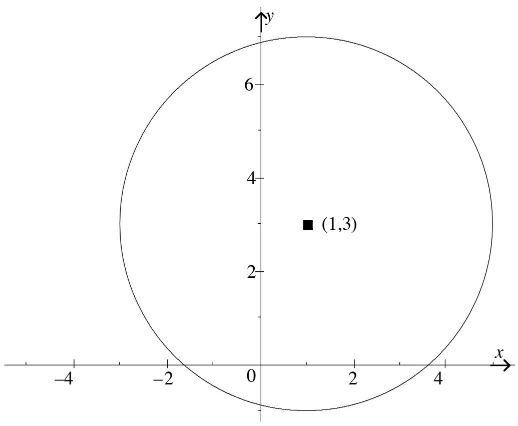
The graph of $x^2 + y^2 = 16$.

b.



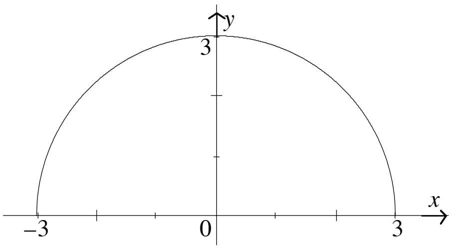
The graph of $x^2 + (y + 2)^2 = 16$.

c.



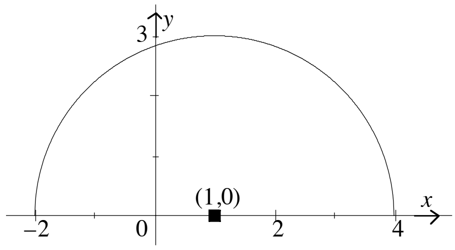
The graph of $(x - 1)^2 + (y - 3)^2 = 16$.

6. a.



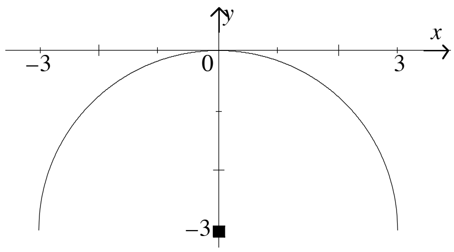
The graph of $y = \sqrt{9 - x^2}$.

b.



The graph of $y = \sqrt{9 - (x-1)^2}$.

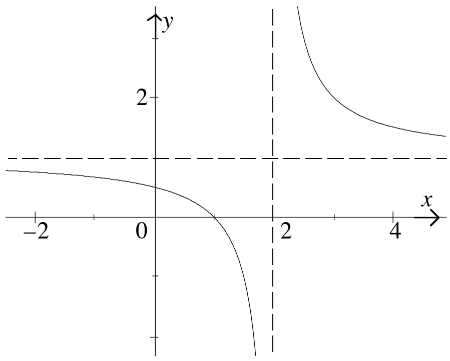
c.



The graph of $y = \sqrt{9 - x^2} - 3$.

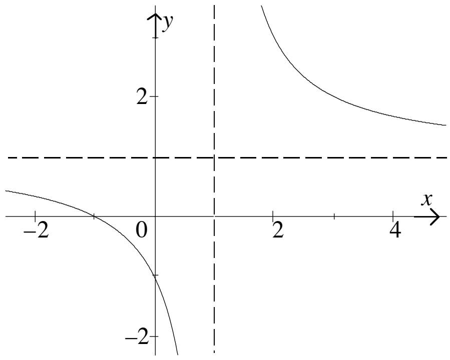
7.

$$\frac{1}{x-2} + 1 = \frac{1 + (x-2)}{x-2} = \frac{x-1}{x-2}$$



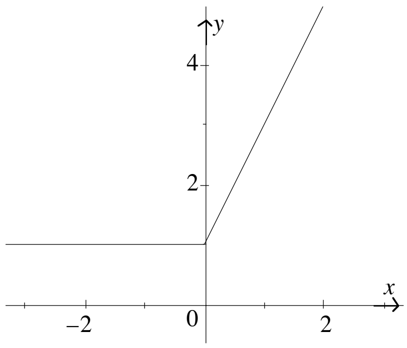
The graph of $y = \frac{x-1}{x-2}$.

8.



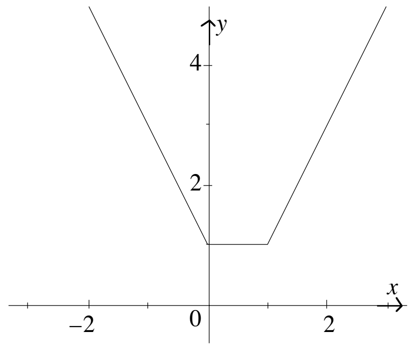
The graph of $y = \frac{x+1}{x-1}$.

9. a.



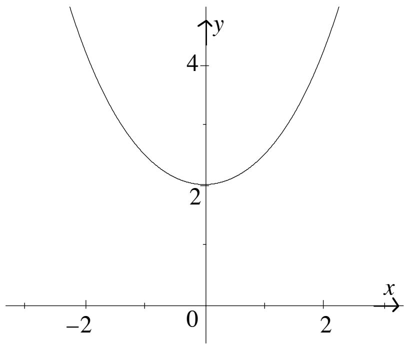
The graph of $y = |x| + x + 1$ for $-2 \leq x \leq 2$.

b.



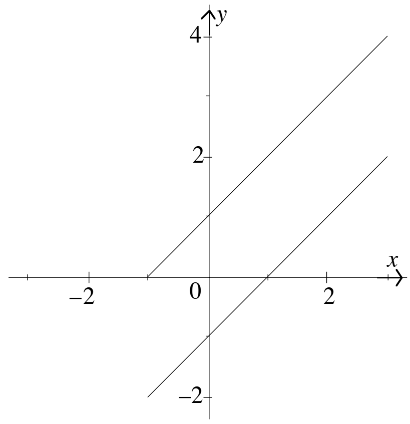
The graph of $y = |x| + |x - 1|$ for $-2 \leq x \leq 3$.

c.



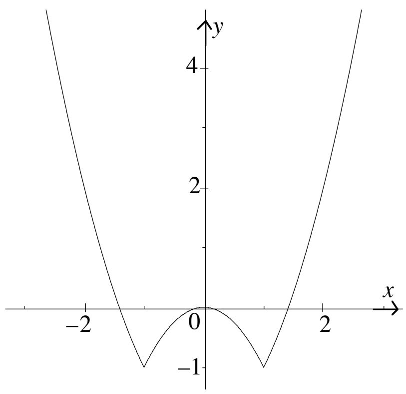
The graph of $y = 2^x + 2^{-x}$ for $-2 \leq x \leq 2$.

d.



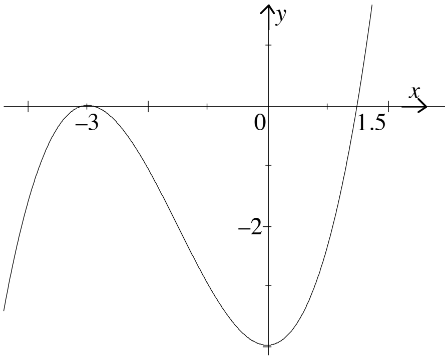
The graph of $|x - y| = 1$ for $-1 \leq x \leq 3$.

10.



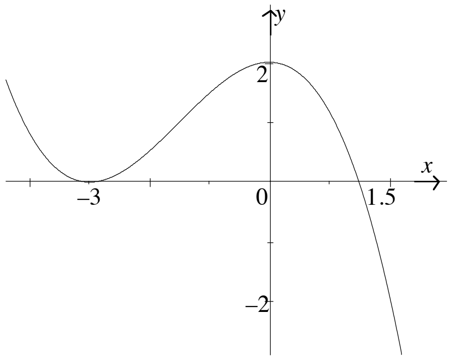
The graph of $f(x) = |x^2 - 1| - 1$.

11. a.



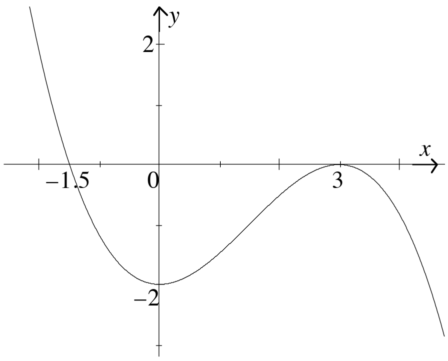
The graph of $y = 2f(x)$.

b.



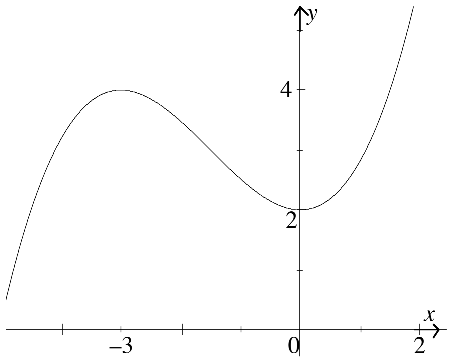
The graph of $y = -f(x)$.

c.



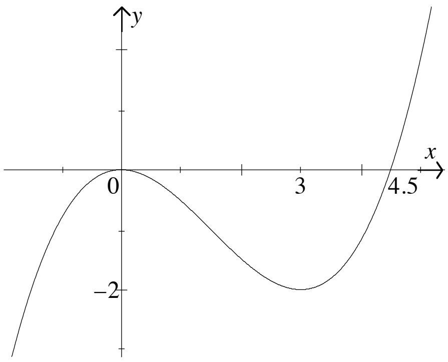
The graph of $y = f(-x)$.

d.



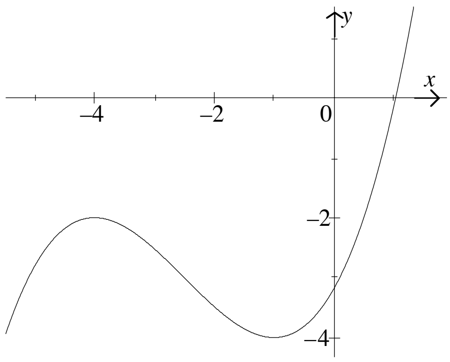
The graph of $y = f(x) + 4$.

e.



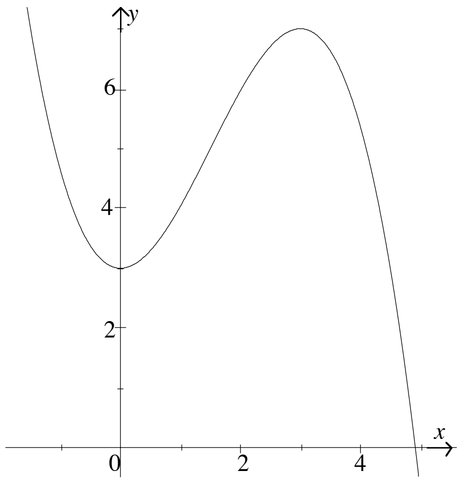
The graph of $y = f(x - 3)$.

f.



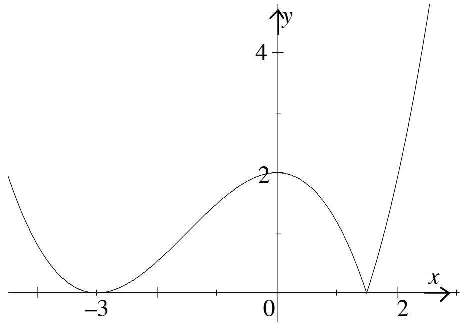
The graph of $y = f(x + 1) - 2$.

g.



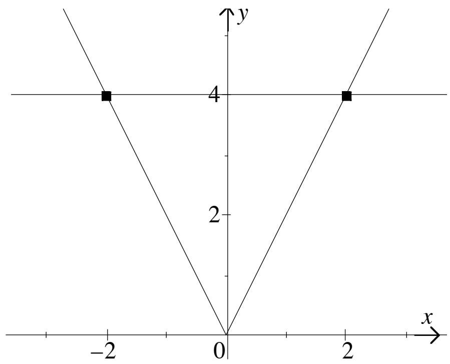
The graph of $y = 3 - 2f(x - 3)$.

h.



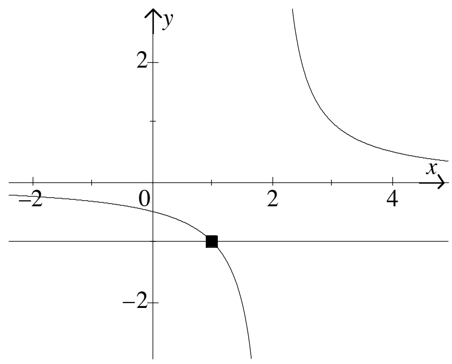
The graph of $y = |f(x)|$.

12. a.



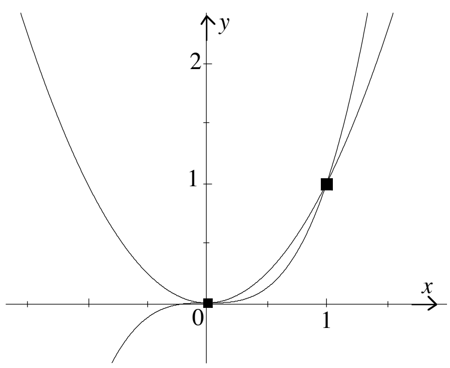
$x = -2$ and $x = 2$ are solutions of the equation $|2x| = 4$.

b.



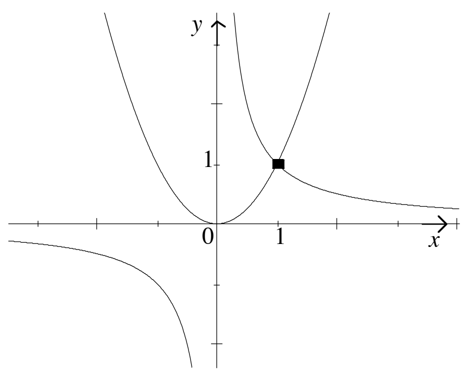
$x = 1$ is a solution of $\frac{1}{x-2} = -1$.

c.



$x = 0$ and $x = 1$ are solutions of the equation $x^3 = x^2$.

d.



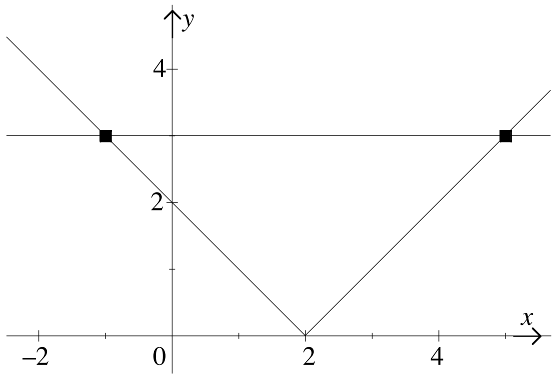
$x = 1$ is a solution of $x^2 = \frac{1}{x}$.

13.

a. For $x \geq 2$, $|x - 2| = x - 2 = 3$. Therefore $x = 5$ is a solution of the inequality. (Note that $x = 5$ is indeed ≥ 2.)

For $x < 2$, $|x - 2| = -(x - 2) = -x + 2 = 3$. Therefore $x = -1$ is a solution. (Note that $x = -1$ is < 2.)

b.

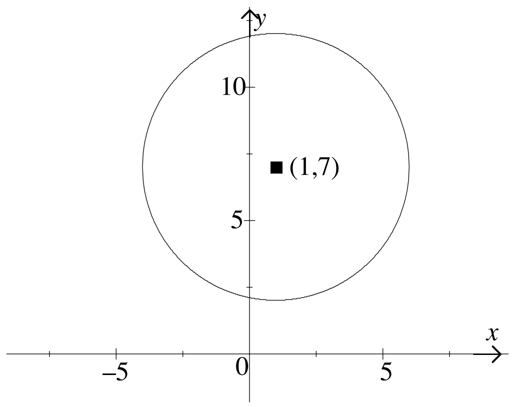


The points of intersection are $(-1, 3)$ and $(5, 3)$.

Therefore the solutions of $|x - 2| = 3$ are $x = -1$ and $x = 5$.

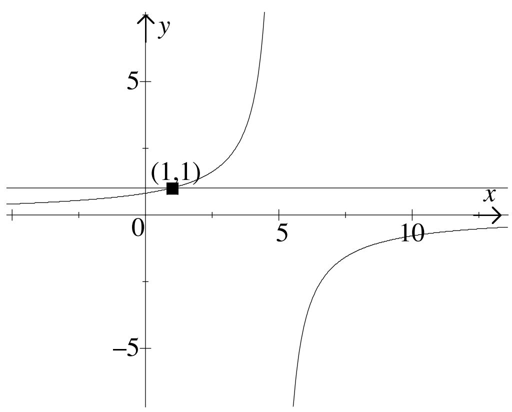
14. The parabolas intersect at $(2, 1)$.

15.



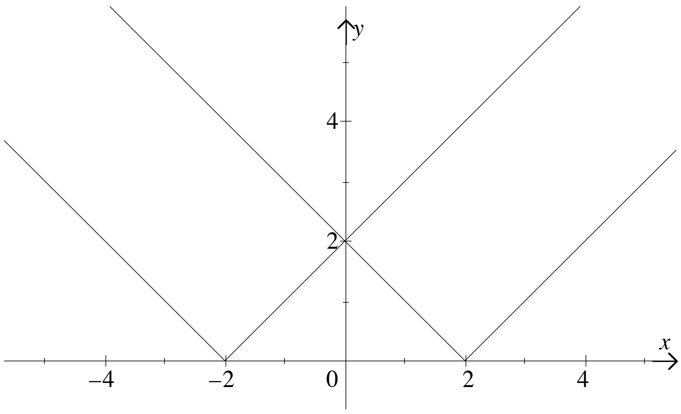
$y = k$ intersects the circle at two distinct points when $2 < k < 12$.

16.



The point of intersection is $(1, 1)$. Therefore the solution of $\frac{4}{5-x} = 1$ is $x = 1$.

17.

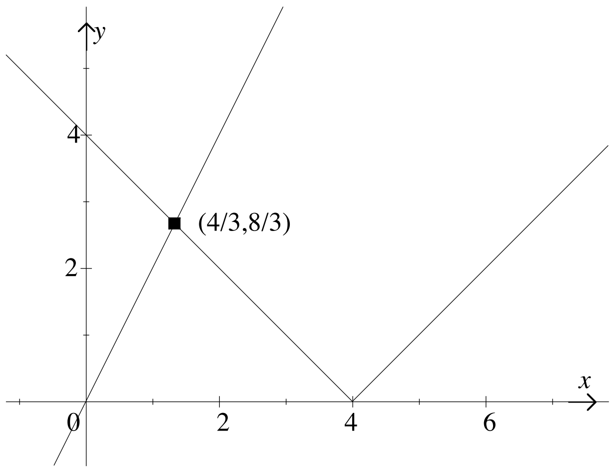


The point of intersection is $(0, 2)$. Therefore the solution of $|x-2| = |x+2|$ is $x = 0$.

18. $n = -1$ or $n = 2$.

19. a. For $x \geq 4$, $|x-4| = x-4 = 2x$ when $x = -4$, but this does not satisfy the condition of $x \geq 4$ so is not a solution.
For $x < 4$, $|x-4| = -x+4 = 2x$ when $x = \frac{4}{3}$. $x = \frac{4}{3}$ is < 4 so is a solution.
Therefore, $x = \frac{4}{3}$ is a solution of $|x-4| = 2x$.

b.



The graph of $y = |x-4|$ and $y = 2x$ intersect at the point $(\frac{4}{3}, \frac{8}{3})$. So the solution of $|x-4| = 2x$ is $x = \frac{4}{3}$.

2.11 Solutions

1. a. The domain is all real x, and the range is all real $y \geq -2$.

b. i $-2 < x < 0$ or $x > 2$

ii $x < -2$ or $0 < x < 2$

c. i $k < -2$

ii There is no value of k for which $f(x) = k$ has exactly one solution.

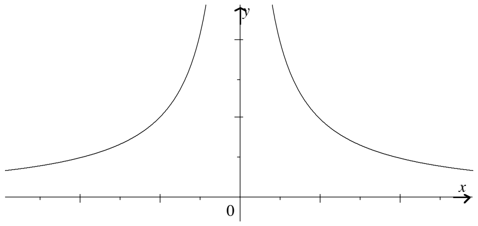
iii $k = 2$ or $k > 0$

iv $k = 0$

v $-2 < k < 0$

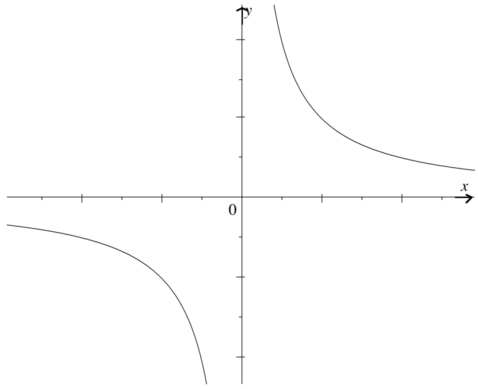
d. $y = f(x)$ is even

2. a.



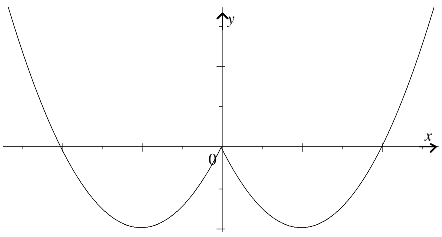
$y = f(x)$ is even.

b.



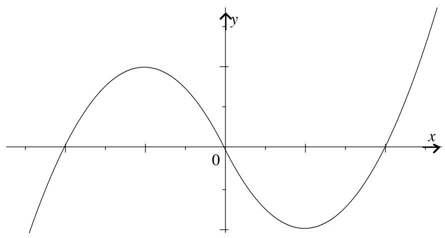
$y = f(x)$ is odd.

a.



$y = g(x)$ is even.

b.



$y = g(x)$ is odd.

3. **a.** even **b.** even **c.** neither **d.** odd **e.** odd

f. even **g.** even **h.** neither **i.** even **j.** even

4. a.

$$\begin{aligned} h(-x) &= f(-x) \cdot g(-x) \\ &= f(x) \cdot -g(x) \\ &= -f(x) \cdot g(x) \\ &= -h(x) \end{aligned}$$

Therefore h is odd.

b.

$$\begin{aligned} h(-x) &= (g(-x))^2 \\ &= (-(g(x))^2 \\ &= (g(x))^2 \\ &= h(x) \end{aligned}$$

Therefore h is even.

c.

$$\begin{aligned} h(-x) &= \frac{f(-x)}{g(-x)} \\ &= \frac{f(x)}{-g(x)} \\ &= -\frac{f(x)}{g(x)} \\ &= -h(x) \end{aligned}$$

Therefore h is odd.

d.

$$\begin{aligned} h(-x) &= f(-x) \cdot (g(-x))^2 \\ &= f(x) \cdot (-g(x))^2 \\ &= f(x) \cdot (g(x))^2 \\ &= h(x) \end{aligned}$$

Therefore h is even.

5. If f is defined at $x = 0$

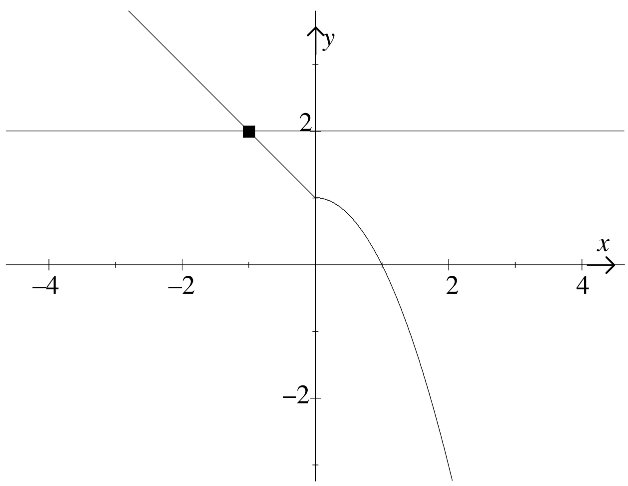
$$\begin{aligned} f(0) &= f(-0) && \text{(since } 0 = -0\text{)} \\ &= -f(0) && \text{(since } f \text{ is odd)} \\ 2f(0) &= 0 && \text{(adding } f(0) \text{ to both sides)} \\ \text{Therefore } f(0) &= 0. \end{aligned}$$

3.2 Solutions

1. **a.** $2f(-1) + f(2) = 2(1 - (-1)) + (1 - (2)^2) = 4 + (-3) = 1.$

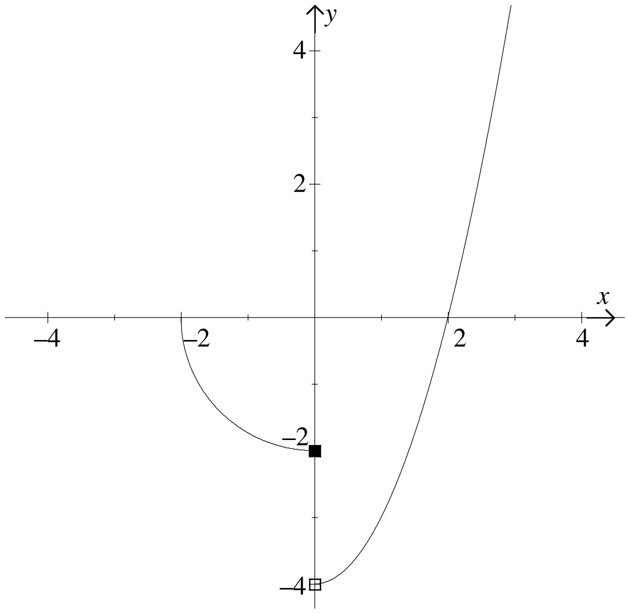
b. $f(a^2) = 1 - (a^2)^2 = 1 - a^4$ since $a^2 \geq 0$.

2. You can see from the graph below that there is one solution to $f(x) = 2$, and that this solution is at $x = -1$.



3. $$g(x) = \begin{cases} \frac{1}{x+1} & \text{for } x < 1 \\ \sqrt{1-x^2} & \text{for } -1 \le x \le 1 \\ -1 & \text{for } x > 1 \end{cases}$$

4. **a.** The domain of f is all real $x \ge -2$.



b. The range of f is all real $y > -4$.

c. i $f(x) = 0$ when $x = -2$ or $x = 2$.

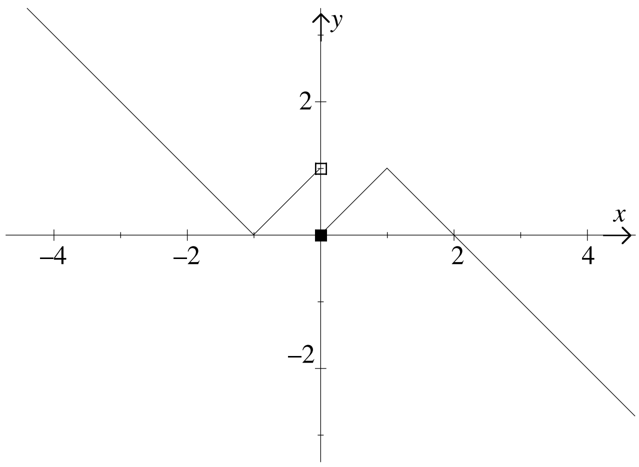
ii $f(x) = -3$ when $x = 1$.

d. i $f(x) = k$ has no solutions when $k \le -4$.

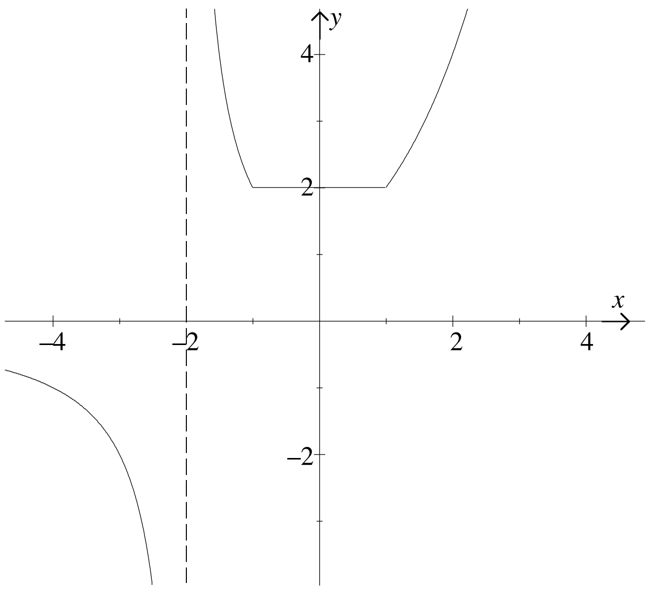
ii $f(x) = k$ has 1 solution when $-4 < k < -2$ or $k > 0$.

iii $f(x) = k$ has 2 solutions when $-2 \le k \le 0$.

5. Note that $f(0) = 0$.



6. The domain of g is all real x, $x \ne -2$.



The range of g is all real $y < 0$ or $y \geq 2$.

7. Note that there may be more than one correct solution.

a. Defining f as

$$f(x) = \begin{cases} x+6 & \text{for } x \leq -3 \\ -x & \text{for } -3 < x < 0 \\ x & \text{for } 0 \leq x \leq 3 \\ -x+6 & \text{for } x > 3 \end{cases}$$

gives a function describing the McMaths burgers' logo using 4 pieces.

b. Defining f as

$$f(x) = \begin{cases} x+6 & \text{for } x \leq -3 \\ |x| & \text{for } -3 < x < 3 \\ -x+6 & \text{for } x \geq 3 \end{cases}$$

gives a function describing the McMaths burgers' logo using 3 pieces.

c. Defining f as

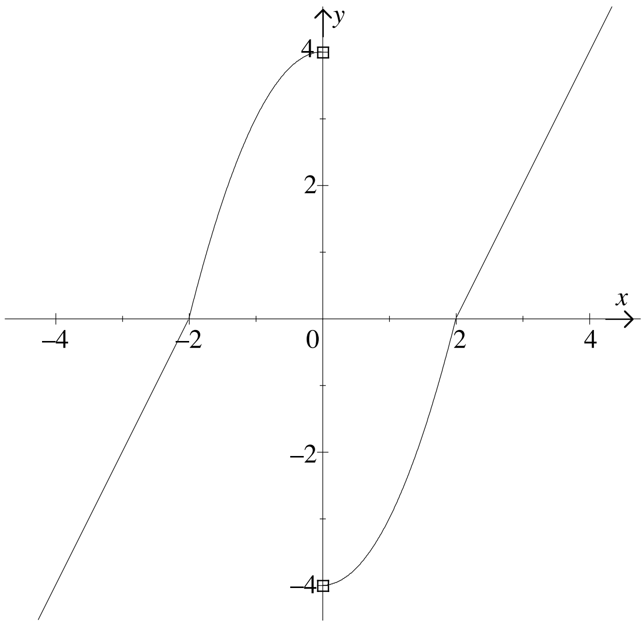
$$f(x) = \begin{cases} 3-|x+3| & \text{for } x \leq 0 \\ 3-|x-3| & \text{for } x > 0 \end{cases}$$

gives a function describing the McMaths burgers' logo using 2 pieces.

8. **a.** Here $a = 1$, $b = -4$, $c = 2$ and $d = -4$. So,

$$f(x) = \begin{cases} x^2-4 & \text{for } 0 < x \leq 2 \\ 2x-4 & \text{for } x > 2 \end{cases}$$

b. Defining f to be an odd function for all real x, $x \neq 0$, we get



c. We can define f as follows

$$f(x) = \begin{cases} 2x+4 & \text{for } x < -2 \\ 4-x^2 & \text{for } -2 \le x < 0 \\ x^2-4 & \text{for } 0 < x \le 2 \\ 2x-4 & \text{for } x > 2 \end{cases}$$

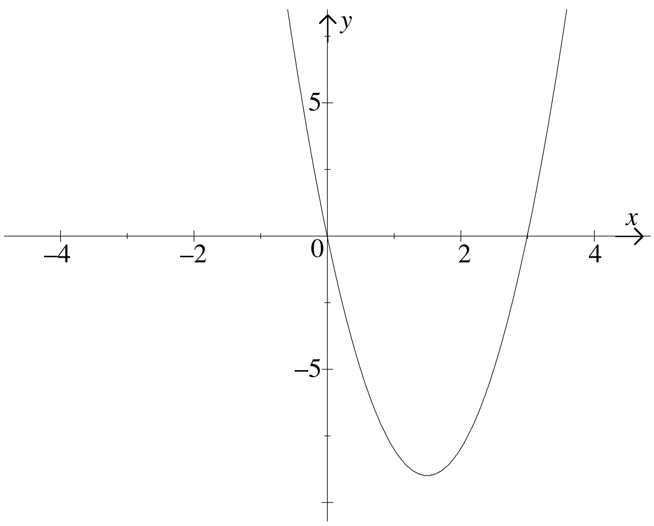
3.4 Solutions

1. **a.** $0 \le x \le 4$

b. $-3 < p \le 1$

c. $x < -4$ or $-3 < x < 3$ or $x > 4$

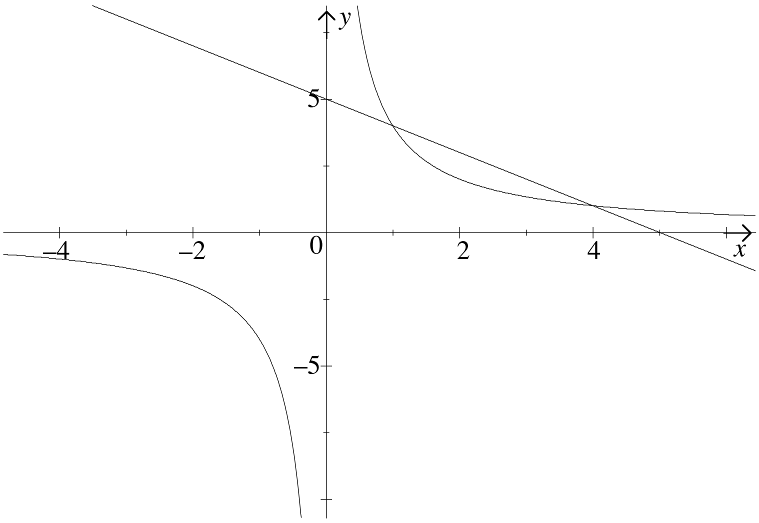
2. **a.** The graph of $y = 4x(x-3)$ is given below



b. From the graph we see that $4x(x-3) \le 0$ when $0 \le x \le 3$.

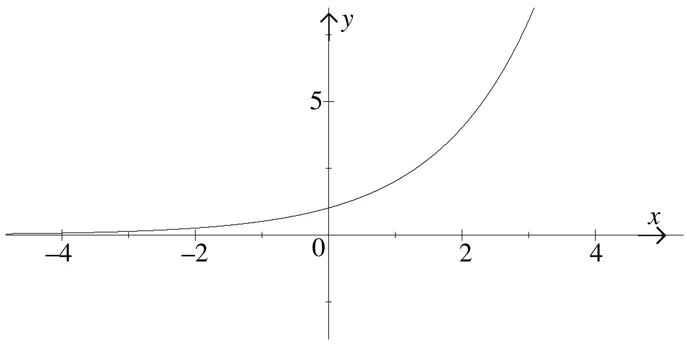
3. **a.** The graphs $y = 5 - x$ and $y = \frac{4}{x}$ intersect at the points $(1, 4)$ and $(4, 1)$.

b. The graphs of $y = 5 - x$ and $y = \frac{4}{x}$



c. The inequality is satisfied for $x < 0$ or $1 < x < 4$.

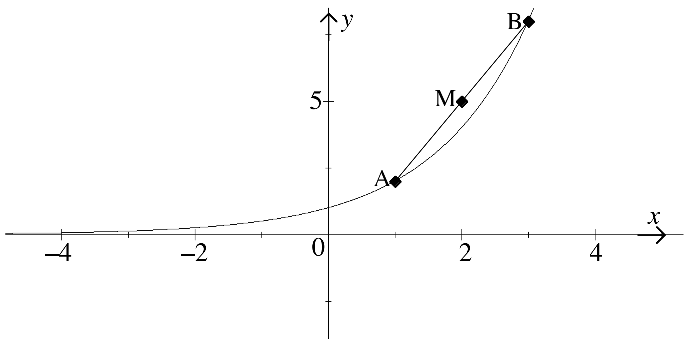
4. **a.** The graph of $y = 2^x$.



b. $2^x < \frac{1}{2}$ when $x < -1$.

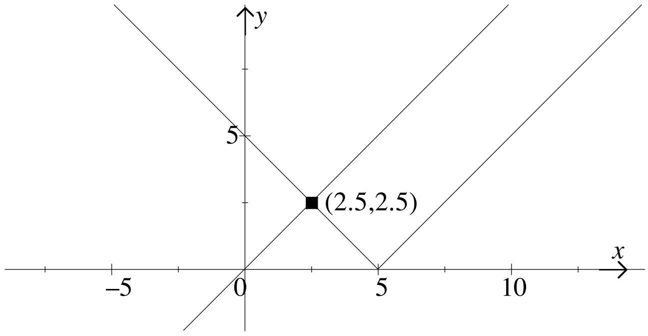
c. The midpoint M of the segment AB has coordinates $(\frac{a+b}{2}, \frac{2^a+2^b}{2})$.
Since the function $y = 2^x$ is concave up, the y-coordinate of M is greater than $f(\frac{a+b}{2})$. So,

$$\frac{2^a + 2^b}{2} > 2^{\frac{a+b}{2}}$$



5.

a.



b. $|x-5| > x$ for all $x < 2.5$.

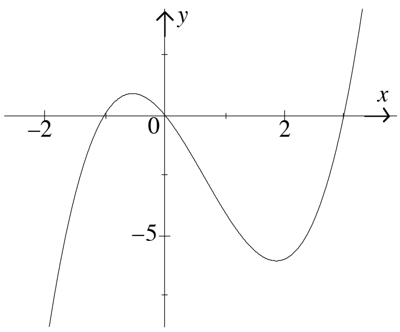
c. i $\quad mx = |x-5|$ has exactly two solutions when $0 < m < 1$.

ii $\quad mx = |x-5|$ has no solutions when $-1 < m < 0$.

6. $-1 \leq x \leq 3$

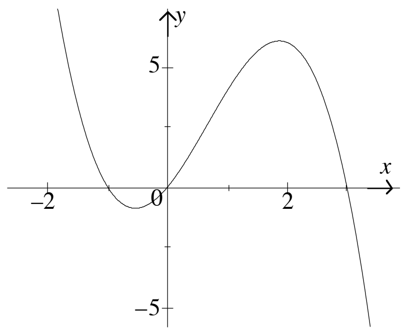
4.3 Solutions

1. a.



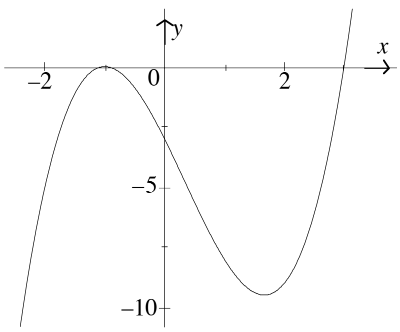
The graph of $P(x) = x(x+1)(x-3)$.

b.



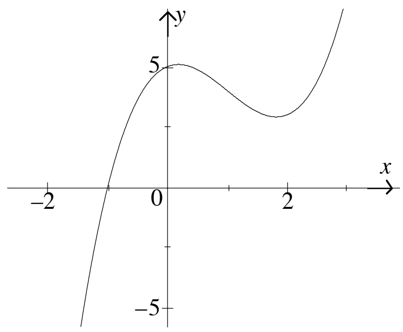
The graph of $P(x) = x(x+1)(3-x)$.

c.



The graph of $P(x) = (x+1)^2(x-3)$.

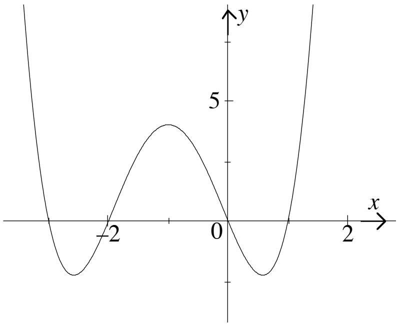
d.



The graph of
$P(x) = (x+1)(x^2-4x+5)$.

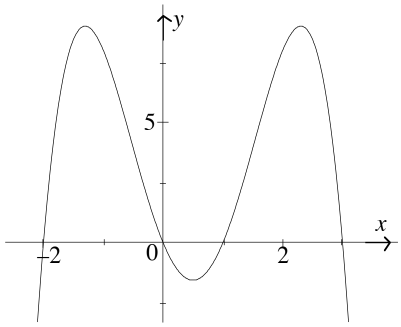
2. **a.** iv. **b.** v. **c.** i. **d.** iii. **e.** ii. **f.** vi.

3. a.



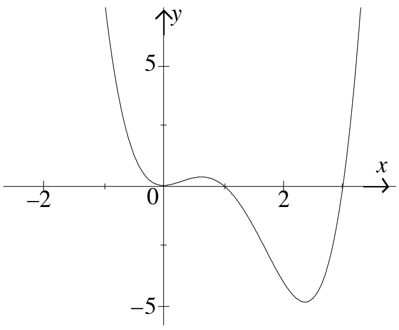
The graph of
$P(x) = x(x-1)(x+2)(x+3)$.

b.



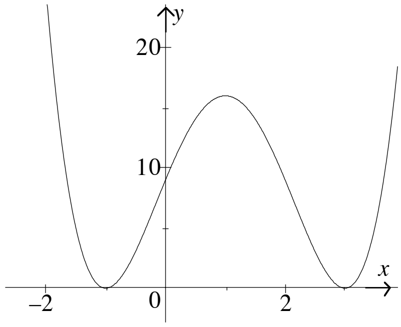
The graph of
$P(x) = x(x-1)(x+2)(3-x)$.

c.



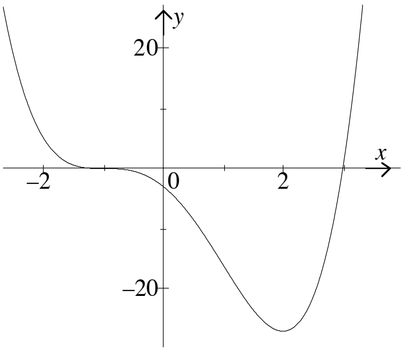
The graph of $P(x) = x^2(x-1)(x-3)$.

d.



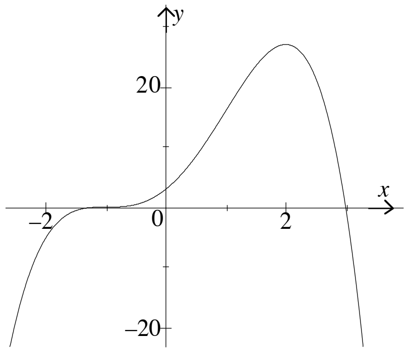
The graph of $P(x) = (x+1)^2(x-3)^2$.

e.



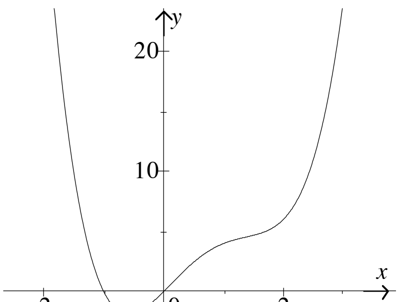
The graph of $P(x) = (x+1)^3(x-3)$.

f.



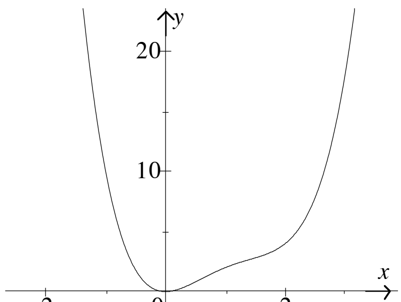
The graph of $P(x) = (x+1)^3(3-x)$.

g.



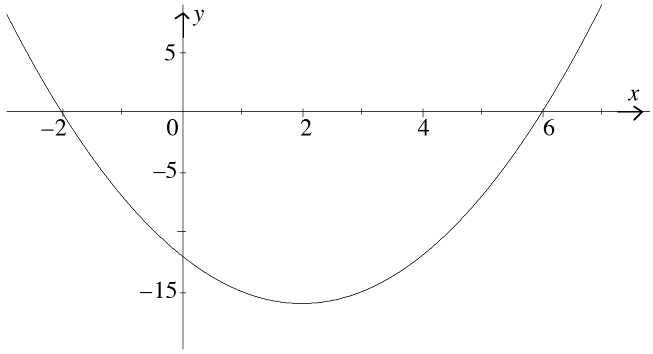
The graph of
$P(x) = x(x+1)(x^2-4x+5)$.

h.



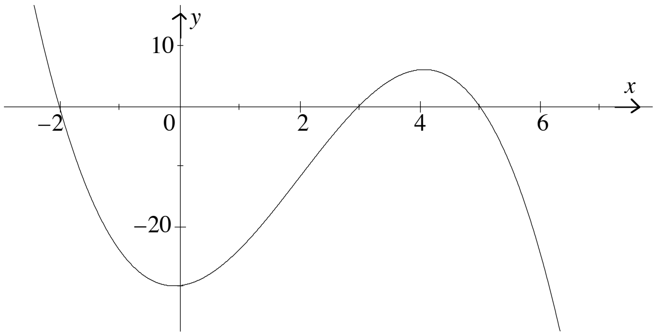
The graph of $P(x) = x^2(x^2-4x+5)$.

4. a.



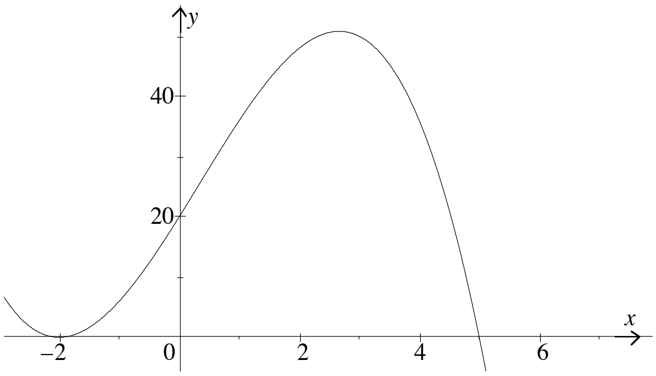
$$x^2 - 4x - 12 < 0 \text{ when } -2 < x < 6.$$

b.



$$(x + 2)(x - 3)(5 - x) > 0 \text{ when } x < -2 \text{ or } 3 < x < 5.$$

c.



$$(x + 2)^2(5 - x) > 0 \text{ when } x < 5.$$

5. $x^2 - 4x - 12 + k \geq 0$ for all real x when $k = 16$.

6. a. $P(x) = x(x - 4)$

b. $P(x) = -x(x - 4)$

c. $P(x) = x^2(x - 4)$

d. $P(x) = \frac{x^3(x-4)}{3}$

e. $P(x) = -x(x - 4)^2$

f. $P(x) = \frac{(x+4)(x-4)^2}{8}$

7. a. The roots of $f(x) = 0$ are $x = -2$, $x = 0$ and $x = 2$.

b. $x = 2$ is the repeated root.

c. The equation $f(x) = k$ has exactly 3 solutions when $k = 0$ or $k = 3.23$.

d. $f(x) < 0$ when $-2 < x < 0$.

e. The least possible degree of the polynomial $f(x)$ is 4.

f. Since $f(0) = 0$, the constant in the polynomial is 0.

g. $f(x) + k \geq 0$ for all real x when $k \geq 9.91$.

4.5 Solutions

1. a. Since $A(x) = (x - a)(x - b)$ is a polynomial of degree 2, the remainder $R(x)$ must be a polynomial of degree < 2. So, $R(x)$ is a polynomial of degree ≤ 1. That is, $R(x) = mx + c$ where m and c are constants. Note that if $m = 0$ the remainder is a constant.

b. Let $P(x) = (x^2 - 5x + 6)Q(x) + (mx + c) = (x - 2)(x - 3)Q(x) + (mx + c)$.
Then

$$\begin{aligned} P(2) &= (0)(-1)Q(2) + (2m + c) \\ &= 2m + c \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} P(3) &= (1)(0)Q(3) + (3m + c) \\ &= 3m + c \\ &= 9 \end{aligned}$$

Solving simultaneously we get that $m = 5$ and $c = -6$. So, the remainder is $R(x) = 5x - 6$.

c. Let $P(x) = (x - a)(x - b)Q(x) + (mx + c)$.
Then

$$\begin{aligned} P(a) &= (0)(a - b)Q(a) + (ma + c) \\ &= am + c \\ &= a^2 \end{aligned}$$

and

$$\begin{aligned} P(b) &= (b - a)(0)Q(b) + (mb + c) \\ &= bm + c \\ &= b^2 \end{aligned}$$

Solving simultaneously we get that $m = a + b$ and $c = -ab$ provided $a \neq b$.
So, $R(x) = (a + b)x - ab$.

2. a.

$$2x^4 + 13x^3 + 18x^2 + x - 4 = (x^2 + 5x + 2)(2x^2 + 3x - 1) - 2$$

b. Let α be a common zero of $f(x)$ and $g(x)$. That is, $f(\alpha) = 0$ and $g(\alpha) = 0$.

Then since $f(x) = g(x)q(x) + r(x)$ we have

$$\begin{aligned} f(\alpha) &= g(\alpha)q(\alpha) + r(\alpha) \\ &= (0)q(\alpha) + r(\alpha) \qquad \text{since } g(\alpha) = 0 \\ &= r(\alpha) \\ &= 0 \qquad \text{since } f(\alpha) = 0 \end{aligned}$$

But, from part **b.** $r(x) = -2$ for all values of x, so we have a contradiction.

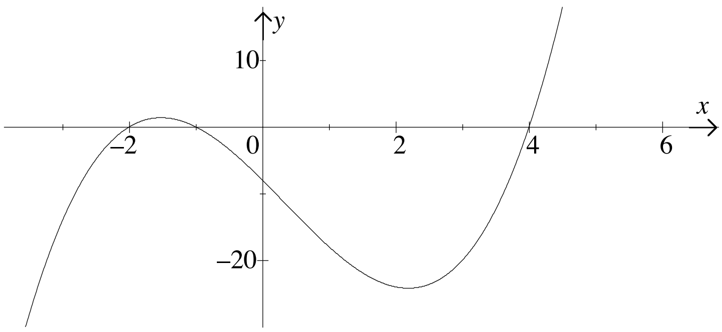
Therefore, $f(x)$ and $g(x)$ do not have a common zero.

This is an example of a proof by contradiction.

3. **a.** i $P(x) = x^3 - x^2 - 10x - 8 = (x + 1)(x + 2)(x - 4)$

ii $x = -1$, $x = -2$ and $x = 4$ are solutions of $P(x) = 0$.

iii

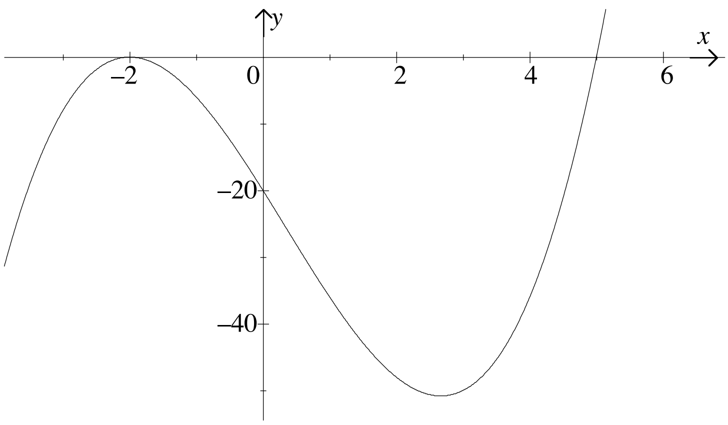


The graph of $P(x) = x^3 - x^2 - 10x - 8$.

b. i $P(x) = x^3 - x^2 - 16x - 20 = (x + 2)^2(x - 5)$.

ii $x = -2$ and $x = 5$ are solutions of $P(x) = 0$. $x = -2$ is a double root.

iii

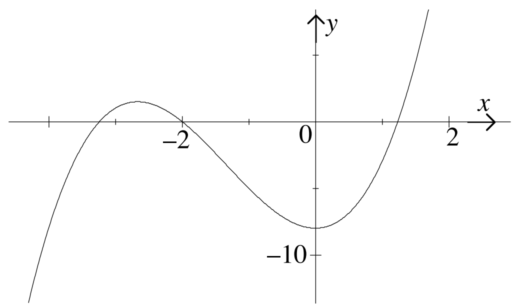


The graph of $P(x) = x^3 - x^2 - 16x - 20$.

c. i $P(x) = x^3+4x^2-8 = (x+2)(x^2+2x-4) = (x+2)(x-(-1+\sqrt{5}))(x-(-1-\sqrt{5}))$

ii $x = -2$, $x = -1 + \sqrt{5}$ and $x = -1 - \sqrt{5}$ are solutions of $P(x) = 0$.

iii



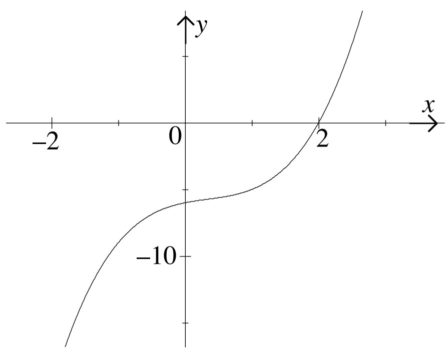
The graph of $P(x) = x^3 + 4x^2 - 8$.

The zeros are $x = -2$, $x = -1 + \sqrt{5}$ and $x = -1 - \sqrt{5}$.

d. i $P(x) = x^3 - x^2 + x - 6 = (x - 2)(x^2 + x + 3)$. $x^2 + x + 3 = 0$ has no real solutions.

ii $x = 2$ is the only real solution of $P(x) = 0$.

iii



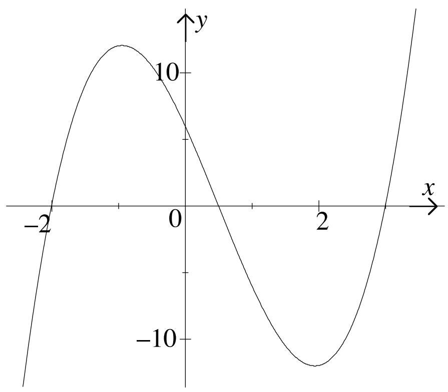
The graph of $P(x) = x^3 - x^2 + x - 6$.

There is only one real zero at $x = 2$.

e. i $P(x) = 2x^3 - 3x^2 - 11 + 6 = (x + 2)(x - 3)(2x - 1)$.

ii $x = -2$, $x = \frac{1}{2}$ and $x = 3$ are solutions of $P(x) = 0$.

iii



The graph of $P(x) = 2x^3 - 3x^2 - 11 + 6$.

DIFFERENTIAL CALCULUS

CHAPTER I : **REAL NUMBER SYSTEM**

Definition 1.1 By Real Number System we mean a non-empty set R, two binary operations addition and multiplication (denoted by + and . respectively) and an order relation $\leq$ defined on R which satisfy the following axioms I, II and III :

I. Algebraic Properties:

For all a,b,c in R,

(1) a+(b+c)=(a+b)+c, a.(b.c)=(a.b).c (associativity),

(2) a+b=b+a,a.b=b.a (commutativity)

(3) there exists 0,1 in R, 0≠1, such that a+0=a,a.1=a (existence of identity for addition and multiplication)

(4) for all a in R ,there is (-a) in R such that a+(-a)=0; for all a (≠0) in R , there exists a-1 in R such that a.a-1 =1 (existence of inverse under addition and multiplication)

(5) a.(b+c)=a.b+a.c (distributivity of . over +)

A system that has more than one element and satisfies these five axioms is called a FIELD. The basic algebraic properties of R can be proved solely on the basis of these field properties. (Field properties will be discussed in detail in a later semester.)

II. ORDER PROPERTIES:

(1) for a,b in R, either a ≤ b or b ≤ a

(2) if a ≤ b and b ≤ a, then a=b

(3) if a ≤ b and b ≤ c, then a ≤ c

(4) if a ≤ b then a+c ≤ b+c

(5) if a ≤ b and 0 ≤ c, then ac ≤ bc

A field satisfying above five properties is called an ordered field. Most of the algebraic and order properties of R can be established for any ordered field (like Q).

III. The Completeness Axiom:

Differentiation of Real Numbers and Rational Numbers

Let $\varphi \neq S \subseteq R$. If a real number M satisfies $s \leq M$ *for* all $s \in S$, then M is called an upper bound of S and S is called bounded above.

If a real number M satisfies (1) M is an upper bound of S and (2) no real number less than M is an upper bound of S, then M is the LEAST UPPER BOUND of S or SUPREMUM of S, written as l.u.b. S or sup S .

Ex: sup $\{\frac{1}{n} : n \in N\} = 0$, sup $\{ 2n : n \in N \}$ does not exist, sup $\{1,2,3\}$=max$\{1,2,3\}$.

The Completeness Axiom for R states that:

Every nonempty subset S of R that is bounded above has a least upper bound in R.

That is, if a nonempty subset of real numbers has an upper bound, then it has a smallest upper bound in R.

Definition: A number is called an algebraic number if it satisfies a polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots \dots .. + a_1 x + a_0 = 0$ where the coefficients a_0 , a_1, ………… , a_n , are integers, $a_n \neq o$, and $n \geq 1$. A real number which is not algebraic (like π, e) is called a transcendental number.

Note: Rational numbers are algebraic numbers since a rational number $x = \frac{m}{n}$, m, n are integers and $n \neq 0$ satisfies the equation $nx - m = 0$.

Theorem: Suppose that a_0 , a_1, ………… , a_n are integers and r is a rational number satisfying the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots \dots .. + a_1 x + a_0 = 0$ where $a_0 , a_n \neq o$, and $n \geq 1$. Write $r = \frac{p}{q}$, where p, q are integers having no common factors and $q \neq 0$. Then q divides a_n and p divides a_0.

Ex: $\sqrt{2}$ cannot represent a rational number.

»By theorem above, the only rational numbers that could possibly be solutions of $x^2 - 2 = 0$ are ± 1 and ± 2. But none of the four numbers ± 1 and ± 2 are solutions of the equation. Since $\sqrt{2}$ represents a solution of $x^2 - 2 = 0$, $\sqrt{2}$ cannot represent a rational number. Note: In , division of an integer p by an integer q is defined iff there exists unique integer r such that $p = qr$ holds.

For $p \neq 0$, division of p by 0 is undefined because there does not exist any integer r such that $p = 0.r$ holds. Division of 0 by 0 is undefined because for any integer $r,\ 0 = 0.r$ holds and uniqueness of r is violated.

The absolute value function and the greatest integer function

For $a \in R$, absolute value of a , denoted as $|a|$, is defined to be the distance between (the points representing) 0 and a:

$|a| = a$, if $a \geq 0$,

$= -a$, if < 0 .

The absolute value function satisfies the following properties:

(1) $|a| = |-a|$, (2) $|a + b| \leq |a| + |b|$, (3) $|ab| = |a||b|$,

(4) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}(if\ b \neq 0)$, (5) $\big||a| - |b|\big| \leq |a - b|$

Note: $|a - b|$ gives distance between the (points representing) real numbers a and b. Hence the statement b lies between $a + 1$ and $a - 1$ can be equivalently put as $|a - b| < 1$.

The greatest integer function is defined on the set R of real numbers as follows:

for real number a satisfying $m \leq a < m + 1$, (m an integer), $[a] = m$.

Thus, for example, $[a] = -1$, for $-1 \leq a < 0$,

$= 0$, for $0 \leq a < 1$

$= 1$, for $1 \leq a < 2$

CHAPTER 2: **SEQUENCES OF REAL NUMBERS**

Definition: A sequence (in R) is a function $a: N \to R$, $a(N)$ is called the n th term of the sequence and denoted by a_n . The sequence a is often denoted by $\{a_1, a_2, \dots \dots, a_n\}$ or ,more compactly, by (a_n). We must distinguish between a sequence (a_n) and its range set $\{a_n\}$: the range set of (1,1/2,1/3,…) is {1,1/2,1/3,…}; that of (1,1,1…) is {1}. We shall use the terms 'bounded','unbounded', 'bounded above' and 'bounded below' for a sequence iff its range set (as a subset of R) has the corresponding property. $(1/n)$ is bounded, (n^2) is bounded below but not above, $(-n^2)$ is bounded above but not below, $((-1)^n n)$ is neither bounded above nor bounded below.

Note: A fraction increases when either the numerator is increased and/or the denominator is decreased.

Ex: Verify whether the following sequences are bounded above and/or bounded below: (1) $\left(\frac{2n+3}{3n+4}\right)$, (2) $\left((-1)^n \frac{3n-1}{n}\right)$, (3) (a_n) where $a_n = \sqrt[3]{n+1} - \sqrt[3]{n}$, (4) (a_n) where $a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$.

Solution: (1) $\frac{2n+3}{3n+4} = \frac{2}{3}\left(\frac{3n+4+\frac{1}{2}}{3n+4}\right) = \frac{2}{3} + \frac{1}{3(3n+4)} < \frac{2}{3} + \frac{1}{3} = 1$, for all natural number n. Thus $0 < \frac{2n+3}{3n+4} < 1$, for all natural n; hence $\left(\frac{2n+3}{3n+4}\right)$ is bounded.

(2) $0 < \left|(-1)^n \frac{3n-1}{n}\right| \le 3 - \frac{1}{n} < 3$for all natural n: hence $\left((-1)^n \frac{3n-1}{n}\right)$ is bounded.

(3) $0 < a_n = \frac{1}{(n+1)^{\frac{2}{3}} + (n+1)^{\frac{1}{3}} n^{\frac{1}{3}} + n^{\frac{2}{3}}} < \frac{1}{3}$ for all n (justify!): hence (a_n) is bounded.

(4) $0 < \frac{n}{\sqrt{n^2+n}} < b_n < \frac{n}{\sqrt{n^2+1}} < \frac{n}{n} = 1$ for all natural n: hence (b_n) is bounded.

Definition: A sequence (a_n) is convergent iff there exists $a \in R$ such that for every $\epsilon > 0$, there exists natural number (ϵ) , in general depending on ϵ , so that $n \ge m$ implies $|a_n - a| < \epsilon$. In this case, we say (a_n) converges to a as $n \to \infty$ or a is a limit of (a_n) and write $\lim_{n\to\infty}(a_n) = a$. A sequence that is not convergent is called divergent.

Ex: Consider the sequence $a = (a_n)$, where $a_n = b$ for all n. Then $\lim(b, b, \dots, b) = b$, since for any $\epsilon > 0$, $|a_n - b| = 0 < \epsilon$ for all $n \geq 1$.

Note that every constant sequence (b,b,…) is convergent to b.

Ex: Consider the sequence (a_n), where $a_n = (-1)^n$, n natural .

$lim(a_n) \neq 1$ since the open interval (0.5 , 1.5) containing 1 does not contain all the infinite number of terms of the sequence with odd suffix . $lim(a_n) \neq -1$ since the open interval (-1.5 ,-0.5) containing -1 does not contain all the infinite number of terms of the sequence with even suffix.

If $a \neq 1, -1$, let $\epsilon = min\{|a-1|, |a+1|\}$. Then $(a-\epsilon, a+\epsilon)$ does not contain any term of the sequence. Hence (a_n)diverges.

Ex: If $a_n = \frac{1}{n}$ for all natural ,

then $\lim_{n\to\infty}(a_n) = 0$: for any $\epsilon > 0$, $|a_n - 0| < \epsilon \Leftarrow n > \frac{1}{\epsilon} \Leftarrow n \geq \left[\frac{1}{\epsilon}\right] + 1 = m$

Note: Observe the direction of implication sign carefully.

Note: If $\lim_{n\to\infty}(a_n) = a$, then the inequality $\big||a_n| - |a|\big| \leq |a_n - a|$, n natural, shows that $\lim_{n\to\infty}|a_n| = |a|$. Converse may not hold as can be seen from the counter example $((-1)^n)$.

Theorem: A convergent sequence is bounded.

»Let $lim(a_n) = a$. Corresponding to $\varepsilon = 1$, there exists positive integer m such that $|a_n - a| < 1$, that is $a - 1 < a_n < a + 1$ for all $n \geq m$. Let $m = min\{a_1, a_2, \dots\dots, a_{m-1}, a - 1\}$ and $M = max\{a_1, a_2, \dots\dots, a_{m-1}, a + 1\}$. Then $m \leq a_n \leq M$ for all natural n. Hence (a_n) is bounded.

Theorem: (Limit Theorem for Sequences) Let $\lim_{n\to\infty}(a_n) = a$ and $\lim_{n\to\infty}(b_n) = b$.

Then (1) $\lim_{n\to\infty}(a_n + b_n) = a + b$, (2) $\lim_{n\to\infty}(ra_n) = ra$, (3) $\lim_{n\to\infty}(a_n b_n) = ab$, (4) if $a \neq 0$, then $\exists\, m \in N$ such that $a_n \neq 0$ for all $n \geq m$ and $\lim_{n\to\infty}\left(\frac{1}{a_n}\right) = \frac{1}{a}$.

Note: Combined applications of different parts ensures that $lim(a_n - b_n) = a - b$ and $lim(a_n/b_n) = a/b$ if $b \neq 0$.

Note: Note that it is possible that $lim(a_n) = a$ and $lim(b_n) = b$, $a_n < b_n$, for all natural n and yet $a = b$. Consider the sequences $(a_n), (b_n)$ where $a_n = 0$ and $b_n = 1/n$ for all n.

Theorem: If (x_n) and (y_n) are convergent sequences of real numbers and if $x_n < y_n$ for all $n \geq m$ (m fixed natural number), then $\lim(x_n) \leq lim\,(y_n)$.

Theorem: (Sandwich Theorem) Let $(a_n), (b_n), (c_n)$ be sequences and $c \in R$ be such that $a_n \leq b_n \leq c_n$, for all $n \geq m$ and $lim(a_n) = lim(c_n) = c$. Then $(b_n) = c$.

Ex: Let $|a| < 1$. Then $lim(a_n) = 0$.

» If $a = 0$, the result is obvious. Let $a \neq 0$. Let $\frac{1}{|a|} = 1 + h$. Then $h > 0$ and $\frac{1}{|a|^n} = (1 + h)^n \geq 1 + nh > nh$ for all natural n. Hence $0 < |a|^n < \frac{1}{nh}$. Since $\lim(\frac{1}{nh}) = 0$, by Sandwich Thorem, $lim(|a|^n) = 0$. Hence $lim(a_n) = 0$.

Ex: Let $a > 0$. Then $lim(a^{\frac{1}{n}}) = 1$.

» If $a = 1$, nothing remains to prove.

Let $a > 1$. Then $a^{\frac{1}{n}} > 1$ for all natural n.

Let $b_n = a^{\frac{1}{n}} - 1$ for natural n. then $= (1 + b_n)^n = 1 + nb_n + \cdots \ldots + (b_n)^n > nb_n$,

so that $0 < b_n < a/n$ for all n.

By Sandwich Theorem, $lim(b_n) = 0$, so that , $lim(a^{\frac{1}{n}}) = 1$.

If $0 < a < 1$, let $b = 1/a$ so that $lim\left(\frac{1}{a^{\frac{1}{n}}}\right) = lim\left(b^{\frac{1}{n}}\right) = 1$, since $b > 1$ and hence $lim(a^{\frac{1}{n}}) = 1$.

<u>Ex:</u> $lim(n^{\frac{1}{n}}) = 1$.

»Since $(a^{\frac{1}{n}}) \geq 1$ for all n, we can write $n^{\frac{1}{n}} = 1 + k_n$ where $k_n \geq 0$ for all n.

Hence $n = (1 + k_n)^n \geq \frac{n(n-1)}{2}(k_n)^2 + 1$.

Thus $0 \leq k_n \leq \sqrt{\frac{2}{n}}$. Thus $lim(k_n) = 0$ and so $lim(n^{\frac{1}{n}}) = 1$.

Example 2.11 Let $b_n = \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+n}$, for natural n. clearly $a_n \leq b_n \leq c_n$ for all natural n , where $b_n = \sum_{k=1}^{n} \frac{n}{n^2+n} = \frac{n^2}{n^2+n}$ and $c_n = \sum_{k=1}^{n} \frac{n}{n^2+1} = \frac{n^2}{n^2+1}$. Since $lim(a_n) = lim(c_n) = 1$, so $lim(b_n) = 1$.

<u>Ex:</u> Use Squeeze Theorem to verify whether following sequences converge: $\left(n^{\frac{1}{n^2}}\right), \left((n!)^{\frac{1}{n^2}}\right)$.

» $1 \leq n^{\frac{1}{n^2}} \leq n^{\frac{1}{n}}$, $1 \leq (n!)^{\frac{1}{n^2}} \leq (n^n)^{\frac{1}{n^2}} = n^{\frac{1}{n}}$ for all natural n.

<u>Note:</u> Every convergent sequence is bounded. <u>The converse may not hold</u> : consider $((-1)^n)$. We now consider a class of sequences for which convergence is equivalent to boundedness.

<u>Definition:</u> A sequence $a: N \to R$ or (a_n) is monotonically increasing iff a is monotonically increasing function, that is, $a_n \leq a_{n+1}$ for all natural n. A sequence $a: N \to R$ or (a_n) is monotonically decreasing iff a is monotonically decreasing function, that is, $a_n \geq a_{n+1}$ for all natural n . A sequence is monotonic iff it is either monotonically increasing or monotonically decreasing.

<u>Theorem :</u> A monotonically increasing sequence is convergent iff it is bounded above.

Ex: Consider (a_n), where $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ for natural n . (a_n) is monotonially increasing. Also $a_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + 2\left(1 - \frac{1}{2^n}\right) < 3$ for all natural n. Thus (a_n) is convergent.

Ex: Consider (a_n), where $a_n = \left(1 + \frac{1}{n}\right)^n$.

Now, $a_n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)(n-2)\ldots 1}{n!}\frac{1}{n^n}$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)$$

$$< 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3.$$

Thus $\quad 2 < a_n < 3$, for all n .

Hence (a_n) is bounded.

Also $\quad a_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\ldots\left(1 - \frac{n-1}{n+1}\right) + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\ldots\left(1 - \frac{n-1}{n+1}\right)\left(1 - \frac{n}{n+1}\right)$.

Thus a_{n+1} is sum of $(n + 2)$ summands whereas an is that of $(n + 1)$ summands and each summand (from beginning) of a_{n+1} is greater than or equal to the corresponding summand of a_n. Hence (a_n) is monotone increasing. Thus (a_n) is convergent. If we denote $lim(a_n)$ by e, then $2 \leq e \leq 3$ since $2 \leq a_n \leq 3$ for all n.

Ex: Consider the sequence (a_n), where $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, for all n. Clearly (a_n) is monotonically increasing. For each n, $a_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \geq 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{n-1}}{2^n} = 1 + \frac{n}{2}$.Hence there is no M such that $a_n < M$, for all n . Thus (a_n) is unbounded above and hence is not convergent.

Ex: Consider the sequence (a_n), where $a_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$. Clearly (an) is monotonically increasing. Also, for each n, $a_n = 1 + \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} = 1 + \left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 2 - \frac{1}{n} < 2$. So (a_n) is bounded above. Hence (a_n) is convergent.

Ex: Consider the sequence (a_n), where $a_n = 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$ for each n. clearly (a_n) is monotonically increasing. We have seen in two examples above that (a_n) is divergent if $p = 1$ and convergent if $p = 2$. This implies that (a_n) is divergent if $p \leq 1$ and convergent if $p \geq 2$, because for each n, $0 < \frac{1}{n} \leq \frac{1}{n^p}$ if $p \leq 1$ while $0 < \frac{1}{n^p} \leq \frac{1}{n^2}$ if $p \geq 2$.

P R O P E R L Y D I V E R G E N T S E Q U E N C E S

Definition: Let (a_n) be a sequence of real numbers. (a_n) tends to ∞ as n tends to ∞ , written as $\lim_{n\to\infty}(a_n) = \infty$ iff for every $M > 0$, there exists natural number m such that $n \geq m$ implies $a_n > M$. Similarly, a sequence (b_n) of real numbers tends to $-\infty$ as n tends to ∞ , written as $\lim_{n\to\infty}(a_n) = -\infty$ iff for every $M > 0$, there exists natural number m such that $n \geq m$ implies $a_n < -M$.

CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE

Theorem: A sequence (a_n) of real numbers is convergent iff for every $\varepsilon > 0$, there exists positive integer m such that for $p, q \geq m$, p, q natural number, $|a_p - a_q| < \varepsilon$ holds.

Ex: Verify that the sequence (a_n) where $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, n natural, does not converge.

»If (a_n) converges, then corresponding to $\varepsilon = \frac{1}{2}$, there exists m such that $|a_p - a_q| < \frac{1}{2}$ for all $p, q \geq m$. In particular, $|a_{2m} - a_m| < \frac{1}{2}$. But $|a_{2m} - a_m| = \frac{1}{m+1} + \cdots + \frac{1}{2m} > \frac{m}{2m} = \frac{1}{2}$, contradiction.

CHAPTER 3: **INFINITE SERIES OF REAL NUMBER**

Definition: An infinite series, or, for short, a series of real numbers is an expression of the form $a_1 + a_2 + \cdots + a_n + \cdots$ or, in more compact notation $\sum_{k=1}^{\infty} a_k$, where a_n is real number , for all n. The sequence (A_n), where $A_n = a_1 + a_2 + \cdots + a_n$, is called the sequence of partial sums corresponding to the series $\sum_{k=1}^{\infty} a_k$.

Definition: We say $\sum_{k=1}^{\infty} a_k$ is convergent iff $lim(A_n)$ exists as real number, that is, iff the sequence of partial sums of the series is convergent. If $\lim_{n\to\infty}(A_n) = A$, then A is the sum of the series $\sum_{k=1}^{\infty} a_k$ and we write $A = \sum_{k=1}^{\infty} a_k$. An infinite series $\sum_{k=1}^{\infty} a_k$ that is not convergent, is divergent. In particular, we say the series $\sum_{k=1}^{\infty} a_k$ diverges to ∞ or to $-\infty$ according as $lim(A_n) = \infty$ or $lim(A_n) = -\infty$.

Note: The convergence of a series is not affected by changing finite number of its terms, although its sum may change by doing so. If $\sum_{k=1}^{\infty} a_k$ and $b_1 + b_2 + \cdots + b_m + \sum_{k=m+1}^{\infty} a_k$ be two series obtained by changing first m number of terms, and if (A_n) and (B_n) be the corresponding sequences of partial sums, then for $n > m, A_n = B_n + \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} b_k$. Thus (A_n)converges iff (B_n)converges and if they converge to A and B respectively, $A = B + \sum_{k=1}^{m} a_k - \sum_{k=1}^{m} b_k$.

Ex: (Geometric Series) For the series $\sum a^k$ (a real),

$A_n = a_1 + a_2 + \cdots + a_{n-1} = \frac{1-a^n}{1-a}$, if $a \neq 1$ and

$A_n = n$, if $a = 1$.

If $-1 < a < 1$, $lim(A_n) = \frac{1}{1-a}$. Hence the series $\sum a^k$ converges for $-1 < a < 1$.

If $a \geq 1$, then $A_n \geq n + 1$ for all n (since $a, a^2, \dots, a^{n-1}$ are ≥ 1); hence (A_n) and thus $\sum a^k$ diverges to ∞.

If $a = -1$, $A_{2n} = 1$ and $A_{2n+1} = 0$ for all n and hence $\sum a^k$ is divergent.

Finally, if $a < -1$, then $lim(A_{2n}) = \infty$ and $lim(A_{2n+1}) = -\infty$; hence $\sum a^k$ is divergent to $-\infty$.

Ex: (Exponential Series) We have seen earlier that the sequence (A_n) given by $A_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ is convergent to a real number $e, 2 \le e \le 3$. Thus $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \ldots\ldots$

Ex: (Harmonic series and its variants) As seen in example 2.17, $1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$ ... is divergent for $p \le 1$ and convergent for $p > 1$. It can be shown that the sequence of partial sums and hence the series $\sum(-1)^{k-1}\frac{1}{k}$ converges.

Note: Since convergence of a series is defined in terms of convergence of its sequence of partial sums, many results about convergence of a series follows from the corresponding results on sequences.

- The sequence of partial sums of a convergent series is bounded.
- Let $\sum a_k = A$, $\sum b_k = B$. Then $\sum(a_k + b_k) = A + B$, $\sum(ra_k) = rA$. If $a_k \le b_k$, for all k , then $A \le B$.
- (Sandwich Theorem) If $(a_k), (b_k), (c_k)$ are sequences of real numbers such that $a_k \le b_k \le c_k$ for each k, and further if $\sum a_k = A = \sum c_k$, then $\sum b_k = A$.

Theorem: If $\sum a_k$ is convergent, then $lim(a_k) = 0$. In other words, if $lim(a_k)$ does not exist or is not equal to zero, then $\sum a_k$ is divergent. Converse may not hold: consider $\sum \frac{1}{k}$.

Ex: $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^p}$ is divergent if $p \le 0$ since $\left|(-1)^{k-1}\frac{1}{k^p}\right| \ge 1$ for all k.

TESTS FOR CONVERGENCE : series of non-negative terms

Theorem: (Comparison Test) Let $(a_k), (b_k)$ be sequences of non-negative real numbers such that $a_k \le b_k$ for all $k \ge$ m, m natural. If $\sum b_k$ is convergent, then $\sum a_k$ is convergent and $\sum a_k \le \sum b_k$. If $\sum a_k$ diverges, then $\sum b_k$ is divergent.

Ex: $\sum \frac{2^k+k}{3^k+k}$ is convergent since $\frac{2^k+k}{3^k+k} \leq 2\left(\frac{2}{3}\right)^k$ for all natural k and $\sum \left(\frac{2}{3}\right)^k$ is convergent.

Ex: $\sum \frac{1}{(1+k^2+k^4)^{\frac{1}{3}}}$ is convergent since $\frac{1}{(1+k^2+k^4)^{\frac{1}{3}}} \leq \frac{1}{3}\frac{1}{k^{\frac{4}{3}}}$ for all natural k and $\sum \frac{1}{k^{\frac{4}{3}}}$ is convergent.

Theorem: (Limit Form of Comparison Test) Let $(a_k), (b_k)$ be sequences of positive real numbers such that $\lim_{n\to\infty}\left(\frac{a_k}{b_k}\right) = L\ (\neq 0)$. Then $\sum a_k$ is convergent iff $\sum b_k$ is convergent. If $\lim_{n\to\infty}\left(\frac{a_k}{b_k}\right) = 0$ and if $\sum b_k$ is convergent, then $\sum a_k$ is convergent. If $\lim_{n\to\infty}\left(\frac{a_k}{b_k}\right) = \infty$ and if $\sum b_k$ is divergent, then $\sum a_k$ is divergent.

Ex: The series $\sum \frac{2^k+k}{3^k-k}$ is convergent by Comparison Test since $\sum \left(\frac{2}{3}\right)^k$ is convergent and $\lim_{n\to\infty}\left(\frac{a_k}{b_k}\right) = 1(\neq 0)$.

Ex: $\sum sin\left(\frac{1}{n}\right)$ diverges by comparison test since $\sum \frac{1}{n}$ is divergent and $\lim_{n\to\infty} \frac{sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1(\neq 0)$

Theorem: (D'Alembert's Ratio Test) Let $\sum a_k$ be a series of positive real numbers and let $\lim_{k\to\infty}\left(\frac{a_{k+1}}{a_k}\right) = x$. Then $\sum a_k$ converges if $x < 1$ and $\sum a_k$ diverges if $x > 1$.

Theorem: (Cauchy's Root Test) Let $\sum a_k$ be a series of positive real numbers and let $\lim_{k\to\infty} a_k^{\frac{1}{k}} = x$. Then $\sum a_k$ converges if $x < 1$ and $\sum a_k$ diverges if $x > 1$.

Theorem: (Raabe's Test) Let $\sum a_k$ be a series of positive real numbers and let $\lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) = x$ .Then $\sum a_k$ converges if $x > 1$ and $\sum a_k$ diverges if $x < 1$.

Ex: Use comparison Test to verify whether the following series converge:

(1) $\sum \frac{n+1}{n^3}$, (2) $\sum \frac{1}{\sqrt{n(n+1)}}(3)\sum \frac{\sqrt{n}}{(n+1)^2}$

»Compare the series respectively with (1) $\sum \frac{1}{n^2}$, (2)$\sum \frac{1}{n}$ and (3)$\sum \frac{1}{n^{3/2}}$

Note: While testing convergence of a series $\sum a_n$ by comparison test, the series is constructed by considering heighest power of n present in numerator and denominator separately .

ALTERNATING SERIES : LEIBNITZ TEST

Definition: A series of the form $\sum(-1)^n a_n$, where $a_n > 0$ for all positive integer n, is an alternating series.

Theorem: (Leibnitz Test) An alternating series $\sum(-1)^n a_n$ is convergent if the sequence $(a_n)$of positive terms is monotonically decreasing and $\lim_{n\to\infty}(a_n) = 0$.

Ex: 1 4 $\sum(-1)^n \frac{1}{2n+3}$ converges by Leibnitz Test.

CHAPTER 4

LIMIT OF A REAL VALUED FUNCTION OF A REAL VARIABLE

Definition: Let $f\colon R \to R$ and $c \in R$.

A real number L is the LEFT(hand) LIMIT of f at c , written as $\lim_{x\to c-} f = L$, iff for every $\epsilon > 0$, there exists $\delta > 0$ such that $c - \delta < x < c$ implies $|f(x) - L| < \epsilon$.

A real number L is the RIGHT(hand) LIMIT of f at c, written as $\lim_{x\to c+} f = L$, iff for every $\epsilon > 0$, there exists $\delta > 0$ such that $c < x < c + \delta$ implies $|f(x) - L| < \epsilon$.

A real number L is limit of f at c , written as $\lim_{x\to c} f = L$, iff
(1) $\lim_{x\to c-} f$ exists and (2) $\lim_{x\to c+} f$ exists and (3) $\lim_{x\to c-} f = L$.

<u>Ex:</u> Prove that $\lim_{x\to 2} 5x = 10$.

»Let $\epsilon > 0$ be given.$|5x - 10| < \epsilon \Leftarrow |x-2| < \frac{\epsilon}{5} = \delta \Leftarrow 0 < |x-2| < \delta, \delta > 0$. Hence the result.

Note: Observe the direction of implication sign carefully.

<u>Ex:</u> $\lim_{\mathrm{x}\to 0} \mathrm{x}^2 \sin\frac{1}{\mathrm{x}} = 0$

»Let $\epsilon > 0$ be given. $\left|\mathrm{x}^2\sin\frac{1}{\mathrm{x}} - 0\right| < \epsilon \Leftarrow |x^2|\left|\sin\frac{1}{\mathrm{x}}\right| \leq |x^2| < \epsilon \Leftarrow |x|^2 < \epsilon \Leftarrow |x| < \sqrt{\epsilon} = \delta \Leftarrow 0 < |x - 0| < \delta$.

<u>Ex:</u> Let $f: R \to R, f(x) = [x]$.

$\lim_{x\to 0+} f = 0$ since for any $\epsilon > 0$, there exists $\delta = 1/2 > 0$ such that $0 < x < 0 + 1/2$ implies $|f(x) - L|$=$|0 - 0| < \epsilon$. Similarly, $\lim_{x\to 0-} f = -1$. Hence $\lim_{x\to 0} f$ does not exist.

<u>Ex:</u> Let $g: R \to R,\ g(x) = e^{\frac{1}{x}},\ x \neq 0,\ g(0) = 0$.

As $x \to 0+$, $\frac{1}{x} \to \infty$ and so $e^{\frac{1}{x}} \to \infty$.

As $x \to 0-$, $-\frac{1}{x} \to \infty$ and so $e^{\frac{1}{x}} = \frac{1}{e^{-\frac{1}{x}}} \to 0$.

So $\lim\limits_{x\to 0+} g$ does not exist while $\lim\limits_{x\to 0-} g = 0$.

<u>Theorem:</u> (Sequential criteria of limits) Let $f: R \to R$ and c be a real number. Then the following are equivalent:

(1) $\lim_{x\to c} f = L$,

(2) for every sequence (x_n) of real numbers that converges to c such that $x_n \neq c$ for all n, the sequence $(f(x_n))$ converges to L.

<u>Theorem:</u> (Divergence criteria) Let $f: R \to R$ and c be a real number.

$\lim_{x\to c} f \neq L$ iff there exists a sequence (x_n) of real numbers that converges to c such that $x_n \neq c$ for all n, $lim(x_n) = c$ but $lim(f(x_n)) \neq L$ (either $lim(f(x_n))$ does not exist or exists but not equal to L).

Alternatively, $\lim_{x\to c} f$ does not exist iff there exist sequences (x_n) and (y_n) of real numbers, $x_n, y_n \neq c$, for all n, $lim(x_n) = lim(y_n) = c$, but $\lim(f(x_n)) \neq \lim(f(x_n))$.

<u>Ex:</u> $\lim_{x\to 0} sin\left(\frac{1}{x}\right)$ does not exist in R.

» The sequences (x_n) and (y_n) both converge to 0 where $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{(2n+1)\frac{\pi}{2}}$ but $\left(sin\left(\frac{1}{x_n}\right)\right) = (sin\, n\pi)$ tends to 0 whereas $(\sin(y_n)) = (\sin(2n + 1)\frac{\pi}{2})$ converges to 1.

<u>Theorem:</u> (Limit Theorem of Functions) Let $f, g: R \to R$ and $b \in R$. If $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then $\lim_{x\to c}(f + g) = L + M$, $\lim_{x\to c}(f - g) = L - M$, $\lim_{x\to c}(fg) = LM$, $\lim_{x\to c}(b\, f) = bL$. If Let $h: R \to R$, $h(x) \neq 0$ for all real x belonging to $(c - \delta, c + \delta)$, for some $\delta > 0$, and if $\lim_{x\to c} h = H \neq 0$, then $\lim(\frac{f}{h}) = \frac{L}{H}$.

<u>Note:</u> If p be a polynomial, then $\lim_{x\to c} p = p(c)$ for any real c.

<u>Theorem:</u> (Squeeze Theorem) Let $f, g, h: R \to R$ and $c \in R$. If $f(x) \leq g(x) \leq h(x)$ for all real x, $x \neq c$, and if $\lim_{x\to c} f = L = \lim_{x\to c} h$, then $\lim_{x\to c} g = L$.

Some Extensions of Limit Concept

<u>Definition:</u> Let Let $f: R \to R$ and $c \in R$. f tends to ∞ as $x \to c$, written as $\lim_{x\to c} f = \infty$, iff for every real M, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$, $f(x) > M$ holds.

<u>Definition:</u> Let $f: R \to R$ and $c \in R$. f tends to $-\infty$ as $x \to c$, written as $\lim_{x\to c} f = -\infty$, iff for every real M, there exists $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta, f(x) < M$ holds.

<u>Ex:</u> $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

»Let $M > 0. \frac{1}{x^2} > M \Leftarrow x2 < \frac{1}{M} \Leftarrow -\frac{1}{\sqrt{M}} < x < \frac{1}{\sqrt{M}} \Leftarrow 0 < |x - 0| < \frac{1}{\sqrt{M}} = \delta$.
Hence the result.

<u>Definition:</u> Let $f: R \to R$. A real number L is a limit of f as $x \to \infty$, written as $\lim_{x\to\infty} f = L$, iff for any given $\epsilon > 0$, there exists K such that $x > K$ implies $|f(x) - L| < \epsilon$.

<u>Definition:</u> Let $f: R \to R$. f tends to ∞ as x tends to ∞, written as $\lim_{x\to\infty} f = \infty$, iff given any real M, there exists K such that $x > K$ implies $f(x) > M$.

E x a m p l e 4 . 8 Evaluate the following limits or show that they do not exist:
(1) $\lim_{x\to 1+} \frac{x}{x-1}$, (2) $\lim_{x\to 0+} \frac{x+2}{\sqrt{x}}$, (3) $\lim_{x\to\infty} \frac{\sqrt{x}-5}{\sqrt{x}+3}$ $(x > 0)$, (4) $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x}$ $(x > 0)$

»(1)Let $M > 1. \frac{x}{x-1} > M \Leftarrow x < \frac{M}{M-1} \Leftarrow 0 < x - 1 < \frac{1}{M-1} = \delta(> 0)$.

Hence $\lim_{x\to 1+} \frac{x}{x-1} = \infty$.

(2) Let $M > 0. \frac{x+2}{\sqrt{x}} > M \Leftarrow \frac{2}{\sqrt{x}} > M \Leftarrow \sqrt{x} < \frac{2}{M} \Leftarrow 0 < x - 0 < \frac{4}{M^2} = \delta$.

Hence $\lim_{x\to 0+} \frac{x+2}{\sqrt{x}} = \infty$.

(3) Let $\varepsilon > 0. \frac{\sqrt{x}-5}{\sqrt{x}+3} > M \Leftarrow \left|\frac{\sqrt{x}-5}{\sqrt{x}+3} - 1\right| < \varepsilon \Leftarrow \frac{8}{\sqrt{x}+3} < \varepsilon \Leftarrow \frac{8}{\sqrt{x}} < \varepsilon \Leftarrow x > \frac{64}{\epsilon^2} = M$.

Hence $\lim_{x\to\infty} \frac{\sqrt{x}-5}{\sqrt{x}+3} = 1$.

Continuous Functions

Definition: Let $f: R \to R$ and let c be a real number. f is continuous at c iff $\lim_{x\to c+} f$, $\lim_{x\to c-} f$ and $f(c)$ exists and $\lim_{x\to c+} f = \lim_{x\to c-} f = f(c)$. If f is continuous at every real c, f is continuous on R.

Theorem: Let $f, g: R \to R$ be continuous at $c \in R$ and k be a real constant. Then the functions $f+g,\ f-g,\ f.g,\ kf$ are continuous at c, where the functions are defined as follows:

$(f+g)(x) = f(x) + g(x)$,

$(f-g)(x) = f(x) - g(x)$,

$(f.g)(x) = f(x).g(x)$,

$(kf)(x) = kf(x)\ for\ x \in R$.

The function $\frac{f}{g}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ is continuous at c if $g(x) \neq 0$ for all x in $(c-\delta, c+\delta)$ for some $\delta > 0$.

Ex: Every constant function $f: R \to R, f(x) = c$ (c real constant) is continuous on R.

Ex: Every polynomial function $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ is continuous on R.

Definition: $f: R \to R$ has removable discontinuity at c iff $\lim_{x\to c+} f = \lim_{x\to c-} f = \lim_{x\to c} f$ exists but either $f(c)$ does not exist or exists but different from the limiting value $\lim_{x\to c} f$.

$f: R \to R$ has discontinuity of first kind at c iff both $\lim_{x\to c+} f$ and $\lim_{x\to c-} f$ exist but $\lim_{x\to c+} f \neq \lim_{x\to c-} f$.

$f: R \to R$ has discontinuity of second kind at c iff at least one of $\lim_{x\to c+} f$ and $\lim_{x\to c-} f$ does not exist.

Ex: Check the continuity of the following functions at the indicated points:

(1) $f(x) = [x] + [-x]\ at\ x = 0$, (2) $f(x) = \frac{x}{1+e^{1/x}}\ at\ x = 0$,

(3) $f(x) = 2 - x, 1 \leq x \leq 2;\ = x - \frac{x^2}{2}, x > 2\ at\ x = 2,$

(4) $f(x) = [x]\ at\ 0,$ (5) $f(x) = \frac{1}{x}, x \neq 0;\ f(0) = 0\ at\ 0.$

» (1) $\lim_{x\to 0-} f$=$\lim_{x\to 0-}(-1+0)$=-1=$\lim_{x\to 0+}(0+(-1))$=$\lim_{x\to 0+} f \neq 0 = f(0)$.

Hence f has removable discontinuity at 0.

(2) $x \to 0- \Rightarrow 1/x \to -\infty \Rightarrow e^{\frac{1}{x}} \to 0 \Rightarrow 1 + e^{\frac{1}{x}} \to 1 \Rightarrow x/(1 + e^{\frac{1}{x}}) \to 0,$
$x \to 0+ \Rightarrow 1/x \to \infty \Rightarrow e^{\frac{1}{x}} \to \infty \Rightarrow x/(1 + e^{\frac{1}{x}}) \to 0.$

Thus $\lim_{x\to 0+} f = \lim_{x\to 0-} f = f(0)$. Hence f is continuous at 0.

(3) $\lim_{x\to 2+} f = \lim\limits_{x\to 2+}\left(\mathrm{x} - \frac{x^2}{2}\right) = 0;$ $\lim_{x\to 2-} f = \lim\limits_{x\to 2-}(2 - \mathrm{x}) = 0$ and $f(2) = 0$. Hence f is continuous at $x = 2$.

(4) $\lim_{x \to 0+} f = 0 \neq -1 = \lim_{x \to 0-} f$; hence f has discontinuity of first kind at 0.

(5) $\lim_{x\to 0+} f$ does not exist as real number; hence f has discontinuity of second kind at 0.

Theorem: If $f: R \to R$ be continuous at c and $g: R \to R$ be continuous at $f(c)$, then $g0f$,defined by $(g0f)(x) = g(f(x))$ for real x, is continuous at c.

Ex: $f: R \to R$, $f(x) = sin\, x$ and $g : R \to R$, $g(x) = x^2$ are continuous on R; hence $fog: R \to R, (fog)(x) = sin(x^2)$ is continuous on R.

Theorem: If f is continuous at c and $f(c) \neq 0$, then there exists $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ for all x in $(c - \delta, c + \delta)$.

Theorem: (Bolano's Intermediate Value Property) Let $f: R \to R$ be continuous and a, b be real numbers, $a < b$, such that $f(a) \neq f(b)$. Let k be a real number between $f(a)$ and $f(b)$. Then there exists c in $[a, b]$ such that $f(c) = k$.

Note: Continuity is sufficient but not necessary for the conclusion in the above theorem to hold: for example, let us define

$f: R \to R, f(x) = x, if\ x \leq 1;$

$= x - 1\ , if\ 1 < x,$ satisfies IVP but is discontinuous at 1.

<u>Corollary:</u> If f(a) and f(b) are of opposite signs, then there exists c in (a,b) such that f(c)=0.

<u>Ex:</u> Prove that $\frac{5}{x-1} + \frac{7}{x-2} + \frac{16}{x-3} = 0$ has a solution between 1 and 2.

»Let f$(x) = 5(x-2)(x-3) + 7(x-1)(x-3) + 16(x-1)(x-2)$. f, being a polynomial function, is continuous at any real number and hence on [1,2]. Also $f(1) > 0$ and $f(2) < 0$. Thus there exists c in (1,2) such that $f(c) = 0$, that is $\frac{5}{c-1} + \frac{7}{c-2} + \frac{16}{c-3} = 0$ has a solution in (1,2).

<u>Ex:</u> Show that the equation $cos\ x = x$ has solution in $(0, \frac{\pi}{2})$.

»Let $f(x) = cos\ x - x\ , x \in [0, \frac{\pi}{2}]$. $f(0) > 0$ and $f(\frac{\pi}{2}) < 0$; also f is continuous on $[0, \frac{\pi}{2}]$. Hence there exists c in $(0, \frac{\pi}{2})$ such that $f\ (c) = 0;\ c$ is a solution of the given equation.

<u>Theorem:</u> Let $f: R \to R$ be continuous and a, b be real numbers, $a < b$. Then there exist c, d in $[a, b]$ such that $f(c) = max\{f(x): x \in [a, b]\}$ and $f(d) = min\{f(x): x \in [a, b]\}$.

<u>Note:</u> The result may not hold if we consider a discontinuous function or a non-closed interval.

Let $f: R \to R, f(x) = \frac{1}{x}$ for $x \neq 0;$

$= 1$ at $x = 0.$

So f is discontinuous at 0. There does not exist c in $[-1,1]$ such that

$f(c) = max\{f(x): x \in [-1,1]\}$. $(0,1) \to R, f(x) = x$ does not have either maximum or minimum on (0,1).

CHAPTER 5

DERIVATIVE OF A REAL VALUED FUNCTION OF A REAL VARIABLE

Definition: Let $f:[a,b] \to R$, $a<c<b$. The left (hand) derivative of f at c, denoted by $f^{(1)}(c-)$, is equal to $\lim_{x\to c-} \frac{f(x)-f(c)}{x-c}$, provided the limit exists. The right(hand) derivativeof f at c, denoted by $f^{(1)}(c+)$, is equal to $\lim_{x\to c+} \frac{f(x)-f(c)}{x-c}$, provided the limit exists.

Ex: Let $f: R \to R$, $f(x)=c$, c real constant. f is derivable at a iff $f^{(1)}(a)$ exists and in that case $f^{(1)}(5)=\lim_{x\to 5}\frac{f(x)-f(5)}{x-5} = \lim_{x\to 5}\frac{c-c}{x-5} = 0$. In general, derivative of a constant function defined on an interval at any point of its domain of definition is zero.

Ex: Let $f:[0,2] \to R$, $f(x) = x^2+x, 0 \le x < 1;$

$$= 2, \text{ at } \quad x=1;$$
$$= 2x^3 - x + 1, \text{ for } 1 < x \le 2.$$

» $f^{(1)}(1+) = lim_{x\to 1+}\frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1-}\frac{(2x^3-x+1)-2}{x-1} = 5$ while $f^{(1)}(1-) = \lim_{x\to 1-}\frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1-}\frac{x^2+x-2}{x-1} = 3$. Since $5 = f^{(1)}(1+) \neq f^{(1)}(1-) = 3$, f is not derivable at 1 . Since $f^{(1)}(2-) = \lim\limits_{x\to 2-}\frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2-}\frac{2x^3-x-14}{x-2} = \lim\limits_{x\to 2-}(2x^2+4x+7) = 23$, so f is differentiable at 2 and $f^{(1)}(2) = 23$.

Theorem: If $f:[a,b] \to R$ be differentiable at $c, a<c<b$, then f is continuous at c.

»$\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}(x-c)\right] = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$. $\lim_{x\to c}(x-c) = f^{(1)}(c) = 0$. Hence $\lim\limits_{x\to c} f(x) = f(c)$: thus f is continuous at c.

Note: Converse may not hold: Consider $f: [-1,1] \to R$, $f(x) = |x|$. f is continuous at 0 but not differentiable at 0 .

Ex: Let $f(x+y) = f(x) + f(y)$ for all real x, y. Let $f(5) = 2$ and $f^{(1)}(0) = 3$; prove that $f^{(1)}(5) = 6$.

» $2 = f(5) = f(5+0) = f(5) + f(0)$; hence $f(0) = 0$. Now $f^{(1)}(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h} = \lim_{h\to 0} \frac{f(5)+f(h)-f(5)}{h} = \frac{f(h)-f(0)}{h} = f^{(1)}(0) = 3$.

Definition: Let $f: [a,b] \to R$. f is (monotonically) increasing on $[a,b]$ iff $x_1, x_2 \in [a,b], x_1 < x_2$ imply $f(x_1) \le f(x_2)$. f is (monotonically) decreasing on $[a,b]$ iff $x_1, x_2 \in [a,b], x_1 < x_2$ imply $f(x_1) \ge f(x_2)$. f is monotonic iff f is either increasing on $[a,b]$ or decreasing on $[a,b]$. f is strictly increasing iff $x_1, x_2 \in [a,b], x_1 < x_2$ imply $f(x_1) \le f(x_2)$. Similarly strictly decreasing function is defined.

Theorem: Let $f: [a,b] \to R$ be a differentiable function. Then

(1) $f^{/}$ is nonnegative throughout $[a,b]$ iff f is monotonically increasing on $[a,b]$,

(2) $f^{/}$ is positive throughout $[a,b]$ implies f is strictly increasing on $[a,b]$.

Ex: Prove that $f(x) = \frac{a \sin x + b \cos x}{c \sin x + d \cos x}$ $(a, b, c, d$ are constants) is either monotonically increasing on R or monotonically decreasing on R.

» $f^{(1)}(x) = \frac{ad-bc}{(c \sin x + d \cos x)^2} \ge 0$ for all x if $ad - bc \ge 0$ and ≤ 0 for all x if $ad - bc \le 0$.

CHAPTER 6

SUCCESIVE DIFFERENTIATION

Definition: Let $f:[a,b] \to R$ be differentiable on $[a,b]$. Then $f^{(1)}:[a,b] \to R$ is a function . This function $f^{(1)}$ may again be differentiable at every point of $[a,b]$, the function $(f^{(1)})^{(1)}$ is denoted by $f^{(2)}$and called second order derivative of f, $f^{(2)}$, in its turn, may have derivative at every point of $[a,b]$, which is denoted by $f^{(3)}$, called third order derivative of f. The process may be continued.

Ex: Let $y = x^m, x > 0, m$ rational. Show that , $y^{(n)} = m(m-1) \dots (m-n+1)x^{m-n}$, for natural n .

» $y^{(1)} = mx^{m-1}$. Result holds for $n = 1$. Let $y^{(n)} = m(m-1) \dots (m-n+1)x^{m-n}$, for some natural n. Then $y^{(n+1)} = (y^{(n)})^{(1)} = m(m-1) \dots (m-n+1)(m-n)x^{m-n-1}$. Thus if the result holds for n , it holds for $n+1$. Thus , by mathematical induction , the result holds for all natural n .

Ex: Let $y = \frac{1}{ax+b}$. Then $y^{(n)} = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$.

Ex: Let $y = ln\, x$. Then $y^{(1)} = x^{-1}$. Hence $y^{(n)} = (y^{(1)})^{(n-1)} = (x^{-1})^{(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$.

Ex: Let $y = sin\, x$. To prove: $y^{(n)} = \sin(\frac{n\pi}{2} + x)$ (n natural).

$y^{(1)} = cos\, x = sin(\frac{\pi}{2} + x)$. Result holds for $n = 1$. Let $y^{(m)} = \sin(\frac{m\pi}{2} + x)$ for some natural m. Then $y^{(m+1)} = \cos\left(\frac{m\pi}{2} + x\right) = sin\left(\frac{\pi}{2} + \frac{m\pi}{2} + x\right) = sin(\frac{(m+1)\pi}{2} + x)$. Hence the result holds by induction.

Ex: Let $y = e^{ax} sin\, bx$.

Then $y(1) = e^{ax}(a\, sin\, bx + b\, cosbx)$.

Let $a = r\, cos\theta$, $b = r\, sin\, \theta, -\pi < \theta \le \pi$.

Then $r = \sqrt{a^2 + b^2}$ and θ satisfies $cos\,\theta = a/r$ and $sin\,\theta = b/r, -\pi < \theta \leq \pi$.

Thus $y^{(1)} = e^{ax} r sin(bx + \theta)$.

Hence $y^{(n)} = e^{ax} r^n \sin(bx + n\,\theta) = e^{ax}(a^2 + b^2)^{n/2} \sin(bx + n\,\theta)$.

Ex: Let $y = \frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$. Equating coefficients of like powers of x in the identity $1 = A(x+2) + B(x-2)$, we get $A = \frac{1}{4}$ $and\ B = -\frac{1}{4}$. Hence $y = \frac{1}{4}(\frac{1}{x-2} - \frac{1}{x+2})$. Thus $y^{(n)} = \frac{(-1)^n n!}{4}\left[\frac{1}{(x-2)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right]$.

Ex: Let $y = \frac{x}{x+1}. Prove\ y^{(5)}(0) = 5!$

» $y = 1 - \frac{1}{x+1}$. Thus $y^{(5)} =-\frac{(-1)^5 5!}{(x+1)^6}$. Hence $y^{(5)}(0) = 5!$

Theorem: (Leibnitz's Theorem) Let f, g be two functions possessing n th order derivatives, then fg is differentiable n times and

$$(fg)^{(n)} = \binom{n}{0} f^{(n)} g + \binom{n}{1} f^{(n-1)} g^{(1)} + \cdots + \binom{n}{r} f^{(n-r)} g^{(r)} + \cdots + \binom{n}{n} f g^{(n)}.$$

Proof: $(fg)^{(1)} = f^{(1)} g + f g^{(1)} = \binom{1}{0} f^{(1)} g + \binom{1}{1} f g^{(1)}$. Thus the result holds for $n = 1$.

Let $(fg)^{(m)} = \binom{m}{0} f^{(m)} g + \binom{m}{1} f^{(m-1)} g^{(1)} + \cdots + \binom{m}{r} f^{(m-r)} g^{(r)} + \cdots + \binom{m}{m} f g^{(m)}$.

Then $(fg)(m+1) = ((fg)^{(m)})^{(1)}$

$$= \left[\binom{m}{0} f^{(m)} g + \binom{m}{1} f^{(m-1)} g^{(1)} + \cdots + \binom{m}{r} f^{(m-r)} g^{(r)} + \cdots + \binom{m}{m} f g^{(m)}\right]^{(1)}$$

$$= [\binom{m}{0} f^{(m+1)} g + \binom{m}{0} f^{(m)} g^{(1)}] + \cdots [\binom{m}{r} f^{(m-r+1)} g^{(r)} + \binom{m}{r} f^{(m-r)} g^{(r+1)}]$$
$$+ \cdots + [\binom{m}{m} f^{(1)} g^{(m)} + \binom{m}{m} f g^{(m+1)}]$$

$= \binom{m+1}{0} f^{(m+1)} g + \binom{m+1}{1} f^{(m)} g^{(1)} + \cdots + \binom{m+1}{r} f^{(m-r+1)} g^{(r)} + \cdots + \binom{m+1}{m+1} f g^{(m+1)}$, since $\binom{m}{r-1} + \binom{m}{r} = \binom{m+1}{r}$) holds. Thus result holds by mathematical induction.

Ex: Let $y = sin(m\, sin^{-1}x)$.

Prove that (1) $(1-x^2)y^{(2)} - xy^{(1)} + m^2y = 0$,

$$(2)\ (1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2-m^2)y^{(n)} = 0.$$

» $y^{(1)} = cos(m\, sin^{-1}x)..\frac{m}{\sqrt{1-x^2}}$.

Squaring and cross multiplying, $(1-x^2)(y^{(1)})^2 = m^2cos^2(m\, sin^{-1}x) = m^2(1-y^2)$ (from given expression).

Hence $2y^{(1)}y^{(2)}(1-x^2) - 2x(y^{(1)})^2 = m^2(-2y)y^{(1)})$.

Since for the given expression $y^{(1)}$ is not identically zero and the relation is to hold for all x, cancelling $2y^{(1)}$ from both sides, we obtain $(1-x^2)y^{(2)} - xy^{(1)} + m^2y = 0$.

Next applying Leibnitz's Theorem (remembering that n th order derivative of sum and difference of a finite number of functions is sum or difference of their n th order derivatives), we get

$[(1-x^2)(y^{(2)})^{(n)} + \binom{n}{1}(y^{(2)})^{(n-1)}(-2x) + \binom{n}{2}(y^{(2)})^{(n-2)}(-2)] - [x(y^{(1)})^{(n)} + \binom{n}{1}(y^{(1)})^{(n-1)} + m^2y^{(1)} = 0$; simplifying we obtain the result.

CHAPTER 7

MEAN VALUE THEOREMS

Theorem: (Rolle's Theorem) If $f: [a,b] \to R$ is continuous on $[a,b]$ and differentiable on (a,b) and if $f(a) = f(b)$, then there is $c \in (a,b)$ such that $f'(c) = 0$.

Ex: If $f(x) = x^3 + px + q$ for $x \in R$, where $p, q \in R$ and $p > 0$, then f has a unique real root. To see this, note that if f had more than one real root, then there would be $a, b \in R$ with $a < b$ and $f(a) = f(b) = 0$. Hence , by Rolle's Theorem, there would be $c \in (a,b)$ such that $f^/(c) = 0$. But $f^/(x) = 3x^2 + p$ is not zero for any $x \in R$ since $p > 0$. On the other hand, $\lim_{x\to-\infty} f = -\infty$ and $\lim_{x\to\infty} f = \infty$.Thus f takes on negative as well as positive values. Hence $f(c) = 0$ for some $c \in R$, since f has the IVP on R. Thus f has a unique real root.

Ex: Consider f:[0,1] $\rightarrow$ R defined by f(x)=x, x $\in$ [0,1) and f(1)=0. Then f is differentiable on (0,1) and f(0)=f(1)=0 but f is not continuous on [0,1]. Note that f'(c)=1 $\neq$ 0 for 0<c<1.

Ex: Consider f:[-1,1] $\rightarrow$R defined by f(x)=|x|. Then f is continuous on [-1,1] and f(-1)=f(1)=1. f '(x)=1 for x>0, =-1, for x<0 and f '(0) does not exist. There does not exist c, -1<c<1, such that f '(0)=0.

Ex: Consider f:[0,1] $\rightarrow$ R defined by f(x)=x for x $\in$ [0,1). Then f is continuous on [0,1]and differentiable on (0,1) but f '(c)=1 $\neq$ 0 for every c $\in$ (0,1). Note that f(0) $\neq$ f(1).

Theorem: (Lagrange's Mean Value Theorem) If f:[a,b] $\rightarrow$R is continuous on [a,b] and differentiable on (a,b), then there exists c $\in$ (a,b) such that f(b)-f(a)=(b-a)f '(c).

Proof: Consider F:[a,b] $\rightarrow$ R defined by F(x)=f(x)-f(a)-s(x-a), where s = $\frac{f(b)-f(a)}{b-a}$. Then F(a)=0 and by our choice of constant s, F(b)=0. So Rolle's Theorem applies to F and, as a result, there exists c $\in$ (a,b) such that F/(c)=0, that is, f '(c)=s, as desired.

Note: If we write b=a+h, then the conclusion of MVT may be stated as follows: f(a+h)=f(a)+hf '(a+θh) for some θ$\in$(0,1).

Corollary: (Mean Value Inequality) If f:[a,b] $\rightarrow$ R is continuous on [a,b] and differentiable on (a,b),and if m,M$\in$R are such that m $\leq$ f '(x) $\leq$ M for all x $\in$(a,b), then m(b-a) $\leq$ f(b)-f(a) $\leq$ M(b-a).

Corollary: Let I be an interval containing more than one point and f: I $\rightarrow$ R be any function. Then f is a constant function on I if and only if f/ exists and is identically zero on I.

Proof: If f is a constant function on I, then it is obvious that $f^{/}$exists on I and $f^{/}(x) = 0$ for all $x \in I$.

Conversely, if $f^{/}$ exists and vanishes identically on I, then for any x_1 , x_2 $\in$I with $x_1 < x_2$, we have $[x_1 , x_2] \subseteq$ I and applying MVT to f on $[x_1 , x_2]$, we obtain $[f(x_2)-f(x_1)]=(x_2 - x_1)f^{/}(c)$ (for some $c \in (x_1, x_2) \subseteq I$) $= 0$ and hence $f(x_1) = f(x_2)$.

Note: The MVT or the Mean Value Inequality may be used to approximate a differentiable function around a point. For example, if m is natural and $f(x) = \sqrt{x}$ for $x \in [m, m+1]$, then $\sqrt{m+1} - \sqrt{m} = f(m+1) - f(m) = f^{/}(c) = \frac{1}{2\sqrt{c}}$ for some $c \in (m, m+1)$. Hence $\frac{1}{2\sqrt{m+1}} < \sqrt{m+1} - \sqrt{m} < \frac{1}{2\sqrt{m}}$. For example, by putting $m = 1$, $\frac{4}{3} = 1 + \frac{1}{3} < 1 + \frac{1}{2\sqrt{2}} < \sqrt{2} < 1 + \frac{1}{2} = \frac{3}{2}$.

Corollary: Let I be an interval containing more than one point and f: I $\to$ R be a differentiable function. Then (1) f ' is nonnegative throughout I iff f is monotonically increasing on I, (2) f ' is positive throughout I implies f is strictly increasing .

Proof: Let $x_1 , x_2 \in I$ with $x_1 < x_2$. Then $[x_1 , x_2] \subseteq I$ and we can apply MVT to the restriction of f to $[x_1 , x_2]$ to obtain $[f(x_2)-f(x_1)]=(x_2 - x_1)f'(c)$ for some $c \in (x_1, x_2)$. Thus $f(x_2) \geq f(x_1)$ iff $f'(c) \geq 0$.

Note: $f: R \to R$, $f(x) = x^3$ is strictly increasing on $[-1,1]$ but $f^{/}(0) = 0$.

Theorem: (Cauchy's MVT) Let f,g: [a,b] $\to$ R are continuous on [a,b] and differentiable on (a,b), then there is c $\in$ (a,b) such that

$g^{(1)}(c)(f(b)-f(a))= f^{(1)}(c)(g(b)-g(a))$

Proof: Consider the function F:[a,b] $\to$ R defined by F(x)= f(x)-f(a) - s[g(x)-g(a)], where $s = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Theorem: (L'Hospital's Rule for $\frac{0}{0}$ Indeterminate Form) Let $c \in R$ and $f, g: (c - r, c) \to R$ be differentiable function such that $\lim_{x \to c-} f = 0$ and $\lim_{x \to c-} g = 0$.

Suppose $g^{(1)}(x) \neq 0$ for all $x \in (c-r, c)$ and $\lim_{x\to c-} \frac{f^{(1)}(x)}{g^{(1)}(x)}$=L. Then $\lim_{x\to c-} \frac{f(x)}{g(x)} = L$.

Here L can be a real number or $\infty\ or - \infty$.

<u>Note:</u> L'Hospital's Rule for $\frac{0}{0}$ Indeterminate Form is also valid for right (hand) limits. The statement is similar to that given above. Combining the two cases follows L'Hospital's Rule for(two-sided)limits of $\frac{0}{0}$ Indeterminate Form, which we may state as follows:

<u>Theorem:</u> Let $c \in R$ and $D = (c-r, c) \cup (c, c+r)$ for some $r > 0$. Let $f, g: D \to R$ be differentiable functions such that $\lim_{x\to c} f = 0$and $\lim_{x\to c} g = 0$. Suppose $g^{(1)}(x) \neq 0$ for all $x \in D$ and $\lim_{x\to c} \frac{f^{(1)}(x)}{g^{(1)}(x)} = L$. Then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$. Here L can be a real number or $\infty\ or - \infty$.

L'Hospital's Rule for $\frac{0}{0}$ Indeterminate Form are also valid if instead of considering limits as $x \to c$, where c is a real number, we consider limits as $x \to \infty$ or as $x \to -\infty$. For example, a statement for limits as $x \to -\infty$ would be as follows:

<u>Theorem:</u> Let a be real number and $f, g: (-\infty, a) \to R$ be differentiable functions such that $\lim_{x\to -\infty} f = 0$ and $\lim_{x\to -\infty} g = 0$. Suppose $g^{(1)}(x) \neq 0$ for all x in $(-\infty, a)$ and $\lim_{x\to -\infty} \frac{f^{(1)}(x)}{g^{(1)}(x)} = L$. Then $\lim_{x\to -\infty} \frac{f(x)}{g(x)} = L$. Here L can be a real number or ∞ or $-\infty$.

<u>Theorem:</u> (L'Hospital's Rule for $\frac{\infty}{\infty}$ Indeterminate Form) Let I be an interval $[a, c)$ where a is real and either c is real with $a < c$ or $c = \infty$. Let $f, g: I \to R$ be differentiable functions such that $\lim_{x\to c-} |g(x)| = \infty$. Suppose $g^{(1)}(x) \neq 0$ for all $x \in I$ and $\lim_{x\to c-} \frac{f^{(1)}(x)}{g^{(1)}(x)} = L$. Then $\lim_{x\to c-} \frac{f(x)}{g(x)} = L$. Here L can be a real number or ∞ or $-\infty$.

<u>Theorem:</u> (Taylor's Theorem) Let n be a nonnegative integer and $f:[a,b] \rightarrow R$ be such that $f^{(1)}, f^{(2)}, \dots \dots, f^{(n)}$ exist on $[a,b]$ and further $f^{(n)}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f^{(1)}(a) + \cdots + (b-a)^n \frac{f^{(n)}(a)}{n!} + (b-a)^{n+1} \frac{f^{(n+1)}(c)}{(n+1)!}.$$

Proof: For $x \in [a,b]$, let $P(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$. Consider $F:[a,b] \rightarrow R$ defined by $F(x) = f(x) - P(x) - s(x-a)^{n+1}$, where $s = \frac{f(b)-P(b)}{(b-a)^{n+1}}$. Then $F(a) = 0$ and our choice of s is such that $F(b) = 0$. So Rolle's Theorem is applicable to F on $[a,b]$ and, as a result, there is $c_1 \in (a,b)$ such that $f^{|}(c_1) = 0$. Next , $f^{(1)}(a) = P^{(1)}(a)$ and so $f^{(1)}(a) = 0$ as well. Now Rolle's Theorem applies to the restriction of $F^{(1)}$ to $[a, c_1]$, and so there is $c_2 \in (a, c_1)$ such that $F^{(2)}(c_2) = 0$. Further , if $n > 1$, then $F^{(2)}(a) = 0$ and so there exists $c_3 \in (a, c_2)$ such that $F^{(3)}(c_3) = 0$. Continuing this way, we see that there is $c = c_{n+1} \in (a, c_n)$ such that $F^{(n+1)}(c) = 0$. Now $P^{(n+1)}$ is identically zero, since P is a polynomial of degree n. In particular, $P^{(n+1)}(c) = 0$. Hence $f^{(n+1)}(c) = s(n+1)!$, which, in turn, yields desired result.

Note: (1)Note that the Lagrange's MVT corresponds to the case n=0 of Taylor's Theorem.

(2) In statement of Taylor's Theorem , the point a was the left end point of the interval on which f was defined. There is an analogous version for right endpoint: if f is as in the statement of Taylor's Theorem, then there exists $c \in (a,b)$ such that $f(a) = f(b) + f^{(1)}(b)(a-b) + \cdots + \frac{f^{(n)}(b)}{n!}(a-b)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(a-b)^{n+1}$. Similarly, it can be proved that if I is any interval, $a \in I$,and $f: I \rightarrow R$ is such that $f^{(1)}, f^{(2)}, \dots \dots, f^{(n)}$ exist on I and $f^{(n+1)}$exists at every interior point of I, then for any $x \in I, x \neq a$, there is c between a and x such that

$$f(x) = f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

The last expression is sometimes referred to as the Taylor formula for f around a. The polynomial given by $P_n(x) = f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the n th Taylor polynomial of f around a.The difference $R_n(x) = f - P_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ is called Lagrange form of remainder after n terms.

(3) Usually, the n^{th} Taylor polynomial of f around a provides a progressively better approximation to f around a as n increases. For example, let us revisit the estimate of $\sqrt{2}$ obtained from Lagrange's MVT and see what happens when we use Taylor's Theorem.

Thus let m be natural and $f: [m, m+1] \to R$, $f(x) = \sqrt{x}$. Applying Taylor formula for f around m, with $n = 1$, we have $f(x) = f(m) + f^{(1)}(m)(x - m) + \frac{f^{(2)}(c)}{2!}(x-m)^2$ for some c between m and x.

In particular, for $x = m+1$, we get $\sqrt{m+1} = \sqrt{m} + \frac{1}{2\sqrt{m}} - \frac{1}{8c\sqrt{c}}$ for some $c \in (m, m+1)$.

For example, by putting $m = 1$, we obtain $1 + \frac{1}{2} - \frac{1}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}}$ and hence

$\frac{11}{8} < \sqrt{2} < 1 + \frac{1}{2} - \frac{1}{16\sqrt{2}} = 35/24$, where in the last inequality we have used the estimate $\sqrt{2} < 3/2$. The resulting bounds $11/8$ and $35/24$ are, in fact, better than the bounds $4/3$ and $3/2$ obtained using MVT.

Taylor's Infinite Series: Let f be a function possessing derivatives of all orders in $[a, a+h]$ for some $h > 0$. The n th Taylor Polynomial around a for different positive integers n are given by

$$f(a+h) = f(a) + hf^{(1)}(a + c_1 h) = f(a) + R_1,$$

$$f(a+h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!}f^{(2)}(a + c_2 h) = f(a) + hf^{(1)}(a) + R_2,$$

$f(a+h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \frac{h^3}{3!} f^{(3)}(a + c_3 h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + R_3$, ………………,

$f(a+h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + c_n h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$, ... where $c_1, c_2, \dots, c_n$, ... are in general different real numbers lying strictly between 0 and 1.

Let us denote $f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$ by S_n. Thus $f(a+h) = S_n + R_n$, where (S_n) is the sequence of partial sums corresponding the infinite series of real numbers
$f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \cdots$. If $Lim(R_n) = 0$, then $Lim(S_n) = f(a+h)$ and we can write

$$f(a+h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!} f^{(2)}(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \cdots$$

Note: Let f be a function possessing derivatives of all orders in $[0, h]$ for some $h > 0$ and let $lim\ (\frac{h^n}{n!} f^{(n)}(\ c_n h)) = 0$. Then the Taylor's infinite series corresponding to f about 0 is :

$f(h) = f(0) + hf^{(1)}(0) + \frac{h^2}{2!} f^{(2)}(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \cdots$ is called Maclaurin's infinite series.

<u>Ex:</u> (The Exponential Series) $f: R \to R$, $f(t) = e^t$, possesses derivatives of every order in $[0, x]$ for an arbitrarily fixedx and $R_n = \frac{x^n}{n!} f^{(n)}(c_n x) \ \to\ 0$ as $n \to \infty$ since $lim\left(\frac{x^n}{n!}\right) = 0\ and\ e^{c_n x}$ is bounded for all positive integer n (since $0 < c_n < 1$ implies $1 < e^{c_n x} < e^x$). Thus e^x can be expanded in Maclaurin's infinite series and

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \cdots.$$

Ex: (The sine and cosine series) Let $f(x) = sin\, x$, then $f^{(n)}(x) = sin\left(\frac{n\pi}{2} + x\right)$ for all natural number n. Thus f possesses derivatives of all orders in $[0, x]$. Also $R_n = \frac{x^n}{n!} sin\left(\frac{n\pi}{2} + c_n x\right) \to 0$ as $n \to \infty$ since $lim\left(\frac{x^n}{n!}\right) = 0$ and $\left|\sin\left(\frac{n\pi}{2} + c_n x\right)\right| \leq 1$ for all n. Thus $sin\, x$ can be expanded in Maclaurin's infinite series $sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

CHAPTER 8

FUNCTIONS OF SEVERAL REAL VARIABLES

Let $f: R^2 \to R$ be a real-valuedfunction of two independent real variables: we shall often write $z = f(x, y)$ where x, y are independent real variables and z is the dependent real variable. Just as we represent function of a single real variable by a planar curve in two-dimensional plane, similarly a real-valued function of two independent real variables is represented by a surface in the three-dimensional space.

Definition: Let $f: R^2 \to R$ and $(a, b) \in R^2$. A real number L is limit of f as $(x, y) \to (a, b)$, written as $\lim_{(x,y)\to(a,b)} f = L$, if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ implies $|f(x, y) - L| < \epsilon$.

Ex: $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0.$

»Let $\epsilon > 0$ be given.

$\left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| < \epsilon \Leftarrow |x||y|\left|\frac{x^2-y^2}{x^2+y^2}\right| < \epsilon \Leftarrow |x||y| < \epsilon$ (since $\left|\frac{x^2-y^2}{x^2+y^2}\right| \leq 1$ for real x, y)

$\Leftarrow |x| < \sqrt{\epsilon}, |y| < \sqrt{\epsilon} \Leftarrow x^2 + y^2 < \epsilon^2$(since $x^2 \geq 0$,$y^2 \geq 0$) $\Leftarrow 0 < \sqrt{(x-0)^2 + (y-0)^2} < \epsilon = \delta$ (say). Hence the result.

Definition: Let $f: R^2 \to R$ and $(a,b) \in R^2$. f is continuous at (a,b) iff
$$\lim_{(x,y)\to(a,b)} f = f(a,b).$$

Note: f is continuous at (a,b) iff for any two sequences $(x_n) \to a$ and $(y_n) \to b$, the sequence $f(x_n, y_n)$ should converge to $f(a,b)$.

Ex: Let $f: R^2 \to R$, $f(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

$\lim_{(x,y)\to(0,0)} f$ does not exist since for sequences $\left(\frac{1}{n}\right)$ and $\left(\frac{1}{n}\right)$ (both of which tend to 0 as $n \to \infty$), $f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \to \frac{1}{2}$ as $n \to \infty$; whereas for the sequences $\left(\frac{1}{n}\right)$ and $\left(-\frac{1}{n}\right)$(both of which tend to 0 as $n \to \infty$), $f\left(\frac{1}{n}, \frac{-1}{n}\right) = -\frac{1}{2} \to -\frac{1}{2}$ as $n \to \infty$. Hence f is not continuous at (0,0).

Definition: Let $f: R^2 \to R$ and $(a,b) \in R^2$. The partial derivative of f with respect to first independent variable x at (a,b), denoted by $f_x(a,b)$, is defined by $\lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$, provided the limit exists. Similarly the partial derivative of f with respect to second independent variable y at (a,b), denoted by $f_y(a,b)$, is defined by $\lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$, provided the limit exists.

Ex: Find $f_x(2,1)$ and $f_y(2,1)$ for $(x,y) = \frac{x+y-1}{x+y+1}$, if they exist.

»$f_x(2,1) = \lim_{h\to 0} \frac{1}{2(4+h)} = \frac{1}{8}$.

Definition: The partial derivatives $f_x(x,y)$ and $f_y(x,y)$of the function $f(x,y)$ are , in turn, functions of x and y. Thus, if the associated limits exist, we may define partial derivatives of higher order for f s follows :

$f_{xx}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b)-f_x(a,b)}{h}$,

$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b)-f_y(a,b)}{h}$, $f_{yx}(a,b) = lim_{h\to 0} \frac{f_x(a,b+h)-f_x(a,b)}{h}$,

$f_{yy}(a,b) = lim_{h\to 0} \frac{f_y(a,b+h)-f_y(a,b)}{h}$.

Here $\quad f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$, $f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$.

Ex: If $u = ln(x^3 + y^3 + z^3 - 3xyz)$, prove that

(1) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$, $\qquad$ (2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3/(x+y+z)^2$

» $x^3 + y^3 + z^3 - 3xyz = = (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.

Thus $u = ln(x+y+z) + ln(x+\omega y+\omega^2 z) + ln(x+\omega^2 y+\omega z)$, where ω is an imaginary cube roots of unity. Now

$\frac{\partial u}{\partial x} = \frac{1}{x+y+z} + \frac{1}{x+\omega y+\omega^2 z} + \frac{1}{x+\omega^2 y+\omega z}$,

$\frac{\partial u}{\partial y} = \frac{1}{x+y+z} + \frac{\omega}{x+\omega y+\omega^2 z} + \frac{\omega^2}{x+\omega^2 y+\omega z}, \frac{\partial u}{\partial z} = \frac{1}{x+y+z} + \frac{\omega^2}{x+\omega y+\omega^2 z} + \frac{\omega}{x+\omega^2 y+\omega z}$.

Thus $\quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$, since $1+\omega+\omega^2 = 0$. Similarly the other part cn be proved .

Homogeneous functions : Euler's Theorem

Definition: $f: R^2 \to R$ is homogeneous of degree n iff $(tx,ty) = t^n f(x,y)$ for all x, y and for every positive t .

$f(x,y) = \frac{xy}{x^2+y^2}$ is homogeneous of degree 0; $f(x,y) = xy + x$ is not homogeneous.

Theorem: (Euler's Theorem for two independent variables) If f is a homogeneous function of degree n having continuous partial derivatives, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

Ex: If $u = tan^{-1}\frac{x^3+y^3}{x-y}$, prove that

(1) $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin\, 2u$,

(2) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(1-4sin2u)sin\,2u.$

» u is not homogeneous but $v=tan\,u=\frac{x^3+y^3}{x-y}$ is homogeneous of degree 2 and partial derivatives $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist and are continuous at all points except at (0,0); hence we can apply Euler's Theorem on v at all points other than $(0,0)$. We get $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=2v$; or, $x\frac{dv}{du}\frac{\partial u}{\partial x}+y\frac{dv}{du}\frac{\partial u}{\partial y}=2sin\,u\,cos\,u,$ or, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2tanu}{sec^2u}=sin\,2u.$

Differentiating partially w.r.t. x and y respectively, $\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=2cos2u\frac{\partial u}{\partial x}$, $x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=2cos2u\frac{\partial u}{\partial y}$. Multiplying the equations by x and y respectively and adding,

$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\;=\;(2\,cos\,2u-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\;=$
$sin2u(2\,cos\,2u-1)$

$=\;(1-4sin2u)sin\,2u.$

<u>Ex:</u> If $u=xf\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$ and f,g have continuous partial derivatives, prove that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$

» $xf\left(\frac{y}{x}\right)$ and $g\left(\frac{y}{x}\right)$ are homogeneous functions of degree 1 and 0 respectively and the functions have continuous partial derivatives; hence $\frac{\partial[\mathrm{xf}\left(\frac{y}{x}\right)]}{\partial x}+\frac{\partial[\mathrm{xf}\left(\frac{y}{x}\right)]}{\partial y}=xf\left(\frac{y}{x}\right)$ and $\frac{\partial[\mathrm{g}\left(\frac{y}{x}\right)]}{\partial x}+\frac{\partial[\mathrm{g}\left(\frac{y}{x}\right)]}{\partial y}=0$. From first relation, $\mathrm{f}\left(\frac{y}{x}\right)+xf^{(1)}\left(\frac{y}{x}\right)((-y)/x^2)+xf^{(1)}\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)=xf\left(\frac{y}{x}\right)$, $g^{(1)}\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)+g^{(1)}\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)=0$. Differentiating the relations w.r.t x and y respectively and multiplying by x and y and adding , we obtain required relation.

Definition: Let $f: R^2 \to R$ possess partial derivatives f_x , f_y at (a, b). f is differentiable at (a, b) iff for all h and k, $f(a+h, b+k) - f(a, b) = fx(a, b)h + fy(a, b)k + A(h, k)h + B(h, k)k$, where $\lim_{(h,k)\to(0,0)} A(h, k) = 0 = \lim_{(h,k)\to(0,0)} B(h, k)$. $df(a, b) = fx(a, b)h + fy(a, b)k$ is called total differential of f.

CHAPTER 9

APPLICATIONS OF DIFFERENTIAL CALCULUS

TANGENTS AND NORMALS TO A PLANE CURVE

Equation of the tangent to a curve in Cartesian form

The equation of tangent to a planar curve whose equation is given in Cartesian explicit form $y = f(x)$ at a point $P : (a, b)$ on the curve is given by: $y - b = \left.\frac{dy}{dx}\right|_{(a,b)} (x - a)$.

The equation of normal to $y = f(x)$ at a point $P : (a, b)$ (that is, a line through (a, b) perpendicular to the tangent at P) is given by: $y - b = -\frac{1}{\left.\frac{dy}{dx}\right|_{(a,b)}}(x - a)$

(In both cases, we assume that the line is not parallel to the y-axis; otherwise, the equation is $y = b$.)

If the equation of the curve is in the implicit form $f(x, y) = 0$, we calculate $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

If the equation of the curve is in the parametric form $x = f(t), y = g(t)$, then $\frac{dy}{dx} = -\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Let T and G be the points of intersection of the tangent PT drawn at P with the x-axis and N be the foot of the perpendicular from P on the x-axis. Then subtangent at P is $NT = \frac{y}{\frac{dy}{dx}}$ and subnormal at P is $NG = y\frac{dy}{dx}$ (NT and NG are the signed distance from N to T and G and may be positive or negative). Length of the tangent PT is the absolute length PT intercepted on the tangent line by the curve and the x-axis and is given by $\frac{y\sqrt{1+y_1^2}}{y_1}$, where $y_1 = \frac{dy}{dx}$.

Angle between two straight lines

If both lines are parallel to y-axis, angle between the lines is 0.

If one of the line is parallel to the y-axis while the other one has angle of inclination $\theta \neq \frac{\pi}{2}$, then the angle between the lines is $\left|\frac{\pi}{2} - \theta\right|$.

If none of the lines is parallel to the y-axis, then the angle between them is tan-1$\left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$.

Angle between two curves at their point(s) of intersection

Angle of intersection of two curves is the angle between the tangents drawn to the two curves at their point of intersection.

Rule of finding equations of tangents to a rational algebraic curve at the origin:

A rational algebraic curve of n th degree is a curve whose equation is of the form

$(a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + \cdots + (a_nx^n + b_nx^{n-1}y + \cdots + k_ny^n) = 0$, where at least one of $a_n, b_n, \dots\dots, k_n$, is nonzero.

The equations of tangent(s) to a rational algebraic curve at (0,0) are obtained by equating to zero the terms of the lowest degree in the equation.

<u>Ex:</u> Show that at any point of the curve $x^{m+n} = k^{m-n}y^{2n}$, the m th power of the subtangent varies as the n th power of the subnormal.

»$(subtangent)^m = \left(\frac{2n}{m+n}\right)^m x^m$ and $(subnormal)^n = \left(\frac{m+n}{2n}\right)^n \frac{x^m}{a^{m+n}}$ (using given equation of the curve).Hence the result.

<u>Ex:</u> Prove that the curves $ax^2 + by^2 = 1$ and $Ax^2 + By^2 = 1$ intersect orthogonally if $\frac{1}{a} - \frac{1}{b} = \frac{1}{A} - \frac{1}{B}$.

»Let (h,k) be a point of intersection of the two curves. Then $x^2+by^2=Ax^2+By^2=1$, whence $\frac{k^2}{h^2}=\frac{a-A}{B-b}$. Also , for the first curve, $\frac{dy}{dx}(h,k)=-\frac{ah}{bk}$ and for the second curve, $dy/dx(h,k)=-Ah/Bk$. Product of slopes of tangents is $\frac{aAh^2}{bBk^2}=\frac{aA}{bB}.\frac{B-b}{a-A}=-1$. Hence the result.

<u>Ex:</u> Show that the points of the curve $y^2=4a(x+asin\frac{x}{a})$ where the tangents are parallel to the x-axis lie on the curve $y^2=4ax$.

»Let (h,k) be a point on the curve $y^2=4a(x+asin\frac{x}{a})$ where the tangents are parallel to the x-axis. Thus $k^2=4a(h+a\ sin\ \frac{h}{a})$. Since tangent to the curve at (h,k) is parallel to x-axis, $\frac{dy}{dx}_{(h,k)}=0$, that is, $sin\frac{h}{a}=0$; hence $k^2=4ah$. Thus (h,k) lies on $y^2=4ax$.

<u>Ex:</u> In the curve $x^m y^n=a^{m+n}$, prove that the portion of the tangent intercepted between the axes is divided at the point of contact into two segments which are in a constant ratio.

» Let the tangent at (h,k) to the curve meet the co-ordinate axes at A and B respectively. From the equation of the curve , $m\ lnx+n\ lny=(m+n)\ ln\ a$. Differentiating , $\frac{dy}{dx}_{(h,k)}=-\frac{mk}{nh}$. The equation of tangent at (h,k) is $y-k=-\frac{mk}{nh}(x-h)$, that is, $\frac{mx}{(m+n)h}+\frac{ny}{(m+n)k}=1$.

Thus A: $\left(\frac{(m+n)h}{m},0\right)$ and B: $\left(0,\frac{(m+n)k}{n}\right)$. It can be easily seen that $AP:PB::n:m$ (ratio independent of (h,k) .

<u>Equation of the tangent to a curve in Polar form</u>

If the equation of the curve is in polar form $r=f(\theta)$, θ is the vectorial angle of a point (r,θ) on the curve and φ is the angle between the radius vector and the tangent vector and if p is the length of the perpendicular dropped from the pole to the tangent at (r,θ), then tan $\varphi=\frac{r}{\frac{dr}{d\theta}}$, $\frac{dr}{d\theta}$ is to be calculated from the equation $r=f(\theta)$ of the curve and evaluated at (r,θ)), $p=r\ sin\ \varphi$.

Angle of intersection between two curves whose polar equations are given

Let the two curves whose equations are $r = f(\theta)$ and $r = g(\theta)$ intersect at P: (r, θ) and let $\varphi 1$ and $\varphi 2$ be the angles made by the tangent vector at P with the radii vector at P of $r = f(\theta)$ and $r = g(\theta)$ respectively. Then $tan\varphi_1 = r/f^{(1)}(\theta)$ and $tan\varphi_2 = r/g^{(1)}(\theta)$ at P. Thus $tan(\varphi_1 - \varphi_2) = \frac{tan\varphi_1 - tan\varphi_2}{1+tan\varphi_1 tan\varphi_2}$ gives the angle of intersection $|\varphi_1 - \varphi_2|$ of the two curves at P.

If p be the length of the perpendicular dropped from the pole onto the tangent to the curve $r = f(\theta)$ at P: (r, θ), then we have the result $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

<u>Ex:</u> Prove that the normal at any point (r, θ) on the curve $r^n = a^n cos\, n\, \theta$ makes an angle $(n + 1)\, \theta$ with the initial line.

» Given equation of the curve is $n\, ln\, r = n\, ln\, a + \, lncos\, n\theta$. Thus $\frac{n}{r}\frac{dr}{d\theta} = -n \tan n\theta$ which gives $tan\, \varphi = -\, cot\, n\, \theta$. Thus $\varphi = \frac{\pi}{2} + n\theta$. Let ψ be the angle that the tangent makes with initial line. Then $\psi - \frac{\pi}{2}$ is the angle made by the normal with the initial line. Now $\psi = \theta + \, \varphi = (n + 1)\, \theta + \frac{\pi}{2}$. Thus $\psi - \frac{\pi}{2} = (n + 1)\, \theta$ is the angle made by the normal with the initial line.

<u>Ex:</u> Show that the curves $r^n = a^n cos\, n\, \theta$ and $r^n = b^n cos\, n\, \theta$ intersect orthogonally.

»Taking natural logarithms and differentiating w.r.t. θ, for the first curve $\frac{n}{r}\frac{dr}{d\theta} = -n \tan n\theta$, which gives $tan\, \varphi_1 = -cot\, n\theta$, that is, $\varphi_2 = \pi/2 + n\theta$. Similarly, for the second curve, $\varphi_2 = n\theta$. Thus the angle between the curves is $|\varphi_1 - \varphi_2| = \frac{\pi}{2}$.

Pedal equation from Cartesian and Polar Equation

A relation between p, the length of the perpendicular from a given point O to the tangent at any point P on a curve and r, the distance of P from O is called pedal equation of the curve w.r.t. O. When nothing is mentioned, O is to be taken as the origin or pole according as the equation is Cartesian or polar.

(A) Pedal Equation from Cartesian

Let $f(x,y)=0$ …………..(1) be the Cartesian equation of the curve.
The equation of tangent at (x,y) being $f_xX+f_yY-\left(xf_x+yf_y\right)=0$,
we have $p^2=\frac{(xf_x+yf_y)^2}{{f_x}^2+{f_y}^2}$ ……………(2)
Also $r^2=x^2+y^2$ …………..(3). E
Eliminating x,y from (1),(2) and (3), we obtain the pedal equation.

(B) Pedal Equation from polar

Let $f(r,\theta)=0$ be the polar equation. We have $tan\,\varphi=\frac{r}{\frac{dr}{d\theta}}$ and $p=r\,sin\,\varphi$. We obtain pedal equation from these equations after eliminating θ and φ.

<u>Ex:</u> Find the pedal equation of the asteroid $x^{2/3}+y^{2/3}=a^{2/3}, a>0$.

»The equation of tangent at a point (x_1,y_1) to the curve is $x{x_1}^{-\frac{1}{3}}+y{y_1}^{-\frac{1}{3}}=a^{\frac{2}{3}}$. Thus

$p^2=\frac{a^{\frac{4}{3}}}{x^{\frac{-2}{3}}+y^{\frac{-2}{3}}}=(axy)^{\frac{2}{3}}.r^2=x^2+y^2=$

$(x^{2/3}+y^{2/3})^3-3x^{2/3}y^{2/3}(x^{2/3}+y^{2/3})=a^2-3(axy)^{\frac{2}{3}}=a^2-3p^2$. Thus the pedal equation is $r^2+3p^2=a^2$

<u>Ex:</u> Find the pedal equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to one of its foci.

»Let $(ae,0)$ be the focus and $P:(h,k)$ be any point on the ellipse. Equation of tangent at P is $y-k=-\frac{b^2h}{a^2k}(x-h)$. Here, $k^2=b^2(1-\frac{h^2}{a^2})=b^2-h^2(1-e^2)=(a^2-x^2)(1-e^2)\ and\ r^2=h^2+k^2=(x-ae)^2+y^2=(a-ex)^2$. Thus $r=a-ex$. Now $(p)^2=$ square of the perpendicular from $(ae,0)$ to the tangent at $P=\frac{(b^2aex-a^2b^2)^2}{a^4y^2+b^4x^2}=\frac{a^2b^4(a-ex)^2}{a^4y^2+b^4x^2}=\frac{b^2r^2}{a^2-e^2x^2}$. Thus $\frac{b^2}{p^2}=\frac{a+ex}{r}=\frac{2a-r}{r}=\frac{2a}{r}-1$.

Ex: Find the pedal equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the centre.

»The parametric equation of the ellipse is $x=a\ cos\varphi, y=b\ sin\varphi$. Thus, $r^2=x^2+y^2=a^2cos2\varphi+b^2sin2\varphi$. Equation of the tangent at $(a\ cos\varphi, b\ sin\varphi)$ is $\frac{x\ cos\varphi}{a}+\frac{ysin\varphi}{b}=1$. Here $p=\frac{ab}{\sqrt{b^2cos^2\varphi+a^2sin^2\varphi}}$. Hence $\frac{a^2b^2}{p^2}=a^2+b^2-r^2$ is the required pedal equation.

Ex: Find the pedal equation of the asteroid $x=a\ cos3\ \theta, y=b\ sin3\ \theta$ with respect to the origin.

»Equation of the tangent at any point 'θ' is $x\sin\theta+\ y\cos\theta=a\sin\theta\cos\theta$. Thus $p=\text{a}\sin\theta\cos\theta$. Now $r^2=x^2+y^2=a^2(cos6\ \theta+sin6\ \theta)=a^2\{(cos2\ \theta+sin2\ \theta)^3-3\ sin2\ \theta\ cos2\ \theta\}=a^2(1-3p^2/a^2)=a^2-3p^2$.

Ex: Find the pedal equation of the parabola $y^2=4a(x+a)$ with respect to the vertex.

»The vertex is at $(-a,0)$. Let $P:(h,k)$ be any point on the parabola.

$r^2=(h+a)^2+k^2=(h+a)^2+4ah+4a^2$. …………………….. (1)

Equation of tangent at (h,k) is $ky=2a(x+h)+4a^2$

The length of the perpendicular from $(-a,0)$ on the tangent at (h,k) is $p=\frac{2a(h-a)+4a^2}{\sqrt{k^2+4a^2}}=\frac{2a(h+a)}{\sqrt{k^2+4a^2}}$ ……………………… (2)

Also $k^2=4a(h+a)$ ………………………….. (3) Eliminating h,k from (1), (2) and (3) , we get the required equation.

CHAPTER 10

CURVATURE

Let L be a fixed line of reference, Γ be a planar curve in the plane of L, A is a fixed point on Γ with reference to which arc length is measured and P,Q be two points on Γ. Let arc AP= s and arc AQ=s+Δs so that arc PQ is Δs. Let the tangents to Γ at P and Q respectively make angles ψ and ψ+Δψ with L respectively. Thus tangent rotates through an angle $\Delta\psi$ for a change Δs in arc length. The curvature (rate of bending of the curve) k of Γ at P is defined as $\lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds}$.

Circle: a special case

Let PQ be an arc of a circle of radius r subtending an angle $\Delta\psi$ (in radian) at the centre of the circle and let Δs be the arc length of the arc. Then $\Delta s = r\Delta\psi$ and thus curvature of the circle at P is

$k = \frac{d\psi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{1}{r}$, independent of the point P on the circle. The radius of the circle $r = \frac{1}{k}$. This prompts the following general definition :

Circle of curvature, radius of curvature

Let Γ be a planar curve, P be a point on the curve such that the curvature k of Γ at P is non-zero. The circle C satisfying the following properties is called the circle of curvature of Γ at P; its centre and radius are called centre of curvature and radius of curvature of Γ at P:

- The radius of the circle is 1/k
- C passes through P and has the same tangent as C at P
- An arc of C containing P and the circle lies on the same side of the tangent at P

Note For a point P on Γ ,curvature of Γ at P is equal to the curvature of its circle of curvature at P .

Formulae for finding radius of curvature ρ

- Intrinsic Equation $s = f(\psi)$: $\rho = \frac{ds}{d\psi}$
- Cartesian Equation $f(x,y) = 0$: $\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}, y_1 = \frac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}$
- Polar Equation $r = f(\theta)$: $\rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$
- Pedal Equation $p = f(r)$: $\rho = r\frac{dr}{dp}$.

Ex: Find the radius of curvature of the curve $y = xe - x$ at its point of local maximum.

» $y_1 = e - x(1 - x) = 0$ gives $x = 1$ $(e - x \neq 0$ for any real $x)$. $y_2(1) = -e - 1 < 0$. Thus curve has local maximum at $(1, e - 1)$. Thus $\rho = |-e| = e$.

Ex: Prove that the radius of curvature of the curve $x = a\,cos3\,\theta, y = a\,sin3\,\theta$, at the point '$\theta$' is $\rho = 3a\,sin\,\theta cos\theta$.

» $y_1 = \frac{3asin^2\theta\,cos\theta}{3acos^2\theta\,sin\,\theta} = tan\theta, y_2 = sec2\theta\frac{d\theta}{dx} = \frac{sec^2\theta}{-3acos^2\theta\,sin\theta} = -\frac{1}{3a}sec^4\theta\,cosec\,\theta$.
Thus $\rho = \left|\frac{sec^3\theta}{-\frac{1}{3a}sec^4\theta\,cosec\theta}\right| = 3a\,sin\theta cos\,\theta$.

Ex: In the curve $r^n = a^n cos\,n\,\theta$, verify that $\rho = \frac{r^2}{(n+1)p} = \frac{a^n}{(n+1)r^{n-1}}$.

»Then $n\,ln\,r = lna^n + lncos\,n\theta$. $\frac{n}{r}\frac{dr}{d\theta} = -n\,tan\,n\theta$. Thus $tan\,\varphi = tan\left(\frac{\pi}{2} + n\theta\right)$. Taking particular solution, $\varphi = \frac{\pi}{2} + n\theta$. Thus $= r\,sin\varphi = r\,cos\,n\theta$. eliminating $cos\,n\theta$, the pedal equation of the curve is

$r^{n+1} = a^n p$. Differentiating w.r.t. p, $\rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)r^{n-1}} = \frac{a^n r^2}{(n+1)r^{n+1}} = \frac{r^2}{(n+1)p}$.

Ex: Show that at any point of the cardioide $r = a(1 - cos\,\theta)$, ρ is numerically equal to $\frac{2}{3}\sqrt{2ar}$.

» $r = a(1 - cos\theta)$. $r_1 = a\,sin\theta$, $r_2 = a\,cos\theta$. $r^2 + r_1^2 = a^2[(1 - cos\theta)^2 + sin2\theta] = 2a^2(1 - cos\theta) = 4a^2sin^2\frac{\theta}{2}$. Thus $(r^2 + r_1^2)^{\frac{3}{2}} = 8a^3sin^3\frac{\theta}{2}$. Also $r^2 + r_1^2 - rr_2 = 6a^2sin^2\frac{\theta}{2}$. Hence $\rho = \frac{4a}{3}sin\frac{\theta}{2} = \frac{2}{3}\sqrt{2ar}$ (from equation of the curve).

Newton's Method of finding radius of curvature

Let Γ be a planar curve, P and Q be two points on Γ. Let C be a circle through P and Q, and having common tangent with Γ at P. If Q approaches P along Γ, the limiting position of C is identical to the circle of curvature to Γ at P. A few important results coming out of this considerations are as follows:

(1) If a curve passes through the origin and the x-axis is tangent at the origin to the curve, then $\lim_{(x,y)\to(0,0)} \frac{x^2}{2y}$ equals the radius of curvature of the curve at the origin.

(2) If a curve passes through the origin and the y-axis is tangent at the origin to the curve, then $\lim_{(x,y)\to(0,0)} \frac{y^2}{2x}$ equals the radius of curvature of the curve at the origin.

(3) If a curve passes through the origin and if ax+by=0 is tangent at the origin to the curve, then $\frac{1}{2}\sqrt{a^2+b^2}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{ax+by}$ equals the radius of curvature of the curve at the origin.

Ex: Show that the radii of curvature of the curve $y^2(a-x) = x^2(a+x)$ at the origin are $\pm\sqrt{2}a$.

»Equating to zero the lowest degree terms of the given equation $a(y+x)(y-x) = x^3 + xy^2$, the tangents at the origin of the given curve are $y+x=0$ and $y-x=0$. The radius of curvature of the branch of the curve to which $y-x=0$ is a tangent,

$$\rho = \frac{1}{2}\sqrt{1^2+(-1)^2}\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y-x} = \frac{1}{2}\sqrt{2}\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)a(y+x)}{x^3+xy^2} =$$

$$\sqrt{2}a \lim_{(x,y)\to(0,0)} \frac{\left\{1+\left(\frac{y}{x}\right)^2\right\}\left\{1+\frac{y}{x}\right\}}{1+\left(\frac{y}{x}\right)^2} = \sqrt{2}a,$$ since $\lim_{(x,y)\to(0,0)} \frac{y}{x}$ (= slope of the tangent $y-x=0$ at the origin)= 1. Similarly , the radius of curvature of the other branch of the curve corresponding to the tangent $y+x=0$ is $-\sqrt{2}a$.

Co-ordinates of Centre of Curvature: Equation of Circle of Curvature, The centre of curvature $(\bar{x},\bar{y})$ of a curve whose Cartesian equation is given is given by

$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$, $\bar{y} = y + \frac{(1+y_1^2)}{y_2}$, where y_1 and y_2 are the first and second order derivative calculated from the given equation of the curve and $y_2 \neq 0$.

Ex: Find the equation of the circle of curvature of $2xy + x + y = 4$ at (1,1).

»From the given equation of the curve, $(y_1)_{(1,1)} = -1$ and $(y_2)_{(1,1)} = 4/3$. Thus $\rho = \frac{3\sqrt{2}}{2}$, $\bar{x} = 5/2$, $\bar{y} = 5/2$. Hence the equation of the circle of curvature at (1,1) is $(x - 5/2)^2 + (y - 5/2)^2 = 9/2$.

Let Γ be a planar curve, $P: (x, y)$ be a point on Γ where y_1, y_2 exist and $y_2 \neq 0$, and $C: (\bar{x}, \bar{y})$ be the corresponding co-odinate of the centre of curvature of Γ at P. Locus Γ_1 of C corresponding to the locus Γ of P is called evolute of Γ and Γ is called **involute** of Γ_1 .

Ex: Find the equation of the evolute of the parabola $y^2 = 12x$.

»$y_1 = \frac{6}{y} = \frac{\sqrt{3}}{\sqrt{x}}$, $y_2 = -\frac{36}{y^3} = -\frac{\sqrt{3}}{2x^{3/2}}$. Hence $\bar{x} = x - \frac{\frac{\sqrt{3}}{\sqrt{x}}\left(1+\frac{3}{x}\right)}{-\frac{\sqrt{3}}{2x^{\frac{3}{2}}}} = 3x + 6$, $\bar{y} = -\frac{y^3}{36}$.

Thus $x = \frac{\bar{x}-6}{3}$, $y = -\sqrt[3]{36\bar{y}}$. Substituting these values in the given equation of parabola, we get $81\bar{y}^2 = 4(\bar{x} - 6)^3$. Changing to current co-ordinates, the equation of the envelope is 81y^2=4(x-6)3.81 $y^2 = 4(x - 6)^3$

Ex: Show that the evolute of the asteroid $x^{2/3} + y^{2/3} = a^{2/3} is\ (x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$.

»Let $x = acos3\theta, y = a\ sin3\ \theta$ be a point on the asteroid. $\bar{x} = acos3\ \theta + 3asin2\ \theta cos\ \theta, \bar{y} = a\ sin3\theta + 3acos2\ \theta sin\ \theta$. Thus $(\bar{x} + \bar{y})^{\frac{2}{3}} + (\bar{x} - \bar{y})^{\frac{2}{3}} = 2a^{2/3}$.

Ex: Find the equation of the evolute of the hyperbola $xy = a^2$.

»$y = a2/x, y(1) = -a2/x2, y(2) = 2a2/x3$. Thus $\bar{x} = \frac{3x}{2} + \frac{y^3}{2a^2}$, , $\bar{y} = \frac{3y}{2} + \frac{x^3}{2a^2}$. Hence $\bar{x} + \bar{y} = \frac{1}{2a^2}(x + y)^3$ and $\bar{x} - \bar{y} = -\frac{1}{2a^2}(x - y)^3$. Thus $(\bar{x} + \bar{y})^{\frac{2}{3}} - (\bar{x} - \bar{y})^{\frac{2}{3}} = (4a)^{2/3}$.

CHAPTER 11

RECTILINEAR ASYMPTOTES

Let Γ be a planar curve whose Cartesian equation is given and let L be a line whose equation is ax+by+c=0. Let P:(α, β) be an arbitrary point on Γ, whose perpendicular distance from L is $\left|\frac{a\alpha+b\beta+c}{\sqrt{a^2+b^2}}\right|$. If $\left|\frac{a\alpha+b\beta+c}{\sqrt{a^2+b^2}}\right|$ tends to zero when either $|\alpha|$ or $|\beta|$ or both tend to ∞, then L is a rectilinear asymptote (or asymptote, in short) of Γ.

<u>Ex:</u> $x = -1$ is an asymptote of $y = \frac{1}{x+1}$ since $\lim_{|y|\to\infty} \frac{x+1}{\sqrt{1^2+0^2}} = \lim_{|y|\to\infty} \frac{1}{y} = 0$. The circle $y = \pm\sqrt{1-x^2}$ does not have any asymptote since for an arbitrary point $P: (\alpha, \beta)$ on the curve, $|\alpha| \le 1$ and $|\beta| \le 1$ and hence neither $|\alpha|$ nor $|\beta|$ tends to ∞. Similarly, $x = \frac{\pi}{2}$ is an asymptote to $y = tan\, x$.

THEOREM: $y = mx + c$ is an asymptote to a planar curve $\Gamma: f(x, y) = 0$ if and only if $m = lim_{|x|\to\infty} \frac{y}{x}$ and $c = lim_{|x|\to\infty}(y - mx)$ [(x, y) is a point of Γ].

Determination of asymptotes not parallel to the y-axis of an algebraic curve

Let Γ be a planar algebraic curve whose equation is $(a_0y^n + a_1y^{n-1}x + \cdots + a_nx^n) + (b_1y^{n-1} + b_2y^{n-2}x + \cdots + b_nx^{n-1}) + \cdots + (c_{n-1}y + c_nx) + d_n = 0$ ……………………………….(1)

which may be written as $x^nf_n\left(\frac{y}{x}\right) + x^{n-1}f_{n-1}\left(\frac{y}{x}\right) + \cdots + f_0\left(\frac{y}{x}\right) = 0$, …………………….. (2) where $f_r\left(\frac{y}{x}\right)$ is a homogeneous polynomial of degree r. If (1) has an asymptote with (unknown) slope m, then $m = lim_{|x|\to\infty} \frac{y}{x}$.

Dividing (2) by x^n and passing to limit as $|x| \to \infty$, we get $f_n(m) = 0$ ………………………(3) whose roots give slopes of possible asymptotes not parallel to y-axis. Let m_1 be a root of $f_n(m) = 0$. Corresponding c value of $y = mx + c$ is given by $c = lim_{|x|\to\infty}(y - mx)$. Let $y - m_1x = k_1$; then $c = lim_{|x|\to\infty} k1$. Thus $\frac{y}{x} = m_1 + \frac{k_1}{x}$. From (2) we get $x^nf_n\left(\mathrm{m}_1 + \frac{k_1}{x}\right) =$

$x^{n-1}f_{n-1}\left(\mathrm{m}_1+\frac{k_1}{x}\right)+\cdots\ldots.+f_0\left(\mathrm{m}_1+\frac{k_1}{x}\right)=0$. By Taylor's Theorem ($fi$'$s$ are polynomials and hence satisfy all conditions of Taylor's Theorem),

$$x^n\left[f_n(m_1)+\frac{k_1}{x}f_n{}^{(1)}(m_1)+\frac{k_1{}^2}{2x^2}f_n{}^{(2)}(m_1)+\cdots\right]+x^{n-1}\left[f_{n-1}(m_1)+\frac{k_1}{x}f_{n-1}{}^{(1)}(m_1)+\frac{k_1{}^2}{2x^2}f_{n-1}{}^{(2)}(m_1)\ +\cdots\right]+\cdots\ldots..=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (4)$$

Arranging in descending powers of x, we get $x^nf_n(m_1)+x^{n-1}\left[k_1f_n^{(1)}(m_1)+f_{n-1}(m_1)\right]+x^{n-2}\left[\frac{k_1^2}{2}f_n^{(2)}(m_1)+k_1f_{n-1}^{(1)}(m_1)+f_{n-2}(m_1)\right]+\cdots\ldots=0.$

Since $f_n(\mathrm{m}_1)=0$ (m_1 is a root of $f_n(\mathrm{m})=0$, dividing by x^{n-1} and taking limit as $|x|\to\infty$, we get $c_1f_n^{(1)}(m_1)+f_{n-1}^{(1)}(m_1)=0$ ………… (5) , or, $c_1=-\frac{f_{n-1}(m_1)}{f_n^{(1)}(m_1)}$, if $f_n^{(1)}(m_1)\neq 0$. Thus $y=m_1x-\frac{f_{n-1}(m_1)}{f_n^{(1)}(m_1)}$ is an asymptote if $f_n^{(1)}(m_1)\neq 0$.

If $f_n^{(1)}(m_1)=0$ but $f_{n-1}(m_1)\neq 0$, then no c value can be obtained from (5); hence there will be no asymptote corresponding to m_1.

If $f_n^{(1)}(m_1)=0$ and $f_{n-1}(m_1)=0$, then (5) becomes an identity which is not acceptable since corresponding to a given slope m_1, infinite number of c values is not possible. From (4) by dividing by x^{n-2} and allowing $|x|\to\infty$, we have $\frac{c_1^2}{2}f_n^{(2)}(m_1)+c_1f_{n-1}^{(1)}(m_1)+f_{n-2}(m_1)=0$ from which two values of $c_1$are obtained. If the roots c_{11}, c_{12} are real and distinct, corresponding to the slope m_1, there are two asymptotes: $y=m_1x+c_{11}$and $y=m_1x+c_{12}$. If the roots are real and equal, say c_{11}, then there is one asymptote $y=m_1x+c_{11}$corresponding to m_1. If the roots are conjugate complex, no asymptote corresponding to slope m_1.

Similarly we proceed, if necessary, to higher powers of c_1.

Note Real roots of $f_n(m)=0$ determines asymptotic directions. It may be that $m_1$is a root of $f_n(m)=0$ but all the corresponding values of c may be complex: then there is no asymptote with slope m_1. If $y=m_1x+c_1$ and $y=m_1x+c_2$ are asymptotes to the same curve, then m is a multiple root of $f_n(m)=0$. Summarising, an algebraic curve of degree n can have at most n asymptotes.

Determination of asymptotes parallel to the y-axis of an algebraic curve

Let $F(x,y) = y^m g(x) + y^{m-1} g_1(x) + \cdots + g_m(x) = 0$, where $g, g_1, \dots, g_m$ are polynomials in x, be the equation of a rational algebraic curve . Dividing by $y^m$and letting $y \to \infty$(since F has asymptote parallel to y-axis, $\lim_{|y|\to\infty} x$ should exist), we get $g(k) = 0$ if $\lim_{|y|\to\infty} x = k$. Thus the vertical asymptotes to $F(x,y) = 0$ are obtained by equating to zero the coefficient of the heighest power of y. No vertical asymptote to the curve exists if the coefficient of highest power of y is a constant or not resolvable in real linear factors.

Ex: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be written in the form $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) + (-1) = 0$, which is of the form $F_2 + F_0 = 0$, where F_2 can be written as product of two real linear factors $\frac{x}{a} + \frac{y}{b}$, $\frac{x}{a} - \frac{y}{b}$ such that $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$ represent two non-parallel lines; hence the asymptotes are given by $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$.

Ex: The parabola $y^2 = 4ax$ has no asymptote parallel to the y-axis; the equation can be written in the form $x^2 f_2(\frac{y}{x}) + x f_1(\frac{y}{x}) = 0$, where $f_2(m) = m_2, f_1(m) = -4a. f_2(m) = 0$ gives $m = 0$; c, if it exists, corresponding to $m = 0$ is given by $c f_2(1)(0) + f_1(0) = 0$, that is, $c.0 - 4a = 0$, contradiction. Hence no asymptote non-parallel to y-axis either: thus no asymptote to the curve.

CHAPTER 12

ENVELOPE OF A FAMILY OF CURVES

A point $P(a,b)$ is a singular point of a curve $f(x,y) = 0$ if $f(a,b) = 0, f_x(a,b) = 0$ and $f_y(a,b) = 0$ holds simultaneously. In contrast P is an ordinary point of the curve $f(x,y) = 0$ if at least one of $f_x(a,b)$ and $f_y(a,b)$ is not equal to zero.

Let $f(x,y,\alpha) = 0$ be a family of curves, where α is a parameter (corresponding to each value of α, there is a curve) . The characteristic points of the family of curves $f(x,y,\alpha) = 0$ are those ordinary points which are lying on each curve $f(x,y,\alpha\ ) = 0$ of the familyand at those points $\frac{\partial f}{\partial \alpha} = 0$ holds simultaneously.

Ex: The characteristic points of the family of circles $(x-\alpha)^2+y^2=a^2$ (α is the parameter)can be obtained by solving simultaneously $f(x,y,\alpha)=(x-\alpha)^2+y^2-a^2$and $\partial f/\partial\alpha=-2(x-\alpha)=0$, which give the two points $(\alpha,\pm a)$. Also $\partial f/\partial y(\alpha,\pm a)=\pm 2a$ is not equal to zero; hence $(\alpha,\pm a)$ are ordinary points and hence are characteristic points of the family.

Characteristic points may not exist for a family of curves : for the family $x^2+y^2=a^2$ of concentric circle, there is no characteristic point.

The envelope of a family of curves $f(x,y,\alpha)=0$ (α parameter)is the locus of isolated characteristic points of the family.

Note: If $f(x,y,\alpha)=0$ and $\partial f/\partial\alpha(x,y,\alpha)=0$ both hold for a point where $f_x=0$ and $f_y=0$, then the point is a singular point and, therefore, not a characteristic point.

Ex: Let us consider the family of curves $x\cos\alpha+y\sin\alpha=a\sin\alpha\cos\alpha$, where α is the parameter, a fixed. The characteristic points are obtained by solving the equations $x\cos\alpha+y\sin\alpha=a\sin\alpha\cos\alpha$ and $-x\operatorname{cosec}\alpha\cot\alpha+y\sec\alpha\tan\alpha=0$ simultaneously.

The envelope, that is, the locus of characteristic points is obtained by eliminating α and is given by $x^{2/3}+y^{2/3}=a^{2/3}$.

Ex: $y=mx+a/m, m$ parameter. Differentiating partially w.r.t. the parameter m, we get 0= x - a/m^2, that is, $m=\pm\sqrt{\frac{a}{x}}$. Substituting in the given equation, $y=\pm 2\sqrt{ax}$, that is, $y^2=4ax$.

Ex: Find the envelope of circles described on the radii vectors of the parabola $y^2=4ax$ as diameter.

»Let $P:(at^2,2at)$ be an arbitrary point on the parabola and O be the origin. Equation of circle with OP as one of its diameter is $(x-0)(x-at^2)+(y-0)(y-2at)=0$, that is, $x^2+y^2-2ayt-at^2x=0$. Differentiating partially w.r.t. t, $-2ay-2atx=0$, that is, $t=-y/x$. Substituting in the equation, $ay^2+x(x^2+y^2)=0$ is the required envelope.

Case of two parameters

Let $f(x, y, \alpha, \beta) = 0$ ------ (1) be the equation of a 2-parameter family of curves where the parameters α, β are connected by $g(\alpha, \beta) = 0$------- (2). For a fixed point (x, y) on the envelope, from given relations, we have $\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \beta}\frac{d\beta}{d\alpha} = 0$ and $\frac{\partial g}{\partial \alpha} + \frac{\partial g}{\partial \beta}\frac{d\beta}{d\alpha} = 0$. Eliminating $\frac{d\beta}{d\alpha}$ from the last two relations, we have $\frac{\frac{\partial f}{\partial \alpha}}{\frac{\partial g}{\partial \alpha}} = \frac{\partial f}{\partial \beta} / \frac{\partial g}{\partial \beta}$ ---------- (3). Eliminating α, β from (1),(2) and (3), we obtain the required envelope.

Ex: Find the envelope of the family of co-axial ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where the parameters a,b are connected by $a^2 + b^2 = c^2$, c fixed.

»Differentiating partially w.r.t. a, $-2\frac{x^2}{a^3} - 2\frac{y^2}{b^3}\frac{db}{da} = 0$ and $2a + 2b\frac{db}{da} = 0$.

Equating values of $\frac{db}{da}$, we get $\frac{x^2}{a^4} = \frac{y^2}{b^4}$, that is, $\frac{\frac{x^2}{a^2}}{a^2} = \frac{\frac{y^2}{b^2}}{b^2} = \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a^2+b^2} = \frac{1}{c^2}$. Hence $a^2 = xc, b^2 = yc$ .Since $a^2 + b^2 = c^2$, the required envelope is $x + y = c$.

STUDY MATERIALS ON INTEGRAL CALCULUS

DEFINITE INTEGRAL

Let f be a real valued **continuous** function defined on a **closed** and **bounded** interval [a,b]. Let us choose a **partition** (collection of finite number of points of [a,b] including a and b) P = {a = $x_0,x_1,x_2,\ldots,x_n$=b} of [a,b] (for example: {0,1/2,1} is a partition for [0,1]).

Let $\delta_r = x_r - x_{r-1}$, r = 1,…,n and $\delta = \max\{\delta_r |\ r = 1,2,\ldots,n\}$. Choose an arbitrary point $c_r \in (x_{r-1},x_r)$ for all r and consider sum of areas of rectangles $\sum_1^n f(c_r)\delta_r$. It can be seen that this sum approaches more closely the actual area under the curve if we make width of the rectangles smaller , that is, if we increase number n of points of subdivision (**sum of areas of two rectangles on $\boldsymbol{\delta_r}$gives a better approximation to the area under the curve than area of a single rectangle**).

Definition: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_1^n f(c_r)\delta_r$, provided the limit exists **independent of choice of points of subdivision x_i and that of c_i , for all i**. It can be proved that for a **continuous function f defined over a closed bounded interval [a,b], $\int_a^b f(x)dx$ exists in above sense**.

Simpler equivalent expression for calculating $\int_a^b f(x)dx$:

We can make choices of x_i and c_i **suitably** so as to obtain equivalent simpler expression of $\int_a^b f(x)dx$.

- Let us choose x_i's equi-spaced, that is , $\delta_1 = \delta_2 = \ldots = \delta_n$ = (b-a)/n. Then $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_1^n f(c_r)$.
- Let us choose c_r = a+rh, r = 1,...,n, where h = (b-a)/n. Then $\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_1^n f(a + r\frac{b-a}{n}) = \lim_{h\to 0} h \sum_1^n f(a + rh)$.

  As a special case, $\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_1^n f\left(\frac{r}{n}\right) = \lim_{h\to 0} h \sum_1^n f(rh)$

  Example: From definition, calculate $\int_0^1 x^2 dx$.

  » $\int_0^1 x^2 dx = \lim_{h\to 0} h \sum_1^n (rh)^2 = \lim_{h\to 0} \frac{(nh)(nh+h)(2nh+h)}{6} = 1/3$, since nh = 1 holds, for every positive integer n and the corresponding h.

  Fundamental Theorem of Integral Calculus

  Theorem: If $\int_a^b f(x)dx$ exists and **if there exists** a function g:[a, b]→R such that $g_1(x)$ = f(x)(suffix denotes order of differentiation) on [a, b], then $\int_a^b f(x)dx$ = g(b) – g(a).

  NOTE: g is called a **primitive** of f. A function f may not possess a primitive on [a,b] but $\int_a^b f(x)dx$ may exist ; in that case, $\int_a^b f(x)dx$ can not be calculated using fundamental theorem. Primitives of f on [a, b] are given by the indefinite integral $\int f(x)dx$: that is the reason why we consider indefinite integrals.

  Example $\int_0^1 x^2 dx$ exists, since x^2 is continuous on [0,1]. Also g(x)= $\int x^2 dx = \frac{x^3}{3}$ +c is a primitive of x^2 on [0,1]. Hence Fundamental Theorem gives $\int_0^1 x^2 dx$=($\frac{1}{3}$+c)-c=$\frac{1}{3}$.

Note $\int_0^1 x^2 dx$ is independent of c though $\int x^2 dx$ involves c.

PROPERTIES OF DEFINITE INTEGRALS

We assume below that the definite integrals exist and whenever we consider $\int_a^b f(x)dx$, a primitive g to f over [a,b] exists, so that we can apply Fundamental Theorem. For a<b, we **define** $\int_b^a f(x)dx$ = - $\int_a^b f(x)dx$.

1. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (irrespective of relative algebraic magnitude of a,b,c)

   Example $\int_3^1 x^2 dx + \int_1^4 x^2 dx =$

   $-\int_1^3 x^2 dx + \int_1^4 x^2 dx = -\left(9 - \frac{1}{3}\right) + \left(\frac{64}{3} - \frac{1}{3}\right) = \frac{37}{7} = \int_3^4 x^2 dx.$
2. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

   Example $\int_0^{\pi/2} \sin x \, dx = \int_0^{\pi/2} \cos x \, dx$
3. $\int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-x)dx$. In particular, if f(a-x) = f(x)for all x in [0,a], then $\int_0^a f(x)dx = 2\int_0^{a/2} f(x)dx$ and if f(a-x) = - f(x) for all x in [0,a], then $\int_0^a f(x)dx$= 0.
4. $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$, if f(a+x) = f(x), n natural.
5. $\int_{-a}^a f(x)dx = \int_0^a \{f(x) + f(-x)\}dx$. If f is even, $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$. If f is odd, $\int_{-a}^a f(x)\, dx$= 0.

   Example $\int_{-1}^1 x^3 \cos 2x \, dx$=0, $\int_{-1}^1 x^4 \cos 2x \, dx$= 2$\int_0^1 x^4 \cos 2x \, dx$

REDUCTION FORMULA

In this chapter, we study how to decrease complexity of some integrals in a stepwise manner by the use of recurrence relation that we derive generally using integration by parts formula.

1. Let $I_n = \int sin^n x\, dx$, n natural.

$I_n = \int sin^{n-1}x \sin x\, dx = \sin^{n-1}x(-\cos x) - (n-1)\int sin^{n-2}x \cos x\,(-\cos x)\, dx = \sin^{n-1}x(-\cos x)+(n-1)\int sin^{n-2}x(1-sin^2 x)dx = -\sin^{n-1}x \cos x+(n-1)I_{n-2}-(n-1)I_n$.

hence $I_n = -\frac{sin^{n-1}x cos\, x}{n} + \frac{n-1}{n} I_{n-2}$.

If we denote $J_n = \int_0^{\pi/2} sin^n x\, dx$, then $J_n = -\frac{sin^{n-1}x cos\, x}{n}\Big|_0^{\pi/2} + \frac{n-1}{n} J_{n-2} = \frac{n-1}{n} J_{n-2}$.

By repeated application of the reduction formula, it can be proved that $J_n = \frac{n-1}{n} J_{n-2} = \ldots = \frac{n-1}{n}\frac{n-3}{n-2}\ldots\frac{1}{2}.J_0 = \frac{n-1}{n}\frac{n-3}{n-2}\ldots\frac{1}{2}.\frac{\pi}{2}$, if n is even natural and $J_n = \frac{n-1}{n}\frac{n-3}{n-2}\ldots\frac{2}{3}J_1 = \frac{n-1}{n}\frac{n-3}{n-2}\ldots\frac{2}{3}$, if n is odd natural.

2. Let $I_n = \int tan^n x\, dx$, n natural.

Then $I_n = \int tan^{n-2}x.tan^2 x\, dx = \int tan^{n-2}x.(sec^2 x - 1)\, dx = \frac{tan^{n-1}x}{n-1} - I_{n-2}$.

Also, $J_n = \int_0^{\pi/4} tan^n x\, dx = \frac{tan^{n-1}x}{n-1}\Big|_0^{\pi/4} - J_{n-2} = \frac{1}{n-1} - J_{n-2}$.

3. Let $I_n = \int sec^n x\, dx$, n natural. Then $I_n = \int sec^{n-2}x.sec^2 x\, dx = \sec^{n-2}x \tan x - (n-2)\int sec^{n-2}x.(sec^2 x - 1)\, dx = \sec^{n-2}x \tan x - (n-2)(I_n - I_{n-2})$. Hence $I_n = \frac{sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$.

4. Let $I_{m,n} = \int sin^m x\, cos^n x\, dx = \int (sin^m x cos\, x)\, cos^{n-1} x\, dx = \frac{sin^{m+1} x\, cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int cos^{n-2} x sin^m x\, sin^2 x\, dx = \frac{sin^{m+1} x\, cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int cos^{n-2} x sin^m x\, (1 - cos^2 x)\, dx = \frac{sin^{m+1} x\, cos^{n-1} x}{m+1} + \frac{n-1}{m+1}$ $I_{m,n-2}$- $\frac{n-1}{m+1} I_{m,n}$. Transposing and simplifying, we get a reduction formula for $I_{m,n}$.

5. Let $I_{m,n} = \int cos^m x \sin nx\, dx =$ -$\frac{cos^m x \cos nx}{n} - \frac{m}{n} \int cos^{m-1} x \sin x \cos nx\, dx =$ -$\frac{cos^m x \cos nx}{n} - \frac{m}{n} \int cos^{m-1} x \sin nx \cos x\, dx + \frac{m}{n} \int cos^{m-1} x \sin(n-1)\, x\, dx$ [since cos nx sin x = sin nx cos x – sin(n-1)x] = -$\frac{cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$. Transposing and simplifying, we get a reduction formula for $I_{m,n}$.

Illustrative examples

1. $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\, dx$ and n>1, show that $I_n + n(n-1)I_{n-2} = n(\frac{\pi}{2})^{n-1}$

   » $I_n = (-x^n \cos x)\Big|_0^{\pi/2} + n \int_0^{\pi/2} x^{n-1} \cos x\, dx =$ n $\left[x^{n-1} \sin x \Big|_0^{\pi/2} - (n-1) \int_0^{\pi/2} x^{n-2} \sin x\, dx \right]$ = n$(\frac{\pi}{2})^{n-1}$-n(n-1)I_{n-2}. Hence the proof.

2. $I_{m,n} = \int x^m (1-x)^n dx = \frac{x^{m+1}}{m+1} (1-x)^n + \frac{n}{m+1} \int x^m [1 - (1-x)](1-x)^{n-1} dx = \frac{x^{m+1}}{m+1} (1-x)^n + \frac{n}{m+1} (I_{m,n-1} - I_{m,n})$. Hence $I_{m,n}$ can be obtained.

3. $I_m = \int_0^{\pi/2} cos^m x \sin mx\, dx$. From 5 above, $I_m = \frac{1}{2m} + \frac{1}{2} I_{m-1} = \frac{1}{2m} + \frac{1}{2}\left[\frac{1}{2(m-1)} + \frac{1}{2} I_{m-2}\right] = \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2} I_{m-2}$. Repeating the use of the reduction formula, it can be proved that $I_m = \frac{1}{2^{m+1}}\left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots + \frac{2^m}{m}\right]$.

IMPROPER INTEGRAL

When we consider the definite integral $\int_a^b f(x)dx$ in earlier standards, we implicitly assume two conditions to hold: **(a)** f is continuous on [a, b] or , to that matter, at least the limit $\boldsymbol{\int_a^b f(x)dx = \lim_{n\to\infty}\sum_1^n f(c_r)\delta_r}$ exists independent of choice of points of subdivision x_i and that of c_i, for all i and **(b)** the interval [a,b] is bounded. We want to extend the definition of $\int_a^b f(x)dx$ when either (a) or (b) or both are **not** met. This extended definition of definite integral is referred to as **Improper Integrals**. Improper integrals can be of two types: (a) **Type 1**: interval of integration is unbounded, (b) **Type 2**: integrand has a finite number of infinite discontinuities in the interval of integration.

Definition of TYPE I improper integral $\int_a^\infty f(x)dx, \int_{-\infty}^a f(x)dx$ and $\int_{-\infty}^\infty f(x)dx$

Let the function f be integrable in [a ,B], for every B>a. If $\lim_{B\to\infty}\int_a^B f(x)dx$ exist finitely, we define $\int_a^\infty f(x)dx = \lim_{B\to\infty}\int_a^B f(x)dx$ and we say $\int_a^\infty f(x)dx$ exists or converges; otherwise $\int_a^\infty f(x)dx$ diverges. Similarly, $\int_{-\infty}^a f(x)dx = \lim_{B\to-\infty}\int_B^a f(x)dx$ (provided the limit exists) and $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$, a is any real, provided $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ exist separately.

Example: $\int_1^\infty \frac{dx}{x^2}$, $\int_1^\infty \frac{dx}{\sqrt{x}}$. The range of integration of the integrals are unbounded. For a>1, $\int_1^a \frac{dx}{x^2}$ = 1- $\frac{1}{a}$ and $\int_1^a \frac{dx}{\sqrt{x}}$ = 2($\sqrt{a}$-1). Since $\lim_{a\to\infty}\left(1-\frac{1}{a}\right)$= 1 exists but $\lim_{a\to\infty}\left(2\left(\sqrt{a}-1\right)\right)$ does not exist, hence the improper integral $\int_1^\infty \frac{dx}{x^2}$ converges and $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges.(compare areas below the curves y = 1/x² and y = 1/$\sqrt{x}$ in diagram below)

Definition of TYPE II improper integral

Let f have an infinite discontinuity only at the point a (that is, $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$) and is continuous in (a, b].Then we define $\int_a^b f(x)dx = \lim_{c\to 0+}\int_{a+c}^b f(x)dx$,0<c<b-a, provided the limit exists. Similarly, if f has an infinite discontinuity only at the point b and is continuous in [a, b), then we define $\int_a^b f(x)dx = \lim_{c\to 0+}\int_a^{b-c} f(x)dx$, 0<c<b-a, provided the limit exists. If f has an infinite discontinuity at d, a<d<b, and is otherwise continuous in [a,b], we define $\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx$, provided both of $\int_a^d f(x)dx$ and $\int_d^b f(x)dx$ exist separately.

Example: $\int_0^1 \frac{dx}{\sqrt{x}}$, $\int_0^1 \frac{dx}{x^2}$. The integrands have an infinite discontinuity at x=0. For 0<a<1, $\int_a^1 \frac{1}{\sqrt{x}}dx = 2(1-\sqrt{a})$ and $\int_a^1 \frac{dx}{x^2} = \frac{1}{a} - 1$. Since $\lim_{a\to 0+} 2(1-\sqrt{a})$ = 2 exists but $\lim_{a\to 0+}\left(\frac{1}{a}-1\right)$ does not exist, so $\int_0^1 \frac{dx}{\sqrt{x}}$ converges whereas $\int_0^1 \frac{dx}{x^2}$ diverges. (Compare areas between x = a,0<a<1, and x = 1, below the curves y = 1/x² and y = 1/$\sqrt{x}$ in diagram above)

Example: $\int_0^{\infty} \frac{dx}{4+9x^2} = \frac{\pi}{12}$

» The integrand $\frac{1}{4+9x^2}$ is continuous everywhere but the interval of integration is unbounded. Let a>0 be fixed. $\int_0^{a} \frac{dx}{4+9x^2} = \frac{1}{9}\int_0^{a} \frac{dx}{x^2+\left(\frac{2}{3}\right)^2} = \frac{1}{6} tan^{-1} \frac{3a}{2}$. Thus $\lim_{a\to\infty} \int_0^{a} \frac{dx}{4+9x^2} = \frac{1}{6} . \frac{\pi}{2} = \frac{\pi}{12}$.

Example: $\int_0^{3} \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2}$.

» The integrand has an infinite discontinuity at x = 3 and is continuous on [0, 3). Let 0<a<3. Then $\int_0^{a} \frac{dx}{\sqrt{9-x^2}}$ = $\sin^{-1}(a/3)$. So $\lim_{a\to 3-} \int_0^{a} \frac{dx}{\sqrt{9-x^2}} = \lim_{a\to 3-} sin^{-1}\left(\frac{a}{3}\right) = \frac{\pi}{2}$. Hence $\int_0^{3} \frac{dx}{\sqrt{9-x^2}} = \frac{\pi}{2}$.

Note : we can apply standard methods of integration, in particular method of substitution, only to a proper integral and not directly to an improper integral. Thus if we substitute z=1/x directly in the improper integral $\int_{-1}^{1} \frac{dx}{x^2}$, we get a value -2 of the integral whereas it can be checked from definition that the improper integral diverges.

TESTS FOR CONVERGENCE OF IMPROPER INTEGRALS

TYPE I INTEGRAL

Theorem: (Comparison test) Let f and g be integrable in [a, B], for every B>a. Let g(x)>0, for all x ≥a. If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ = c≠0, then the integrals $\int_a^{\infty} f(x)dx$ and $\int_a^{\infty} g(x)dx$ either both converge or both diverge. If c = 0 and $\int_a^{\infty} g(x)dx$ converges , then $\int_a^{\infty} f(x)dx$ converges.

Theorem: (**μ Test**) Let f be integrable in [a, B], for every B>a. Then $\int_a^\infty f(x)dx$ converges if $\lim_{x\to\infty} x^\mu f(x)$ exists with $\mu > 1$ and $\int_a^\infty f(x)dx$ diverges if $\lim_{x\to\infty} x^\mu f(x)$ exists and ≠0 with $\mu \le 1$.

Example: $\int_0^\infty \frac{dx}{e^x+1}$ converges **by comparison test** , since $0 \le \frac{1}{e^x}$ for all x≥0, $\lim_{x\to\infty} \frac{e^x}{e^x+1} = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1$ and $\int_0^\infty \frac{dx}{e^x}$ converges (need to prove!).

Example: $\int_0^\infty e^{-x^2} dx$ converges **by μ Test** since $\lim_{x\to\infty}(x^2 e^{-x^2}) = 0$ (verify using L'Hospital's rule) , $\mu = 2 > 1$ and e^{-x^2} is continuous, and hence integrable, in [0,B] for B>0.

Example: $\int_0^\infty \frac{x^{3/2}}{3x^2+5} dx$ diverges, since $\lim_{x\to\infty}\left(x^{1/2}\left(\frac{x^{3/2}}{3x^2+5}\right)\right) = 1/3$, $\mu = ½ < 1$ and $\frac{x^{3/2}}{3x^2+5}$ is continuous, and hence integrable, in [0,B] for B>0.

TYPE II INTEGRAL

Theorem: (Comparison test) Let f and g be integrable in [c, b], for every c, a<c<b. Let g(x)>0 , for all x, a<x≤b. If $\lim_{x\to a+} \frac{f(x)}{g(x)} \neq 0$, then $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both converge or both diverge. If $\lim_{x\to a+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)dx$converges, then $\int_a^b f(x)dx$ converges.

Theorem: (**μ Test**) Let f be integrable in[c, b], for every c, a<c<b. Then $\int_a^b f(x)dx$ converges if $\lim_{x\to a+}(x-a)^\mu$ f(x) exists for $0 < \mu < 1$ and $\int_a^b f(x)dx$ diverges if $\lim_{x\to a+}(x-a)^\mu$f(x) exists (≠0) for $\mu \ge 1$.

Example: $\int_0^1 \frac{\mathrm{dx}}{(1+\mathrm{x})\sqrt{\mathrm{x}}}$ converges , since $\lim_{x\to 0+}(x-0)^{\frac{1}{2}}\frac{1}{(1+x)\sqrt{x}} = 1$, for μ<1 and $\frac{1}{(1+\mathrm{x})\sqrt{\mathrm{x}}}$ is continuous, and hence integrable, in [c,1]for 0<c<1.

Example: $\int_{1/2}^1 \frac{\mathrm{dx}}{\sqrt{\mathrm{x}(1-\mathrm{x})}}$ converges, since $\lim_{x\to 1-}(1-x)^{1/2}\frac{1}{\sqrt{x(1-x)}} = 1$, for $\mu = \frac{1}{2} < 1$ and $\frac{1}{\sqrt{\mathrm{x}(1-\mathrm{x})}}$ is continuous, and hence integrable, in [1/2, c]for 1/2<c<1.

THE GAMMA AND BETA FUNCTIONS

Definition (Gamma function) For n>0, Γ(n) = $\int_0^\infty e^{-x}x^{n-1}dx$.

NOTE: Gamma function is an improper integral of type I. If 0<n<1, Γ(n) is also an improper integral of type II. We shall assume convergence of the gamma function in our course of study.

Definition (Beta function) For m, n>0, β(m,n) = $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

NOTE: Beta function is an improper integral of type II if either m or n or both lies between 0 and 1 strictly; otherwise it is a proper integral.

Properties of Gamma and Beta functions

1. For any a>0, $\int_0^\infty e^{-ax}x^{n-1}dx$ = Γ(n)/a^n.

   » let 0<c<d. consider the proper integral I = $\int_c^d e^{-ax}x^{n-1}dx$. Let y = ax. Then I = $\int_{ac}^{ad} e^{-y}\frac{y^{n-1}}{a^{n-1}}.\frac{1}{a}dy = \frac{1}{a^n}\int_{ac}^{ad} e^{-y}y^{n-1}dy$. Thus lim I = $\frac{\Gamma(\mathrm{n})}{a^n}$ as $c \to 0+$ and d→ ∞.

2. Γ(n+1) = n Γ(n)

» Let 0<c<d. using integration by parts on the proper integral I = $\int_c^d e^{-x}x^n dx$, we get I = $\left(-\frac{x^n}{e^x}\right)|_c^d + n\int_c^d e^{-x}x^{n-1}dx$ = $(\frac{c^n}{e^c} - \frac{d^n}{e^d})+ n\int_c^d e^{-x}x^{n-1}dx$, which tends to n Γ(n) as c→ 0+ and d→ ∞ (by use of L'Hospital's rule). Hence the result.

3 Γ(1) = 1 (can be verified easily)

4 Γ(n+1) = n!, for a natural n (follows from property 2 and 3)

4 β(m,n) = β(n,m) (follows using a substitution y = 1-x after passing to a proper integral)

5 β(m,n) =$2\int_0^{\pi/2} sin^{2m-1}\theta\, cos^{2n-1}\theta\, d\theta$ (follows using a substitution x = $\sin^2\theta$ after passing to a proper integral)

6 $\beta(\frac{1}{2},\frac{1}{2})=\pi$ (follows from definition)

7 $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

8 $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

9 For 0<m<1, Γ(m) Γ(1-m) = π cosec(m π)

Example: $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

» The range of integration of the given integral is unbounded but the integrand is continuous everywhere. For 0<a, $\int_0^a e^{-x^2} dx = \frac{1}{2}\int_0^{a^2} y^{-\frac{1}{2}}e^{-y}dy$ (substituting y = x^2 in the proper integral). Thus $\lim_{a\to\infty}\int_0^a e^{-x^2} dx$ = $\frac{1}{2}\lim_{a\to\infty}\int_0^{a^2} y^{-\frac{1}{2}}e^{-y}dy = \frac{1}{2}\int_0^\infty e^{-y}y^{\frac{1}{2}-1}\,dy = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}/2$.

Example: $\int_0^1 \frac{\mathrm{dx}}{(1-\mathrm{x}^6)^{\frac{1}{6}}}$ dx.

» The integrand has an infinite discontinuity at x = 1.Let $0<c<1$. Substituting $x^3 = \sin\theta$ in the proper integral $\int_0^c \frac{dx}{(1-x^6)^{1/6}}$, $\int_0^c \frac{dx}{(1-x^6)^{1/6}} = \frac{1}{3}\int_0^{sin^{-1}c^3} cos^{5/3-1}\ \theta\ sin^{1/3-1}\theta\ d\theta$. Since $\lim_{c\to 1-} \int_0^c \frac{dx}{(1-x^6)^{1/6}} = \frac{1}{6}\ \beta(5/3,1/3) = \frac{1}{6}\frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{3}+\frac{1}{3}\right)} = \frac{1}{6}.\frac{2}{3}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{1} = \frac{1}{9}\ \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{1}{27}\pi cosec\left(\frac{\pi}{3}\right)$.

DOUBLE INTEGRAL

Let f(x,y) be a bounded function of two independent variables x and y defined over a closed rectangular region R: $a \le x \le b$; $c \le y \le d$. we take partitions $\{a = x_0, x_1, \dots, x_{r-1}, x_r, \dots, x_n = b\}$ of [a,b] and $\{c = y_0, y_1, \dots, y_{s-1}, y_s, \dots, y_m = d\}$. These partitions divides the rectangle R into mn number of subrectangles R_{ij}($1 \le i \le n$, $1 \le j \le m$). Let us choose arbitrarily $(\alpha_i, \beta_j) \in$ Rij where $\alpha_i \in [x_{i-1}, x_i]$ and $\beta_j \in [y_{j-1}, y_j]$, $1 \le i \le n$, $1 \le j \le m$. the volume of the parallelepiped with base R_{ij} and altitude $f(\alpha_i, \beta_j)$ is $f(\alpha_i, \beta_j)(x_i - x_{i-1})(y_j - y_{j-1})$. $\sum_{i,j} f(\alpha_i, \beta_j)(x_i - x_{i-1})(y_j - y_{j-1})$, sum of the volumes of all the parallelepipeds erected over all of the R_{ij}'s, gives an approximation of the volume enclosed by the curve and the planes x = a,x = b, y = c, y = d and z = 0. The approximation can be improved by increasing number of subrectangles into which R is divided into. Thus the limit $\lim_{m\to\infty, n\to\infty} \sum_{i,j} f(\alpha_i, \beta_j)(x_i - x_{i-1})(y_j - y_{j-1})$, provided it exists, gives the volume and is represented by $\iint_R\ f(x,y)dxdy$.

NOTE: Every continuous function is integrable over any rectangle.

Theorem: (equivalence of double integrals with repeated integrals) If $\iint_R f(x,y)dxdy$ exists over a rectangle R: a≤x≤b; c≤y≤d and $\int_a^b f(x,y)dx$ exists for each value of y in [c,d], then the repeated integral $\int_c^d dy \int_a^b f(x,y)dx$ exists and is equal to $\iint_R f(x,y)dxdy$.

Example: Evaluate $\iint_R sin(x+y)dx\,dy$ over R:0≤x≤$\frac{\pi}{2}$, 0≤y≤$\frac{\pi}{2}$.

Sol: sin(x+y) is continuous on R, so the double integral $\iint_R sin(x+y)dx\,dy$ exists . Evaluating given double integral in terms of repeated integrals,

$$\iint_R sin(x+y)dx\,dy = \int_0^{\pi/2} dx \int_0^{\pi/2} \sin(x+y)dy = \int_0^{\pi/2}[-\cos(x+y)|_0^{\frac{\pi}{2}}dx = \int_0^{\frac{\pi}{2}}(\cos x + \sin x)\,dx = 2.$$

EVALUATION OF AREA

Cartesian co-ordinate

It has already been seen that area of the region bounded by the curve y = f(x), lines x = a, x = b and y = 0 is given by $\int_a^b f(x)dx$, provided it exists. Similarly area of the region bounded by the curve x = g(y), lines y = c, y = d and x = 0 is given by $\int_c^d g(y)dy$, provided it exists. We can define F:[a,b]→ R by F(t)=$\int_a^t f(x)dx$,a≤t≤b.

Example: Find the area of the bounded region bounded by the curves y = x^2 and x = y^2.

Sol: On solving the given equations of the curves, the point of intersection of the two curves are (0,0) and (4,4). Thus required area = $\int_0^4 \sqrt{4x}dx - \int_0^4 \frac{x^2}{4}dx.$

Example: Find the area of the loop formed by the curve $y^2 = x(x-2)^2$

Sol: The abscissa of points of intersection of the curve with the x-axis are given by y = 0, that is, x = 0,2,2. For x<0, no real value of y satisfy the equation. Hence no part of the curve exist corresponding to x<0. Corresponding to each x-value satisfying 0<x<2, there exist two values of y, equal in magnitude and opposite in sign. Thus between x = 0 and x = 2, the curve is symmetric about the x-axis and a loop is formed thereby. For x>2, y→ ∞ as x→ ∞. the required area = $2\int_0^2 (x-2)\sqrt{x}dx$ (by symmetry of the curve about x-axis).

Example: Prove that area included in a circle of radius r unit is πr^2 square unit.

Sol: We can choose two perpendicular straight lines passing through the centre of the circle as co-ordinate axes. With reference to such a co-ordinate system, equation of the circle is y = $\pm\sqrt{r^2 - x^2}$.Curve is symmetric about the axes .Thus required area = $4\int_0^r \sqrt{r^2 - x^2}$ dx.

Polar Coordinates:

The area of the region bounded by the curve r = f(θ) , the radius vector $\theta = \alpha, \theta = \beta$ is given by $\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$.

Example: Find the area enclosed by the cardioide r = a(1+ cos θ)

Sol: As θ varies from 0 to $\frac{\pi}{2}$, r decreases continuously from 2a to a. When θ further increases from $\frac{\pi}{2}$ to π, r decreases further from a to 0. Also the curve is symmetric about the initial line (since the equation of the curve remains unaffected on replacing θ by $-\theta$). Hence the area enclosed by the curve = 2.$\frac{a^2}{2}\int_0^{\pi}(1 + cos\theta)^2 d\theta$.

Example: Find the area enclosed by the cardioide r = a(1+ cos θ) and r = a(1- cos θ)

Sol: The vectorial angle corresponding the points of intersection of the curves are $\theta = \frac{\pi}{2}$ and θ=-$\frac{\pi}{2}$. Because of the symmetry of the curves about the initial line, $\theta = \frac{\pi}{2}$, θ=-$\frac{\pi}{2}$ and $\theta = \pi$, required area is 4.$\frac{a^2}{2}\int_0^{\pi/2}(1 - \cos\theta)^2 \, d\theta$.

AN INTRODUCTION TO STATISTICS

"Status"-"State"->"Statistics".

STEPS AT A GLANCE :
Collection of Data-> Summarisation of Data-> Analysis of Data-> Interpretation of Data towards a VALID DECISION.

WHAT IS THE MAIN PROBLEM IN STATISTICS?
Given a sample(a set of outcomes),we are to say(infer) about the population or the model.Statistics primarily deals with situations in which the occurrence of some event can't. be predicted with certainty.

WHAT ARE THE MAJOR OBJECTIVES OF STATISTICS?
1. To make inference about a population from an analysis of information contained in the sample data.
2. To make assessments of the extent of uncertainty involved in these inferences.
A third objective,no less important,is to design the process & the extent of sampling so that the observations from a basis for drawing valid & accurate inferences.

GIVE THE DEFINITION OF STATISTICS?
"STATISTICS" is a science of decision making on the basis of sample observations drawn from a population under uncertainty. That is, it is a mathematical discipline concerned with collection of data,summarisation of data,analysis of data & interpretation of data toward a valid decision.
Encyclopaedia Americana :-
As a name of a field of study, Statistics refers to the science & arts of obtaining & analysing quantitative data with a view to make sound inferences in the face of uncertainty.
Encyclopaedia Britannica:-
As is commonly understood now a days, Statistics is a mathematical discipline concerned with the study of masses of quantitative data of any kind.

WHAT IS THE MEANING OF THE TERM 'STATISTICS'?
As a singular noun it refers the science of collecting,analysing & interpreting numerical data relating to an aggregate of individuals. As a plural noun it denotes the numerical & quantitative information,e.g., labour statistics,vital statistics.

IS STATISTICS A SCIENCE?
Any Science has for its objectives the formulation of laws for explaining phenomena in some part of the real world with a deterministic view-point.
As Kendall explained ," Statistics is the branch of scientific method which deals with the data obtained by counting or measuring the properties of population of natural phenomena".
Indeed, we can call Statistical Methodology as Scientific Method. It is noted that STATISTICS is sometimes called the study of variation, i.e., a population or group without any variation &

uncertainty is no interest to Statistics. So, Statistics is the scientific methodology which deals with the collection,classification & tabulation of numerical facts as a basis for explanation,description & comparison of social phenomena.

DEFINE THE TERMS: STATISTICAL DATA,POPULATION,SAMPLE.

Statistical Data:- The numerical data or measurements obtained in case of an enquiry into a phenomenon,marked by uncertainty & variability,constitute Statistical Data. Uncertainty & variability are two major characteristics of Statistical data. Not all quantitative data is statistical data. Example of statistical data--Suppose we study the 'Heights of students in a particular college'.Here we can't predict the height of an individual with certainty & there will be variation in heights of students. Counter Example:--Multiplication table in a tabular form is a quantitative data,but since there is no uncertainty & variability involved in the data so it's not a Statistical Data.

Population:- A set or group of observations relating to a phenomenon under statistical investigation is known as statistical population or simply population. However,the term 'population' implies an aggregate or collection of measurements on a given variable(s). Population is said to be finite or infinite according to whether the set contains a finite or infinite number of observations.

Example-Measurements of heights in your college.

Note that: I. The characteristics of a population are called parameters. II.A population contains finite or infinite no of observations on a given variable(s).

Sample :- The set of data actually collected through a process of observation from selected items of any source is called a Sample. However, "Sample" is a subset of population or a true representation of population.

Example- Measurements of heights of students of Statistics department in your college.

Note that:- I. The characteristics of sample are called as Statistic. II. A sample is taken in order to gather information about a population.

WRITE DOWN THE DRAWBACKS OF STATISTICS?

Limitations of Statistics: I. Statistics deals with quantitative data only. II. Statistical law holds good only for aggregate of items or average individuals. It may not true for a particular individual or item. III. Inadequate knowledge of data interpretation may lead to invalid decision.

There are some sayings: "There are there kinds of lies--lies,white lies & Statistics" , "Figure won't lie,but liars figure" , "Statistics is like a clay of which you can make a god or a devil".

COMMENT ON THE FOLLOWING: "In a study of ages & professions of deceased men, it was found that the profession with the lowest average age of death was 'student' .So it appears that student profession is very dangerous."

It is obvious that every professional must have some basic education & it happens that the average age of every professional men must be higher than the age of the students. But it can happen that profession with lowest age of death was student . So the given statement is TRUE. But the conclusion made from the fact is incorrect . It can never be told that the student profession is dangerous. To conclude properly,we must have data for computing the proportions or percentages of deceased men in different professions. Therefore the conclusion

made here is absurd.
This is an example of the situation where inadequate information takes into bad decisions. Statistics is a science of DECISION MAKING. So,wrong data interpretation will show some absurd decision,might be harmful for society.

EXPLAIN THE STATEMENT : "Blindly using any data happened to be available can lead to misleading information & bad decision."
There are two kinds of people: Some of them believe that the inferences based on statistical data are very reliable & trusty. And others don't believe statistical results at all,they think it as a damned lies. But the fact is statistics is sometimes misused either deliberately or often due to lack of knowledge. Making conclusions based on inadequate information,deliberate manipulation & personal bias may lead to bad decision. Statistics are not to be blamed for all these. Statistical methods are most danger tool in the hand of non-experts.Lastly Statistics is like a clay of which you can make a god or a devil.

MEASURES OF CENTRAL TENDENCY

A Frequency Distribution corresponding to a variable specifies the values the variable takes and the frequencies or the number of times each variate value is taken.

Following are the marks obtained by 60 students in an examination:

22,47,9,42,31,17,13,15,18,13,2,21,27,38,15,0,33,10,34,29,26,16,25,33,36,10,24,2,26,19,14,36,18, 25,21,33,35,25,18,28,25,17,38,10,3,31,24,3,12,16,33,18,26,29,27,29,28,35,26,27.

Here the variable is the ‘number of marks’. The data in the above form is called **raw** or **ungrouped data**. This representation of the data does not furnish any useful information and is rather confusing to mind. To make the data more compact and understandable, we arrange the data from the array in ascending or descending order of magnitude to obtain a **Frequency Table.** Take each mark from the data and place a bar(|) or tally mark against the number when it occurs. Tally marks are recorded in batches of five, the fifth occurrence is shown by putting a cross tally(/) on the first four bars ||||/. We get the following frequency table of marks:

Frequency Table of Marks in an Examination

<table>
<tr><td>Marks:</td><td>0</td><td>2</td><td>3</td><td>9</td><td>10</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td><td>17</td><td>18</td></tr>
<tr><td>Tally marks:</td><td>|</td><td>|</td><td>||</td><td>|</td><td>|||</td><td>|</td><td>||</td><td>|</td><td>||</td><td>||</td><td>||</td><td>||||</td></tr>
<tr><td>Frequency:</td><td>1</td><td>1</td><td>2</td><td>1</td><td>3</td><td>1</td><td>2</td><td>1</td><td>2</td><td>2</td><td>2</td><td>4</td></tr>
</table>

<table>
<tr><td>Marks:</td><td>19</td><td>21</td><td>22</td><td>24</td><td>25</td><td>26</td><td>27</td><td>28</td><td>29</td><td>31</td><td>33</td><td>34</td></tr>
<tr><td>Tally marks:</td><td>|</td><td>||</td><td>||</td><td>||</td><td>||||</td><td>||||</td><td>|||</td><td>||</td><td>|||</td><td>||</td><td>||||</td><td>|</td></tr>
</table>

Frequency: 1 2 2 2 4 4 3 2 3 2 4 1

Marks: 35 36 38 42 47

Tally marks: || || || | |

Frequency: 2 2 2 1 1

If the identity of the individuals about whom a particular information is taken is not relevant , nor the order in which the observation arise, then the first real step of condensation of data is achieved by arranging the data into groups:

Frequency Table of Marks in an Examination

<table>
<tr><th>Marks (Class)</th><th>Tally Marks</th><th>No.of students (frequency)</th><th>Cumulative frequency(less than)</th><th>Cumulative frequency(greater than)</th></tr>
<tr><td>0-5</td><td>///</td><td>4</td><td>4</td><td>60</td></tr>
<tr><td>6-10</td><td>/</td><td>1</td><td>5</td><td>59</td></tr>
<tr><td>11-15</td><td>////| //</td><td>7</td><td>12</td><td>58</td></tr>
<tr><td>16-20</td><td>////| ////| /</td><td>11</td><td>23</td><td>52</td></tr>
<tr><td>21-25</td><td>////| /</td><td>6</td><td>29</td><td>45</td></tr>
<tr><td>26-30</td><td>////| ////| ////| /</td><td>16</td><td>45</td><td>29</td></tr>
<tr><td>31-35</td><td>////| //</td><td>7</td><td>52</td><td>23</td></tr>
<tr><td>36-40</td><td>////| /</td><td>6</td><td>58</td><td>12</td></tr>
<tr><td>41-45</td><td>/</td><td>1</td><td>59</td><td>5</td></tr>
<tr><td>46-50</td><td>/</td><td>1</td><td>60</td><td>4</td></tr>
</table>

This type of representation of frequencies is called a **grouped frequency distribution.** The groups 0-5, 6-10,... are called **classes;** 0 and 5 are called the **lower limit** and **upper limit** of the class 0-5 respectively. The difference 5-0=5 between the upper and lower limits of a class is called the **width** of the class. The value $\frac{0+5}{2}$=2.5 which lies midway between the lower and the upper limits is called the **mid-value** or **central value** of the class. The **less-than cumulative frequency** (**greater-than cumulative frequency** resp.) corresponding to a class is the total number of observations less than or equal to the upper limit (greater than or equal to the lower limit) of the class. Following points need be kept in mind while classifying given data:

- **Classes should be clearly defined and should not lead to any ambiguity**
- **Classes should be exhaustive**(each of given value should be included in one of the class**es)**
- **Classes should be mutually exclusive and non-overlapping**
- **Classes should be of equal width**

- **Number of classes should neither be too large nor too small; preferably it should lie between 5 and 15.**

NOTE: A variable, which can take any numerical value within certain range , is called a **continuous variable**. Consider frequency distribution of the continuous variable of ages in years of students in a college. We cannot arrange the data in age groups 16-20,21-25 etc. since there can be students having ages between 20 and 21 years. If the original 'inclusive' class intervals(of the form [a,b]) are ,say, 16-20,21-25,..., we calculate the adjustment $\frac{1}{2}$(lower limit of succeeding class-upper limit of a class)= $\frac{1}{2}$(21-20)=.5 and change the class intervals to 'exclusive' type([a,b)): 15.5-20.5, 20.5-25.5,.... It is understood that age of students whose age is ≥ 15.5 and < 20.5 are included in the class interval 15.5-20.5.

Comparison of Frequency Distributions

It is frequently necessary to compare two frequency distributions. If they are of different types , a precise comparison is difficult and is usually not required. If they are of same type, a comparison can be made in terms of values of the following four types of measures:

- **Measure of location** or **central tendency** gives a single value around which largest number of values of the variate tend to cluster.
- **The scale parameter** or **measure of dispersion** gives the degree of scatter about the central value. It measures variability or lack of homogeneity of data.
- **Measure of skewness** measuring degree of departure from symmetry
- **Measure of Kurtosis** measuring degree of 'flatness' of the 'top' as compared with the 'normal' curve.

Characteristics of a good measure of Central Tendency

- ✓ It should be based on all observations
- ✓ It should not be affected much by extreme values
- ✓ It should be rigidly defined
- ✓ It should be easily understandable and easy to calculate
- ✓ It should be amenable to algebraic treatment
- ✓ It should be least affected by fluctuation of sampling: if a number of samples of same size are drawn from a population, the measure of central tendency having minimum variation among the different calculated values should be preferred.

Different Measures of Central Tendency

- Arithmetic Mean
- Geometric Mean
- Harmonic Mean
- Median and Quartiles
- Mode

Arithmetic Mean

If a variate X takes values $x_1,\ldots,x_n$, then the A.M. of the set of observations $x_1,\ldots,x_n$, is defined by $\bar{x} = \frac{x_1+\cdots+x_n}{n}$. If the variate-values are not of equal 'importance', we may attach to them 'weights' $w_1,\ldots,w_n$ as measures of their importance; the corresponding weighted mean is defined by

$\bar{x} = \frac{w_1x_1+\cdots+w_nx_n}{w_1+\cdots+w_n}$.

In particular, if the variate-value x_1 occurs f_1 times, x_2 occurs f_2 times,…, then $\bar{x} = \frac{f_1x_1+\cdots+f_nx_n}{f_1+\cdots+f_n} = \frac{1}{N}\sum_1^n f_kx_k$, where $N=f_1+\ldots+f_n$ is the total frequency.

Note the A.M. of a grouped or continuous frequency distribution is computed by above formula where x's denote the mid-values of the corresponding class intervals.

Example 1.1 Find the A.M. of following frequency distribution:

x:	1	2	3	4	5	6	7
f:	5	9	12	17	14	10	6

- **Computation of Mean**

x	f	fx
1	5	5
2	9	18
3	12	36
4	17	68
5	14	70
6	10	60
7	6	42
Total	73	299

Thus $\bar{x} = \frac{1}{N}\sum_1^n f_kx_k = \frac{299}{73}$.

Example 1.2 Find the A.M. of following frequency distribution:

Marks:	0-10	10-20	20-30	30-40	40-50	50-60
No. of students:	12	18	27	20	17	6

- **Computation of Mean**

Marks	No. of students(f)	Mid-point(x)	fx
0-10	12	5	60
10-20	18	15	270
20-30	27	25	675

30-40	20	35	700
40-50	17	45	765
50-60	6	55	330
Total	**100**		**2800**

A.M.= $\bar{x} = \frac{1}{N}\sum_1^n f_k x_k = \frac{2800}{100} = 28.$

Change of origin and scale

Let x and u be two variates related by u=$\frac{x-a}{h}$. Then $\sum fx = \sum af + \sum hfu$. Thus $\bar{x} = \frac{\sum fx}{\sum f} = a + h\frac{\sum fu}{\sum f} = a + h\bar{u}$.

Thus mean is dependent on both change of origin a and scale h.

Example 1.3 Find the A.M. of following frequency distribution:

Marks:	0-10	10-20	20-30	30-40	40-50	50-60
No. of students:	12	18	27	20	17	6

- Let us take the origin a=300 and scale h=50 so that u=$\frac{x-300}{50}$.

Properties of A.M.

- Algebraic sum of deviations of a set of variate values from their arithmetic mean is zero.
  - $\sum f_i(x_i - \bar{x})$=$\sum f_i x_i - \bar{x}\sum f_i$=N$\bar{x}$-N$\bar{x}$ =0, where N=$\sum f_i$.

- (Mean of the combined distribution) If $\overline{x_1}$,..., $\overline{x_k}$ be the A.M.s of k distributions with respective frequencies n_1,...,n_k, then the mean $\bar{x}$ of the combined distribution of frequency N=$\sum n_i$ is given by: $\bar{x}$=$\frac{1}{N}\sum_{i=1}^k n_i\overline{x_i}$.

Example 1.4 The average salary of male employees in a firm was Rs. 5200 and that of females was Rs. 4200. The mean salary of all the employees was Rs. 5000. Find the percentage of male and female employees.

- Let n_1 and n_2 denote respectively the number of male and female employees and $\overline{x_1}$ and $\overline{x_2}$ denote their average salary (in Rs.). Then 5000(n_1+n_2)=5200 n_1+4200 n_2 implying n_1:n_2::4:1. Thus percentage of male and female employees in the firm is 80% and 20% respectively.

Geometic Mean

If n positive values $x_1,\ldots,x_n$ occur $f_1,\ldots,f_n$ times respectively, then geometric mean(G.M.) G of the set of observations is defined by G=$[{x_1}^{f_1}\ldots{x_n}^{f_n}]^{\frac{1}{N}}$, where N=$\sum_1^n f_i$.

Harmonic Mean

The harmonic mean H of n non-zero variate values x_i with frequencies f_i is given by H=$\frac{\sum f_i}{\sum\frac{f_i}{x_i}}$.

Relation between A.M., G.M. and H.M.

If A,G,H stand for the A.M., G.M. and H.M. respectively of a finite series of positive values **of a variate**, then it can be proved that A$\geq G \geq H$.

MEDIAN

Mean can not be calculated whenever there is frequency distribution with open end classes. Also the mean is affected to a great extent by presence of extreme value in the set of observations. For instance, if salary of 8 persons be Rs. 150,225,240,260,275,290,300 and 1500, the mean salary is Rs. 405, which is not a good measure of central tendency because out of the 8 people, seven get Rs. 300 or less.

Median of a finite set of variate values is the value of the variate which divides it into two equal parts. It is the value which exceeds and is exceeded by same number of observations. Median is thus a positional average.

In case of ungrouped data, if the number of observations is odd then median is the middle value after the values have been arranged in ascending or descending order of magnitude. In case of even number of observations, there are two middle terms and median is taken to be the arithmetic mean of the middle terms. Thus , median of the values 25,20,15,35,18, that is, of 15,18,20,25,35 is 20 and the median of 8,20,50,25,15,30, that is, of 8,15,20,25,30,50 is (20+25)/2=22.5.

In case of discrete frequency distribution, median is obtained as follows:

- Find N/2, where N=$\sum f_i$.
- Find the cumulative frequency (less than type) just greater than N/2
- The corresponding value of the variate is the median.

Example 1.5 Obtain the median for the following frequency distribution:

x:	1	2	3	4	5	6	7	8	9
f:	8	10	11	16	20	25	15	9	6

- **Calculation of Median**

x	f	c.f.
1	8	8
2	10	18

3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120=N

N/2=60. The c.f. just greater than N/2 is 65 and the value of x corresponding to 65 is 5. Thus , median is 5.

In case of grouped frequency distribution, median is obtained as follows:

Let us consider the grouped frequency distribution:

Class intervals	frequency	cumulative frequency
$x_1 - x_2$	f_1	F_1
$x_2 - x_3$	f_2	F_2
…	….	….
$x_p - x_{p+1}$	f_p	F_p
….	…..	…
$x_n - x_{n+1}$	f_n	F_n

where $F_k=\sum_{i=1}^{k} f_i$. Let the smallest c.f. greater then N/2 is F_p. Then the **median class** is $x_p - x_{p+1}$. We assume that frequency of a class is uniformly distributed over the class interval. Let the c.f. for the class just above the median class be c . Thus (N/2-c) is the frequency of the interval between the median and the lower limit of the median class . the length of the interval corresponding to the frequency (N/2-c) is $\frac{\frac{N}{2}-c}{f}$ I, where f is frequency of the median class, I is the length of the class interval of the median class . Hence the median is $L_0+\frac{\frac{N}{2}-c}{f}$ I, where L_0 is lower limit of the median class.

Properties of Median

- Median is a positional average and hence is not influenced by extreme values
- Median can be calculated even in the case of open end intervals
- Median can be located even if the data is incomplete
- It is not a good representative of data if the number of observations is small
- It is not amenable to algebraic treatment
- It is susceptible to sampling fluctuations

Quartiles are thsose variate values which divide the total frequency into four equal parts; **deciles** and **percentiles** divide into ten and hundred equal parts respectively. Suppose the values of the variate have been arranged in ascending order of magnitude, then the value of the quartile having the position between the lower extreme and the median , is the first quartile Q_1 and that between the median and the upper extreme is the third quartile Q_3. The median is the second quartile Q_2, is the fifth decile D_5 and the fiftieth percentile P_{50}. For a grouped frequency distribution, the quartiles, deciles and percentiles are given by

$$Q_i = l + \frac{\frac{iN}{4} - C}{f} h, \ i=1,2,3$$

$$D_j = l + \frac{\frac{jN}{10} - C}{f} h, \ j=1,\ldots,9$$

$$P_k = l + \frac{\frac{kN}{100} - C}{f} h, \ k=1,\ldots,99$$

where l is the lower limit of the class in which the particular quartile/decile/percentile lies, f is the frequency of the class , h is the width of this class, C is the cumulative frequency upto and including the class preceding the class in which the particular quartile/decile/percentile lies and N is the total frequency.

Example 1.6 Calculate the three quartiles for the following frequency distribution of the number of marks obtained by 49 students in a class:

Marks	**No. of students**	**Marks**	**No. of students**
5-10	5	25-30	5
10-15	6	30-35	4
15-20	15	35-40	2
20-25	10	40-45	2

Cumulative Frequency Table

Class	**Frequency**	**Cumulative Frequency(less than)**
5-10	5	5
10-15	6	11
15-20	15	26
20-25	10	36
25-30	5	41
30-35	4	45

35-40	2	47
40-45	2	49=N

The cumulative frequency immediately greater than N/4=49/4 is 26; hence to find Q_1,

L=15, h=15-10=5, C=11, f=15. Thus $Q_1= 15+\frac{\frac{49}{4}-11}{15}5= 15.47$ marks.

For median, N/2= 24.5 . Thus the median class is 15-20. Median = $15+\frac{\frac{49}{2}-11}{15}5=19.5$ marks.

To find Q_3, we have 3N/4=147/4 . Hence Q_3 lies in the class 25-30. L=25, C=36,f=5,h=5. Hence $Q_3=25+\frac{\frac{147}{4}-36}{5}5=25.75$.

Example 1.7 In a factory employing 3000 persons, in a day 5% work less than 3 hours, 580 work from 3.01 to 4.50 hours, 30% work from 4.51 to 6.00 hours, 500 work from 6.01 to 7.50 hours, 20% work from 7.51 to 9.00 hours and the rest work 9.01 or more hours. What is the median hours of work?

- **Calculation for Median Wages**

Work Hours	No. of employees(f)	Less than c.f.	Class Boundaries
Less than 3	5/100 x 3500=150	150	Below 3.005
3.01-4.50	580	730	3.005-4.505
4.51-6.00	30/100 x 3000=900	1630	4.505-6.005
6.01-7.50	500	2130	6.005-7.505
7.51-9.00	20/100 x 3000=600	2730	7.505-9.005
9.01 and above	3000-2730=270	3000	9.005 and above

N=3000. The c.f. just greater than N/2=1500 is 1630.the corresponding class 4.51-6.00, whose class boundaries are 4.505-6.005, is the median class. Hence median=$l+\frac{h}{2}\left(\frac{N}{2}-c\right)=4.505+\frac{1.5}{900}(1500-730)=4.505+1.283=5.79$(approx.).

Example 1.8 An incomplete frequency distribution is given as follows:

Variable	**Frequency**	**Variable**	**Frequency**
10-20	12	50-60	?
20-30	30	60-70	25
30-40	?	70-80	18
40-50	65	Total(N)	**229**

Given that the median value is 46, determine the missing frequencies.

- Let the frequency of the class 30-40 be f_1 and that of 50-60 be f_2. Then $f_1+f_2=229 - (12+30+65+25+18)= 79$.
  Since median is given to be 46, 40-50 is the median class. Using formula for median , we get
  $46=40+\frac{114.5-(12+30+f_1)}{65}$ x 10. Hence f_1= 33.5= 34(approx.). Hence f_2=45.

Mode

Let us consider the following statements: The average height of an Indian is 5'6"; the average size of shoes sold in a shop is 7; an average student in a hostel spends Rs. 750 per month. In all the above statements,the average referred to is mode. Mode is the value of the variate which occurs most frequently in a set of observations and around which the other members of the set cluster densely. In other words, mode is the value of the variable which is predominant in the given set of values. In case of discrete frequency distribution, mode is the value of the variable corresponding to maximum frequency. In the following distribution:

x:	1	2	3	4	5	6	7	8
f:	4	9	16	25	22	15	7	3

value of x corresponding to maximum frequency viz. 25 is 4. Hence mode is 4.

In any one of the following cases, mode is determined by the method of grouping:

- If the maximum frequency is repeated
- If the maximum frequency occurs in the very beginning or at the end of the distribution
- If there are irregularities in the distribution

Example 1.9 Find the mode of the following frequency distribution:

Size(x):	1	2	3	4	5	6	7	8	9	10	11	12
Frequency:	3	8	15	23	35	40	32	28	20	45	14	6

- The distribution is not regular since the frequencies are increasing steadily upto 40 and then decrease but the frequency 45 after 20 does not seem to be consistent with the distribution. We cannot say that since the maximum frequency is 45, mode is 10. Here we locate mode by the method of grouping as explained below:

Size			Frequency			
(x)	**(i)**	**(ii)**	**(iii)**	**(iv)**	**(v)**	**(vi)**
1	3					
		11				
2	8			26		
			23			
3	15				46	
		38				
4	23					
			58			
5	35			98		73
		75				
6	40				107	
			72			
7	32					
		60				
8	28			80		100
			48			
9	20					
		65			93	
10	45					79
			59			
11	14			65		
		20				
12	6					

The frequencies in column (i) are the original frequencies. Column (ii) is obtained by combining the frequencies two by two.If we leave the first frequency and combine the remaining frequencies two by two, we get column (iii).Combining the frequencies two by two after leaving the first two frequencies results in a repetition of column (ii). Hence, we proceed to combine the frequencies three by three , thus getting column (iv). The combination of frequencies three by three after leaving the first frequency results in column (v) and after leaving the first two frequencies results in column (vi).

Analysis Table

Column No.	Maximum Frequency(1)	Value or combination of values of x giving max. frequency in (1) (2)
(i)	45	10
(ii)	75	5,6
(iii)	72	6,7
(iv)	98	4,5,6
(v)	107	5,6,7
(vi)	100	6,7,8

Mode for Continuous frequency distribution

Mode= l+ $\frac{f_m - f_1}{2f_m - f_1 - f_2}$ h

Where l is the lower limit of the modal class(class having maximum frequency), f_m is the maximum frequency, f_1 and f_2 are the frequencies of the classes preceding and following modal class.

Example 1.10 The median and mode of the following wage distribution are known to be Rs. 3350 and Rs. 3400 respectively.Find the values of f_3,f_4,f_5:

Wages (in Rs.)	No. of employees	Wages (in Rs.)	No. of employees
0-1000	4	4000-5000	f_5
1000-2000	16	5000-6000	6
2000-3000	f_3	6000-7000	4
3000-4000	f_4	**Total**	230

- **Calculation for median and mode**

Wages (in Rs.)	frequency(f)	less than c.f.
0-1000	4	4
1000-2000	16	20
2000-3000	f_3	$20+f_3$
3000-4000	f_4	$20+f_3+f_4$
4000-5000	f_5	$20+f_3+f_4+f_5$
5000-6000	6	$26+f_3+f_4+f_5$
6000-7000	4	$30+f_3+f_4+f_5=N$

$N=30+f_3+f_4+f_5=230$. Thus $f_3+f_4+f_5=200$.

Since median is 3350, which lies in 3000-4000, 3000-4000 is the median class. Using median formula,

$3350=3000+\frac{1000}{f_4}[115-(20+f_3)]$. Thus $f_3=95-0.35f_4$.

Mode being 3400, modal class is 3000-4000. Using formula for mode,

$3400=3000+\frac{1000(f_3-f_4)}{2f_4-f_3-f_5}$; hence $\frac{3400-3000}{1000}=\frac{f_4+0.35f_4-95}{2f_4-(200-f_4)}$. Thus $f_4=100$. Hence $f_3=95-0.35\text{x}100=60$, $f_5=40$.

Note: For a symmetrical distribution, mean, median and mode coincide. If the distribution is **moderately asymmetrical**, they obey the following empirical relationship: **mode = 3 median – 2 mean**

MEASURES OF DISPERSION

A measure of central tendency alone is not enough to have a clear idea about the data unless all observations are almost the same. Moreover two sets of observations may have the same central tendencies whereas variability of data within the sets may vary widely . Consider

Set A:	30	30	30	30	30
Set B:	28	29	30	31	32
Set C:	3	5	30	37	75

All the three sets have same mean and mode; but the amount of variation differs widely amongst the sets.

Characteristics of an ideal measure of dispersion

- It should be rigidly defined
- It should be easily understandable and easy to calculate
- It should be based on all observations
- It should be amenable to further mathematical treatment
- It should be least affected by fluctuation of sampling

Different measres of dispersion

Range: Range is the difference between the maximum and the minimum values of the variate. It is easily understood and easy to calculate but depends only on the two extreme values which themselves are subject to sampling fluctuation; hence range is not a reliable measure of dispersion.

Quartile Deviation: quartile deviation or semi-interquartile range is given by $\frac{1}{2}(Q_3-Q_1)$, where Q_1 and Q_3 are the first and the third quartiles of the frequency distribution. Quartile deviation is definitely a better measure than the range as it makes use of 50% of data. But since ignores the other 50% of data, it cannot be regarded as a reliable measure.

Mean Deviation If $x_i|f_i$, i=1,...,n be the frequency distribution, then mean deviation from A (usually mean, median or mode) is defined by s= $\frac{1}{N}\sum_{i=1}^{n} f_i|x_i - A|$, $\sum_{i=1}^{n} f_i$=N.

Since mean deviation is based on all the observations, it is a better measure of dispersion as compared to range and quartile deviation. But use of absolute value renders it useless for further mathematical treatment.

Example 2.1 Calculate Q.D. and M.D. from mean, for the following data:

Marks: 0-10 10-20 20-30 30-40 40-50 50-60 60-70

No. of students: 6 5 8 15 7 6 3

- **Calculation for Q.D. and M.D. from mean**

Marks	Mid-value	f	d=(x-35)/10	fd	$\lvert x-\bar{x}\rvert$	f$\lvert x-\bar{x}\rvert$	c.f. (less)
0-10	5	6	-3	-18	28.4	170.4	6
10-20	15	5	-2	-10	18.4	92.0	11
20-30	25	8	-1	-8	8.4	67.2	19
30-40	35	15	0	0	1.6	24.0	34
40-50	45	7	1	7	11.6	81.2	41
50-60	55	6	2	12	21.6	129.6	47
60-70	65	3	3	9	31.6	94.8	50
Total				-8		659.2	

(1) Here N=50, N/4=12.75, 3N/4=37.25

The c.f.(less than) just greater than 12.75 is 19. Hence the corresponding class 20-30 contains Q_1.

$Q_1=20+\frac{10}{8}(12.75-11)=22.19$

The c.f.(less than) just greater than 37.25 is 41. Hence the corresponding class 40-50 contains Q_3.

$Q_3=40+\frac{10}{7}(37.25-34)=44.64$.

Hence Q.D. $=\frac{1}{2}(Q_3-Q_1)=\frac{1}{2}(44.64-22.19)=11.23$.

(2) $\bar{x}=$ A+$\frac{h\sum fd}{N}$=35+$\frac{10X(-8)}{50}$ = 33.4 marks. Thus M.D. (from mean) $=\frac{1}{N}\sum f\lvert x-\bar{x}\rvert=\frac{659.2}{50}$=13.184.

Standard Deviation, Variance

For the frequency distribution $x_i|f_i$, i=1,...,n , S.D. σ is defined by: $\sigma=\sqrt{\frac{1}{N}\sum f_i(x_i-\bar{x})^2}$, where $\bar{x}$ is the A.M. of the distribution (non-negative value of the square root is considered). σ^2 is called the variance.

Note $s^2=\frac{1}{N}\sum f_i(x_i-A)^2=\frac{1}{N}\sum f_i(x_i-\bar{x}+\bar{x}-A)^2=\frac{1}{N}\sum f_i\{(x_i-\bar{x})^2+(\bar{x}-A)^2+2(\bar{x}-A)(x_i-\bar{x})\}$

$=\frac{1}{N}\sum f_i(x_i-\bar{x})^2+(\bar{x}-A)^2\frac{1}{N}\sum f_i+2(\bar{x}-A)\frac{1}{N}\sum f_i(x_i-\bar{x})^2=\sigma^2+(\bar{x}-A)^2$, where $d=\bar{x}-A$.

Thus s^2 is least when d=0.that is, when $\bar{x}$=A. Thus M.D. is least when A=$\bar{x}$ and S.D. is least value of M.D.

Note (1) $\sigma^2=\frac{1}{N}\sum f_i(x_i-\bar{x})^2=\frac{1}{N}\sum f_i(x_i^2+\bar{x}^2-2x_i\bar{x})=\frac{1}{N}\sum f_i x_i^2+\bar{x}^2\frac{1}{N}\sum f_i-2\bar{x}\frac{1}{N}\sum f_i x_i=\frac{1}{N}\sum f_i x_i^2+\bar{x}^2-2\bar{x}^2=\frac{1}{N}\sum f_i x_i^2-\left(\frac{1}{N}\sum f_i x_i\right)^2$. This expression is often used for calculation of σ^2.

(2) If n_1 and n_2 are the sizes, $\overline{x_1}$ and $\overline{x_2}$ be the means and σ_1 and σ_2 be the S.D. s of two series, then the S.D. σ of the combined series of n_1+n_2 observations is given by: $\sigma^2=\frac{1}{n_1+n_2}[n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)]$, where $d_1=\overline{x_1}-\bar{x}$, $d_2=\overline{x_2}-\bar{x}$ and $\bar{x}=\frac{n_1\overline{x_1}+n_2\overline{x_2}}{n_1+n_2}$ is the mean of the combined series.

Example 2.2 For a group of 200 candidates , the mean and S.D. of scores were found to be 40 and 15 respectively. Later on it was discovered that the scores 43 and 35 were misread as 34 and 53 respectively. Find the corrected mean and S.D. corresponding to the corrected figures.

- Let x be the given variable. Given n=200, $\bar{x}$=40 and σ=15. Now 40= $\bar{x}=\frac{1}{200}\sum x_i$ gives $\sum x_i$=8000.
  $\sigma^2=\frac{1}{n}\sum x_i^2-\bar{x}^2$ gives $\sum x_i^2$=200(225+1600)=365000.
  Corrected $\sum x_i$=8000-34-53+43+35=7991, corrected mean $=\frac{7991}{200}$=39.995
  Corrected $\sum x_i^2$= 365000-34^2-53^2+43^2+35^2=364109
  Corrected $\sigma^2=\frac{364109}{200}-(39.995)^2$=224.14. Thus corrected $\sigma=\sqrt{224.14}$=14.97.

Example 2.3 The first of two samples has 100 items with mean 15 and s.d. 3.if the whole group has 250 items with mean 15.6 and s.d. $\sqrt{13.44}$. find s.d. of the second group.

- Here n_1=100,$\overline{x_1}$=15, σ_1=3, n=n_1+n_2=250, $\bar{x}$=15.6, $\sigma=\sqrt{13.44}$
  $\bar{x}=\frac{n_1\overline{x_1}+n_2\overline{x_2}}{n_1+n_2}$ gives $\overline{x_2}$=16. Hence $d_1=\overline{x_1}-\bar{x}$=15-15.6=-0.6, $d_2=\overline{x_2}-\bar{x}$=16-15.6=0.4
  From $\sigma^2=\frac{1}{n_1+n_2}[n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)]$, σ_2=4.

Moments

The r th (raw) moment of a variable x about any point A , denoted by $\mu_r^/$, is given by $\mu_r^/=\frac{1}{N}\sum f_i(x_i-A)^r$.

The r th (central) moment of a variable x about mean $\bar{x}$, denoted by μ_r , is given by μ_r=$\frac{1}{N}\sum f_i(x_i-\bar{x})^r$.

In particular, μ_0=$\frac{1}{N}\sum f_i(x_i-\bar{x})^0$=$\frac{1}{N}\sum f_i$=1, μ_1=$\frac{1}{N}\sum f_i(x_i-\bar{x})^1$=0, μ_2=$\frac{1}{N}\sum f_i(x_i-\bar{x})^2$= σ^2.

Relation between raw and central moments

μ_r=$\frac{1}{N}\sum f_i(x_i-\bar{x})^r$=$\frac{1}{N}\sum f_i(x_i-A+A-\bar{x})^r$=$\frac{1}{N}\sum f_i(d_i+A-\bar{x})^r$, where d_i=x_i-A.

$\bar{x}$ =A+ $\frac{1}{N}\sum f_i d_i$ =A+ μ_1' . Hence $\mu_r=\frac{1}{N}\sum f_i(d_i-\mu_1')^r=\frac{1}{N}\sum f_i(d_i^r-C_1^r\,d_i^{r-1}\mu_1'+C_2^r d_i^{r-2}(\mu_1')^2$ -...+(-1)$^r(\mu_1')^r)$

=μ_r'-$C_1^r\mu_{r-1}'\mu_1'$+$C_2^r\mu_{r-2}'(\mu_1')^2$-...+(-1)$^r(\mu_1')^r$.

In particular, on putting r=2,3,4 and simplifying, we get μ_2=μ_2'-${\mu_1'}^2$, μ_3=μ_3'-3$\mu_2'\mu_1'$+2${\mu_1'}^3$, μ_4=$\mu_4'-4\mu_3'\mu_1'$+6$\mu_2'{\mu_1'}^2$-3${\mu_1'}^4$.

Effect of change of origin and scale on Moments

Let u=$\frac{x-A}{h}$. Then $\bar{x}=A+h\bar{u}$. Thus x-$\bar{x}$=h(u-$\bar{u}$). Thus $\mu_r(x)$=$\frac{1}{N}\sum f_i(x_i-\bar{x})^r$ =$\frac{1}{N}\sum f_i\{h(u_i-\bar{u})\}^r$ =$h^r\mu_r(u)$.

Symmetrical and Skew Distributions

A distribution is symmetrical when the frequencies are symmetrically distributed about the mean , that is, when the values of the variate equidistant from mean have equal frequencies. For example, the following distribution is symmetrical about its mean 5:

x:	1	2	3	4	5	6	7	8	9
f:	3	4	6	9	10	9	6	4	3

It can be seen that if n is odd, $\frac{1}{N}\sum_1^n f_i(x_i-\bar{x})^n$ =0 since all the terms cancel in pairs, n being odd and f_1=f_n, f_2=f_{n-1},.... Thus μ_n=0 , for n odd. Hence for a symmetrical distribution, $\beta_1=\frac{\mu_3^2}{\mu_2^3}$ = 0. Thus β_1 is a measure of departure from symmetry.

Also for a symmetrical distribution, the mean, median and mode coincide. Further, in the case of such distribution, median lies halfway between the two quartiles.

Skewness means lack in symmetry. It indicates whether the frequency curve is inclined more to one side than the other, that is , whether the frequency curve has a longer tail on one side. **Skewness is positive** if the curve is more elongated to the right side, that is, if the mean of the distribution is greater than the mode; in the reverse case, it is negative. Skewness gives an idea about the direction in which also the extent to which the distribution is distorted from the symmetrical distribution.

For distribution of moderate skewness, an empirical relation holds: mean-mode= 3(mean-median).

Karl Pearson's coefficient of skewness is given by : coefficient of skewness=$\frac{mean-mode}{standard\ deviation}$=$\frac{3(mean-median)}{standard\ deviation}$.

It is a pure number since the numerator and denominator have the same dimension. It has value zero for a symmetrical distribution.

Bowley's measure of skewness is $\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}$.

Example 2.4 Find out a coefficient of dispersion based on quartile deviation and a measure of skewness from the following table giving wages of 230 persons:

Wages(in Rs)	**f**	**c.f.**	**Wages(in Rs)**	**f**	**c.f.**
140-160	12	12	220-240	50	157
160-180	18	30	240-260	45	202
180-200	35	65	260-280	20	222
200-220	42	107	280-300	8	230

- Here N/2=115 and the 115 person has a wage in the class 220-240. Hence median=Q_2= 220+ $\frac{115-107}{50}$ x 20 = Rs. 223.20. Similarly, Q_1=180+ $\frac{57.5-30}{35}$ x 20=Rs. 195.71, Q_3=240+$\frac{172.5-157}{45}$x 20=Rs. 246.88. It can be shown that mean=Rs. 220.87, S.D.= Rs. 34.52.
  Coefficient of dispersion based on quartile deviation= $\frac{Q_3-Q_1}{Q_3+Q_1}$=$\frac{51.17}{442.59}$=0.1156.
  Measure of skewness = $\frac{mean-mode}{S.D.}$=$\frac{220.87-233.00}{34.52}$= - 0.3514
  Second measure of skewness=$\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}$=- 0.07446.

Pearson's $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$- Coeficients

$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$, $\gamma_1 = \sqrt{\beta_1}$, $\beta_2 = \frac{\mu_4}{\mu_2^2}$, γ_2=β_2-3.

The values of the two coefficients β_1, β_2 enable us to know whether the given distribution is symmetrical and whether it is relatively more or less flat than the normal curve. β_1 gives a measure of departure from symmetry. Kurtosis measures whether the given frequency curve is relatively more or less flat –topped compared to the normal curve (to be studied later). For a normal distribution , β_2=3. Curves with values of β_2 less than 3 are called Platykurtic whereas those with values of β_2 greater than 3 are called Lettokurtic. Curves with $\beta_2$value equal to 3 are called Mesokurtic.

BASIC PROBABILITY THEORY

Basic Terminology

Random Experiment : If in each trial(repetition) of an experiment conducted under identical conditions, the outcome is not unique , but may be any one of possible outcomes, then such an experiment is called a random experiment. Examples of random experiments are: tossing a coin, throwing a die, selecting a card from a pack of playing cards etc. in all these cases, there are a number of possible outcomes which can occur but there is an uncertainty as to which of them will actually occur.

A piece of Information: A pack of cards consists of four suits called Spades, Hearts and Clubs. Each suit consists of 13 cards, of which nine cards are numbered from 2 to 10, an ace, a king, a queen and a jack(or knave).Spades and clubs are black-faced cards while hearts and diamonds are red-faced cards.

Outcome: result of a random experiment is called an outcome.

Sample Space, Events: The collection S of all possible outcomes of a random experimepernt is called sample space of the random experiment; any subset of S is an event; a singleton subset of S is an elementary(simple) event. For example, in an experiment which consists of throwing a six-faced die, possible outcomes are 1,2,3,4,5,6. Thus sample space of this experiment is {1,2,3,4,5,6}, {1} is an elementary event; getting an even number, {2,4,6} is an event of this experiment.

Exhaustive Events: events $E_1,\ldots,E_k$ of a random experiment with sample space S are called exhaustive iff $S = E_1 \cup \ldots \cup E_k$. in the case of throwing of a die, getting even points(that is, 2,4, or 6) and getting odd points (that is, 1,3, or 5) are exhaustive events.

Equally Likely Events: Events are equally likely if there is no reason to expect any one of them compared to others. In the trial of drawing a card from a well-shuffled pack of cards, any of the 52 cards may appear, so that the 52 elementary events are equally likely.

Exclusive Events: Events are exclusive if the occurrence of any one of them precludes the occurrence of all others. On the contrary, events are compatible if it is possible for them to happen simultaneously. For instance, in the rolling of two dice, the cases of the face marked 5 appearing on both dice are compatible.

Favourable Events: The trials which entail the happening of an event are favourable to the event. For example, in the tossing of a dice, the number of favourable events to the appearance of a multiple of 3 are two viz. getting 3 and 6.

Classical (a priori) definition of probability

If a random trial results in n exhaustive, mutually exclusive and equally likely outcomes, out of which m are favourable to the occurrence of an event E, then the probability of occurrence of E , denoted by P(E), is given by P(E)=$\frac{m}{n}$.

It is clear from definition that 0≤p≤1. Since the number of cases in which event A will not happen is n-m, the probability q that the event A will not happen is given by P($\bar{A}$)=$\frac{n-m}{n}$=1-$\frac{m}{n}$=1-P(A).

An event A is **certain** to happen if all the trials are favourable to it and then the probability of its happening is unity; for an event which is certain not to happen, the probability is zero.

Example 3.1 Find the chance that if a card is drawn at random from an ordinary pack, it is one of the court cards.

- Court cards are kings, queens, jacks and their number in a pack of 52 cards is 12, so that the number of favourable events is 12. Hence the probability is 12/52=3/13.

Example 3.2 What is the chance that a leap year selected at random will contain 53 Sundays?

- A leap year which contains 366 days has 52 Sundays corresponding to 52 weeks and 2 more days.There are following seven possibilities: (1) Sunday, Monday, (2) Monday,Tuesday,(3) Tuesday, Wednesday, (4) Wednesday, Thursday, (5) Thursday, Friday, (6) Friday , Saturday, (7) Saturday, Sunday. Out of these seven possibilities, there are two favourable outcomes, namely (1) and (8). Thus the required probability is 2/7.

Example 3.3 An urn contains 9 balls, two of which are red, three blue and four black.Three balls are drawn from the urn at random. What is the chance that (1) three balls are of different colours, (2) two balls are of the same colour and third is different, (3) the balls are of the same colour?

- (1) Three balls can be drawn from 9 balls in C_3^9=84 ways and these are equally likely, exhaustive and mutually likely cases. A red ball can be drawn in 2 ways, a blue in 3 and a black in 4 ways, so that three differently coloured balls can be drawn in 2 x 3x 4=24 ways. Hence the probability is 24/84=2/7.

  (2)two blue balls can be drawn in C_2^3 ways and then a red or black ball in 6 ways so that the two blue balls and a different coloured ball can be drawn in 6xC_2^3=18 ways. Two black balls and a different coloured ball can be drawn in 5xC_2^4=30 ways. Similarly the number of ways in which two red balls and a different coloured ball can be drawn in 7xC_2^2=7 ways. Thus the number of ways two balls of same colour and a ball of different colour can be drawn is 18+30+7=55. Thus required probability is 55/84.

(3)Three blue balls can be drawn in 1 way and 3 black balls in C_3^4 or 4 ways so that the corresponding probability is 5/84.

Limitation of Classical Definition

This definition breaks down in the following cases:

- If the various outcomes of the random experiment are not equally likely
- If the number of exhaustive outcomes of the random experiment is infinite or unknown

Von Mises's statistical (or empirical) definition of probability

If trials be repeated a great number of times under essentially same conditions, then the limit of the ratio of the number of times that an event happens to the total number of trials, as the number of trials increases indefinitely, is the probability of the happening of the event,provided the ratio approaches a finite and a unique limit.

Axiomatic definition of Probability

To an event A(that is, a subset of sample space S) is assigned a real number P(A), called probability of A, satisfying the following properties:

- (Axiom of non-negativity)P(A)≥0
- (Axiom of certainty) P(S)=1
- (Axiom of Additivity) If $\{A_n\}$ is any finite or infinite sequence of **disjoint** events, then $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$.

Note P, **the probability function,** is otherwise unspecified except it is to satisfy above three axioms.

Notation for two events A,B of a sample space S,A∪B={x∈S: x∈A or x∈B}, A∩B={x∈S: x∈A and x∈B},$\bar{A}$ = { x∈S: x∉A}, A-B={ x∈S: x∈ A and x∉B}, A∪B can be denoted by A+B, if A and B are disjoint; AΔB=($\bar{A}$ ∩B)+(A∩ $\bar{B}$).

Example 3.4 Let A,B,C are three arbitrary events. Find expression for the following events:

(1)Only A occurs, (2) Both A,B but not C , occur, (3) All three events occur, (4) At least one occurs, (5) At least two occur, (6) one and no more occurs, (7) two and no more occur, (8) none occurs.

➢ (1) $A\cap \bar{B} \cap \bar{C}$, (2) $A\cap B \cap \bar{C}$, (3) $A\cap B \cap C$, (4)$A\cup B \cup C$, (5) $(A\cap B \cap \bar{C})\cup(A\cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$, (6) $(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$, (7) $(A \cap B \cap \bar{C})\cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$, (8) $\overline{A \cup B \cup C}$.

Some Theorems on Probability

Theorem3.1 Probability of impossible event is zero: P(∅)=0.

- $S = S \cup \emptyset$ and S, $\emptyset$ are disjoint events. Using axiom of additivity, $P(S)=P(S \cup \emptyset)=P(S)+P(\emptyset)$; hence $P(\emptyset)=0$.

Note: P(A)=0 does not necessarily mean A is impossible event. In case of continuous random variable X, the probability at a point is always zero: $P(X=c)=0$.

Theorem3.2 $P(\bar{A})=1-P(A)$.

- $1=P(S)=P(A\cup \bar{A})=P(A)+P(\bar{A})$ (since $A\cap \bar{A}=\emptyset$).

Corollary $0\le P(\bar{A})=1-P(A)$; hence $0\le P(A)\le 1$.

Lemma For two events A,B, $P(\bar{A}\cap B) = P(B) - P(A\cap B)$

- $\bar{A}\cap B(\subseteq \bar{A})$ and $A\cap B(\subseteq A)$ are disjoint events and $B=B\cap S=B\cap(A\cup \bar{A})=(A\cap B)\cup(\bar{A}\cap B)$; hence by axiom of additivity, $P(\bar{A}\cap B) = P(B) - P(A\cap B)$.

Corollary (1) If $A\subseteq B$, then $P(\bar{A}\cap B) = P(B)-P(A)$, (2) $P(A)\le P(B)$.

Theorem3.3 (Addition Theorem of Probability) If A,B are any two events, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$.

- $A\cup B=A\cup(\bar{A}\cap B)$ and A, $\bar{A}\cap B$ are disjoint. Hence $P(A\cup B)=P(A)+ P(\bar{A}\cap B)= P(A)+P(B)-P(A\cap B)$.

Generalising , for three events A,B,C, we have

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$.

Example 3.5 If $p_1=P(A)$, $p_2=P(B)$, $p_3=P(A\cap B)$, express the following in terms of p_1,p_2,p_3: (1) $P(\overline{A\cup B})$, (2) $P(\bar{A}\cup \bar{B})$, (3)$P(\bar{A}\cap B)$,(4) $P(\bar{A}\cup B)$

- (1) $P(\overline{A\cup B})=1-P(A\cup B)=1-[P(A)+P(B)-P(A\cap B)]=1-p_1-p_2-p_3$.(2) $P(\bar{A}\cup \bar{B})=P(\overline{A\cap B})=1-P(A\cap B)=1-p_3$. (3) $P(\bar{A}\cap B)=P(B)- P(A\cap B)=p_2-p_3$.(4) $P(\bar{A}\cup B)=P(\bar{A})+P(B)- P(\bar{A}\cap B)=1-p_1+p_2-(p_2-p_3)=1-p_1+p_3$.

Example 3.6 It two dice are thrown, what is the probability that the sum is (1) greater than 8, (2)neither 7 nor 11?

- (1)If X denotes the sum on the two dice, then we want $P(X>8)$. The required event can happen in the following mutually exclusive cases: X=9, X=10,X=11,X=12. Hence by addition theorem on probability, $P(X>8)=P(X=9)+P(X=10)+P(X=11)+P(X=12)$. In a throw of two dice, the sample space contains $6^2=36$ points. The number of favourable cases can be enumerated as:
  X=9: (3,6),(6,3),(4,5),(5,4)
  X=10: (4,6),(6,4),(5,5)
  X=11: (5,6),(6,5)

X=12:(6,6).

Thus P(X>8)=$\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}$=$\frac{5}{18}$.

(2)Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11 with a pair of dice.

X=7: (1,6),(6,1),(2,5),(5,2),(3,4),(4,3)

X=11: (5,6),(6,5)

Required probability= P($\bar{A} \cap \bar{B}$)=1-P(A∪B)=1-[P(A)+P(B)] (since A and B are disjoint events)=1-$\frac{1}{6}-\frac{1}{18}$=$\frac{7}{9}$.

Example 3.7 Two dice are tossed. Find the probability of getting an even number on the first die or a total of 8.

➢ Let A be the event of getting an even number on the first dice and B be the event that the sum of points obtained on the two dice is 8. The events are represented by the following subsets of the sample space S:
A={2,4,6} X {1,2,3,4,5,6}, B={(2,6),(6,2),(3,5),(5,3),(4,4)}. Here A∩B={(2,6),(6,2),4,4)}.
Required probability is P(A∪B)=P(A)+P(B)-P(A∩B)=$\frac{18}{36}+\frac{5}{36}-\frac{3}{36}$=$\frac{5}{9}$.

Example 3.8 An integer is chosen at random from first two hundred natural numbers. What is the probability that the integer is divisible by 6 or 8?

➢ Sample space of the random experiment is {1,2,…,200}. The event A ' integer chosen is divisible by 6' is given by {6,12,…,198}; the event B' integer chosen is divisible by 8' is given by {8,16,…,200}. LCM of 6 and 8 is 24. Hence a number is divisible by 6 and 8 iff it is divisible by 24. Thus A ∩ B={24,48,…,192}. Hence required probability is P(A∪B)=P(A)+P(B)-P(A∩B)=$\frac{33}{200}+\frac{25}{200}-\frac{8}{200}$=$\frac{1}{4}$.

Example 3.9 The probability that a student passes Physics test is 2/3 and the probability that he passes both Physics test and English test is 14/45.The probability that he passes at least one test is 4/5. What is the probability that he passes English test?

➢ Let A be the event that the student passes the Physics test and B be the event that he passes English test. Given P(A)=$\frac{2}{3}$, P(A∩B)= $\frac{14}{45}$, P(A∪B)= $\frac{4}{5}$. We want P(B). From P(A∪B)=P(A)+P(B)-P(A∩B), we get P(B)=$\frac{4}{9}$.

Example 3.10 An investment consultant predicts that the odds against the price of a certain stock will go up during the next week are 2:1 and the odds in favour of the price remaining the same are 1:3.What is the probability that the price of the stock will go down during the next week?

- Let A denote the event ‘ stock price will go up’ and B be the event ‘stock price will remain same’. Then P(A)= $\frac{1}{2+1}$, P(B)= $\frac{1}{1+3}$. Thus P(A∪B)=P(A)+P(B)= $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$. Hence the probability that the stock price will go down is given by P($\bar{A}\cap\bar{B}$)=1- P(A∪B)=1-$\frac{7}{12}$=$\frac{5}{12}$.

Example 3.11 An MBA applies for a job in two firms X and Y. The probability of his being selected in firm X is 0.7 and being rejected at Y is 0.5. The probability of at least one of his applications being rejected is 0.6. What is the probability that he will be selected in one of his firms?

- Let A and B denote the events that the person is selected in firms X and Y respectively. Then P(A)=0.7, P($\bar{B}$)=0.5. Thus P($\bar{A}$)=1-0.7=0.3, P(B)=1-0.5=0.5 and 0.6= P($\bar{A}\cup\bar{B}$)= P($\bar{A}$)+ P($\bar{B}$) − P($\bar{A}\cap\bar{B}$). The probability that the person will be selected in one of the twofirms X or Y is given by: P(A∪B)=1- P($\bar{A}\cap\bar{B}$)=1-[P($\bar{A}$) + P($\bar{B}$) − P($\bar{A}\cup\bar{B}$)]=1-(0.3+0.5-0.6)=0.8.

Example 3.12 Three newspapers A,B and C are published in a certain city. It is estimated from a survey that 20% read A, 16% read B, 14% read C, 8% read both A and B, 5% read both A and C, 4% read both B and C, 2% read all three. Find what percentage read at least one of the papers?

- Let E,F,G denote the events that a person reads newspapers A,B and C respectively. Then we are given: P(E)=$\frac{20}{100}$, P(F)= $\frac{16}{100}$, P(G)= $\frac{14}{100}$, P(E∩F)= $\frac{8}{100}$, P(E∩G)= $\frac{5}{100}$, P(G∩F)= $\frac{4}{100}$, P($E\cap F\cap G$)= $\frac{2}{100}$.
  The required probability that a person reads at least one of the newspapers is given by P($E\cup F\cup G$)=P(E)+P(F)+P(G)- P(E∩F)- P(E∩G)- P(G∩F)+ P($E\cap F\cap G$)= $\frac{35}{100}$=0.35.

Example 3.13 A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

- Let A,B and C denote the event ‘card drawn is a king’, ‘card drawn is a heart’ and ‘card drawn is a red card’ respectively. Then A,B,C are not mutually exclusive.
  A∩B: card drawn is king of hearts ; n(A∩B)=1
  B∩ C=B(since B⊆C): card drawn is a heart ;n(B∩ C)=13
  A∩ C: card drawn is a red king; n(A∩ C)=2
  A∩ B ∩ C= A∩ B: card drawn is the king of hearts; n(A∩ B ∩ C)=1.
  Thus P(A)= $\frac{4}{52}$, P(B)= $\frac{13}{52}$, P(C)= $\frac{26}{52}$, P(A∩B)= $\frac{1}{52}$, P(B∩ C)= $\frac{13}{52}$, P(A∩ C)= $\frac{2}{52}$, P(A∩ B ∩ C)= $\frac{1}{52}$. Thus required probability is P($A\cup B\cup C$)=P(A)+P(B)+P(C)- P(A∩B)- P(B∩ C)- P(A∩ C)+ P(A∩ B ∩ C)= $\frac{7}{13}$.

Conditional Probability

In many situations we have the information about the occurance of an event A and are required to find out the probability of the occurrence of another event B. This is denoted by P(B/A). For example, if we know that a card drawn from a pack is black, we may need to calculate the probability that it is the ace of spade.

Let us take the problem of throwing a fair die twice. Suppose same number of spots do not appear in both the throws and we are required to find the probability that the sum of number of spots in the two throws is six.

A patient comes to a doctor with his family history that his elders suffered from high blood pressure. He wants to know the probability of the event that he will also suffer from high blood pressure.

Definition Let A and B be two events. The **conditional probability** of event B supposing event A has occurred, is defined by P(B/A)=$\frac{P(A\cap B)}{P(A)}$, if P(A)>0.

Note Let $B_1,\ldots,B_k$ be mutually exclusive events. The conditional probability of $\cup_1^k B_i$ given that event A has occurred is given by P$\left(\frac{\cup_1^k B_i}{A}\right) = \frac{P[(\cup_1^k B_i)\cap A]}{P(A)}$=$\frac{\sum_1^k P(B_i\cap A)}{P(A)}$.

Example 3.14 If a card drawn from a pack is black, represented by event A, find the probability of the event B that the card drawn is ace of spade .

- Number of black cards in a pack of 52 cards is 26. P(A)=26/52=1/2. Out of 26 black cards, only one is ace of spade. The event A∩B contains only one point; thus $P(A \cap B) = \frac{1}{52}$. Hence P(B/A)= $\frac{P(A\cap B)}{P(A)}$=$\frac{\frac{1}{52}}{\frac{1}{2}}$=$\frac{1}{26}$.

Example 3.15 An experiment is conducted by throwing a fair dice twice. Let A be the event that same number of spots do not turn up in two throws and B be the event that sum of the spots is 6. Find P(B/A).

- A includes all 36 points of the sample space except (1,1),(2,2),(3,3),(4,4),(5,5) and (6,6). Thus P(A)=30/36=5/6. Points favourable to event B are (1,5),(2,4),(3,3),(4,2),(5,1). Points common to A and B , that is A∩B , are (1,5),(2,4),(4,2),(5,1). Thus P(A∩B)=4/36. Thus P(B/A)= $\frac{\frac{4}{36}}{\frac{30}{36}}$=$\frac{4}{30}$=0.133.

Example 3.16 10% of patients feel they suffer and are really suffering from TB, 30% feel they suffer but actually do not suffer, 25% do not feel they are suffering but are suffering and remaining 35% neither feel nor suffering from TB. Find the probility of events E_1,E_2,E_3,E_4, where E_1: person who suffers from TB and feels he suffering from TB, E_2: person has TB and does not feel, E_3:person feels he has TB and does not suffer from TB, E_4: person feels and has TB.

- Let us define events: A: person feels he has TB, B: person suffers from TB. P(A∩B)=0.1, P(A∩ $\bar{B}$)=.3, P($\bar{A}$ ∩B)=.25, P($\bar{A} \cap \bar{B}$)=.35. Thus P(A)= P(A∩B)+ P(A∩ $\bar{B}$)=0.1+0.3=0.4, P(B)= P(A∩B)+ P($\bar{A}$ ∩B)=.10+.25=.35, P($\bar{A}$)= P($\bar{A}$ ∩B)+P($\bar{A} \cap \bar{B}$)=0.25+0.35=0.6,P($\bar{B}$)= P(A∩ $\bar{B}$)+ P($\bar{A} \cap \bar{B}$)=0.3+0.35=0.65. Hence
  P(E_1)=P(B/A)=$\frac{P(A\cap B)}{P(A)} = \frac{0.1}{0.4} = 0.25$, P($E_2$)=P(B/ $\bar{A}$)=$\frac{P(B\cap \bar{A})}{P(\bar{A})}$=0.417,P($E_3$)= P(A/ $\bar{B}$)=0.462, P(E_4)=P(A/B) = 0.286.

Example 3.17 There are two lots of manufactured item. Let one contain 40 pieces whereas lot two contains 50 pieces.it is known that former lot contains 25% defective pieces and the later one 10%. We flip a coin and select a piece from a lot one if it turns with head up; otherwise we select a piece from lot 2. Find the probability that a selected piece will be defective.

- Let A,B stand for the event 'piece is selected from lot 1' and 'piece is selected from lot 2'. Since the probability of turning head up=1/2, we have P(A)=1/2 and P($\bar{A}$)=1/2. Lot 1 has 10 defective and 30 non-defective pieces; lot 2 has 5 defective and 45 non-defective pieces. Given P(B/A)=1/4, P(B/ $\bar{A}$) =1/10. Thus P(A∩B)=P(B/A)P(A)=1/8, P($\bar{A}$ ∩B)= P(B/$\bar{A}$) P($\bar{A}$)=1/20. Hence the probability that the selected item is defective is P(A∩B)+ P($\bar{A}$ ∩B)=0.175.

Independent Events

If we draw two cards from a pack of cards in succession, then the results of the two draws are independent if the cards are drawn **with replacement** and are not independent if the cards are drawn **without replacement**.

Definition An event A is independent of another event B iff P(A/B)=P(A). This definition is meaningful when P(A/B) is defined, that is, when P(B)≠0.

Theorem3.4 If two events A and B are such that P(A) ≠0, P(B) ≠0 and A is independent of B, then B is independent of A.

- P(A/B)=P(A)⇒$\frac{P(A\cap B)}{P(B)} = P(A) \Rightarrow P(A\cap B)$=P(A)P(B) ⇒P(B/A)= $\frac{P(A\cap B)}{P(A)}$=$\frac{P(A)P(B)}{P(A)}$=P(B).

Theorem3.5 If A,B are events with positive probilities, then A and B are independent iff $P(A\cap B)$=P(A)P(B).

Theorem3.6 If A and B are independent, then (1) A and $\bar{B}$, (2) $\bar{A}$ and B , (3) $\bar{A}$, $\bar{B}$ are independent.

- Since A and B are independent, $P(A\cap B)$=P(A)P(B). P(A∩ $\bar{B}$)=P(A)- $P(A\cap B)$=P(A)-P(A)P(B)=P(A)P($\bar{B}$). P($\bar{A} \cap \bar{B}$)=P($\overline{A \cup B}$)=1-P(A∪B)=1-[P(A)+P(B)- $P(A\cap B)$]=1-P(A)-P(B)+P(A)P(B)=[1-P(A)][1-P(B)]=P($\bar{A}$)P($\bar{B}$).

Example 3.18 If $A \cap B = \emptyset$, then show that $P(A) \le P(\bar{B})$

- $A=(A \cap B) \cup (A \cap \bar{B})=\emptyset \cup (A \cap \bar{B})= A \cap \bar{B} \Rightarrow A \subseteq \bar{B} \Rightarrow P(A) \le P(\bar{B})$.

Example 3.19 Let A and B be two events such that P(A)=3/4, P(B)=5/8. Show that (a) $P(A \cup B) \ge \frac{3}{4}$, (b) $\frac{3}{8} \le P(A \cap B) \le \frac{5}{8}$.

- (a) $A \subseteq A \cup B \Rightarrow \frac{3}{4} \le P(A) \le P(A \cup B)$.
  (b) $A \cap B \subseteq B \Rightarrow P(A \cap B) \le P(B)= \frac{5}{8}$. Also, $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1 \Rightarrow \frac{3}{4} + \frac{5}{8} - 1 \le P(A \cap B) \Rightarrow \frac{3}{8} \le P(A \cap B)$

BAYES' THEOREM

Theorem3.7 If $E_1, E_2, \dots, E_n$ are mutually disjoint events with $P(E_i) \neq 0 (i=1,\dots,n)$, then for any arbitrary event A which is a subset of $\cup_1^n E_i$ such that P(A)>0,

$$P(E_i/A)=\frac{P(E_i)P(\frac{A}{E_i})}{\sum_1^n P(E_i)P(\frac{A}{E_i})} = \frac{P(E_i)P(\frac{A}{E_i})}{P(A)}$$

Example 3.20 Suppose that a product is produced in three factories X,Y, and Z. It is known that factory X produces thrice as many items as factory Y and that factory Y and Z produces same number of items. 3% of the items produced by each of the factories X and Z are defective and 5% of those manufactured in Y are defective. All the items in the three factories are stocked and an item of the product is selected at random. (1) What is the probability that the item selected is defective? (2) if an item selected at random is found to be defective, what is the probability that it was produced in factory X,Y,Z respectively?

- Let the number of items produced by factories X,Y, and Z be 3n,n,n respectively. Let E_1, E_2, E_3 be the events that the items are produced by factory X,Y and Z respectively and let A be the event that the item being defective. Then $P(E_1)=\frac{3n}{3n+n+n}=0.6$, $P(E_2)=0.2$, $P(E_3)=0.2$. Also, $P(A/E_1)= P(A/E_3)=0.03$, $P(A/E_2)=0.05$(given).
  (1) The probability that an item selected at random from the stock is defective is given by $P(A)=\sum_1^3 P(A \cap E_i) = \sum_1^3 P(E_i)P(\frac{A}{E_i})$=.6 x .03+.2x.05+.2x.03=.034.
  (2) By Bayes' rule, the required probabilities are given by :
  $P(E_1/A)=\frac{P(E_1)P(\frac{A}{E_1})}{P(A)} = \frac{.6 X .03}{.034}=\frac{9}{17}$, $P(E_2/A)=\frac{P(E_2)P(\frac{A}{E_2})}{P(A)} = \frac{.2 X .05}{.034}=\frac{5}{17}$, $P(E_3/A)=\frac{P(E_3)P(\frac{A}{E_3})}{P(A)} = \frac{3}{17}$.

Example 3.21 In 2002 there will be three candidates for the position of the principal –Mr. x, Mr. y and Mr.z—whose chances of getting the appointment are in the ratio 4:2:3 respectively. The probability that Mr. x if selected would introduce co-education in the college is 0.3. The corresponding probabilities for Mr. y and Mr.z are 0.5 and 0.8. (1) What is the probability that there will be co-education in 2003? (2) if there is co-education in 2003, what is the probability that Mr. z is the principal?

- Let us define the events
  A: introduction of co-education, E_1: Mr. x is selected as principal

  E_2: Mr. y is selected as principal, E_3: Mr. z is selected as principal.

  $P(E_1)=4/9$, $P(E_2)=2/9$, $P(E_3)=3/9$, $P(A/E_1)=3/10$, $P(A/E_2)=5/10$, $P(A/E_3)=8/10$.

  (1) The required probability that there will be coeducation in the college in 2003 is
  $P(A)=P[(A\cap E_1)\cup(A\cap E_2)\cup(A\cap E_3)]=P(A\cap E_1)+P(A\cap E_2)+P(A\cap E_3)$
  $=P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)=\frac{4}{9}.\frac{3}{10}+\frac{2}{9}.\frac{5}{10}+\frac{3}{9}.\frac{8}{10}=\frac{23}{45}$.

  (2) The required probability is given by Bayes' rule:

  $P(E_3/A)=\frac{P(E_3)P(\frac{A}{E_3})}{P(A)}=\frac{\frac{3}{9}X\frac{8}{10}}{\frac{46}{90}}=\frac{12}{23}$.

RANDOM VARIABLES

In many random experiments, we are interested not in knowing which of the outcomes has occurred but in the numbers associated with them. For example, when n coins are tossed, one may be interested in knowing the number of heads obtained. When a pair of dice are tossed, one may seek information about the sum of points. Thus, we associate a real number with each outcome of a random experiment. In other words, we are considering a function whose domain is the set of all possible outcomes and whose range is a subset of the set of reals.

Definition Let S be the sample space associated with a given random experiment. A real-valued function X: S→ $(-\infty,\infty)$ is called a one-dimensional random variable(**r.v**.).

Notation If x is a real number, the set of all w∈S such that X(w)=x is denoted by X=x. Thus P(X=x)=P{w: X(w)=x}. Similarly $P(X \le a)=P\{w \in S: X(w)\in(-\infty,a]\}$, $P(a<X\le b)=P\{w:X(w)\in(a,b]\}$

Example 4.1 Consider the random experiment of tossing a coin. Then $S=\{w_1,w_2\}$, $w_1=H$, $w_2=T$. Define $X:\{w_1,w_2\}\to\{0,1\}$ by $X(w_1)=1$, $X(w_2)=0$. X is a r.v.

A function $X:S\to R^2$ is a two-dimensional random variable.

Example 4.2: If a dart is thrown at a circular target, the sample space S is the set of all points w on the target.By imagining a co-ordinate system placed on the target with the origin at the centre, we can consider a two-dimensional random variable X which assigns to every point w of the circular region , its rectangular co-ordinates (x,y)

Example 4.3 If a pair of dice is tossed , then S={1,2,3,4,5,6}X{1,2,3,4,5,6} .Let X be the random variable defined by X(i,j)=max{i,j}. Then

P(X=1)=P{(i,j):X(i,j)=1}=P{(1,1)}=1/36, P(X=2)=P{(1,2),(2,2),(2,1)}=3/36.

Distribution Function

Definition Let X be a random variable(r.v.). The function F : $(-\infty,\infty)\to[0,1]$ defined by $F(x)=P\{t: X(t)\leq x\}$ is the **distribution function** (d.f.) of the r.v. X.

Note: To emphasize the r.v. X, we sometimes denote F(x) by $F_X(x)$.

Properties of Distribution Function

(1) If F is the d.f. of the r.v. X and if a<b, then $P(a<X\leq b)=F(b)-F(a)$.

- The events $a<X\leq b$' and $X\leq a$ are disjoint and their union is the event $X\leq b$. Hence, by addition theorem of probability,
  $P(a<X\leq b)+P(X\leq a)=P(X\leq b)$. Hence the result.

Corollary: $P(a\leq X\leq b)=P\{(X=a)\ (a<X\leq b)\}=P(X=a)+ P(a<X\leq b) =$

$P(X=a)+ [F(b)-F(a)]$. When $P(X=a)=0$, the events

$a\leq X\leq b$ and $a<X\leq b$ have same probability $F(b)-F(a)$

(2) $0\leq F(x) \leq 1$. If x<y, then $F(x) \leq F(y)$.

Discrete Random Variable

A r.v. which can assume only at most countable number of real values is a discrete random variable. Example of discrete random variable are marks obtained in a test, number of accidents per month etc.

Probability Mass Function

If X is a one-dimensional discrete r.v. taking at most a countable number of values $x_1,x_2,\ldots$, then the probabilistic behaviour of X at each x_i is described by its probability mass function.

Definition If X is a discrete r.v. having distinct values $x_1,x_2,\ldots$, then the function $p_X(x)$, or simply p(x), defined by $p(x)=P(X=x_i)=p_i$, if $x=x_i$ and =0, if $x\neq x_i$, i=1,2,... is called probability mass function(p.m.f.) of r.v. X.

Note (1)The set $\{(x_1,p_1),(x_2,p_2),\ldots\}$ specifies the probability distribution of the r.v. X.

(2)$P(x_i)\geq 0$, for all i and $\sum_1^{\infty} p(x_i) = 1$.

Example 4.4 Let S={H,T} be the sample space corresponding to the random experiment of tossing of a 'fair' coin. Let X be the r.v. defined by X(H)=1, X(T)=0. X has only two distinct values, namely, 0 and 1. The corresponding p.m.f. is given by: $p(1)=P(X=1)=P(H)=1/2$, $p(0)=1/2$.

Example 4.5 A r.v. X has the following p.m.f.:

x_i:	0	1	2	3	4	5	6	7
p_i:	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

(1)Find k, (2)Evaluate $P(X<6)$,$P(X\geq 6)$ and $P(0<X<5)$, (3) If $P(X\leq a)>1/2$, find the minimum value of a, (4) determine the p.d.f. of X.

- (1) Since , $k+2k+2k+3k+k^2+2k^2+7k^2+k=1$ giving $10k^2+9k-1=0$, which gives $k=1/10$ or -1. Since $p_2=k$ cannot be negative, -1 is rejected and $k=1/10$.

(2)$P(X<6)=P(X=0)+P(X=1)+\ldots+P(X=5)=1/10+2/10+2/10+3/10+1/100=81/100$. Now $P(X\geq 6)=1-P(X<6)=1-81/100=19/100$.

(3) $P(X\leq a)>1/2$. By trial, we get a=4.
(4) The p.d.f. of X is given by:

X:	0	1	2	3
F(x)=P(X≤x):	0	k=1/10	3k=3/10	5k=5/10
X:	4	5	6	7
F(x)	8k=4/5	$8k+k^2$	$8k+3k^2$	$9k+10k^2$

Example 4.6 If $p(x)=x/15$, x=1,2,3,4,5; =0, elsewhere be the p.m.f. of a r.v.X. Find (1) P{X=1 or 2}, (2) $P\{\frac{1}{2}<X<\frac{5}{2}\,|\,X>1\}$.

- (1) P{X=1 or 2}=P(X=1)+P(X=2)=1/15+2/15=1/5.

(3) $P\{\frac{1}{2}<X<\frac{5}{2}\,|\,X>1\}=\frac{P\{\left(\frac{1}{2}<X<\frac{5}{2}\right)\cap(X>1)\}}{P(X>1)}=\frac{P((X=1\ or 2)\cap(X>1))}{P(X>1)}=\frac{P(X=2)}{1-P(X=1)}=\frac{\frac{2}{15}}{1-\frac{1}{15}}=\frac{1}{7}$.

Example 4.7 An experiment consists of three independent tosses of a fair coin. Let X=the number of heads, Y=the number of head runs,Z=the length of head runs, a head run being defined as consecutive occurrence of at least two heads, its length then being the number of heads occurring together in three tosses of the coin. Find the probability function of (1)X, (2) Y, (3)Z,(4)X+Y and (5)XY.

Elementary Event	**Random Variables**				
	X	**Y**	**Z**	**X+Y**	**XY**
HHH	3	1	3	4	3
HHT	2	1	2	3	2
HTH	2	0	0	2	0
HTT	1	0	0	1	0
THH	2	1	2	3	2
THT	1	0	0	1	0
TTH	1	0	0	1	0
TTT	0	0	0	0	0

(1)Obviously X is a r.v. which can take the values 0,1,2, and 3. p(3)=P(HHH)=$\left(\frac{1}{2}\right)^3=\frac{1}{8}$, p(2)=P(HHT $\cup HTH \cup THH$)=P(HHT)=P(HTH)+P(THH)=1/8+1/8+1/8=3/8. Similarly p(1)=3/8, p(0)=1/8.

(2) probability distribution of Y: p(0)=5/8, p(1)=3/8.

(3)probability distribution of Z: p(0)=5/8, p(1)=0,p(2)=2/8,p(3)=1/8.

(4) probability distribution of U=X+Y:p(0)=1/8, p(1)=3/8,p(2)=1/8,p(3)=2/8,p(4)=1/8.

(5) probability distribution of V=XY: p(0)=5/8,p91)=0,p(2)=2/8,p(3)=1/8.

Continuous Random Variable

Definition A r.v. X is continuous iff X takes all values between two unequal real numbers.

Probability Density Function

Consider a small interval (x, x+dx) of length dx about x. Let f(x) be any continuous function of x so that f(x)dx represents the probability that X falls in the infinitesimal interval (x, x+dx). Symbolically, P(x$\leq X \leq x+dx$)=f_X(x)dx. f_X is called probability density function (p.d.f.) of the r.v. X.

The probability for a variate value to lie in the interval [a,b] is P(a≤X≤b)=$\int_a^b f_X(x)dx$.

The p.d.f. f(x) of a r.v. X has the properties:

f(x)≥0, $\int_{-\infty}^{\infty} f(x)dx$=1 (since $\int_{-\infty}^{\infty} f(x)dx$ gives total probability), P(X=c)= $\int_c^c f_X(x)dx$=0(where c is any value of the variate X)

Various measures of central tendency, dispersion, skewness and kurtosis for continuous probability distribution

The formulae for these measures in case of discrete frequency distribution can be easily extended to the case of continuous probability distribution by simply replacing $p_i=f_i/N$ by f(x)dx, x_i by x and summation over 'i' by integration over the specified range of the variable X.

Let f(x) be the p.d.f. of a r.v. X, [a,b] being the range of X. Then

A.M. $\bar{x}$=$\int_a^b xf(x)dx$, μ_r(central)=$\int_a^b (x-\bar{x})^r f(x)dx$, $\mu_r^{/}$(about x=A)=$\int_a^b (x-A)^r f(x)dx$

Median is the point which divides the total area into two equal parts: if M be the median, then $\int_a^M f(x)dx = \int_M^b f(x)dx = 1/2$.

Mode is the value of x for whixh f(x) is maximum. Mode is the solution of $f^{/}$(x)=0 and $f^{//}$(x)<0, provided it lies in [a,b].

Example 4.8 The diameter of an electric cable , say X, is assumed to be a continuous random variable with p.d.f. f(x)=6x(1-x), 0≤x≤1. (1) Check that f(x) is a p.d.f., (2) determine the median b of the distribution.

- (1) $\int_0^1 f(x)dx = 1$ (by direct calculation); hence f(x) is p.d.f. of r.v. X. (2) P(X<b)=P(X>b)⇒6 $\int_0^b f(x)dx = 6\int_b^1 f(x)dx$⇒ b=1/2, lying in [0,1].

Example 4.9 Suppose that the life in hours of a certain part of radio tube is a continuous random variable X with p.d.f. given by f(x)=100/x^2, when x≥100; =0, elsewhere. (1) What is the probability that all of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation?(2)What is the probability that none of the original tubes will have to be replaced during the first 150 hours of operation?(3)what is the probability that a tube will last less than 200 hours if it is known that the tube is still functioning after 150 hours of service?

- (1)p=P(X≤150)=$\int_{100}^{150} f(x)dx = \int_{100}^{150} \frac{100}{x^2} dx = \frac{1}{3}$. By compound probability theorem, the probability that all three of original tubes will have to be replaced during the first 150 hours =p^3=1/27.
  (2)The probability that a tube is not replaced during the first 150 hours of operation is P(X>150)=1-P(X≤150)=1-p=2/3. By compound probability theorem, the probability that none of the three tubes will have to be replaced during the first 150 hours =q^3=8/27.
  (3)Probability that a tube will last less than 200 hours given that the tube is still functioning after 150 hours is P(X<200|X>150)=$\frac{P(150<X<200)}{P(X>150)}$=$\frac{\int_{150}^{200}\frac{100}{x^2}dx}{\int_{150}^{\infty}\frac{100}{x^2}dx} = \frac{1}{6} X \frac{3}{2}$=0.25.

Example 4.10 The amount of bread (in hundreds of pounds) x that a certain bakery is able to sell in a day is found to be a numerical valued random phenomenon with a probability function specified by the p.d.f. f(x) given by f(x)=kx, 0≤x<5; =k(10-x), 5≤x<10; =0, otherwise. (1) find the value of k such that f(x) is a p.d.f., (2)what is the probability that the number of pounds of bread that will be sold tomorrow is (a) more than 500 pounds, (b) less than 500 pounds and (c) between 250 and 750 pounds? (3) Denoting by A,B,C the events that the pounds of bread sold are as in (2)(a),(2)(b) and (2)(c) respectively, find P(A|B),P(A|C). Are (1) A,B independent, (2) A,C independent?

- (1)$\int_{-\infty}^{\infty} f(x)dx$=1 gives k=1/25.
  (2)(a)P(5≤ X ≤10)=$\int_5^{10} \frac{1}{25}(10-x)dx$=0.5
  (b)P(0≤ $X < 5) = \int_0^5 \frac{1}{25} xdx$ =0.5
  (c)P(2.5≤ $X \le 7.5) = \int_{2.5}^5 \frac{1}{25} x\, dx + \int_5^{7.5} \frac{1}{25}(10-x)dx$=3/4
  (3)From (2)(a),(b),(c), P(A)=0.5, P(B)=0.5, P(C)=3/4. The events A∩B and A∩ C are given by: A∩B=∅, A∩ C:5<X<7.5. Thus P(A∩B)=0,P(A∩C)= $\frac{1}{25}\int_5^{7.5} \frac{1}{25}(10-x)dx$=3/8.

$P(A)P(C)=1/2 \times 3/4=3/8=P(A\cap C)$, $P(A)P(B)=1/4 \neq 0 =, B\ are\ not\ independ$ $P(A\cap B)$. Thus A,C are independent and A,B are not independent.

Expectation of a r.v.

Let the r.v. X take values $x_1,\dots,x_n$ with probabilities $p_1,\dots,p_n$. Let X take value x_i , f_i number of times; let $N=f_1+\dots+f_n$. Mean of X is given by $\frac{f_1x_1+\cdots+f_nx_n}{N} = \frac{f_1}{N}x_1 + \cdots + \frac{f_n}{N}x_n$. Let $N\to\infty$. Using the statistical definition of probability, limiting value of mean of X , $\bar{X} = p_1x_1 + \cdots + p_nx_n$.

Definition Expectation of X, $E(X)=\sum_1^n p_i x_i$.

Thus E(X) may be regarded as the limiting value of the average value of X realized in N random experiments as $N\to\infty$. Generalising, if f(X) is a function of X, f(X) will take values $f(x_1),\dots,f(x_n)$ with frequencies $f_1,\dots,f_n$ and the average value of f(X) in N experiments is $\frac{f_1}{N}f(x_1)+\dots+\frac{f_n}{N}f(x_n)$ and as $N\to\infty$, this approaches to $E(f(X))=p_1f(x_1)+\dots+p_nf(x_n)$. In particular, $\mu_r^{/}(a)=E[(X-a)^r]$, $\mu_r = E[(X-\bar{X})^r]$, $\mu_2 = E[(X-\bar{X})^2]=\sigma^2$.

Example 4.11 What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success in a trial?

- If X denotes the number of failures preceding the first success , we find that X takes the values 0,1,2,3,... with probabilities $p, qp, q^2p, q^3p,\dots$, where $q=1-p$. Thus probability density function is $f(x)=q^rp$, $r=0,1,2,\dots$. Hence $E(X)=0.p+1.qp+2.q^2p+3.q^3p+\dots=pq(1+2q+3q^2+\dots)=pq(1-q)^{-2}$ (since $q<1$)$=q/p = 1/p-1$.

Properties of Expectation

1. Addition Theorem of Expectation: If X,Y are r.v., then E(X+Y)=E(X)+E(Y).
2. Multiplication Theorem of Expectation: If X,Y are independent r.v., E(XY)=E(X)E(Y).
3. If X is a r.v. and a,b are constants, then E(aX+b)=aE(X)+b, provided all the expectations exist.
4. If $X\geq 0$,then $E(X)\geq 0$.
5. If X,Y are r.v. and $X(t)\geq Y(t)$, forall t, then $E(X)\geq E(Y)$, provided all expectations exist.

Example 4.12 Let X be a r.v. with the following probability distribution:

x:	-3	6	9
P(X=x):	1/6	½	1/3

Find E(X) and E(X²) and using laws of expectation, evaluate $E(2X+1)^2$.

- $E(X)=\sum xp(x)=(-3)\frac{1}{6} + 6\frac{1}{2} + 9\frac{1}{3}=\frac{11}{2}$, $E(X^2)=\sum x^2p(x) = \frac{93}{2}$. Then $E(2X+1)^2=4E(X^2)+4E(X)+1=209$.

Example 4.13 Two unbiased dice are thrown. Find the expected values of the sum of numbers of points on them.

- The probability function of X(sum of number of heads on two dice) is

x:	2	3	4	…	12
P(X=x):	1/36	2/36	3/36	…	1/36

$E(X)=\sum px = \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12) = \frac{1}{36}252=7.$

SOME IMPORTANT DISTRIBUTIONS

Binomial Distribution

Let a series of n trials be performed in which occurance of an event is called a 'success' and its a non-occurrence is called a 'failure'.Let p be the probability of a success and q=1-p is the probability of a failure. We shall assume that trials are independent and probability p of success is same in every trial. The number of successes in n trials may be 0,1,2,…,n and is a randam variate. The probability of x succeses and n-x failures in a series of n independent trials in a specified order(say) SSFSFFF… FSF (S represents success and F represents failure) is given by compound probability theorem: P(SSFSFFF… FSF)=P(S)…P(F)=p..p(x factors)q…q(n-x factors)=$p^x q^{n-x}$. But x successes in n trials can occur in C_x^n ways and the probability for each one of these ways are same, viz. $p^x q^{n-x}$. Hence the probability of x successes in n trials *in any order* is given by the addition theorem of probability by the expression $C_x^n p^x q^{n-x}$. The probability distribution of the number of successes so obtained is called **Binomial probability distribution**, for the obvious reason that the probabilities of 0,1,…,n successes viz. q^n, $C_1^n p^1 q^{n-1}$, $C_2^n p^2 q^{n-2}$,…,p^n are the successive terms of the binomial expansion of $(q+p)^n$.

Definition A random variable X is said to follow binomial distribution with parameters n and p, written as $X \sim B(n,p)$, if it assumes only non-negative values and its p.m.f. is given by P(X=x)=p(x)= $C_x^n p^x q^{n-x}$, x=0,1,…,n, q=1-p; =0,otherwise. A random variable which follows binomial distribution is called a **binomial variate**.

For a binomial distribution, following conditions must hold:

- Number of trials n is finite
- Trials are independent
- Probability of success p is constant for each trial
- Each trial results in one of two mutually exclusive and exhaustive outcomes, termed 'success' and 'failure'

Example 5.1 Ten coins are tossed. Find the probability of getting at least seven heads.

- P=q=1/2, n=10. The probability of getting x heads in a random throw of 10 coins is p(x)= $C_x^{10}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x}$ =$C_x^{10}\left(\frac{1}{2}\right)^{10}$, x=0,1,...,10. Hence probability of getting at least 7 heads is P(X≥7)= p(7)+p(8)+p(9)+p(10)= $\left(\frac{1}{2}\right)^{10}\left[C_7^{10}+C_8^{10}+C_9^{10}+C_{10}^{10}\right]=\frac{176}{1024}$.

Example 5.2 A and B play a game in which their chances of winning are in the ratio 3:2. Find A's chance of winning at least three games out of the five games played.

- Let p be the probability that A wins the game. N=5,p=3/5, q=2/5. The probability that out of 5 games played, A wins 'x' games is given by:P(X=x)= $C_x^5\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{5-x}$, x=0,1,...,5.
  The required probability that A wins at least three games is given by P(X≥3)=$\sum_{r=3}^{5} C_r^5\,\frac{3^r 2^{5-r}}{5^5}$= 0.68.

Example 5.3 A multiple-choice test consists of 8 questions with 3 answers to each question(of which only one is correct). A student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 and the third answer if he gets 5 or 6. To get a distinction, the syudent must secure at least 75% correct answers. If there is no negative marking, what is the probability that the student secures a distinction?

- Since there are three answers to each question, out of which only one is correct, the probability of getting a correct answer to a question is p=1/3,so that q=2/3. The probability of getting r correct answers in a8-question set is P(X=x)=p(x)= $C_x^8\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{8-x}$, x=0,1,...,8.
  Hence the required probability of securing a distinction (that is, to get correct answers to at least 6 out of 8 questions) is given by: p96)+p(7)+p(8)= $C_6^8\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^{8-6}+C_7^8\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^{8-7}+C_8^8\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^{8-8}$=0.0197.

Example 5.4 The probability of a man hitting a target is ¼. (1) if he fires 7 times , what is the probability of his hitting the target at least twice? (2) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3?

- p=probability of the man hitting the target =1/4, q=1-p=3/4.
  p(x)=probability of getting x hits in 7 shots=$C_x^7\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{7-x}$, x=0,1,...,7.
  (1) Probability of at least 2 hits=1-[p(0)+p(1)]=1-[$C_0^7\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{7-0}+C_1^7\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{7-1}$]=$\frac{4547}{8192}$.

(2) Probabilty of at least one hit in n shots=1-p(0)=1-$\left(\frac{3}{4}\right)^n$. It is required to find n such that 1-$\left(\frac{3}{4}\right)^n$ >$\frac{2}{3}$, that is, $\frac{1}{3}$>$\left(\frac{3}{4}\right)^n$. Taking logarithms on both sides and simplifying, n>$\frac{0.4771}{0.1250}$=3.8. Thus required number of shots is 4.

Example 5.5 In a Binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter 'p' of the distribution.

- Let X~B(n,p) where n=5, p(1)=0.4096, p(2)=0.2048. P(X=x)= $C_x^5 p^x(1-p)^{5-x}$, x=0,1,..,5. Given p(1)= $C_1^5 p^1(1-p)^{5-1}$ = 0.4096,p(2)= $C_2^5 p^2(1-p)^{5-2}$=0.2048. Dividing, we get $\frac{5(1-p)}{10p}$ =2, p=0.2.

Moments of Binomal Distribution

$$\mu_1^{/} = E(X) = \sum_{x=0}^{n} x\, C_x^n p^x q^{n-x} = np \sum_{1}^{n} C_{x-1}^{n-1} p^{x-1} q^{n-x} = np(q+p)^{n-1} = np.$$

Thus mean of Binomial distribution is np. It can be verified that $\mu_2^{/}$ =n(n-1)p²+np, μ_2(central)=npq, μ_3=npq(q-p).

Note If X~B(n,p), then mean=np, variance=npq. Hence variance<mean for a Binomial variate.

Example 5.6 The mean and the variance of a binomial distribution are 4 and 4/3 respectively. Find P(X≥1).

- Let X~B(n,p). Then np=4, npq=4/3. q=1/3.p=1-q=2/3. Hence n=4/p=6. Thus P(X≥1)=1-P(X=0)=1-q^n=1-$\left(\frac{1}{3}\right)^6$=0.99863.

Poisson Distribution

Poisson Distribution is a limiting case of Binomial Distribution under the following conditions:

- n, the number of trials , is indefinitely large, that is, n→ ∞
- p, the constant probability of success for each trial is indefinitely small, that is, p→0
- np=λ (say) is finite.

Definition A r.v. X is said to follow a Poisson distribution if it assumes only non-negative values and its p.m.f. is given by p(x,γ)=P(X=x)=$\frac{e^{-\lambda}\lambda^x}{x!}$, x=0,1,…,λ>0; =0,otherwise.

λ is known as the parameter of the distribution; we write X~P(λ) to denote X is a Poisson variate with parameter λ.

Following are some examples of Poisson variate:

- the number of typographical errors per page in typed material
- the number of defective screws per box of 100 screws
- the number of bacterial colonies in a given culture per unit area of microscope slab
- the number of deaths in a district in one year by a rare disease

Moments of Poisson Distribution

$\mu_1^/ = E(X) = \lambda$, $\mu_2^/$ =E(X²)= $\lambda^2+\lambda$, $\mu_2=\mu_2^/ - (\mu_1^/)^2=\lambda$, $\mu_3=\lambda$.

Example 5.7 Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 percent of such fuses are defective.

➢ n=200, p=probability of defective fuses=2%=.02. Since p is small, we may use oisson distribution. λ =mean number of defective pins=np=200(.02)=4. Thus required probability =P(X≤5)=$\sum_{x=0}^{5}\frac{e^{-4}4^x}{x!}$=$e^{-4}\left[1+4+\frac{4^2}{2}+\frac{4^3}{6}+\frac{4^4}{24}+\frac{4^5}{120}\right]$=.785.

Example 5.8 Six coins are tossed 6400 times. Using Poisson distribution, find the approximate probability of getting six heads r times.

➢ The probability of getting six heads in one throw of six coins (a single trial) is p=$\left(\frac{1}{2}\right)^6$, assuming head and tail are equally probable. λ =np=6400$\left(\frac{1}{2}\right)^6$=100. Thus required probability of getting 6 heads r number of times is P(X=r)=$\frac{e^{-100}100^r}{r!}$, r=0,1,2,...

Example 5.9 In a book of 520 pages, 390 typographical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

➢ The average number of typographical error per page in the book is λ=390/520=0.75.
Using Poisson probability law,the probability of x errors per page is given by P(X=x)=$\frac{e^{-\lambda}\lambda^x}{x!}$=$\frac{e^{-0.75}(0.75)^x}{x!}$, x=0,1,2,.... The required probability that a random sample of 5 pages will contain no error is given by $[P(X=0)]^5=(e^{-0.75})^5=e^{-3.75}$.

Normal Distribution

Definition A r.v. X is said to have a normal distribution **with parameters μ** (called 'mean') and σ^2(called 'variance') if its p.d.f. is given by the probability law: $f(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, -\infty<x<\infty, -\infty<\mu<\infty, \sigma>0$.

Note (1) When a r.v.is normally distributed with mean μ and standard deviation σ, it is customary to write X~N(μ,σ^2). If X~N(μ,σ^2), then Z=$\frac{X-\mu}{\sigma}$~N(0,1); Z is called corresponding standard normal variate. The p.d.f. of standard normal variate Z is given by $\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$. The corresponding distribution function, denoted by $\Phi(z)=P(Z\leq z) = \int_{-\infty}^{z} \emptyset(u)\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}}\, du$.

Few Properties of distribution function of standard normal variate

- $\Phi(-z)=1- \Phi(z)$
- $P(a\leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$, where $X\sim N(\mu, \sigma^2)$.

Chief Characteristics of the Normal Distribution and Normal Probability curve

The normal probability curve with mean μ and s.d. σ is given by $f(x)=\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$. The curve has the following properties:

- The curve is bell-shaped and symmetrical about the line $x= \mu$
- Mean,median and mode of the distribution coincide
- As x increases numerically, f(x) decreases rapidly, the maximum probability $[p(x)]_{max}=\frac{1}{\sigma\sqrt{2\pi}}$ occurring at $x= \mu$.
- $\beta_1 = 0, \beta_2 = 3$
- Since f(x) (being probability)≥ 0, for all x, no portion of the curve lies below the x-axis
- x-axis is an asymptote to the curve f(x)
- $\mu_{2r+1}=0$, r=0,1,2,...
- mean deviation about mean=4 $\sigma/5$ (approx.), quartile deviation=2 $\sigma/3$(approx.)
- Area property: $P(\mu - \sigma < X < \mu + \sigma) = 0.6826$, $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$, $P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3) = 0.9973$.

Example 5.10 For a certain normaldistribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. What is the arithmetic mean and s.d. of the distribution?

- $\mu_1^{/}$(about X=10)=40. Thus mean=10+$\mu_1^{/}$=50. Also $\mu_4^{/}$(about X=50)=48, that is, μ_4=48(since mean =50). But for a normal distribution with s.d. σ, $\mu_4=3\sigma^4=48$ giving $\sigma=2$.

Example 5.11 $X \sim N(12,4)$. (a) Find the probability of (1) X≥20, (2)X≤20, (3)0≤X≤ 12. (b) Find x such that P(X>x)=0.24.

- (a) For X=20, $Z= \frac{20-12}{4}$ =2. Thus $P(X \geq 20)=P(Z \geq 2)=0.5-P(0 \leq Z \leq 2)$ =0.5-0.4772=0.0228.
  P(X≤20)=1- P(X≥20)=1-.0228=.9722.
  P(0≤X≤ 12)=P(-3≤Z≤0)=P(0≤ $Z \leq 3$)=0.49865.

(b)When X=x, Z=$\frac{x-12}{4}$=z_1(say).GivenP(X>x)=P(Z>z_1)=0.24;thus P(0<Z<z_1)=0.26. From normal table, z_1=0.71(approx..) Hence $\frac{x-12}{4}$=0.71 giving x=14.84.

Example 5.12 The mean yield for one-acre plot is 662 kilos with s.d. 32 kilos. Assuming normal distribution, how many one acre plots in a batch of 1000 plots would you expect to have yield (1) over 700 kilos,(2)below 650 kilos and (3) what is the lowest yield of the best 100plots?

- If the r.v. X denotes the yield (in kilos) for one-acre plot, then X~ N(μ, σ^2), with $\mu = 662$, σ=32.
  (1) The probability that a plot has a yield over 700 kilos is given by P(X>700)=P(Z>1.19) [z=$\frac{700-662}{32} = 1.19$] = 0.5-P(0≤Z≤1.19)=0.5-0.3830=0.1170. Hence in a batch of 1000 plots, the expected number of plots with yield over 700 kilos is 1000 x 0.117=117.
  (2) Required number of plots with yield below 650 kilos isgiven by 1000 x P(X<650)=1000 X P(Z<-0.38)[z= $\frac{650-662}{32}$] =1000 x P(Z>0.38)=1000 X [0.5-P(0≤ Z ≤0.38)]=1000[0.5-0.1480]=352.
  (3) The lowest yield , say, x,of best 100 plots is given by:P(X>x)=$\frac{100}{1000}$=0.1. When X=x, Z=$\frac{x-662}{32}$=z_1 (say) such that P(Z>z_1)=0.1⇒ P(0≤Z≤ z_1)=0.4⇒z_1=1.28(approx.) (from normal tables). Thus x=662+32z_1=702.96. Hence the best 100 plots have yield over 702.96 kilos.

Example 5.13 The marks obtained by a number of students for a certain subject are assumed to be approximately normally distributed with mean value 65 and s.d. 5. If 3 students are taken at random from this list, what is the probability that exactly 2 of them will have marks over 70?

- Let the r.v. X denote the marks obtained by the given set of students in the given subject. Given that X~ N(μ, σ^2), with $\mu = 65$, σ=5. The probability that a randomly selected student from the given set gets marks over 70 is given by p=P(X>70)=P(Z>1)=0.5-P(0≤Z≤ 1)=0.5-0.3413=0.1587. Since this probability is same for each student of the set, the required probability that out of 3 students selected at random from the set, exactly 2 will have marks over 70, is given by the binomial probability law: C_2^3p²(1-p)=3 x (0.1587)² x (0.8413)=.06357.

Often we come across situations in which our focus is simultaneously on two or more possibly related variables . If change in one variable affects a change in the other variable, the variables are said to be correlated. If increase in values of one variable results in increase in the corresponding values of the other variable, variables are said to be positively correlated; if increase in one variable results in decrement of values of the other variable, variables are negatively correlated. Correlation is said to be perfect if deviation in one variable is followed by a corresponding and proportional deviation in the other variable.

Scatter Diagram

It is simplest diagrammatic representation of bivariate data. Thus for the bivariate distribution (x_i,y_i), i=1,…,n; if the values of the variables X and Y are plotted along the x-axis and y-axis respectively in the x-y plane, the diagram of dots so obtained is known as scatter diagram. From the scatter diagram, we can form a fairly good, though vague, idea whether the variables are correlated or not: if the points are very close to each other, we should expect high correlation between the variables.

Karl Pearson's coefficients of correlation

Let (x_i,y_i),i=1,…,n be a bivariate distribution of two r.v.s X and Y. Correlation coefficient between X,Y , usually denoted by r(X,Y) or by r_{XY}, is a numerical measure of **linear relationship** between them and is defined as : r(X,Y)=$\frac{Cov(X,Y)}{\sigma_X\sigma_Y}$, where Cov(X,Y)=E[{X-E(X)}{Y-E(Y)}]=$\frac{1}{n}\sum_1^n(x_i-\bar{x})\,(y_i-\bar{y})=\frac{1}{n}\sum x_iy_i-\bar{y}\frac{1}{n}\sum x_i-\bar{x}\frac{1}{n}\sum y_i+\bar{x}\bar{y}\;=\frac{1}{n}\sum x_iy_i-\bar{x}\bar{y}$, $\sigma_X^2=\frac{1}{n}\sum(x_i-\bar{x})^2=\frac{1}{n}\sum x_i^2-\bar{x}^2$, $\sigma_Y^2=\frac{1}{n}=\frac{1}{n}\sum(y_i-\bar{y})^2=\frac{1}{n}\sum y_i^2-\bar{y}^2$.

Note r(X,Y) is independent of units of measurement of X,Y.

Karl Pearson's correlation coefficient is based on the **assumptions**:

- There is a linear relationship between the r.v.s, that is, if the paired observations of both the variables are plotted on a scatter diagram, the plotted points will approximately be concurrent
- The variations in the two variables follow the normal law.

Limits for value of correlation coefficient

r(X,Y)=$\frac{Cov(X,Y)}{\sigma_X\sigma_Y}=\frac{\frac{1}{n}\sum_1^n(x_i-\bar{x})(y_i-\bar{y})}{\left[\frac{1}{n}\sum(x_i-\bar{x})^2\frac{1}{n}\sum(y_i-\bar{y})^2\right]^{\frac{1}{2}}}\Rightarrow r^2=\frac{(\sum a_ib_i)^2}{(\sum a_i^2)(\sum b_i^2)}$, where $a_i=x_i-\bar{x}$, $b_i=y_i-\bar{y}$. Now by Schwartz inequality, $(\sum a_ib_i)^2\le(\sum a_i^2)(\sum b_i^2)$. Hence $r^2\le1$. Thus $-1\le r\le 1$.

Note if r=+1, correlation is perfect and positive; if r=-1, correlation is perfect and negative.

Effect of change of origin and scale of reference on correlation coefficient

If U=$\frac{X-a}{h}$, $V=\frac{Y-b}{k}$, then $r_{XY}=\frac{hk}{\sqrt{h^2k^2}}r_{UV}$. Hence if h,k are of the same sign, then $r_{XY}=r_{UV}$; if h,k are of opposite sign, $r_{XY}=-r_{UV}$.

Note r is independent of origin a,b.

Note two independent variables are uncorrelated; converse may not hold.

- If X,Y are independent, Cov(X,Y)=0; hence r(X,Y)= $\frac{Cov(X,Y)}{\sigma_X\sigma_Y}$=0. Two uncorrelated variables may not be independent:

X:	-3	-2	-1	1	2	3
Y:	9	4	1	1	4	9
XY:	-27	-8	-1	1	8	27

r(X,Y)=0 but X,Y are dependent: Y=X^2.

Example 6.1 Calculate the correlation coefficient for the following heights(in inches) of fathers (X) and their sons(Y):

X:	65	66	67	67	68	69	70	72
Y:	67	68	65	68	72	72	69	71

- **Calculation for correlation coefficient**

X	**Y**	**X^2**	**Y^2**	**XY**
65	67	4225	4489	4355
66	68	4356	4624	4488
67	65	4489	4225	4355
67	68	4489	4624	4556
68	72	4624	5184	4896
69	72	4761	5184	4968
70	69	4900	4761	4830
72	71	5184	5041	5112
544	552	37028	38132	37560

$$\bar{X}=\frac{1}{n}\sum X=\frac{544}{8}=68, \bar{Y}=\frac{1}{n}\sum Y=\frac{552}{8}=69$$

r(X,Y)= $\frac{Cov(X,Y)}{\sigma_X\sigma_Y}=\frac{\frac{1}{n}\sum XY-\bar{X}\bar{Y}}{\sqrt{\left(\frac{1}{n}\sum X^2-\bar{X}^2\right)\left(\frac{1}{n}\sum Y^2-\bar{Y}^2\right)}}=\frac{\frac{1}{8}X\ 37560-68\ X\ 69}{\sqrt{\left\{\frac{37028}{8}-68^2\right\}\left\{\frac{38132}{8}-69^2\right\}}}$=0.603.

Short-cut Method

X	Y	U=X-68	V=Y-69	U^2	V^2	UV
65	67	-3	-2	9	4	6
66	68	-2	-1	4	1	2
67	65	-1	-4	1	16	4
67	68	-1	-1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
TOTAL		0	0	36	44	24

$\bar{U}$ =0, $\bar{V}$ =0, Cov(U,V)=$\frac{1}{n}\sum UV - \bar{U}\bar{V} = \frac{1}{8}\ X\ 24$ =3, $\sigma_U^2 = \frac{1}{n}\sum U^2 - \bar{U}^2 = \frac{1}{8}\ X\ 36$=4.5, $\sigma_V^2 = \frac{1}{n}\sum V^2 - \bar{V}^2 = \frac{1}{8}\ X 44$=5.5. Thus r_{UV}=$\frac{Cov(U,V)}{\sigma_U \sigma_V}$=$\frac{3}{\sqrt{4.5\ X\ 5.5}}$=0.603.

Example 6.2 A computer while calculating correlation coefficient between two variables X and Y from 25 pairs of observations obtained the following results: n=25,$\sum X = 125, \sum X^2 = 650, \sum Y = 100, \sum Y^2 = 460, \sum XY = 508$. It was however later discovered at the time of checking that he had copied down two pairs as (6,14),(9,6) while the correct values are (8,12),(6,8). Obtain the correct value of correlation coefficient.

- Corrected $\sum X$ =125-6-8+8+6=125,corrected $\sum Y$ =100-14-6+12+8=100
  Corrected $\sum X^2$=650-6^2-8^2+8^2+6^2=650, corrected $\sum Y^2$=460-14^2-6^2+12^2+8^2=436,
  Corrected $\sum XY$=508-6 X 14-8 X 6+8 X 12+6 X 8=520.
  Corrected $\bar{X}$=$\frac{1}{25}$ x 125=5, Corrected $\bar{Y}$=$\frac{1}{25}$ x100=4.
  Cov(X,Y)= $\frac{1}{n}\sum XY - \bar{X}\bar{Y}$=4/5. σ_X^2=1, σ_Y^2=36/25. Hence corrected r_{XY}=0.67.

Regression Analysis

Regression Analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of data.

If the variables in a bivariate distribution are related, the corresponding points in the scatter diagram will cluster round some curve called' curve of regression'. If the curve is a straight line, it is called 'line of regression' and there is said to be linear regression between the variables.

The line of regression is the line which gives the best estimate to the value of one variable for any specific value of the other variable. Thus 'line of regression' is the line of best fit and is obtained by principle of least squares.

Let us suppose that in the bivariate distribution (x_i,y_i), i=1,...,n, X is independent and Y is dependent variable. Let the line of regression of Y on X be Y=a+bX **(1)**. (1) represents a family of straight lines for different values of a and b. The problem is to find a and b corresponding to the line of 'best fit'.

According to the principle of least squares, we have to find a,b so that E=$\sum_1^n (y_i - a - bx_i)^2$ is minimum. From the principle of maxima and minima, 0=$\frac{\partial E}{\partial a} = -2\sum_1^n (y_i - a - bx_i)$ and 0=$\frac{\partial E}{\partial b} = -2\sum_1^n x_i(y_i - a - bx_i)$ which gives $\sum_1^n y_i = na +$ b $\sum_1^n x_i$ **(2)**, $\sum_1^n x_i y_i = a\sum_1^n x_i + b\sum_1^n x_i^2$**(3)**

Equations (2) and (3) are known as **normal equations** for estimating a and b. From (2), on dividing by n, we get $\bar{y} = a + b\bar{x}$**(4)** Thus the line of regression of Y on X passes through $(\bar{x}, \bar{y})$.

Now A(say)=Cov(X,Y)=$\frac{1}{n}\sum x_i y_i - \bar{x}\bar{y} \Rightarrow$ $\frac{1}{n}\sum x_i y_i$=A+$\bar{x}\bar{y}$. **(5)**

$\sigma_X^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2 \Rightarrow$ $\frac{1}{n}\sum x_i^2 = \sigma_X^2 + \bar{x}^2$.**(6)**

From (3),(5) and (6), A+$\bar{x}\bar{y}$=a$\bar{x}$+b$(\sigma_X^2 + \bar{x}^2)$. From (4) and (6), A=bσ_X^2 giving b=A/σ_X^2.

Since the regression line of Y on X passes through $(\bar{x}, \bar{y})$ and has slope b=A/σ_X^2, its equation is

Y-$\bar{y}$ = A/σ_X^2(X-$\bar{x}$), that is, Y-$\bar{y}$ =r$\frac{\sigma_Y}{\sigma_X}$(X-$\bar{x}$).

Similarly the equation of line of regression of X on Y is given by X-$\bar{x}$=r$\frac{\sigma_X}{\sigma_Y}$(Y-$\bar{y}$).

Note In case of perfect correlation r=$\pm$1 and in that case the equations of two regression lines coincide: $\frac{Y-\bar{y}}{\sigma_Y}$=$\pm\frac{X-\bar{x}}{\sigma_X}$.

Regression Coefficients

b_{YX}= r$\frac{\sigma_Y}{\sigma_X}$ and b_{XY}= r$\frac{\sigma_X}{\sigma_Y}$ are called regression coefficient of Y on X and of X on Y respectively.

Properties of regression coefficients

- b_{YX} b_{XY}=r^2. Thus r=$\pm\sqrt{b_{XY} b_{YX}}$. Since r=$\frac{A}{\sigma_X \sigma_Y}$, b_{YX}=$\frac{A}{\sigma_X^2}$, b_{XY}=$\frac{A}{\sigma_Y^2}$, it may be noted that sign of correlation coefficient is same as that of regression coefficients, since the sign of each depends on that of A. Thus, if the regression coefficients are positive, r is positive; if the regression coefficients are negative, r is negative. Hence the sign to be taken before the square root is that of the regression coefficients.

- If one of the regression coefficients is >1, then the other must be <1: b_{YX} b_{XY}=$r^2 \leq 1$; if b_{YX}=$\frac{1}{b_{XY}}$>1, then b_{XY}<1.

- Regression coefficients are independent of change of origin but not of scale: if U=$\frac{X-a}{h}$, V=$\frac{Y-b}{k}$, then b_{XY}=$\frac{h}{k}$$b_{UV}$.

Example 6.3 Obtain the equations of two lines of regressions for the following data. Also obtain the estimate of X for Y=70.

X:	65	66	67	67	68	69	70	72
Y:	67	68	65	68	72	72	69	71

➢ Let U=X-68, V=Y-69. Then $\bar{U} = 0, \bar{V} = 0$, $\sigma_U^2 = 4.5, \sigma_V^2 = 5.5$, Cov(U,V)=3,r(U,V)=0.6. Since correlation coefficient is independent of change of origin, r=r(X,Y)=r(U,V)=0.6.
$\bar{X}$=68+$\bar{U}$=68, $\bar{Y}$=69+$\bar{V}$=69,σ_X=σ_U=$\sqrt{4.5}$=2.12, $\sigma_Y = \sigma_V$=$\sqrt{5.5}$=2.35.
Equation of line of regression of Y on X is: Y-$\bar{Y}$=r$\frac{\sigma_Y}{\sigma_X}$(X-$\bar{X}$), or, Y=0.665 X+23.78
Equation of line of regression of X on Y is: X-$\bar{X}$=r$\frac{\sigma_X}{\sigma_Y}$(Y-$\bar{Y}$), or, X=0.54Y+30.74
To estimate X for given Y, we use line of regression of X on Y.If Y=70, estimated value of X is given by $\hat{X}$=0.54 X 70+30.74=68.54.

Example 6.4 In a partially destroyed laboratory, record of an analysis of correlation data, the following results only are available: Variance of X=9, Regression equations: 8X-10Y+66=0,40X-18Y=214. What are values of(1) $\bar{X}, \bar{Y}$, (2)r_{XY}, (3) σ_Y?

➢ (1)Since ($\bar{X}, \bar{Y}$) is the point of intersection of the lines of regression, solving given equations of lines of regression simultaneously, we get $\bar{X}$=13, $\bar{Y}$=17.
(2) Comparing given equations of regression lines Y=$\frac{8}{10}X + \frac{66}{10}$, X=$\frac{18}{40}Y + \frac{214}{10}$, we get b_{YX}=$\frac{8}{10} = \frac{4}{5}$, b_{XY}=$\frac{18}{40} = \frac{9}{20}$. Hence r²= b_{YX}. b_{XY}=$\frac{9}{25}$. Hence r=$\pm\frac{3}{5}$. Since both the regression coefficients are positive, r=$\frac{3}{5}$=0.6.
(3)we have b_{YX}=r.$\frac{\sigma_Y}{\sigma_X}$; hence $\frac{4}{5}$=$\frac{3}{5}$ $\frac{\sigma_Y}{3}$, giving σ_Y=4.

Example 6.5 Find the most likely price in Mumbai corresponding to the price of Rs. 70 at Kolkata from the following:

	Kolkata	Mumbai
Average price	65	67
Standard Deviation	2.5	3.5

Correlation coefficient between the prices of commodities in the two cities is 0.8.

- Let the prices (in Rs.) in Kolkata and Mumbai be denoted by X and Y respectively. Given $\bar{X}$=65, $\bar{Y}$=67, σ_X=2.5, σ_Y=3.5, r=0.8. We want Y corresponding to X=70.
  Line of regression of Y on X is: Y-$\bar{Y}$= r.$\frac{\sigma_Y}{\sigma_X}$(X-$\bar{X}$) ,or, Y=67+0.8 X $\frac{3.5}{2.5}$(X-65).
  When X=70, $\hat{Y}$=67+0.8 X $\frac{3.5}{2.5}$(70-65)=72.6.
  Thus most likely price in Mumbai corresponding to the price of Rs .70 in Kolkata is Rs. 72.60.

Example 6.6 Can Y=5+2.8 X and X=3-0.5Y be the estimated regression equations of Y on X and of X on Y respectively?

- Line of regression of Y on X is: Y=5+2.8X ;thus b_{YX}=2.8
  Line of regression of X on Y is:X=3-0.5Y; thus b_{XY}=-0.5
  This is not possible, since the regression coefficients b_{YX}, b_{XY} must be of the same sign. Hence given equations can not be taken as lines of regression.

Curvilinear Regression

In many situations, variables X and Y may be related non-linearly. Extending the method of finding regression lines using method of least square, we like to fit a parabolic curve $Y=a+b_1X+b_2X^2$ to the given set $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ of n observations.

Using principle of least squares, we have to determine the constants a,b_1,b_2 so that E=$\sum_1^n(y_i - a - b_1x_i - b_2x_i^2)^2$ is minimum. Equating to zero the partial derivatives of E w.r.t. a,b_1,b_2, we obtain the **normal equations** :

0= $\frac{\partial E}{\partial a}$ =-2 $\sum_1^n(y_i - a - b_1x_i - b_2x_i^2)$, 0= $\frac{\partial E}{\partial b_1} = -2\sum_1^n x_i(y_i - a - b_1x_i - b_2x_i^2)$, 0=$\frac{\partial E}{\partial b_2} = -2\sum_1^n x_i^2(y_i - a - b_1x_i - b_2x_i^2)$.

Simplifying, $\sum y_i = na + b_1\sum x_i + b_2\sum x_i^2$, $\sum x_iy_i = a\sum x_i + b_1\sum x_i^2 + b_2\sum x_i^3$, $\sum x_i^2y_i = a\sum x_i^2 + b_1\sum x_i^3 + b_2\sum x_i^4$. Solving these equations simultaneously, we get a,b_1,b_2 corresponding to the curve of best fit.

Example 6.7 For 10 randomly selected observations, following data were recorded:

Overtime hours(X):	1	1	2	2	3	3	4	5	6	7
Additional units(Y):	2	7	7	10	8	12	10	14	11	14

Fit a parabolic curve to above data using method of least squares.

Serial No.	X	Y	X^2	X^3	X^4	XY	X^2Y
1	1	2	1	1	1	2	2
2	1	7	1	1	1	7	7
3	2	7	4	8	16	14	28

4	2	10	4	8	16	20	40
5	3	8	9	27	81	24	72
6	3	12	9	27	81	36	108
7	4	10	16	64	256	40	160
8	5	14	25	125	625	70	350
9	6	11	36	216	1296	66	396
10	7	14	49	343	2401	98	686
Total	34	95	154	820	4774	377	1849

Corresponding normal equations are: $10a+34b_1+154b_2=95, 34a+154b_1+820b_2=377, 154a+820b_1+4774b_2=1849$. Solving , $a=1.80, b_1=3.48, b_2=-0.27$. Thus regression equation of Y on X is: $Y=1.80+3.48X-0.27X^2$.

INDEX NUMBERS

An index number may be defined as a measure of the average change in a group of related variables over two different situations. The group of variables may be the prices of a specified set of commodities, the volumes of production in different sectors of an industry, the marks obtained by a student in different subjects and so on. The two different 'situations' may be either two different times or two different places.

The most commonly used index number is the index number of prices. Let p_o and p_1 denote the prices of a commodity in suitable units in two different situations denoted by '0'and '1'. Any change in the price of the commodity from '0' to '1' may be expressed either in absolute or relative terms. The absolute change is p_1-p_o; the relative change is given by p_1/p_o, which is called a price relative. The problem is to combine these various individual changes in prices and get a measure of the overall change in the prices of the set of commodities. A **price index number** is a sort of average of these individual price relatives, and it measures the price changes of all the commodities collectively.

Although different commodities may have peculiar characteristics in their price fluctuations, it has been empirically found that , taken as a whole, the distribution of price relatives is bell-shaped with a marked central tendency, provided the base period is in the recent past. Hence we are justified in taking an appropriate measure of central tendency in combining the different price relatives.

Let us denote by p_{oi} the price of i th commodity in the base period and by p_{1i} the price of this commodity in the current period (i=1,...,k). If we use the arithmetic mean of price relatives for constructing the index number, then $I_{01}=\frac{\sum_i p_{1i}/p_{0i}}{k}$ is a simple or unweighted index number.

Choice of weights

The commodities included in the index number are not all of equal importance. For instance, in constructing a wholesale price index for India, 'rice' should have greater importance than 'tobacco'. So the problem of weighting different commodities included in the index number according to their importance deserves attention. If we ignore weights, we get an inappropriately weighted index. If w_i be the weight attached to the price relative for the i th commodity, then we get the weighted A.M.

$I_{01}=\frac{\sum \frac{p_{1i}}{p_{0i}} w_i}{\sum w_i}$. Choosing different weight system, we get different index numbers:

- Choosing $w_i=q_{oi}$ (the base period quantities) we get Laspeyres' index: $I_{01}=\frac{\sum_i p_{1i}q_{0i}}{\sum_i p_{0i}q_{0i}}$.
- Choosing $w_i=q_{1i}$ (the current period quantities) we get Paasche's index: $I_{01}=\frac{\sum_i p_{1i}q_{1i}}{\sum_i p_{0i}q_{1i}}$
- Choosing $w_i=(q_{1i}+q_{oi})/2$, we get Edgeworth-Marshall index: $I_{01}=\frac{\sum_i p_{1i}(q_{1i}+q_{0i})}{\sum_i p_{0i}(q_{1i}+q_{0i})}$
- Fisher's 'ideal' index: $I_{01}=\sqrt{\frac{\sum_i p_{1i}q_{0i}}{\sum_i p_{0i}q_{0i}} \frac{\sum_i p_{1i}q_{1i}}{\sum_i p_{0i}q_{1i}}}$

Example 7.1 Table below gives the wholesale prices (p) and quantities produced (q) of a number of commodities in Delhi. Calculate Laspeyres', Paasche's, Edgeworth-Marshall and Fisher's index numbers for the year 1985 , with the year 1982 as base.

Commodity	**1982**		**1985**	
	p	**q**	**p**	**q**
Rice	277.92	1.1	366.67	6.2
Wheat	176.25	106.0	186.58	116.9
Jowar	151.00	4.2	182.57	5.5
Barley	121.83	2.4	181.25	1.0
Bajra	156.75	13.1	155.75	6.1
Gram	273.00	1.0	498.83	0.6

- Let p_{oi},q_{oi} and p_{1i},q_{1i} denote the prices and quantities for 1982 and 1985, respectively. Then

$\sum_i p_{0i}q_{0i}$=22241.229, $\sum_i p_{1i}q_{0i}$=23921.766, $\sum_i p_{0i}q_{1i}$=24399.034, $\sum_i p_{1i}q_{1i}$=26519.314.

Thus Laspeyres' Inex=$\frac{23921.766}{22241.229} x\ 100$=107.56

Paasche's Index=$\frac{26519.314}{24399.034} x\ 100$=108.69

Edgeworth-Marshall Index=$\frac{50441.08}{46640.263}$ x 100=108.15

and Fisher's 'ideal' index number=$\sqrt{107.56\ x\ 108.69}$=108.12.

TIME SERIES ANALYSIS

Time series is a series of observations recorded at different points or intervals of time. Maximum temperature of a place for different days of a month, yearly production of coal for last 20 years, monthly sales figure of some product are examples of time series data.

Let y_t denotes the value of the variable y at time t (t=1,..,n).In case the figures relate to n successive periods (and not points of time), t is to be taken as the mid-point of the t th period.

Components of time series

A graphical representation of a time series shows continual change over time, giving us an overall impression of haphazard movement. A critical study of the series will, however, reveal that the change is not totally haphazard and a part of it, at least, can be accounted for. The systematic part which can be accounted for may be attributed to several broad factors: (1)**secular trend**, (2)**seasonal variation**, (3)**cyclical variation**. Separation of the different components of a time series is of importance, because it may be that we are interested in a particular component of the systematic variation or that we want to study the series after eliminating the effect of a particular component. It may be noted that it is the systematic part of the time series which may be used in forecasting.

Secular Trend or trend of a time series is the smooth, regular, long-term movement of the series if observed long enough. Sudden or frequent changes are incompatible with the idea of trend .

Seasonal variation

Seasonal variation stands for a periodic movement in a time series where the period is not longer than one year. It is the component which recurs or repeats at regular intervals of time. Example of seasonal fluctuation may be found in the passenger traffic during the 24 hours of a day, sales of a departmental store during the 12months of a year etc. The study and measurement of this component is of prime importance in certain cases. The efficient running of any departmental store , for example, would necessitate a careful study of seasonal variation in the demand of the goods.

Cyclical Fluctuation

By cyclical fluctuation we mean the oscillatory movement in a time series, the period of oscillation being more than a year. One complete period is called a cycle. The cyclical fluctuations are not necessarily periodic, since the length of the cycle as also the intensity of fluctuations may change from one cycle to another.

Irregular Fluctuation

This component is either wholly unaccountable or are caused by such unforeseen events as wars, floods, strikes etc.

Estimation of secular trend in a time series by elimination of seasonal and cyclical fluctuation

In order to measure the trend , we are to eliminate from the time series the other three components. If the period of seasonal fluctuations be a year, then the yearly totals or tearly averages will be free from the seasonal effect. Thus, in determining the trend from monthly data, it is customary to start with the yearly totals or averages, which are free from seasonal effects. The monthly trend values can be obtained from the annual trend values by interpolation.to eliminate the other two components, viz. the cyclical and the irregular, we may consider the following methods:

Method of moving averages

The simple moving average of period k of a time series gives a new series of arithmetic means, each of k successive observations of the time series. We start with the first k observations. At the next stage, we leave the first and include the (k+1)th observation. This process is repeated until we arrive at the last k observations. Each of these means is centered against the time which is the mid-point of the time interval included in the calculation of the moving average. Thus when k, the period of moving average, is odd, the moving average values correspond to tabulated time values for which the time series is given. When k is even,the moving average falls midway between two tabulated values. In this case, we calculate a subsequent two-item moving average to make the resulting moving average values correspond to the tabulated time periods.

The interpolation ofsimple moving averages is very simple. A k-point moving average may be interpreted as the estimated value for the middle of the period covered from successive linear curves fitted through the first k points, through the 2^{nd} to the (k+1)th values and so on, and lastly through the last k points.

Consider the first k points $y_1,\ldots,y_k$. Let the origin be shifted to the middle of the period so that $\sum t_i=0$. The normal equations for fitting a curve Y=a+bt through $y_1,\ldots,y_k$ are

$\sum y_i = ka + b\sum t_i, \sum t_i y_i = a\sum t_i + b\sum t_i^2$

So that $\hat{a} = \frac{\sum y_i}{k} = \bar{y}, \hat{b} = \frac{\sum t_i y_i}{\sum t_i^2}$. Hence the estimated value for the middle of the period covered ,that is, for t=0, from the curve $Y=\hat{a}+\hat{b}t$ is $\hat{a}$, which is the first moving average value. Similarly it can be shown that the estimated value from the fitted linear curve through $y_2,\ldots,y_{k+1}$ would be $\frac{1}{k}\sum_{i=2}^{k+1} y_i$, the second moving average value and so on.

A moving average with a properly selected period will smooth out cyclical fluctuations from the series and give an estimate of the trend. The central problem in this method is thus the selection of an appropriate method which will eliminate all fluctuations that d raw the series away from the trend.

Example 8.1 Apply 3-year moving average to the following data on production of cements. Plot the data and the trend values on the same graph.

Year:	1992	1993	1994	1995	1996	1997	1998	1999
Output(in '000 tons)	1542	1447	1552	2102	2612	3195	3597	3567

- **Determination of trend values by 3-yearly moving average**

year	Output(in '000 tons)	3-yearly moving total	3-yearly moving average(Trend values)
1992	1542	..	..
1993	1447	4541	1513.7
1994	1552	5101	1700.3
1995	2102	6266	2088.7
1996	2612	7909	2636.3
1997	3195	9344	3114.7
1998	3537	10299	3433.0
1999	3567	..	..

Example 8.2 Work out the trend values by 4-yearly moving average from the following data on production of iron ore(in '000 tons)

Year:	1983	'84	'85	'86	'87	'88	'89	'90	'91	'92	'93
Production:	110	125	118	134	121	132	145	155	159	148	162

Year	production	4-yearly moving total	4-yearly moving average	4-yearly(centered) moving average
1983	110			
1984	125			
		487	121.75	
1985	118			123.125
		498	124.5	
1986	134			125.370
		505	126.25	
1987	121			129.620
		532	133.00	
1988	132			135.625
		553	138.25	

1989	145			143.000
		591	147.75	
1990	155			149.750
		607	151.75	
1991	159			153.870
		624	156.00	
1992	148			
1993	162			

Example 8.3 Calculate the trend values by the method of moving averages from the following data on quarterly production(in ‘000 tons):

	year		
Quarter	**1995**	**1996**	**1997**
I	15	15	20
II	19	22	21
III	21	23	25
IV	18	20	20

Is it possible to find the trend value for the first quarter of 1998 by the above method? Justify.

Year	Quarter	Production (in’000 tons)	4-Quarter moving total	4-Quarter moving average	4-Quarter(centered) moving average
	I	15			----
1995	II	19			----
			73	18.25	
	III	21			18.250
			73	18.25	
	IV	18			18.625
			76		
	I	15		19.00	
			78		19.250

1996	II	22		19.50	
			80		19.750
	III	23		20.00	
			85		20.625
	IV	20		21.25	
			84		21.125
	I	20		21.00	
			86		21.250
1997	II	21		21.50	
			86		21.500
	III	25		21.50	

	IV	20		-----	

It is not possible to find the trend values for the 1st quarter of 1998 by the moving average method since there is no specific mathematical equation which can be used for interpolation or prediction purposes.

Method of mathematical curves

The trend values obtained by the method of moving averages , even though fairly smooth, is not representable by a simple mathematical formula.Since there does not exist any mathematically expressed trend equation, the method fails to achieve the main objective of trend analysis, that is, the interpolation and extrapolation of trend values.Therefore, attempt is made to fit the observed time series with a fairly simple mathematical curve. The fitting of mathematical curve has two parts: (1) determination of the appropriate trend curve, (2) determination of unknown parameters involved in the equation. From the graphical representation of the given time series , an investigator may guess the nature of the which fits the data best. The method is subjective in this sense. Determination of unknown constants appearing in the trend equation can be done by method of least squares.

Example 8.4 Following table gives the number of hospital beds in West Bengal for the years 1979 to 1986. Plot of year versus no. of beds suggest that a linear trend will be appropriate to fit to the given data. The necessary data are done in table below:

Year	No.of beds	t=2(year-mid-period)	ty_t	t^2	$T_t=a_0+a_1t$
1979	55477	-7	-388339	49	55938

1980	58045	-5	-290225	25	57365
1981	58448	-3	-175344	9	58792
1982	59876	-1	-59876	1	60219
1983	61894	1	61894	1	61646
1984	63734	3	191202	9	63073
1985	64667	5	323335	25	64500
1986	65319	7	457233	49	65927
Total	487460	0	119880	168	

Since $\sum t=0$, the normal equations are $487460=8a_0$, $119880=168a_1$ so that $a_0=60932.5$, $a_1=713.57$. The linear trend equation is , therefore, $T_t=60932.5+713.57t$.

Example 8.5 table below shows the data on passenger-kilometer(millions) for Indian Railways during 1983 to 1989. Fit a quadratic trend :

Year	Pass-Kilo	T=year-1986	ty_t	t^2y_t	t^2	t^4
1983	6096	-3	-18288	54864	9	81
1984	6379	-2	-12758	25516	4	16
1985	6774	-1	-6774	6774	1	1
1986	7327	0	0	0	0	0
1987	7516	1	7516	7516	1	1
1988	7863	2	15726	31452	4	16
1989	8427	3	25281	75843	9	81
Total	50382	0	10703	201965	28	196

Here $\sum t=0, \sum t^3=0$. Hence the normal equations are

$0=\sum y=7a_0+28a_1$, $10703=\sum yt=a_2\sum t^2$, $201965=28a_0+196a_1$. Solving, $a_0=7176.63$, $a_1=5.20$, $a_2=382.25$. Thus trend equation is $T_t=7176.63+5.20t+382.25t^2$.

Quadratic Trend fitted to the data

Year	T=year-1986	a_2t	a_1t^2	Trend $T_t=a_0+a_2t+a_1t^2$
1983	-3	-1146.75	46.80	6076.68
1984	-2	-764.5	20.80	6432.93
1985	-1	-382.25	5.20	6799.58
1986	0	0	0	7176.63
1987	1	382.25	5.20	7564.08
1988	2	764.5	20.80	7961.93
1989	3	1146.75	46.80	8370.18

LINEAR PROGRAMMING

CHAPTER I

Mathematical formulation of Linear Programming Problem

Let us consider two real life situations to understand what we mean by a programming problem. For any industry, the objective is to earn maximum profit by selling products which are produced with limited available resources, keeping the cost of production at a minimum. For a housewife the aim is to buy provisions for the family at a minimum cost which will satisfy the needs of the family.
All these type of problems can be done mathematically by formulating a problem which is known as a programming problem. Some restrictions or constraints are to be adopted to formulate the problem. The function which is to be maximized or minimized is called the objective function. If in a programming problem the constraints and the objective function are of linear type then the problem is called a linear programming problem. There are various types of linear programming problems which we will consider through some examples.

Examples

1. (Production allocation problem) Four different type of metals, namely, iron, copper, zinc and manganese are required to produce commodities A, B and C. To produce one unit of A, 40kg iron, 30kg copper, 7kg zinc and 4kg manganese are needed. Similarly, to produce one unit of B, 70kg iron, 14kg copper and 9kg manganese are needed and for producing one unit of C, 50kg iron, 18kg copper and 8kg zinc are required. The total available quantities of metals are 1 metric ton iron, 5 quintals copper, 2 quintals of zinc and manganese each. The profits are Rs 300, Rs 200 and Rs 100 by selling one unit of A, B and C respectively. Formulate the problem mathematically.
   Solution: Let z be the total profit and the problem is to maximize z(called the objective function). We write below the given data in a tabular form:

	Iron	Copper	Zinc	Manganese	Profit per unit in Rs
A	40kg	30kg	7kg	4kg	300
B	70kg	14kg	0kg	9kg	200
C	60kg	18kg	8kg	0kg	100
Available quantities→	1000kg	500kg	200kg	200kg	

To get maximum profit, suppose x_1 units of A, x_2 units of B and x_3 units of C are to be produced. Then the total quantity of iron needed is $(40x_1 + 70x_2 + 60x_3)$kg. Similarly, the total quantity of copper, zinc and manganese needed are $(30x_1 + 14x_2 + 18x_3)$kg , $(7x_1 + 0x_2 + 8x_3)$kg and $(4x_1 + 9x_2 + 0x_3)$kg respectively. From the conditions of the problem we have,

$$40x_1 + 70x_2 + 60x_3 \leq 1000$$

$$30x_1 + 14x_2 + 18x_3 \leq 500$$

$$7x_1 + \ 0x_2 + \ \ 8x_3 \leq 200$$

$$4x_1 + \ 9x_2 + \ \ 0x_3 \leq 200$$

The objective function is $z = 300x_1 + 200x_2 + 100x_3$ which is to be maximized. Hence the problem can be formulated as,

Maximize $$z = 300x_1 + 200x_2 + 100x_3$$

Subject to

$$40x_1 + 70x_2 + 60x_3 \leq 1000$$

$$30x_1 + 14x_2 + 18x_3 \leq 500$$

$$7x_1 + \ 0x_2 + \ \ 8x_3 \leq 200$$

$$4x_1 + \ 9x_2 + \ \ 0x_3 \leq 200$$

As none of the commodities produced can be negative, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

All these inequalities are known as constraints or restrictions.

2. (Diet problem) A patient needs daily 5mg, 20mg and 15mg of vitamins A, B and C respectively. The vitamins available from a mango are 0.5mg of A, 1mg of B, 1mg of C, that from an orange is 2mg of B, 3mg of C and that from an apple is 0.5mg of A, 3mg of B, 1mg of C. If the cost of a mango, an orange and an apple be Rs 0.50, Rs 0.25 and Rs 0.40 respectively, find the minimum cost of buying the fruits so that the daily requirement of the patient be met. Formulate the problem mathematically.

Solution: The problem is to find the minimum cost of buying the fruits. Let z be the objective function. Let the number of mangoes, oranges and apples to be bought so that the cost is minimum and to get the minimum daily requirement of the vitamins be x_1, x_2, x_3 respectively. Then the objective function is given by

$$z = 0.50\, x_1 + 0.25\, x_2 + 0.40\, x_3$$

From the conditions of the problem

$$0.5x_1 + 0x_2 + 0.5x_3 \geq 5$$
$$x_1 + 2x_2 + \quad 3x_3 \geq 20$$
$$x_1 + 3x_2 + \ x_3 \geq 15 \quad \text{and}$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Hence the problem is

Minimize $z = 0.50\, x_1 + 0.25\, x_2 + 0.40\, x_3$.

Subject to $\quad 0.5x_1 + 0x_2 + 0.5x_3 \geq 5$

$$x_1 + 2x_2 + \quad 3x_3 \geq 20$$

$$x_1 + \ 3x_2 + \quad x_3 \geq 15$$

and $\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

3. (Transportation problem) Three different types of vehicles A, B and C have been used to transport 60 tons of solid and 35 tons of liquid substance. Type A vehicle can carry 7 tons solid and 3 tons liquid whereas B and C can carry 6 tons solid and 2 tons liquid and 3 tons solid and 4 tons liquid respectively. The cost of transporting are Rs 500, Rs 400 and Rs 450 respectively per vehicle of type A, B and C respectively. Find the minimum cost of transportation. Formulate the problem mathematically.

   Solution: Let z be the objective function. Let the number of vehicles of type A, B and C used to transport the materials so that the cost is minimum be x_1, x_2, x_3 respectively. Then the objective function is $= 500x_1 + 400x_2 + 450x_3$. The quantities of solid and liquid transported by the vehicles are $7x_1 + 6x_2 + 3x_3$ tons and $3x_1 + 2x_2 + 4x_3$ tons respectively. By the conditions of the problem, $7x_1 + 6x_2 + 3x_3 \geq 60$ and $3x_1 + 2x_2 + 4x_3 \geq 35$. Hence the problem is

   Minimize $z = 500x_1 + 400x_2 + 450x_3$

   Subject to $7x_1 + 6x_2 + 3x_3 \geq 60$

   $3x_1 + 2x_2 + 4x_3 \geq 35$

   And $x_1, x_2, x_3 \geq 0$

4. An electronic company manufactures two radio models each on a separate production line. The daily capacity of the first line is 60 radios and that of the second line is 75 radios. Each unit of the first model uses 10 pieces of a certain electronic component, whereas each unit of the second model uses 8 pieces of the same component. The maximum daily availability of the special component is 800 pieces. The profit per unit of models 1 and 2 are Rs 500 and Rs 400 respectively. Determine the optimal daily production of each model.

   Solution: This is a maximization problem. Let x_1, x_2 be the number of two radio models each on a separate production line. Therefore the objective function is $z = 500x_1 + 400x_2$ which is to be maximized. From the conditions of the problem we have $x_1 \leq 60$, $x_2 \leq 75$, $10x_1 + 8x_2 \leq 800$. Hence the problem is

   Maximize $z = 500x_1 + 400x_2$

Subject to $\quad x_1 \le 60$

$x_2 \le 75$

$10x_1 + 8x_2 \le 800$

And $\quad x_1, x_2 \ge 0$

5. An agricultural firm has 180 tons of Nitrogen fertilizers, 50 tons of Phosphate and 220 tons of Potash. It will be able to sell 3:3:4 mixtures of these substances at a profit of Rs 15 per ton and 2:4:2 mixtures at a profit of Rs 12 per ton respectively. Formulate a linear programming problem to determine how many tons of these two mixtures should be prepares so as to maximize profit.

Solution: Let the 3:3:4 mixture be called A and 2:4:2 mixture be called B. Let x_1 , x_2 tons of these two mixtures be produced to get maximum profit. Thus the objective function is $z = 15x_1 + 12x_2$ which is to be maximized. Let us denote Nitrogen, Phosphate and Potash as N Ph and P respectively.

Then in the mixture A , $\frac{N}{3} = \frac{Ph}{3} = \frac{P}{4} = k_1$(say).

$$\Longrightarrow N = 3k_1\,, Ph = 3k_1\,, P = 4k_1$$

$$\Longrightarrow x_1 = 10k_1\,.$$

Similarly for the mixture B , $N = 2k_2\,, Ph = 4k_2\,, P = 2k_2k_1$

$$\Longrightarrow x_2 = 8k_2\,.$$

Thus the constraints are $\frac{3}{10}x_1 + \frac{1}{4}x_2 \le 180$ [since in A, amount of nitrogen = $\frac{3k_1}{10k_1}x_1 = \frac{3}{10}x_1$] Similarly $\frac{3}{10}x_1 + \frac{1}{2}x_2 \le 250$ and $\frac{2}{5}x_1 + \frac{1}{4}x_2 \le 220$. Hence the problem is

Maximize $z = 15x_1 + 12x_2$

Subject to $\quad \frac{3}{10}x_1 + \frac{1}{4}x_2 \le 180$

$\frac{3}{10}x_1 + \frac{1}{2}x_2 \le 250$

$\frac{2}{5}x_1 + \frac{1}{4}x_2 \le 220$

And $\quad x_1, x_2 \ge 0.$

6. A coin to be minted contains 40% silver, 50% copper, 10% nickel. The mint has available alloys A, B, C and D having the following composition and costs, and availability of metals:

	% silver	% copper	% nickel	Costs per Kg
A	30	60	10	Rs 11
B	35	35	30	Rs 12
C	50	50	0	Rs 16
D	40	45	15	Rs 14
Availability of metals →	Total 1000 Kgs			

Present the problem of getting the alloys with specific composition at minimum cost in the form of a L.P.P.

Solution: Let $x_1, x_2, x_{3\,,}\ x_4$ Kg s be the quantities of alloys A, B, C, D used for the purpose. By the given condition $x_1 + x_2 + x_3 + x_4 \leq 1000$.

The objective function is $z = 11x_1 + 12x_2 + 16x_3 + 14x_4$

and the constraints are $0.3x_1 + 0.35x_2 + 0.5x_3 + \ 0.4x_4 \leq 400$ for silver

$0.6x_1 + 0.35x_2 + 0.5x_3 + 0.45x_4 \leq 500$ for copper

$0.1x_1 + \ 0.3x_2 + \qquad + 0.15x_4 \leq 100$ for nickel

Thus the L.P.P is Minimize $z = 11x_1 + 12x_2 + 16x_3 + 14x_4$

Subject to
$$0.3x_1 + 0.35x_2 + 0.5x_3 + \ 0.4x_4 \leq 400$$
$$0.6x_1 + 0.35x_2 + 0.5x_3 + 0.45x_4 \leq 500$$
$$0.1x_1 + \ 0.3x_2 + \qquad + 0.15x_4 \leq 100$$
$$x_1 + \quad x_2 + \quad x_3 + \quad x_4 \leq 1000$$
And $x_1\,, x_2\,, x_3 \geq 0$

7. A hospital has the following minimum requirement for nurses.

Period	Clock time (24 hours day)	Minimum number of nurses required
1	6A.M-10A.M	60
2	10A.M-2P.M	70
3	2P.M-6P.M	60
4	6P.M-10P.M	50
5	10P.M-2A.M	20
6	2A.M-6A.M	30

Nurses report to the hospital wards at the beginning of each period and work for eight consecutive hours. The hospital wants to determine the minimum number of nurses so that there may be sufficient number of nurses available for each period. Formulate this as a L.P.P.

Solution: This is a minimization problem. Let $x_1, x_2, \dots\dots, x_6$ be the number of nurses required for the period 1, 2, ……, 6. Then the objective function is

Minimize, $z = x_1 + x_2 + \cdots \dots + x_6$ and the constraints can be written in the following manner.

x_1 nurses work for the period 1 and 2 and x_2 nurses work for the period 2 and 3 etc. Thus for the period 2,

$x_1 + x_2 \geq 70.$

Similarly, for the periods 3, 4, 5, 6, 1 we have,

$x_2 + x_3 \geq 60$

$x_3 + x_4 \geq 50$

$x_4 + x_5 \geq 20$

$x_5 + x_6 \geq 30$

$x_6 + x_1 \geq 60$, $x_j \geq 0, j = 1, 2, \dots \dots, 6$

Mathematical formulation of a L.P.P

From the discussion above, now we can mathematically formulate a general Linear Programming Problem which can be stated as follows.

Find out a set of values $x_1, x_2, \dots \dots, x_n$ which will optimize (either maximize or minimize) the linear function

$z = c_1x_1 + c_2x_2 + \cdots \dots + c_nx_n$

Subject to the restrictions

$a_{11}x_1 + a_{12}x_2 + \cdots \dots + a_{1n}x_n (\leq = \geq) b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots \dots + a_{2n}x_n (\leq = \geq) b_2$

………………………………………………..

$a_{m1}x_1 + a_{m2}x_2 + \cdots \dots + a_{mn}x_n (\leq = \geq) b_m$

And the non-negative restrictions $x_j \geq 0, j = 1, 2, \dots \dots, n$ where a_{ij}, c_j, b_i $(i = 1, 2, \dots \dots, m,\ j = 1, 2, \dots \dots., n)$ are all constants and x_j , $(j = 1, 2, \dots \dots, n)$ are variables. Each of the linear expressions on the left hand side connected to the corresponding constants on the right side by only one of the signs $\leq$, = and $\geq$,is known as a constraint. A constraint is either an equation or an inequation.

The linear function $z = c_1x_1 + c_2x_2 + \cdots \dots + c_nx_n$ is known as the objective function.

By using the matrix and vector notation the problem can be expressed in a compact form as

Optimize $z = c^Tx$ subject to the restrictions $Ax \leq = \geq b, x \geq 0,$

where $A = [a_{ij}]$ is a m x n coefficient matrix.,

$c = (c_1, c_2, \dots \dots, c_n)^T$ is a n-component column vector, which is known as a cost or price vector,

$x = (x_1, x_2, \dots \dots, x_n)^T$ is a n-component column vector, which is known as decision variable vector or legitimate variable vector and

$b = (b_1, b_2, \dots \dots, b_m)^T$ is a m-component column vector, which is known as requirement vector.

In all practical discussions, $b_i \geq 0 \; \forall i$. If some of them are negative, we make them by multiplying both sides of the inequality by (-1).

If all the constraints are equalities, then the L.P.P is reduced to

Optimize $z = c^T x$ subject to $Ax = b, x \geq 0$.

This form is called the standard form.

Feasible solution to a L.P.P: A set of values of the variables, which satisfy all the constraints and all the non-negative restrictions of the variables, is known as the feasible solution (F.S.) to the L.P.P.

Optimal solution to a L.P.P: A feasible solution to a L.P.P which makes the objective function optimal is known as the optimal solution to the L.P.P

There are two ways of solving a linear programming problem: (1) Geometrical method and (2) Algebraic method.

A particular L.P.P is either a minimization or a maximization problem. The problem of minimization of the objective function z is nothing but the problem of maximization of the function $(-z)$ and vice versa and $\min z = -\max(-z)$ with the same set of constraints and the same solution set.

Graphical or Geometrical Method of Solving a Linear Programming Problem

We will illustrate the method by giving examples.

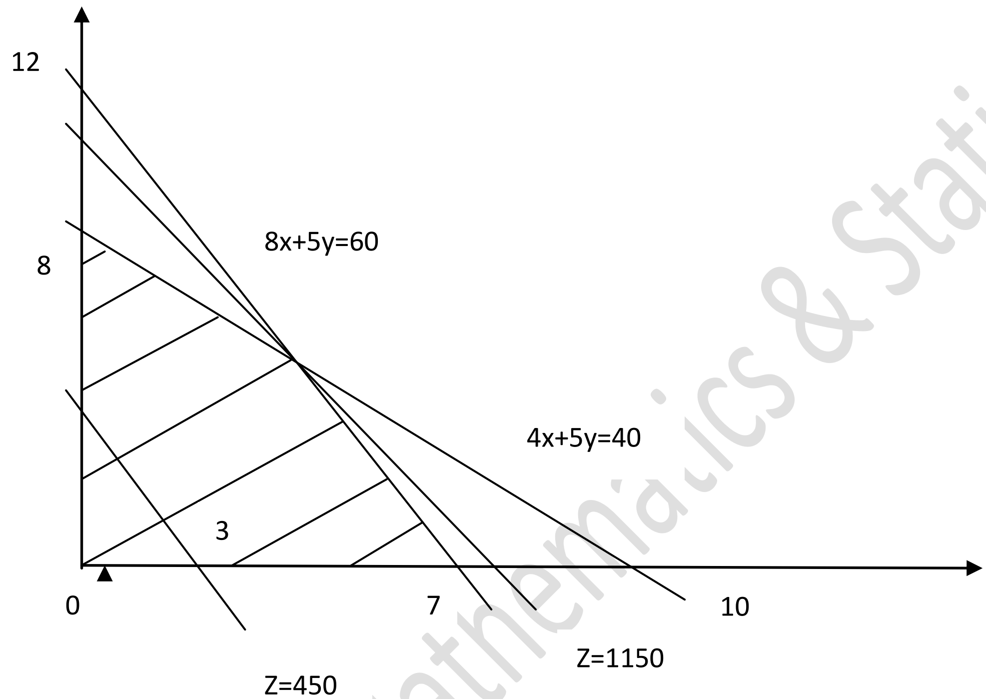
Examples

Solve the following problems graphically.

1. Maximize $z = 150x + 100y$

   Subject to $8x + 5y \leq 60$

   $4x + 5y \leq 40$, $x, y \geq 0$.



The constraints are treated as equations along with the non negativity relation. We confine ourselves to the first quadrant of the xy plane and draw the lines given by those equations. Then the directions of the inequalities indicate that the striped region in the graph is the feasible region. For any particular value of z, the graph of the objective function regarded as an equation is a straight line (called the profit line in a maximization problem) and as z varies, a family of parallel lines is generated. We have drawn the line corresponding to z=450. We see that the profit z is proportional to the perpendicular distance of this straight line from the origin. Hence the profit increases as this line moves away from the origin. Our aim is to find a point in the feasible region which will give the maximum value of z. In order to find that point we move the profit line away from origin keeping it parallel to itself. By doing this we find that (5,4) is the last point in the feasible region which the moving line encounters. Hence we get the optimal solution $z_{max} = 1150$ for $= 5, y = 4$.

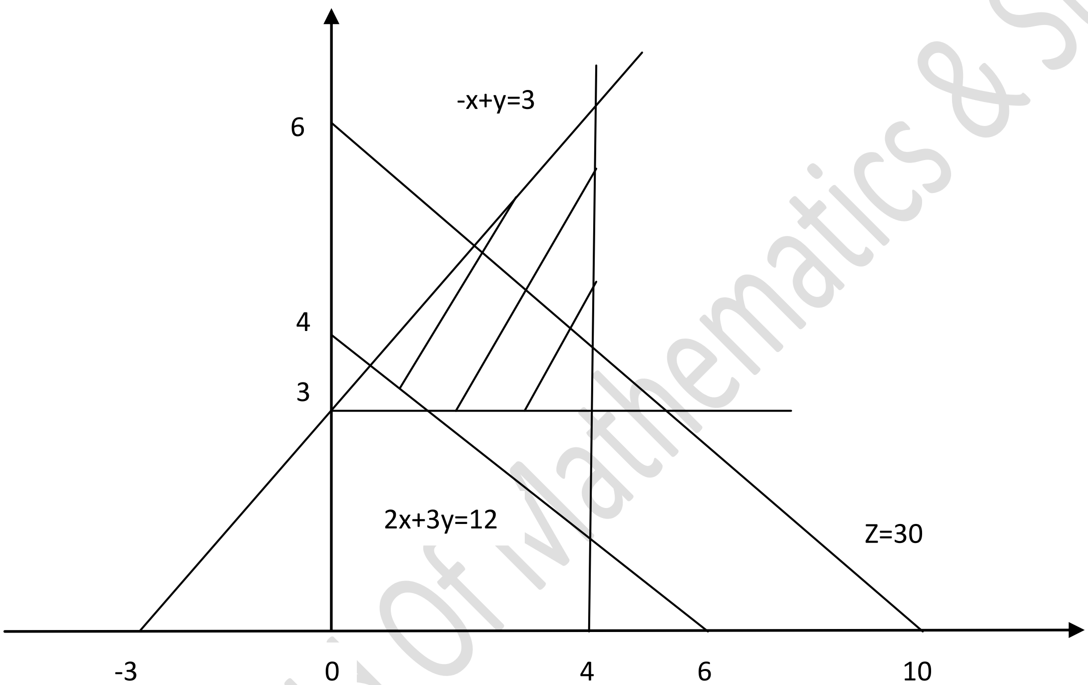
Note: If we have a function to minimize, then the line corresponding to a particular value of the objective function (called the cost line in a minimization problem) is moved towards the origin.

2. Solve graphically:

   Minimize $z = 3x + 5y$

   Subject to $2x + 3y \geq 12$

$$-x + y \leq 3$$
$$x \leq 4$$
$$y \leq 4$$



Here the striped area is the feasible region. We have drawn the cost line corresponding to z=30. As this is a minimization problem the cost line is moved towards the origin and the cost function takes its minimum at $z_{min} = 19.5$ for $= 1.5, y = 3$.

In both the problems above the L.P.P. has a unique solution.

3.

Solve graphically:

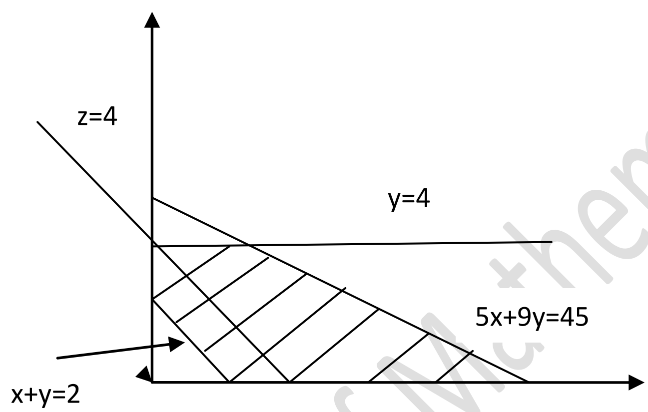
Minimize $z = x + y$

Subject to $5x + 9y \leq 45$

$$x + y \geq 2$$

$$y \leq 4 \quad , x, y \geq 0$$

Here the striped area is the feasible region. We have drawn the cost line corresponding to z=4. As this is a minimization problem the cost line when moved towards the origin coincides with the boundary line $x + y = 2$ and the optimum value is attained at all points lying on the line segment joining (2,0) and (0,2) including the end points. Hence there are an infinite number of solutions. In this case we say that alternative optimal solution exists.



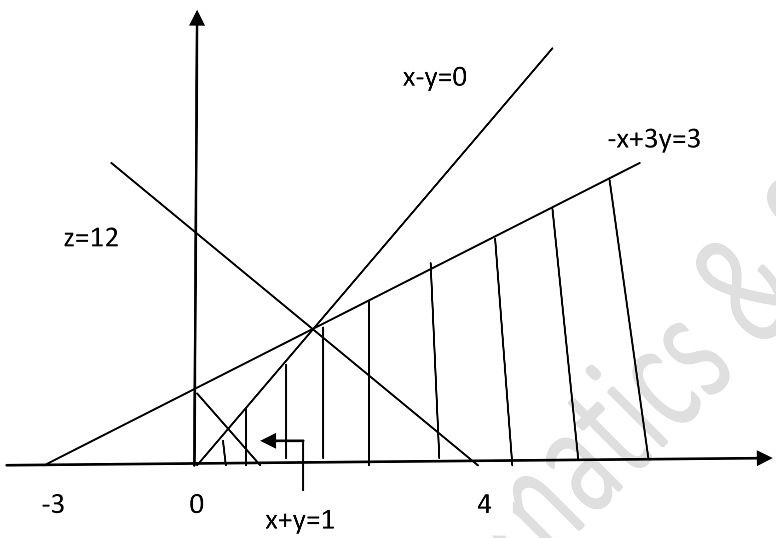
4. Solve graphically

   Maximize $z = 3x + 4y$

   Subject to $x - \quad y \geq 0$

   $x + \quad y \geq 1$

   $-x + 3y \leq 3\ ,\ x, y \geq 0$



The striped region in the graph is the feasible region which is unbounded.. For any particular value of z, the graph of the objective function regarded as an equation is a straight line (called the profit line in a maximization problem) and as z varies, a family of parallel lines is generated. We have drawn the line corresponding to z=12. We see that the profit z is proportional to the perpendicular distance of this straight line from the origin. Hence the profit increases as this line moves away from the origin. As we move the profit line away from origin keeping it parallel to itself we see that there is no finite maximum value of z.

Ex: Keeping everything else unaltered try solving the problem as a minimization problem.

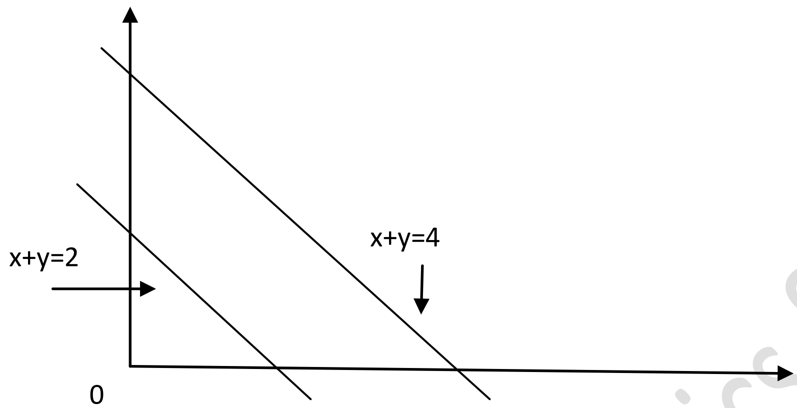
5. Solve graphically

Maximize $z = 2x - 3y$

Subject to $x + \ y \leq \ 2$

$x + \ y \geq 4$

$x, y \ \geq 0$



It is clear that there is no feasible region.

In algebraic method, the problem can be solved only when all constraints are equations. We now show how the constraints can be converted into equations.

Slack and Surplus Variables

When the constraints are inequations connected by the sign " ≤ " , in each inequation a variable is added on the left hand side of it to convert ind sidet into an equation. For example, the constraint

$$x_1 - 2x_2 + 7x_3 \leq 4$$

is connected by the sign ≤ . Then a variable x_4 is added to the left hand side and it is converted into an equation

$$x_1 - 2x_2 + 7x_3 + x_4 = 4$$

From the above it is clear that the slack variables are non-negative quantities.

If the constraints are connected by " $\geq$ " ,in each inequation a variable is subtracted from the left hand side to convert it into an equation. These variables are known as surplus variables. For example,

$$x_1 - 2x_2 + 7x_3 \geq 4$$

is converted into an equation by subtracting a variable x_4 from the left hand side.

$$x_1 - 2x_2 + 7x_3 - x_4 = 4$$

The surplus variables are also non-negative quantities.

Let a general L.P.P containing r variables and m constraints be

Optimize $z = c_1x_1 + c_2x_2 + \cdots \ldots + c_rx_r$

subject to $a_{i1}x_1 + a_{i2}x_2 + \cdots \ldots + a_{in}x_n \leq = \geq b_i$, $i = 1,2, \ldots \ldots, m,$ $x_j \geq 0, j = 1,2, \ldots \ldots, r,$

where one and only one of the signs $\leq, =, \geq$ holds for each constraint, but the signs may vary from one constraint to another. Let k constraints out of the m be inequations ($0 \leq k \leq m$). Then introducing k slack or surplus variables $x_{r+1}, x_{r+2}, \ldots \ldots, x_n, n = r + k$, one to each of the inequations, all constraints can be converted into equations containing n variables. We further assume that $n \geq m.$ The objective function is similarly accommodated with k slack or surplus variables $x_{r+1}, x_{r+2}, \ldots \ldots, x_n$, the cost components of these variables are assumed to be zero. Then the adjusted objective function is

$z_{ad} = c_1x_1 + c_2x_2 + \cdots \ldots + c_rx_r + 0x_{r+1} + 0x_{r+2} + \cdots \ldots + 0x_n$, and then the problem can be written as

Optimize $z_{ad} = c^Tx$ subject to $Ax = b, x \geq 0,$

where A is an mxn matrix , known as coefficient matrix given by

$A = (a_1, a_2, \ldots \ldots, a_n),$

where $a_j = (a_{1j}, a_{2j}, \ldots \ldots, a_{mj})^T$ is a column vector associated with the vector x_j , $j = 1,2, \ldots \ldots, n$.

$c = (c_1, c_2, \ldots \ldots, c_r, 0,0, \ldots, 0)^T$ is a n-component column vector,

$x = (x_1, x_2, \ldots \ldots, x_r, x_{r+1}, x_{r+2}, \ldots \ldots, x_n)^T$ is a n-component column vector, and

$b = (b_1, b_2, \ldots \ldots, b_m)^T$ is a m-component column vector.

The components of b can be made positive by proper adjustments.

It is worth noting that the column vectors associated with the slack variables are all unit vectors. As the cost components of the slack and surplus variables are all zero, it can be verified easily that the solution set which optimizes z_{ad} also optimizes z. Hence to solve the original L.P.P it is sufficient to solve the standard form of the L.P.P. So, for further discussions we shall use the same notation for z_{ad} and z.

Problems

1. Transform the following Linear Programming Problems to the standard form:

(i) Maximize $z = 2x_1 + 3x_2 - 4x_3$

Subject to
$$\begin{aligned} 4x_1 + 2x_2 - x_3 &\le 4 \\ -3x_1 + 2x_2 + 3x_3 &\ge 6 \\ x_1 + x_2 - 3x_3 &= 8, \quad x_j \ge 0, j = 1,2,3. \end{aligned}$$

Solution: First constraint is $\le$ type and the second one is a $\ge$ type, so adding a slack and a surplus variable respectively, the two constraints are converted into equations. Hence the transformed problem can be written as

Maximize $z = 2x_1 + 3x_2 - 4x_3 + 0x_4 + 0x_5$

Subject to
$$\begin{aligned} 4x_1 + 2x_2 - x_3 + x_4 \qquad &= 4 \\ -3x_1 + 2x_2 + 3x_3 \qquad - x_5 &= 6 \\ x_1 + x_2 - 3x_3 \qquad\qquad &= 8, \quad x_j \ge 0, j = 1,2,3,4,5. \end{aligned}$$

(ii) Maximize $z = x_1 - x_2 + x_3$

Subject to
$$\begin{aligned} x_1 + x_2 - 3x_3 &\ge 4 \\ 2x_1 - 4x_2 + x_3 &\ge -5 \\ x_1 + 2x_2 - 2x_3 &\le 3, \quad x_j \ge 0, j = 1,2,3. \end{aligned}$$

Solution: The problem can be transformed as

Maximize $z = x_1 - x_2 + x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to
$$
\begin{aligned}
x_1 + \ x_2 - 3x_3 + x_4 \qquad\qquad &= \ \ 4\\
2x_1 - 4x_2 + \ x_3 \qquad - x_5 \qquad &= -5\\
x_1 + \ 2x_2 \ - 2x_3 \qquad\qquad - x_6 &= \ \ 3
\end{aligned}
\quad ,
$$
$x_j \geq 0, j = 1,2,3,4,5,6$.

x_4, x_5 are surplus and x_6 is a slack variable. Making the second component of b vector positive , the second equation can be written as

$-2x_1 + 4x_2 - \ x_3 \qquad + x_5 \qquad = 5$

and in that case the surplus variable is changed into a slack variable.

2. Express the following minimization problem as a standard maximization problem by introducing slack and surplus variables.

Minimize $\qquad z = 4x_1 - x_2 + 2x_3$

Subject to
$$
\begin{aligned}
4x_1 + \ \ x_2 \ - x_3 &\leq \ 7\\
2x_1 - 3x_2 + \ x_3 &\leq 12\\
x_1 + \ \ x_2 \ + \ x_3 &= \ 8\\
4x_1 + \ 7x_2 - x_3 &\geq 16 \quad , x_j \geq 0, j = 1,2,3 .
\end{aligned}
$$

Solution: After introducing slack variables in the first two constraints and a surplus in the fourth, the converted problem is, Minimize $z^* = (-z) = 4x_1 - x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to
$$
\begin{aligned}
4x_1 + \ \ x_2 \ - x_3 + x_4 \qquad\qquad &= \ \ 7\\
2x_1 - 3x_2 + \ x_3 \qquad + x_5 \qquad &= 12\\
x_1 + \ \ x_2 \ + \ x_3 \qquad\qquad\qquad &= \ \ 8\\
4x_1 + \ 7x_2 - x_3 \qquad\qquad - x_6 &= 16 \quad , \ x_j \geq 0, j =
\end{aligned}
$$
$1,2, \ldots ,6$.

Writing the above problem as a standard maximization problem

Maximize $\quad z^* = (-z) = 4x_1 - x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$

Subject to
$$
\begin{aligned}
4x_1 + \ \ x_2 \ - x_3 + x_4 \qquad\qquad &= \ \ 7\\
2x_1 - 3x_2 + \ x_3 \qquad + x_5 \qquad &= 12\\
x_1 + \ \ x_2 \ + \ x_3 \qquad\qquad\qquad &= \ \ 8\\
4x_1 + \ 7x_2 - x_3 \ - x_6 &= 16 \quad , x_j \geq 0, j = 1,2, \ldots ,6 .
\end{aligned}
$$

Variable unrestricted in sign

If a variable x_j is unrestricted in sign, then it can be expressed as a difference of two non-negative variables, say, x_j', x_j'' as $x_j = x_j' - x_j''$, $x_j' \geq 0, x_j'' \geq 0$. If $x_j' > x_j''$, then $x_j > 0$, if $x_j' = x_j''$, then $x_j = 0$ and if $x_j' < x_j''$, then $x_j < 0$. Hence x_j is unrestricted in sign.

3. Write down the following L.P.P in the standard form.

$$\text{Maximize} \quad z = 2x_1 + 3x_2 - x_3$$

$$\begin{aligned} \text{Subject to} \quad & 4x_1 + x_2 + x_3 \geq 4 \\ & 7x_1 + 4x_2 - x_3 \leq 25 , \end{aligned}$$

$x_j \geq 0, j = 1,3$, x_2 unrestricted in sign .

Solution: Introducing slack and surplus variables and writing $x_2 = x_2' - x_2''$, where $x_2' \geq 0, x_2'' \geq 0$,

the problem in the standard form is

$$\text{Maximize} \quad z = 2x_1 + 3x_2' - 3x_2'' - x_3 + 0x_4 + 0x_5$$

$$\begin{aligned} \text{Subject to} \quad & 4x_1 + x_2' - x_2'' + x_3 - x_4 = 4 \\ & 7x_1 + 4x_2' - 4x_2'' - x_3 + x_5 = 25 , \end{aligned}$$

$x_1 . x_2', x_2'', x_3 \geq 0$.

TOP 10 PLACES FOR M.A./M.SC ECONOMICS IN INDIA:

1. Indian Statistical Institute
2. Delhi School of Economics
3. Indira Gandhi Institute of Development Research
4. Jawaharlal Nehru University
5. Centre for Development Studies
6. Madras School of Economics
7. South Asian University, Delhi
8. Gokhale Institute of Politics and Economics
9. University of Hyderabad
10. Jamia Millia Islamia