MATHEMATICS CONTESTS

THE AUSTRALIAN SCENE 2017

COMBINED (MCYA AND AMOC)

A Di Pasquale, N Do and KL McAvaney

It is not in the stars to hold our destiny but in ourselves

William Shakespeare

Australian Mathematical Olympiad Committee

A department of the Australian Mathematics Trust

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The Mathematics/ Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

The Australian Mathematical Olympiad Committee (AMOC) also acknowledges the significant financial support it has received from the Australian Government towards the training of our Olympiad candidates and the participation of our team at the International Mathematical Olympiad (IMO).

The views expressed here are those of the authors and do not necessarily represent the views of the government.

Special thanks

With special thanks to the Australian Mathematical Society, the Australian Association of Mathematics Teachers and all those schools, societies, families and friends who have contributed to the expense of sending the 2017 IMO team to Rio de Janeiro.

ACKNOWLEDGEMENTS

The Australian Mathematical Olympiad Committee (AMOC) sincerely thanks all sponsors, teachers, mathematicians and others who have contributed in one way or another to the continued success of its activities. The editors sincerely thank those who have assisted in the compilation of this book, in particular the students who have provided solutions to the 2017 IMO. Thanks also to members of AMOC and Challenge Problems Committees, Chief Mathematician Mike Clapper, Chief Executive Officer Nathan Ford, staff of the Australian Mathematics Trust and others who are acknowledged elsewhere in the book.

FROM THE AMT CHIEF EXECUTIVE OFFICER

This year's IMO was the most challenging in the history of the competition. Even so, the Australian team performed admirably, attaining three Silver and two Bronze medals and an Honourable Mention. Linus Cooper was one of only two competitors (out of 615) to correctly solve problem 3 on Day 1.

In 2017, the Australian Mathematics Trust applied to participate in the EGMO competition in 2018. We were fortunate enough to have our application approved and we have appointed a Team Leader (Ms Thanom Shaw) and a Deputy Leader (Ms Michelle Chen). A number of girls attended the AMOC School of Excellence in Melbourne and commenced their training for EGMO Florence 2018. A team will be selected following the process for the IMO team. We wish them every success in their inaugural event.

There are a great many enthusiastic staff and volunteers that help support our Mathematical Olympiad programs.
On behalf of the Australian Mathematics Trust, I would like to thank:

Emeritus Professor Cheryl Praeger AM, Australian Mathematical Olympiad Committee (AMOC), Chair

Dr Angelo Di Pasquale, AMOC Director of Training and International Mathematical Olympiad (IMO) Team Leader

Mr Andrew Elvey-Price, IMO Deputy Team Leader

Mr Mike Clapper, AMT Chief Mathematician

Dr Norman Do, Chair, AMOC Senior Problems Committee

Dr Kevin McAvaney, Chair, MCYA Challenge Committee

Members of the AMT Board and AMOC Committee

AMOC tutors, mentors, volunteers and ex-Olympians

Nathan Ford
December 2018

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BACKGROUND NOTES ON THE IMO AND AMOC

The Australian Mathematical Olympiad Committee

In 1980, a group of distinguished mathematicians formed the Australian Mathematical Olympiad Committee (AMOC) to coordinate an Australian entry in the International Mathematical Olympiad (IMO).

Since then, AMOC has developed a comprehensive program to enable all students (not only the few who aspire to national selection) to enrich and extend their knowledge of mathematics. The activities in this program are not designed to accelerate students. Rather, the aim is to enable students to broaden their mathematical experience and knowledge.

The largest of these activities is the MCYA Challenge, a problem-solving event held in second term, in which thousands of young Australians explore carefully developed mathematical problems. Students who wish to continue to extend their mathematical experience can then participate in the MCYA Enrichment Stage and pursue further activities leading to the Australian Mathematical Olympiad and international events.

Originally AMOC was a subcommittee of the Australian Academy of Science. In 1992 it collaborated with the Australian Mathematics Foundation (which organises the Australian Mathematics Competition) to form the Australian Mathematics Trust. The Trust, a not-for-profit organisation under the trusteeship of the University of Canberra, is governed by a Board which includes representatives from the Australian Academy of Science, Australian Association of Mathematics Teachers and the Australian Mathematical Society.

The aims of AMOC include:

(1) giving leadership in developing sound mathematics programs in Australian schools
(2) identifying, challenging and motivating highly gifted young Australian school students in mathematics
(3) training and sending Australian teams to future International Mathematical Olympiads.

AMOC schedule from August until July for potential IMO team members

Each year hundreds of gifted young Australian school students are identified using the results from the Australian Mathematics Competition, the Mathematics Challenge for Young Australians program and other smaller mathematics competitions, including the Australian Intermediate Mathematics Olympiad. A network of dedicated mathematicians and teachers has been organised to give these students support during the year either by correspondence sets of problems and their solutions or by special teaching sessions. After participation in other invitational competitions, about 45 of these outstanding students are identified and invited to attend the residential AMOC School of Excellence held in November/December.

In February approximately 100 students are invited to attempt the Australian Mathematical Olympiad. The best 20 or so of these students are then invited to represent Australia in the correspondence Asian Pacific Mathematics Olympiad in March. About 12 students are selected for the AMOC Selection School in April and about 15 younger students are also invited to this residential school. Here, the Australian team of six students plus one reserve for the International Mathematical Olympiad, held in July each year, is selected. A personalised support system for the Australian team operates during May and June.

It should be appreciated that the AMOC program is not meant to develop only future mathematicians. Experience has shown that many talented students of mathematics choose careers in engineering, computing, and the physical and life sciences, while others will study law or go into the business world. It is hoped that the AMOC Mathematics Problem-Solving Program will help the students to think logically, creatively, deeply and with dedication and perseverance; that it will prepare these talented students to be future leaders of Australia.

The International Mathematical Olympiad

The IMO is the pinnacle of excellence and achievement for school students of mathematics throughout the world. The concept of national mathematics competitions started with the Eötvos Competition in Hungary during 1894. This idea was later extended to an international mathematics competition in 1959 when the first IMO was held in Romania. The aims of the IMO include:

(1) discovering, encouraging and challenging mathematically gifted school students
(2) fostering friendly international relations between students and their teachers
(3) sharing information on educational syllabi and practice throughout the world.

It was not until the mid-sixties that countries from the western world competed at the IMO. The United States of America first entered in 1975. Australia has entered teams since 1981.

Students must be under 20 years of age at the time of the IMO and have not enrolled at a tertiary institution. The Olympiad contest consists of two four-and-a-half hour papers, each with three questions.

Australia has achieved varying successes as the following summary of results indicate. HM (Honorable Mention) is awarded for obtaining full marks in at least one question.

The IMO will be held in Cluj-Napoca, Romania, in 2018.

Summary of Australia's achievements at previous IMOs

Year	City	Gold	Silver	Bronze	HM	Rank
1981	Washington			1		23 out of 27 teams
1982	Budapest			1		21 out of 30 teams
1983	Paris		1	2		19 out of 32 teams
1984	Prague		1	2		15 out of 34 teams
1985	Helsinki	1	1	2		11 out of 38 teams
1986	Warsaw			5		15 out of 37 teams
1987	Havana		3			15 out of 42 teams
1988	Canberra	1		1	1	17 out of 49 teams
1989	Braunschweig		2	2		22 out of 50 teams
1990	Beijing		2	4		15 out of 54 teams
1991	Sigtuna			3	2	20 out of 56 teams
1992	Moscow	1	1	2	1	19 out of 56 teams
1993	Istanbul	1	2	3		13 out of 73 teams
1994	Hong Kong		2	3	3	12 out of 69 teams
1995	Toronto		1	4	1	21 out of 73 teams
1996	Mumbai		2	3		23 out of 75 teams
1997	Mar del Plata	2	3	1		9 out of 82 teams
1998	Taipei		4	2		13 out of 76 teams
1999	Bucharest	1	1	3	1	15 out of 81 teams
2000	Taejon	1	3	1		16 out of 82 teams
2001	Washington D.C.	1		4		25 out of 83 teams
2002	Glasgow	1	2	1	1	26 out of 84 teams
2003	Tokyo		2	2	2	26 out of 82 teams
2004	Athens	1	1	2	1	27 out of 85 teams
2005	Merida			6		25 out of 91 teams
2006	Ljubljana		3	2	1	26 out of 90 teams
2007	Hanoi		1	4	1	22 out of 93 teams
2008	Madrid		5	1		19 out of 97 teams
2009	Bremen	2	1	2	1	23 out of 104 teams
2010	Astana	1	3	1	1	15 out of 96 teams
2011	Amsterdam		3	3		25 out of 101 teams
2012	Mar del Plata		2	4		27 out of 100 teams
2013	Santa Marta	1	2	3		15 out of 97 teams
2014	Cape Town	1 Perfect Score by Alexander Gunning	3	2		11 out of 101 teams
2015	Chiang Mai	2	4			6 out of 104 teams
2016	Hong Kong		2	4		25 out of 109 teams
2017	Rio de Janeiro		3	2	1	34 out of 111 teams

MATHEMATICS CHALLENGE FOR YOUNG AUSTRALIANS

The Mathematics Challenge for Young Australians (MCYA) started on a national scale in 1992. It was set up to cater for the needs of the top 10 percent of secondary students in Years 7–10, especially in country schools and schools where the number of students may be quite small. Teachers with a handful of talented students spread over a number of classes and working in isolation can find it very difficult to cater for the needs of these students. The MCYA provides materials and an organised structure designed to enable teachers to help talented students reach their potential. At the same time, teachers in larger schools, where there are more of these students, are able to use the materials to better assist the students in their care.

The aims of the Mathematics Challenge for Young Australians include:

encouraging and fostering

– a greater interest in and awareness of the power of mathematics

– a desire to succeed in solving interesting mathematical problems

– the discovery of the joy of solving problems in mathematics

identifying talented young Australians, recognising their achievements nationally and providing support that will enable them to reach their own levels of excellence

providing teachers with

– interesting and accessible problems and solutions as well as detailed and motivating teaching discussion and extension materials

– comprehensive Australia-wide statistics of students' achievements in the Challenge.

There are three independent stages in the Mathematics Challenge for Young Australians:

- Challenge (three weeks during the period March–June)
- Enrichment (April–September)
- Australian Intermediate Mathematics Olympiad (September).

Challenge

Challenge consists of four levels. Middle Primary (Years 3–4) and Upper Primary (Years 5–6) present students with four problems each to be attempted over three weeks, students are allowed to work on the problems in groups of up to three participants, but each must write their solutions individually. The Junior (Years 7–8) and Intermediate (Years 9–10) levels present students with six problems to be attempted over three weeks, students are allowed to work on the problems with a partner but each must write their solutions individually.

There were 13,649 submissions (1728 Middle Primary, 3716 Upper Primary, 5401 Junior, 2804 Intermediate) for the Challenge in 2017. The 2017 problems and solutions for the Challenge, together with some statistics, appear later in this book.

Enrichment

This is a six-month program running from April to September, which consists of seven different parallel stages of comprehensive student and teacher support notes. Each student participates in only one of these stages.

The materials for all stages are designed to be a systematic structured course over a flexible 12–14 week period between April and September. This enables schools to timetable the program at convenient times during their school year.

Enrichment is completely independent of the earlier Challenge; however, they have the common feature of providing challenging mathematics problems for students, as well as accessible support materials for teachers.

Ramanujan (years 4–5) includes estimation, special numbers, counting techniques, fractions, clock arithmetic, ratio, colouring problems, and some problem-solving techniques. There were 342 entries in 2017.

Newton (years 5–6) includes polyominoes, fast arithmetic, polyhedra, pre-algebra concepts, patterns, divisibility and specific problem-solving techniques. There were 732 entries in 2017.

Dirichlet (years 6–7) includes mathematics concerned with tessellations, arithmetic in other bases, time/distance/speed, patterns, recurring decimals and specific problem-solving techniques. There were 1292 entries in 2017.

Euler (years 7–8) includes primes and composites, least common multiples, highest common factors, arithmetic sequences, figurate numbers, congruence, properties of angles and pigeonhole principle. There were 1853 entries in 2017.

Gauss (years 8–9) includes parallels, similarity, Pythagoras' Theorem, using spreadsheets, Diophantine equations, counting techniques and congruence. Gauss builds on the Euler program. There were 1218 entries in 2017.

Noether (top 10% years 9–10) includes expansion and factorisation, inequalities, sequences and series, number bases, methods of proof, congruence, circles and tangents. There were 692 entries in 2017.

Polya (top 10% year 10) Topics include angle chasing, combinatorics, number theory, graph theory and symmetric polynomials. There were 271 entries in 2017.

Australian Intermediate Mathematics Olympiad

This four-hour competition for students up to Year 10 offers a range of challenging and interesting questions. It is suitable for students who have performed well in the AMC (Distinction and above), and is designed as an endpoint for students who have completed the Gauss or Noether stage. There were 2251 entries for 2017 and 11 perfect scores.

MEMBERSHIP OF MCYA COMMITTEES

Mathematics Challenge for Young Australians Committee 2017

Director

Dr K McAvaney, Victoria

Challenge Committee

Adj Prof M Clapper, Australian Mathematics Trust, ACT
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Mr A Edwards, Queensland Studies Authority
Mr B Henry, Victoria
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Mrs L Mottershead, New South Wales
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Mr P Swain, Victoria
Dr P Swedosh, The King David School, VIC
Mrs A Thomas, New South Wales
Ms K Trudgian, Queensland

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Mr J Dowsey, University of Melbourne, VIC
Dr M Evans, International Centre of Excellence for Education in Mathematics, VIC
Mr B Henry, Victoria
Dr D Mathews, Monash University, VIC

Enrichment

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Dr M Evans, International Centre of Excellence for Education in Mathematics, VIC
Mr K Hamann, South Australia
Mr B Henry, Victoria
Dr K McAvaney, Victoria
Dr A M Storozhev, Attorney General’s Department, ACT
Emeritus Prof P Taylor, Australian Capital Territory
Dr O Yevdokimov, University of Southern Queensland

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Deputy Chair

Prof Andrew Hassall, Australian National University, ACT

Chief Executive Officer

Mr Nathan Ford, Australian Mathematics Trust, ACT

Chief Mathematician

Mr Mike Clapper, Australian Mathematics Trust, ACT

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Chair, Senior Problems Committee

Dr N Do, Monash University, VIC

Chair, Challenge

Dr K McAvaney, VIC

Director of Training and IMO Team Leader

Dr A Di Pasquale, University of Melbourne, VIC

IMO Deputy Team Leader

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Dr G Gamble, University of Western Australia
Dr Ian Roberts, Northern Territory
Assoc Prof D Daners, University of Sydney, NSW
Mr D Martin, South Australia
Dr A Offer, Queensland
Dr P Swedosh, The King David School, VIC
Dr Chris Wetherell, Radford College, ACT

Representatives

Ms J McIntosh, Challenge Committee
Ms A Nakos, Challenge Committee
Prof M Newman, Challenge Committee

AMOC TIMETABLE FOR SELECTION OF THE TEAM TO THE 2018 IMO

August 2017—July 2018

Hundreds of students are involved in the AMOC programs which begin on a state basis. The students are given problem-solving experience and notes on various IMO topics not normally taught in schools.

The students proceed through various programs with the top 25 students, including potential team members and other identified students, participating in a 10-day residential school in November/December.

The selection program culminates with the March Selection School during which the team is selected.

Team members then receive individual coaching by mentors prior to assembling for last minute training before the IMO.

Month	Activity
August	Outstanding students are identified from AMC results, MCYA, other competitions and recommendations; and eligible students from previous training programs AMOC state organisers invite students to participate in AMOC programs Various state-based programs AMOC Senior Contest
September	Australian Intermediate Mathematics Olympiad
November/ December	AMOC School of Excellence
January	Summer Correspondence Program for those who attended the School of Excellence
February	Australian Mathematical Olympiad
March	Asian Pacific Mathematics Olympiad
March	AMOC Selection School
May–June	Personal Tutor Scheme for IMO team members
July	Short mathematics school for IMO team members 2018 IMO in Cluj-Napoca, Romania

ACTIVITIES OF AMOC SENIOR PROBLEMS COMMITTEE

This committee has been in existence for many years and carries out a number of roles. A central role is the collection and moderation of problems for senior and exceptionally gifted intermediate and junior secondary school students. Each year the Problems Committee provides examination papers for the AMOC Senior Contest and the Australian Mathematical Olympiad. In addition, problems are submitted for consideration to the Problem Selection Committees of the annual Asian Pacific Mathematics Olympiad and the International Mathematical Olympiad.

AMOC Senior Problems Committee October 2016–September 2017

Adj Prof M Clapper, Australian Mathematics Trust, ACT
Dr A Devillers, University of Western Australia, WA
Dr A Di Pasquale, University of Melbourne, VIC
Dr N Do, Monash University, VIC, (Chair)
Dr I Guo, University of Sydney, NSW
Assoc Prof D Hunt, University of NSW
Dr J Kupka, Monash University, VIC
Dr K McAvaney, Deakin University, VIC
Dr D Mathews, Monash University, VIC
Dr A Offer, Queensland
Dr C Rao, NEC Australia, VIC
Dr B B Saad, Monash University, VIC
Assoc Prof J Simpson, Curtin University of Technology, WA
Dr I Wanless, Monash University, VIC

1. 2017 Australian Mathematical Olympiad

The Australian Mathematical Olympiad (AMO) consists of two papers of four questions each and was sat on 14 and 15 February. There were 105 participants including 12 from New Zealand, eight more participants than 2017. Three students, Matthew Cheah, William Hu and Guowen Zhang, achieved perfect scores and eight other students were awarded Gold certificates,17 students were awarded Silver certificates and 26 students were awarded Bronze certificates.

2. 2017 Asian Pacific Mathematics Olympiad

On Monday 13 and Tuesday 14 March students from nations around the Asia-Pacific region were invited to write the Asian Pacific Mathematics Olympiad (APMO). Of the top ten Australian students who participated, there were 1 Gold, 2 Silver and 4 Bronze certificates awarded. Australia finished in 14th place overall.

3. 2017 International Mathematical Olympiad, Rio De Janeiro, Brazil.

The IMO consists of two papers of three questions worth seven points each. They were attempted by teams of six students from 111 countries on 18 and 19 July in Rio de Janeiro, Brazil, with Australia being placed 34th. The results for Australia were three Silver, two Bronze medals and one Honourable Mention.

4. 2017 AMOC Senior Contest

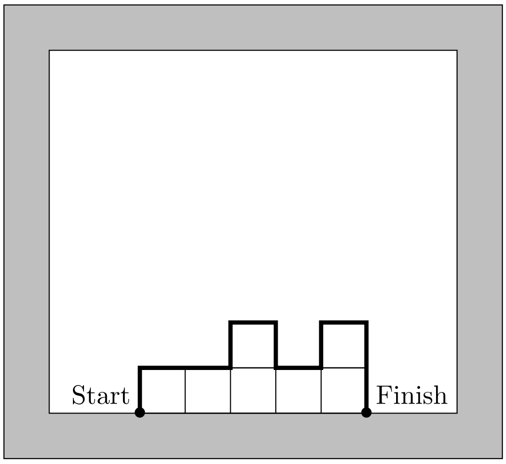
Held on Tuesday 8 August, the Senior Contest was sat by 99 students (compared to 79 in 2016). There were ten students who obtained Gold certificates with perfect scores and three other students who also obtained a Gold certificates. Fourteen students obtained Silver certificates and 24 students obtained Bronze certificates.

CHALLENGE PROBLEMS – MIDDLE PRIMARY

Students may work on each of these four problems in groups of up to three, but must write their solutions individually.

MP1 Annabel's Ants

Annabel made a shape by placing identical square tiles in a frame as shown in the diagram below. The tiles are arranged in columns. Each column touches the base but no column touches the sides or top. There are no empty gaps between columns. The frame can be enlarged as needed.



Annabel notices an ant walking along the edge of the shape made by the tiles. Beginning at the start, the ant follows the thick line. It walks a total of 11 tile edges to reach the finish.

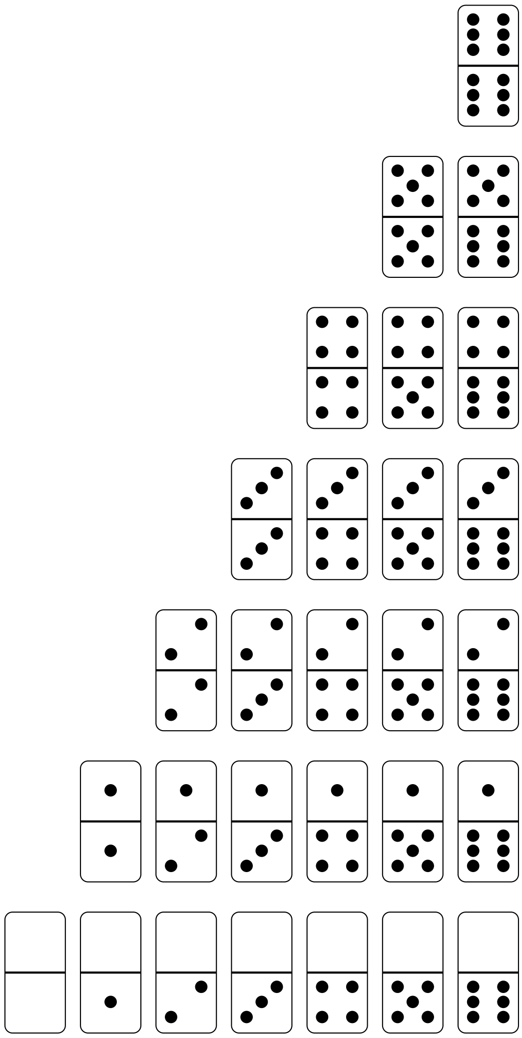
a Draw a diagram to show how to arrange the 7 tiles in the frame so that the ant walks a total of 15 tile edges to get from the start to the finish.

b Draw a diagram to show how to arrange the 7 tiles so that the ant will walk exactly 8 tile edges.

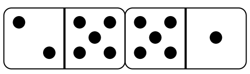
c Show that it is possible to arrange the 7 tiles so that the ant walks exactly 9, 10, 11, 12, 13, 14 tile edges.

MP2 Domino Chains

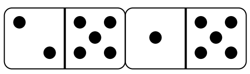
Dominoes are rectangular tiles, twice as long as they are wide, with two sets of dots separated by a line as shown below. The number of dots in each set varies from 0 to 6. There are 28 different dominoes.



In a game of dominoes, they are placed end-to-end to form a chain, always matching the number of dots where the dominoes join. For example, the [2,5] and [1,5] dominoes can be placed as in the first diagram, but not the second.



allowed



not allowed

a Make a chain of nine dominoes that includes all the dominoes that have at least one set of six dots.

Each domino has a *domino product*, that is, the product of the number of dots at one end with the number of dots at the other. For example, the domino product of [2,5] is $2 \times 5 = 10$.

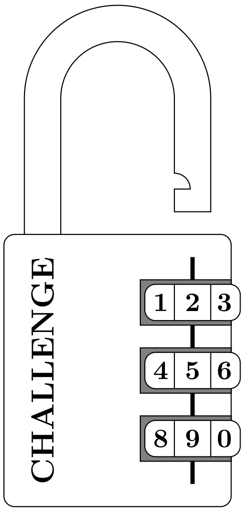
b How many dominoes have an odd domino product? Justify your answer.

c List the ten dominoes which have the largest domino products.

d Make a chain of nine dominoes so that the sum of their domino products is 196.

MP3 Lock Out

A combination padlock has three wheels, each with digits 0 to 9 engraved on it in order. The wheels can be turned *independently* in either direction without stopping. A wheel clicks each time it is turned so that whichever digit is on the marker line changes to either digit next to it on the wheel. The lock can be closed or opened when the correct three digits line up on the marker line. This lock shows the number 259 at the marker line.



a What is the least number of clicks needed to change 259 to 961?

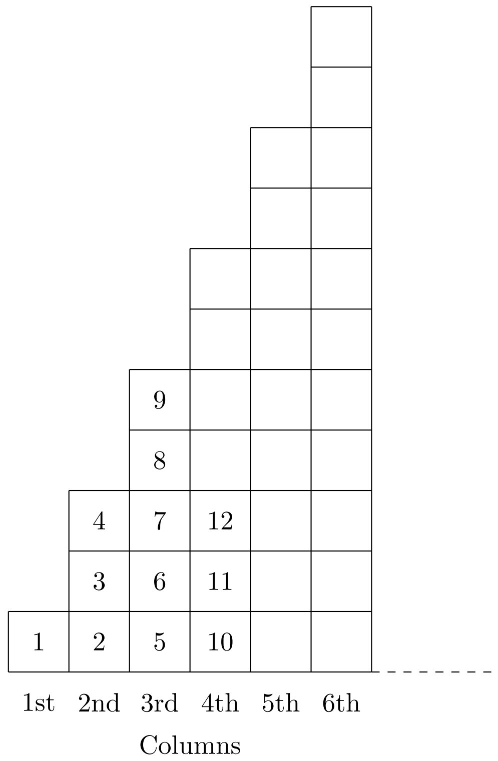
b From 259, the top wheel is clicked 5 times in one direction, the middle wheel 5 times in one direction, and the bottom wheel 4 times in one direction. List all numbers that could then appear.

c From 259, how many different numbers can result with only one click? List them all.

d From 259, how many different numbers can result if each wheel is clicked once? List them all.

MP4 Steps to Infinity

The numbers 1, 2, 3, ... are arranged in steps as shown.



a Which column contains the number 41?

b The bottom number in a column is 65. What is the top number in that column?

c Which column contains the number 1000?

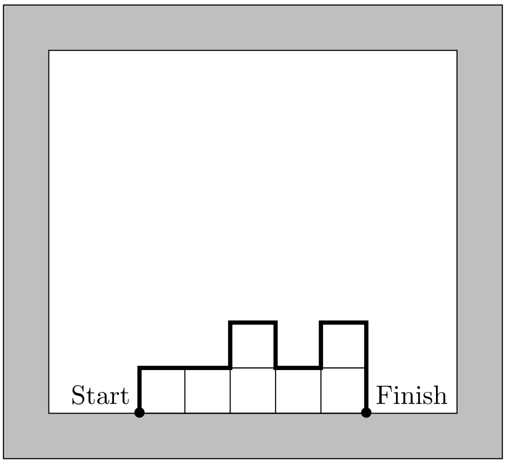
d How many numbers are in the column that contains the number 1000?

CHALLENGE PROBLEMS – UPPER PRIMARY

Students may work on each of these four problems in groups of up to three, but must write their solutions individually.

UP1 Annabel's Ants

Annabel made a shape by placing identical square tiles in a frame as shown in the diagram below. The tiles are arranged in columns. Each column touches the base but no column touches the sides or top. There are no empty gaps between columns. The frame can be enlarged as needed.



Annabel notices an ant walking along the edge of the shape made by the tiles. Beginning at the start, the ant follows the thick line. It walks a total of 11 tile edges to reach the finish.

a Draw a diagram to show how to arrange the 7 tiles so that the ant will walk 8 tile edges.

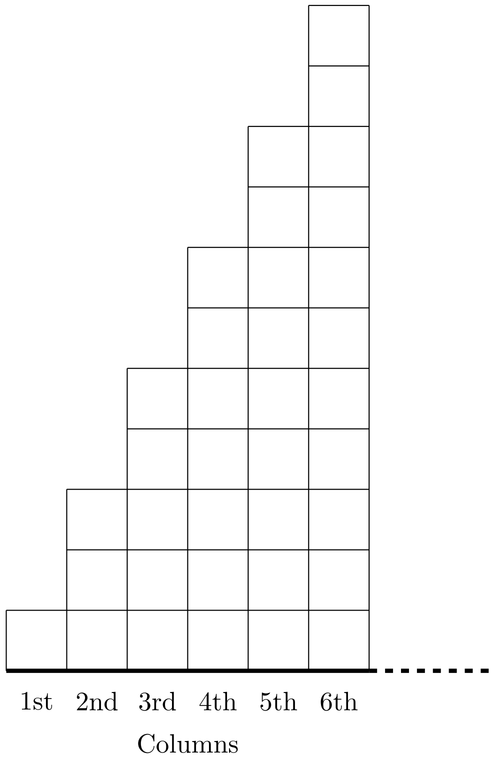
b Draw three different arrangements of the 7 tiles, no two with the same maximum column height, so that the ant will walk 9 tile edges.

c Show that it is possible to arrange the 7 tiles so that the ant walks exactly 10, 11, 12, 13, 14, 15 tile edges.

d Show that it is possible to arrange 49 tiles so that the ant walks fewer than 21 tile edges.

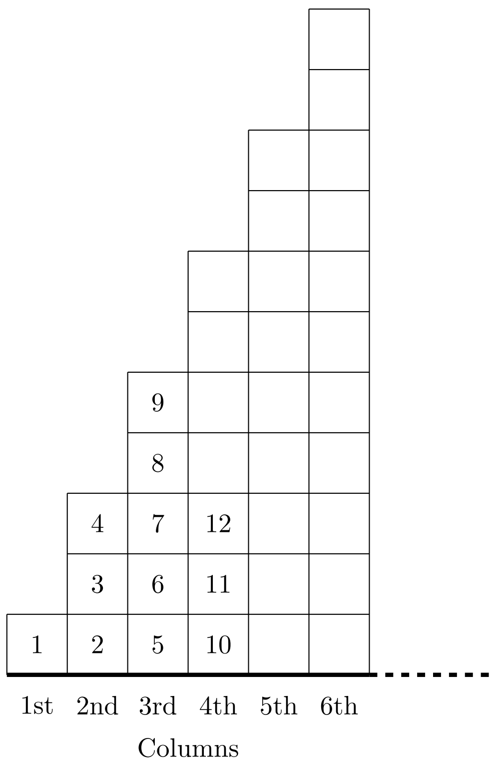
UP2 Steps to Infinity

Each student in the class is given a diagram of a staircase like this:



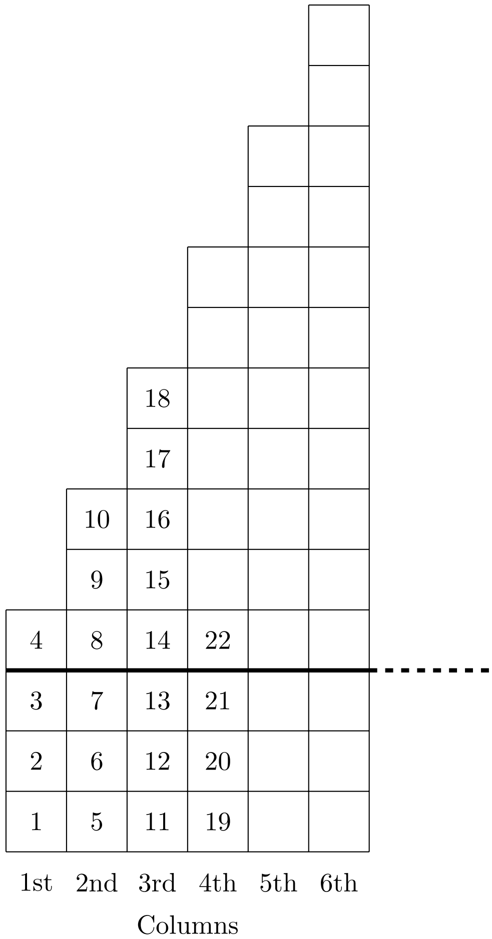
Some students add extra rows at the bottom of this blank staircase so that each added row has its leftmost square in column 1. They then write the numbers 1, 2, 3, etc. in the squares, starting with 1 in the bottom-left square of their staircase and moving up each successive column without missing any squares.

a Ahmed doesn't add any extra rows and places the numbers in the squares of his blank staircase as shown.



Continuing Ahmed's pattern, one of the columns would have the number 145 in its bottom square. What would be the top number in that column?

b Basil takes his blank staircase, adds three extra rows at the bottom and numbers the squares as shown.



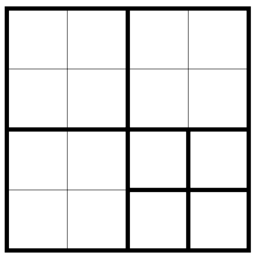
Continuing Basil's pattern, what would be the top number in the 10th column of Basil's staircase?

c Chelsea takes her blank staircase, adds a certain number of extra rows to the bottom and then numbers the squares. The top number in the 15th column is 405. How many rows did Chelsea add to her blank staircase?

d Davina adds a certain number of extra rows to the bottom of her blank staircase and then numbers the squares. The top number in one of the columns is 51. How many rows could Davina have added to her blank staircase? Give all possible answers.

UP3 Square Parts

Sal cuts a square of integer (whole number) side length into smaller squares of integer side length. For example, she might cut a 4×4 square into four 1×1 squares and three 2×2 squares, giving a total of seven square pieces.



a Draw a diagram to show how Sal can cut a 5×5 square into 10 square pieces.

b Draw a diagram to show how Sal can cut a 5×5 square into 11 square pieces.

c Sal has two 4×4 squares, three 3×3 squares, four 2×2 squares, and five 1×1 squares. Draw a diagram to show how she could place some or all of these squares together without gaps or overlaps to make a square that is as large as possible. Explain why she cannot construct a larger square.

UP4 Bracelets

Sam makes bracelets by threading beads on a chain. The beads are identical except for colour. After threading the beads, he closes the chain by twisting together the two links at the ends to form a continuous chain. Two bracelets are considered the same if one is a rotation or reflection (flip) of the other.

a Sam chose to make several bracelets each having 4 beads: 1 red and each of the others either black or white. How many different bracelets could he make?

b Next Sam makes several bracelets each having 4 beads and each bead is either black or white. How many different bracelets could he make?

c Next Sam makes several bracelets each having 5 beads: 1 red and each of the others either black or white. How many different bracelets could he make?

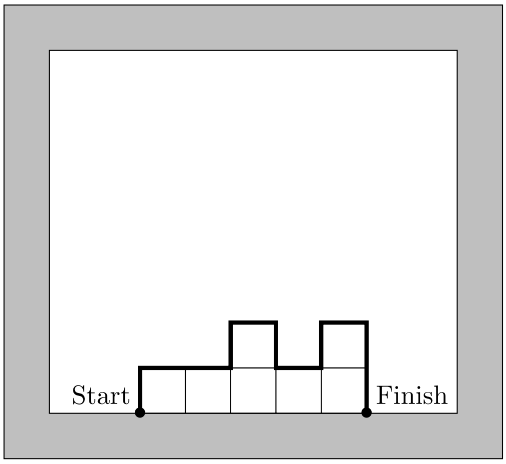
d Next Sam makes several bracelets each having 5 beads and each bead is either black or white. How many different bracelets could he make?

CHALLENGE PROBLEMS – JUNIOR

Students may work on each of these six problems with a partner but each must write their solutions individually.

J1 Annabel's Ants

Annabel made a shape by placing identical square tiles in a frame as shown in the diagram below. The tiles are arranged in columns. Each column touches the base but no column touches the sides or top. There are no empty gaps between columns. The frame can be enlarged as needed.



Annabel notices an ant walking along the edge of the shape made by the tiles. Beginning at the start, the ant follows the thick line. It walks a total of 11 tile edges to reach the finish.

a Show that it is possible to arrange 7 tiles so that the ant walks exactly 8, 9, 10, 11, 12, 13, 14, 15 tile edges.

b Show six ways of arranging 7 tiles so that the ant walks a total of 9 tile edges.

c Show that it is possible to arrange 49 tiles so that the ant walks a total of less than 21 tile edges.

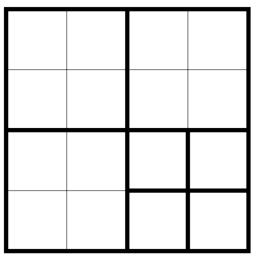
d Show four arrangements of 137 tiles, each arrangement with a different maximum height, so that the ant walks a total of 34 tile edges.

J2 Steps to Infinity

See Upper Primary Problem 2.

J3 Square Parts

Sal cuts a square of integer (whole number) side length into smaller squares of integer side length. For example, she might cut a 4×4 square into four 1×1 squares and three 2×2 squares, giving a total of 7 square pieces.



a Draw a diagram to show how Sal can cut a 5×5 square into 11 square pieces.

b Show that Sal cannot cut a 4×4 square into 11 square pieces.

c Sal has two 4×4 squares, three 3×3 squares, four 2×2 squares, and four 1×1 squares. Draw a diagram to show how she could place some or all of these squares together without gaps or overlaps to make a square that is as large as possible. Explain why she cannot construct a larger square.

J4 Tribonacci Sequences

A *tribonacci sequence* is a sequence of numbers such that each term from the fourth onwards is the sum of the previous three terms. The first three terms in a tribonacci sequence are called its *seeds*. For example, if the three seeds are 6, 19, 22, then the next few terms are 47 $(6 + 19 + 22)$, 88 $(19 + 22 + 47)$, 157 $(22 + 47 + 88)$, and 292 $(47 + 88 + 157)$.

a Find the smallest 5-digit term in the sequence above.

b The 5th, 6th, 7th terms of a tribonacci sequence are respectively 36, 71, 135. What are the three seeds for this sequence?

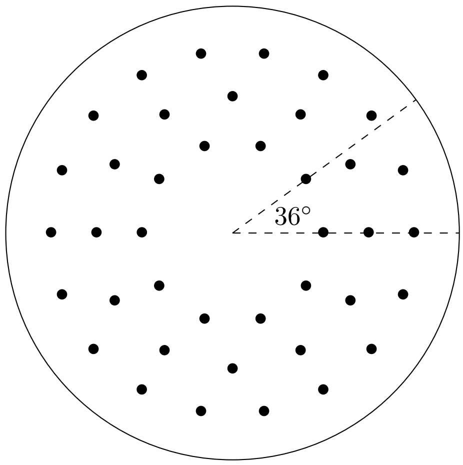
c The seeds of a tribonacci sequence are 20, 17, 2017. Is the 2017th term even or odd? Explain.

d If a tribonacci sequence has 20 as its second seed and 17 as its third seed, find all positive integers that can be its first seed so that 2017 appears as a term somewhere in the sequence.

J5 Shower Heads

The jets (or holes) on a shower head are arranged in circles that are concentric with the rim. The jets are equally spaced on each circle and there is at least one radius of the shower head that intersects every circle at a jet.

The *angular separation* of two jets on a circle is the size of the angle formed by the two radii of the circle that pass through the jets. All angular separations are integers. For example, on the shower head shown, there are 10 jets on the inner circle. Hence the angular separation of adjacent jets on the inner circle is $360°/10 = 36°$.



a A shower head has three circles of jets: an inner circle with 12 jets, a middle circle with 18 jets, and an outer circle with 36 jets. What is the angular separation of adjacent jets in each circle?

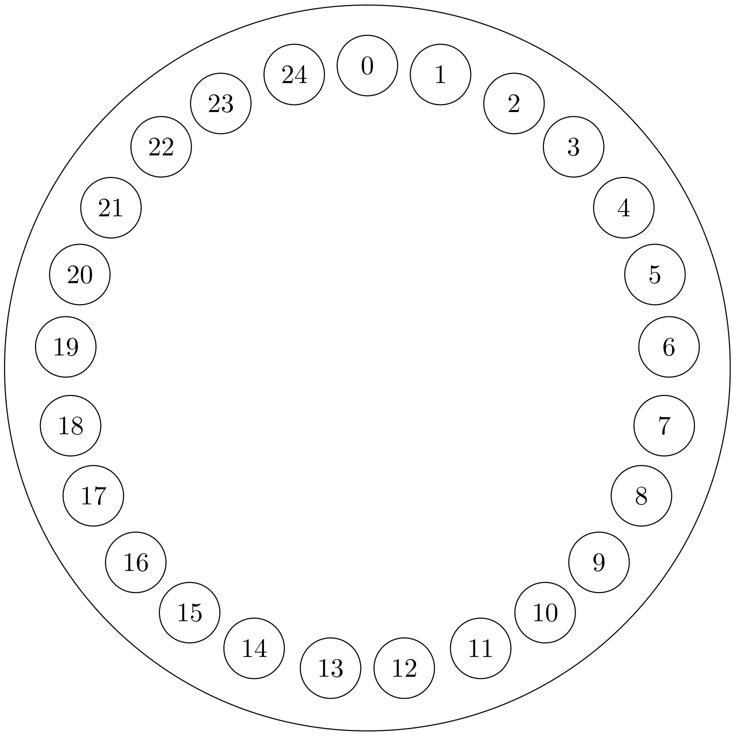
b For the shower head in Part **a**, how many radii of the shower head pass through three jets?

c For the shower head in Part **a**, how many radii of the shower head pass through just two jets?

d Another shower head has four circles with 10, 20, 30, and 45 jets respectively. Explain why no diameter of the shower head passes through eight jets.

J6 Circle Hopscotch

A hopping circuit is painted on a school playground pavement. It consists of 25 small circles arranged in a large circle and numbered 0 to 24.



Each student starts at 0 and hops clockwise either 3 places (a 3-hop) or 4 places (a 4-hop) on each turn. For example, a student's first hop from 0 will end on either position 3 or 4. Students must go twice around the circuit and end back at 0 to complete a game. All students list in order the numbers they land on and record the total number of hops they take.

a In one game a student took 13 hops. Write down a possible list of numbers he landed on.

b Find all possible combinations of the number of 3-hops and the number of 4-hops in a game.

c What is the smallest number of different numbers a student can land on in one game? Explain your answer.

d Jo and Mike decide to play a longer version of the game according to the following rules. They take each of their hops at the same time starting with both on 0. Whenever Jo takes a 4-hop, Mike takes a 3-hop; whenever Jo takes a 3-hop, Mike takes a 4-hop.

Jo's first five full hops on each lap are 4-hops. After that, she takes 3-hops until she next reaches or passes 0. How many laps will each of them have completed when they next meet at 0?

CHALLENGE PROBLEMS – INTERMEDIATE

Students may work on each of these six problems with a partner but each must write their solutions individually.

I1 Tribonacci Sequences

A *tribonacci sequence* is a sequence of numbers such that each term from the fourth onwards is the sum of the previous three terms. The first three terms in a tribonacci sequence are called its *seeds*. For example, if the three seeds are 6, 19, 22, then the next few terms are 47 $(6+19+22)$, 88 $(19+22+47)$, 157 $(22+47+88)$, and 292 $(47+88+157)$.

a Find the smallest 5-digit term of the sequence above.

b The seeds of a tribonacci sequence are 20, 17, 2017. Is the 2017th term even or odd? Explain.

c A tribonacci sequence has second seed 20, third seed 17 and its eighth term is zero. Find its first seed.

d If a tribonacci sequence has 20 as its second seed and 17 as its third seed, find all positive integers that can be its first seed so that 2017 appears as a term somewhere in the sequence.

I2 Rowing Machine

A person exercising on a rowing machine is kept informed about his/her progress by various numbers displayed by the machine. Three numbers of particular importance are the total distance rowed in metres, the total time rowed in minutes and seconds, and the current pace, which is the time it would take in minutes and seconds to row 500 metres.

For example, the diagram shows that this rower has covered a distance of 1000 metres, has been rowing for 5 minutes and 50 seconds, and would take 2 minutes and 40 seconds to row 500 metres at the current pace.

Distance	1000
Time	5:50
Pace	2:40

a Thomas and Jack start rowing at the same time. Each aims to get to 1000 m first. Thomas' strategy is to row the whole distance at a 3 minute pace. Jack decides to row at a slower 3 minute 20 second pace for the first 3 minutes and then row faster at a 2 minute 40 second pace for the rest of the distance. Who gets to 1000 m first?

b Thomas and Jack repeat the task of rowing to 1000 m first. Thomas maintains his plan of rowing 1000 m at a 3 minute pace. Jack changes his strategy to row the first 600 m at a 3 minute 20 second pace, then the next 400 m at a 2 minute 40 second pace. Who gets to 1000 m first?

c Jack plans to row 3000 m in 15 minutes. He sets out at a 2 minute 38 second pace, but after rowing 600 m realises that he will not be able to cover 3000 m in 15 minutes at his current pace. At what pace does he need to row the remainder of his training session to reach his goal?

d Thomas rows 1000 m at a 3 minute pace. Jack rows at a 3 minute 20 second pace at the beginning, and then he rows at a 2 minute 40 second pace. For what distance should Jack row at a 3 minute 20 second pace to finish the 1000 m at the same time as Thomas?

I3 Blocking Circles

Three students each draw a circle of radius 12 cm with centre O. Then they draw four circles inside this circle which may touch but not overlap.

a Student A draws the first interior circle with a radius of 4 cm so that it touches the large circle. She then draws the remaining three interior circles in turn. Each has a radius of 4 cm and touches the large circle and the most recently drawn small circle. When viewed from O, what fraction of the circumference of the large circle is blocked by the interior circles?

b Student B also draws the four interior circles with radius 4 cm. Their centres are all the same distance from O, and the circumference of the large circle is entirely blocked by the interior circles when viewed from O. How far are the centres of the interior circles from O?

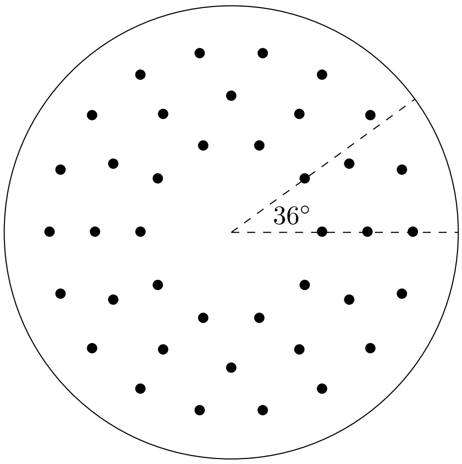
c Student C draws two circles of radius 4 cm with their centres on a diameter of the large circle and each at distance 5 cm from O. She then draws two more circles, this time each of radius 3 cm, so that the circumference of the large circle is entirely blocked by all four interior circles when viewed from O.

What are the minimum and maximum distances from O to the centre of a circle of radius 3 cm?

I4 Shower Heads

The jets (or holes) on a shower head are arranged in circles that are concentric with the rim. The jets are equally spaced on each circle and there is at least one radius of the shower head that intersects every circle at a jet.

The *angular separation* of two jets on a circle is the size of the angle formed by the two radii of the circle that pass through the jets. All angular separations are integers. For example, on the shower head shown, there are 10 jets on the inner circle. Hence the angular separation of adjacent jets on the inner circle is $360°/10 = 36°$.



a A shower head has three circles of jets: an inner circle with 12 jets, a middle circle with 18 jets, and an outer circle with 36 jets. How many radii of the shower head pass through just two jets?

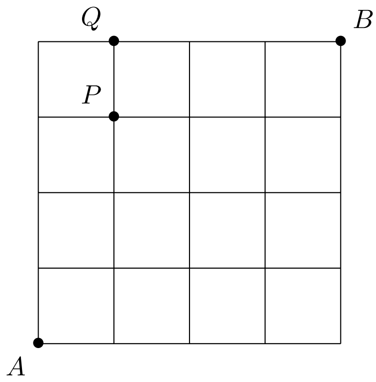
b Another shower head has three circles with 20, 30, and 45 jets respectively. Explain why no diameter of the outer circle passes through six jets.

c A shower head with three circles of jets has a different number of jets in each circle. Exactly two diameters of the shower head pass through six jets. If the total number of jets is 100, how many jets are there in each circle? Find all combinations.

d Find the maximum number of jets in a shower head with three circles if only one radius of the shower head passes through three jets, no radius passes through just two jets, and each circle has at least two jets.

I5 Chance Encounters

The bottom-left corner of a 4×4 grid is labelled A and the diagonally opposite corner B. Two other grid points are labelled P and Q as shown.



A counter is placed at A. Each second the counter moves along a grid line to an adjacent grid point, always increasing its distance from A. It continues to do this until it reaches point B after 8 seconds. Where there are two possible moves, they are equally likely.

a Explain why there is a 1 in 4 chance that the counter starting at A will be at P after 4 seconds.

b What is the probability that the counter starting at A will be at Q after 5 seconds?

A second counter is placed at B. Each second the counter moves along a grid line to an adjacent grid point, always increasing its distance from B. It continues to do this until it reaches point A after 8 seconds. Where there are two possible moves, they are equally likely.

The two counters start moving at the same time.

c Indicate on a diagram all the grid points at which the counters can meet. Explain why there are no other grid points at which they could meet.

d What is the probability that the counters meet?

I6 Unequal Partitions

A partition of an integer into distinct positive parts (no two are equal) is called an *unequal partition*. For example, there are seven unequal partitions of 12 into 3 parts: $12 = 9+2+1 = 8+3+1 = 7+4+1 = 7+3+2 = 6+5+1 = 6+4+2 = 5+4+3$.

The *span* of a partition is the difference between the smallest and largest number in the partition. For example, the spans of the unequal partitions of 12 into 3 parts listed above are respectively 8, 7, 6, 5, 5, 4, 2. Thus 2 is the smallest span of an unequal partition of 12 into 3 parts.

a Find at least two unequal partitions of 2017 into 5 parts with a span of 7.

b Find an unequal partition of 2017 into 5 parts with the smallest possible span.

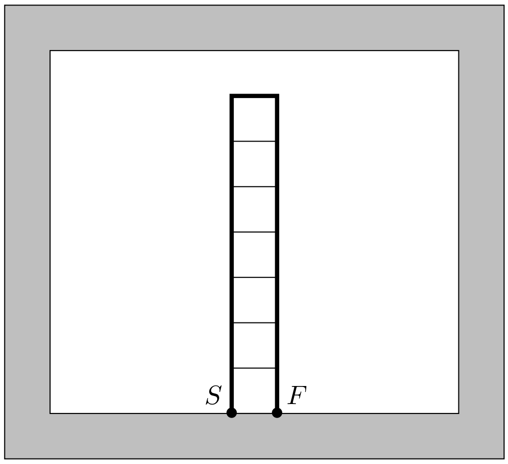
c Find the smallest number that has an unequal partition into 5 parts. Describe the set of all numbers that have an unequal partition into 5 parts with a span of 4.

d Show that for each number that has an unequal partition into 5 parts, the smallest span for such partitions is 4 or 5.

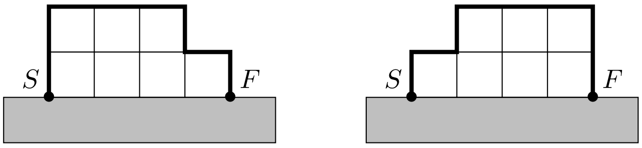
CHALLENGE SOLUTIONS – MIDDLE PRIMARY

MP1 Annabel's Ants

a

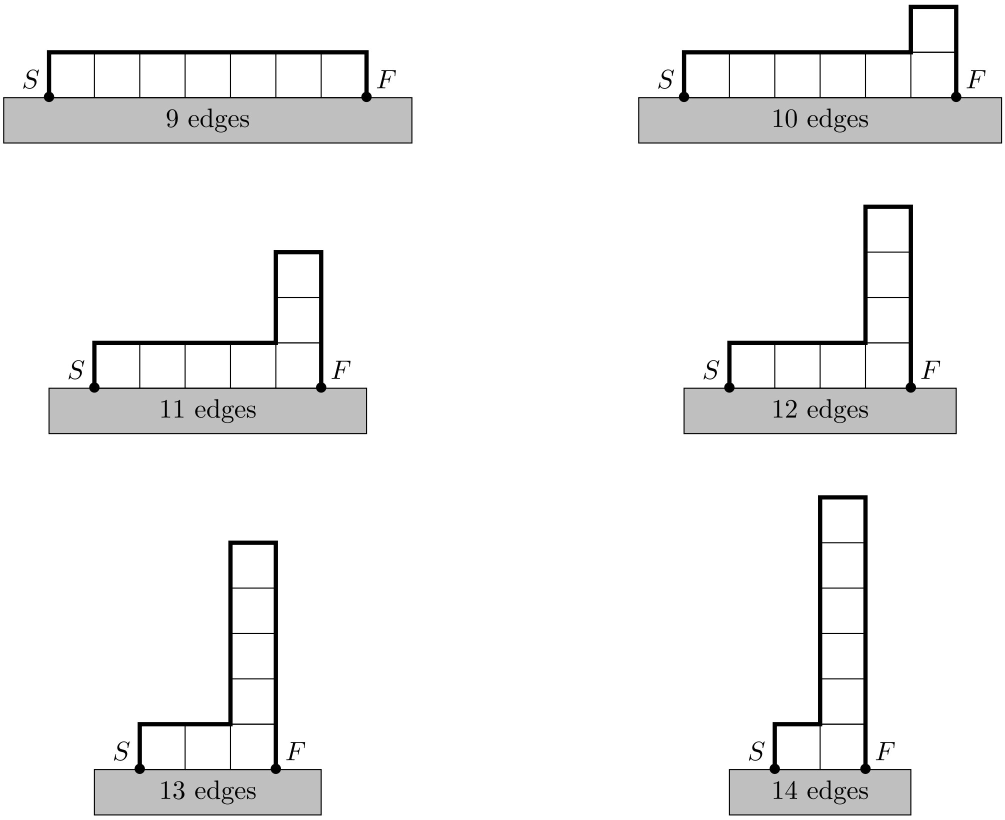


b Either one of the two possible arrangements.



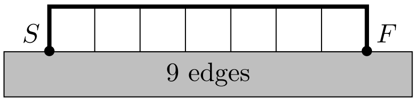
c Alternative i

Here is an arrangement for each total number of tile edges from 9 to 14. In each case there is at least one other valid arrangement.

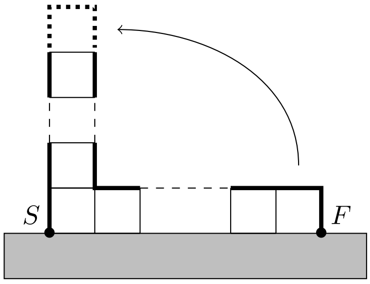


Alternative ii

If 7 tiles are placed horizontally, then the ant walk has 9 tile edges.



One at a time, place the right tile above the left tile so as to make a column on top of the left tile. Each time this is done, the horizontal edge on top of the left column is removed from the ant walk but two new (vertical) edges are added to the ant walk, one on the tile that is moved and one on the tile that was next to it.



So the number of edges in the ant walk increases by exactly 1. At least 5 tiles can be moved in this way. So 7 tiles can be arranged to produce ant walks with 9, 10, 11, 12, 13, 14 tile edges.

MP2 Domino Chains

a There are seven dominoes with six dots at either end. The following chain of nine dominoes contains all seven: [0,6] [6,6] [6,5] [5,4] [4,6] [6,3] [3,2] [2,6] [6,1].

There are other valid chains.

b The product of an even number with any other number is even. The product of two odd numbers is odd. Hence the only dominoes with odd domino products are [1,1] [1,3] [1,5] [3,3] [3,5] [5,5]. Thus there are exactly six dominoes with an odd domino product.

c The following table shows the domino product for each domino.

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1		1	2	3	4	5	6
2			4	6	8	10	12
3				9	12	15	18
4					16	20	24
5						25	30
6							36

The 10 largest products, including repetitions, are 36, 30, 25, 24, 20, 18, 16, 15, 12, 12.

Hence the required dominoes are [6,6] [5,6] [5,5] [4,6] [4,5] [3,6] [4,4] [3,5] [3,4] [2,6].

d Adding various domino products from the table in Part **c**, we find the nine largest domino products 36, 30, 25, 24, 20, 18, 16, 15, 12 sum to 196. These domino products come from dominoes [6,6] [5,6] [5,5] [4,6] [4,5] [3,6] [4,4] [3,5], along with either [3,4] or [2,6].

Choosing [2,6], we get the following chain: [2,6] [6,6] [6,5] [5,5] [5,4] [4,4] [4,6] [6,3] [3,5].

There are other acceptable chains.

MP3 Lock Out

a The diagram shows the digits either side of 2, 5, 9.

7	8	9	0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7	8	9	0
4	5	6	7	8	9	0	1	2	3	4

The least number of clicks needed to change 2 to 9 is 3.
The least number of clicks needed to change 5 to 6 is 1.
The least number of clicks needed to change 9 to 1 is 2.

So the least number of clicks needed to change 259 to 961 is $3 + 1 + 2 = 6$.

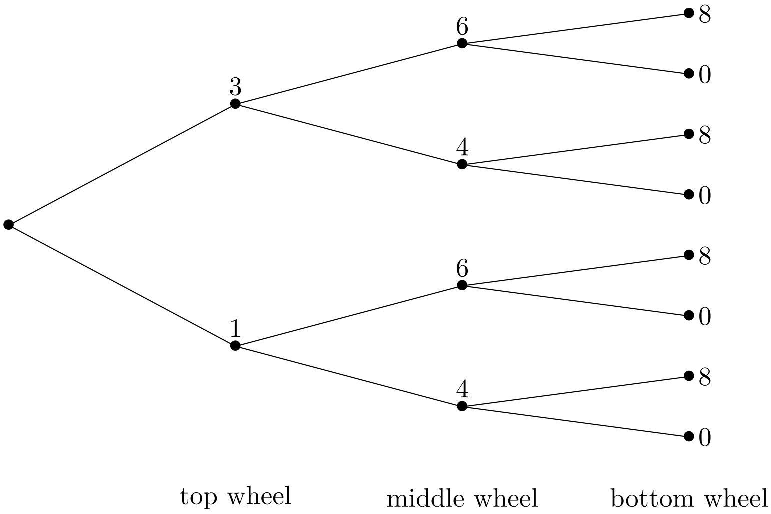
b Five clicks of the top wheel in either direction gives digit 7.
Five clicks of the middle wheel in either direction gives digit 0.
Four clicks on the bottom wheel gives digits 3 or 5.

So the only numbers that could result are 703 and 705.

c One click would change 2 to 1 or 3, resulting in 159 or 359.
One click would change 5 to 4 or 6, resulting in 249 or 269.
One click would change 9 to 0 or 8, resulting in 250 or 258.

So six numbers could result: 159, 249, 250, 258, 269, 359.

d The tree diagram shows the digits at the marker line after one click for each wheel.



So eight numbers could result: 140, 148, 160, 168, 340, 348, 360, 368.

MP4 Steps to Infinity

a In column order, the number of squares in the columns are 1, 3, 5, 7, $\ldots$. So the numbers in the top squares in column order are 1, $1+3=4$, $1+3+5=9$, $1+3+5+7=16$, $1+3+5+7+9=25$, $1+3+5+7+9+11=36$, $1+3+5+7+9+11+13=49$, $\ldots$. Hence 41 is in column 7.

b As in Part **a**, the numbers in the top squares in column order are 1, $1+3=4$, $4+5=9$, $9+7=16$, $16+9=25$, $25+11=36$, $36+13=49$, $49+15=64$, $64+17=81$, $\ldots$. So the top number in the column that has 65 at the bottom is 81.

c As in Part **a**, the number at the top of column c is the sum of the first c odd numbers.

From a spreadsheet or calculator, the sum of the first 31 odd numbers is 961 and the sum of the first 32 odd numbers is 1024.

(Alternatively, the 32nd odd number is 1 less than the 32nd even number, that is, $2\times 32-1=63$. Now add the first 32 odd numbers in an order that is easy to calculate, such as low numbers with high numbers: $(1+63)+(3+61)+(5+59)+\cdots+(29+35)+(31+33)=16\times 64=1024$. Hence the sum of the first 31 odd numbers is $1024-63=961$.)

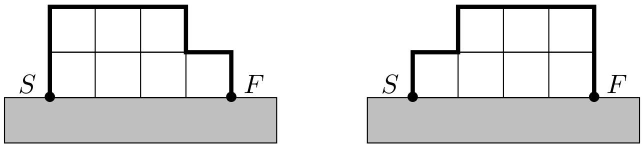
So 1000 occurs in the 32nd column.

d From Part **c**, 1000 is in the 32nd column. As in Part **a**, the number of numbers in the 32nd column is the 32nd odd number, which is 63.

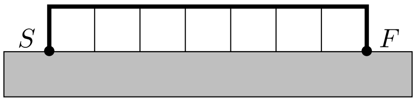
CHALLENGE SOLUTIONS – UPPER PRIMARY

UP1 Annabel's Ants

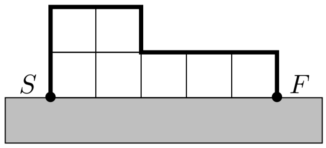
a There are two possible arrangements.



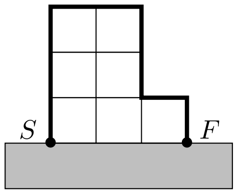
b There is only one arrangement with maximum column height 1.



There are four arrangements with maximum column height 2. Here is one arrangement.

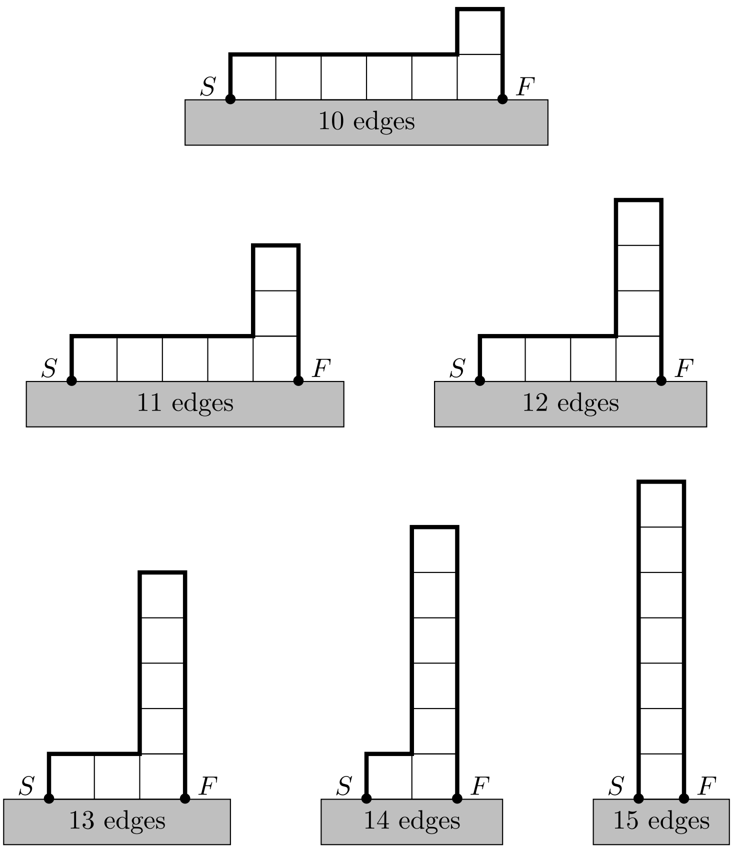


There are five arrangements with maximum column height 3. Here is one arrangement.



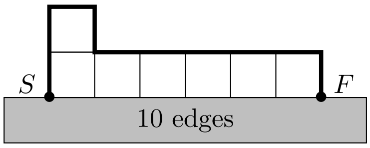
c Alternative i

Here is one arrangement for each total number of tile edges from 10 to 15. There are several other arrangements.

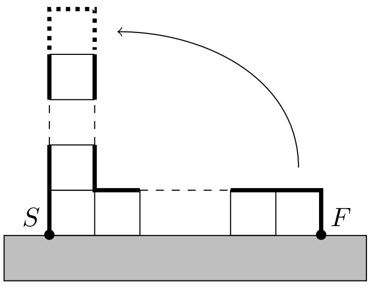


Alternative ii

In this arrangement of 7 tiles the ant walk has 10 tile edges.

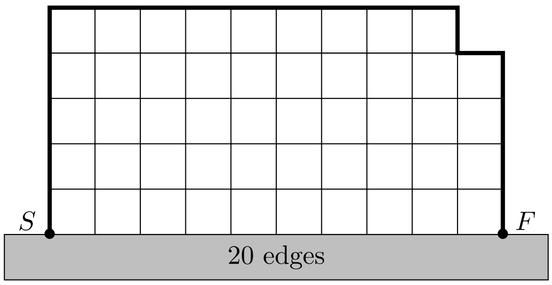


One at a time, place the right tile above the left tile so as to make a column on top of the left tile. Each time this is done, the horizontal edge on top of the left column is removed from the ant walk but two new (vertical) edges are added to the ant walk, one on the tile that is moved and one on the tile that was next to it.



So the number of edges in the ant walk increases by exactly 1. Five tiles can be moved in this way. So 7 tiles can be arranged to produce ant walks with 10, 11, 12, 13, 14, 15 tile edges.

d There are only two possible arrangements: this one and its reflection about a vertical line.



UP2 Steps to Infinity

a From one column to the next, the number of squares increases by 2. So, in column order, the number of squares in the columns of Ahmed's staircase are the odd numbers 1, 3, 5, 7, ... From this we can calculate the numbers at the top of various columns, as shown in this table.

Column	Top number	Column	Top number
1	1	8	$49 + 15 = 64$
2	$1 + 3 = 4$	9	$64 + 17 = 81$
3	$4 + 5 = 9$	10	$81 + 19 = 100$
4	$9 + 7 = 16$	11	$100 + 21 = 121$
5	$16 + 9 = 25$	12	$121 + 23 = 144$
6	$25 + 11 = 36$	13	$144 + 25 = 169$
7	$36 + 13 = 49$	14	$169 + 27 = 196$

So 145 is at the bottom of column 13 and the top number of column 13 is 169.

b Alternative i

From Part **a**, the number at the top of the 10th column of Ahmed's staircase is 100. Given Basil added 3 rows, the number at the top of the 10th column of his staircase would be $100 + 3 \times 10 = 130$.

Alternative ii

From one column to the next, the number of squares increases by 2. So, in column order, the number of squares in the columns of Basil's staircase are the even numbers 4, 6, 8, ... From this we can calculate the numbers at the top of various columns, as shown in this table.

Column	Top number	Column	Top number
1	4	6	$40 + 14 = 54$
2	$4 + 6 = 10$	7	$54 + 16 = 70$
3	$10 + 8 = 18$	8	$70 + 18 = 88$
4	$18 + 10 = 28$	9	$88 + 20 = 108$
5	$28 + 12 = 40$	10	$108 + 22 = 130$

So the top number in the 10th column would be 130.

c Continuing the calculations in Part **a**, the number at the top of the 15th column of Ahmed's staircase would have been 225. The top number of the 15th column in Chelsea's staircase is 405. Since $405 - 225 = 180$, the number of rows that Chelsea added was $180 \div 15 = 12$.

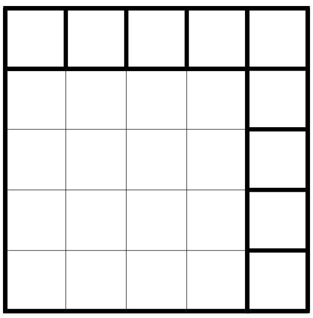
d The top number in column n of Davina's staircase is the top number in the same column n of Ahmed's staircase plus a multiple of n. So the last column in which 51 could be the top number is column 7. In the following table, A is the top number in column n in Ahmed's table. We check if $51 - A$ is a multiple of n.

n	A	$51 - A$	Multiple of n?
1	1	50	yes
2	4	47	no
3	9	42	yes
4	16	35	no
5	25	26	no
6	36	15	no
7	49	2	no

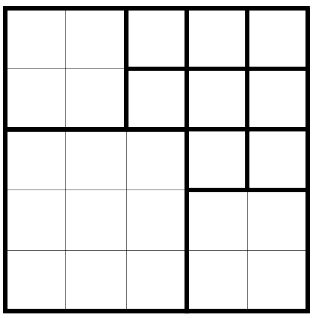
Since $50 \div 1 = 50$, Davina could have added 50 rows and then 51 would have been at the top of column 1. Since $42 \div 3 = 14$, Davina could have added 14 rows and then 51 would have been at the top of column 3. These are the only options available to Davina.

UP3 Square Parts

a One 4×4 piece plus nine 1×1 pieces.

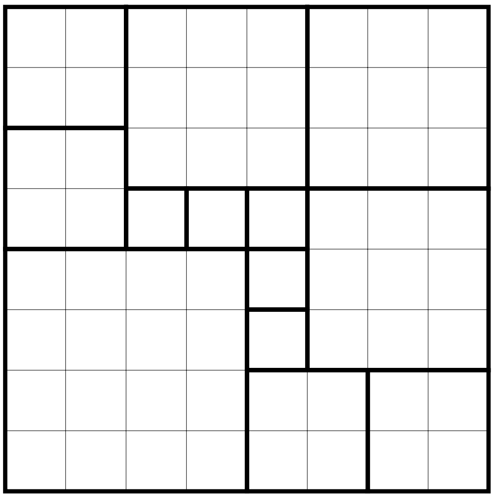


b One 3×3 piece, two 2×2 pieces, and eight 1×1 pieces.



c The total area of the pieces is $2 \times (4 \times 4) + 3 \times (3 \times 3) + 4 \times (2 \times 2) + 5 \times (1 \times 1) = 80$.
Since $8 \times 8 = 64$ and $9 \times 9 = 81$, the largest square that could be constructed from these pieces cannot be larger than 8×8.

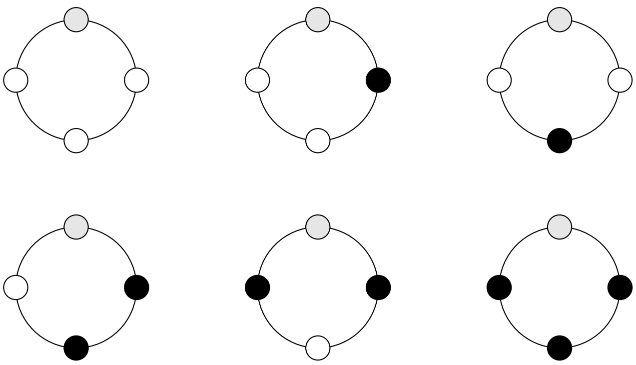
Here is an 8×8 square, made of all the available pieces except one 4×4.



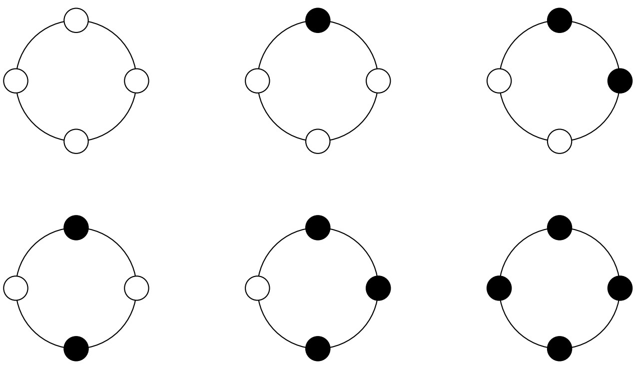
So the largest square that Sal can make is 8×8.

UP4 Bracelets

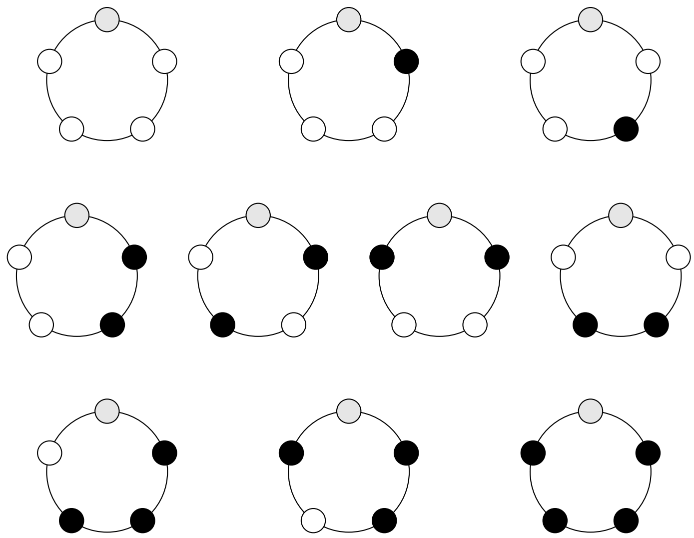
a There are 6 different bracelets as shown in the diagram. The grey dot represents the red bead.



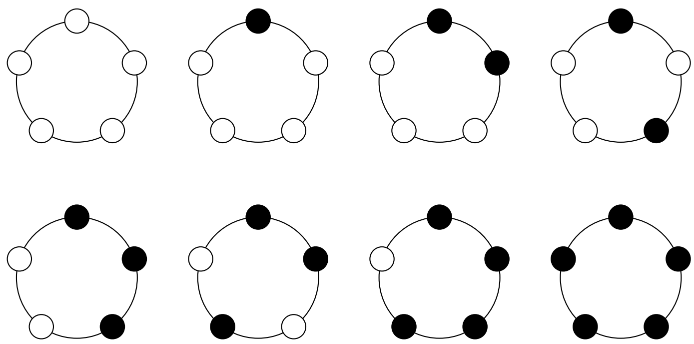
b There are 6 different bracelets.



c There are 10 different bracelets.

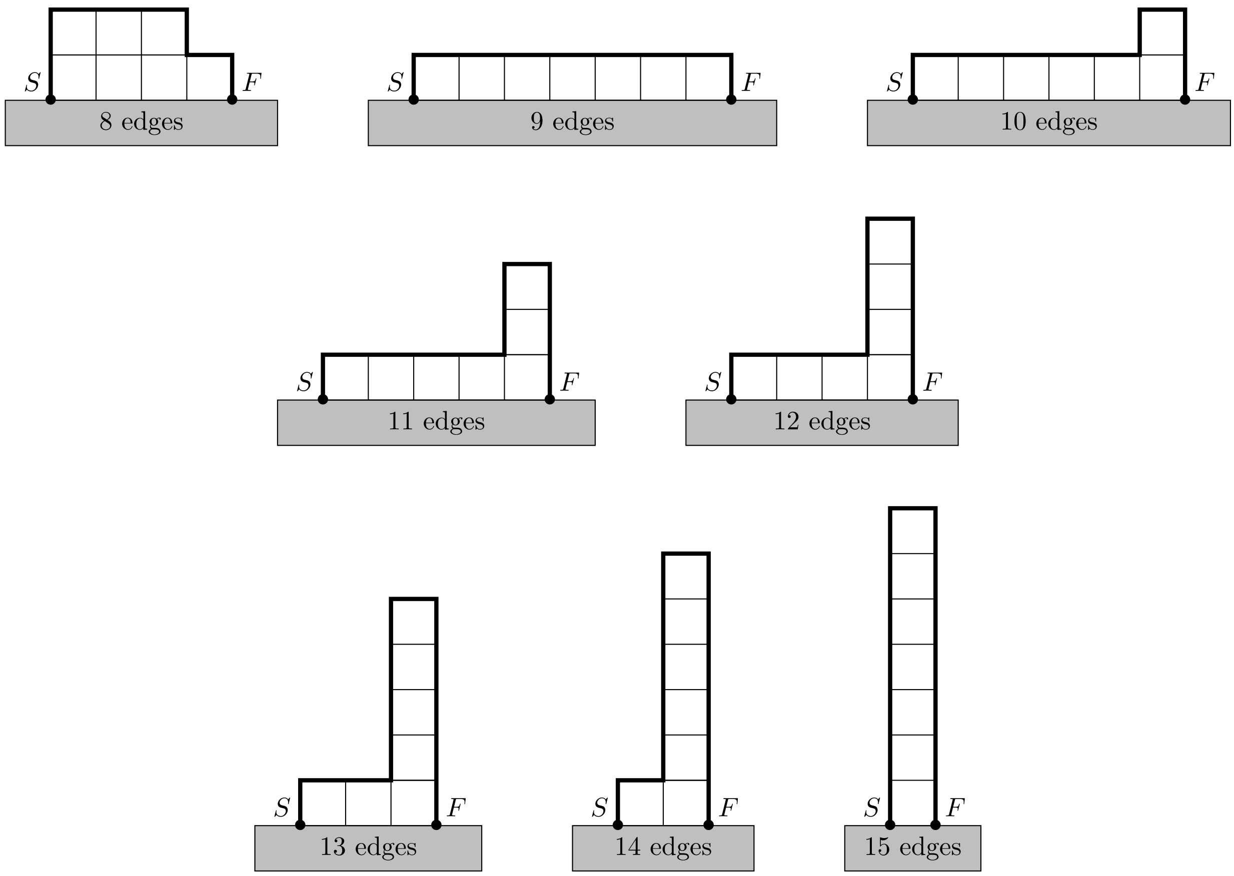


d There are 8 different bracelets.



J1 Annabel's Ants

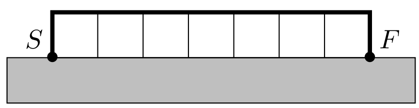
a Here is one arrangement for each total number of edges from 8 to 15. There are several other arrangements.



b There are 10 arrangements as we see in the following discussion. Any six of these are acceptable.

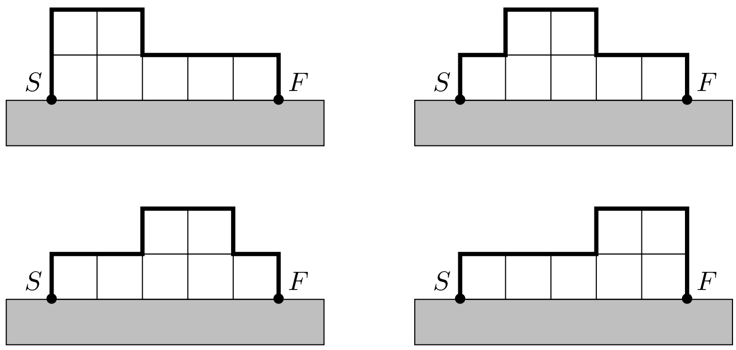
We sort the arrangements according to the maximum height for the tile columns in the frame. Notice that since the ant starts and finishes at the same height, the number of vertical edges that it crawls up must equal the number of vertical edges that it crawls down. So the number of vertical edges in any ant walk is even. Also notice that the number of horizontal edges in any ant walk equals the number of columns in which the tiles are placed.

- *Maximum column height of 1.* There is only one arrangement.

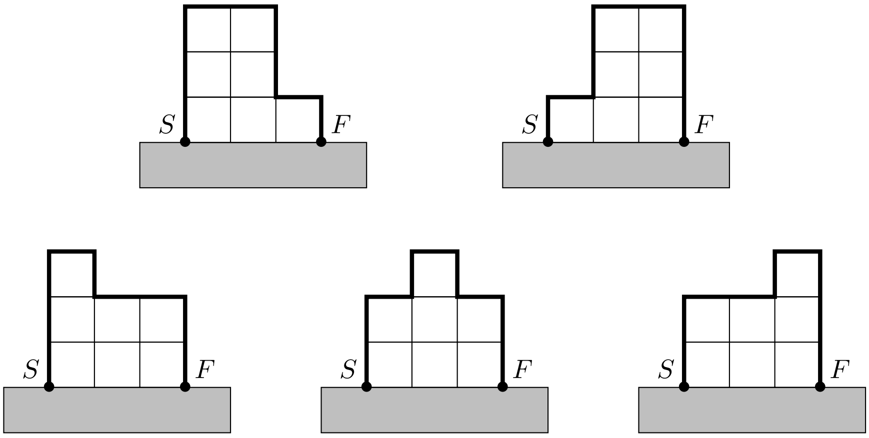


- *Maximum column height of 2.* There are at least 4 columns and at most 6 columns. If there are 4 or 6 columns, then the number of tile edges in the walk is even so cannot be 9.

If there are 5 columns, then there are just 4 arrangements.



• *Maximum column height of 3.* There are at least 3 columns and at most 5 columns. If there are 3 columns, then there are just 5 arrangements.

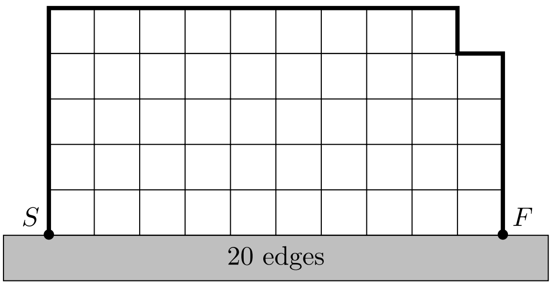


If there are 4 or 5 columns, then the number of tile edges in the walk is at least $4 + 2 \times 3 = 10$, which is greater than 9.

• *Maximum column height of 4, 5, or 6.* There are at least 2 columns. So the number of tile edges in the walk is at least $2 + 2 \times 4 = 10$, which is greater than 9.

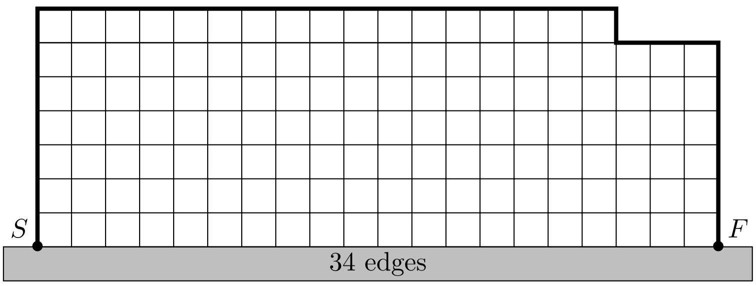
• *Maximum column height of 7.* The number of tile edges in the walk is $1 + 2 \times 7 = 15$, which is greater than 9.

c There are only two possible arrangements: this one and its reflection about any vertical line.

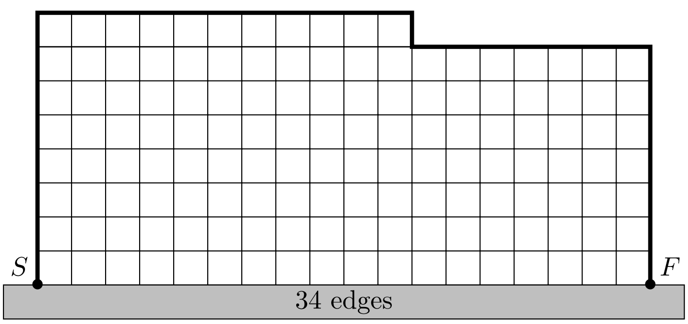


d Here are four arrangements. Their reflections about any vertical line are also acceptable.

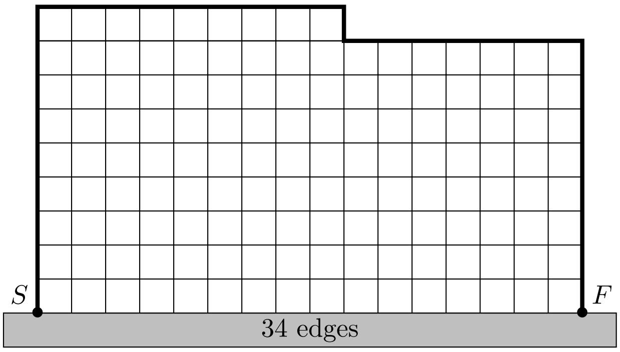
Maximum height 7.



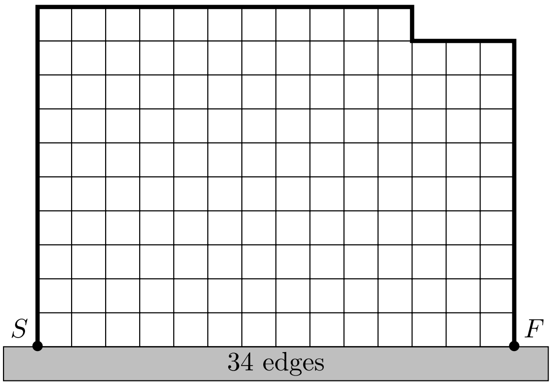
Maximum height 8.



Maximum height 9.



Maximum height 10.

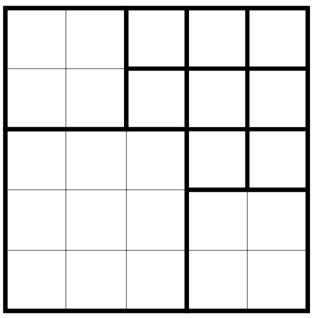


J2 Steps to Infinity

See Upper Primary Problem 2.

J3 Square Parts

a The diagram shows one way to cut a 5×5 square into 11 square pieces. The 11 pieces are: one 3×3 piece, two 2×2 pieces, and eight 1×1 pieces. There are several other possible partitions. For example, see Extension **1**.



b Alternative i

The table shows all possible combinations of squares that form a 4×4 square.

Largest square(s)	Other squares	Total
one 4×4	none	1
one 3×3	seven 1×1	8
four 2×2	none	4
three 2×2	four 1×1	7
two 2×2	eight 1×1	10
one 2×2	twelve 1×1	13
sixteen 1×1	none	16

No combination totals 11 squares.

Alternative ii

Suppose a 4×4 square is divided into smaller square pieces. If all of the pieces were 1×1, then we would have 16 pieces, which is more than 11. So, if we want exactly 11 pieces, then at least one piece must be bigger than 1×1.

If one of the pieces is 3×3 or bigger, then the number of pieces is at most $1 + (16 - 9) = 8$, which is less than 11. So each piece is 1×1 or 2×2.

If two or more pieces are 2×2, then the number of pieces is at most $2 + (16 - 8) = 10$, which is less than 11. So only one piece is 2×2 and the number of pieces is $1 + (16 - 4) = 13$ pieces, which is more than 11.

Hence it is impossible to divide a 4×4 square into 11 smaller square pieces.

Alternative iii

Suppose a 4×4 square is divided into 11 smaller square pieces. The table shows, for each number of 1×1 pieces, the least area covered by all 11 pieces.

No. of 1×1	Least area of 11 pieces
0	$0 + 11 \times 4 = 44$
1	$1 + 10 \times 4 = 41$
2	$2 + 9 \times 4 = 38$
3	$3 + 8 \times 4 = 35$
4	$4 + 7 \times 4 = 32$
5	$5 + 6 \times 4 = 29$
6	$6 + 5 \times 4 = 26$
7	$7 + 4 \times 4 = 23$
8	$8 + 3 \times 4 = 20$
9	$9 + 2 \times 4 = 17$

In all cases the total area of the 11 pieces is more than 16. So there are either 10 or 11 pieces that are 1×1. If all 11 pieces are 1×1, then their total area is less than 16. Hence there must be exactly 10 pieces that are 1×1. But that leaves an area of 6 to be divided into two squares larger than 1×1. This is impossible. So a 4×4 square cannot be divided into 11 smaller square pieces.

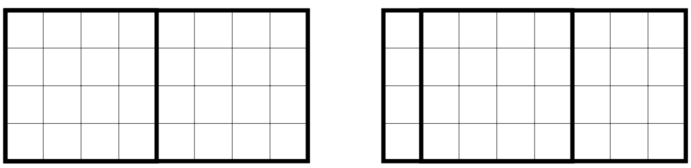
Alternative iv

Suppose a 4×4 square is divided into 11 smaller square pieces. All pieces must be smaller than 4×4. Let the number of 3×3 squares be a. Let the number of 2×2 squares be b. Let the number of 1×1 squares be c. Then $a + b + c = 11$ and $9a + 4b + c = 16$. Subtracting these two equations gives $8a + 3b = 5$. Hence $a = 0$. Therefore $3b = 5$, which is impossible. So a 4×4 square cannot be divided into 11 smaller square pieces.

c The total area of the pieces is $2 \times (4 \times 4) + 3 \times (3 \times 3) + 4 \times (2 \times 2) + 4 \times (1 \times 1) = 79$.
So the largest square that could be constructed from these pieces must be 8×8 or smaller. Since $79 - 64 = 15$, pieces with total area 15 must be removed before an 8×8 square could be made. The only combinations of square numbers that total 15 are $9 + 4 + 1 + 1$, and $4 + 4 + 4 + 1 + 1 + 1$.

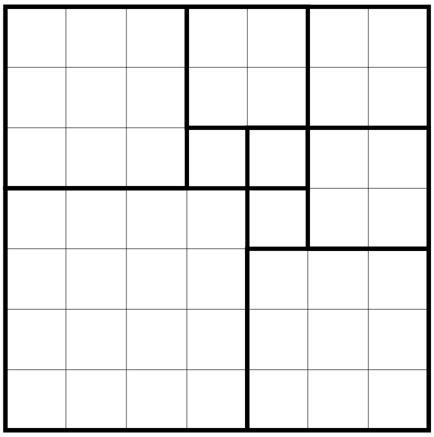
Hence the only combinations of pieces that have total area 64 are
$2 \times (4 \times 4) + 2 \times (3 \times 3) + 3 \times (2 \times 2) + 2 \times (1 \times 1)$, and
$2 \times (4 \times 4) + 3 \times (3 \times 3) + 1 \times (2 \times 2) + 1 \times (1 \times 1)$.

If two 4×4 squares occupy an 8×8 square, then each must be against the boundary of the 8×8 square. So they are in separate 8×4 rectangular halves of the 8×8 square. (The boundary between these two rectangular halves is either horizontal or vertical.) Since at least two 3×3 squares must be used, at most one can cross the line dividing these half-regions. So an entire 3×3 square must occupy the vacant space of one of these half-regions. Therefore the 4×4 square in that half-region occupies a corner of that half-region or leaves a 1×4 space between the 4×4 square and an end boundary of that half-region.



In the first case there will be a 1×3 space between the 3×3 square and either the 4×4 square or the boundary of the 8×8 square. So three 1×1 squares are required to fill that 1×3 space. In the second case, four 1×1 squares are required to fill the 1×4 space. Since at most two 1×1 squares can be used, an 8×8 square cannot be constructed from the given pieces.

Here is a 7×7 square made from some of the available pieces.



So the largest square that Sal can make is 7×7.

J4 Tribonacci Sequences

a Stopping at the smallest 5-digit number, the sequence is

6, 19, 22, 47, 88, 157, 292, 537, 986, 1815, 3338, 6139, 11292.

b Working backwards gives the earlier terms in the sequence.

Since the 4th term + the 5th term + the 6th term = the 7th term,
the 4th term is $135 - 71 - 36 = 28$.
Similarly, the 3rd term is $71 - 36 - 28 = 7$,
the 2nd term is $36 - 28 - 7 = 1$,
and the 1st term is $28 - 7 - 1 = 20$.

So the seeds are 20, 1, 7.

c The sum of one even and two odd numbers is even. The sum of two even and one odd number is odd. So the tribonacci sequence 20, 17, 2017, ... , gives the following sequence of even and odd numbers:

even, odd, odd, even, even, odd, odd, even, ...

Thus the sequence 'even, odd, odd, even' repeats endlessly. Since $2017 = (4 \times 504) + 1$, the 2017th term will correspond to the first term in the 505th occurrence of the sequence 'even, odd, odd, even'. So the 2017th term is an even number.

d Let a be the first seed. The sequence becomes

a, 20, 17, $a + 37$, $a + 74$, $2a + 128$, $4a + 239$, $7a + 441$, $13a + 808$, $24a + 1488$, $44a + 2737$, ...

We set each term equal to 2017 to see if there is a solution for a.

The first term gives $a = 2017$.

The fourth term gives $a = 1980$.

The fifth term gives $a = 1943$.

The sixth term would give $2a = 1889$, which is impossible since 1889 is odd.

The seventh term would give $4a = 1778$, which is impossible because 4 does not divide 1778.

The eighth term would give $7a = 1576$, which is impossible because 7 does not divide 1576.

The ninth term gives $13a = 1209$, which gives $a = 93$.

The tenth term would give $24a = 529$, which is impossible because 529 is odd.

Since a is positive, the eleventh term is more than 2017 and each term after the eleventh is bigger than its predecessor. So all terms from the eleventh onwards are more than 2017.

So the only positive integer values of a are 2017, 1980, 1943, 93.

J5 Shower Heads

a The angular separations of adjacent jets in the inner, middle, and outer circles are $360°/12 = 30°$, $360°/18 = 20°$, $360°/36 = 10°$ respectively.

b The angular separation between radii that pass through three jets is the lowest common multiple of 30, 20, and 10 degrees, that is, $60°$. Hence the number of radii that pass through three jets is $360/60 = 6$.

c Alternative i

From Part **b**, exactly six radii pass through a jet on all three circles.

The lowest common multiple of 30 and 10 is 30. Hence the number of radii that pass through a jet on the inner and outer circles is $360/30 = 12$. Of these, six pass through all three circles. So the number of radii that pass through a jet on the inner and outer circles but not on the middle circle is $12 - 6 = 6$.

The lowest common multiple of 20 and 10 is 20. Hence the number of radii that pass through a jet on the middle and outer circles is $360/20 = 18$. Of these, six pass through all three circles. So the number of radii that pass through a jet on the middle and outer circles but not the inner circle is $18 - 6 = 12$.

The lowest common multiple of 30 and 20 is 60. Hence all radii that pass through a jet on the inner and middle circles also pass through a jet on the outer circle.

So the total number of radii that pass through just two jets is $6 + 12 = 18$.

Alternative ii

From Part **b**, the pattern of jets on the three circles repeats every $60°$. The table shows the angles from $0°$ to $60°$ on each circle at which a radius passes through a jet.

outer circle	0	10	20	30	40	50	60
middle circle	0		20		40		60
inner circle	0			30			60

Thus in $60°$ exactly 3 radii pass through just two jets. So the number of radii in $360°$ that pass through just two jets is $6 \times 3 = 18$.

d Alternative i

Since the outer circle has an odd number of jets, no diameter passes through two jets on this circle. Hence no diameter of the shower head passes through 8 jets.

Alternative ii

The angular separations for the four circles are $360/10 = 36°$, $360/20 = 18°$, $360/30 = 12°$, $360/45 = 8°$. The lowest common multiple of 36, 18, 12, 8 is 72. Since 180 is not a multiple of 72, there is no diameter of the shower head that passes through 8 jets.

J6 Circle Hopscotch

a Completing two laps requires a student to move a total of 50 places. Thirteen 4-hops moves 52 places around the circle, which is 2 places too many. Hence twelve 4-hops and one 3-hop is 1 place too far. But eleven 4-hops and two 3-hops will cover exactly 50 places. There are many acceptable orderings of these 13 hops. For example: 0, 3, 6, 10, 14, 18, 22, 1, 5, 9, 13, 17, 21, 0.

b From the solution to Part **a**, the maximum number of 4-hops in a total of 50 places is 11. If we reduce the number of 4-hops then they must be replaced by 3-hops to keep the total of 50 places. The lowest common multiple of 4 and 3 is 12. So this table shows the only possible combinations of 4-hops and 3-hops in a game.

4-hops	11	8	5	2
3-hops	2	6	10	14

c The total length of any sequence of 6 hops is at most $6 \times 4 = 24$ places. So in order to go around the circuit, a student must land on at least 7 different numbers.

One way to land on only 7 different numbers is to begin by going around the board once with 4 hops of length 4 and 3 hops of length 3. With this sequence of hops the student lands on 4, 8, 12, 16, 19, 22, 0. If the same sequence of hops is repeated, the student lands on the same numbers again.

So 7 is the smallest number of different numbers a student can land on.

d If Jo takes five 4-hops, she is at position 20. Two 3-hops will then move her to 1. On her second lap, the five 4-hops will take her to 21. Then two 3-hops will take her to 2. On her third lap, five 4-hops takes her to 22 and one 3-hop will bring her back to 0. After this, her pattern will repeat, so Jo will only be at 0 at the end of every three laps that she completes.

We can see that in completing 3 laps, Jo takes fifteen 4-hops and five 3-hops. So, in the same time, Mike will take fifteen 3-hops and five 4-hops. This is a total of $15 \times 3 + 5 \times 4 = 65$ places, which means he will be at position 15 when Jo completes her three laps, that is, 10 places behind. Because Mike follows Jo's pattern, he will advance 15 places after every 3 laps which Jo completes. The table shows Mike's position after every 3 laps by Jo, up to lap 15.

Laps by Jo	3	6	9	12	15
Mike's position	15	5	20	10	0

So the first time they meet at 0 after they start is when Jo has completed 15 laps. At that time Mike is behind by $5 \times 10 = 50$ places, which is 2 laps. So Mike completes 13 laps.

CHALLENGE SOLUTIONS – INTERMEDIATE

I1 Tribonacci Sequences

a Stopping at the smallest 5-digit number, the sequence is

6, 19, 22, 47, 88, 157, 292, 537, 986, 1815, 3338, 6139, 11292.

b The sum of one even and two odd numbers is even. The sum of two even and one odd number is odd. So the tribonacci sequence 20, 17, 2017, ... , gives the following sequence of even and odd numbers:

even, odd, odd, even, even, odd, odd, even, ...

Thus the sequence 'even, odd, odd, even' repeats endlessly. Since $2017 = (4 \times 504) + 1$, the 2017th term will correspond to the first term in the 505th occurrence of the sequence 'even, odd, odd, even'. So the 2017th term is an even number.

c Let a be the first seed. The sequence becomes a, 20, 17, $a + 37$, $a + 74$, $2a + 128$, $4a + 239$, $7a + 441$, ...

The eighth term is $7a + 441 = 0$. Then $7a = -441$ and $a = -63$.

d Let a be the first seed. The sequence becomes

a, 20, 17, $a + 37$, $a + 74$, $2a + 128$, $4a + 239$, $7a + 441$, $13a + 808$, $24a + 1488$, $44a + 2737$, ...

We set each term equal to 2017 to see if there is a solution for a.

The first term gives $a = 2017$.

The fourth term gives $a = 1980$.

The fifth term gives $a = 1943$.

The sixth term would give $2a = 1889$, which is impossible since 1889 is odd.

The seventh term would give $4a = 1778$, which is impossible because 4 does not divide 1778.

The eighth term would give $7a = 1576$, which is impossible because 7 does not divide 1576.

The ninth term gives $13a = 1209$, which gives $a = 93$.

The tenth term would give $24a = 529$, which is impossible because 529 is odd.

Since a is positive, the eleventh term is more than 2017 and each term after the eleventh is bigger than its predecessor. So all terms from the eleventh onwards are more than 2017.

So the only positive integer values of a are 2017, 1980, 1943, 93.

I2 Rowing Machine

a Alternative i

At 3 minute pace, Thomas rows 1000 m in $2 \times 3 = 6$ minutes.

At 3 minute 20 second pace, Jack rows 500 m in $3\frac{1}{3} = \frac{10}{3}$ minutes. So in 3 minutes Jack rows $500 \times (3 \div \frac{10}{3}) = 450$ m. Hence Jack needs to row $1000 - 450 = 550$ m at his second pace of 2 minute 40 seconds $= \frac{8}{3}$ minutes. The time this takes is $\frac{550}{500} \times \frac{8}{3} = \frac{44}{15}$ minutes. Jack's total time is $3 + \frac{44}{15} = 5\frac{14}{15}$ minutes = 5 minutes 56 seconds.

So Jack reaches 1000 m first.

Alternative ii

At 3 minute pace, Thomas rows 1000 m in $2 \times 3 = 6$ minutes.

At 3 minute 20 second pace, Jack rows 500 m in $3 \times 60 + 20 = 200$ seconds. So the distance Jack rows at this pace in 3 minutes is $500 \times \frac{180}{200} = 450$ m. So Jack needs to row $1000 - 450 = 550$ m at his second pace of 2 minutes 40 seconds. At this pace he rows 500 m in $2 \times 60 + 40 = 160$ seconds, hence 50 m in 16 seconds. Jack's total time is $180 + 160 + 16 = 356$ seconds = 5 minutes 56 seconds.

So Jack reaches 1000 m first.

Alternative iii

At 3 minute pace, Thomas rows 1000 m in $2 \times 3 = 6$ minutes.

At 3 minute 20 second pace, Jack rows 500 m in 3 minutes 20 seconds, which is $500/(3\frac{1}{3}) = 150$ m per minute.

At 2 minute 40 second pace, Jack rows 500 m in 2 minute 40 seconds, which is $500/(2\frac{2}{3}) = 187.5$ m per minute.

So in 6 minutes Jack would row $3 \times 150 + 3 \times 187.5 = 450 + 562.5 = 1012.5$ m.

So Jack reaches 1000 m first.

Alternative iv

At 3 minute pace, Thomas rows 1000 m in $2 \times 3 = 6$ minutes.

If Jack rowed 500 m at 3 minute 20 second pace and 500 m at 2 minute 40 second pace, then he would row 1000 m in 6 minutes.

If he rows at 3 minute 20 second pace for just 3 minutes and the remaining distance at 2 minute 40 second pace, then he will row less than 500 m at the slower rate and more than 500 m at the faster rate, and therefore reach 1000 m in less than 6 minutes.

So Jack reaches 1000 m first.

b Alternative i

At 3 minute pace, Thomas rows 1000 m in $2 \times 3 = 6$ minutes.

At 3 minute 20 second pace, Jack rows 600 m in $\frac{600}{500} \times \frac{10}{3} = 4$ minutes. At 2 minute 40 second pace, Jack rows 400 m in $\frac{400}{500} \times \frac{8}{3} = 2\frac{2}{15}$ minutes. Jack's total time is $4 + 2\frac{2}{15} = 6\frac{2}{15}$ minutes = 6 minutes and 8 seconds.

So Thomas reaches 1000 m first.

Alternative ii

Thomas's pace is $(3 \times 60)/5 = 36$ seconds per 100 m.

Jack's first pace is $(3 \times 60 + 20)/5 = 200/5 = 40$ seconds per 100 m. So Jack takes $6 \times (40 - 36) = 24$ seconds longer than Thomas to complete the first 600 m.

Jack's second pace is $(2 \times 60 + 40)/5 = 160/5 = 32$ seconds per 100 m. So Thomas takes $4 \times (36 - 32) = 16$ seconds longer than Jack to complete the last 400 m.

Hence Jack takes $24 - 16 = 8$ seconds longer than Thomas to complete 1000 m. So Thomas reaches 1000 m first.

Alternative iii

At 3 minute pace, Thomas rows 1000 m in $2 \times 3 = 6$ minutes.

If Jack rowed 500 m at 3 minute 20 second pace and 500 m at 2 minute 40 second pace, then he would row 1000 m in 6 minutes.

If he rows at 3 minute 20 second pace for 600 m and the next 400 m at 2 minute 40 second pace, then he will row more than 500 m at the slower rate and less than 500 m at the faster rate, and therefore reach 1000 m in more than 6 minutes.

So Thomas reaches 1000 m first.

c Alternative i

Rowing at 2:38 pace Jack covers every 100 m in $(2 \times 60 + 38)/5 = 158/5 = 31.6$ seconds. So he covers 600 m in $6 \times 31.6 = 189.6$ seconds.

He wishes to row 3000 m in 15 minutes, so must cover the remaining 2400 m in $15 \times 60 - 189.6 = 710.4$ seconds. So he must row every 100 m in $710.4/24 = 29.6$ seconds. This means 500 m will take $5 \times 29.6 = 148$ seconds, or 2:28 pace.

Alternative ii

Jack wishes to row 3000 m in 15 minutes, that is, row at pace $15/6 = $ 2:30. If Jack rows at 2:38 pace he will be 8 seconds slower per 500 m than required. After 600 m he has rowed 1/5 of the total distance, so has 4/5 of the total distance remaining. So he has 4 times the distance to regain the time lost. This means the time lost (8 seconds per 500 m) must be regained at one quarter the pace, that is 2 seconds per 500m. So Jack must row the remaining distance at a pace of 2:28.

Alternative iii

Jack wishes to row 3000 m in $15 \times 60 = 900$ seconds.

Rowing at 2:38 pace Jack covers 600 m in $(600/500) \times (2 \times 60 + 38) = (6/5) \times 158 = 189.6$ seconds.

Suppose he rows the remaining 2400 m at a pace of x. Then $189.6 + (2400/500)x = 900$. So $24x/5 = 900 - 189.6 = 710.4$ and $x = (5 \times 710.4)/24 = 3552/24 = 148$ seconds = 2:28.

d Alternative i

The average of the Jack's two paces is $(3\frac{1}{3} + 2\frac{2}{3})/2 = 6/2 = 3$ minutes, which is the same pace as Thomas. If Jack rows for the same distance for each of his two paces, then his average pace for the total distance is the average of the two paces. So if Jack rows at a 3 minute 20 second pace for the first 500 m and at a 2 minutes 40 second pace for the last 500 m, he will finish at the same time as Thomas.

Alternative ii

At 3 minute pace, Thomas rows 1000 m in $2 \times 3 = 6$ minutes.

Suppose Jack rows for x metres at a 3 minute 20 second pace and $1000 - x$ metres at a 2 minute 40 second pace. The total time in minutes that Jack takes to row 1000 m is

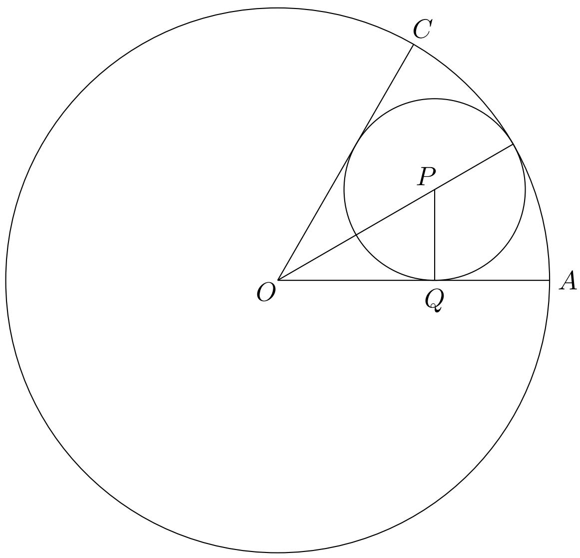
$$\left(\frac{x}{500} \times \frac{10}{3}\right) + \left(\frac{1000 - x}{500} \times \frac{8}{3}\right) = \frac{10x}{1500} + \frac{8000 - 8x}{1500} = \frac{8000 + 2x}{1500}$$

So $(8000 + 2x)/1500 = 6$, $8000 + 2x = 9000$, and $x = 500$. Thus Jack must row at a 3 minute 20 second pace for 500 m.

I3 Blocking Circles

a Alternative i

Each of the small circles blocks the same amount of circumference from O. In the diagram, OA and OC are tangents to the small circle with centre P.

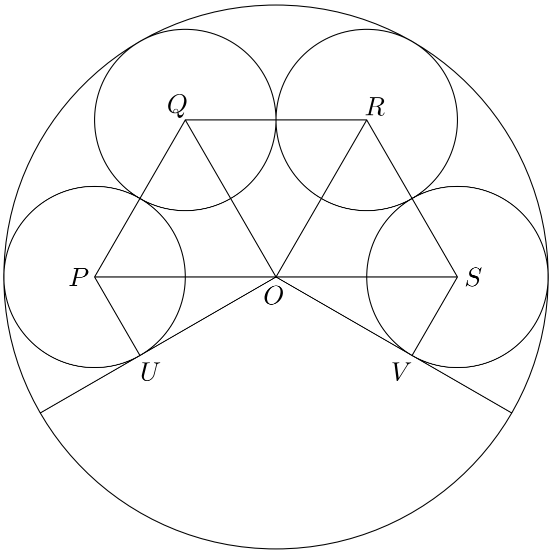


So $\angle OQP = 90°$. Also $PQ = 4$ cm and $OP = 12 - 4 = 8$ cm. Hence reflecting $\triangle OQP$ about the line OQ produces an equilateral triangle (with side length 8). Therefore $\angle AOP = 30°$. Similarly, $\angle COP = 30°$.

So the fraction of the circumference of the large circle that is blocked from O by one small circle is $(30+30)/360 = 60/360 = 1/6$. Hence the fraction of the circumference of the large circle that is blocked from O by all four small circles is $4/6 = 2/3$.

Alternative ii

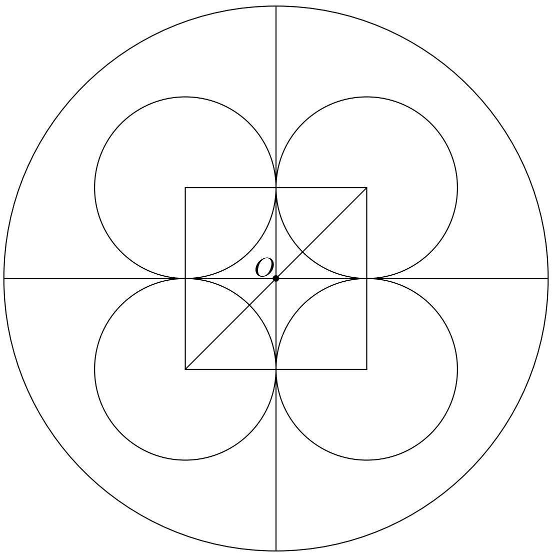
In the diagram, P, Q, R, S are the centres of the four small circles and OU, OV are tangents to two of them.



Since $OP = OQ = 12 - 4 = 8\,\text{cm}$ and $PQ = 4 + 4 = 8\,\text{cm}$, $\triangle OPQ$ is equilateral. Hence $\angle POQ = 60°$. Similarly, $\angle QOR = 60°$ and $\angle ROS = 60°$. Since $PU = 4\,\text{cm}$ and $\angle PUO = 90°$, $\triangle OPU$ is half an equilateral triangle. Therefore $\angle UOP = 30°$. Similarly, $\angle SOV = 30°$.

Hence the fraction of the circumference of the large circle that is blocked from O by the four small circles is $(30 + 60 + 60 + 60 + 30)/360 = 240/360 = 2/3$.

b Each of the small circles blocks the same length of circumference from O. So each blocks exactly one quarter of the circumference. Hence each small circle touches two perpendicular diameters of the large circle. So their centres form a square of side length 8 cm as shown.

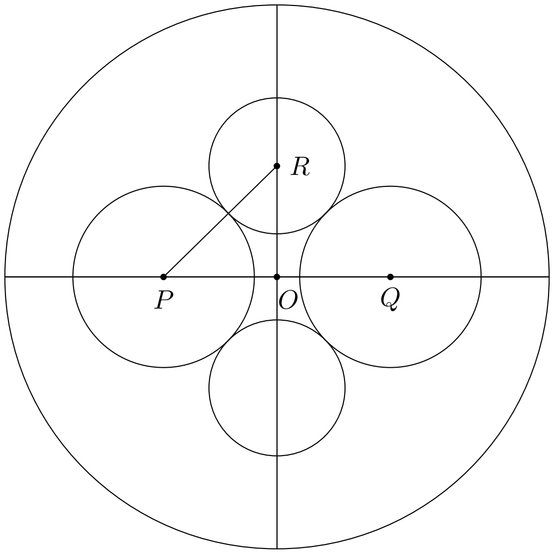


From Pythagoras' theorem, the diagonal of the square is $\sqrt{64 + 64} = 8\sqrt{2}\,\text{cm}$. So the distance from O to the centre of each small circle is $4\sqrt{2}\,\text{cm}$.

c We first find the minimum distance from the centre O to the centre of a circle of radius 3 cm.

Let P and Q be the centres of the circles of radius 4 cm. Since the interior circles cannot overlap, a circle of radius 3 cm will be closest to the centre of the large circle when it touches both circles of radius 4 cm. Let R be the centre of the circle of radius 3 cm. Then $RP = 3 + 4 = 7\,\text{cm} = RQ$.

Since $OP = OQ$, triangles ROP and ROQ are congruent. So R will be on the diameter of the large circle that is perpendicular to the diameter through P and Q. In this configuration, the circumference of the large circle is entirely blocked by all four interior circles when viewed from O.

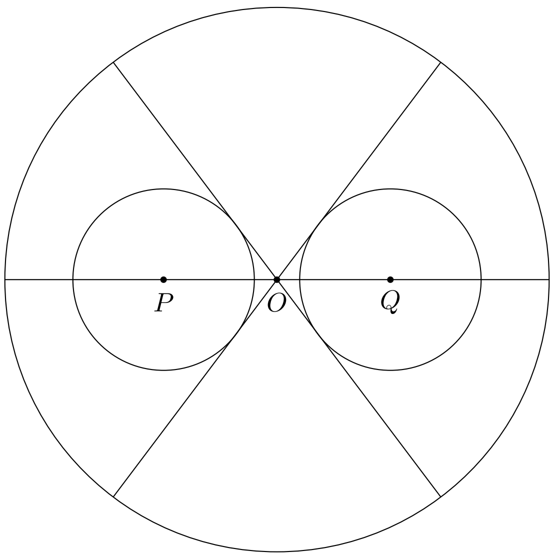


Applying Pythagoras' theorem in $\triangle OPR$ gives $OR = \sqrt{PR^2 - PO^2} = \sqrt{49 - 25} = \sqrt{24} = 2\sqrt{6}\,\text{cm}$.

So the minimum distance from the centre O to the centre of a circle of radius 3 cm is $2\sqrt{6}\,\text{cm}$.

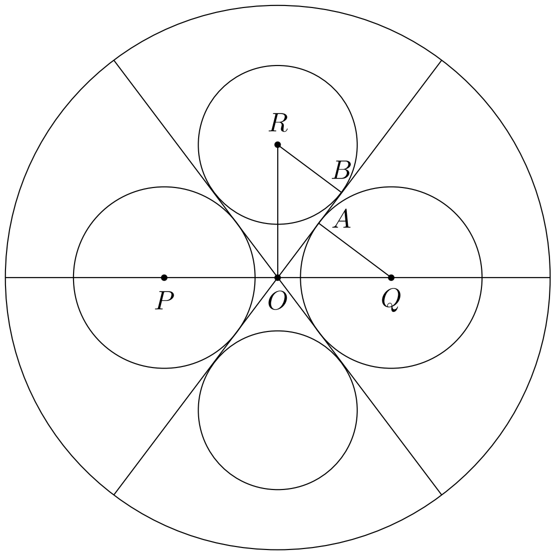
Now we find the maximum distance from the centre O to a circle of radius 3 cm.

Let P and Q be the centres of the circles of radius 4 cm. Draw the tangents to these circles through O. The diameter through P and Q bisects two of the angles between these tangents.



For the circumference of the circle of radius 12 cm to be entirely blocked, the two circles of radius 3 cm must touch these tangents. So the centre of each of the circles of radius 3 cm is equidistant to these tangents and therefore on their angle bisector. Hence the centres of each of the circles of radius 3 cm must lie on the diameter of the large circle that is perpendicular to the diameter through P and Q.

Let R be the centre of a circle of radius 3 cm as shown. The tangent from O to the circle of radius 4 cm meets that circle at A and the circle of radius 3 cm at B. The diagram is not drawn to scale.



Since $\angle ROQ = 90°$ and triangles OBR and OAQ are right-angled, $\angle BOR = 90° - \angle AOQ = \angle AQO$. Applying Pythagoras' theorem to $\triangle OAQ$ gives $OA^2 = OQ^2 - AQ^2 = 25 - 16 = 9$. So $OA = 3\,\text{cm} = BR$. Hence triangles BOR and AQO are congruent. Therefore $OR = OQ = 5\,\text{cm}$.

So the maximum distance from the centre O to the centre of a circle of radius 3 cm is 5 cm.

I4 Shower Heads

a The angular separations for the three circles are: $360/12 = 30°$, $360/18 = 20°$, $360/36 = 10°$.

The angular separation between radii that pass through three jets is the lowest common multiple of 30, 20, and 10 degrees, that is, 60°. Hence the number of radii that pass through three jets is $360/60 = 6$.

Alternative i

The lowest common multiple of 30 and 10 is 30. Hence the number of radii that pass through a jet on the inner and outer circles is $360/30 = 12$. Of these, six pass through all three circles. So the number of radii that pass through a jet on the inner and outer circles but not on the middle circle is $12 - 6 = 6$.

The lowest common multiple of 20 and 10 is 20. Hence the number of radii that pass through a jet on the middle and outer circles is $360/20 = 18$. Of these, six pass through all three circles. So the number of radii that pass through a jet on the middle and outer circles but not the inner circle is $18 - 6 = 12$.

The lowest common multiple of 30 and 20 is 60. Hence all radii that pass through a jet on the inner and middle circles also pass through a jet on the outer circle.

So the total number of radii that pass through just two jets is $6 + 12 = 18$.

Alternative ii

The pattern of jets on the three circles repeats every 60°. The table shows the angles from 0° to 60° on each circle at which a radius passes through a jet.

outer circle	0	10	20	30	40	50	60
middle circle	0		20		40		60
inner circle	0			30			60

Thus in 60° exactly 3 radii pass through just two jets. So the number of radii in 360° that pass through just two jets is $6 \times 3 = 18$.

b Alternative i

Since the outer circle has an odd number of jets, no diameter passes through two jets on this circle. Hence no diameter of the shower head passes through six jets.

Alternative ii

The angular separations for the three circles are $360/20 = 18°$, $360/30 = 12°$, $360/45 = 8°$. The lowest common multiple of 18, 12, 8 is 72. Since 180 is not a multiple of 72, there is no diameter of the shower head that passes through 6 jets.

c The lowest common multiple of the three angular separations must be $90°$. So the angular separations must be different factors of 90. They must also be greater than $3°$ otherwise the number of jets will exceed 100. The following table shows all acceptable angular separations for jets on a circle and the corresponding number of jets in that circle.

Angle (degrees)	5	6	9	10	15	18	30	45	90
Jets	72	60	40	36	24	20	12	8	4

The only combinations of three jet numbers that total 100 are:
72, 24, 4; 72, 20, 8; 60, 36, 4; 40, 36, 24.
It is easy to check for each combination that the lowest common multiple of its corresponding angular separations is indeed $90°$.

d **Alternative i**

Since there is only one radius that passes through 3 jets, the three angular separations must have a lowest common multiple of $360°$. Since no radius passes through just 2 jets, every pair of angular separations must have a lowest common multiple of $360°$.

Since $360 = 2^3 \times 3^2 \times 5$, we are seeking three angular separations less than 360 of the form $2^x 3^y 5^z$, where $0 \le x \le 3$, $0 \le y \le 2$, $0 \le z \le 1$.

Two of the angular separations must have factor 8, otherwise there are two angular separations that are not multiples of 8 and they will therefore have a lowest common multiple less than 360. Similarly, two of the angular separations must have factor 9 and two must have factor 5.

If two angular separations have factor 8 then both can't have factor 9, otherwise one of them will also have factor 5 and that angular separation will therefore equal 360. Similarly, two angular separations can't both have factors 8 and 5, and two can't both have factors 9 and 5. So each angular separation has just two of the factors 8, 9, 5. The three smallest angular separations that satisfy these conditions are $5 \times 8 = 40$, $5 \times 9 = 45$, and $8 \times 9 = 72$. Hence the maximum number of jets in the shower head is $360/40 + 360/45 + 360/72 = 9 + 8 + 5 = 22$.

Alternative ii

The angular separation of adjacent jets in a circle multiplied by the number of jets in that circle is 360. Since $360 = 2^3 \times 3^2 \times 5$, the number of jets in a circle is a multiple of one or more of 2, 3, or 5.

If a circle has an even number of jets, then the angular separation is a divisor of 180. Hence, when two circles have an even number of jets, there are at least two radii that pass through two jets, which is disallowed.

If the number of jets in a circle is a multiple of 3, then the angular separation is a divisor of 120. Hence, when the number of jets in two circles is a multiple of 3, there are at least three radii that pass through two jets, which is disallowed.

If the number of jets in a circle is a multiple of 5, then the angular separation is a divisor of 72. Hence, when the number of jets in two circles is a multiple of 5, there are at least five radii that pass through two jets, which is disallowed.

So at most one circle has an even number of jets, at most one circle has a multiple of three jets, and at most one circle has a multiple of five jets.

No jet can have a multiple of 6 jets, otherwise the other circles would both have 5 jets. Similarly, no jet has a multiple of ten jets and none has a multiple of 15 jets. So one jet has 2, 4, or 8 jets, another has 3 or 9 jets, and the third circle has 5 jets.

Hence the total number of jets in the shower head is at most $8 + 9 + 5 = 22$. The angular separations in circles with 8, 9, and 5 jets are respectively $9 \times 5 = 45°$, $8 \times 5 = 40°$, and $8 \times 9 = 72°$. The lowest common multiple of 45, 40, 72 is 360 and the lowest common multiple of any two of these is also 360. Therefore only one radius of the shower head passes through three jets and no radius passes through just two jets. So the maximum number of jets in the shower head is 22.

I5 Chance Encounters

a Alternative i

Each path from A to P has 4 grid lines: one horizontal (H) and three vertical (V) in some order. The probability of each grid line being chosen is $\frac{1}{2}$. So the probability of each path is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$. There are 4 paths from A to P: HVVV, VHVV, VVHV, VVVH.

So the probability of arriving at P is $4 \times \frac{1}{16} = \frac{1}{4}$.

Alternative ii

Each path that takes 4 seconds has 4 grid lines: some horizontal (H) and some vertical (V). There are 16 such paths from A:
VVVV,
HVVV, VHVV, VVHV, VVVH,
HHVV, HVHV, HVVH, VHHV, VHVH, VVHH,
HHHV, HHVH, HVHH, VHHH,
HHHH

In each of these paths each grid line has probability $\frac{1}{2}$ of being chosen. So the probability for each path is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$. Four of these paths end at P. So the probability of arriving at P is $4 \times \frac{1}{16} = \frac{1}{4}$.

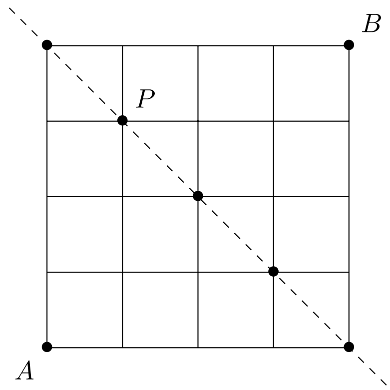
b The counter can reach Q either from the grid point to the left of Q or from P.

There is only one path from A to Q via the grid point to the left of Q. This path comprises four vertical grid lines, each taken with probability $\frac{1}{2}$, and one horizontal grid line taken with probability 1 (no choice). Hence, the probability that the counter takes this path is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{16}$.

From Part **a**, the probability the counter will be at P after 4 seconds is $\frac{1}{4}$. The probability that the counter moves from P to Q is $\frac{1}{2}$. Hence the probability the counter reaches Q via P is $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.

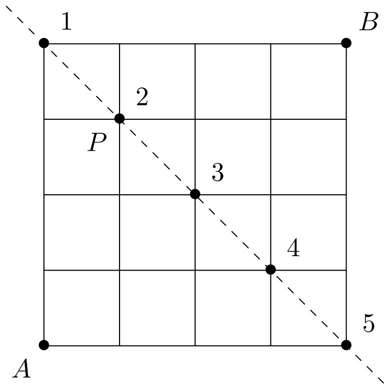
Therefore the probability that the counter will be at Q after 5 seconds is $\frac{1}{16} + \frac{1}{8} = \frac{3}{16}$.

c The counters will meet if they are in the same place at the same time. Each of the grid points on the diagonal through P can be reached in 4 seconds from A and in 4 seconds from B. Hence the counters could meet at any of these five grid points.



To reach a grid point above this diagonal, the counter moving from A will take more than 4 seconds and the counter moving from B will take less than 4 seconds. To reach a grid point below this diagonal, the counter moving from A will take less than 4 seconds and the counter moving from B will take more than 4 seconds. So it is not possible for the counters to meet at any grid point other than those on the diagonal through P.

d Label the grid points on the diagonal through P from top to bottom 1, 2, 3, 4, 5.



The number of paths from A to each of the grid points 1, 2, 3, 4, 5 is respectively 1, 4, 6, 4, 1.

Each of these paths has 4 grid lines and each grid line has probability $\frac{1}{2}$ of being chosen. So each path has probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$. Hence the probability of the counter from A reaching grid points 1, 2, 3, 4, 5 is respectively $\frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}$.

By symmetry the same probabilities apply to the counter from B.

So the probability the counters meet is $(\frac{1}{16})^2 + (\frac{4}{16})^2 + (\frac{6}{16})^2 + (\frac{4}{16})^2 + (\frac{1}{16})^2 = (1+16+36+16+1)/256 = 70/256 = 35/128$.

I6 Unequal Partitions

a Any two of the following partitions:

$$\begin{aligned} 2017 &= 400 + 401 + 403 + 406 + 407 \\ &= 400 + 401 + 404 + 405 + 407 \\ &= 400 + 402 + 403 + 405 + 407 \\ &= 399 + 403 + 404 + 405 + 406. \end{aligned}$$

b Since there are five parts, the span must be at least 4. To have a span of 4, the parts must be consecutive. Since

$$\begin{aligned} 401 + 402 + 403 + 404 + 405 &= 2015 \\ 402 + 403 + 404 + 405 + 406 &= 2020 \end{aligned}$$

no unequal partition of 2017 has span 4. Hence the span must be at least 5. Since $401+402+403+405+406 = 2017$ and $406 - 401 = 5$, $401 + 402 + 403 + 405 + 406$ is an unequal partition of 2017 with the smallest possible span.

c The smallest number that is the sum of 5 unequal parts is $1 + 2 + 3 + 4 + 5 = 15$.

An unequal partition with 5 parts has a span of 4 if and only if its parts are consecutive. So all the numbers that have an unequal partition into 5 parts with a span of 4 are of the form

$$n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5n + 10$$

with $n \geq 1$.

d If an unequal partition has 5 parts, then its span is at least 4. The smallest number with a 5-part unequal partition is $1 + 2 + 3 + 4 + 5 = 15$.

Every integer greater than or equal to 15 has one of the forms $5n + 10$, $5n + 11$, $5n + 12$, $5n + 13$, $5n + 14$ with $n \geq 1$.

Each of the following 5-part partitions has a span of 4 or 5:
$5n + 10 = n + (n + 1) + (n + 2) + (n + 3) + (n + 4)$
$5n + 11 = n + (n + 1) + (n + 2) + (n + 3) + (n + 5)$
$5n + 12 = n + (n + 1) + (n + 2) + (n + 4) + (n + 5)$
$5n + 13 = n + (n + 1) + (n + 3) + (n + 4) + (n + 5)$
$5n + 14 = n + (n + 2) + (n + 3) + (n + 4) + (n + 5)$

So the smallest 5-part span for any integer is 4 or 5.

CHALLENGE STATISTICS – MIDDLE PRIMARY

Mean Score/School Year/Problem

Year	Number of Students	Mean				
		Overall	Problem			
			1	2	3	4
3	589	9.6	3.1	2.2	2.3	2.3
4	1048	10.6	3.3	2.6	2.6	2.5
*ALL YEARS	1645	10.3	3.3	2.4	2.5	2.4

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem			
	1 Annabel's Ants	2 Domino Chains	3 Lock Out	4 Steps to Infinity
Did not attempt	1%	3%	4%	6%
0	4%	10%	10%	10%
1	5%	14%	14%	13%
2	12%	21%	19%	26%
3	20%	25%	25%	18%
4	59%	26%	28%	27%
Mean	3.3	2.4	2.5	2.4
Discrimination Factor	0.4	0.7	0.7	0.7

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean top score'.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean bottom score'.

(4) The discrimination factor = $\dfrac{\text{mean top score} - \text{mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

CHALLENGE STATISTICS – UPPER PRIMARY

Mean Score/School Year/Problem

Year	Number of Students	Mean				
		Overall	Problem			
			1	2	3	4
5	1537	9.3	3.3	2.4	2.3	1.6
6	1890	10.5	3.5	2.6	2.7	1.9
7	104	11.0	3.7	2.8	3.0	1.6
*ALL YEARS	3543	10.0	3.4	2.5	2.6	1.7

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem			
	1 Annabel's Ants	2 Steps to Infinity	3 Square Parts	4 Bracelets
Did not attempt	0%	2%	4%	6%
0	1%	9%	9%	32%
1	4%	11%	5%	12%
2	10%	24%	36%	14%
3	24%	26%	17%	18%
4	61%	28%	30%	18%
Mean	3.4	2.5	2.6	1.7
Discrimination Factor	0.3	0.6	0.6	0.8

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean top score'.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean bottom score'.

(4) The discrimination factor = $\dfrac{\text{mean top score – mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

CHALLENGE STATISTICS – JUNIOR

Mean Score/School Year/Problem

Year	Number of Students	Mean						
		Overall	Problem					
			1	2	3	4	5	6
7	2522	11.7	2.8	2.4	2.1	2.0	2.2	1.5
8	2557	14.2	3.1	2.8	2.6	2.5	2.6	2.0
*ALL YEARS	5101	13.0	2.9	2.6	2.4	2.2	2.4	1.7

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem					
	1 Annabel's Ants	2 Steps to Infinity	3 Square Parts	4 Tribonacci Sequences	5 Shower Heads	6 Circle Hopscotch
Did not attempt	2%	5%	6%	11%	16%	17%
0	4%	8%	5%	10%	9%	15%
1	7%	11%	20%	15%	16%	23%
2	19%	21%	26%	22%	15%	21%
3	30%	26%	18%	28%	19%	16%
4	38%	30%	24%	14%	24%	8%
Mean	2.9	2.6	2.4	2.2	2.4	1.7
Discrimination Factor	0.5	0.7	0.7	0.7	0.8	0.7

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean top score'.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean bottom score'.

(4) The discrimination factor = $\dfrac{\text{mean top score} - \text{mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

CHALLENGE STATISTICS – INTERMEDIATE

Mean Score/School Year/Problem

Year	Number of Students	Mean						
		Overall	Problem					
			1	2	3	4	5	6
9	1785	13.0	2.9	3.1	2.0	2.0	2.0	2.6
10	853	14.9	3.1	3.4	2.5	2.2	2.1	2.9
*ALL YEARS	2649	13.6	3.0	3.2	2.2	2.1	2.0	2.7

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem					
	1 Tribonacci Sequences	2 Rowing Machine	3 Blocking Circles	4 Shower Heads	5 Chance Encounters	6 Unequal Partitions
Did not attempt	2%	4%	16%	15%	12%	15%
0	3%	5%	16%	12%	10%	8%
1	8%	5%	12%	15%	17%	9%
2	17%	11%	19%	29%	30%	15%
3	32%	20%	17%	17%	23%	21%
4	38%	55%	21%	13%	8%	32%
Mean	3.0	3.2	2.2	2.1	2.0	2.7
Discrimination Factor	0.5	0.6	0.8	0.7	0.6	0.8

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean top score'.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean bottom score'.

(4) The discrimination factor = $\dfrac{\text{mean top score} - \text{mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD

1. The number x is 111 when written in base b, but it is 212 when written in base $b-2$. What is x in base 10?

[**2 marks**]

2. A triangle ABC is divided into four regions by three lines parallel to BC. The lines divide AB into four equal segments. If the second largest region has area 225, what is the area of ABC?

[**2 marks**]

3. Twelve students in a class are each given a square card. The side length of each card is a whole number of centimetres from 1 to 12 and no two cards are the same size. Each student cuts his/her card into unit squares (of side length 1 cm). The teacher challenges them to join all their unit squares edge to edge to form a single larger square without gaps. They find that this is impossible.

Alice, one of the students, originally had a card of side length a cm. She says, 'If I don't use any of my squares, but everyone else uses their squares, then it is possible!'

Bob, another student, originally had a card of side length b cm. He says, 'Me too! If I don't use any of my squares, but everyone else uses theirs, then it is possible!'

Assuming Alice and Bob are correct, what is ab?

[**3 marks**]

4. Aimosia is a country which has three kinds of coins, each worth a different whole number of dollars. Jack, Jill, and Jimmy each have at least one of each type of coin. Jack has 4 coins totalling \$28, Jill has 5 coins worth \$21, and Jimmy has exactly 3 coins. What is the total value of Jimmy's coins?

[**3 marks**]

5. Triangle ABC has $AB = 90$, $BC = 50$, and $CA = 70$. A circle is drawn with centre P on AB such that CA and CB are tangents to the circle. Find $2AP$.

[**3 marks**]

6. In quadrilateral $PQRS$, $PS = 5$, $SR = 6$, $RQ = 4$, and $\angle P = \angle Q = 60^\circ$. Given that $2PQ = a + \sqrt{b}$, where a and b are unique positive integers, find the value of $a + b$.

[**4 marks**]

7. Dan has a jar containing a number of red and green sweets. If he selects a sweet at random, notes its colour, puts it back and then selects a second sweet, the probability that both are red is 105% of the probability that both are red if he eats the first sweet before selecting the second. What is the largest number of sweets that could be in the jar?

[**4 marks**]

8. Three circles, each of diameter 1, are drawn each tangential to the others. A square enclosing the three circles is drawn so that two adjacent sides of the square are tangents to one of the circles and the square is as small as possible. The side length of this square is $a + \dfrac{\sqrt{b} + \sqrt{c}}{12}$ where a, b, c are integers that are unique (except for swapping b and c). Find $a + b + c$.

[**4 marks**]

9. Ten points $P_1, P_2, \ldots, P_{10}$ are equally spaced around a circle. They are connected in separate pairs by 5 line segments. How many ways can such line segments be drawn so that only one pair of line segments intersect?

[**5 marks**]

10. *Ten-dig* is a game for two players. They try to make a 10-digit number with all its digits different. The first player, A, writes any non-zero digit. On the right of this digit, the second player, B, then writes a digit so that the 2-digit number formed is divisible by 2. They take turns to add a digit, always on the right, but when the nth digit is added, the number formed must be divisible by n. The game finishes when a 10-digit number is successfully made (in which case it is a *draw*) or the next player cannot legally place a digit (in which case the other player *wins*).

Show that there is only one way to reach a draw.

[**5 marks**]

Investigation

Show that if A starts with any non-zero even digit, then A can always win no matter how B responds.

[**4 bonus marks**]

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD SOLUTIONS

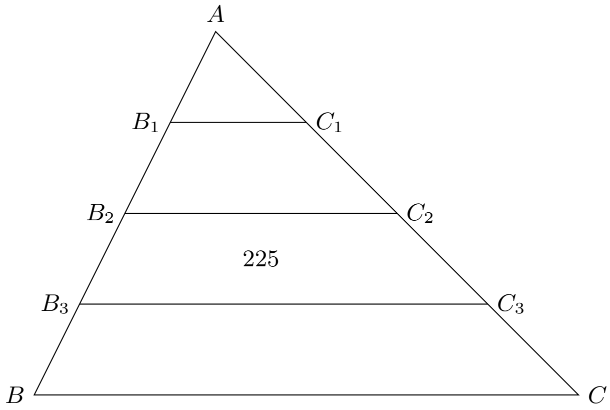
1. We have $x = b^2 + b + 1$ and $x = 2(b-2)^2 + (b-2) + 2 = 2(b^2 - 4b + 4) + b = 2b^2 - 7b + 8$. $\boxed{1}$

Hence $0 = (2b^2 - 7b + 8) - (b^2 + b + 1) = b^2 - 8b + 7 = (b-7)(b-1)$.

From the given information, $b - 2 > 2$. So $b = 7$ and $x = 49 + 7 + 1 = \mathbf{57}$. $\boxed{1}$

2. *Method 1*

Let B_1C_1, B_2C_2, B_3C_3, be the lines parallel to BC as shown. Then triangles ABC, AB_1C_1, AB_2C_2, AB_3C_3 are equiangular, hence similar. Region $B_3C_3C_2B_2$ has area 225.



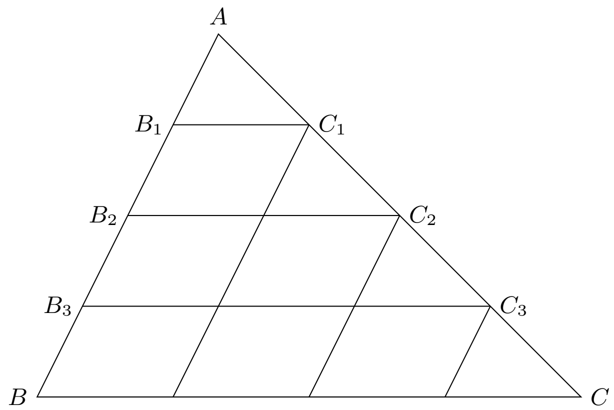
$\boxed{1}$

Since the lines divide AB into four equal segments, the sides and altitudes of the triangles are in the ratio 1:2:3:4. So their areas are in the ratio 1:4:9:16.

Let the area of triangle AB_1C_1 be x. Then $225 = |AB_3C_3| - |AB_2C_2| = 9x - 4x = 5x$ and the area of triangle ABC is $16x = 16 \times \frac{225}{5} = 16 \times 45 = \mathbf{720}$. $\boxed{1}$

Method 2

Let B_1C_1, B_2C_2, B_3C_3, be the lines parallel to BC. Draw lines parallel to AB as shown. This produces 4 small congruent triangles and 6 small congruent parallelograms.



$\boxed{1}$

Drawing the diagonal from top left to bottom right in any parallelogram produces two triangles that are congruent to the top triangle. Thus triangle ABC can be divided into 16 congruent triangles. The region $B_3C_3C_2B_2$ has area 225 and consists of 5 of these triangles. Hence $225 = \frac{5}{16} \times |ABC|$ and $|ABC| = \frac{16}{5} \times 225 = \mathbf{720}$. $\boxed{1}$

3. *Method 1*

Firstly, we note that the combined area of the 12 student cards is
$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144 = 650$. $\boxed{1}$
(Alternatively, use $1 + 2^3 + 3^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.)

According to Alice and Bob, $650 - x^2 = y^2$ for some integers x and y, where $1 \leq x \leq 12$.
So $y^2 \geq 650 - 144 = 506$ and $y^2 \leq 650 - 1 = 649$. Therefore $23 \leq y \leq 25$. $\boxed{1}$

If $y = 23$, then $x = 11$. If $y = 24$, then x is not an integer. If $y = 25$, then $x = 5$.

Thus $a = 5$ and $b = 11$ or vice versa. So $ab = 5 \times 11 = \mathbf{55}$. $\boxed{1}$

Method 2

Firstly, we note that the combined area of the 12 student cards is
$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + 121 + 144 = 650$. $\boxed{1}$
(Alternatively, use $1 + 2^3 + 3^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.)

According to Alice, $650 - a^2 = c^2$ for some integer c. Since 650 is even, a and c must both be even or odd. If a and c are even, then a^2 and c^2 are multiples of 4. But 650 is not a multiple of 4, so a and c are odd. $\boxed{1}$

We try odd values for a from 1 to 11.

$650 - 1^2 = 649$, which is not a perfect square.
$650 - 3^2 = 641$, which is not a perfect square.
$650 - 5^2 = 625$, which is 25^2, giving one of the solutions.
$650 - 7^2 = 601$, which is not a perfect square.
$650 - 9^2 = 569$, which is not a perfect square.
$650 - 11^2 = 529$, which is 23^2, giving the second solution.

Thus $a = 5$ and $b = 11$ or vice versa. So $ab = 5 \times 11 = \mathbf{55}$. $\boxed{1}$

4. *Method 1*

Let the value of the three types of coin be a, b, c and let Jack's collection be $2a + b + c = 28$. Then, swapping b with c if necessary, Jill's collection is one of:

$$3a + b + c, \quad 2a + 2b + c, \quad a + 2b + 2c, \quad a + 3b + c. \qquad \boxed{1}$$

Since $3a + b + c$ and $2a + 2b + c$ are greater than 28, Jill's collection is either $a + 2b + 2c$ or $a + 3b + c$. If $a + 2b + 2c = 21$, then adding $2a + b + c = 28$ gives $3(a + b + c) = 49$, which is impossible since 3 is not a factor of 49. $\boxed{1}$

So $a + 3b + c = 21$. Subtracting from $2a + b + c = 28$ gives $a = 2b + 7$, which means a is odd and at least 9. If $a = 9$, then $b = 1$ and $c = 9$. But a, b, c must be distinct, so a is at least 11. Since $b + c \geq 3$, we have $2a \leq 25$ and $a \leq 12$. Hence $a = 11$, $b = 2$, $c = 4$ and $a + b + c = \mathbf{17}$. $\boxed{1}$

Method 2

Let the value of the three types of coin be a, b, c. Then Jill's collection is one of:

$$2a + 2b + c, \quad 3a + b + c.$$

And Jack's collection is one of:

$$2a + b + c, \quad a + 2b + c, \quad a + b + 2c. \qquad \boxed{1}$$

Suppose Jill's collection is $2a+2b+c = 21$. Since $2a+b+c$ and $a+2b+c$ are less than $2a+2b+c$, Jack's collection must be $a + b + 2c = 28$. Adding this to Jill's yields $3(a + b + c) = 49$, which is impossible since 3 is not a factor of 49. $\boxed{1}$

So Jill's collection is $3a + b + c = 21$. Since $2a + b + c$ is less than $3a + b + c$, Jack's collection must be $a + 2b + c = 28$ or $a + b + 2c = 28$. Swapping b with c if necessary, we may assume that $a + 2b + c = 28$. Subtracting $3a + b + c = 21$ gives $b = 2a + 7$ and $c = 14 - 5a$. So $a \leq 2$. If $a = 1$, then $b = 9 = c$. Hence $a = 2$, $b = 11$, $c = 4$ and $a + b + c = \mathbf{17}$. $\boxed{1}$

Method 3

Let the value of the three types of coin be a, b, c, where $1 \leq a < b < c$. Then Jack's collection is one of:

$$2a + b + c, \quad a + 2b + c, \quad a + b + 2c.$$

And Jills' collection is one of:

$$3a + b + c, \quad 2a + 2b + c, \quad 2a + b + 2c, \quad a + 3b + c, \quad a + 2b + 2c, \quad a + b + 3c.$$

All of Jill's possible collections exceed $2a + b + c$, so Jack's collection is $a + 2b + c$ or $a + b + 2c$. All of Jill's possible collections exceed $a+2b+c$, except possibly for $3a+b+c$. If $3a+b+c = 21$, then subtracting from $a+2b+c = 28$ gives $b = 7+2a \geq 9$. But then $a+2b+c \geq 1+18+10 > 28$. $\boxed{1}$

So Jack's collection is $a+b+2c = 28$. Then $a+b$ is even, hence $b \geq 3$, $a+b \geq 4$, $2c = 28-a-b \leq 24$, and $c \leq 12$. Of Jill's possible collections, only $3a + b + c$, $2a + 2b + c$, and $a + 3b + c$ could be less than $a + b + 2c$. If $a + 3b + c = 21$, then subtracting from $a + b + 2c = 28$ gives $c = 7 + 2b$, which means $c \geq 13$. If $2a + 2b + c = 21$, then subtracting from $2a + 2b + 4c = 56$ gives $3c = 35$, which means c is a fraction. $\boxed{1}$

So $3a + b + c = 21$. Subtracting from $a + b + 2c = 28$ gives $c = 7 + 2a$, which means c is odd and at least 9. If $c = 9$, then $a = 1$ and $b = 9 = c$. So $c = 11$, $a = 2$, $b = 4$ and $a + b + c = \mathbf{17}$. $\boxed{1}$

Method 4

Let the value of the three types of coin be a, b, c, where $1 \le a < b < c$.

Then Jack's collection is $28 = a + b + c + d$ where d equals one of a, b, c. Since $a + b \ge 3$, $c + d \le 25$. So $d \le 25 - c \le 25 - d$. Then $2d \le 25$, hence $d \le 12$, which implies $a + b + c \ge 16$.

Jills' collection is $21 = a+b+c+e$ where e is the sum of two of a, b, c with repetition permitted. So $e \ge 2a \ge 2$. Hence $a + b + c \le 19$. $\boxed{1}$

From $a + b + c + d = 28$ and $16 \le a + b + c \le 19$, we get $9 \le d \le 12$.
If $d = a$, then $a + b + c + d > 4d \ge 36$. If $d = b$, then $a + b + c + d > 1 + 3d \ge 28$. So $d = c$.

From $21 = a + b + c + e \ge 16 + e \ge 16 + 2a$ we get $2a \le 5$, hence $a \le 2$. $\boxed{1}$

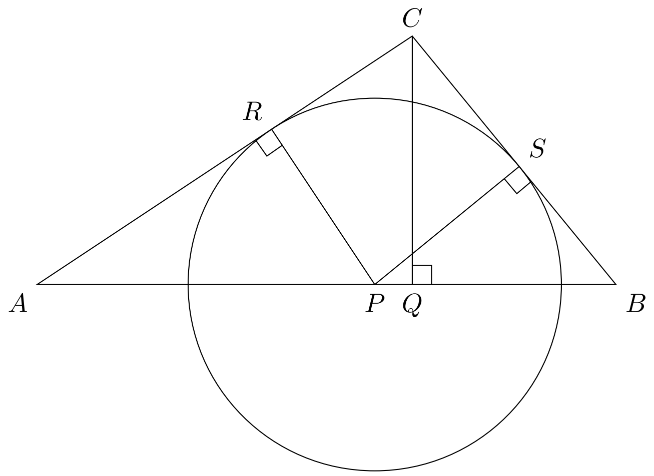
The following table lists all cases. Note that each of x and y equals one of a, b, c.

a	$a+b+c$	d	c	b	e	comment
1	16	12	12	3	5	$e \ne x + y$
1	17	11	11	5	4	$e \ne x + y$
1	18	10	10	7	3	$e \ne x + y$
1	19	9	9	9	2	$b = c$
2	16	12	12	2	5	$a = b$
2	17	11	11	4	4	$e = 2a$
2	18	10	10	6	3	$e \ne x + y$
2	19	9	9	8	2	$e \ne x + y$

So $a = 2$, $b = 4$, $c = 11$, $d = 11$, $e = 4$, and $a + b + c =$ **17**. $\boxed{1}$

5. *Method 1*

Let CA touch the circle at R and CB touch the circle at S. Let Q be a point on AB so that CQ and AB are perpendicular.



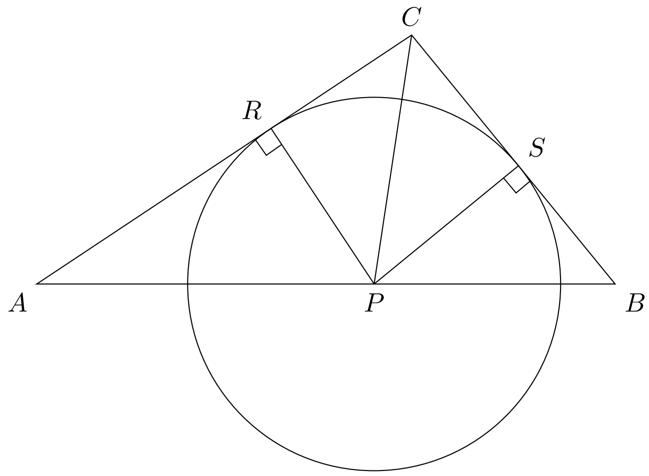
$\boxed{1}$

Let r be the radius of the circle.
From similar triangles AQC and ARP, $CQ/r = 70/AP$.
From similar triangles BQC and BSP, $CQ/r = 50/BP = 50/(90 - AP)$. $\boxed{1}$

Hence $7(90 - AP) = 5AP$, $630 = 12AP$, $2AP =$ **105**. $\boxed{1}$

Method 2

Let CA touch the circle at R and CB touch the circle at S.



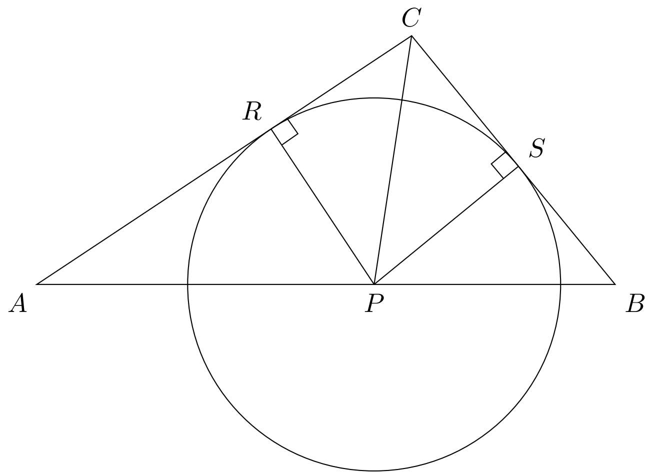
$\boxed{1}$

The radius of the circle is the height of triangle APC on base AC and the height of triangle BPC on base BC. So ratio of the area of APC to the area of BPC is $AC : PC = 7 : 5$. $\boxed{1}$

Triangles APC and BPC also have the same height on bases AP and BP. So the ratio of their areas is $AP : (90 - AP)$. Hence $5AP = 7(90 - AP)$, $12AP = 630$, and $2AP = \mathbf{105}$. $\boxed{1}$

Method 3

Let CA touch the circle at R and CB touch the circle at S.



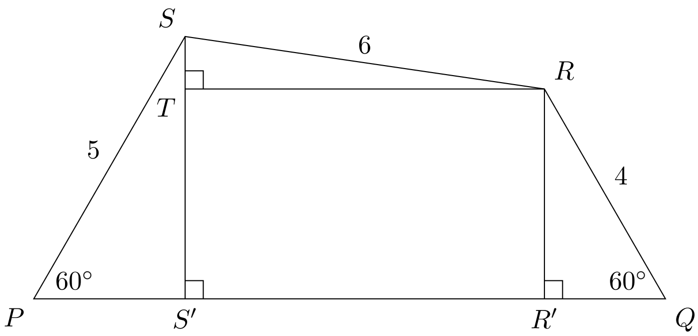
$\boxed{1}$

Since $PR = PS$, right-angled triangles PRC and PSC are congruent. Hence CP bisects $\angle ACB$. $\boxed{1}$

From the angle bisector theorem, $AP/PB = AC/BC = 7/5$. Hence $5AP = 7(90 - AP)$, $12AP = 630$, and $2AP = \mathbf{105}$. $\boxed{1}$

6. *Method 1*

Let SS' and RR' be perpendicular to PQ with S' and R' on PQ. Let RT be perpendicular to SS' with T on SS'.



$\boxed{1}$

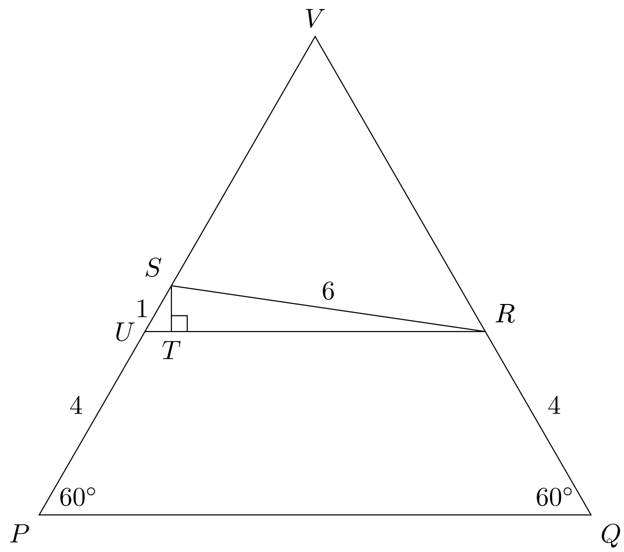
Since $\angle P = 60°$, $PS' = 5/2$ and $SS' = 5\sqrt{3}/2$.
Since $\angle Q = 60°$, $QR' = 2$ and $RR' = 2\sqrt{3}$. $\boxed{1}$

Hence $ST = SS' - TS' = SS' - RR' = 5\sqrt{3}/2 - 2\sqrt{3} = \sqrt{3}/2$.
Applying Pythagoras' theorem to $\triangle RTS$ gives $RT^2 = 36 - \frac{3}{4} = 141/4$. $\boxed{1}$

So $PQ = PS' + S'R' + R'Q = PS' + TR + R'Q = 5/2 + \sqrt{141}/2 + 2$.
Hence $a + \sqrt{b} = 2PQ = 9 + \sqrt{141}$. An obvious solution is $a = 9$, $b = 141$.
Given that a and b are unique, we have $a + b = \mathbf{150}$. $\boxed{1}$

Method 2

Let U be the point on PS so that UR is parallel to PQ. Let T be the point on RU so that ST is perpendicular to RU. Extend PS and QR to meet at V.



$\boxed{1}$

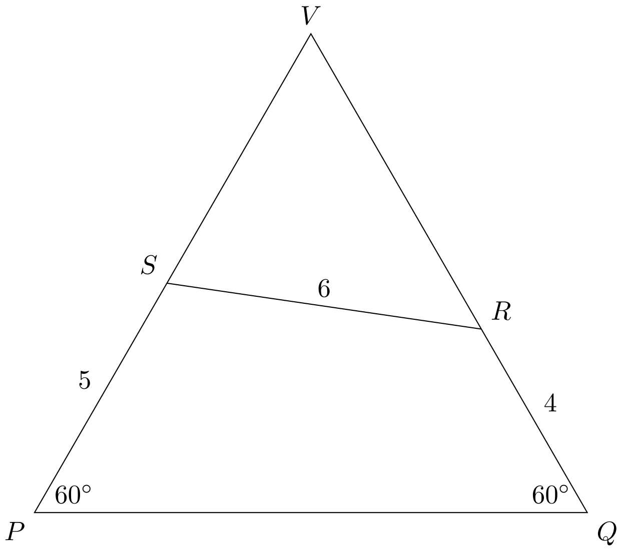
Triangle PQV is equilateral. Since $UR \parallel PQ$, $\triangle URV$ is equilateral and $PU = QR = 4$.
So $US = 1$, $UT = \frac{1}{2}$, $ST = \frac{\sqrt{3}}{2}$. $\boxed{1}$

Applying Pythagoras' theorem to $\triangle RTS$ gives $RT^2 = 36 - \frac{3}{4} = 141/4$. $\boxed{1}$

We also have $RT = RU - UT = RV - \frac{1}{2} = QV - \frac{9}{2} = PQ - \frac{9}{2}$.
So $2PQ = 9 + \sqrt{141} = a + \sqrt{b}$. Given that a and b are unique, we have $a + b = \mathbf{150}$. $\boxed{1}$

Method 3

Extend PS and QR to meet at V.



$\boxed{1}$

Triangle PQV is equilateral. Let $PQ = x$. Then $VS = x - 5$ and $VR = x - 4$. $\boxed{1}$

Applying the cosine rule to $\triangle RVS$ gives

$$\begin{aligned} 36 &= (x-4)^2 + (x-5)^2 - 2(x-4)(x-5)\cos 60^\circ \\ &= (x^2 - 8x + 16) + (x^2 - 10x + 25) - (x^2 - 9x + 20) \\ 0 &= x^2 - 9x - 15 \end{aligned} \qquad \boxed{1}$$

Hence $2x = 9 + \sqrt{81 + 60} = a + \sqrt{b}$. Given that a and b are unique, we have $a + b = \mathbf{150}$. $\boxed{1}$

Comment

We can prove that a and b are unique as follows. We have $(a-9)^2 = 141 + b - 2\sqrt{141b}$. So $2\sqrt{141b}$ is an integer, hence $141b$ is a perfect square. Since $141 = 3 \times 47$ and 3 and 47 are prime, $b = 141m^2$ for some integer m. Hence $|a - 9| = \sqrt{141}|m - 1|$. If neither side of this equation is 0, then we can rewrite it as $r = \sqrt{141}s$ where r and s are coprime integers, giving $r^2 = 141s^2 = 3 \times 47 \times s^2$. So 3 divides r^2. Then 3 divides r, 9 divides r^2, 9 divides $3s^2$, 3 divides s^2, hence 3 divides s, a contradiction. So both sides of the equation are 0. Therefore $a = 9$ and $b = 141$.

7. *Method 1*

Let there be r red sweets and g green sweets. We may assume $r \geq 2$. If Dan puts the first sweet back, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r}{r+g}. \qquad \boxed{1}$$

If Dan eats the first sweet, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r-1}{r+g-1}. \qquad \boxed{1}$$

The first probability is 105% of the second, so dividing and rearranging gives

$$\begin{aligned}
\frac{r}{r+g} \times \frac{r+g-1}{r-1} &= \frac{105}{100} = \frac{21}{20} \\
20\left(\frac{r+g-1}{r+g}\right) &= 21\left(\frac{r-1}{r}\right) \\
20\left(1-\frac{1}{r+g}\right) &= 21\left(1-\frac{1}{r}\right) \\
\frac{21}{r} &= 1+\frac{20}{r+g} > 1
\end{aligned} \qquad \boxed{1}$$

So $r < 21$. If $r = 20$, then $\frac{1}{20} = \frac{20}{r+g}$, and $r + g = 400$.
If $r + g$ increases, then $1 + \frac{20}{r+g}$ and therefore $\frac{21}{r}$ decrease, so r increases.
Since r cannot exceed 20, $r + g$ cannot exceed 400.
So the largest number of sweets in the jar is **400**. $\boxed{1}$

Method 2

Let there be r red sweets and g green sweets. We may assume $r \geq 2$. If Dan puts the first sweet back, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r}{r+g}. \qquad \boxed{1}$$

If Dan eats the first sweet, then the probability that the two selected sweets are red is

$$\frac{r}{r+g} \times \frac{r-1}{r+g-1}. \qquad \boxed{1}$$

The first probability is 105% of the second, so dividing and rearranging gives

$$\begin{aligned}
\frac{r}{r+g} \times \frac{r+g-1}{r-1} &= \frac{105}{100} = \frac{21}{20} \\
20r(r+g-1) &= 21(r+g)(r-1) \\
20r(r+g) - 20r &= 21r(r+g) - 21(r+g) \\
r + 21g &= r(r+g) \\
r + g &= 1 + 21g/r
\end{aligned} \qquad \boxed{1}$$

If $r \geq 21$, then $r + g \geq 21 + g$ and $1 + 21g/r \leq 1 + g$, a contradiction. So $r \leq 20$.

If $r = 20$, then $20 + g = 1 + 21g/20$, hence $g = 400 - 20 = 380$ and $r + g = 400$.

We also have the equation $(21 - r)g = r(r - 1)$.
If $r < 20$, then $g < (21 - r)g = r(r - 1) < 20 \times 19 = 380$, hence $r + g < 20 + 380 = 400$.
So the largest number of sweets in the jar is **400**. $\boxed{1}$

Method 3

Let there be r red sweets and g green sweets. We may assume $r \geq 2$. Let $n = r + g$. Then the probability of selecting two red sweets if the first sweet is put back is

$$\frac{r}{n} \times \frac{r}{n} \qquad \boxed{1}$$

and the probability if Dan eats the first sweet before selecting the second is

$$\frac{r}{n} \times \frac{r-1}{n-1}. \qquad \boxed{1}$$

The first probability is 105% of the second, so dividing and rearranging gives

$$\begin{aligned} \frac{r}{n} \times \frac{n-1}{r-1} &= \frac{105}{100} = \frac{21}{20} \\ 20r(n-1) &= 21n(r-1) \\ 21n - nr - 20r &= 0 \\ (n+20)(21-r) &= 420 \qquad \boxed{1} \end{aligned}$$

Since $n + 20$ is positive, $21 - r$ is positive.
Hence n is largest when $21 - r = 1$ and then $n + 20 = 420$.
So the largest number of sweets in the jar is **400**. $\boxed{1}$

Comment

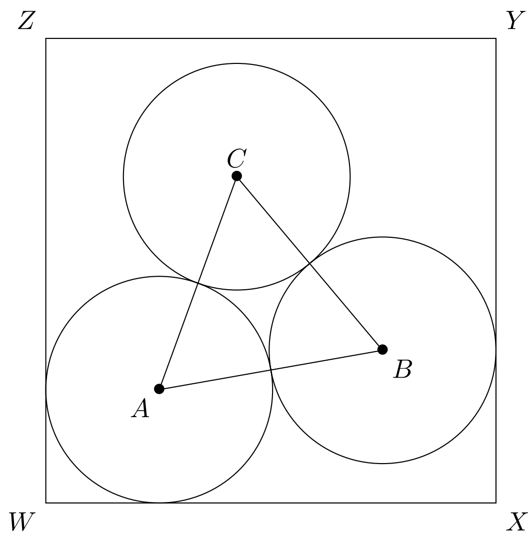
Since $21 - r$ is a factor of 420 and $2 \leq r \leq 20$, the following table gives all possible values of r, n, g.

$21 - r$	$n + 20$	r	n	g
1	420	20	400	380
2	210	19	190	171
3	140	18	120	102
4	105	17	85	68
5	84	16	64	48
6	70	15	50	35
7	60	14	40	26
10	42	11	22	11
12	35	9	15	6
14	30	7	10	3
15	28	6	8	2

.

8. Let $WXYZ$ be a square that encloses the three circles and is as small as possible. Let the centres of the three given circles be A, B, C. Then ABC is an equilateral triangle of side length 1. We may assume that A, B, C are arranged anticlockwise and that the circle with centre A touches WX and WZ. We may also assume that WX is horizontal.

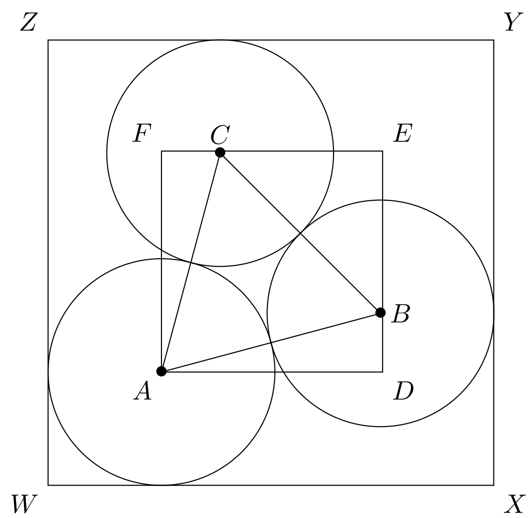
Note that if neither YX nor YZ touch a circle, then the square can be contracted by moving Y along the diagonal WY towards W. So at least one of YX and YZ must touch a circle and it can't be the circle with centre A. We may assume that XY touches the circle with centre B.



If YZ does not touch a circle, then the 3-circle cluster can be rotated anticlockwise about A allowing neither YX nor YZ to touch a circle. So YZ touches the circle with centre C. $\boxed{1}$

Method 1

Let $ADEF$ be the rectangle with sides through C and B parallel to WX and WZ respectively.



$\boxed{1}$

Since $AF = WZ - 1 = WX - 1 = AD$, $ADEF$ is a square.
Since $AC = 1 = AB$, triangles AFC and ADB are congruent. So $FC = DB$ and $CE = BE$.
Let $x = AD$. Since $AB = 1$ and triangle ADB is right-angled, $DB = \sqrt{1 - x^2}$.

Since CBE is right-angled isosceles with $BC = 1$, we have $BE = 1/\sqrt{2}$.
So $x = DE = \sqrt{1 - x^2} + 1/\sqrt{2}$. $\boxed{1}$

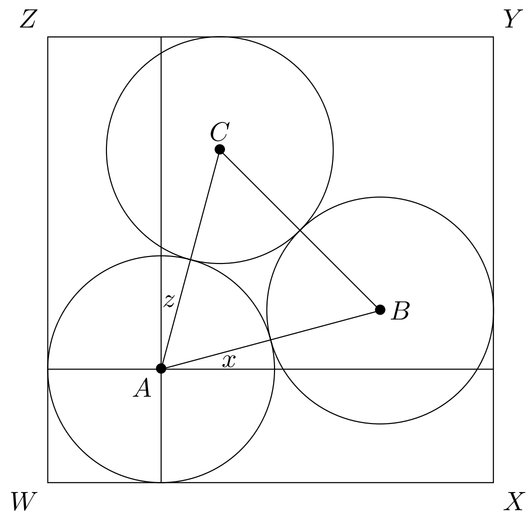
Squaring both sides of $x - 1/\sqrt{2} = \sqrt{1 - x^2}$ gives

$$\begin{aligned} 1 - x^2 = (x - 1/\sqrt{2})^2 &= x^2 - \sqrt{2}x + 1/2 \\ 0 &= 2x^2 - \sqrt{2}x - 1/2 \\ x &= (\sqrt{2} \pm \sqrt{2+4})/4 \end{aligned}$$

Since $x > 0$, we have $x = (\sqrt{2} + \sqrt{6})/4 = (\sqrt{18} + \sqrt{54})/12$. Hence $WX = 1 + (\sqrt{18} + \sqrt{54})/12$. We are told that $WX = a + (\sqrt{b} + \sqrt{c})/12$ where a, b, c are unique integers. This gives $a + b + c = 1 + 18 + 54 = \mathbf{73}$. $\boxed{1}$

Method 2

Draw lines through A parallel to WX and WZ.



$\boxed{1}$

With angles x and z as shown, we have

$$\begin{aligned} WX &= \frac{1}{2} + AB\cos x + \frac{1}{2} = 1 + \cos x \\ WZ &= \frac{1}{2} + AC\cos z + \frac{1}{2} = 1 + \cos z \end{aligned}$$

Since $WX = WZ$, $x = z$. Since $x + 60^\circ + z = 90^\circ$, we have $x = 15^\circ$. So $\boxed{1}$

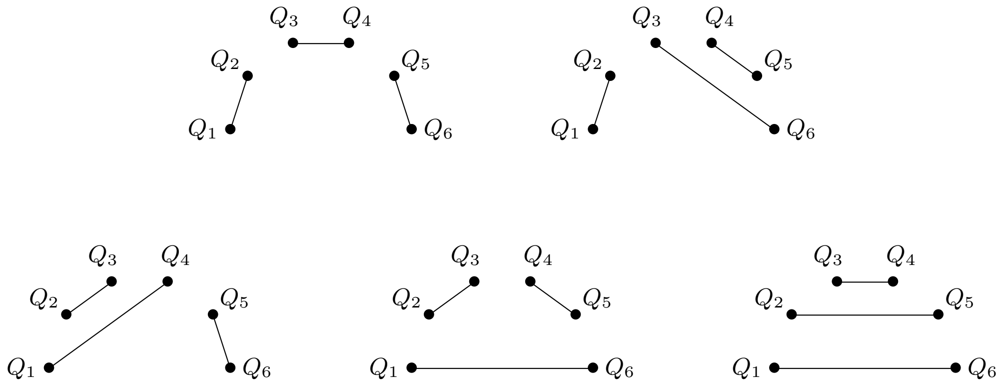
$$\begin{aligned} WX &= 1 + \cos 15^\circ = 1 + \cos(45^\circ - 30^\circ) \\ &= 1 + \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\ &= 1 + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} \\ &= 1 + \frac{1 + \sqrt{3}}{2\sqrt{2}} = 1 + \frac{\sqrt{2} + \sqrt{6}}{4} = 1 + \frac{\sqrt{18} + \sqrt{54}}{12} \end{aligned}$$

We are told that $WX = a + (\sqrt{b} + \sqrt{c})/12$ where a, b, c are unique integers. This gives $a + b + c = 1 + 18 + 54 = \mathbf{73}$. $\boxed{1}$

9. *Method 1*

Let the pair of intersecting lines be AC and BD where A, B, C, D are four of the ten given points. These lines split the remaining six points into four subsets S_1, S_2, S_3, S_4. For each i, each line segment beginning in S_i also ends in S_i, otherwise AC and BD would not be the only intersecting pair of lines. Thus each S_i contains an even number of points, from 0 to 6. $\boxed{1}$

If S_i contains 2 points, then there it has only 1 line segment. If S_i contains 4 points, then there are precisely 2 ways to connect its points in pairs by non-crossing segments. If S_i contains 6 points, let the points be $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ in clockwise order. To avoid crossing segments, Q_1 must be connected to one of Q_2, Q_4, Q_6. So, as shown, there are precisely 5 ways to connect the six points in pairs by non-crossing segments.



$\boxed{1}$

In some order, the sizes of S_1, S_2, S_3, S_4 are $\{6, 0, 0, 0\}$, $\{4, 2, 0, 0\}$, or $\{2, 2, 2, 0\}$. We consider the three cases separately.

In the first case, by rotation about the circle, there are 10 ways to place the S_i that has 6 points. Then there are 5 ways to arrange the line segments within that S_i. So the number of ways to draw the line segments in this case is $10 \times 5 = 50$. $\boxed{1}$

In the second case, in clockwise order, the sizes of the S_i must be (4, 2, 0, 0), (4, 0, 2, 0) or (4, 0, 0, 2). In each case, by rotation about the circle, there are 10 ways to place the S_i. Then there are 2 ways to arrange line segments within the S_i that has 4 points, and there is 1 way to arrange the line segment within the S_i that has 2 points. So the number of ways to draw the line segments in this case is $3 \times 10 \times 2 \times 1 = 60$. $\boxed{1}$

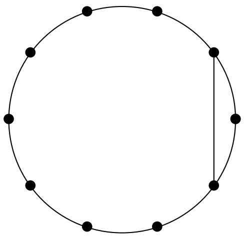
In the third case, in clockwise order, the sizes of the S_i must be (2, 2, 2, 0). By rotation about the circle, there are 10 ways to place the S_i. Then there is only 1 way to arrange the line segment within each S_i that has 2 points. So there are 10 ways to arrange the line segments in this case.

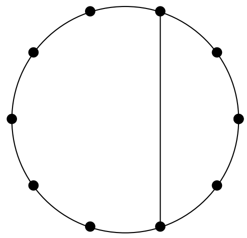
In total, the number of ways to arrange the line segments is $50 + 60 + 10 = \mathbf{120}$. $\boxed{1}$

Method 2

The pair of intersecting lines partition the circle into four arcs. In order to allow the remaining points to be paired up without further crossings, we require each such arc to contain an even number of points. $\boxed{1}$

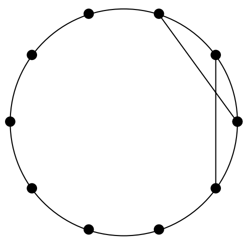
So each line of a crossing pair partitions the circle into two arcs, each of which contain an odd number of points. Disregarding rotation of the circle, a crossing line is one of only two types.



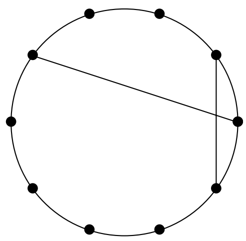


$\boxed{1}$

So, disregarding rotations, there are only four ways to have the pair of crossing lines. Underneath each diagram we list the number of ways of joining up the remaining pairs of points without introducing more crossings. (The number 5 is justified in Method 1.)



5



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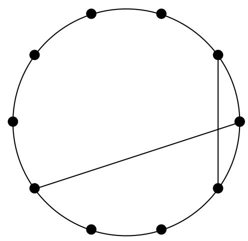

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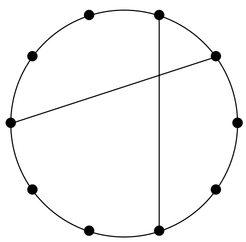
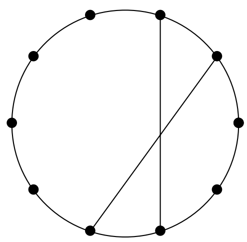


1



2

$\boxed{2}$

So, counting rotations, the number of pairings with a single crossing is
$10 \times (5 + 2 + 2 + 1 + 2) = \mathbf{120}$. $\boxed{1}$

10. Since the 2nd, 4th, 6th, 8th and 10th digits must be even, the other digits must be odd. Since the last digit must be 0, the fifth digit must be 5. $\boxed{1}$

Let a be the 3rd digit and b be the 4th digit. If b is 4 or 8, then 4 divides b but does not divide $10a$ since a is odd. Hence 4 does not divide $10a + b$. So the 4th digit is 2 or 6.

Now let a, b, c be the 6th, 7th, 8th digits respectively. If c is 8, then 8 divides $100a + c$ but does not divide $10b$ since b is odd. Hence 8 does not divide $100a + 10b + c$. If c is 4, then 8 divides $100a$ but does not divide $10b + c = 2(5b + 2)$ since b is odd. Hence 8 does not divide $100a + 10b + c$. So the 8th digit is 2 or 6.

So each of the 2nd and 6th digits is 4 or 8. $\boxed{1}$

Since 3 divides the sum of the first three digits and the sum of the first six digits, it also divides the sum of the 4th, 5th, and 6th digits. So the 4th, 5th, and 6th digits are respectively 2 5 8 or 6 5 4. Thus we have two cases with a, b, c, d equal to 1, 3, 7, 9 in some order. $\boxed{1}$

Case 1. a 4 b 2 5 8 c 6 d 0

Since 3 divides $a + 4 + b$, one of a and b equals 1 and the other is 7. Since 8 divides 8 c 6, c is 9. So we have 1 4 7 2 5 8 9 6 d 0 or 7 4 1 2 5 8 9 6 d 0. But neither 1 4 7 2 5 8 9 nor 7 4 1 2 5 8 9 is a multiple of 7. $\boxed{1}$

Case 2. a 8 b 6 5 4 c 2 d 0

Since 8 divides 4 c 2, c is 3 or 7.

If $c = 3$, then, because 3 divides $a + 8 + b$, we have one of:

1 8 9 6 5 4 3 2 d 0, 7 8 9 6 5 4 3 2 d 0, 9 8 1 6 5 4 3 2 d 0, 9 8 7 6 5 4 3 2 d 0.

But none of 1 8 9 6 5 4 3, 7 8 9 6 5 4 3, 9 8 1 6 5 4 3, 9 8 7 6 5 4 3 is a multiple of 7.

If $c = 7$, then, because 3 divides $a + 8 + b$, we have one of:

1 8 3 6 5 4 7 2 d 0, 1 8 9 6 5 4 7 2 d 0, 3 8 1 6 5 4 7 2 d 0, 9 8 1 6 5 4 7 2 d 0.

None of 1 8 3 6 5 4 7, 1 8 9 6 5 4 7, 9 8 1 6 5 4 7 is a multiple of 7.

This leaves 3 8 1 6 5 4 7 2 9 0 as the only draw. $\boxed{1}$

Investigation

Note that B must play an even digit on each turn.

If A starts with 2, then B can only respond with 20, 24, 26, or 28. A may then leave one of 204, 240, 261, 285. B cannot respond to 261. The other numbers force respectively 20485, 24085, 28560. B cannot respond to any of these. $\boxed{\text{bonus 1}}$

If A starts with 4, then B can only respond with 40, 42, 46, or 48. A may then leave one of 408, 420, 462, 480. B cannot respond to 408 and 480. Each of the other numbers force one of 42085, 46205, 46280, 46285. B cannot respond to any of these. $\boxed{\text{bonus 1}}$

If A starts with 6, then B can only respond with 60, 62, 64, or 68. A may then leave one of 609, 621, 648, 684. B cannot respond to 621. The other numbers force respectively 60925, 64805, 68405. B can only respond with 609258. Then A may reply with 6092583, to which B has no response. $\boxed{\text{bonus 1}}$

If A starts with 8, then B can only respond with 80, 82, 84, or 86. A may then leave one of 804, 825, 840, 864. B cannot respond to 804 and 840. The other numbers force respectively 82560, 86405. B cannot respond to either of these. $\boxed{\text{bonus 1}}$

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD STATISTICS

Distribution of Awards/School Year

Year	Number of Students	Number of Awards				
		Prize	High Distinction	Distinction	Credit	Participation
8	531	5	24	76	146	280
9	649	16	71	130	159	273
10	529	25	92	116	141	155
Other	542	2	9	44	117	370
All Years	2251	48	196	366	563	1078

Number of Correct Answers Questions 1–8

Year	Number Correct/Question							
	1	2	3	4	5	6	7	8
8	263	277	378	358	34	19	36	6
9	390	361	484	490	117	88	84	28
10	324	301	451	427	81	98	112	43
Other	250	206	287	306	22	12	16	5
All Years	1227	1145	1600	1581	254	217	248	82

Mean Score/Question/School Year

Year	Number of Students	Mean Score			Overall Mean
		Question			
		1–8	9	10	
8	531	7.8	0.4	0.8	8.6
9	649	9.7	0.6	1.1	10.9
10	529	11.5	0.7	1.2	12.9
Other	542	5.8	0.3	0.5	6.4
All Years	2251	8.7	0.5	0.9	9.7

2017 AMOC SENIOR CONTEST

Tuesday, 8 August 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. For each pair of real numbers (r, s), prove that there exists a real number x that satisfies at least one of the following two equations.

$$x^2 + (r+1)x + s = 0$$
$$rx^2 + 2sx + s = 0$$

2. Let $ABCD$ be a quadrilateral with AB not parallel to CD. The circle with diameter AB is tangent to the side CD at X. The circle with diameter CD is tangent to the side AB at Y.

Prove that the quadrilateral $BCXY$ is cyclic.

3. Let $a_1 < a_2 < \cdots < a_{2017}$ and $b_1 < b_2 < \cdots < b_{2017}$ be positive integers such that

$$\left(2^{a_1}+1\right)\left(2^{a_2}+1\right)\cdots\left(2^{a_{2017}}+1\right) = \left(2^{b_1}+1\right)\left(2^{b_2}+1\right)\cdots\left(2^{b_{2017}}+1\right).$$

Prove that $a_i = b_i$ for $i = 1, 2, \ldots, 2017$.

4. Find all positive integers $n \geq 5$ for which we can place a real number at each vertex of a regular n-sided polygon, such that the following two conditions are satisfied.

- None of the n numbers is equal to 1.
- For each vertex of the polygon, the sum of the numbers at the nearest four vertices is equal to 4.

5. Let n be a positive integer. Consider $2n$ points equally spaced around a circle. Suppose that n of the points are coloured blue and the remaining n points are coloured red. We write down the distance between each pair of blue points in a list, from shortest to longest. We write down the distance between each pair of red points in another list, from shortest to longest. (Note that the same distance may occur more than once in a list.)

Prove that the two lists of distances are the same.

AMOC SENIOR CONTEST SOLUTIONS

1. For each pair of real numbers (r, s), prove that there exists a real number x that satisfies at least one of the following two equations.

$$x^2 + (r+1)x + s = 0$$
$$rx^2 + 2sx + s = 0$$

Solution 1 (Norman Do)

In order to obtain a contradiction, suppose that there does not exist a real number x that satisfies at least one of the two equations. The discriminants of the two quadratic equations are $(r+1)^2 - 4s$ and $4s^2 - 4rs$, respectively. Therefore, we have

$$(r+1)^2 - 4s < 0 \qquad \text{and} \qquad 4s^2 - 4rs < 0.$$

Adding these two inequalities, we obtain

$$(r+1)^2 - 4s + 4s^2 - 4rs < 0 \qquad \Rightarrow \qquad (r+1-2s)^2 < 0.$$

Since the square of a real number cannot be negative, this yields a contradiction. It follows that there must exist a real number x that satisfies at least one of the two equations.

Solution 2 (Alice Devillers, Angelo Di Pasquale, Ivan Guo, Dan Mathews, Chaitanya Rao and Ian Wanless)

If $s \leq 0$, then the discriminant $(r+1)^2 - 4s$ of the first quadratic equation is a sum of two non-negative numbers. Hence, it is non-negative and the first equation has a real solution.

If $s > 0$ and $s \geq r$, then the discriminant $4s^2 - 4rs$ of the second quadratic equation is non-negative. Hence, the second equation has a real solution.

The only case left to consider is $0 < s < r$. Then the discriminant of the first quadratic equation is

$$(r+1)^2 - 4s > (s+1)^2 - 4s = (s-1)^2 \geq 0.$$

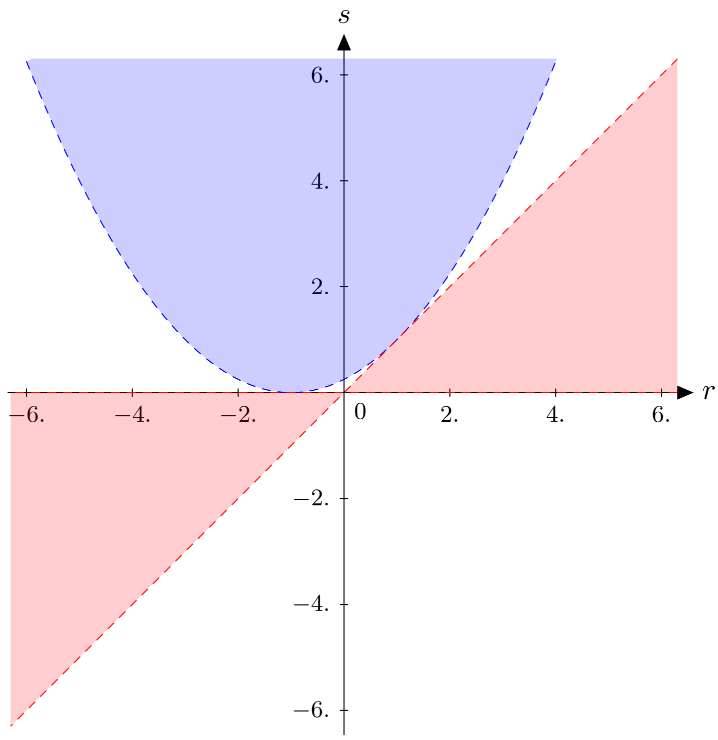
Hence, the first equation has a real solution.

Solution 3 (Angelo Di Pasquale)

As in Solution 1, we wish to show that there do not exist real numbers r and s such that

$$(r+1)^2 - 4s < 0 \qquad \text{and} \qquad 4s^2 - 4rs < 0.$$

One can simply observe this from the graphs of these two inequalities, which are shown in the figure below.



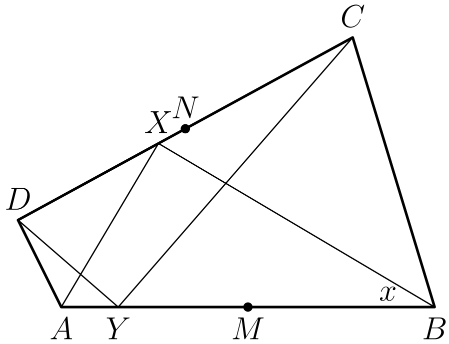
The parabola is clearly tangent to the r-axis. It only remains to see that it is tangent to the line $s = r$. Computing the intersection of these two curves, we see that there is only one intersection point and it occurs at $(1, 1)$. Since the line $s = r$ is not parallel to the s-axis, it follows that it must be tangent to the parabola.

2. Let $ABCD$ be a quadrilateral with AB not parallel to CD. The circle with diameter AB is tangent to the side CD at X. The circle with diameter CD is tangent to the side AB at Y.

Prove that the quadrilateral $BCXY$ is cyclic.

Solution 1 (Norman Do)

Let M and N be the midpoints of AB and CD, respectively. Then MX is perpendicular to CD, since MX is the radius of a circle to which CD is tangent. Similarly, NY is perpendicular to AB. It follows that the points M, N, X, Y lie on a circle.



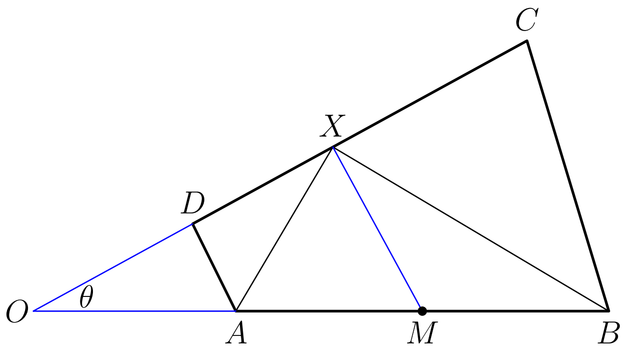
Let $\angle ABX = x$ and note that $\angle AMX = 2x$, since it is the angle subtended at the centre of the circumcircle of triangle ABX. It follows that $\angle YNX = \angle YMX = \angle AMX = 2x$, where we have used the fact that $MNXY$ is a cyclic quadrilateral.

So $\angle CNY = 180° - \angle YNX = 180° - 2x$. However, note that triangle CNY is isosceles with $CN = NY$. Therefore, $\angle NCY = \angle NYC = x$. Since $\angle XCY = \angle XBY = x$, we have deduced that the quadrilateral $BCXY$ is cyclic.

A second case arises when X and Y are on different sides of the line MN, in which case we have the equality $\angle YNX = 180° - \angle YMX$ rather than $\angle YNX = \angle YMX$. This can be handled in an analogous manner or with the use of directed angles.

Solution 2 (Norman Do)

Suppose that the lines AB and CD meet at O, and let $\angle AOD = \theta$. If M is the midpoint of AB, then the angle sum in right-angled triangle MXO yields $\angle XMO = 90° - \theta$. Therefore, the angle subtended by AX in the circle with diameter AB is $\angle ABX = \frac{1}{2}\angle AMX = 45° - \frac{\theta}{2}$.



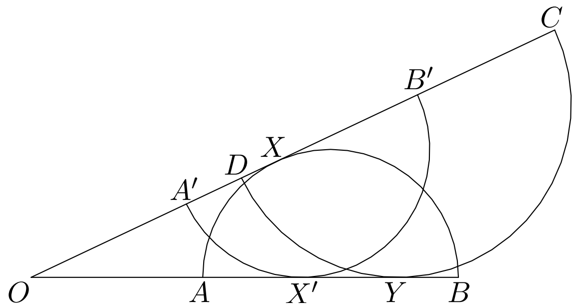
An analogous calculation using the circle with diameter CD yields $\angle DCY = 45^\circ - \frac{\theta}{2}$ as well. Since $\angle XBY = \angle XCY = 45^\circ - \frac{\theta}{2}$, we have deduced that the quadrilateral $BCXY$ is cyclic.

Solution 3 (Angelo Di Pasquale)

Suppose that the lines AB and CD meet at O. The reflection of the semicircle AXB in the bisector of $\angle AOD$, results in a semicircle $A'X'B'$, as shown in the diagram below. This semicircle is a dilation of the semicircle DYC with centre of dilation O. So using the fact that $OB' = OB$ and $OX' = OX$, we find that

$$\frac{OB'}{OC} = \frac{OX'}{OY} \quad \Rightarrow \quad \frac{OB}{OC} = \frac{OX}{OY} \quad \Rightarrow \quad OB \cdot OY = OC \cdot OX.$$

By the power of a point theorem, this implies that the quadrilateral $BCXY$ is cyclic.



Solution 4 (Angelo Di Pasquale)

Suppose that the lines AB and CD meet at a point O. There is an orientation-reversing similarity transformation that sends semicircle AXB to semicircle DYC. Since AB is not parallel to CD, it is the composition of the reflection in the bisector of $\angle AOD$ followed by the dilation by factor $\frac{DC}{AB}$ with centre O.

This implies that triangle ABX is similar to triangle DCY. Hence, $\angle OBX = \angle OCY$ and it follows that the quadrilateral $BCXY$ is cyclic.

Solution 5 (Ivan Guo)

Suppose that BX and CY intersect at Z. By considering the angle sum in triangles BYZ and CXZ, we have $\angle BXC + \angle YCX = \angle CYB + \angle XBY$. Now we invoke the alternate segment theorem to rewrite this as $\angle XAB + \angle YCX = \angle YDC + \angle XBY$. Using the fact that AB and CD are diameters, we have $90^\circ - \angle XBA + \angle YCX = 90^\circ - \angle YCD + \angle XBY$, which simplifies to $\angle YCX = \angle XBY$. Therefore, the quadrilateral $BCXY$ is cyclic.

Solution 6 (Daniel Mathews)

Let $\angle CXB = a$, $\angle XBY = b$, $\angle XCY = c$ and $\angle CYB = d$. We will show that $a = d$ and $b = c$, from which it follows that the quadrilateral $BCXY$ is cyclic.

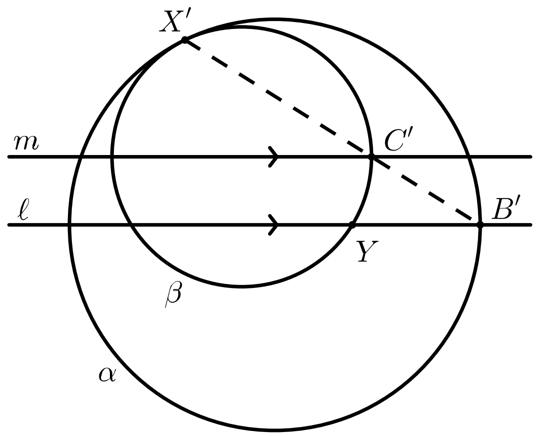
Since $180° - a - c = 180° - b - d$ is the angle between BX and CY, we have $a + c = b + d$. We have $a = \angle CXB = \angle XAB$ by the alternate segment theorem, and we also have $b = \angle XBY = 90° - \angle XAB$ from the right-angled triangle ABX. Therefore, $a + b = 90°$.

By the same argument, we have $d = \angle CYB = \angle CDY$ by the alternate segment theorem, and we also have $c = \angle XCY = 90° - \angle CDY$ from the right-angled triangle CDY. Therefore, $c + d = 90°$.

We now have $a + b = 90°$, $c + d = 90°$ and $a + c = b + d$, from which it follows that $a + c = b + d = 90°$. Hence, $a = 90° - b = d$ and $b = 90° - a = c$, giving the desired result.

Solution 7 (Alan Offer)

Consider the effect of an inversion about a circle centred at Y: the line ℓ through A and B is fixed; the circle with diameter AB maps to a circle α with centre on ℓ; the circle with diameter CD maps to a line m parallel to ℓ; the line through C and D maps to a circle β through Y with centre on m and internally tangent to α at X', the image of X; and the images B' and C' of B and C, respectively, lie in the same direction from X' on their respective circles.



Now circle α is related to β by a dilation about X', which maps m through the centre of β to the parallel line ℓ through the centre of α, so C' is mapped to B'. Hence, B', C' and X' are collinear, which under the inversion reveals that $BCXY$ is cyclic.

3. Let $a_1 < a_2 < \cdots < a_{2017}$ and $b_1 < b_2 < \cdots < b_{2017}$ be positive integers such that

$$\left(2^{a_1}+1\right)\left(2^{a_2}+1\right)\cdots\left(2^{a_{2017}}+1\right) = \left(2^{b_1}+1\right)\left(2^{b_2}+1\right)\cdots\left(2^{b_{2017}}+1\right).$$

Prove that $a_i = b_i$ for $i = 1, 2, \ldots, 2017$.

Solution 1 (Norman Do)

Suppose that there is some $i \in \{1, 2, \ldots, 2017\}$ for which $a_i \neq b_i$. Then if we cancel out equal factors on both sides of the equation, we obtain an equation of the form

$$(2^{A_1}+1)(2^{A_2}+1)\cdots(2^{A_n}+1) = (2^{B_1}+1)(2^{B_2}+1)\cdots(2^{B_n}+1),$$

where we may assume that $A_1 < A_2 < \cdots < A_n$, $B_1 < B_2 < \cdots < B_n$ and $A_1 < B_1$, without loss of generality.

Expanding both sides of the equation yields an equation of the form

$$1 + 2^{A_1} + [\,\text{higher powers of } 2\,] = 1 + 2^{B_1} + [\,\text{higher powers of } 2\,],$$

from which we obtain

$$2^{A_1} + [\,\text{higher powers of } 2\,] = 2^{B_1} + [\,\text{higher powers of } 2\,].$$

However, note that 2^{B_1} divides the right hand side but not the left hand side, which yields the desired contradiction. It follows that $a_i = b_i$ for $i = 1, 2, \ldots, 2017$.

Solution 2 (Alice Devillers, Dan Mathews and Kevin McAvaney)

We will prove the following statement for all positive integers n by induction. If $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$ are positive integers such that

$$\left(2^{a_1}+1\right)\left(2^{a_2}+1\right)\cdots\left(2^{a_n}+1\right) = \left(2^{b_1}+1\right)\left(2^{b_2}+1\right)\cdots\left(2^{b_n}+1\right),$$

then $a_i = b_i$ for $i = 1, 2, \ldots, n$.

The statement is clearly true for $n = 1$, since $2^{a_1} + 1 = 2^{b_1} + 1$ implies that $a_1 = b_1$.

Now assume that the statement is true for $n = k-1$ where $k \geq 2$ is an integer and consider the case $n = k$. Suppose that $a_1 < a_2 < \ldots < a_k$ and $b_1 < b_2 < \ldots < b_k$ are positive integers such that

$$\left(2^{a_1}+1\right)\left(2^{a_2}+1\right)\cdots\left(2^{a_k}+1\right) = \left(2^{b_1}+1\right)\left(2^{b_2}+1\right)\cdots\left(2^{b_k}+1\right).$$

If $a_1 \neq b_1$, we may assume without loss of generality that $a_1 < b_1$. We know that $a_i \geq a_1+1$ for all $2 \leq i \leq k$ and $b_j \geq a_1 + 1$ for all $1 \leq j \leq k$. Now consider the equation above modulo 2^{a_1+1}. The left side is $2^{a_1} + 1$, while the right side is 1, which yields the desired contradiction.

So we have deduced that $a_1 = b_1$ and hence,

$$\left(2^{a_2}+1\right)\left(2^{a_3}+1\right)\cdots\left(2^{a_k}+1\right) = \left(2^{b_2}+1\right)\left(2^{b_3}+1\right)\cdots\left(2^{b_k}+1\right)$$

with $a_2 < a_3 < \cdots < a_k$ and $b_2 < b_3 < \cdots < b_k$. By the induction hypothesis, we know that $a_i = b_i$ for all $2 \leq i \leq k$. This completes the proof of the statement by induction and we recover the original problem in the case $n = 2017$.

4. Find all positive integers $n \geq 5$ for which we can place a real number at each vertex of a regular n-sided polygon, such that the following two conditions are satisfied.

   - None of the n numbers is equal to 1.
   - For each vertex of the polygon, the sum of the numbers at the nearest four vertices is equal to 4.

Solution 1 (Angelo Di Pasquale)

The answer is any even $n \geq 6$.

If n is even, the conditions are clearly satisfied if we alternate $0.5, 1.5, 0.5, 1.5, \ldots$ around the polygon.

Now suppose that $n = 2m+1$ is odd and let the numbers be $x_1, x_2, \ldots, x_n$ in order around the polygon. Here and throughout this proof, we consider the subscripts modulo n. Then for each i, we have

$$\begin{aligned} & x_i + x_{i+1} + x_{i+3} + x_{i+4} = x_{i+1} + x_{i+2} + x_{i+4} + x_{i+5} \\ \Rightarrow \quad & x_i + x_{i+3} = x_{i+2} + x_{i+5}. \end{aligned} \tag{$*$}$$

For each i, let $A_i = x_i + x_{i+3}$. Then equation (*) may be written as $A_i = A_{i+2}$. Thus, the sequence $A_1, A_2, A_3, \ldots$ has period 2. However, it also has period n and hence, it has period $\gcd(2, n) = 1$. In particular, we have $A_i = A_{i+3}$, which implies that $x_i = x_{i+6}$.

So we have deduced that the sequence x_1, x_2, x_3, . . . has period 6. However, it also has period n and hence, it has period $\gcd(6, n) = 1$ or 3. In either case, equation (*) simplifies to $x_i = x_{i+2}$. So the sequence x_1, x_2, x_3, . . . has period 2. However, it also has period n and hence, it has period $\gcd(2, n) = 1$. It follows that all of the x_i are equal to 1, which yields the desired contradiction.

Solution 2 (Dan Mathews and Ian Wanless)

The answer is any even $n \geq 6$.

If n is even, the conditions are clearly satisfied if we alternate $0.5, 1.5, 0.5, 1.5, \ldots$ around the polygon.

Now suppose that $n = 2m+1$ is odd and let the numbers be $x_1, x_2, \ldots, x_n$ in order around the polygon. Here and throughout this proof, we consider the subscripts modulo n.

Define $y_i = x_i + x_{i+1}$ and observe that by the given conditions, we have $y_i = 4 - y_{i+3}$ for each i. By repeated application of this rule, we can deduce that $y_i = 4 - y_{i+3n}$, since n is odd. However, we have $y_{i+3n} = y_i$, so it follows that $y_i = 2$ for all i.

Hence, $x_i + x_{i+1} = 2 = x_{i+1} + x_{i+2}$, from which we deduce that $x_i = x_{i+2}$. So the sequence $x_1, x_2, x_3, \ldots$ has period 2. However, it also has period n and hence, it has period $\gcd(2, n) = 1$. It follows that all of the x_i are equal to 1, which yields the desired contradiction.

5. Let n be a positive integer. Consider $2n$ points equally spaced around a circle. Suppose that n of the points are coloured blue and the remaining n points are coloured red. We write down the distance between each pair of blue points in a list, from shortest to longest. We write down the distance between each pair of red points in another list, from shortest to longest. (Note that the same distance may occur more than once in a list.)

   Prove that the two lists of distances are the same.

Solution 1 (Kevin McAvaney)

The distance between two of the points is uniquely determined by the number of points between them on the circle. So if two of the points have $k-1$ points between them where $k \leq n$, we say that their *chord length* is k.

If n consecutive points are coloured blue, then the remaining n consecutive points are coloured red. Due to the symmetry of this configuration, the two lists of distances are the same.

If there are no n consecutive red points, then one can obtain n consecutive red points by repeatedly switching colours on adjacent pairs of points. We show that the lists of chord lengths are the same after one such switch if and only if they are the same before the switch.

Consider a pair of adjacent points X and Y, where X is red and Y is blue. Draw a diameter of the circle perpendicular to XY. For each point U on the same side of the diameter as X, there is a corresponding point V on the same side of the diameter as Y such that the chord lengths XU and YV are equal to k for some $1 \leq k \leq n-2$.

For each $1 \leq k \leq n-2$, there are four possibilities for the colours of U and V — namely, red-red, red-blue, blue-red and blue-blue.

- In the first case, a red-red chord of length k is changed to a red-red chord of length $k+1$ and a red-red chord of length $k+1$ is changed to a red-red chord of length k.
- In the second case, a red-red chord of length k is changed to a red-red chord of length $k+1$ and a blue-blue chord of length k is changed to a blue-blue chord of length $k+1$.
- In the third case, a red-red chord of length $k+1$ is changed to a red-red chord of length k and a blue-blue chord of length $k+1$ is changed to a blue-blue chord of length k.
- In the fourth case, a blue-blue chord of length k is changed to a blue-blue chord of length $k+1$ and a blue-blue chord of length $k+1$ is changed to a blue-blue chord of length k.

Thus, the lists of chord lengths are the same after a switch if and only if they are the same before the switch. It follows that the lists of distances are the same for any colouring.

Solution 2 (Alice Devillers, Kevin McAvaney and Ian Wanless)

We use the notion of chord length defined in Solution 1.

Let b_k the number of pairs of blue points whose chord length is k. Let r_k be the number of pairs of red points whose chord length is k. Let m_k be the number of pairs of points, one blue and one red, whose chord length is k.

For $1 \leq k < n$, the number of blue points is equal to $\frac{2b_k+m_k}{2} = n$. Note that we divide by 2 here, as each point is a member of two pairs whose chord length is k. Similarly, we obtain that the number of red points is equal to $\frac{2r_k+m_k}{2} = n$. It immediately follows that $b_k = r_k$.

Furthermore, the number of blue points is equal to $2b_n + m_n = n$. Note that we do not need to divide by 2 here, as each point is a member of only one pair whose chord length is n. Similarly, we obtain that the number of red points is equal to $2r_n + m_n = n$. It immediately follows that $b_n = r_n$.

Since we have shown that $b_k = r_k$ for $1 \leq k \leq n$, it follows that the lists of distances are the same.

AMOC SENIOR CONTEST STATISTICS

Distribution of Awards/School Year

School Year	Number of Students	Gold	Silver	Bronze	HM	Participation
10	30	2	4	6	7	11
11	45	9	8	12	7	9
Other	24	2	2	6	6	8
Total	99	13	14	24	20	28

Score Distribution/Problem

Number of Students/Score

Problem Number	0	1	2	3	4	5	6	7	Mean
1	14	1	7	5	3	6	7	54	5.1
2	22	15	0	4	1	0	1	51	4.2
3	49	2	0	0	1	0	3	34	2.9
4	9	23	10	2	8	2	2	35	3.8
5	43	10	1	1	0	0	7	16	2.2

Note: These counts do not include students who did not attempt the problem.

Mean Score/Problem/School Year

School Year	Number of Students	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Overall Mean
10	30	4.5	3.6	2.2	3.2	2.1	14.7
11	45	6.0	4.6	3.7	4.5	2.5	19.9
Other	24	4.0	4.2	2.4	3.4	1.5	13.6
Total	99	5.1	4.2	2.9	3.8	2.2	16.8

AMOC SCHOOL OF EXCELLENCE

The 2016 AMOC School of Excellence was held on 1–10 December at Newman College, University of Melbourne. The main qualifying exams for this are the AIMO and the AMOC Senior Contest.

A total of 31 students from around Australia attended the school.

The students are divided into a senior group and a junior group. There were 17 junior students, 13 of whom were attending for the first time. There were 14 students making up the senior group, 5 of whom were first-time seniors.

The program covered the four major areas of number theory, geometry, combinatorics and algebra. Each day would start at 8:30am with lectures or an exam and go until 12:30pm. After a one-hour lunch break they would resume the lecture program at 1:30pm. By 4pm participants would usually have free time, followed by dinner at 6pm. Finally, each evening would round out with a problem session, topic review, or exam review from 6:45pm until 8:30pm.

Many thanks to Adrian Agisilaou, Ross Atkins, Alexander Chua, and Andrew Elvey Price, who served as live-in staff.

My thanks also go to Thomas Baker, Michelle Chen, Aaron Chong, Yong See Foo, Ivan Guo, Patrick He, Alfred Liang, Daniel Mathews, Kim Ramchen, and Chaitanya Rao, who assisted in lecturing and marking.

Angelo Di Pasquale
Director of Training, AMOC

AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

2017 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 14 February 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. For which integers $n \geq 2$ is it possible to write the numbers $1, 2, 3, \ldots, n$ in a row in some order so that any two numbers written next to each other in the row differ by 2 or 3?

2. Given five distinct integers, consider the ten differences formed by pairs of these numbers. (Note that some of these differences may be equal.)

Determine the largest integer that is certain to divide the product of these ten differences, regardless of which five integers were originally given.

3. Determine all functions f defined for real numbers and taking real numbers as values such that

$$f(x^2 + f(y)) = f(xy)$$

for all real numbers x and y.

4. Suppose that S is a set of 2017 points in the plane that are not all collinear.

Prove that S contains three points that form a triangle whose circumcentre is not a point in S.

AUSTRALIAN MATHEMATICS TRUST

The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

Australian Mathematical Olympiad Committee

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

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2017 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 2
Wednesday, 15 February 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Determine the number of positive integers n less than 1 000 000 for which the sum

$$\frac{1}{2 \times \lfloor\sqrt{1}\rfloor + 1} + \frac{1}{2 \times \lfloor\sqrt{2}\rfloor + 1} + \frac{1}{2 \times \lfloor\sqrt{3}\rfloor + 1} + \cdots + \frac{1}{2 \times \lfloor\sqrt{n}\rfloor + 1}$$

is an integer.

(Note that $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

6. The circles K_1 and K_2 intersect at two distinct points A and M. Let the tangent to K_1 at A meet K_2 again at B, and let the tangent to K_2 at A meet K_1 again at D. Let C be the point such that M is the midpoint of AC.

Prove that the quadrilateral $ABCD$ is cyclic.

7. There are 1000 athletes standing equally spaced around a circular track of length 1 kilometre.

(a) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 335 metres apart around the track?

(b) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 336 metres apart around the track?

8. Let $f(x) = x^2 - 45x + 2$.

Find all integers $n \geq 2$ such that exactly one of the numbers

$$f(1), f(2), \ldots, f(n)$$

is divisible by n.

Australian Mathematics Trust

The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

1. **Comment** This problem was solved by 80 of the 104 contestants. We present some of the many different approaches that were possible in arriving at a complete solution.

Solution 1

Answer: All integers $n \geq 4$.

Clearly $n = 2$ does not work because 1 and 2 differ by 1. Also $n = 3$ does not work because 2 cannot go next to either 1 or 3. It remains to show that all $n \geq 4$ work.

Using 2,4,1,3 as a starting point, at each step we simply write the next number on either side of the list with the odd numbers on the left and the even numbers on the right until we reach the number n as follows.

$$\ldots, 9, 7, 5, \underbrace{2, 4, 1, 3}, 6, 8, 10, \ldots. \qquad \square$$

Comments The above solution seems to be the simplest. This was the most frequent solution found by the contestants.

In the remaining solutions, we only demonstrate that all $n \geq 4$ work.

Solution 2

For $n = 4$ and 5, we have working sequences 2,4,1,3 and 3,5,2,4,1, respectively.

For $n \geq 6$ we have the following two cases.

Case 1 n is even

We have the working sequence

$$\underbrace{1, 3, 5, \ldots, n-3, n-5}_{\text{area 1}}, n-2, n, n-3, n-1, \underbrace{n-4, n-6, \ldots, 6, 4, 2}_{\text{area 2}}.$$

Here area 1 consists of the odd numbers from 1 up to $n-5$, while area 2 consists of the even numbers from $n-4$ down to 2.

Case 2 n is odd

We have the working sequence

$$\underbrace{2, 4, 6, \ldots, n-3, n-5}_{\text{area 1}}, n-2, n, n-3, n-1, \underbrace{n-4, n-6, \ldots, 5, 3, 1}_{\text{area 2}}.$$

Here area 1 consists of the even numbers from 2 up to $n-5$, while area 2 consists of the odd numbers from $n-4$ down to 1. $\square$

Comment This solution is closely related to the first solution. Can you explain the connection?

Solution 3

For $n = 4$ and 5, we have working sequences 2,4,1,3 and 3,5,2,4,1, respectively.

In each of the three cases that follow, k is any integer greater than or equal to 2.

Case 1 $n = 3k$

We have the working sequence

$$1, \underbrace{3, 6, 9, \ldots, 3k}_{\text{area 1}}, \underbrace{3k-2, 3k-5, 3k-8, \ldots, 4}_{\text{area 2}}, \underbrace{2, 5, 8, \ldots, 3k-1}_{\text{area 3}}.$$

Here the numbers in area 1 go up by 3 each time, the numbers in area 2 go down by 3 each time, and the numbers in area 3 go up by 3 each time.

Case 2 $n = 3k + 1$

We simply append $3k+1$ to the above working sequence for $n = 3k$ as shown below.

$$1, \underbrace{3, 6, 9, \ldots, 3k}_{\text{area 1}}, \underbrace{3k-2, 3k-5, 3k-8, \ldots, 4}_{\text{area 2}}, \underbrace{2, 5, 8, \ldots, 3k-1}_{\text{area 3}}, 3k+1$$

Case 3 $n = 3k + 2$

We have the working sequence

$$1, \underbrace{3, 6, 9, \ldots, 3k}_{\text{area 1}}, \underbrace{3k+2, 3k-1, 3k-4, \ldots, 2}_{\text{area 2}}, \underbrace{4, 7, 10, \ldots, 3k+1}_{\text{area 3}}.$$

Again the numbers in area 1 go up by 3 each time, the numbers in area 2 go down by 3 each time, and the numbers in area 3 go up by 3 each time. $\square$

Solution 4

Consider the following repeating pattern of groups of four numbers.

$$\underbrace{2,4,1,3}_{\text{group 1}},\underbrace{6,8,5,7}_{\text{group 2}},\underbrace{10,12,9,11}_{\text{group 3}},\underbrace{14,16,13,15}_{\text{group 4}},\ldots \tag{1}$$

Note that each group of four is formed by adding 4 to each member of the previous group of four. In each of the four cases that follow, k is any positive integer.

Case 1 $n = 4k$

The first k groups in (1) form a working sequence.

Case 2 $n = 4k + 1$

Take the first k groups in (1) to get a working sequence for $n = 4k$ whose last member is $4k - 1$. Appending $4k + 1$ at the end yields a working sequence for $n = 4k + 1$.

Case 3 $n = 4k + 2$

Take the first $k - 1$ groups in (1) to get a working sequence for $n = 4k - 4$ whose last member is $4k - 5$. Appending

$$4k-3, 4k-1, 4k+1, 4k-2, 4k, 4k+2$$

at the end yields a working sequence for $n = 4k + 2$.

Case 4 $n = 4k + 3$

Take the first $k - 1$ groups in (1) to get a working sequence for $n = 4k - 4$ whose last member is $4k - 5$. Appending

$$4k-3, 4k-1, 4k+2, 4k, 4k-2, 4k+1, 4k+3$$

at the end yields a working sequence for $n = 4k + 3$. $\square$

Solution 5

We say that a working sequence for n is *helpful* if its rightmost term is $n - 1$. We shall prove that there is a helpful working sequence for each integer $n \geq 4$.

To start with, we have the following helpful working sequences for $n = 4, 5, 6, 7, 8$.

$$
\begin{aligned}
n = 4: &\quad 2, 4, 1, 3 \\
n = 5: &\quad 1, 3, 5, 2, 4 \\
n = 6: &\quad 1, 3, 6, 4, 2, 5 \\
n = 7: &\quad 2, 5, 7, 4, 1, 3, 6 \\
n = 8: &\quad 1, 3, 6, 8, 5, 2, 4, 7
\end{aligned}
$$

Moreover, we can transform any helpful working sequence for n into a helpful working sequence for $n + 5$ by appending

$$n + 1, n + 3, n + 5, n + 2, n + 4$$

at the end.

The result now follows. $\square$

Solution 6

Call a working sequence for n *useful* if $n-2$ and n are adjacent terms of the sequence. We shall prove that there is a useful working sequence for each integer $n \geq 4$.

To start with, we have the following useful working sequences for $n = 4, 5, 6, 7, 8$.

$$\begin{aligned}
n = 4: &\quad \boxed{2, 4}, 1, 3 \\
n = 5: &\quad 1, \boxed{3, 5}, 2, 4 \\
n = 6: &\quad 1, 3, \boxed{6, 4}, 2, 5 \\
n = 7: &\quad 2, \boxed{5, 7}, 4, 1, 3, 6 \\
n = 8: &\quad 2, 4, 7, 5, \boxed{8, 6}, 3, 1
\end{aligned}$$

Moreover, we can transform any useful working sequence for n into a useful working sequence for $n + 5$ by inserting

$$n + 1, n + 4, n + 2, \boxed{n + 5, n + 3}$$

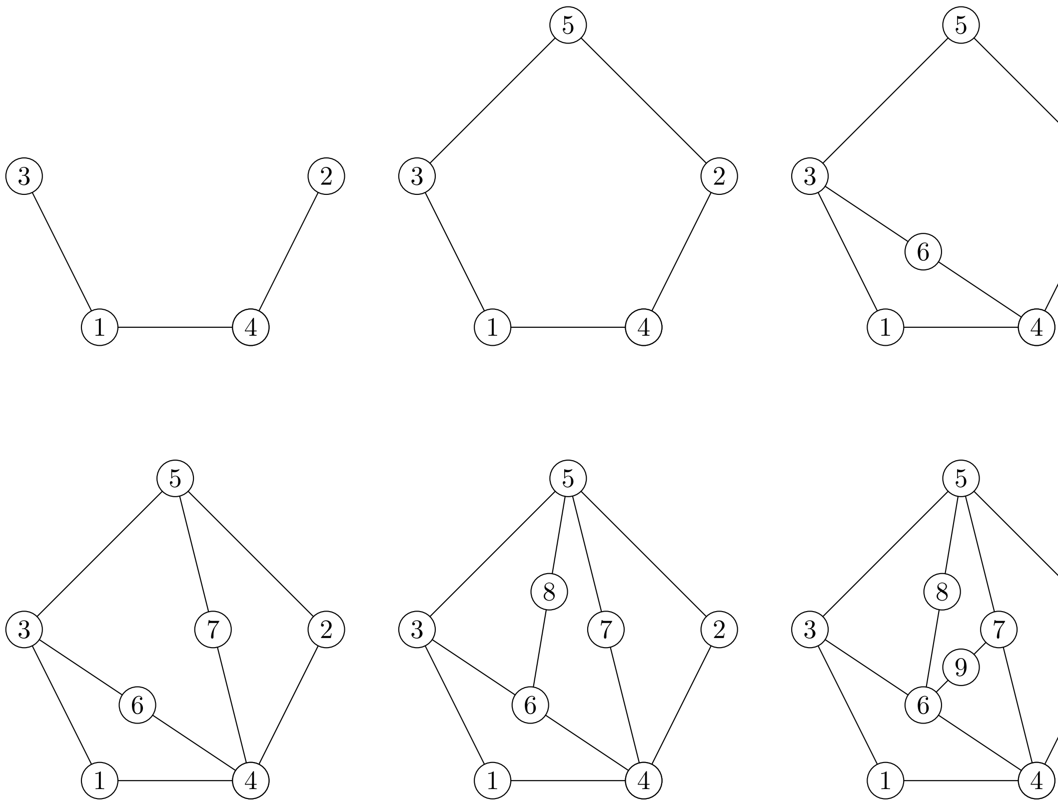
or its reverse in between $n - 2$ and n.

The result now follows. $\square$

Comment 1 We invite the reader to find answers to the following variations on the given problem.

- The same problem with the additional requirement that the sequence starts with 1.[1]
- The same problem with the additional requirements that the sequence starts with 1 and ends with n.
- The same problem with the additional requirement that the sequence is circular, that is, the difference between the first and last term is also 2 or 3.

Comment 2 It is convenient to visualise the situation using a graph G. Each of the positive integers $1, 2, \ldots, n$ corresponds to a vertex of G, and two vertices are connected by an edge if and only if they differ by two or three. It is fairly simple to draw the graph by adding one vertex at a time as depicted below for $n = 4, 5, 6, 7, 8, 9$.



Finding a valid sequence is equivalent to finding a path in the corresponding graph that visits each vertex exactly once.

[1] Actually this was already done in solution 3.

2. **Solution**

Answer: 288

For the integers 1, 2, 3, 4, 5, we calculate directly that the product of the ten differences is $2^5 \times 3^2 = 288$. Hence the answer is a factor of 288.

To show that the answer is 288, we shall show that 2^5 and 3^2 divide the product, P say, of the ten differences.

Part 1 Show that 2^5 divides P.

By the pigeonhole principle, at least three of the five numbers have the same parity.

If four or more of the five numbers have the same parity, then each of the $\binom{4}{2} = 6$ differences between them is even. Thus 2^6, and hence 2^5 divides P.

If, on the other hand, exactly three of the numbers, say a, b, and c are of one parity, and the other two, say d and e are of the other parity, then each of the differences $a - b$, $a - c$, $b - c$, and $d - e$ are even. Since a, b, and c have the same parity, either

$$a, b, c \in \{0, 2\} \pmod 4 \quad \text{or} \quad a, b, c \in \{1, 3\} \pmod 4.$$

In each case at least one of the differences $a - b$, $a - c$, or $b - c$ is divisible by 4. Hence again 2^5 divides P.

Part 2 Show that 3^2 divides P.

By the pigeonhole principle, two of the five numbers are congruent modulo 3.

If three of the numbers are congruent modulo 3, then each of the $\binom{3}{2} = 3$ differences between them is divisible by 3. Thus 3^3, and hence 3^2 divides P.

If at most two numbers are congruent to each other modulo 3, say $a \equiv b \pmod 3$, then the other three numbers must come from the other two congruence classes. Again by the pigeohole principle, it follows that two of the other three numbers are congruent modulo 3, say $c \equiv d \pmod 3$. It follows that 3^2 divides P. $\square$

3. **Solution 1**

Answers: $f(x) = c$ for any real constant c.

For reference we are given

$$f(x^2 + f(y)) = f(xy) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Put $y = 0$ in (1) to find

$$f(x^2 + f(0)) = f(0) \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

Observe that $x^2 + f(0)$ covers all real numbers that are greater than or equal to $f(0)$. It follows that

$$f(x) = f(0) \quad \text{whenever } x \geq f(0). \tag{3}$$

Next choose $a \geq f(0)$ with $a \neq 0$. Put $y = a$ into (1) and use (3) and then (2) to find

$$f(xa) = f(x^2 + f(a)) = f(x^2 + f(0)) = f(0).$$

Since $a \neq 0$, the expression xa covers all real numbers as x ranges over the reals. Hence $f(x) = f(0)$ for all $x \in \mathbb{R}$. Thus f is a constant function. It is readily seen that any such function satisfies (1). $\square$

Solution 2

For reference we are given

$$f(x^2 + f(y)) = f(xy) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Put $y = 0$ in (1) to deduce

$$f(x^2 + f(0)) = f(0) \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

Put $x = 0$ in (1) to deduce

$$f(f(y)) = f(0) \quad \text{for all } y \in \mathbb{R}. \tag{3}$$

Replacing y with $f(y)$ in (1), and using (3) and then (2), we have for all $x, y \in \mathbb{R}$

$$f(xf(y)) = f(x^2 + f(f(y)) = f(x^2 + f(0)) = f(0).$$

If $f(y) = 0$ for all $y \in \mathbb{R}$, then it is readily checked that this function satisfies (1).

If, on the other hand, there is a real number y such that $f(y) \neq 0$, then $xf(y)$ covers all real numbers as x ranges over the reals. Hence $f(x) = f(0)$ for all $x \in \mathbb{R}$. So f is a constant function. It is readily seen that any such function satisfies (1). $\square$

4. **Solution 1**

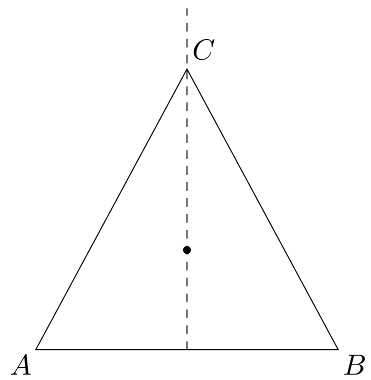
Let A and B be two points of S whose distance apart is minimal. Let ℓ be the perpendicular bisector of AB.

Case 1 The line ℓ contains no point of S.

Since not all points of S are collinear, there exists a point X not on the line AB. Consequently, the circumcentre of $\triangle ABX$, which lies on ℓ, is not in S, as required.

Case 2 The line ℓ contains at least one point of S.

Let C be a point in S, lying on ℓ, and of minimal distance to the line AB.



Recall AB is the minimal distance between points of S. Thus $CA = CB \geq AB$. Therefore $\angle ACB$ is the smallest angle in $\triangle ABC$, and so $\angle ACB \leq 60°$. Since $\triangle ABC$ is isosceles with $CA = CB$, we also have $\angle BAC = \angle CBA < 90°$. Hence $\triangle ABC$ is acute.

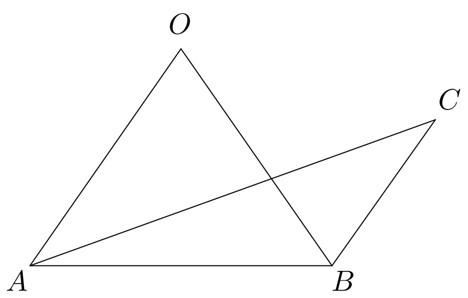
Acute triangles contain their circumcentres, so the circumcentre of $\triangle ABC$, which lies on ℓ, also lies inside $\triangle ABC$ and is therefore closer to the line AB than the point C. Since no point in S, that is also on ℓ, lies closer to AB than C, it follows that the circumcentre of $\triangle ABC$ is not in S, as required. $\square$

Solution 2

Let A and B be any two consecutive points on the convex hull of S. Orient the diagram so that AB is horizontal and no point of S lies below the line AB. Of all points in S that lie strictly above the line AB, let C be a point such that $\angle ACB$ is maximal. We shall prove that the circumcentre, O say, of $\triangle ABC$ is not in S.

Case 1 $0^\circ < \angle ACB < 90^\circ$

We have $\angle AOB = 2\angle ACB$. Hence $0^\circ < \angle ACB < \angle AOB < 180^\circ$. Thus O lies above the line AB and satisfies $\angle AOB > \angle ACB$. From the maximality of $\angle ACB$, we conclude that O is not in S, as desired.



Case 2 $\angle ACB = 90^\circ$

The point O is the midpoint of AB. Since A and B were chosen to be consecutive points on the convex hull of S, it follows that O is not in S, as desired.

Case 3 $90^\circ < \angle ACB < 180^\circ$

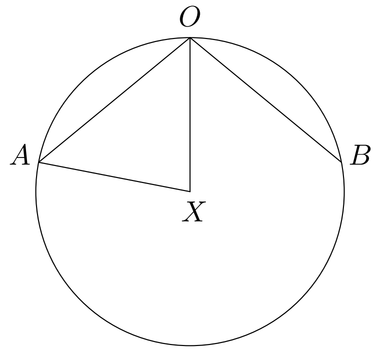
The point O lies below the line AB and so is not in S, as desired. $\square$

Solution 3

Let ABC be a triangle formed from points in S that has minimal circumradius, R say, and let O be the circumcentre of $\triangle ABC$.

If O is not in S, then we are done.

If O is in S, then since the three directed angles (modulo $360°$) $\angle AOB$, $\angle BOC$, and $\angle COA$ sum to $360°$, we may suppose without loss of generality that $\angle AOB \leq 120°$. Let X be the circumcentre of $\triangle AOB$. To complete the proof we shall show that X is not in S.



By symmetry OX bisects $\angle AOB$. Hence $\angle XAO = \angle AOX = \frac{1}{2}\angle AOB \leq 60°$.

If $\angle XAO = \angle AOX < 60°$, then $\angle OXA > 60°$ is strictly the largest angle in $\triangle AOX$. Thus OA is strictly the largest side of $\triangle AOX$. So $OX < OA$. Hence $\triangle AOB$ has circumradius $OX < OA = R$, which contradicts the minimality of R.

If, on the other hand, $\angle XAO = \angle AOX = 60°$, then $\triangle AOX$ is equilateral and has circumradius equal to $OA/\sqrt{3} < OA = R$. Since $\triangle ABC$ has minimal circumradius formed from points in S, it follows that X is not in S, as desired. $\square$

Solution 4

Let ABC be a triangle formed from points in S that has minimal circumradius, and let O be its circumcentre. For any triangle UVW, let R_{UVW} denote its circumradius.

If O is not in S, then we are done.

If O is in S, let α, β, and γ be the angles at A, B, and C, respectively, in $\triangle ABC$.

Case 1 $\triangle ABC$ is acute.

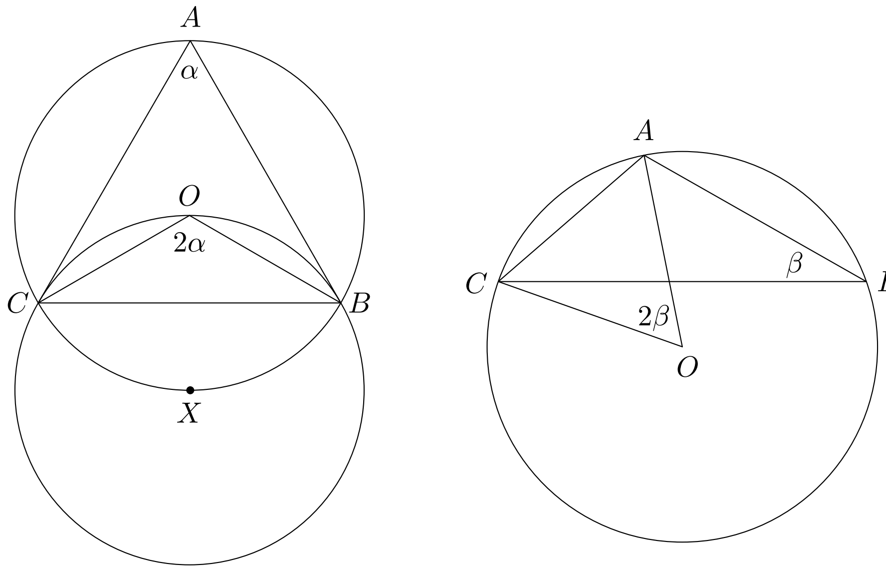
Note that $\angle BOC = 2\alpha$. By the minimality of R_{ABC}, we have $R_{ABC} \leq R_{BOC}$. Using the extended sine rule in $\triangle ABC$ and $\triangle BOC$, to compute these circumradii, we have

$$\frac{BC}{\sin\alpha} \leq \frac{BC}{\sin 2\alpha} = \frac{BC}{2\sin\alpha\cos\alpha} \quad \Rightarrow \quad \cos\alpha \leq \frac{1}{2}.$$

Since $0^\circ < \alpha < 90^\circ$ it follows that $\alpha \geq 60^\circ$. Similarly, $\beta \geq 60^\circ$ and $\gamma \geq 60^\circ$. But $\alpha + \beta + \gamma = 180^\circ$. Hence $\alpha = \beta = \gamma = 60^\circ$, and so $\triangle ABC$ is equilateral.

Let X be the circumcentre of $\triangle BOC$. If X is not in S, then we are done.

If X is in S, then $\angle OXC = 2\angle OBC = 60^\circ$, and $XO = XC$, so that $\triangle OXC$ is equilateral. However, equilateral $\triangle OXC$ is clearly smaller than equilateral $\triangle ABC$. Thus $R_{OXC} < R_{ABC}$, which contradicts the minimality of R_{ABC}.



Case 2 $\triangle ABC$ is not acute.

Without loss of generality we may suppose that $\alpha \geq 90^\circ$. It follows that $\beta + \gamma \leq 90^\circ$. Without loss of generality we may assume that $\angle\beta \leq 45^\circ$.

Note that $\angle AOC = 2\beta \leq 90^\circ$. By the minimality of R_{ABC}, we have $R_{ABC} \leq R_{AOC}$. Using the extended sine rule in $\triangle ABC$ and $\triangle AOC$, to compute these circumradii, we have

$$\frac{AC}{\sin\beta} \leq \frac{AC}{\sin 2\beta} \quad \Rightarrow \quad \sin 2\beta \leq \sin\beta.$$

However this is a contradiction because the sine function is strictly increasing on the interval $[0, 90^\circ]$. Hence this case does not occur. $\square$

Solution 5 (Ivan Guo, AMOC Senior Problems Committee)

We will prove a stronger statement, namely, that S contains three non-collinear points whose circumcircle does not contain a point of S *anywhere* in its interior.

Let A and B be two points of S separated by minimal distance. Consider the circle with diameter AB. Note that this circle does not contain any other points of S.

One side of the line AB contains at least one point of S. Continuously expand the circle towards that side, while making sure that it still passes through A and B. Eventually, it must hit a third point of S. At this stage, the circle meets (at least) three points of S but contains no points of S in its interior. $\square$

Comment (Adrian Agisilaou, AMO marker)

Let S be any set of points that are not all collinear, and let H be the convex hull of S. It is always possible to find a triangulation[4] of H with the property that the circumcircle of each such triangle does not contain any points of S strictly in its interior. Such a triangulation is called a *Delaunay triangulation*. See `https://en.wikipedia.org/wiki/Delaunay_triangulation` for more details. Such triangulations are a topic in contemporary mathematical research.

[4]By a *triangulation*, we mean a partition of H into triangles so that for each such triangle T, its three vertices are points of S and no other point on the perimeter of T is a point of S.

5. **Solution 1**

Answer: 999

For reference the given sum is

$$S = \frac{1}{2\lfloor\sqrt{1}\rfloor + 1} + \frac{1}{2\lfloor\sqrt{2}\rfloor + 1} + \cdots + \frac{1}{2\lfloor\sqrt{n}\rfloor + 1}.$$

Let m be the unique positive integer satisfying $m^2 \leq n < (m+1)^2$.

For each positive integer r, consider how many positive integers x there are such that $\lfloor\sqrt{x}\rfloor = r$. We require $r^2 \leq x < (r+1)^2$. So there are exactly $(r+1)^2 - r^2 = 2r+1$ values of x with $\lfloor\sqrt{x}\rfloor = r$. It follows that

$$\frac{1}{2\lfloor\sqrt{r^2}\rfloor+1} + \frac{1}{2\lfloor\sqrt{r^2+1}\rfloor+1} + \cdots + \frac{1}{2\left\lfloor\sqrt{(r+1)^2-1}\right\rfloor+1} = (2r+1) \times \frac{1}{2r+1} = 1. \quad (1)$$

Using the above results, we can split S up into the following m smaller sums.

$$\begin{aligned}
S = &\frac{1}{2\lfloor\sqrt{1^2}\rfloor + 1} + \frac{1}{2\lfloor\sqrt{1^2+1}\rfloor + 1} + \frac{1}{2\lfloor\sqrt{2^2-1}\rfloor + 1} \\
&+ \frac{1}{2\lfloor\sqrt{2^2}\rfloor + 1} + \frac{1}{2\lfloor\sqrt{2^2+1}\rfloor + 1} + \cdots + \frac{1}{2\lfloor\sqrt{3^2-1}\rfloor + 1} \\
&+ \frac{1}{2\lfloor\sqrt{3^2}\rfloor + 1} + \frac{1}{2\lfloor\sqrt{3^2+1}\rfloor + 1} + \cdots + \frac{1}{2\lfloor\sqrt{4^2-1}\rfloor + 1} \\
&\vdots \\
&+ \frac{1}{2\left\lfloor\sqrt{(m-1)^2}\right\rfloor + 1} + \frac{1}{2\left\lfloor\sqrt{(m-1)^2+1}\right\rfloor + 1} + \cdots + \frac{1}{2\lfloor\sqrt{m^2-1}\rfloor + 1} \\
&+ \frac{1}{2\lfloor\sqrt{m^2}\rfloor + 1} + \frac{1}{2\lfloor\sqrt{m^2+1}\rfloor + 1} + \cdots + \frac{1}{2\lfloor\sqrt{n}\rfloor + 1}.
\end{aligned}$$

From (1), each of the first $m-1$ lines in the above has sum equal to 1. From the discussion immediately preceding (1), the last line has sum equal to

$$\underbrace{\frac{1}{2m+1} + \frac{1}{2m+1} + \cdots + \frac{1}{2m+1}}_{n-m^2+1 \text{ terms}} = \frac{n-m^2+1}{2m+1}.$$

However, since $m^2 \leq n < (m+1)^2$ we see that

$$0 < \frac{n-m^2+1}{2m+1} \leq 1.$$

Hence S is an integer if and only if $n - m^2 + 1 = 2m+1$, that is, $n = (m+1)^2 - 1$.

Since n is a positive integer less than one million, the sum S is an integer precisely when $n = 2^2-1, 3^2-1, \ldots, 1000^2-1$. Thus there are exactly 999 values of n for which S is an integer. $\square$

Solution 2

For each positive integer n let

$$f(n)=\frac{1}{2\left\lfloor\sqrt{1}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{2}\right\rfloor+1}+\cdots+\frac{1}{2\left\lfloor\sqrt{n}\right\rfloor+1}.$$

Note that f is strictly increasing.

We claim that $f(k^2-1)=k-1$ for each integer $k\geq 2$. We prove this by induction.

For the base case $k=2$, we calculate

$$f(3)=\frac{1}{2\left\lfloor\sqrt{1}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{2}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{3}\right\rfloor+1}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1.$$

Hence the base case checks out.

For the inductive step, let us assume that $f(k^2-1)=k-1$ for some integer $k\geq 2$. We calculate

$$\begin{aligned}f((k+1)^2-1)&=f(k^2-1)+\frac{1}{2\left\lfloor\sqrt{k^2}\right\rfloor}+1+\frac{1}{2\left\lfloor\sqrt{k^2+1}\right\rfloor}+\cdots+\frac{1}{2\left\lfloor\sqrt{k^2+2k}\right\rfloor}\\&=k-1+\underbrace{\frac{1}{2k+1}+\frac{1}{2k+1}+\cdots+\frac{1}{2k+1}}_{2k+1\text{ terms}}\\&=k.\end{aligned}$$

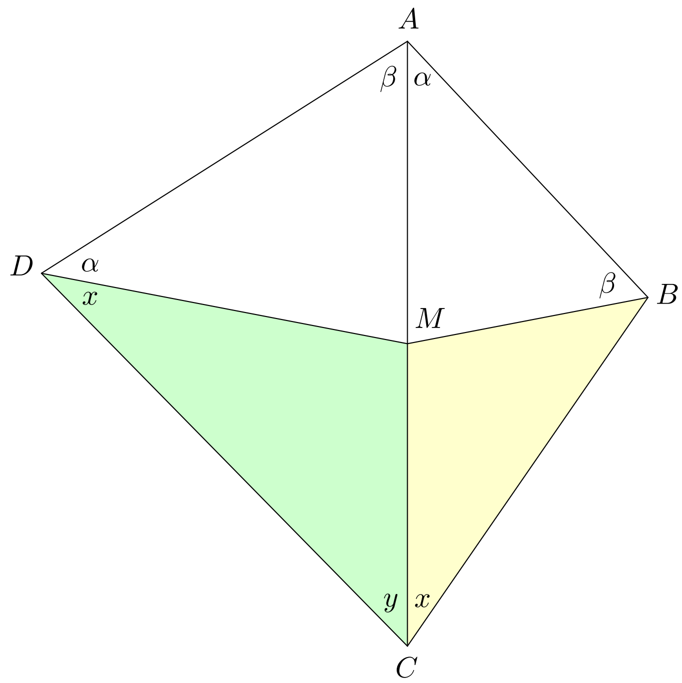
Note that the second line of the above calculation follows from the first because all of $k^2,k^2+1,\ldots,k^2+2k$ are greater than or equal to k^2 but strictly less than $(k+1)^2$. This completes the induction, thus establishing the claim.

Since f is strictly increasing, the claim implies that $f(n)$ is an integer if and only if $n=k^2-1$ for some integer $k\geq 2$.

Recall n is a positive integer less than one million. Hence $f(n)$ is an integer precisely when $n=2^2-1,3^2-1,\ldots,1000^2-1$. Thus there are exactly 999 values of n for which S is an integer. □

6. **Solution 1**

By the alternate segment theorem, since circle AMD is tangent to AB, we may let $\angle MAB = \angle MDA = \alpha$. Analogously, we may let $\angle DAM = \angle ABM = \beta$.



Therefore, $\triangle AMD \sim \triangle BMA$ (AA). Hence

$$\frac{DM}{MA} = \frac{AM}{MB}.$$

Since $AM = MC$, it follows from the above equality that

$$\frac{DM}{MC} = \frac{CM}{MB}. \tag{1}$$

The external angle sums in triangles AMD and AMB yield

$$\angle DMC = \alpha + \beta = \angle CMB.$$

Combining this with (1) implies $\triangle DMC \sim \triangle CMB$ (PAP). Hence we may let $\angle CDM = \angle BCM = x$. Also let $\angle MCD = y$. These allows us to directly compute

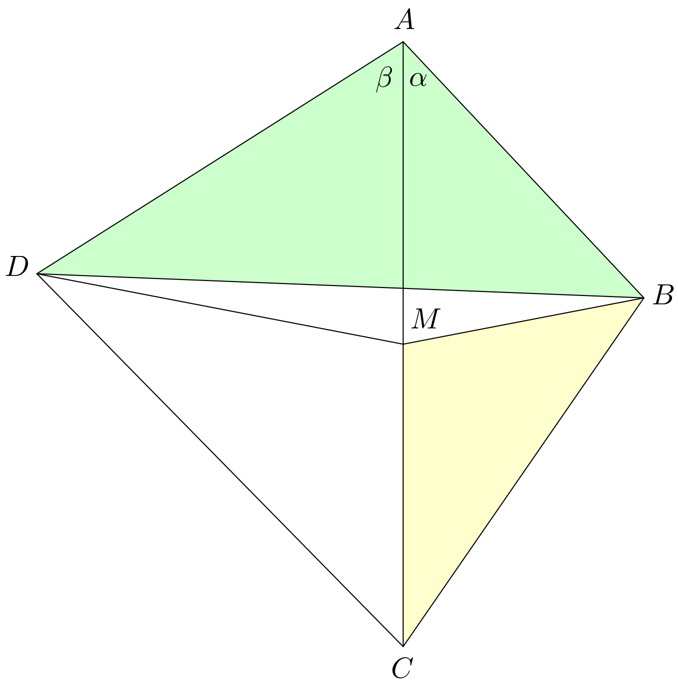
$$\angle DAB + \angle BCD = \alpha + \beta + x + y = 180^\circ,$$

where the last equality is due to the angle sum in $\triangle DMC$.

Since $\angle DAB + \angle BCD = 180^\circ$, we conclude that $ABCD$ is cyclic. $\square$

Solution 2

As in solution (1), we may let $\angle MAB = \angle MDA = \alpha$ and $\angle DAM = \angle ABM = \beta$.



Therefore, $\triangle AMD \sim \triangle BMA$ (AA). Hence

$$\frac{DA}{AM} = \frac{AB}{BM}.$$

Since $AM = MC$, it follows from the above equality that

$$\frac{DA}{CM} = \frac{AB}{MB}. \tag{2}$$

The exterior angle sum in $\triangle AMB$ yields $\angle CMB = \alpha + \beta = \angle DAB$. Combining this with (2) implies $\triangle DAB \sim \triangle CMB$ (PAP). It follows that

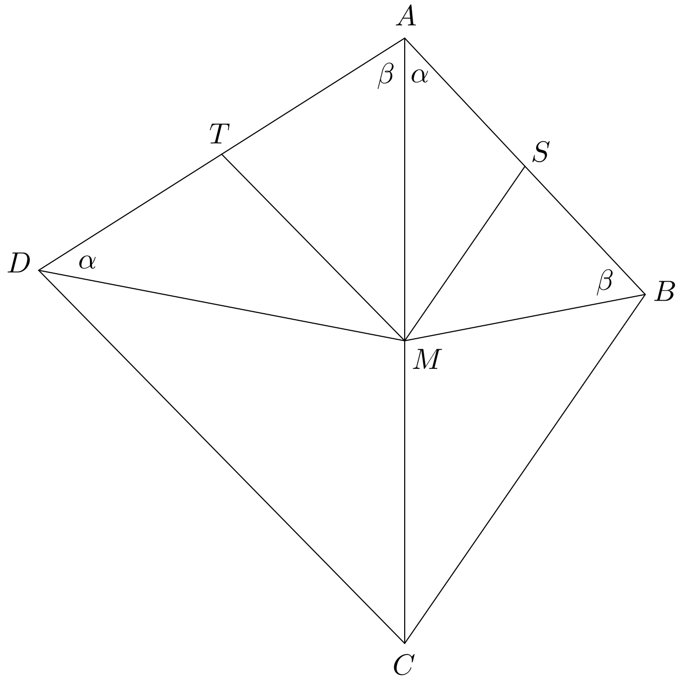
$$\angle BDA = \angle BCM = \angle BCA.$$

Since $\angle BDA = \angle BCA$, it follows that $ABCD$ is cyclic. $\square$

Solution 3

As in solution (1), we have $\triangle AMD \sim \triangle BMA$.

Let S and T be the midpoints of AB and AD, respectively.



Since S and T are corresponding points in similar triangles BMA and AMD, it follows that $\angle MSB = \angle MTA$. This implies that quadrilateral $ASMT$ is cyclic.

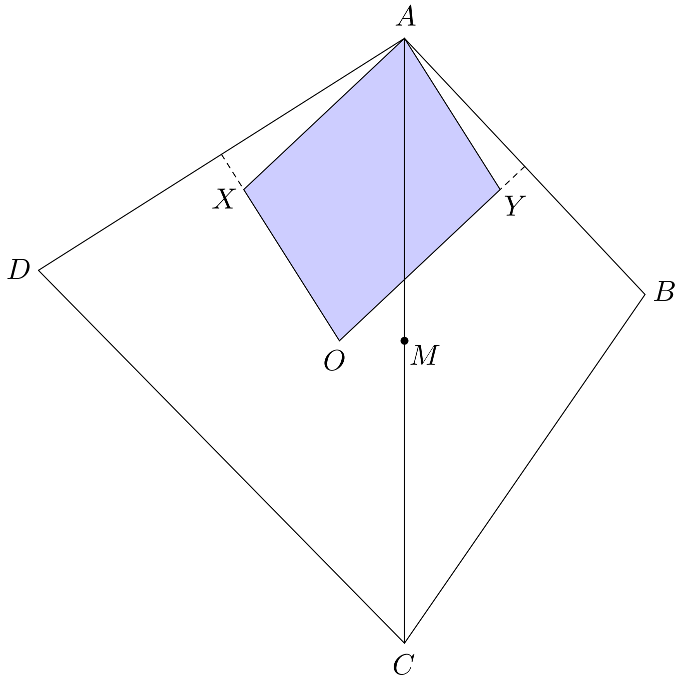
However, since quadrilateral $ABCD$ is the image of $ASMT$ under a dilation of factor 2 about A, it follows that $ABCD$ is also cyclic. $\square$

Comment Here is an alternative way of explaining the above solution. Since $\triangle MDA \sim \triangle MAB$, there is a spiral symmetry centred at M that sends DA to AB.[5] Since T and S are midpoints of DA and AB, respectively, the same spiral symmetry sends T to S, and so sends $\triangle MTA$ to $\triangle MSB$. Thus $\angle MTA = \angle MSB$ which implies that $ASMT$ and hence also $ABCD$ is cyclic.

[5]See the section entitled *Similar Switch* in chapter 5 of *Problem Solving Tactics* published by the AMT.

Solution 4

Let X and R_X denote the centre and radius, respectively, of K_1, and let Y and R_Y denote the centre and radius, respectively, of K_2. Let O be the intersection of the perpendicular bisectors of AD and AB. We shall prove that O is the circumcentre of quadrilateral $ABCD$.



Since AX is a radius of K_1 and AB is a tangent of K_1, we know that $AX \perp AB$. However we also have $YO \perp AB$. Hence $AX \parallel YO$. Similarly $AY \parallel XO$. Hence $AXOY$ is a parallelogram. Thus $YO = AX = R_X$ and $XO = AY = R_Y$.

Consider the reflection in the perpendicular bisector of XY. Let O' be the image of O under this reflection. We claim that $O' = M$. To see this, observe that the segment $O'X$ is the image of OY under the reflection. Hence $O'X = OY = R_X$. Hence O' lies on K_1. Similarly, O' lies on K_2. Hence O' is one of the intersection points of K_1 and K_2. Since O' lies on the same side of XY as O, we have $O' \neq A$. Thus $O' = M$. Furthermore, since $XY \perp AM$, we also have $OM \perp AM$.

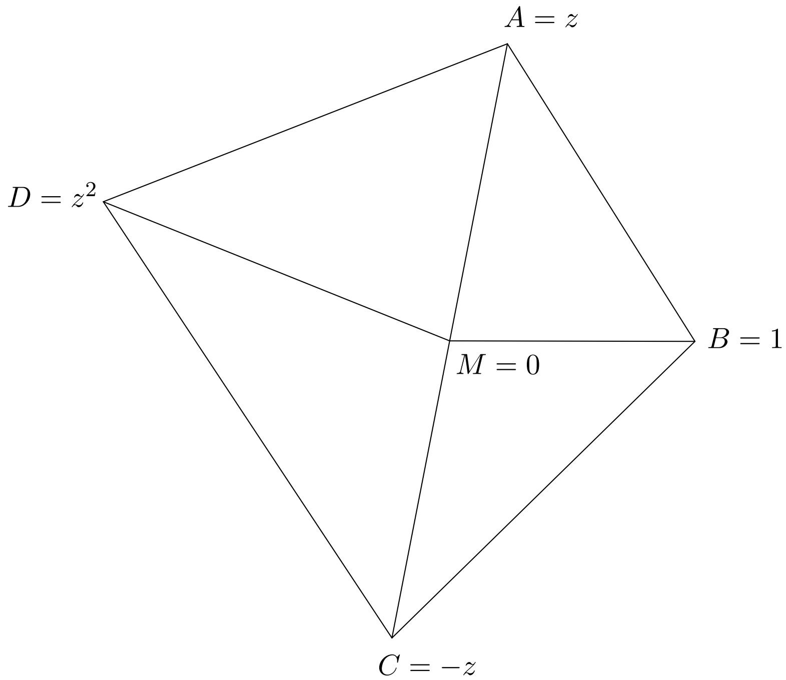
Recall that M is the midpoint of AC. Thus OM is the perpendicular bisector of AC. But OX is the perpendicular bisector of AD. Hence O is the circumcentre of $\triangle ADC$. Similarly O is the circumcentre of $\triangle ABC$. These two deductions imply that O is the circumcentre of quadrilateral $ABCD$, as claimed. $\square$

Solution 5 (Angelo Di Pasquale, Director of Training, AMOC)

This is a computational solution via complex numbers.

Without loss of generality we may assume that points M, B, and A are situated at the complex numbers 0, 1, and z, respectively.

As in solution 1, we have $\triangle BMA \sim \triangle AMD$. Hence D is situated at z^2. Also since M is the midpoint of AC, the point C is situated at $-z$.



Quadrilateral $ABCD$ is cyclic if and only if $\angle DAB + \angle BCD = 180°$. We compute

$$\angle DAB = \arg\left(\frac{B-A}{D-A}\right) = \arg\left(\frac{1-z}{z^2-z}\right) = \arg\left(-\frac{1}{z}\right) = \arg(-1) - \arg z,$$

and

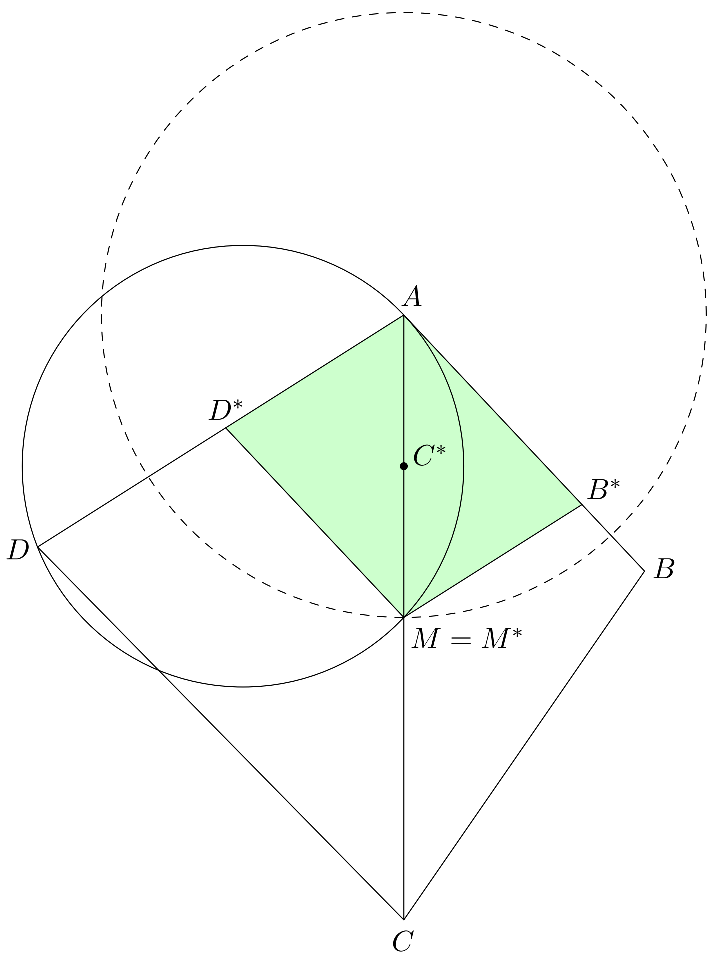
$$\angle BCD = \arg\left(\frac{D-C}{B-C}\right) = \arg\left(\frac{z^2+z}{1+z}\right) = \arg(z).$$

Therefore $\angle DAB + \angle BCD = \arg(-1) = 180°$, as desired. $\square$

Solution 6

This is a solution via inversion.

Consider the inversion about A of radius AM. For any point Z, let Z^* denote its image under the inversion. Clearly $M^* = M$, and C^* is the midpoint of AM.



The lines AD, AM, and AB all pass through A, so they remain fixed under the inversion. Circle ADM is tangent to the line AB. Thus circle ADM is mapped to the line through M that is parallel to AB. From this we deduce that D^* is the intersection of AD and the line through M parallel to AB. Similarly B^* is the intersection of AB and the line through M parallel to AD. Hence AB^*MD^* is a parallelogram.

Any parallogram's diagonals bisect each other. Since C^* is the midpoint of AM, it is also the midpoint of B^*D^*. In particular, B^*, C^*, and D^* are collinear. This implies that circle BCD passes through A. Therefore quadrilateral $ABCD$ is cyclic, as required. $\square$

7. **Solution**

Answers: (a) 32 (b) 0

Let the athletes be $A_1, A_2, \ldots, A_{1000}$ in that order, clockwise around the track.

(a) Starting from A_1 and proceeding clockwise around the track in intervals of 335 metres we meet athletes in the order:

$$A_1, A_{336}, A_{671}, A_6, A_{341}, A_{676}, A_{11}, \ldots, A_{331}, A_{666}, (A_1).$$

Note that since 335 and 1000 are both divisible by 5, only athletes A_i with $i \equiv 1 \pmod 5$ can occur in the above list. Furthermore, the directed distance between every third athlete in the above list is five metres. Thus the above list contains precisely all A_i with $i \equiv 1 \pmod 5$. Hence the list contains exactly 200 different athletes.

Athlete A_1 can either be paired with A_{336} or A_{666}. But once this pairing is chosen, all of the rest of the pairings are forced. Thus there are exactly two ways of pairing up all the athletes A_i for $i \equiv 1 \pmod 5$.

There is nothing particularly special about starting a list with A_1. We could have started four other lists with A_2, A_3, A_4, and A_5, respectively. Analogous arguments show that there are exactly two ways of pairing up the athletes in each such list. Since the five lists are independent, the total number of pairings is $2^5 = 32$. $\square$

(b) Starting from A_1 and proceeding clockwise around the track in intervals of 336 metres we meet athletes in the order:

$$A_1, A_{337}, A_{673}, A_9, A_{345}, A_{681}, A_{17}, \ldots, A_{329}, A_{665}, (A_1).$$

Note that since 336 and 1000 are both divisible by 8, only athletes A_i with $i \equiv 1 \pmod 8$ can occur in the above list. Furthermore, the directed distance between every third athlete in the above list is eight metres. Thus the above list contains precisely all A_i with $i \equiv 1 \pmod 8$. Hence the list contains exactly 125 different athletes. However, since 125 is odd, it is not possible to pair everyone up from the above list. Hence no such pairing is possible. $\square$

Comment (Ivan Guo, AMOC Senior Problems Committee)

For the general problem of n (where n is an even positive integer) athletes standing equally spaced around a circular track of length n metres, it can be shown that the number of ways of dividing the athletes into $\frac{n}{2}$ pairs such that the members of each pair are k metres apart is

$$\begin{cases} 2^{\gcd(k,n)}, & \text{if } \frac{n}{\gcd(k,n)} \text{ is even,} \\ 0, & \text{if } \frac{n}{\gcd(k,n)} \text{ is odd.} \end{cases}$$

8. This was the most difficult problem of the 2017 AMO. Just six contestants managed to solve it completely.

Solution 1

Answer: $n = 2017$

We observe that the function $f(x) = x^2 - 45x + 2$ is a parabola that is symmetric about $x = 22\frac{1}{2}$. Hence for any real number x we have

$$f(x) = f(45 - x). \tag{1}$$

Suppose that $n \mid f(k)$ where k is an integer satisfying $1 \leq k \leq n$. Using (1), it follows that $n \mid f(45 - k)$. However, since f is a polynomial, we know that

$$a \equiv b \pmod{n} \quad \Rightarrow \quad f(a) \equiv f(b) \pmod{n}. \tag{2}$$

Hence if $j \equiv 45 - k \pmod{n}$, where $1 \leq j \leq n$, then from (2) we have

$$f(j) \equiv f(45 - k) \equiv 0 \pmod{n}.$$

Since exactly one of the numbers $f(1), f(2), \ldots, f(n)$ is divisible by n, we have

$$\begin{aligned} & k \equiv 45 - k \pmod{n} \\ \Rightarrow \quad & 2k \equiv 45 \pmod{n}. \end{aligned} \tag{3}$$

We also know that $n \mid f(k)$, and so we may compute as follows.

$$\begin{aligned} & k^2 - 45k + 2 \equiv 0 \pmod{n} \\ \Rightarrow \quad & (2k)^2 - 90 \times 2k + 8 \equiv 0 \pmod{n} \\ \Rightarrow \quad & (45)^2 - 90 \times 45 + 8 \equiv 0 \pmod{n} \quad \text{(from (3))} \\ \Leftrightarrow \quad & -2017 \equiv 0 \pmod{n} \end{aligned}$$

Hence $n \mid 2017$. Since $n \geq 2$ and 2017 is prime, we have $n = 2017$. However, we still must check whether or not $n = 2017$ actually works.

Suppose that $2017 \mid f(k)$, where $1 \leq k \leq 2017$. We compute as follows.

$$\begin{aligned} & k^2 - 45k + 2 \equiv 0 \pmod{2017} \\ \Leftrightarrow \quad & 4k^2 - 180k + 8 \equiv 0 \pmod{2017} \quad \text{(since 2017 is odd)} \\ \Leftrightarrow \quad & (2k - 45)^2 \equiv 0 \pmod{2017} \\ \Leftrightarrow \quad & (2k - 45) \equiv 0 \pmod{2017} \quad \text{(since 2017 is prime)} \\ \Leftrightarrow \quad & 2k \equiv 45 \pmod{2017} \\ & \equiv 2062 \pmod{2017} \\ \Leftrightarrow \quad & k \equiv 1031 \pmod{2017} \quad \text{(since 2017 is odd)} \\ \Leftrightarrow \quad & k = 1031 \quad \text{(since } 1 \leq k \leq 2017) \end{aligned}$$

This shows that $k = 1031$ is the only integer with $1 \leq k \leq 2017$ such that $f(k)$ is divisible by 2017. $\square$

Solution 2 (Angelo Di Pasquale, Director of Training, AMOC)

Note that if $x \equiv y \pmod{n}$, then it follows that $f(x) \equiv f(y) \pmod{n}$. Therefore, we are seeking all n such that $f(x) \equiv 0 \pmod{n}$ has a unique solution modulo n.

Suppose that $f(k) = an$ for some integer a. Using the quadratic formula, we find that

$$k = \frac{45 \pm \sqrt{2017 + 4an}}{2}. \tag{1}$$

Hence, $2017 + 4an$ is an odd perfect square. So if one root of the quadratic is an integer, then so is the other. By the condition of the problem, this implies that

$$\frac{45 + \sqrt{2017 + 4an}}{2} \equiv \frac{45 - \sqrt{2017 + 4an}}{2} \pmod{n}.$$

Transferring everything in the above congruence to the left yields

$$\sqrt{2017 + 4an} \equiv 0 \pmod{n}.$$

Squaring the above yields $2017 \equiv 0 \pmod{n}$. Since 2017 is prime and $n \geq 2$, it follows that $n = 2017$.

Conversely, if $n = 2017$, then the quadratic formula (1) tells us that for k to be an integer, we require $1 + 4a = 2017j^2$ for some odd integer $j = 2i + 1$. Substituting this into the equation yields $k = 1031 + 2017i$ or $k = -986 - 2017i$. So the only such value of k in the required range is $k = 1031$, which corresponds to $i = 0$, $j = 1$ and $a = 504$. $\square$

Score Distribution/Problem

Number of Students/Score	Problem Number							
	1	2	3	4	5	6	7	8
0	0	4	34	81	4	33	21	82
1	6	9	16	5	0	14	2	10
2	7	5	12	0	1	8	1	1
3	0	1	7	0	3	1	1	0
4	1	5	0	0	4	1	5	3
5	3	7	2	3	0	1	22	2
6	7	26	7	2	37	1	0	0
7	80	47	26	13	55	45	52	6
Average	6.2	5.4	2.8	1.2	6.1	3.5	4.8	0.7

XXIX Asian Pacific Mathematics Olympiad

March, 2017

Time allowed: 4 hours *Each problem is worth 7 points*

The contest problems are to be kept confidential until they are posted on the official APMO website `http://apmo.ommenlinea.org`.
Please do not disclose nor discuss the problems over online until that date. The use of calculators is not allowed.

Problem 1. We call a 5-tuple of integers *arrangeable* if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \ldots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

Problem 2. Let ABC be a triangle with $AB < AC$. Let D be the intersection point of the internal bisector of angle BAC and the circumcircle of ABC. Let Z be the intersection point of the perpendicular bisector of AC with the external bisector of angle $\angle BAC$. Prove that the midpoint of the segment AB lies on the circumcircle of triangle ADZ.

Problem 3. Let $A(n)$ denote the number of sequences $a_1 \geq a_2 \geq \ldots \geq a_k$ of positive integers for which $a_1 + \cdots + a_k = n$ and each $a_i + 1$ is a power of two $(i = 1, 2, \ldots, k)$. Let $B(n)$ denote the number of sequences $b_1 \geq b_2 \geq \ldots \geq b_m$ of positive integers for which $b_1 + \cdots + b_m = n$ and each inequality $b_j \geq 2b_{j+1}$ holds $(j = 1, 2, \ldots, m-1)$.

Prove that $A(n) = B(n)$ for every positive integer n.

Problem 4. Call a rational number r *powerful* if r can be expressed in the form $\frac{p^k}{q}$ for some relatively prime positive integers p, q and some integer $k > 1$. Let a, b, c be positive rational numbers such that $abc = 1$. Suppose there exist positive integers x, y, z such that $a^x + b^y + c^z$ is an integer. Prove that a, b, c are all *powerful.*

Problem 5. Let n be a positive integer. A pair of n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with integer entries is called an *exquisite pair* if

$$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

Determine the maximum number of distinct n-tuples with integer entries such that any two of them form an exquisite pair.

1. **Solution**

Answer: $n_1 = n_2 = \cdots = n_{2017} = 29$

It is easy to see that the above is a valid 2017-tuple. We claim there are no others.

Note that in this solution to this problem all subscripts are taken modulo 2017.

We say that a 5-tuple of integers is *k-arrangeable* if its elements can be labelled a, b, c, d, e in some order such that $a - b + c - d + e = k$.
We say that a 2017-tuple $n_1, n_2, \ldots, n_{2017}$ of integers is k-good if every 5-tuple $n_i, n_{i+1}, n_{i+2}, n_{i+3}, n_{i+4}$ $(i = 1, 2, \ldots, 2017)$ is k-arrangeable. We are asked to determine all 2017-tuples of integers that are 29-good.

For any integers a, b, c, d, e, observe that

$$a - b + c - d + e = 29$$

if and only if

$$(a - 29) - (b - 29) + (c - 29) - (d - 29) + (e - 29) = 0.$$

Thus $n_1, n_2, \ldots, n_{2017}$ is 29-good if and only if $m_1, m_2, \ldots, m_{2017}$ is 0-good where $m_i = n_i - 29$ for $i = 1, 2, \ldots, 2017$. We shall prove that the only 0-good sequence is $m_1 = m_2 = \cdots = m_{2017} = 0$.

Suppose there is a 0-good sequence with the property that not all m_i are 0. Choose $m_1, m_2, \ldots, m_{2017}$ to be such a sequence that minimises

$$|m_1| + |m_2| + \cdots + |m_{2017}|. \tag{1}$$

Note that if $a - b + c - d + e = 0$, then $a + b + c + d + e = 2(b + d) \equiv 0 \pmod 2$. Hence for each i we have

$$m_i + m_{i+1} + m_{i+2} + m_{i+3} + m_{i+4} \equiv 0 \pmod 2. \tag{2}$$

Replacing i with $i + 1$ in (2) yields

$$m_{i+1} + m_{i+2} + m_{i+3} + m_{i+4} + m_{i+5} \equiv 0 \pmod 2. \tag{3}$$

Subtracting (2) from (3) yields

$$m_i \equiv m_{i+5} \pmod 2. \tag{4}$$

Since $\gcd(5, 2017) = 1$, equation (4) implies that all the m_i are congruent to the same thing modulo 2. From (2) we see that all the m_i are even.

Consider the sequence of integers $\frac{m_1}{2}, \frac{m_2}{2}, \ldots, \frac{m_{2017}}{2}$. It is also 0-good. However,

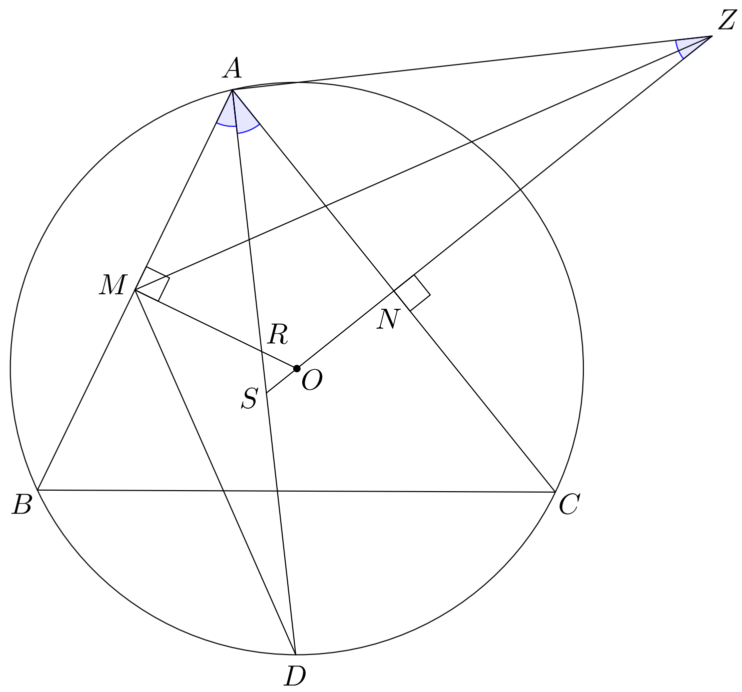
$$0 < \left|\frac{m_1}{2}\right| + \left|\frac{m_2}{2}\right| + \cdots + \left|\frac{m_{2017}}{2}\right| < |m_1| + |m_2| + \cdots + |m_{2017}|.$$

This contradicts the minimality of the expression at (1).

Thus we have shown that the only 0-good sequence is $m_1 = m_2 = \cdots = m_{2017} = 0$. This corresponds to $n_1 = n_2 = \cdots = n_{2017} = 29$. $\square$

2. **Solution 1**

Let $\alpha = \angle BAD = \angle DAC$. Let M and N be the midpoints of AB and AC, respectively. Let O be the circumcentre of triangle ABC. Thus O lies on the perpendicular bisectors of AB, AC, and AD. Let line AD intersect lines MO and NO at R and S, respectively.



Since the internal and external bisectors of an angle are perpendicular, we have $\angle CAZ = 90° - \alpha$. The angle sum in $\triangle ANZ$ yields $\angle AZN = \alpha$. The angle sum in $\triangle SAN$ yields

$$\angle OSR = \angle NSA = 90° - \alpha.$$

Summing the angles in $\triangle MAR$ yields

$$\angle SRO = \angle ARM = 90° - \alpha = \angle OSR.$$

If ℓ is the line through O that is perpendicular to AD, then R and S are symmetric in ℓ. Since O is the centre of circle ABC, we have $OA = OD$. So A and D are also symmetric in ℓ. It follows that $AS = RD$.

Recall $\angle MAR = \alpha = \angle AZN$, and $\angle RMA = 90° = \angle SAZ$. It follows that $\triangle RMA \sim \triangle SAZ$ (AA). Therefore

$$\frac{MR}{MA} = \frac{AS}{AZ} = \frac{RD}{AZ}.$$

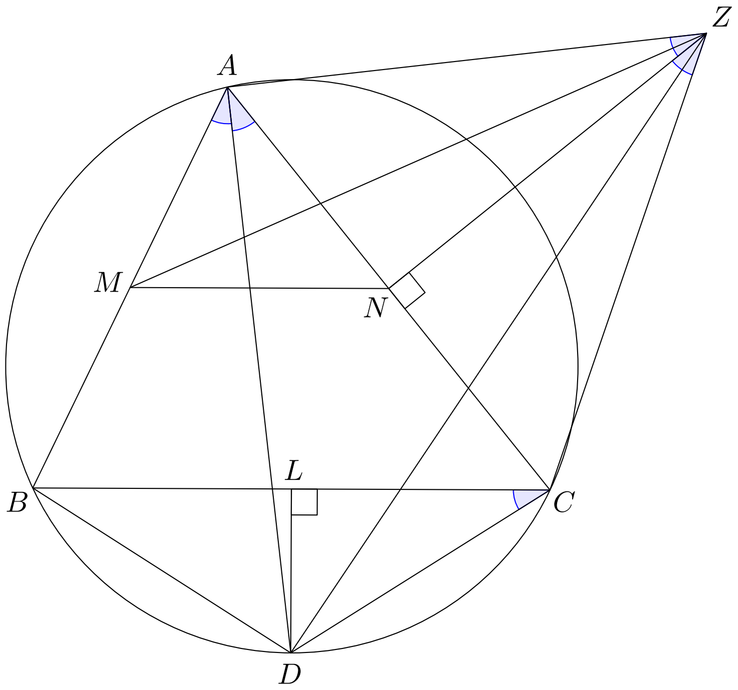
Furthermore,

$$\angle MRD = 180° - \angle ARM = 90° + \alpha = \angle SAZ + \angle MAR = \angle MAZ.$$

Hence $\triangle MRD \sim \triangle MAZ$ (PAP). Therefore, $\angle AZM = \angle RDM = \angle ADM$. Thus $MAZD$ is cyclic. Therefore circle ADZ passes through M, as required. $\square$

Solution 2

Let $\alpha = \angle BAD = \angle DAC$. Let L, M, and N be the midpoints of BC, AB, and AC, respectively.



Claim 1 $\triangle DLC \sim \triangle CNZ$.

Proof Since the internal and external bisectors of an angle are perpendicular, we have $\angle CAZ = 90° - \alpha$. The angle sum in $\triangle ANZ$ yields $\angle AZN = \alpha$. Since ZN is the perpendicular bisector of AC, we have $\triangle ANZ \equiv \triangle CNZ$. Thus $\angle NZC = \alpha$.

We know AD bisects $\angle BAC$. Hence $DB = DC$. Since L is the midpoint of BC, and $\triangle BCD$ is isosceles with $DB = DC$, it follows that $DL \perp BC$. From circle $ABDC$, we have $\angle BCD = \angle BAD = \alpha = \angle NZC$. We also have $\angle DLC = 90° = \angle CNZ$. Hence $\triangle DLC \sim \triangle CNZ$ (AA). ☐

Claim 2 $\triangle DCZ \sim \triangle MNZ$.

Proof Since M and N are the midpoints of AB and AC, respectively, we have $MN \parallel BC$, and $MN = \frac{1}{2}BC = LC$.

The angle sum in $\triangle CNZ$ yields $\angle ZCN = 90° - \alpha$. Hence

$$\angle ZCD = 90° - \alpha + \angle ACB + \alpha = 90° + \angle ANM = \angle ZNM.$$

From $\triangle DLC \sim \triangle CNZ$, we also have

$$\frac{DC}{CZ} = \frac{LC}{NZ} = \frac{MN}{NZ}.$$

Hence $\triangle DCZ \sim \triangle MNZ$ (PAP). ☐

Finally, from $\triangle DCZ \sim \triangle MNZ$ we have

$$\angle MZD = \angle NZD + \angle MZN = \angle NZD + \angle DZC = \angle NZC = \alpha = \angle MAD.$$

Hence $MAZD$ is cyclic, and so circle ADZ passes through M, as required. ☐

Solution 3

This is a computational solution via complex numbers. Refer to the diagram in solution 1.

Without loss of generality we may assume that circle $ABCD$ is the unit circle in the complex plane. For convenience, we suppose that A, B, C, and D are positioned at complex numbers a^2, b^2, c^2, and d^2, where $|a| = |b| = |c| = |d| = 1$.

If O is the centre of circle $ABCD$, then $\angle BOD = 2\angle BAD = 2\angle DAC = \angle DOC$. Thus OD bisects $\angle BOC$. Hence $d^2 = b^2c^2$, and so $d = \pm bc$. By changing b to $-b$, if necessary, we may assume that $(A, B, C, D) = (a^2, b^2, c^2, bc)$.

Let z and m be the complex numbers representing Z and M, respectively. Note that $m = \frac{a^2+b^2}{2}$ because M is the midpoint of AB. The place where the internal and external bisectors of an angle of a triangle meet the triangle's circumcircle, are diametrically opposite on the circumcircle. Hence the line AZ meets circle $ABCD$ at the point represented by the complex number $-bc$. Since a^2, z, and $-bc$ are collinear, we have $\frac{z-a^2}{-bc-a^2}$ is a real number. Therefore

$$\frac{z-a^2}{a^2+bc} = \overline{\left(\frac{z-a^2}{a^2+bc}\right)} = \frac{\bar{z}-\bar{a}^2}{\bar{a}^2+\bar{b}\bar{c}} = \frac{\bar{z}-\frac{1}{a^2}}{\frac{1}{a^2}+\frac{1}{bc}} = \frac{a^2bc\left(\bar{z}-\frac{1}{a^2}\right)}{a^2+bc}.$$

Here we have used the fact that if $|u| = 1$, then $\bar{u} = \frac{1}{u}$. Tidying up yields

$$z = a^2 - bc + a^2bc\bar{z}. \tag{1}$$

Next, ZO is the perpendicular bisector of AC. So $\frac{z}{a^2-c^2}$ is a pure imaginary number. Hence

$$\frac{z}{a^2-c^2} = -\overline{\left(\frac{z}{a^2-c^2}\right)} = \frac{\bar{z}}{\bar{c}^2-\bar{a}^2} = \frac{\bar{z}}{\frac{1}{c^2}-\frac{1}{a^2}} = \frac{a^2c^2\bar{z}}{a^2-c^2}.$$

Therefore

$$\bar{z} = \frac{z}{a^2c^2}. \tag{2}$$

Substituting (2) into (1) and solving for z yields

$$z = \frac{c(a^2-bc)}{c-b}.$$

To prove that $MAZD$ is cyclic, it suffices to show that $\arg\left(\frac{m-d}{z-d}\right) = \arg\left(\frac{m-a}{z-a}\right)$. This is equivalent to showing that $E = \frac{(m-d)(z-a)}{(z-d)(m-a)}$ is real. First we simplify matters by writing everything in terms of a, b, and c. We compute

$$E = \frac{\left(\frac{a^2+b^2}{2}-bc\right)\left(\frac{c(a^2-bc)}{c-b}-a^2\right)}{\left(\frac{c(a^2-bc)}{c-b}-bc\right)\left(\frac{a^2+b^2}{2}-a^2\right)} = \frac{(a^2+b^2-2bc)(a^2b-bc^2)}{(a^2c+b^2c-2bc^2)(b^2-a^2)} = \frac{b(a^2-c^2)}{c(b^2-a^2)}$$

Finally, we show E is real by showing $\overline{E} = E$. We have

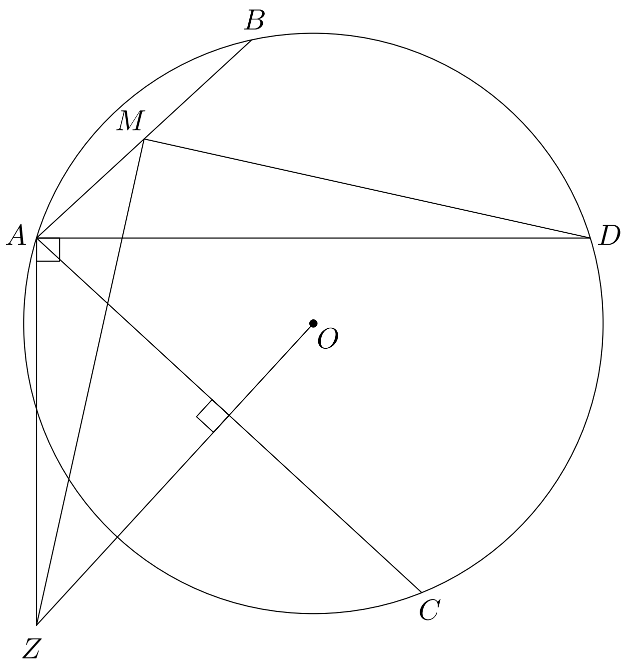
$$\overline{E} = \overline{\left(\frac{b(a^2-c^2)}{c(b^2-a^2)}\right)} = \frac{\bar{b}(\bar{a}^2-\bar{c}^2)}{\bar{c}(\bar{b}^2-\bar{a}^2)} = \frac{\frac{1}{b}\left(\frac{1}{a^2}-\frac{1}{c^2}\right)}{\frac{1}{c}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)} = \frac{b(a^2-c^2)}{c(b^2-a^2)} = E,$$

as required. $\square$

Solution 4

This is a computational solution via coordinate geometry.

Toss the problem onto the Cartesian plane so that circle $ABCD$ is centred at the origin $O = (0, 0)$ and has radius 1. Rotate the figure so that AD is horizontal.



We have

$$A = (-a, b) \quad \text{and} \quad D = (a, b)$$

for some $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$.

Let m be the gradient of line AB. Therefore $B = (-a + c, b + cm)$ for some $c \neq 0$. Since B is on the unit circle we have $(-a+c)^2+(b+cm)^2 = 1$. Expanding this out, remembering that $a^2 + b^2 = 1$ and $c \neq 0$, we deduce $c = \frac{2a-2bm}{m^2+1}$. Hence

$$B = \left(-a + \frac{2a - 2bm}{m^2 + 1}, b + \frac{m(2a - 2bm)}{m^2 + 1}\right).$$

Since AD bisects $\angle BAC$, the gradient of AC is $-m$. Hence we find C as per B using $-m$ instead of m.

$$C = \left(-a + \frac{2a + 2bm}{m^2 + 1}, b - \frac{m(2a + 2bm)}{m^2 + 1}\right)$$

Let M be the midpoint of AB. Thus

$$M = \left(-a + \frac{a - bm}{m^2 + 1}, b + \frac{m(a - bm)}{m^2 + 1}\right).$$

The line AZ is vertical because the internal and external bisectors of an angle are perpendicular. Since AZ contains $A = (-a, b)$, the equation of AZ is $x = -a$.

The perpendicular bisector of AC passes through the centre O of the unit circle. The gradient of the line AC is $-m$. Since $OZ \perp AC$, the gradient of line OZ is $\frac{1}{m}$. Hence the equation of OZ is $y = \frac{1}{m}x$

It is straightforward to compute that the intersection of lines AZ and OZ is

$$Z = \left(-a, -\frac{a}{m}\right).$$

The gradient of ZM is given by

$$\frac{b + \frac{m(a-bm)}{m^2+1} + \frac{a}{m}}{-a + \frac{a-bm}{2m^2+1} + a} = \frac{bm + a(m^2+1)}{m(a-bm)}.$$

The gradient of MD is given by

$$\frac{b + \frac{m(a-bm)}{m^2+1} - b}{-a + \frac{a-bm}{m^2+1} - a} = -\frac{m(a-bm)}{bm + a(2m^2+1)}.$$

The product of the gradients of ZM and MD is equal to -1. Thus $ZM \perp MD$. Hence $\angle ZMD = 90° = \angle ZAD$. It follows that $MAZD$ is cyclic. $\square$

3. **Solution 1**

Let A and B denote the set of sequences counted by $A(n)$ and $B(n)$, respectively. We show $A(n) = B(n)$ by showing there is a bijection between A and B.

Step 1 Produce a map from B to A.

Let $b_1 \geq b_2 \geq \cdots \geq b_m$ be a sequence in B. For convenience, let $b_{m+1} = 0$. Since $b_i \geq 2b_{i+1}$ we can define non-negative integers $x_m, x_{m-1}, \ldots, x_1$ by

$$x_i = b_i - 2b_{i+1} \quad \text{for } i = m, m-1, \ldots, 1. \tag{1}$$

Note that $x_m = b_m > 0$.

Consider the sequence $a_1 \geq a_2 \geq \cdots \geq a_k$ consisting of x_m copies of $2^m - 1$, followed by x_{m-1} copies of $2^{m-1} - 1$, and so on, down to x_1 copies of 1. We calculate

$$\begin{aligned}
\sum_{i=1}^{k} a_k &= \sum_{i=1}^{m} x_i(2^i - 1) && (2)\\
&= \sum_{i=1}^{m} b_i(2^i - 1) - \sum_{i=1}^{m} 2b_{i+1}(2^i - 1) && (3)\\
&= b_1 + \sum_{i=2}^{m} b_i(2^i - 1) - \sum_{i=2}^{m+1} b_i(2^i - 2) && (4)\\
&= \sum_{i=1}^{m} b_i \quad (\text{since } b_{m+1} = 0)\\
&= n.
\end{aligned}$$

Hence $a_1 \geq a_2 \geq \cdots \geq a_k$ is a sequence in A.

Step 2 Produce a map from A to B.

Let $a_1 \geq a_2 \geq \cdots \geq a_k$ be a sequence in A. For each positive integer i let x_i denote the number of terms of that sequence that are equal to $2^i - 1$. So we have

$$\sum_{i=1}^{m} x_i(2^i - 1) = n,$$

where x_m is the last nonzero term of the sequence $x_1, x_2, \ldots$.

For convenience, let $b_{m+1} = 0$. Define $b_m, b_{m-1}, \ldots, b_1$ recursively using (1). Observe that $b_i \geq 2b_{i+1}$ for $i = 1, 2, \ldots, m-1$. Calculating similarly to step 1, we have

$$\sum_{i=1}^{m} b_i = \text{RHS}(4) = \text{RHS}(3) = \text{RHS}(2) = n.$$

Hence $b_1, b_2, \ldots, b_m$ is a sequence in B.

Step 3 Note that steps 1 and 2 both use the invertible system of equations (1), but in opposite directions. Hence the mappings from steps 1 and 2 are inverses of each other, and are therefore bijections. Hence $A(n) = B(n)$. ☐

Solution 2

Let A and B denote the set of sequences counted by $A(n)$ and $B(n)$, respectively. We show $A(n) = B(n)$ by showing there is a bijection between A and B.

Step 1 Produce a mapping from A to B.

Let $a_1 \geq a_2 \geq \cdots \geq a_k$ be a sequence in A. For each positive integer i let x_i denote the number of terms of that sequence that are equal to $2^i - 1$. So we have

$$\sum_{i=1}^{m} x_i(2^i - 1) = n,$$

where x_m is the last nonzero term of the sequence $x_1, x_2, \ldots$.

From the sequence $x_1, x_2, \ldots, x_m$, construct the sequence $b_1, b_2, \ldots, b_m$ as follows. For convenience, let $b_{m+1} = 0$ and define $b_m, b_{m-1}, \ldots, b_1$ recursively by

$$b_i = x_i + 2b_{i+1} \quad \text{for } i = m, m-1, \ldots, 1. \tag{1}$$

Observe that $b_i \geq 2b_{i+1}$ for $i = 1, 2, \ldots, m-1$. Using the above recurrence and remembering that $b_{m+1} = 0$, we calculate

$$\begin{aligned}
\sum_{i=1}^{m} b_i &= \sum_{i=1}^{m} x_i + 2\sum_{i=2}^{m} b_i \\
&= \sum_{i=1}^{m} x_i + 2\sum_{i=2}^{m} x_i + 4\sum_{i=3}^{m} b_i \\
&= \sum_{i=1}^{m} x_i + 2\sum_{i=2}^{m} x_i + 4\sum_{i=3}^{m} x_i + 8\sum_{i=4}^{m} b_i \\
&\;\vdots \\
&= \sum_{i=1}^{m} x_i + 2\sum_{i=2}^{m} x_i + 4\sum_{i=3}^{m} x_i + \cdots + 2^{m-1}\sum_{i=m}^{m} x_i \\
&= \sum_{i=1}^{m} x_i(1 + 2 + 2^2 + \cdots + 2^{i-1}) \\
&= \sum_{i=1}^{m} x_i(2^i - 1) \qquad (2) \\
&= n.
\end{aligned}$$

Thus we have shown that $b_1 \geq b_2 \geq \cdots \geq b_m$ is a sequence in B.

Step 2 Produce a mapping from B to A.

Let $b_1 \geq b_2 \geq \cdots \geq b_m$ be a sequence in B. For convenience, let $b_{m+1} = 0$. Since $b_i \geq 2b_{i+1}$ we define non-negative integers $x_1, x_2, \ldots, x_m$ using (1).

Note that $x_m = b_m > 0$. We run a calculation very similar to one the above except that we start with $n = \sum_{i=1}^{m} b_i$ and end at (2). From (2) we produce a sequence in A by listing out x_m terms all equal to $2^m - 1$, then x_{m-1} terms all equal to $2^{m-1} - 1$, and so on, down to x_1 terms all equal to 1.

Step 3 Observe that the mappings from steps 1 and 2 are inverses of each other, and are therefore bijections. Hence $A(n) = B(n)$. $\square$

Solution 3

Let A and B denote the set of sequences counted by $A(n)$ and $B(n)$, respectively. We show $A(n) = B(n)$ by showing there is a bijection between A and B. We do this indirectly by constructing a set of matrices M and showing there is a bijection between A and M, and a bijection between B and M. But first we give an example to illustrate how we will do this.

Given a sequence in A, we construct a corresponding sequence in B as follows. Write each number of the sequence in A as a sum of consecutive powers of 2 and then stack them in rows to form a rectangular matrix. Summing the columns of the matrix yields a sequence in B. For example, from the sequence $31, 31, 15, 3, 3, 3, 1$ in A, we derive the corresponding sequence $47, 23, 10, 5, 2$ in B as shown.

$$\begin{array}{cc}
\begin{array}{c} 31 \\ 31 \\ 15 \\ 3 \\ 3 \\ 3 \\ 1 \end{array} &
\begin{pmatrix} 16 & 8 & 4 & 2 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 8 & 4 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
 & \begin{array}{ccccc} 47 & 23 & 10 & 5 & 2 \end{array}
\end{array}$$

Note that as shown in the above example, except for the rows corresponding to the maximal members of the sequence, we have filled out each row using 0s once all the powers of 2 have been used.

Step 1 Describe the set M.

We call a matrix X *useful* if it satisfies the following conditions.

(i) The entries in the top row from left to right are $2^x, 2^{x-1}, 2^{x-2}, \ldots, 1$ for some non-negative integer x.

(ii) The entries in each row from left to right are $2^y, 2^{y-1}, \ldots, 1, 0, \ldots, 0$ for some non-negative integer $y \leq x$, and where there are $x - y$ trailing 0s.

(iii) The row sums of X form a weakly decreasing sequence from top to bottom.

(iv) The sum of the entries of X is n.

For example, if $n = 87$ then the following matrices X_1 and X_2 are useful.

$$X_1 = \begin{pmatrix} 16 & 8 & 4 & 2 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 8 & 4 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 32 & 16 & 8 & 4 & 2 & 1 \\ 8 & 4 & 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2 Produce a bijection from A to M.

Let $a_1 \geq a_2 \geq \cdots \geq a_k$ be a sequence in A. Thus $a_i = 2^{x_i} - 1$ for some positive integers $x_1 \geq x_2 \geq \cdots \geq x_k$. Let X be the matrix with k rows and x_i columns such that the entries in the ith row from left to right are $2^{x_i-1}-1, 2^{x_i-2}-1, \ldots, 1, 0, \ldots, 0$ where there are $x_1 - x_i$ trailing 0s.

For example, if $n = 87$ and $(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (31, 31, 15, 3, 3, 3, 1)$ then we produce the matrix X_1 shown above.

Observe that by construction, any such produced matrix X is useful. Conversely, any useful matrix X can be turned into a sequence in A by letting a_i be the sum of the entries in the ith row of X. Furthermore, these two operations are mutually inverse. Hence each of these operations are bijections.

Step 3 Produce a bijection from B to M.

Let $b_1 \geq b_2 \geq \cdots \geq b_m$ be a sequence in B. Form a matrix X with m columns and $b_1 - b_2 - b_3 - \cdots - b_m$ rows as follows.

While $b_m > 0$, repeat the following loop.

- The next row of X from left to right is $2^{m-1}, 2^{m-2}, \ldots, 1$.
- Replace each b_i that is positive by $b_i - 2^{m-i}$.

It is easy to check that the new sequence has the property that the ith term is at least double the $(i+1)$st term.

Once $b_m = 0$, we move on to b_{m-1} and do a similar thing.

While $b_{m-1} > 0$, repeat the following loop.

- The next row of X from left to right is $2^{m-2}, 2^{m-3}, \ldots, 1$.
- Replace each b_i that is positive by $b_i - 2^{m-1-i}$.

We similarly keep progressing through all the b_i until $b_1 = 0$.

For example, starting with $(b_1, b_2, b_3, b_4, b_5) = (47, 23, 10, 5, 2)$ we compute as follows.

$$\begin{array}{rrrrrr}
 & 47 & 23 & 10 & 5 & 2 \\
\to & (16 & 8 & 4 & 2 & 1) \\
 & 31 & 15 & 6 & 3 & 1 \\
\to & (16 & 8 & 4 & 2 & 1) \\
 & 15 & 7 & 2 & 1 & 0 \\
\to & (8 & 4 & 2 & 1 & 0) \\
 & 7 & 3 & 0 & 0 & 0 \\
\to & (2 & 1 & 0 & 0 & 0) \\
 & 5 & 2 & 0 & 0 & 0 \\
\to & (2 & 1 & 0 & 0 & 0) \\
 & 3 & 1 & 0 & 0 & 0 \\
\to & (2 & 1 & 0 & 0 & 0) \\
 & 1 & 0 & 0 & 0 & 0 \\
\to & (1 & 0 & 0 & 0 & 0) \\
 & 0 & 0 & 0 & 0 & 0
\end{array}
\quad \longrightarrow \quad
\begin{pmatrix}
16 & 8 & 4 & 2 & 1 \\
16 & 8 & 4 & 2 & 1 \\
8 & 4 & 2 & 1 & 0 \\
2 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

By construction, any such produced matrix X is useful and has the property that the sum of the entries in the jth column is b_j for $j = 1, 2, \ldots, m$. Hence we have produced a map from B to M.

Conversely, we produce a map from M to B as follows. Given any useful matrix X, let b_j be the sum of the elements in the jth column of X. Note that $b_j \geq 2b_{j+1}$

because each entry of X is at least double that of its neighbour to the right. Thus the b_j form a sequence in B.

It is obvious that the composition of the mappings from B to M to B is the identity map on B. To show that the individual mappings from B to M and from M to B are bijections, it is enough to show that the mapping from M to B is injective. But this is obvious because b_m determines the rightmost column of X. After this b_{m-1} determines the $(m-1)$st column of X, and so on, until b_1 determines the first column of X.

Step 4 Since there are bijections between M and each of A and B, we conclude that $|A| = |B| = |M|$, as required. □

Comment The resulting mappings between A and B in all the solutions presented are actually the same mappings. It is just that the exposition of solution is different in each case.

4. **Solution**

Let $(a,b,c)=\left(\dfrac{d}{g},\dfrac{e}{h},\dfrac{f}{i}\right)$, where $\gcd(d,g)=\gcd(e,h)=\gcd(f,i)=1$.

We have

$$a^x+b^y+c^z=\frac{d^xh^yi^z+g^xe^yi^z+g^xh^yf^z}{g^xh^yi^z}. \tag{1}$$

Since LHS(1) is an integer, we have $g^x \mid g^xh^yi^z \mid d^xh^yi^z+g^xe^yi^z+g^xh^yf^z$. Hence $g^x \mid d^xh^yi^z$. Since $\gcd(d,g)=1$, it follows that

$$g^x \mid h^yi^z. \tag{2}$$

Similarly, we find

$$h^y \mid g^xi^z \tag{3}$$

We are given $abc=1$, and so

$$def=ghi. \tag{4}$$

Suppose p is a prime factor of f. From (4) and $\gcd(f,i)=1$, we deduce that $p \mid gh$. Without loss of generality $p \mid g$. From (2) and $\gcd(f,i)=1$, we have $p \mid h$.

For a positive integer n and a prime number p, the notation $\nu_p(n)$ denotes the exponent of p in the prime factorisation of n. Put another way, if $k=\nu_p(n)$, this means that $p^k \parallel n$. That is, $p^k \mid n$ but $p^{k+1} \nmid n$.

We have $p \mid f,g,h$ and $p \nmid d,e,i$ from $\gcd(d,g)=\gcd(e,h)=\gcd(f,i)=1$. Hence from (4), we have

$$\nu_p(f)=\nu_p(g)+\nu_p(h). \tag{5}$$

Recall that $p \nmid i$. So from (2), we have $\nu_p(g^x)\le\nu_p(h^y)$. And from (3), we have $\nu_p(h^y)\le\nu_p(g^x)$. Thus $\nu_p(g^x)=\nu_p(h^y)$. Therefore

$$x\nu_p(g)=y\nu_p(h). \tag{6}$$

Combining (5) and (6), we find

$$y\nu_p(f)=(x+y)\nu_p(g).$$

This may be rearranged as

$$\frac{\nu_p(f)}{\nu_p(g)}=\frac{(x+y)/\gcd(x+y,y)}{y/\gcd(x+y,y)}. \tag{7}$$

The fraction in RHS(7) is in lowest terms. So $\nu_p(f)$ is divisible by $k=\frac{x+y}{\gcd(x+y,y)}>1$. But this is true for all primes p dividing f. Hence $f=m^k$ for some integer m, and so $c=\frac{f}{i}=\frac{m^k}{i}$. Thus c is powerful.

Similar reasoning shows that a and b are also powerful. $\square$

5. Answer: The maximum is $n^2 + n + 1$.

   There are two parts to the solution of this problem.

   (a) Exhibit $n^2 + n + 1$ n-tuples such that each pair is exquisite.

   (b) Show that any collection containing more than $n^2 + n + 1$ n-tuples has a pair that is not exquisite.

Solution to part (a)

Consider the following five types of n-tuples as shown in the table

Type	Tuples	Number of
A	$(0, 0, \ldots, 0)$	1
B	$(0, 0, \ldots, 0, 1, 0, \ldots, 0)$	n
C	$(0, 0, \ldots, 0, -1, 0, \ldots, 0)$	n
D	$(0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots 0)$	$\binom{n}{2}$
E	$(0, 0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots 0)$	$\binom{n}{2}$

Type A consists of the n-tuple all of whose entries are 0.

Type B consists of all n-tuples with $n - 1$ 0s and a 1.

Type C consists of all n-tuples with $n - 1$ 0s and a -1.

Type D consists of all n-tuples with $n - 2$ 0s and two 1s.

Type E consists of all n-tuples with $n - 2$ 0s and a 1 and a -1 such that the -1 is to the left of the 1.

It is easy to see that if any of the above n^2+n+1 n-tuples is paired with an n-tuple from A, B, or C, the result is an exquisite pair. Also any two members from D form an exquisite pair, as do any two members from E. Finally we observe that if we select one member from each of D and E we always form an exquisite pair. □

Solution to part (b) (Problem Selection Committee)

Let us recast the problem in the language of vectors. An n-tuple is simply an n-dimensional vector with integer coordinates. Two such vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ form an exquisite pair if and only if their inner product satisfies $|\boldsymbol{a} \cdot \boldsymbol{b}| \leq 1$.

Let us call a set A of vectors exquisite if every pair of vectors in A is exquisite. We shall show that if A is an exquisite set of n-dimensional integer vectors, then $|A| \leq n^2 + n + 1$.

Claim If A_i is any exquisite set of n-dimensional integer vectors such that the last non-zero coordinate of each vector is in the ith position, then $|A_i| \leq 2i$.

Note that the result follows immediately from the claim as follows. For each positive integer i, with $1 \leq i \leq n$, we let A_i denote the set of vectors in A whose last non-zero coordinate is in the ith position. Since the A_i partition the non-zero vectors of A, the claim tells us that the number of non-zero vectors in A is at most

$$\sum_{i=1}^{n} |A_i| \leq \sum_{i=1}^{n} 2i = n^2 + n.$$

Hence it suffices to prove the claim. To help us, we first prove the following lemma.

Lemma Let i be any positive integer. For any set of $2i + 1$ different non-zero i-dimensional vectors, there exist two of them, $\boldsymbol{a}$ and $\boldsymbol{b}$ say, such that $\boldsymbol{a} \cdot \boldsymbol{b} > 0$. Here the vectors are only required to have real, not necessarily integer, coordinates.

Proof We proceed by induction. The base case $i = 1$ is trivial because for any three different non-zero real numbers, two of them have the same sign.

Assume the lemma is true for $i - 1$ and consider any set of $2i + 1$ i-dimensional vectors. Since the inner product is invariant under rotation, we may rotate our $2i + 1$ vectors in i-dimensional space so that one of them lies on the ith coordinate axis, and so has the form $(0, 0, \ldots, 0, r)$. Next scale all of the vectors by a factor of $-\frac{1}{r}$. This reduces us to the case where one of the vectors is $\boldsymbol{a} = (0, 0, \ldots, 0, -1)$.

If one of the remaining vectors $\boldsymbol{b}$ has negative ith coordinate, then $\boldsymbol{a} \cdot \boldsymbol{b} > 0$, as desired. So we may assume that each of the remaining vectors has a non-negative last coordinate.

For each of our $2i+1$ i-dimensional vectors $\boldsymbol{x}$, let $\boldsymbol{x'}$ be the $(i-1)$-dimensional vector obtained from $\boldsymbol{x}$ by truncating its last coordinate. Let S be the *multiset* consisting of these $2i + 1$ truncated vectors.

Note that $\boldsymbol{a'}$ consists entirely of zeros. If $\boldsymbol{a'} = \boldsymbol{b'} = \boldsymbol{c'}$ for different $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, then two of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ would have entries of the same sign in their ith coordinates and therefore have positive inner product. Hence we may assume that at most two members of S are the zero vector. Let T be the multiset of non-zero vectors of S. Thus $|T| \geq 2i-1$.

If two members, $\boldsymbol{b'}$ and $\boldsymbol{c'}$ say, of T were equal, then since $\boldsymbol{b}$ and $\boldsymbol{c}$ both have non-negative last coordinate their inner product would be positive. Hence T must consist of at least $2i - 1$ *distinct* non-zero vectors. Applying the inductive assumption to T yields $\boldsymbol{b'} \cdot \boldsymbol{c'} > 0$ for some two different members of T. And since $\boldsymbol{b}$ and $\boldsymbol{c}$ both have non-negative last coordinate, it follows that $\boldsymbol{b} \cdot \boldsymbol{c} > 0$, as desired. $\square$

To complete the proof of the given problem, we prove the claim as follows. First observe that the claim is trivial for $i = 1$, so we restrict ourselves to $i \geq 2$.

If A_i contains three or more vectors, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ say, whose only nonzero coordinate is in the ith position, then one of them, $\boldsymbol{a}$ say, has a number in the ith coordinate that is an integer whose absolute value is at least 2. It then follows that $|\boldsymbol{a} \cdot \boldsymbol{b}| \geq 2$, which is a contradiction. Hence there are at most two such vectors in A_i. Let us remove them from A_i to form the exquisite set B_i. It suffices to show that $|B_i| \leq 2i - 2$.

For each vector $\boldsymbol{a}$ in B_i with a negative ith coordinate, let us replace $\boldsymbol{a}$ with $-\boldsymbol{a}$. Since $|(-\boldsymbol{x}) \cdot \boldsymbol{y}| = |\boldsymbol{x} \cdot \boldsymbol{y}|$, for any vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, this does not change whether or not each pair in B_i is exquisite. Hence we may assume without loss of generality that the ith coordinate of each vector in B_i is positive.

Case 1 There are two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in B_i that agree in their first $i-1$ coordinates.

Let $\boldsymbol{a} = (r_1, r_2, \ldots, r_{i-1}, a)$ and $\boldsymbol{b} = (r_1, r_2, \ldots, r_{i-1}, b)$. Then

$$\boldsymbol{a} \cdot \boldsymbol{b} = r_1^2 + r_2^2 + \cdots + r_{i-1}^2 + ab.$$

Since $r_1^2 + r_2^2 + \cdots + r_{i-1}^2 \geq 1$ and $ab \geq 1$, the result follows.

Case 2 No two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in B_i agree in their first $i - 1$ coordinates.

For each vector $\boldsymbol{x} \in B_i$, let $\boldsymbol{x'}$ be the $(i - 1)$-dimensional vector obtained from $\boldsymbol{x}$ by truncating its last coordinate. Let C_{i-1} be the set of such truncated vectors.

Therefore C_{i-1} is a *set* of $(i-1)$-dimensional integer vectors satisfying $|C_{i-1}| = |B_i|$. (Note that C_{i-1} is not necessarily exquisite.) It suffices to show that $|C_{i-1}| \leq 2i-2$.

Suppose, for the sake of contradiction, that $|C_{i-1}| \geq 2i-1$. From lemma 1 with i replaced by $i-1$, there exist two vectors $\boldsymbol{a'}$ and $\boldsymbol{b'}$ in C_i such that $\boldsymbol{a'} \cdot \boldsymbol{b'} > 0$.

Let $\boldsymbol{a'} = (a_1, a_2, \ldots, a_{i-1})$ and $\boldsymbol{b'} = (b_1, b_2, \ldots, b_{i-1})$. Reversing the truncation yields vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_{i-1}, a_i)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_{i-1}, b_i)$ in B_i. Hence

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + \cdots + a_{i-1}b_{i-1} + a_ib_i = \boldsymbol{a'} \cdot \boldsymbol{b'} + a_ib_i.$$

However we know that $a_ib_i > 0$ and $\boldsymbol{a'} \cdot \boldsymbol{b'} > 0$. Since all coordinates of the vectors are integers it follows that $a_ib_i \geq 1$ and $\boldsymbol{a'} \cdot \boldsymbol{b'} \geq 1$. Hence $\boldsymbol{a} \cdot \boldsymbol{b} \geq 2$, which contradicts that $\boldsymbol{a}$ and $\boldsymbol{b}$ form an exquisite pair. Hence $|C_{i-1}| \leq 2i-2$, as desired. $\square$

ASIAN PACIFIC MATHEMATICS OLYMPIAD RESULTS

Country scores

Rank	Country	Number of Contestants	Score	Gold	Silver	Bronze	Hon.Men
1	USA	10	295	1	2	4	3
2	Republic of Korea	10	294	1	2	4	3
3	Thailand	10	270	1	2	4	3
4	Japan	10	264	1	2	4	3
5	Russia	10	263	1	2	4	3
6	Islamic Republic of Iran	10	253	1	2	4	3
7	Taiwan	10	241	1	2	4	3
8	Canada	10	230	1	2	4	3
9	Singapore	10	208	1	2	4	3
10	Brazil	10	207	1	2	4	3
11	Philippines	10	190	1	2	4	3
12	Hong Kong	10	176	1	2	4	3
13	Indonesia	10	173	1	2	4	3
14	Australia	10	170	1	2	4	3
15	Peru	10	158	1	2	4	3
16	India	10	157	0	3	4	3
17	Mexico	10	149	1	2	4	3
18	Argentina	10	132	0	3	3	2
19	Saudi Arabia	10	116	1	0	3	4
20	Bangladesh	10	109	0	1	2	7
21	New Zealand	10	93	0	0	2	8
22	Tajikistan	10	92	0	1	0	9
23	Kazakhstan	10	81	0	0	1	9
24	Turkmenistan	10	67	0	0	1	6
25	Nicaragua	8	65	0	1	1	4
26	Macedonia	8	63	0	0	1	6
27	Kyrgyzstan	7	58	0	0	2	3
28	Malaysia	10	57	0	1	2	0
29	Syria	9	55	0	0	1	5
30	El Salvador	9	48	0	0	2	2

Rank	Country	Number of Contestants	Score	Gold	Silver	Bronze	Hon.Men
31	Cambodia	10	44	0	0	0	4
32	Sri Lanka	9	43	0	0	1	3
33	Pakistan	10	28	0	0	0	2
34	Ecuador	10	24	0	0	1	0
35	Colombia	5	23	0	0	1	0
36	Bolivia	7	19	0	0	0	1
37	Trinidad and Tobago	9	17	0	0	0	1
38	Costa Rica	1	8	0	0	0	1
39	Panama	1	8	0	0	0	1
	Total	353	4948	17	42	92	129

AMOC SELECTION SCHOOL

The 2017 AMOC Selection School was held on 16–25 April at Robert Menzies College, Macquarie University, Sydney. The main qualifying exams are the AMO and the APMO.

A total of 30 students from around Australia attended the school.

The routine is similar to that for the AMOC School of Excellence; however, there is the added interest of the actual selection of the Australian IMO team. This year the IMO would be held in Rio de Janeiro, Brazil.

The students are divided into a junior group and a senior group. This year there were 16 juniors and 14 seniors. It is from the seniors that the team of six for the IMO plus one reserve team member is selected. This year we wanted to explore the possibility of sending a team to the European Girls' Mathematical Olympiad the following year, so five of the junior places were reserved for girls. I am pleased to report that all Australian states, plus the ACT were represented among the students.

Many thanks to Adrian Agisilaou, Ross Atkins, Michelle Chen, Alexander Chua, and Andrew Elvey Price, who assisted me as live-in staff members.

My thanks also go to Peter Brown, Stephen Farrar, Victor Khou, Vickie Lee, Johnny Lim, Seyoon Ragavan, Andy Tran, Gareth White, Rachel Wong, Sampson Wong, Kevin Xian, and Jonathan Zheng, all of whom came in to give lectures or help with the marking of exams.

Angelo Di Pasquale
Director of Training, AMOC

2017 Australian IMO Team

Name	School	Year
James Bang	Baulkham Hills High School NSW	10
Matthew Cheah	Penleigh and Essendon Grammar School VIC	12
Linus Cooper	James Ruse Agricultural High School NSW	11
William Hu	Christ Church Grammar School WA	11
Hadyn Tang	Trinity Grammar School VIC	8
Guowen Zhang	St Joseph's College (Gregory Terrace) QLD	11
Reserve		
Haowen Gao	Knox Grammar School NSW	11

The 2017 Australian IMO team was completely new. No team member had been to an IMO before. The last time this occurred was in 2007.

From left: James Bang, Matthew Cheah, Linus Cooper, Minister for Industry, Innovation and Science, Senator the Hon Arthur Sinodinos AO, William Hu, Haydn Tang and Guowen Zhang.

IMO TEAM PREPARATION SCHOOL

For the week preceding the IMO, our team met with our British counterparts for a final dose of training and acclimatisation to the IMO conditions. This year we stayed at a lovely hotel in Itaipava, about 70 km north of central Rio. I was accompanied by Nathan Ford for the entire camp, and Angelo Di Pasquale for the first few days, after which he left to do his leader duties at the IMO. The British delegation at the camp consisted of the six team members, their deputy leader, Dominic, and observer, Jill. The British brought an extremely experienced team, who already collectively owned nine IMO medals which was quite a contrast to our entirely new team.

As it happened, I had been visiting the UK just before the IMO, so I travelled from London with the British team, who were all at the airport well before the flight. I hear that this was not the case for our own team. It came down to the wire for one or two of the Australians, but fortunately everyone managed to board their flights, and we all arrived in Brazil without any more drama.

After we arrived everyone had a full free day to adjust to the time zone and mingle. Our students somewhat reluctantly went outside for a while, but otherwise played a variety of indoor games.

The next five days all started the same way, with an IMO style exam for all 12 students. There was a very loud rooster next door, which was excellent for training everyone to ignore such distractions, as you never know what to expect at the IMO. We considered introducing a variety of other distractions for the students but in the end decided that the rooster was distraction enough. After the exam each morning, the teams were given free time for the rest of the day while Angelo, Dominic and I marked their exams. During the free time they mostly played games or visited some local attractions.

As tradition dictates, the fifth and final exam was designated the mathematical ashes, in which our teams compete for glory and an urn containing burnt remains of some British scripts from 2008. The UK had an extremely strong team this year, and sadly it showed as they defeated us 83–63, thereby extending their hold on the ashes to 9 consecutive years. I have a good feeling about next year though...

Once again, this camp was a great way to wind up both teams' training before the IMO. The co-training experience was hugely beneficial for all involved, as the students learnt a lot from each other, and as trainers we learnt a bit too. This tradition will hopefully continue well into the future. Many thanks to everyone who made this a success, in particular, the UKMT, the entire UK delegation and our local guides in Itaipava.

Andrew Elvey Price
IMO Deputy Leader

THE MATHEMATICS ASHES

1. Point A_1 lies inside acute scalene triangle ABC and satisfies

$$\angle A_1AB = \angle A_1BC \quad \text{and} \quad \angle A_1AC = \angle A_1CB.$$

   Points B_1 and C_1 are similarly defined. Let G and H be the centroid and orthocentre, respectively, of triangle ABC.

   Prove that A_1, B_1, C_1, G, and H all lie on a common circle.

2. (a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b < \sqrt{n} + 1$ and $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$.

   (b) Prove there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b < \sqrt{n}$ and $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$.

3. Let n be a given positive integer. Determine the smallest positive integer k with the following property:

   > It is possible to mark k cells on a $2n \times 2n$ square array so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

THE MATHEMATICS ASHES RESULTS

The 10th Mathematics Ashes competition at the joint pre-IMO training camp in Rio de Janeiro was won by the UK. The results for the two teams were as follows, with Australia scoring a total of 63 and the UK scoring a total of 83:

Australia

	Q1	Q2	Q3	
AUS 1	7	7	0	14
AUS 2	7	3	0	10
AUS 3	0	7	0	7
AUS 4	2	3	2	7
AUS 5	2	5	2	9
AUS 6	7	7	2	16
Total	25	32	6	63

United Kingdom

	Q1	Q2	Q3	
UNK 1	7	6	2	15
UNK 2	7	7	0	14
UNK 3	7	7	0	14
UNK 4	7	7	2	16
UNK 5	6	2	0	8
UNK 6	7	7	2	16
Total	41	36	6	83

IMO TEAM LEADER'S REPORT

The 58th International Mathematical Olympiad (IMO) was held on 12–23 July 2017 in Rio de Janeiro, Brazil. This was the largest IMO in history with a record number of 615 high school students from 111 countries participating. Of these, 62 were girls.

Each participating country may send a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the Jury. This Jury was ably chaired by Nicolau Saldanha.

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 32 problems from 150 problem proposals submitted by 51 of the participating countries from around the world. During the Jury meetings three of the shortlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. Eventually, the Jury finalised the exam questions and then made translations into the 57 languages required by the contestants.

The six questions that ultimately appeared on the IMO contest are described as follows.

1. An easy number theoretic sequence problem proposed by South Africa.
2. A medium to difficult functional equation proposed by Albania.
3. A difficult game theory problem with incomplete information proposed by Austria.
4. A relatively easy classical geometry problem proposed by Luxembourg.
5. A medium to difficult combinatorics problem reminiscent of the Erd}os-Szekeres theorem. It was proposed by Russia.
6. A difficult problem, somewhat reminiscent of Lagrange interpolation, combining number theory and polynomials. It was proposed by the United States of America.

These six questions were posed in two exam papers held on Tuesday 18 July and Wednesday 19 July. Each paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

For many years now there has been an opening ceremony prior to the first day of competition. Following the formal speeches there was the parade of the teams and the 2017 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes, which had been agreed to earlier. A local team of markers called Coordinators also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant's exam script that is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader's country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury.

Problem 1 turned out to be the easiest problem in the IMO for many years[1] with an average score of 5.94. In contrast, problem 3 ended up being the most difficult problem ever in the IMO's 58 year history. It averaged only 0.04. Just two students managed to score full marks on it, while 608 students were unable to score a single point.

The medal cuts were set at 25 for gold, 19 for silver and 16 for bronze. Consequently, there were 291 (=47.3%) medals awarded. The medal distributions[2] were 44 (=7.8%) gold, 90 (=14.6%) silver and 153 (=24.9%) bronze. These awards were presented at the closing ceremony. Of those who did not get a medal, a further 222 contestants received an honourable mention for solving at least one question perfectly.

1 We have to go back to the IMO in 1981 to find problems with higher average scores.

2 The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of gold, silver, and bronze medals should be approximately in the ratio 1:2:3.

No contestant was able to achieve a perfect score of 42. The top score was 35 which was obtained by the following three students.

Amirmojtaba Sabour, Iran

Yuta Takaya, Japan

Hũu Quốc Huy Hoàng, Vietnam

In an effort to encourage female participation at the IMO, five girls were each given a special award at the closing ceremony.

Congratulations to the Australian IMO team on their solid performance this year. They finished 34th in the rankings,[3] bringing home three Silver and two Bronze medals.

The three Silver medallists were Matthew Cheah, year 12, Penleigh and Essendon Grammar School, VIC, Linus Cooper, year 11, James Ruse Agricultural High School, NSW, and Guowen Zhang, year 11, St Joseph's College, QLD.

The Bronze medallists were James Bang, year 10, Baulkham Hills High School, NSW, and William Hu, year 11, Christ Church Grammar School, WA.

Hadyn Tang, year 8, Trinity Grammar School, VIC received an Honourable Mention for his complete solution to problem 1.

The 2017 IMO was organised by Brazil's National Institute of Pure and Applied Mathematics (IMPA), and the Brazilian Mathematical Society.

The 2018 IMO is scheduled to be held July 3–14 in Cluj-Napoca, Romania. Venues for future IMOs have been secured up to 2022 as follows.

2019 United Kingdom

2020 Russia

2021 United States

2022 Norway

Much of the statistical information found in this report can also be found at the official website of the IMO.
www.imo-official.org

Angelo Di Pasquale
IMO Team Leader, Australia

3 The ranking of countries is not officially part of the IMO general regulations. However, countries are ranked each year on the IMO's official website according to the sum of the individual student scores from each country.

INTERNATIONAL MATHEMATICAL OLYMPIAD

IMO 2017
RIO DE JANEIRO - BRAZIL
58th International Mathematical Olympiad

English (eng), day 1

Tuesday, July 18, 2017

Problem 1. For each integer $a_0 > 1$, define the sequence $a_0,\ a_1,\ a_2,\ \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \qquad \text{for each } n \geqslant 0.$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

IMO 2017
RIO DE JANEIRO - BRAZIL
58th International Mathematical Olympiad

English (eng), day 2

Wednesday, July 19, 2017

Problem 4. Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 5. An integer $N \geqslant 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

(N) no one stands between the two shortest players.

Show that this is always possible.

Problem 6. An ordered pair (x, y) of integers is a *primitive point* if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

INTERNATIONAL MATHEMATICAL OLYMPIAD SOLUTIONS

1. **Solution 1** (Hadyn Tang, year 8, Trinity Grammar School, VIC. Hadyn achieved an honourable mention with the 2017 Australian IMO team.)

   Answer: All positive integers n that are multiples of 3.

   Case 1 $a_0 \equiv 0 \pmod 3$

   Clearly all terms of the sequence are multiples of 3. Let a_i be term of the sequence having minimal value.

   Suppose for the sake of contradiction that $a_i > 9$. Let $x \geq 1$ be the largest integer such that $3^{2^x} < a_i$. The sequence $a_i, a_{i+1}, a_{i+2}, \ldots$ is formed by adding 3 each time until a perfect square, a_j say, is reached. Note that a_j cannot exceed $3^{2^{x+1}}$ because $3^{2^{x+1}}$ is also a perfect square that is a multiple of 3. It follows that

   $$a_{j+1} = \sqrt{a_j} \leq \sqrt{3^{2^{x+1}}} = 3^{2^x} < a_i,$$

   which contradicts the minimality of a_i.

   Hence $a_i \leq 9$. It follows that the sequence enters the cycle

   $$3 \to 6 \to 9 \to 3 \to 6 \to 9 \to \cdots$$

   which contains the number 3 infinitely many times.

   Case 2 $a_0 \equiv 2 \pmod 3$

   Since no perfect square is congruent to 2 modulo 3, it follows inductively that $a_{n+1} = a_n + 3$ for all non-negative integers n. As the sequence is strictly increasing, it cannot contain the same value infinitely many times.

   Case 3 $a_0 \equiv 1 \pmod 3$

   Clearly no term of the sequence is a multiple of 3.

   If $a_i \equiv 2 \pmod 3$ for some positive integer i, then by a similar argument as given in case 2, the sequence contains the same value infinitely many times.

   We are left to discuss the case $a_n \equiv 1 \pmod 3$ for all non-negative integers n. Let a_i be a term of the sequence having minimal value.

   Suppose for the sake of contradiction that $a_i > 16$. Let $x \geq 1$ be the largest integer such that $2^{2^x} < a_i$. The sequence $a_i, a_{i+1}, a_{i+2}, \ldots$ is formed by adding 3 each time until a perfect square, a_j say, is reached. Note that a_j cannot exceed $2^{2^{x+1}}$ because $2^{2^{x+1}}$ is also a perfect square that is congruent to 1 modulo 3. It follows that

   $$a_{j+1} = \sqrt{a_j} \leq \sqrt{2^{2^{x+1}}} = 2^{2^x} < a_i,$$

   which contradicts the minimality of a_i.

   Hence $a_i \leq 16$. It is easy to see inductively that $a_0 > 1$ implies $a_n > 1$ for all positive integers n. And since

   $$7 \to 10 \to 13 \to 16 \to 4 \to 2,$$

   we see that any such sequence eventually reaches the number 2 which is not congruent to 1 modulo 3. This final contradiction concludes the proof. ☐

   Comment The final contradiction in the above proof might lead one to think that if $a_0 \equiv 1 \pmod 3$, then the resulting sequence always contains the number 2. But this does not follow from the above logic. Rather, the final contradiction merely shows is that it is impossible to have $a_i \equiv 1 \pmod 3$ for all non-negative integers i. For example, if $a_0 = 19$, then $a_1 = 22$, $a_2 = 25$, and $a_3 = 5 \equiv 2 \pmod 3$.

Solution 2 (William Hu, year 11, Christ Church Grammar School, WA. William was a Bronze medallist with the 2017 Australian IMO team.)

Case 1 $a_0 \equiv 0 \pmod 3$

Clearly all terms of the sequence are multiples of 3. Let $9x^2$ be the smallest square number that is a multiple of 3 and that is greater than a_0.

We claim that $a_n \leq 9x^2$ for all positive integers n. If this is not true, let $i \geq 1$ be the smallest index such that $a_i > 9x^2$.

Since $3 \mid a_i$, this implies that $a_i \geq 9x^2 + 3$. Furthermore, since i is the smallest index with $a_i > 9x^2$, we cannot have $a_{i-1} = a_i^2$. Hence $a_{i-1} = a_i - 3 \geq 9x^2$. Again since i is the smallest index with $a_i > 9x^2$, it follows that $a_{i-1} = 9x^2$. But then $a_i = \sqrt{a_{i-1}} = 3x < 9x^2$, which contradicts $a_i \geq 9x^2 + 3$.

From the above, it follows that the sequence is bounded above by $9x^2$. Since all terms of the sequence are positive integers with an upper bound, by the infinite pigeonhole principle at least one of the numbers $1, 2, 3, \ldots, 9x^2$ occurs an infinite number of times in the sequence, as desired.

For the remaining cases, note that if $a_0 \not\equiv 0 \pmod 3$, then $a_i \not\equiv 0 \pmod 3$ for all non-negative integers i.

Case 2 $a_i \equiv 2 \pmod 3$ for some non-negative integer i.

Since no perfect square is congruent to 2 modulo 3, it follows inductively that $a_{n+1} = a_n + 3 \equiv 2 \pmod 3$ for each integer $n \geq i$. Hence the sequence is strictly increasing from the ith term onward and so attains no value infinitely many times.

Case 3 $a_i \equiv 1 \pmod 3$ for all non-negative integers i.

Let m be the smallest value that the sequence attains. Thus $a_i = m$ for some i. Since $m \equiv 1 \pmod 3$ we also have $(m-2)^2 \equiv 1 \pmod 3$.

The sequence $a_i, a_{i+1}, a_{i+2}, \ldots$ is formed by adding 3 each time until a perfect square, a_j say, is reached.

If $(m-2)^2 \geq m$ then a_j cannot exceed $(m-2)^2$ because $(m-2)^2$ is also a perfect square that is congruent to 1 modulo 3. In this case we would have

$$a_{j+1} = \sqrt{a_j} \leq m - 2 < a_i$$

which contradicts the minimality of m.

Thus $(m-2)^2 < m$, and so $m < 4$. Since $m \equiv 1 \pmod 3$, we have $m = 1$. However it is easy to see inductively that $a_0 > 1$ implies $a_n > 1$ for all positive integers n. So case 3 never occurs. $\square$

2. **Solution** (Based on the presentation of Guowen Zhang, year 11, St Joseph's College, QLD. Guowen was a Silver medallist with the 2017 Australian IMO team.)

Answers: $f(x) = 0$, $f(x) = x - 1$, and $f(x) = 1 - x$. It is straightforward to verify that these satisfy the given functional equation.

For reference we are given

$$f(f(x)f(y)) + f(x + y) = f(xy) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Putting $x = y = 0$ in (1) yields $f(f(0)^2) = 0$. Hence 0 is in the range of f.

Case 1 There exists a real number $a \neq 1$ such that $f(a) = 0$.

Putting $x = a$ in (1) yields

$$f(0) + f(a + y) = f(ay).$$

Since $a \neq 1$, we may solve the equation $a + y = ay$ for y. This allows us to deduce $f(0) = 0$. Then putting $x = 0$ in (1) yields $f(y) = 0$ for all $y \in \mathbb{R}$.

Case 2 We have $f(1) = 0$ and $f(a) \neq 0$ for all $a \neq 1$.

Since $f(f(0)^2) = 0$, it follows that $f(0)^2 = 1$. Note that $f(x)$ solves (1) if and only if $-f(x)$ solves (1). So it suffices to consider only the case $f(0) = 1$.

Putting $y = 0$ into (1) yields

$$f(f(x)) + f(x) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

Replacing x with $f(x)$ in (2) yields

$$f(f(f(x))) + f(f(x)) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{3}$$

Subtracting (2) from (3) yields

$$f(f(f(x))) = f(x) \quad \text{for all } x \in \mathbb{R}. \tag{4}$$

If f were injective then (4) would imply

$$f(f(x)) = x \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

Putting (5) into (2) would then yield $f(x) = 1 - x$ for all $x \in \mathbb{R}$.

It only remains to prove that f is injective.

Recall that $f(x)$ is a solution if and only if $-f(x)$ is. So we may return to the earlier situation where we had $f(0)^2 = 1$ and only consider the case $f(0) = -1$.

Putting $y = 1$ in (1) yields $f(x + 1) = f(x) + 1$ for all $x \in \mathbb{R}$. An easy induction yields.

$$f(x + n) = f(x) + n \quad \text{for all } x \in \mathbb{R} \text{ and for all } n \in \mathbb{N}^+. \tag{6}$$

Comment The remainder of this proof is by the Problem Selection Committee.

Recall that it only remains to prove that f is injective.

Suppose that $f(a) = f(b)$ for some $a, b \in \mathbb{R}$. From (6) we see that

$$f(a + n + 1) = f(b + n) + 1 \quad \text{for all } n \in \mathbb{N}^+. \tag{7}$$

Consider the following the system of equations.

$$x + y = a + n + 1 \tag{8a}$$
$$xy = b + n \tag{8b}$$

By Vieta's formulas, x and y are the roots of

$$z^2 - (a + n + 1)z + b + n = 0.$$

This has real solutions if and only if

$$(a + n + 1)^2 \geq 4(b + n),$$

which is true provided that n is sufficiently large.

So choose n sufficiently large so that there exist real x and y solving (8a) and (8b). Putting these x and y into (1) yields

$$\begin{aligned}
& & f(f(x)f(y)) + f(a + n + 1) &= f(b + n) & & \\
&\Rightarrow & f(f(x)f(y)) + 1 &= 0 & & \text{(from (7))} \\
&\Rightarrow & f(f(x)f(y) + 1) &= 0 & & \text{(from (6))} \\
&\Rightarrow & f(x)f(y) + 1 &= 1 & & \text{(we are in case 2)} \\
&\Rightarrow & f(x)f(y) &= 0. & &
\end{aligned}$$

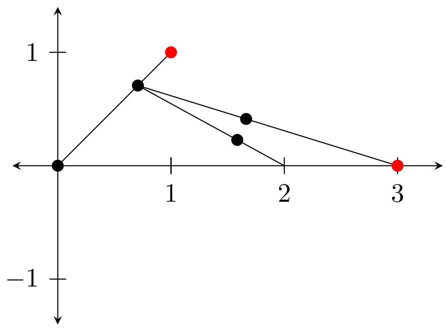
By symmetry, we may assume without loss of generality that $f(x) = 0$. Since we are in case 2, we have $x = 1$. Thus (8b) yields $y = b + n$, and then (8a) yields $a = b$. It follows that f is injective, which concludes the proof. ☐

3. This was by far the hardest problem of the 2017 IMO. In fact it was the hardest problem ever set on an IMO. Its average score was just 0.04 out of 7.[1] Only two[2] of the 615 contestants were able to solve[3] this problem and only a further 5 contestants managed to achieve a nonzero[4] score on it.

   Discussion Let us call the point being reported by the tracking device on each move a *ping*.[5] Part of the difficulty of the problem is that it is very tempting to think that since the rabbit is invisible, the hunter's best strategy is simply to follow the ping on each move. However, this is simply not the case as the following concrete example shows.

   Without loss of generality we may assume that initially both the hunter and the rabbit are at the origin of the Cartesian plane. Suppose that the first ping is at $(1, 1)$. In following the ping, the hunter arrives at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on her first move.

   Suppose that the second ping is at $(3, 0)$. Then it is obvious that the only place the rabbit could be after its second move is at $(2, 0)$, and the hunter is capable of deducing this. So the best strategy for the hunter on her second move is not to follow the ping and move towards $(3, 0)$, but instead to follow the rabbit by moving towards $(2, 0)$ because she knows exactly where it is!



   The astute reader might counter that the above scenario only arises because the second ping makes it possible to determine the exact location of the rabbit. However if the second ping were instead at $(2.99, 0)$, then the hunter cannot determine the exact location of the rabbit. Yet the hunter can still deduce that the rabbit is very much closer to $(2, 0)$ than to $(2.99, 0)$. So even in this case where the exact location of the rabbit is not known, the hunter can still sufficiently narrow down the location of the rabbit to see that following the ping is still not necessarily the best strategy.

   What is the fundamental flaw in assuming that following the ping is the best strategy for the hunter? It is that we are basically assuming that the hunter is handicapped with some sort of amnesia that prevented her from remembering any information about previous pings. If the hunter did have such amnesia, then yes, following the ping would be her best strategy. However, we are not entitled to assume this. Consequently, any attempt at the problem that implicitly assumed the hunter had such amnesia was awarded 0 points.

[1] The previous record low average was for problem 6 in the 2007 IMO which averaged 0.15 out of 7.

[2] Problem 3 in the 2007 IMO was also only solved by two contestants that year, however its average score was 0.3 out of 7.

[3] The two contestants who solved this problem were Linus Cooper from Australia and Mikhail Ivanov from Russia.

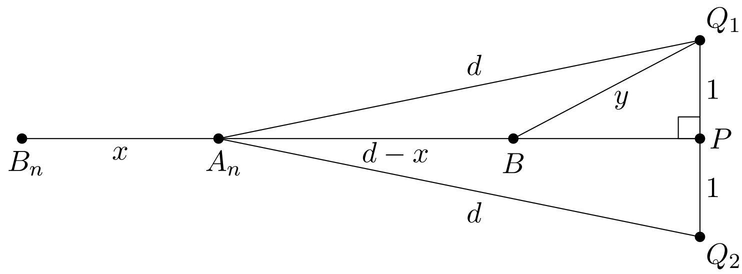
[4] So 608 of the 615 contestants scored 0 points for this problem. No other problem in the history of the IMO has scored so many 0s.

[5] We are imagining that the tracking device is a sort of radar.

Solution (Linus Cooper, year 11, James Ruse Agricultural High School, NSW. Linus was a Silver medallist with the 2017 Australian IMO team.)

We shall prove that there is no strategy for the hunter that guarantees that the distance between her and the rabbit is at most 100 after 10^9 rounds.

The key to the problem is to consider the scenario illustrated in the diagram below. Recall that A_n and B_n denote the respective positions of the rabbit and the hunter after n rounds. Let ℓ denote the line B_nA_n, and let x be the distance between B_n and A_n. Points Q_1 and Q_2 are at distance 1 from the line ℓ and are at distance d from the point A_n, where $d > x$ is a positive integer to be chosen later.



Imagine the rabbit at A_n flipping a coin. If it comes up heads, the rabbit proceeds directly to Q_1 in the next d rounds. If it comes up tails, then the rabbit proceeds directly to Q_2 in the next d rounds.[6] Suppose further that for each round, the tracking device reports the point that is at the foot of the perpendicular from the rabbit's location to the line ℓ. Note that these reported points are consistent with the rabbit's path irrespective of whether it heads to Q_1 or Q_2.

Turning out attention to the hunter, one possibility is that she moves one unit to the right along ℓ per round. This would place her at point B which is at distance d from B_n. In this case, after d rounds the distance between the hunter and the rabbit would be $y = BQ_1 = BQ_2$. Note that if the hunter does anything else she will end up strictly to the left of B. If she ends up on or above ℓ, then the distance between her and Q_2 would be more than y. If she ends up below ℓ, then the distance between her and Q_1 would be more than y. So no choice of the hunter can guarantee that the distance between her and the rabbit is less than y after d rounds.

Let us compute a lower bound for y. Pythagoras' theorem applied to $\triangle A_nPQ_1$ yields $AP = \sqrt{d^2 - 1}$. Thus $BP = \sqrt{d^2 - 1} - (d - x)$. Pythagoras' theorem applied to $\triangle BPQ_1$ yields

$$\begin{aligned}
y^2 &= 1^2 + (x + \sqrt{d^2 - 1} - d)^2 \\
&= x^2 + 2d^2 - 2d\sqrt{d^2 - 1} - 2x(d - \sqrt{d^2 - 1}) \\
&= x^2 + 2(d - x)(d - \sqrt{d^2 - 1}) \\
&= x^2 + \frac{2(d - x)}{d + \sqrt{d^2 - 1}} \\
&> x^2 + \frac{2(d - x)}{d + \sqrt{d^2}} \\
&= x^2 + 1 - \frac{x}{d}.
\end{aligned}$$

[6] Alternatively, we could loosely think of the rabbit as being a quantum rabbit which has the two possible quantum states of either being at Q_1 or Q_2 after d rounds.

Choosing $d = 2\lceil x \rceil$ yields $y^2 > x^2 + \frac{1}{2}$. But for $x \geq 1$ we have $x^2 + \frac{1}{2} > (x + \frac{1}{5x})^2$. It follows that

$$y > x + \frac{1}{5x}.$$

To summarise, we have shown the following lemma.

Lemma If at some stage the distance between the hunter and the rabbit is $x > 1$, then after a further $2\lceil x \rceil$ rounds, the distance between the hunter and the rabbit potentially exceeds $x + \frac{1}{5x}$.

To clarify terminology, when we say that the distance between the hunter and the rabbit is potentially something, we mean that the hunter cannot guarantee that the distance between her and the rabbit is less than the said potential.

We call a set of moves in the lemma that the rabbit might make a *swoop* if it results in the distance between the hunter and the rabbit potentially exceeding $x + \frac{1}{5x}$.

We use the lemma and the swoop concept to finish the problem.

To start with, note that after one round, the distance between the hunter and the rabbit is potentially equal to 2. This is because the tracking device might simply report the original starting position of the rabbit and hunter, which gives no new information to the hunter. So whatever direction the hunter moves in for round 1, the rabbit might have gone in the opposite direction.

Next suppose that after some rounds, the distance between the hunter and the rabbit is at least $x \geq 2$. Let $n = \lfloor x \rfloor$. We claim that after a further $10(n+1)^2$ rounds the potential distance between the hunter and the rabbit is at least $n + 1$.

Assume for the sake of contradiction that this is not the case. Hence by the lemma, each swoop potentially increases the distance between the hunter and the rabbit by more than $\frac{1}{5(n+1)}$. Hence after at most $5(n+1)$ swoops, the distance between the hunter and the rabbit is potentially increased by more than 1, and so is potentially at least $n+1$. Since each swoop requires no more than $2(n+1)$ rounds, at total of at most $2(n+1) \times 5(n+1) = 10(n+1)^2$ rounds have been used in the process. This contradiction establishes the claim.

From the claim we may calculate an upper bound U for the number of rounds needed to ensure that the distance between the hunter and the rabbit is potentially at least 101. It is given by

$$\begin{aligned} U &\leq 1 + 10 \cdot 3^2 + 10 \cdot 4^2 + \cdots + 10 \cdot 101^2 \\ &\ll 10 \cdot 100 \cdot 101^2 \\ &\ll 10^9. \end{aligned}$$

So the distance between the hunter and the rabbit potentially exceeds 100 in well under 10^9 rounds. Once this occurs the rabbit could simply hop directly away from the hunter until all 10^9 rounds have occurred. In this way the hunter cannot guarantee that the distance between her and the rabbit is at most 100. $\square$

Comment The number 100 in the given problem is nowhere near sharp. Taking a little more care in the above final calculation would show that after 10^9 rounds, the distance between the rabbit and the hunter is potentially at least 668.

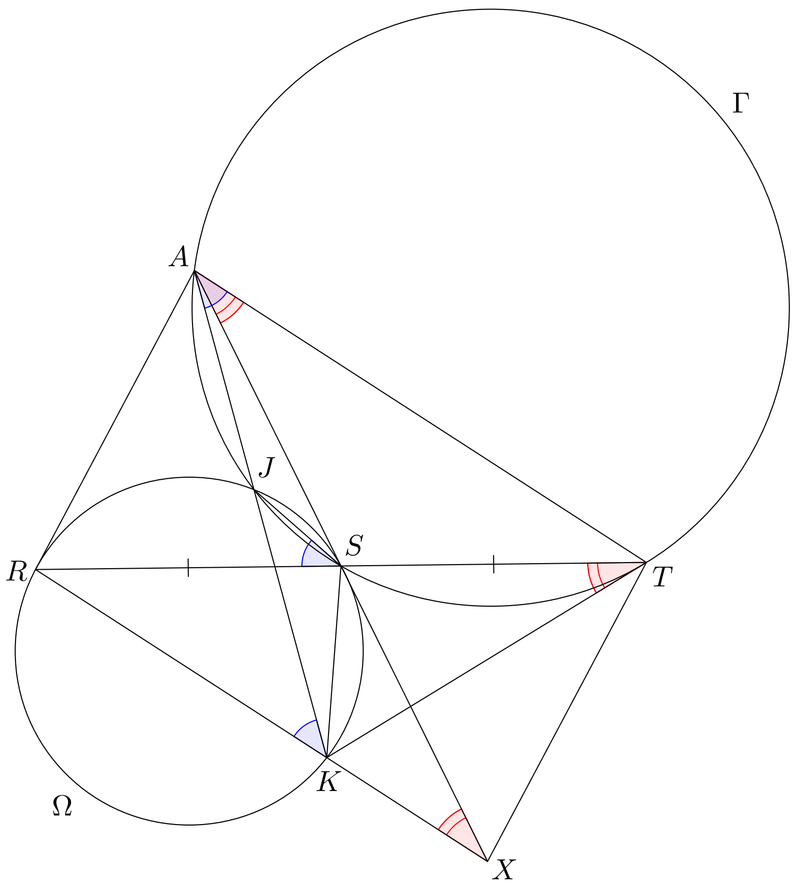
4. **Solution 1** (Found independently by Matthew Cheah, year 12, Penleigh and Essendon Grammar School, VIC, and William Hu, year 11, Christ Church Grammar School, WA. Matthew was a Silver medallist and William was a Bronze medallist with the 2017 Australian IMO team.)

From cyclic quadrilaterals $RKSJ$ and $STAJ$ we have

$$\angle JKR = \angle JSR = \angle JAT.$$

Since A, J, and T are collinear, it follows that $RK \parallel AT$.

Let X be the intersection of lines AS and RK. Since $RX \parallel AT$ and S is the midpoint of RT, it follows that $RXTA$ is a parallelogram.[7]



From parallelogram $RXTA$ and using the alternate segment theorem on circle Ω and line AR, we have

$$\angle SKR = \angle SRA = \angle XTS.$$

Hence quadrilateral $KXTS$ is cyclic.

This along with parallelogram $RXTA$ imply

$$\angle STK = \angle SXK = \angle SAT.$$

Therefore, by the alternate segment theorem, KT is tangent to Γ at T. $\square$

[7] An easy way to see this is to note that $\triangle SRX \equiv \triangle STA$ (AAS), and so $RX = TA$.

Solution 2 (James Bang, year 10, Baulkham Hills High School, NSW. James was a Bronze medallist with the 2017 Australian IMO team.)

From cyclic quadrilaterals $RKSJ$ and $STAJ$ we have

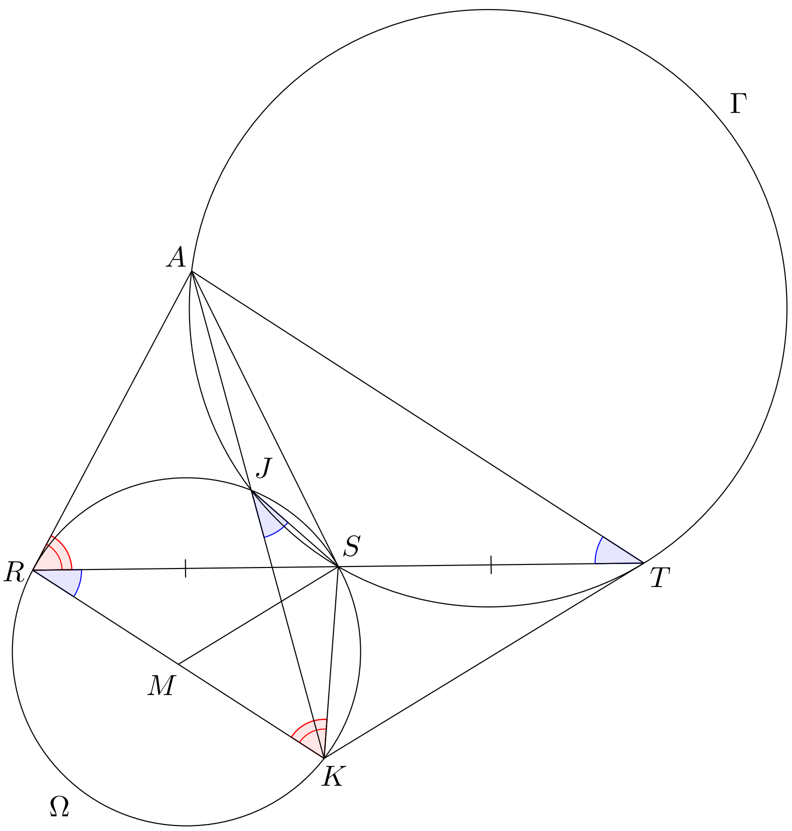
$$\angle KRS = \angle KJS = \angle ATS.$$

Since AR is tangent to Ω at R, by the alternative segment theorem we have

$$\angle SKR = \angle TRA.$$

Hence $\triangle SKR \sim \triangle ART$ (AA).

Let M be the midpoint of KR. Note that since S is also the midpoint of RT, we have $MS \parallel KT$.



Since M and S are the corresponding midpoints of KR and RT in similar triangles SKR and ART, it follows that $SKMR \sim ARST$. Using this and parallel lines SM and KT, we deduce

$$\angle SAT = \angle RSM = \angle STK.$$

Therefore, by the alternate segment theorem, KT is tangent to Γ at T. $\square$

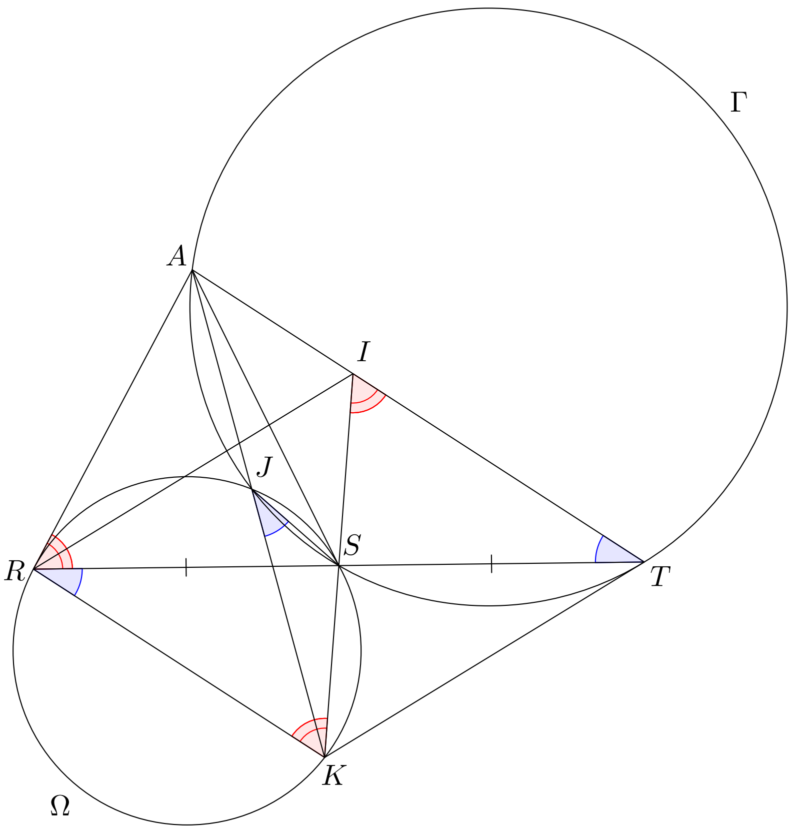
Solution 3 (Guowen Zhang, year 11, St Joseph's College, QLD. Guowen was a Silver medallist with the 2017 Australian IMO team.)

From cyclic quadrilaterals $RKSJ$ and $STAJ$ we have

$$\angle KRS = \angle KJS = \angle ATS.$$

Hence $RK \parallel AT$.

Let I be the intersection of lines KS and AT. Since $RK \parallel IT$ and S is the midpoint of RT, it follows that $RKTI$ is a parallelogram.[8]



From parallelogram $RKTI$ and using the alternate segment theorem on circle Ω and line AR, we have

$$\angle SIT = \angle SKR = \angle SRA.$$

Hence quadrilateral $RSIA$ is cyclic.

This along with parallelogram $RKTI$ imply

$$\angle SAT = \angle SRI = \angle STK.$$

Therefore, by the alternate segment theorem, KT is tangent to Γ at T. $\square$

[8] An easy way to see this is to note that $\triangle SRK \equiv \triangle STI$ (AAS), and so $RK = TI$.

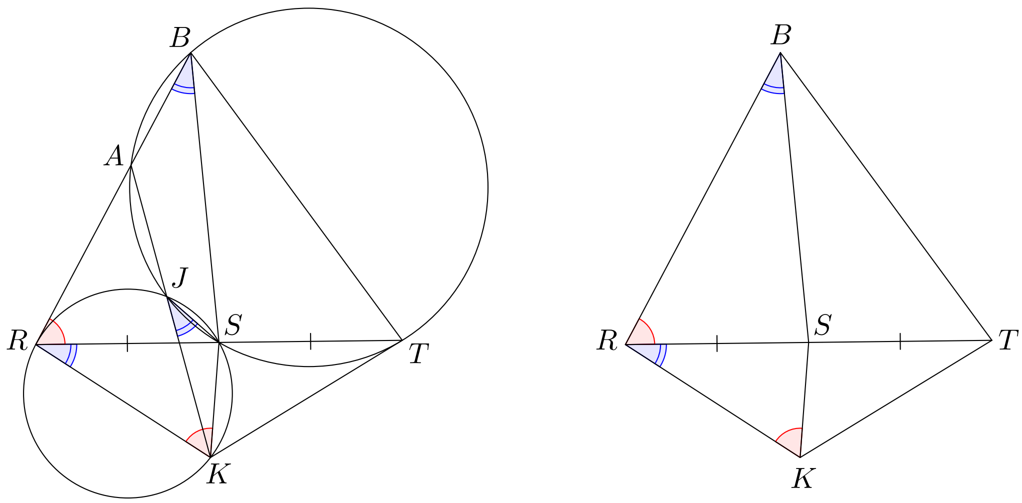
Solution 4 (Angelo Di Pasquale, Leader of the 2017 Australian IMO team)

Since AR is tangent to Ω at R, by the alternative segment theorem we have

$$\angle SKR = \angle TRA.$$

Let B be the second intersection point of line RA with circle Γ. Then

$$\angle KRS = \angle KJS = \angle ABS.$$



From here on, we focus on the part of the diagram shown on the right above.

We have $\triangle KRS \sim \triangle RBS$ (AA). Hence

$$\frac{KS}{SR} = \frac{SR}{SB}.$$

Since $SR = ST$, it follows from the above equality that

$$\frac{KS}{ST} = \frac{ST}{SB}. \tag{1}$$

The external angle sums in triangles KRS and RBS yield

$$\angle KST = \angle SKR + \angle KRS = \angle SRB + \angle RBS = \angle TSB.$$

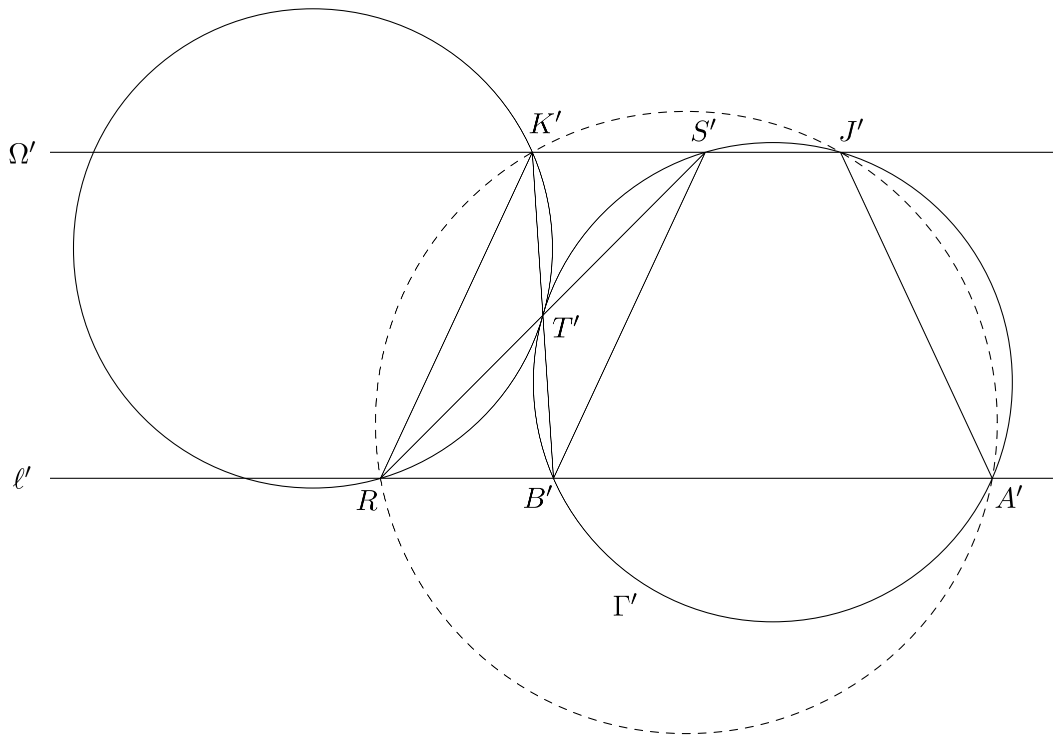
Combining this with (1) implies $\triangle KST \sim \triangle TSB$. Therefore $\angle STK = \angle SBT$. Hence by the alternate segment theorem, circle BST is tangent to KT at T. Since circle BST is Γ, the result follows. $\square$

Comment This solution shows a connection between this problem and problem 6 of the 2017 Australian Mathematical Olympiad (AMO). See solution 1 of the AMO problem found earlier in this document. It is readily seen that points A, B, C, D, and M in the AMO problem correspond to points R, B, T, K, and S in this solution to the IMO problem. The IMO Jury was made aware of the this connection but decided that the two problems were sufficiently dissimilar to permit the use of the proposed IMO problem.

Solution 5 (Dan Carmon, Leader of the 2017 Israeli IMO team)

Consider an inversion of arbitrary radius about R. For any object Z, let Z' denote its image under the inversion. We determine the images of the objects given in the problem statement as follows.

- Since S is the midpoint of RT, it follows that T' is the midpoint of RS'.
- Circle Γ through S, T, A, and J becomes circle Γ' through S', T', A', and J'.
- Line ℓ through R and A becomes line ℓ' through R and A'.
- Since the circle Ω through R, K, S, and J is tangent to the line ℓ at R, it follows that Ω becomes the line Ω' through K', S', and J', and is parallel to ℓ'.
- The line through A, J, and K becomes a circle through R, A', J', and K'.
- The line through K and T becomes a circle through R, K', and T'.
- It is required to prove that circle $RK'T'$ is tangent to Γ'.



Let B' be the second intersection point of circle $S'T'A'J'$ with line RA'.[9]

We have $\angle A'RK' = 180^\circ - \angle S'J'A' = \angle A'B'S'$. Thus $K'R \parallel S'B'$. Since also $K'S' \parallel RB'$, we see that $RK'S'B'$ is a parallelogram.

The diagonals of any parallelogram bisect each other. But T' is the midpoint of RS'. So it is also the midpoint of $K'B'$.

The half turn about T' interchanges points R and K' with points S' and B', respectively. So it interchanges circle $T'RK$ with circle $TS'B'$. Since T' is common to both these circles, it follows that the two circles are tangent at T', as desired. $\square$

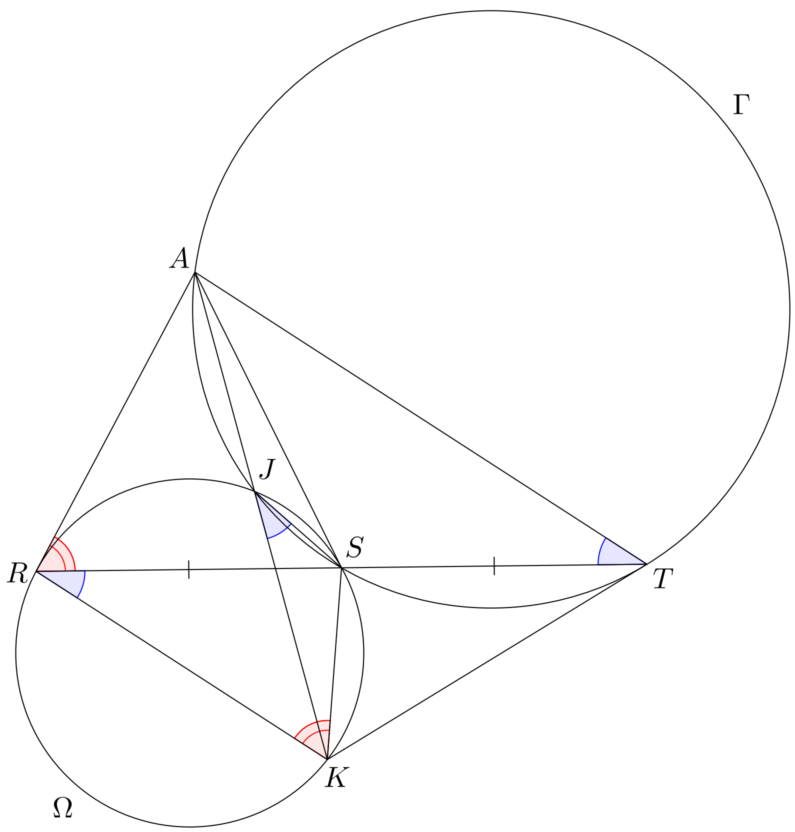
[9]The point B' happens to be the image under the inversion of point B from solution 4.

Solution 6 (Problem Selection Committee)

As in solution 2, we deduce the equal angles as marked in the diagram so that $\triangle ART \sim \triangle SKR$ (AA). Hence

$$\frac{RT}{KR} = \frac{AT}{SR} = \frac{AT}{ST}.$$

Since also $\angle KRT = \angle ATS$, it follows that $\triangle KRT \sim \triangle STA$ (PAP). Hence $\angle RTK = \angle SAT$. The result now follows from the alternate segment theorem. $\square$



5. **Solution** (Matthew Cheah, year 12, Penleigh and Essendon Grammar School, VIC. Matthew was a Silver medallist with the 2017 Australian IMO team.)

Let us give T-shirts in n different colours to the players as follows. The shortest $n+1$ players wear T-shirts with colour 1, the next shortest $n+1$ players wear T-shirts with colour 2, and so on up to the tallest $n+1$ players who wear T-shirts with colour n. It suffices to show that we can remove $n(n-1)$ players such that each colour T-shirt is represented twice among the remaining $2n$ players, and for each such colour, the two players who wear the T-shirt of that colour have no other players between them.

We present the formal proof along with an example that illustrates what is happening at each stage. The example is the case $n = 4$, where the T-shirt colours from left to right are as follows.

$$4\ 2\ 1\ 2\ 3\ 2\ 4\ 3\ 1\ 1\ 4\ 1\ 3\ 2\ 3\ 4\ 4\ 1\ 2\ 3$$

We start by scanning from left to right until we identify two T-shirts of the same colour for the first time. Note that this will happen within the first $n+1$ players. Suppose that colour X is the first colour to occur twice. In our example $X = 2$. Remove each player to the left of the second X who is wearing a T-shirt that does not have colour X. In our example this yields the following.

$$\cancel{4}\ \underline{2}\ \cancel{1}\ \underline{2}\ 3\ 2\ 4\ 3\ 1\ 1\ 4\ 1\ 3\ 2\ 3\ 4\ 4\ 1\ 2\ 3$$

Next, remove every occurrence of X that is to the right of the second X. In our example this yields the following.

$$2\ 2\ 3\ \cancel{2}\ 4\ 3\ 1\ 1\ 4\ 1\ 3\ \cancel{2}\ 3\ 4\ 4\ 1\ \cancel{2}\ 3$$

Note that colour X now occurs exactly twice, and that the two instances of X lie to the left of everything else. Also every other colour occurs either n or $n+1$ times.

Next, for each colour that is not X and that occurs $n+1$ times, remove one instance of that colour at random. In our example we only need to remove one 3. We choose one at random to yield the following.

$$2\ 2\ 3\ 4\ \cancel{3}\ 1\ 1\ 4\ 1\ 3\ 3\ 4\ 4\ 1\ 3$$

Thus we reach the situation where we have XX followed by a group G of $n(n-1)$ players. Note that in G, there are $n-1$ different coloured T-shirts, each occurring n times. This shows that the truth of the case with n different coloured T-shirts where each colour occurs $n+1$ times follows inductively from the truth of the case with $n-1$ different coloured T-shirts where each colour occurs n times. To complete the proof, we need only to remark that the base case $n = 1$ is trivially true. $\square$

6. **Solution** (Dan Carmon, Leader of the 2017 Israeli IMO team)

We are asked to prove that for any finite set S of primitive points there exists a homogeneous polynomial $p(x, y)$ of positive degree whose coefficients are integers such that $p(u, v) = 1$ whenever (u, v) is a point in S.

A simplifying step is to show that, at any stage, we can assume that one of the points in S is $(1, 0)$. We justify this as follows. If (r, s) is a point in S, then since $\gcd(r, s) = 1$, there are integers a and b such that $ar + bs = 1$. Consider the transformation

$$t\colon (x, y) \mapsto (ax + by, ry - sx).$$

Let $S_1 = \{t(u, v)\colon (u, v) \in S\}$. Note that $t(r, s) = (1, 0) \in S_1$.

It is readily checked that the inverse of t is given by

$$t^{-1}\colon (x, y) \mapsto (rx - by, sx + ay).$$

This implies that each point of S_1 is primitive.

Now if p_1 is a polynomial satisfying the requirements of the problem for $S_1 = t(S)$, then the polynomial $p = p_1 \circ t$ is a polynomial satisfying the requirements of the problem for S. This completes our justification of the simplifying step.

The simplifying step will help us to complete the proof by using induction on the size of S.

For the base case, if S consists of the single point $(1, 0)$, then $p(x, y) = x + y$ satisfies the requirements of the problem.

For the inductive step, assume that a required polynomial exists whenever S has size at most n. Consider any S with $n + 1$ points. By our simplifying step we may assume that $(1, 0)$ is in S. Let $S' = \{(u_i, v_i)\colon i = 1, 2, \ldots, n\}$ be the set obtained by removing $(1, 0)$ from S. By the inductive assumption there is a homogeneous polynomial $f(x, y) = a_0x^m + a_1x^{m-1}y + \cdots + a_my^m$ with integer coefficients such that

$$f(u_i, v_i) = 1 \quad \text{for all } (u_i, v_i) \in S'. \tag{1}$$

Consider the homogeneous polynomial

$$g(x, y) = (v_1x - u_1y)(v_2x - u_2y) \cdots (v_nx - u_ny). \tag{2}$$

Observe that $g(u_i, v_i) = 0$ for $i = 1, 2, \ldots, n$.

We claim that the homogeneous polynomial

$$p(x, y) = f(x, y)^j - cx^{mj-n}g(x, y)$$

satisfies the conditions of the problem for appropriately chosen integers j and c.

From (1) and (2) we have $p(u_i, v_i) = 1$ for all $(u_i, v_i) \in S'$. To complete the inductive step, we require $p(1, 0) = 1$. Since $p(1, 0) = f(1, 0)^j - cg(1, 0)$, this is the same as

$$a_0^j - 1 = cv_1v_2 \cdots v_n. \tag{3}$$

If $d \mid a_0$ and $d \mid v_i$, then $d \mid f(u_i, v_i) = 1$. Hence $\gcd(a_0, v_i) = 1$. It follows that $\gcd(a_0, v_1v_2 \cdots v_n) = 1$.

If $v_1v_2 \cdots v_n = 0$, then $|a_0| = 1$. Thus $j = 2$ and $c = 0$ satisfies (3).

If $v_1v_2 \cdots v_n \neq 0$, let j be a multiple of $\varphi(|v_1v_2 \cdots v_n|)$ such that $mj - n \geq 0$. Then $a_0^j \equiv 1 \pmod{v_1v_2 \cdots v_n}$ by Euler's theorem. This allows us to find c satisfying (3).

In either case we have completed the inductive step and hence also the proof. $\square$

INTERNATIONAL MATHEMATICAL OLYMPIAD RESULTS

Mark distribution by question

Mark	Q1	Q2	Q3	Q4	Q5	Q6
0	40	183	608	47	451	557
1	16	110	3	93	46	24
2	17	26	0	42	47	9
3	5	138	0	14	9	5
4	12	79	1	15	0	4
5	54	10	1	4	2	2
6	25	8	0	6	1	0
7	446	61	2	394	59	14
Total	615	615	615	615	615	615
Mean	5.94	2.30	0.04	5.03	0.97	0.29

Australian scores at the IMO

Name	Q1	Q2	Q3	Q4	Q5	Q6	Score	Award
James Bang	7	3	0	7	0	0	17	Bronze
Matthew Cheah	7	1	0	7	7	0	22	Silver
Linus Cooper	7	2	7	1	2	0	19	Silver
William Hu	7	0	0	7	2	0	16	Bronze
Hadyn Tang	7	0	1	2	0	0	10	Honourable Mention
Guowen Zhang	7	4	0	7	0	1	19	Silver
Totals	42	10	8	31	11	1	103	
Australian average	7.0	1.67	1.33	5.17	1.83	0.17	17.17	
IMO average	5.94	2.30	0.04	5.03	0.97	0.29	14.58	

The medal cuts were set at 25 for gold, 19 for silver and 16 for bronze.

Some country totals

Rank	Country	Total
1	South Korea	170
2	China	159
3	Vietnam	155
4	United States of America	148
5	Iran	142
6	Japan	134
7	Singapore	131
7	Thailand	131
9	Taiwan	130
9	United Kingdom	130
11	Russia	128
12	Georgia	127
12	Greece	127
14	Belarus	122
14	Czech Republic	122
14	Ukraine	122
17	Philippines	120
18	Bulgaria	116
18	Italy	116
18	Netherlands	116
18	Serbia	116
22	Hungary	115
22	Poland	115
22	Romania	115
25	Kazakhstan	113
26	Argentina	111
26	Bangladesh	111
26	Hong Kong	111
29	Canada	110
30	Peru	109
31	Indonesia	108
32	Israel	107
33	Germany	106
34	Australia	103
35	Croatia	102
35	Turkey	102
37	Brazil	101
37	Malaysia	101
39	France	100
39	Saudi Arabia	100

Distribution of awards at the 2017 IMO

Country	Total	Gold	Silver	Bronze	HM
Albania	67	0	0	1	5
Algeria	70	0	0	1	4
Argentina	111	1	2	1	2
Armenia	99	0	2	2	1
Australia	103	0	3	2	1
Austria	74	0	2	0	2
Azerbaijan	98	0	0	4	2
Bangladesh	111	0	2	2	2
Belarus	122	1	1	4	0
Belgium	80	0	1	2	2
Bolivia	41	0	0	0	4
Bosnia and Herzegovina	95	0	0	4	2
Botswana	19	0	0	0	1
Brazil	101	0	2	1	3
Bulgaria	116	0	4	2	0
Cambodia	11	0	0	0	1
Canada	110	1	2	2	1
Chile	67	0	0	1	4
China	159	5	1	0	0
Colombia	81	0	0	1	5
Costa Rica	58	0	0	0	5
Croatia	102	0	2	3	1
Cuba	13	0	0	0	1
Cyprus	93	0	0	5	1
Czech Republic	122	1	2	2	1
Denmark	77	0	0	1	5
Ecuador	66	0	0	1	4
Egypt	3	0	0	0	0
El Salvador	57	0	0	1	3
Estonia	72	0	1	0	4
Finland	56	0	0	0	6
France	100	0	2	2	2
Georgia	127	1	2	3	0
Germany	106	0	1	3	2
Ghana	6	0	0	0	0
Greece	127	1	4	1	0

Country	Total	Gold	Silver	Bronze	HM
Guatemala	20	0	0	0	1
Honduras	12	0	0	0	0
Hong Kong	111	1	1	3	1
Hungary	115	2	1	1	1
Iceland	45	0	0	0	3
India	90	0	0	3	3
Indonesia	108	0	2	3	1
Iran	142	2	3	1	0
Iraq	13	0	0	0	1
Ireland	80	0	0	2	4
Israel	107	0	3	2	0
Italy	116	2	1	1	2
Ivory Coast	11	0	0	0	0
Japan	134	2	2	2	0
Kazakhstan	113	1	2	1	1
Kenya	8	0	0	0	0
Kosovo	55	0	0	1	2
Kyrgyzstan	75	0	0	2	3
Latvia	84	0	0	3	2
Liechtenstein	22	0	0	0	2
Lithuania	69	0	0	2	3
Luxembourg	45	0	0	1	1
Macau	94	1	0	0	5
Macedonia (FYR)	77	0	0	1	4
Malaysia	101	0	2	2	2
Mexico	96	0	1	2	3
Moldova	83	0	1	0	4
Mongolia	93	0	1	2	3
Montenegro	42	0	0	1	2
Morocco	75	0	0	1	4
Myanmar	15	0	0	0	1
Nepal	3	0	0	0	0
Netherlands	116	1	2	1	1
New Zealand	94	0	0	3	3
Nicaragua	44	0	0	1	2
Nigeria	51	0	0	0	4
Norway	71	0	0	2	3
Pakistan	58	0	0	1	3

Country	Total	Gold	Silver	Bronze	HM
Panama	15	0	0	0	1
Paraguay	48	0	0	0	2
Peru	109	0	2	3	1
Philippines	120	0	3	3	0
Poland	115	1	0	5	0
Portugal	89	0	0	2	2
Puerto Rico	55	0	0	0	4
Romania	115	0	3	2	1
Russia	128	1	3	2	0
Saudi Arabia	100	0	2	2	1
Serbia	116	0	4	2	0
Singapore	131	2	1	2	1
Slovakia	75	0	0	1	5
Slovenia	90	0	0	2	4
South Africa	81	0	0	2	4
South Korea	170	6	0	0	0
Spain	86	0	0	3	2
Sri Lanka	80	0	0	3	3
Sweden	91	0	1	2	3
Switzerland	83	0	0	1	5
Syria	85	0	1	0	5
Taiwan	130	1	4	1	0
Tajikistan	95	0	0	3	3
Tanzania	5	0	0	0	0
Thailand	131	3	0	2	1
Trinidad and Tobago	15	0	0	0	1
Tunisia	59	0	0	1	3
Turkey	102	0	1	3	2
Turkmenistan	93	0	0	2	4
Uganda	22	0	0	0	1
Ukraine	122	1	2	2	1
United Kingdom	130	3	0	2	1
United States of America	148	3	3	0	0
Uruguay	43	0	0	0	3
Uzbekistan	69	0	1	0	4
Venezuela	59	0	0	2	2
Vietnam	155	4	1	1	0
Total (111 teams, 615 contestants)		48	90	153	222

ORIGIN OF SOME QUESTIONS

AMOC Senior Contest 2017

Questions 1, 2 and and 3 were submitted by Norman Do.

Question 4 was submitted by Angelo Di Pasquale.

Question 5 was submitted by Kevin McAvaney.

Australian Mathematical Olympiad 2017

Questions 1, 2, 5 and 6 were submitted by Norman Do.

Questions 3, 4 and 8 were submitted by Angelo Di Pasquale.

Question 7 was submitted by Ivan Guo.

Asian Pacfic Mathematics Olympiad 2017

Question 3 was composed by Norman Do and submitted by the AMOC Senior Problems Committee.

International Mathematical Olympiad 2017

Although no problem submitted by Australia appeared on the final IMO papers, three appeared on the IMO 2017 shortlist, which will be available to the public after the conclusion of IMO 2018.

- Problem A7 on the 2017 IMO shortlist was composed by Alan Offer. Alan is the AMOC State Director for Queensland and a member of the AMOC Senior Problems Committee. He represented Australia at the 1989 IMO, where he was awarded a Silver medal.
- Problem C2 on the 2017 IMO shortlist was composed by Ross Atkins. Ross represented Australia at the 2003 IMO, where he was awarded a Bronze medal.
- Problem G8 on the 2017 IMO shortlist was composed by Ivan Guo. Ivan is a member of the AMOC Senior Problems Committee. He represented Australia at the 2003 IMO and the 2004 IMO, where he was awarded a Bronze medal and a Gold medal, respectively.

MATHEMATICS CHALLENGE FOR YOUNG AUSTRALIANS HONOUR ROLL

Because of changing titles and affiliations, the most senior title achieved and later affiliations are generally used, except for the Interim committee, where they are listed as they were at the time.

Problems Committee for Challenge

Dr K McAvaney	Victoria, (Director)	12 years; 2006–2017
	Member	1 year; 2005-2006
Mr B Henry	Victoria (Director)	17 years; 1990–2006
	Member	12 years; 2006–2017
Prof P J O'Halloran	University of Canberra, ACT	5 years; 1990–1994
Dr R A Bryce	Australian National University, ACT	23 years; 1990–2012
Mr M Clapper	Australian Mathematics Trust, ACT	5 years; 2013–2017
Ms L Corcoran	Australian Capital Territory	3 years; 1990–1992
Ms B Denney	New South Wales	8 years; 2010–2017
Mr J Dowsey	University of Melbourne, VIC	8 years; 1995–2002
Mr A R Edwards	Department of Education, QLD	28 years; 1990–2017
Dr M Evans	Scotch College, VIC	6 years; 1990–1995
Assoc Prof H Lausch	Monash University, VIC	24 years; 1990–2013
Ms J McIntosh	AMSI, VIC	16 years; 2002–2017
Mrs L Mottershead	New South Wales	26 years; 1992–2017
Miss A Nakos	Temple Christian College, SA	25 years; 1993–2017
Dr M Newman	Australian National University, ACT	28 years; 1990–2017
Ms F Peel	St Peter's College, SA	2 years; 1999, 2000
Dr I Roberts	Northern Territory	5 years; 2013–2017
Ms T Shaw	SCEGGS, NSW	5 years; 2013–2017
Ms K Sims	New South Wales	19 years; 1999–2017
Dr A Storozhev	Attorney General's Department, ACT	23 years; 1994–2016
Prof P Taylor	Australian Mathematics Trust, ACT	20 years; 1995–2014
Mrs A Thomas	New South Wales	18 years; 1990–2007
Dr S Thornton	reSolve, ACT	20 years; 1998–2017
Miss G Vardaro	Wesley College, VIC	24 years: 1993–2006, 2008–2017

Visiting members

Prof E Barbeau	University of Toronto, Canada	1991, 2004, 2008
Prof G Berzsenyi	Rose Hulman Institute of Technology, USA	1993, 2002
Dr L Burjan	Department of Education, Slovakia	1993
Dr V Burjan	Institute for Educational Research, Slovakia	1993
Mrs A Ferguson	Canada	1992
Prof B Ferguson	University of Waterloo, Canada	1992, 2005
Dr D Fomin	St Petersburg State University, Russia	1994
Prof F Holland	University College, Ireland	1994
Dr A Liu	University of Alberta, Canada	1995, 2006, 2009
Prof Q Zhonghu	Academy of Science, China	1995
Dr A Gardiner	University of Birmingham, United Kingdom	1996
Prof P H Cheung	Hong Kong	1997
Prof R Dunkley	University of Waterloo, Canada	1997
Dr S Shirali	India	1998
Mr M Starck	New Caledonia	1999
Dr R Geretschläger	Austria	1999, 2013
Dr A Soifer	United States of America	2000
Prof M Falk de Losada	Colombia	2000
Mr H Groves	United Kingdom	2001
Prof J Tabov	Bulgaria	2001, 2010

Prof A Andzans	Latvia	2002
Prof Dr H-D Gronau	University of Rostock, Germany	2003
Prof J Webb	University of Cape Town, South Africa	2003, 2011
Mr A Parris	Lynwood High School, New Zealand	2004
Dr A McBride	University of Strathclyde, United Kingdom	2007
Prof P Vaderlind	Stockholm University, Sweden	2009, 2012
Prof A Jobbings	United Kingdom	2014
Assoc Prof D Wells	United States of America	2015
Dr P Neumann	United Kingdom	2016

Moderators for Challenge

Mr W Akhurst	New South Wales
Ms N Andrews	ACER, Camberwell, VIC
Mr L Bao	Leopold Primary School, VIC
Prof E Barbeau	University of Toronto, Canada
Mr R Blackman	Victoria
Ms J Breidahl	St Paul's Woodleigh, VIC
Ms S Brink	Glen Iris, VIC
Prof J C Burns	Australian Defence Force Academy, ACT
Mr A. Canning	Queensland
Mrs F Cannon	New South Wales
Mr J Carty	ACT Department of Education, ACT
Dr E Casling	Australian Capital Territory
Mr B Darcy	South Australia
Ms B Denney	New South Wales
Mr J Dowsey	Victoria
Mr S Ewington	Sydney Grammar School, NSW
Br K Friel	Trinity Catholic College, NSW
Dr D Fomin	St Petersburg University, Russia
Mrs P Forster	Penrhos College, WA
Mr T Freiberg	Queensland
Mr W Galvin	University of Newcastle, NSW
Mr S Gardiner	University of Sydney, NSW
Mr M Gardner	North Virginia, USA
Ms P Graham	Tasmania
Mr B Harridge	University of Melbourne, VIC
Ms J Hartnett	Queensland
Mr G Harvey	Australian Capital Territory
Ms I Hill	South Australia
Ms N Hill	Victoria
Dr N Hoffman	Edith Cowan University, WA
Prof F Holland	University College, Ireland
Mr D Jones	Coff's Harbour High School, NSW
Ms R Jorgenson	Australian Capital Territory
Dr T Kalinowski	University of Newcastle, NSW
Assoc Prof H Lausch	Victoria
Mr J Lawson	St Pius X School, NSW
Mr R Longmuir	China
Ms K McAsey	Victoria
Dr K McAvaney	Victoria
Ms J McIntosh	AMSI, VIC
Ms N McKinnon	Victoria
Ms T McNamara	Victoria
Mr G Meiklejohn	Queensland School Curriculum Council, QLD

Moderators for Challenge *continued*

Mr M O'Connor	AMSI, VIC
Mr J Oliver	Northern Territory
Mr S Palmer	New South Wales
Dr W Palmer	University of Sydney, NSW
Mr G Pointer	South Australia
Prof H Reiter	University of North Carolina, USA
Mr M Richardson	Yarraville Primary School, VIC
Mr G Samson	Nedlands Primary School, WA
Mr J Sattler	Parramatta High School, NSW
Mr A Saunder	Victoria
Mr W Scott	Seven Hills West Public School, NSW
Mr R Shaw	Hale School, WA
Ms T Shaw	New South Wales
Dr B Sims	University of Newcastle, NSW
Dr H Sims	Victoria
Ms K Sims	New South Wales
Prof J Smit	The Netherlands
Mrs M Spandler	New South Wales
Mr G Spyker	Curtin University, WA
Ms C Stanley	Queensland
Dr E Strzelecki	Monash University, VIC
Mr P Swain	Victoria
Dr P Swedosh	The King David School, VIC
Prof J Tabov	Academy of Sciences, Bulgaria
Mrs A Thomas	New South Wales
Ms K Trudgian	Queensland
Ms J Vincent	Melbourne Girls Grammar School, VIC
Prof J Webb	University of Capetown, South Africa
Dr D Wells	USA

Mathematics Enrichment Development

Enrichment Committee — Development Team (1992–1995)

Mr B Henry	Victoria (Chairman)
Prof P O'Halloran	University of Canberra, ACT (Director)
Mr G Ball	University of Sydney, NSW
Dr M Evans	Scotch College, VIC
Mr K Hamann	South Australia
Assoc Prof H Lausch	Monash University, VIC
Dr A Storozhev	Australian Mathematics Trust, ACT

Polya Development Team (1992–1995)

Mr G Ball	University of Sydney, NSW (Editor)
Mr K Hamann	South Australia (Editor)
Prof J Burns	Australian Defence Force Academy, ACT
Mr J Carty	Merici College, ACT
Dr H Gastineau-Hill	University of Sydney, NSW
Mr B Henry	Victoria
Assoc Prof H Lausch	Monash University, VIC
Prof P O'Halloran	University of Canberra, ACT
Dr A Storozhev	Australian Mathematics Trust, ACT

Polya Development Team (2013–2015)

Adj Prof M Clapper	Australian Mathematics Trust, ACT (Editor)
Dr R Atkins	New Zealand
Dr A di Pasquale	University of Melbourne, VIC

Dr R Geretschläger	Austria
Dr D Mathews	Monash University, VIC
Dr K McAvaney	Australian Mathematics Trust, ACT

Euler Development Team (1992–1995)

Dr M Evans	Scotch College, VIC (Editor)
Mr B Henry	Victoria (Editor)
Mr L Doolan	Melbourne Grammar School, VIC
Mr K Hamann	South Australia
Assoc Prof H Lausch	Monash University, VIC
Prof P O'Halloran	University of Canberra, ACT
Mrs A Thomas	Meriden School, NSW

Gauss Development Team (1993–1995)

Dr M Evans	Scotch College, VIC (Editor)
Mr B Henry	Victoria (Editor)
Mr W Atkins	University of Canberra, ACT
Mr G Ball	University of Sydney, NSW
Prof J Burns	Australian Defence Force Academy, ACT
Mr L Doolan	Melbourne Grammar School, VIC
Mr A Edwards	Mildura High School, VIC
Mr N Gale	Hornby High School, New Zealand
Dr N Hoffman	Edith Cowan University, WA
Prof P O'Halloran	University of Canberra, ACT
Dr W Pender	Sydney Grammar School, NSW
Mr R Vardas	Dulwich Hill High School, NSW

Noether Development Team (1994–1995)

Dr M Evans	Scotch College, VIC (Editor)
Dr A Storozhev	Australian Mathematics Trust, ACT (Editor)
Mr B Henry	Victoria
Dr D Fomin	St Petersburg University, Russia
Mr G Harvey	New South Wales

Newton Development Team (2001–2002)

Mr B Henry	Victoria (Editor)
Mr J Dowsey	University of Melbourne, VIC
Mrs L Mottershead	New South Wales
Ms G Vardaro	Annesley College, SA
Ms A Nakos	Temple Christian College, SA
Mrs A Thomas	New South Wales

Dirichlet Development Team (2001–2003)

Mr B Henry	Victoria (Editor)
Mr A Edwards	Ormiston College, QLD
Ms A Nakos	Temple Christian College, SA
Mrs L Mottershead	New South Wales
Mrs K Sims	Chapman Primary School, ACT
Mrs A Thomas	New South Wales

Ramanujan Development Team (2014–2016)

Mr B Henry	Victoria (Editor)
Adj Prof M Clapper	Australian Mathematics Trust, ACT
Ms A Nakos	Temple Christian College, SA
Mr A Edwards	Department of Education, QLD
Dr K McAvaney	Australian Mathematics Trust, ACT
Dr I Roberts	Charles Darwin University, NT

Australian Intermediate Mathematics Olympiad Committee

Dr K McAvaney	Victoria (Chair)	11 years; 2007–2017
Mr M Clapper	Australian Mathematics Trust, ACT	4 years; 2014–2017
Mr J Dowsey	University of Melbourne, VIC	19 years; 1999–2017
Dr M Evans	AMSI, VIC	19 years; 1999–2017
Mr B Henry	Victoria (Chair)	8 years; 1999–2006
	Member	11 years; 2007–2017
Assoc Prof H Lausch	Monash University, VIC	17 years; 1999–2015
Mr R Longmuir	China	2 years; 1999–2000
Dr D Mathews	Monash University, VIC	1 year; 2017

AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE HONOUR ROLL

Because of changing titles and affiliations, the most senior title achieved and later affiliations are generally used, except for the Interim committee, where they are listed as they were at the time.

Interim Committee 1979–1980

Mr P J O'Halloran	Canberra College of Advanced Education, ACT, Chair
Prof A L Blakers	University of Western Australia
Dr J M Gani	Australian Mathematical Society, ACT,
Prof B H Neumann	Australian National University, ACT,
Prof G E Wall	University of Sydney, NSW
Mr J L Williams	University of Sydney, NSW

The Australian Mathematical Olympiad Committee was founded at a meeting of the Australian Academy of Science at its meeting of 2–3 April 1980.

* denotes Executive Position

Chair*

Prof B H Neumann	Australian National University, ACT	7 years; 1980–1986
Prof G B Preston	Monash University, VIC	10 years; 1986–1995
Prof A P Street	University of Queensland	6 years; 1996–2001
Prof C Praeger	University of Western Australia	16 years; 2002–2017

Deputy Chair*

Prof P J O'Halloran	University of Canberra, ACT	15 years; 1980–1994
Prof A P Street	University of Queensland	1 year; 1995
Prof C Praeger,	University of Western Australia	6 years; 1996–2001
Assoc Prof D Hunt	University of New South Wales	14 years; 2002–2015
Prof A Hassell	Australian National University	2 years; 2016–2017

Executive Director*

Prof P J O'Halloran	University of Canberra, ACT	15 years; 1980–1994
Prof P J Taylor	University of Canberra, ACT	18 years; 1994–2012
Adj Prof M G Clapper	University of Canberra, ACT	4 years; 2013–2016

Chief Executive Officer*

Mr J N Ford	Australian Mathematics Trust, ACT	1 year; 2017

Secretary

Prof J C Burns	Australian Defence Force Academy, ACT	9 years; 1980–1988
Vacant		4 years; 1989–1992
Mrs K Doolan	Victorian Chamber of Mines, VIC	6 years; 1993–1998

Treasurer*

Prof J C Burns	Australian Defence Force Academy, ACT	8 years; 1981–1988
Prof P J O'Halloran	University of Canberra, ACT	2 years; 1989–1990
Ms J Downes	CPA	5 years; 1991–1995
Dr P Edwards	Monash University, VIC	8 years; 1995–2002
Prof M Newman	Australian National University, ACT	6 years; 2003–2008
Dr P Swedosh	The King David School, VIC	9 years; 2009–2017

Director of Mathematics Challenge for Young Australians*

Mr J B Henry	Deakin University, VIC	17 years; 1990–2006
Dr K McAvaney	Deakin University, VIC	12 years; 2006–2017

Chair, Senior Problems Committee

Prof B C Rennie	James Cook University, QLD	1 year; 1980

Mr J L Williams	University of Sydney, NSW	6 years; 1981–1986
Assoc Prof H Lausch	Monash University, VIC	27 years; 1987–2013
Dr N Do	Monash University, VIC	4 years; 2014–2017

Director of Training*

Mr J L Williams	University of Sydney, NSW	7 years; 1980–1986
Mr G Ball	University of Sydney, NSW	3 years; 1987–1989
Dr D Paget	University of Tasmania	6 years; 1990–1995
Dr M Evans	Scotch College, VIC	3 months; 1995
Assoc Prof D Hunt	University of New South Wales	5 years; 1996–2000
Dr A Di Pasquale	University of Melbourne, VIC	17 years; 2001–2017

Team Leader

Mr J L Williams	University of Sydney, NSW	5 years; 1981–1985
Assoc Prof D Hunt	University of New South Wales	9 years; 1986, 1989, 1990, 1996–2001
Dr E Strzelecki	Monash University, VIC	2 years; 1987, 1988
Dr D Paget	University of Tasmania	5 years; 1991–1995
Dr A Di Pasquale	University of Melbourne, VIC	15 years; 2002–2010, 2012–2017
Dr I Guo	University of New South Wales	1 year; 2011

Deputy Team Leader

Prof G Szekeres	University of New South Wales	2 years; 1981–1982
Mr G Ball	University of Sydney, NSW	7 years; 1983–1989
Dr D Paget	University of Tasmania	1 year; 1990
Dr J Graham	University of Sydney, NSW	3 years; 1991–1993
Dr M Evans	Scotch College, VIC	3 years; 1994–1996
Dr A Di Pasquale	University of Melbourne, VIC	5 years; 1997–2001
Dr D Mathews	University of Melbourne, VIC	3 years; 2002–2004
Dr N Do	University of Melbourne, VIC	4 years; 2005–2008
Dr I Guo	University of New South Wales	4 years; 2009–10, 2012–2013
Mr G White	University of Sydney, NSW	1 year; 2011
Mr A Elvey Price	Melbourne University, VIC	4 years; 2014–2017

State Directors

Australian Capital Territory

Prof M Newman	Australian National University	1 year; 1980
Mr D Thorpe	ACT Department of Education	2 years; 1981–1982
Dr R A Bryce	Australian National University	7 years; 1983–1989
Mr R Welsh	Canberra Grammar School	1 year; 1990
Mrs J Kain	Canberra Grammar School	5 years; 1991–1995
Mr J Carty	ACT Department of Education	17 years; 1995–2011
Mr J Hassall	Burgmann Anglican School	2 years; 2012–2013
Dr C Wetherell	Radford College	4 years; 2014–2017

New South Wales

Dr M Hirschhorn	University of New South Wales	1 year; 1980
Mr G Ball	University of Sydney, NSW	16 years; 1981–1996
Dr W Palmer	University of Sydney, NSW	20 years; 1997–2016
Assoc Prof D Daners	University of Sydney, NSW	1 year; 2017

Northern Territory

Dr I Roberts	Charles Darwin University	4 years; 2014–2017

Queensland

Dr N H Williams	University of Queensland	21 years; 1980–2000
Dr G Carter	Queensland University of Technology	10 years; 2001–2010
Dr V Scharaschkin	University of Queensland	4 years; 2011–2014
Dr A Offer	Queensland	3 years; 2015 –2017

South Australia/Northern Territory

Mr K Hamann	SA Department of Education	19 years; 1980–1982, 1991–2005, 2013
Mr V Treilibs	SA Department of Education	8 years; 1983–1990
Dr M Peake	Adelaide	8 years; 2006–2013
Dr D Martin	Adelaide	4 years; 2014–2017

Tasmania

Mr J Kelly	Tasmanian Department of Education	8 years; 1980–1987
Dr D Paget	University of Tasmania	8 years; 1988–1995
Mr W Evers	St Michael's Collegiate School	9 years; 1995–2003
Dr K Dharmadasa	University of Tasmania	14 years; 2004–2017

Victoria

Dr D Holton	University of Melbourne	3 years; 1980–1982
Mr B Harridge	Melbourne High School	1 year; 1982
Ms J Downes	CPA	6 years; 1983–1988
Mr L Doolan	Melbourne Grammar School	9 years; 1989–1998
Dr P Swedosh	The King David School	20 years; 1998–2017

Western Australia

Dr N Hoffman	WA Department of Education	3 years; 1980–1982
Assoc Prof P Schultz	University of Western Australia	14 years; 1983–1988, 1991–1994, 1996–1999
Assoc Prof W Bloom	Murdoch University	2 years; 1989–1990
Dr E Stoyanova	WA Department of Education	7 years; 1995, 2000–2005
Dr G Gamble	University of Western Australia	12 years; 2006–2017

Editor

Prof P J O'Halloran	University of Canberra, ACT	1 year; 1983
Dr A W Plank	University of Southern Queensland	11 years; 1984–1994
Dr A Storozhev	Australian Mathematics Trust, ACT	15 years; 1994–2008

Editorial Consultant

Dr O Yevdokimov	University of Southern Queensland	9 years; 2009–2017

Other Members of AMOC (showing organisations represented where applicable)

Mr W J Atkins	Australian Mathematics Foundation	18 years; 1995–2012
Dr S Britton	University of Sydney, NSW	8 years; 1990–1998
Prof G Brown	Australian Academy of Science, ACT	10 years; 1980, 1986–1994
Dr R A Bryce	Australian Mathematical Society, ACT	9 years; 1991–1998
	Mathematics Challenge for Young Australians	14 years; 1999–2012
Mr G Cristofani	Department of Education and Training	2 years; 1993–1994
Ms L Davis	IBM Australia	4 years; 1991–1994
Dr W Franzsen	Australian Catholic University, ACT	9 years; 1990–1998
Dr J Gani	Australian Mathematical Society, ACT	1980
Assoc Prof T Gagen	ANU AAMT Summer School	6 years; 1993–1998
Ms P Gould	Department of Education and Training	2 years; 1995–1996
Prof G M Kelly	University of Sydney, NSW	6 years; 1982–1987
Ms J McIntosh	Australian Mathematical Sciences Institute,VIC	
	Mathematics Challenge for Young Australians	6 years; 2012–2017
Prof R B Mitchell	University of Canberra, ACT	5 years; 1991–1995
Ms Anna Nakos	Mathematics Challenge for Young Australians	15 years; 2003–2017
Mr S Neal	Department of Education and Training	4 years; 1990–1993
Prof M Newman	Australian National University, ACT	15 years; 1986–1998
	Mathematics Challenge for Young Australians	19 years; 1999–2017,
	(Treasurer during the interim)	2003–2008
Prof R B Potts	University of Adelaide, SA	1 year; 1980
Mr H Reeves	Australian Association of Maths Teachers	15 years; 1988–1998
	Australian Mathematics Foundation	2014–2017

Mr N Reid	IBM Australia	3 years; 1988–1990
Mr M Roberts	Tasmania	4 years; 2014–2017
Mr R Smith	Telecom Australia	5 years; 1990–1994
Prof P J Taylor	Australian Mathematics Foundation	6 years; 1990–1994, 2013
Prof N S Trudinger	Australian Mathematical Society, ACT	3 years; 1986–1988
Assoc Prof I F Vivian	University of Canberra, ACT	1 year; 1990
Dr M W White	IBM Australia	9 years; 1980–1988

Associate Membership (inaugurated in 2000)

Mr G Ball	17 years; 2000–2016
Ms S Britton	17 years; 2000–2016
Dr M Evans	18 years; 2000–2017
Dr W Franzsen	17 years; 2000–2016
Prof T Gagen	17 years; 2000–2016
Mr H Reeves	15 years; 2000–2014

AMOC Senior Problems Committee

Current members

Dr N Do	Monash University, VIC (Chair)	4 years; 2014–2017
	(member)	11 years; 2003–2013
Mr M Clapper	Australian Mathematics Trust	5 years; 2013–2017
Dr A Devillers	University of Western Australia, WA	2 years; 2016–2017
Dr A Di Pasquale	University of Melbourne, VIC	17 years; 2001–2017
Dr I Guo	University of Sydney, NSW	10 years; 2008–2017
Dr J Kupka	Monash University, VIC	14 years; 2003–2016
Dr K McAvaney	Deakin University, VIC	22 years; 1996–2017
Dr D Mathews	Monash University, VIC	17 years; 2001–2017
Dr A Offer	Queensland	6 years; 2012–2017
Dr C Rao	Telstra, VIC	18 years; 2000–2017
Dr B B Saad	Monash University, VIC	24 years; 1994–2017
Dr J Simpson	Curtin University, WA	19 years; 1999–2017
Dr I Wanless	Monash University, VIC	18 years; 2000–2017

Previous members

Mr G Ball	University of Sydney, NSW	16 years; 1982–1997
Mr M Brazil	LaTrobe University, VIC	5 years; 1990–1994
Dr M S Brooks	University of Canberra, ACT	8 years; 1983–1990
Dr G Carter	Queensland University of Technology	10 years; 2001–2010
Dr M Evans	Australian Mathematical Sciences Institute, VIC	27 years; 1990–2016
Dr J Graham	University of Sydney, NSW	1 year; 1992
Dr M Herzberg	Telecom Australia	1 year; 1990
Assoc Prof D Hunt	University of New South Wales	29 years; 1986–2014
Dr L Kovacs	Australian National University, ACT	5 years; 1981–1985
Assoc Prof H Lausch	Monash University, VIC (Chair)	27 years; 1987–2013
	(member)	2 years; 2014–2015
Dr D Paget	University of Tasmania	7 years; 1989–1995
Prof P Schultz	University of Western Australia	8 years; 1993–2000
Dr L Stoyanov	University of Western Australia	5 years; 2001–2005
Dr E Strzelecki	Monash University, VIC	5 years; 1986–1990
Dr E Szekeres	University of New South Wales	7 years; 1981–1987
Prof G Szekeres	University of New South Wales	7 years; 1981–1987
Em Prof P J Taylor	Australian Capital Territory	1 year; 2013
Dr N H Williams	University of Queensland	20 years; 1981–2000

Mathematics School of Excellence

Dr S Britton	University of Sydney, NSW (Coordinator)	2 years; 1990–1991
Mr L Doolan	Melbourne Grammar, VIC (Coordinator)	6 years; 1992, 1993–1997
Mr W Franzsen	Australian Catholic University, ACT (Coordinator)	2 years; 1990–1991
Dr D Paget	University of Tasmania (Director)	5 years; 1990–1994
Dr M Evans	Scotch College, VIC	1 year; 1995
Assoc Prof D Hunt	University of New South Wales (Director)	4 years; 1996–1999
Dr A Di Pasquale	University of Melbourne, VIC (Director)	18 years; 2000–2017

International Mathematical Olympiad Selection School

Mr J L Williams	University of Sydney, NSW (Director)	2 years; 1982–1983
Mr G Ball	University of Sydney, NSW (Director)	6 years; 1984–1989
Mr L Doolan	Melbourne Grammar, VIC (Coordinator)	3 years; 1989–1991
Dr S Britton	University of Sydney, NSW (Coordinator)	7 years; 1992–1998
Mr W Franzsen	Australian Catholic University, ACT (Coordinator)	8 years; 1992–1996, 1999–2001
Dr D Paget	University of Tasmania (Director)	6 years; 1990–1995
Assoc Prof D Hunt	University of New South Wales (Director)	5 years; 1996–2000
Dr A Di Pasquale	University of Melbourne, VIC (Director)	17 years; 2001–2017