



An Roinn Oideachais
agus Scileanna

**JUNIOR
CERTIFICATE**



MATHEMATICS

SYLLABUS

FOUNDATION, ORDINARY & HIGHER LEVEL

For examination from 2016

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MATHEMATICS

Junior Certificate Mathematics

Introduction

Mathematics may be seen as the study of quantity, structure, space and change. What does that mean in the context of learning mathematics in post-primary school? In the first instance the learner needs essential skills in numeracy, statistics, basic algebra, shape and space, and technology to be able to function in society. These skills allow learners to make calculations and informed decisions based on information presented and to solve problems they encounter in their everyday lives. The learner also needs to develop the skills to become a good mathematician. Someone who is a good mathematician will be able to compute and then evaluate a calculation, follow logical arguments, generalise and justify conclusions, problem solve and apply mathematical concepts learned in a real life situation.

Mathematical knowledge and skills are held in high esteem and are seen to have a significant role to play in the development of the knowledge society and the culture of enterprise and innovation associated with it. Mathematics education should be appropriate to the abilities, needs and interests of learners and should reflect the broad nature of the subject and its potential for enhancing their development.

The elementary aspects of mathematics, use of arithmetic and the display of information by means of a graph, are an everyday occurrence. Advanced mathematics is also widely used, but often in an unseen and unadvertised way. The mathematics of error-correcting codes is applied to CD players and to computers. The stunning pictures of far away planets and nebulae sent by Voyager II and Hubble could not have had their crispness and quality without such mathematics. Statistics not only provides the theory and methodology for the analysis of wide varieties of data but is essential in medicine for analysing data on the causes of illness and on the utility of new drugs. Travel by aeroplane would not be possible without the mathematics of airflow and of control systems. Body scanners are the expression of subtle mathematics discovered in the 19th century, which makes it possible to construct an image of the inside of an object from information on a number of single X-ray views of it.

Through its application to the simple and the everyday, as well as to the complex and remote, it is true to say that mathematics is involved in almost all aspects of life and living.

Aims

Junior Certificate Mathematics aims to

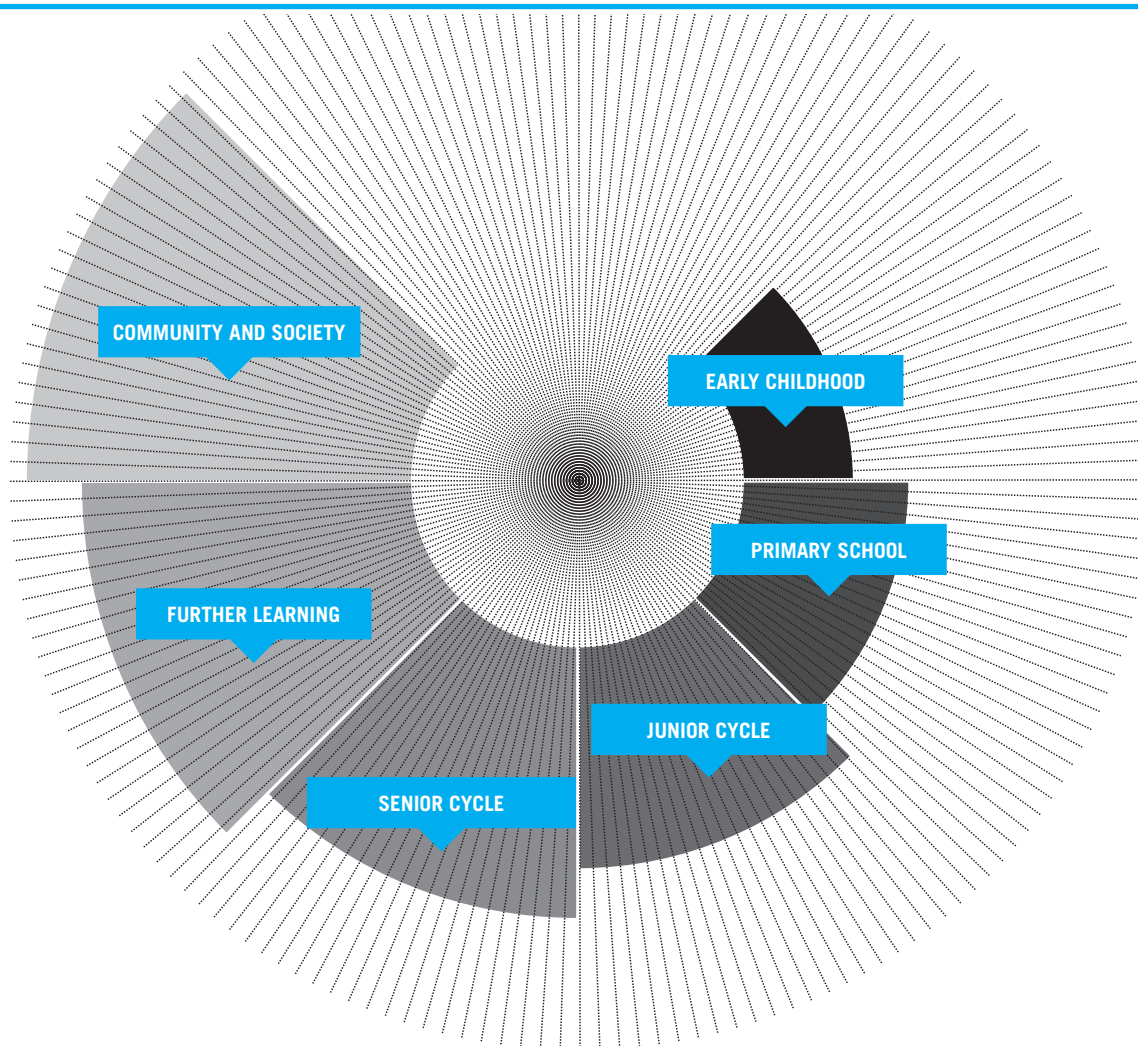
- develop the mathematical knowledge, skills and understanding needed for continuing education, for life and for work
- develop the skills of dealing with mathematical concepts in context and applications, as well as in solving problems
- support the development of literacy and numeracy skills
- foster a positive attitude to mathematics in the learner.

Objectives

The objectives of Junior Certificate Mathematics are that learners develop mathematical proficiency, characterised as

- *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
- *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*—ability to formulate, represent, and solve mathematical problems in both familiar and unfamiliar contexts
- *adaptive reasoning*—capacity for logical thought, reflection, explanation, justification and communication
- *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence, perseverance and one's own efficacy.

Related learning



Mathematical learning is cumulative with work at each level building on and deepening what students have learned at the previous level to foster the overall development of understanding. The study of Junior Certificate Mathematics encourages the learner to use the numeracy and problem solving skills developed in early childhood education and primary mathematics. The emphasis is on building connected and integrated mathematical understanding. As learners progress through their education, mathematical skills, concepts and knowledge are developed when they work in more demanding contexts and develop more sophisticated approaches to problem solving.

Mathematics is not learned in isolation. It has significant connections with other curriculum subjects. Many elements of Science have a quantitative basis and learners are expected to be able to work with data, produce graphs, and interpret patterns and trends. In Technical Graphics, drawings are used in the analysis and solution of 2D and 3D problems through the rigorous application of geometric principles. In Geography, learners use ratio to determine scale and in everyday life people use timetables, clocks and

currency conversions to make life easier. Consumers need basic financial awareness and in Home Economics learners use mathematics when budgeting and making value for money judgements. In Business Studies learners see how mathematics can be used by business organisations in budgeting, consumer education, financial services, enterprise, and reporting on accounts.

Mathematics, Music and Art have a long historical relationship. As early as the fifth century B.C., Pythagoras uncovered mathematical relationships in music; many works of art are rich in mathematical structure. The modern mathematics of fractal geometry continues to inform composers and artists.

Senior cycle and junior cycle mathematics have been developed simultaneously to allow for strong links to be established between the two. The strands structure allows a smooth transition from junior cycle to a similar structure in senior cycle mathematics. The pathways in each strand are continued, allowing the learner to see ahead and appreciate the connectivity between junior and senior cycle mathematics.

Bridging Framework for Mathematics

Post-primary mathematics education builds on and progresses the learner's experience of mathematics in the Primary School Curriculum. This is achieved with reference not only to the content of the syllabuses but also to the teaching and learning approaches used.

Mathematics in the Primary School Curriculum is studied by all children from junior infants to sixth class. Content is presented in two-year blocks but with each class level clearly delineated. The Mathematics Curriculum is presented in two distinct sections.

It includes a skills development section which describes the skills that children should acquire as they develop mathematically. These skills include

- applying and problem solving
- communicating and expressing
- integrating and connecting
- reasoning
- implementing
- understanding and recalling.

It also includes a number of strands which outline content that is to be included in the mathematics programme at each level. Each strand includes a number of strand units. Depending on the class level, strands can include

- early mathematical activities
- number
- algebra
- shape and space
- measures
- data.

The adoption of a strands structure in Junior Certificate Mathematics continues the pathways which different topics of mathematics follow as the learner progresses from primary school. To facilitate a smooth transition between mathematics in the primary school and in junior cycle a Bridging Framework has been developed. This contains three elements, a *Common Introductory Course*, a *bridging content document* and a *bridging glossary*.

The *Common Introductory Course*, will be studied by all learners as a minimum (see page 33). It is designed so that all of the strands are engaged with to some extent in the first year, so ensuring that the range of topics which have been studied in fifth and sixth classes are revisited.

The *bridging content* document has been developed to illustrate to both primary and post-primary teachers the pathways for learners in each strand. Another element of the Bridging Framework is a *bridging glossary* of common terminology for use in upper primary school and early junior cycle. Sample bridging activities have also been developed to assist teachers of fifth and sixth classes in primary school in their planning. These can be used by post-primary mathematics teachers to support learners in the transition to junior cycle mathematics. These documents can be viewed at www.ncca.ie/projectmaths.

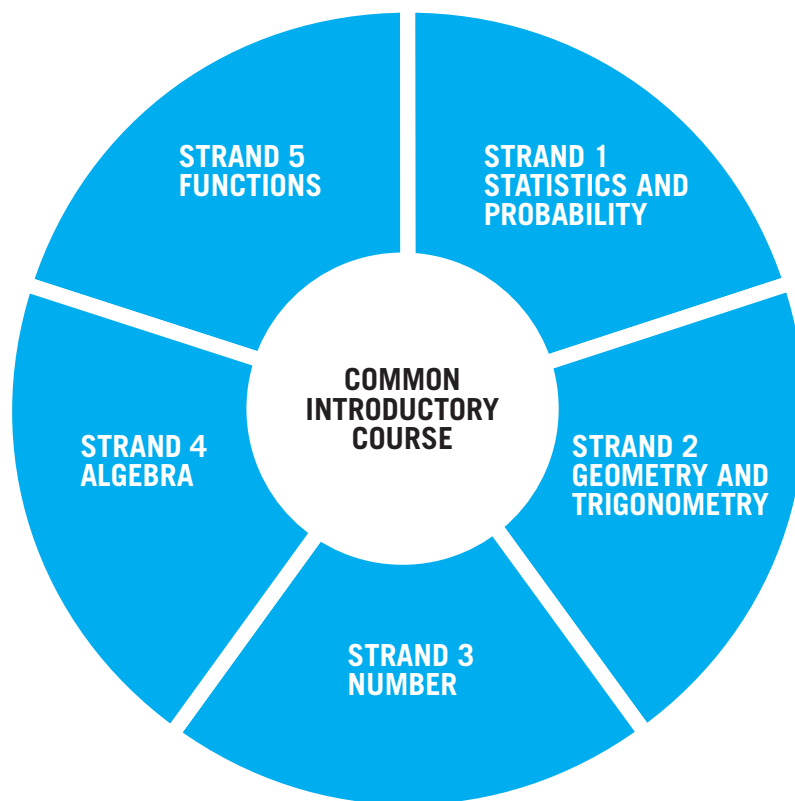
The Bridging Framework for Mathematics provides a lens through which teachers in primary school can view post-primary mathematics syllabuses and post-primary teachers can also view mathematics in the Primary School Curriculum. It facilitates improved continuity between mathematics in primary and post-primary schools.



SYLLABUS OVERVIEW

Syllabus overview

Junior Certificate Mathematics



Structure

The Junior Certificate Mathematics syllabus comprises five strands:

1. Statistics and Probability
2. Geometry and Trigonometry
3. Number
4. Algebra
5. Functions

The selection of topics and learning outcomes in each strand is presented in tabular form, and Ordinary level is a subset of Higher level (**HL**). **Material for Higher level only is shown in bold text.**

The strand structure of the syllabus should not be taken to imply that topics are to be studied in isolation. Where appropriate, connections should be made within and across the strands and with other areas of learning.

Time allocation

The Junior Certificate Mathematics syllabus is designed as a 240 hour course of study.

Problem solving

Problem solving means engaging in a task for which the solution is not immediately obvious. Problem solving is integral to mathematical learning. In day-to-day life and in the workplace the ability to problem solve is a highly advantageous skill.

In the mathematics classroom problem solving should not be met in isolation, but should permeate all aspects of the teaching and learning experience. Problems may concern purely mathematical matters or some applied context.

In a mathematics problem-solving environment it is recognised that there are three things learners need to do:

- make sense of the problem
- make sense of the mathematics they can learn and use when doing the problem
- arrive at a correct solution to the problem.

However, in the mathematics classroom, the focus is on the mathematical knowledge and skills that can be learned in the process of obtaining an answer, rather than on the answer itself. The emphasis, therefore, is on generating discussion and on the reasoning and sense-

making opportunities the problem affords the learners as they engage with the mathematics involved. They learn to analyse the problem and break it down into manageable steps, to reflect on their strategies and those of others and to adjust their own approaches where necessary.

Teachers play an important role in helping students develop these kinds of skills. By encouraging learners to share, explain and justify their solution strategies, those that work as well as those that don't work, teachers can help learners to develop robust and deep mathematical understanding as well as confidence in their mathematical ability.

The quality of the tasks that learners engage with play an important role in a problem-solving environment. A task must engage learners and present them with a challenge that requires exploration. Problem-solving tasks activate creative mathematical thinking processes as opposed to imitative thinking processes activated by routine tasks. Reasoning mathematically about tasks empowers learners to make connections within mathematics and to develop deep conceptual understanding

Teaching and learning

In each strand, and at each syllabus level, emphasis should be placed on making connections between the strands and on appropriate contexts and applications of mathematics so that learners can appreciate its relevance to current and future life. The focus should be on the learner understanding the concepts involved, building from the concrete to the abstract and from the informal to the formal. As outlined in the syllabus objectives and learning outcomes, the learner's experiences in the study of mathematics should contribute to the development of problem-solving skills through the application of mathematical knowledge and skills.

The use of context-based tasks and a collaborative approach to problem solving can support learners in developing their literacy and numeracy skills. Through discussing ideas about the tasks and their solutions, learners develop the ability to explain and justify their thinking and so gain confidence in their ability to communicate mathematical ideas.

The learner builds on knowledge constructed initially through exploration of mathematics in primary school. This is facilitated by the study of the *Common Introductory Course* at the start of post-primary schooling, which facilitates both continuity and progression in mathematics. Particular emphasis is placed on promoting the learner's confidence in themselves (that they can 'do' mathematics)

and in the subject (that mathematics makes sense). Through the use of meaningful contexts, opportunities are presented for the learner to achieve success.

The variety of activities engaged in enables learners to take charge of their own learning by setting goals, developing action plans and receiving and responding to assessment feedback. As well as varied teaching strategies, varied assessment strategies will provide information that can be used as feedback so that teaching and learning activities can be modified in ways which best suit individual learners.

Careful attention must be paid to the learner who may still be experiencing difficulty with some of the material covered at primary level. The experience of post-primary mathematics must therefore help the learner to construct a clearer knowledge of, and to develop improved skills in basic mathematics and to develop an awareness of its usefulness. Appropriate new material should also be introduced, so that the learner can feel that progress is being made. At junior cycle, the course pays attention to consolidating the foundation laid in the primary school and to addressing practical issues; but it should also cover new topics and underpin progress to the further study of mathematics in each of the strands.

Differentiation

Students learn at different rates and in different ways. Differentiation in teaching and learning and in the related assessment arrangements is essential in order to meet the needs of all learners. In junior cycle syllabuses, differentiation is primarily addressed in three areas: the content and learning outcomes of the syllabus; the process of teaching and learning; the assessment arrangements associated with examinations. For exceptionally able students, differentiation may mean extending and/or enriching some of the topics or learning outcomes. This should supplement, but not replace, the core work being undertaken. For students with general learning difficulties, differentiation may mean teaching at a different pace, having varied teaching methodologies or having a variety of ways of assessing them.

Junior Certificate Mathematics is offered at two levels, Ordinary and Higher level. There is no separate course for Foundation level. The Higher level learning outcomes are indicated in bold in each strand. Learners at Higher level will engage with all of the learning outcomes for Ordinary level as well as those designated for Higher level only.

In each strand, learners at Foundation level will engage appropriately with all of the learning outcomes at Ordinary

level. This allows them to have a broad experience of mathematics. More importantly, it will also allow them to follow the Ordinary level course at senior cycle, should they choose to do so.

Mathematics at Ordinary level is geared to the needs of learners who are beginning to deal with abstract ideas. However, learners may go on to use and apply mathematics in their future careers, and will meet the subject to a greater or lesser degree in daily life.

The Ordinary level, therefore, must start by offering mathematics that is meaningful and accessible to learners at their present stage of development. It should also provide for the gradual introduction of more abstract ideas, leading learners towards the use of academic mathematics in the context of further study at senior cycle.

Mathematics at Ordinary level places particular emphasis on the development of mathematics as a body of knowledge and skills that makes sense, and that can be used in many different ways as an efficient system for solving problems and finding answers. Alongside this, adequate attention must be paid to the acquisition and consolidation of fundamental mathematical skills, in the absence of which the learner's development and progress will be hindered. The Ordinary level is intended to equip learners with the knowledge and skills required in everyday life, and it is also intended to lay the groundwork for those who may proceed to further studies in areas in which specialist mathematics is not required.

Mathematics at Higher level is geared to the needs of learners who will proceed with their study of mathematics at Leaving Certificate and beyond. However, not all learners taking the course are future specialists or even future users of academic mathematics. Moreover, when they start to study the material, some of them are only beginning to deal with abstract concepts.

Junior Certificate Mathematics is designed for the wide variety and diversity of abilities and learners. On the one hand it focuses on material that underlies academic mathematical studies, ensuring that learners have a chance to develop their mathematical ability and interest to a high level. On the other, it addresses the practical and obviously applicable topics that learners meet in life outside of school. At Higher level, particular emphasis can be placed on the development of powers of abstraction and generalisation and on the idea of rigorous proof, hence giving learners a feeling for the great mathematical concepts that span many centuries and cultures. Problem solving can be addressed in both mathematical and applied contexts.



STRANDS OF STUDY

Strand 1: Statistics and Probability

In Junior Certificate Mathematics, learners build on their primary school experience and continue to develop their understanding of data analysis by collecting, representing, describing, and interpreting numerical data. By carrying out a complete investigation, from formulating a question through to drawing conclusions from data, learners gain an understanding of data analysis as a tool for learning about the world. Work in this strand focuses on engaging learners in this process of data investigation: posing questions, collecting data, analysing and interpreting this data in order to answer questions.

Learners advance in their knowledge of chance from primary school to deal more formally with probability. The *Common Introductory Course* (see appendix), which draws on a selection of learning outcomes from this strand, enables learners to begin the process of engaging in a more formal manner with the concepts and processes involved.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- use a variety of methods to represent their data
- explore concepts that relate to ways of describing data
- develop a variety of strategies for comparing data sets
- complete a data investigation of their own
- encounter the language and concepts of probability.

Strand 1: Statistics and Probability

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
1.1 Counting	Listing outcomes of experiments in a systematic way, such as in a table, using sample spaces, tree diagrams.	<ul style="list-style-type: none"> – list all possible outcomes of an experiment – apply the fundamental principle of counting
1.2 Concepts of probability	<p>The probability of an event occurring: students progress from informal to formal descriptions of probability.</p> <p>Predicting and determining probabilities.</p> <p>Difference between experimental and theoretical probability.</p>	<ul style="list-style-type: none"> – decide whether an everyday event is likely or unlikely to occur – recognise that probability is a measure on a scale of 0-1 of how likely an event is to occur – use set theory to discuss experiments, outcomes, sample spaces – use the language of probability to discuss events, including those with equally likely outcomes – estimate probabilities from experimental data – recognise that, if an experiment is repeated, there will be different outcomes and that increasing the number of times an experiment is repeated generally leads to better estimates of probability – associate the probability of an event with its long-run, relative frequency
1.3 Outcomes of simple random processes	Finding the probability of equally likely outcomes.	<ul style="list-style-type: none"> – construct sample spaces for two independent events – apply the principle that, in the case of equally likely outcomes, the probability is given by the number of outcomes of interest divided by the total number of outcomes (examples using coins, dice, spinners, containers with different coloured objects, playing cards, sports results, etc.) – use binary / counting methods to solve problems involving successive random events where only two possible outcomes apply to each event
1.4 Statistical reasoning with an aim to becoming a statistically aware consumer	<p>Situations where statistics are misused and learn to evaluate the reliability and quality of data and data sources.</p> <p>Different types of data.</p>	<ul style="list-style-type: none"> – engage in discussions about the purpose of statistics and recognise misconceptions and misuses of statistics – work with different types of data: <ul style="list-style-type: none"> – categorical: nominal or ordinal – numerical: discrete or continuous in order to clarify the problem at hand – evaluate reliability of data and data sources
1.5 Finding, collecting and organising data	<p>The use of statistics to gather information from a selection of the population with the intention of making generalisations about the whole population.</p> <p>Formulating a statistics question based on data that vary, allowing for distinction between different types of data.</p>	<ul style="list-style-type: none"> – clarify the problem at hand – formulate questions that can be answered with data – explore different ways of collecting data – generate data, or source data from other sources including the internet – select a sample from a population (Simple Random Sample) – recognise the importance of representativeness so as to avoid biased samples – design a plan and collect data on the basis of above knowledge – summarise data in diagrammatic form including data presented in spreadsheets

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
1.6 Representing data graphically and numerically	<p>Methods of representing data.</p> <p>Students develop a sense that data can convey information and that organising data in different ways can help clarify what the data have to tell us. They see a data set as a whole and so are able to use proportions and measures of centre to describe the data.</p> <p>Mean of a grouped frequency distribution.</p>	<p>Graphical</p> <ul style="list-style-type: none"> – select appropriate methods to represent and describe the sample (univariate data only) – evaluate the effectiveness of different displays in representing the findings of a statistical investigation conducted by others – use pie charts, bar charts, line plots, histograms (equal intervals), stem and leaf plots and back-to-back stem and leaf plots to display data – use appropriate graphical displays to compare data sets <p>Numerical</p> <ul style="list-style-type: none"> – use a variety of summary statistics to describe the data: central tendency – mean, median, mode – variability – range, quartiles and inter-quartile range – recognise the existence of outliers
1.7 Analysing, interpreting and drawing conclusions from data	<p>Drawing conclusions from data; limitations of conclusions.</p>	<ul style="list-style-type: none"> – interpret graphical summaries of data – relate the interpretation to the original question – recognise how sampling variability influences the use of sample information to make statements about the population – draw conclusions from graphical and numerical summaries of data, recognising assumptions and limitations
Students learn about Students should be able to		
1.8 Synthesis and problem-solving skills	<ul style="list-style-type: none"> – explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions. 	

Strand 2: Geometry and Trigonometry

The synthetic geometry covered in Junior Certificate Mathematics is selected from *Geometry for Post-primary School Mathematics*, including terms, definitions, axioms, propositions, theorems, converses and corollaries. The formal underpinning for the system of post-primary geometry is that described by Barry (2001)¹.

The geometrical results listed in the following pages should first be encountered by learners through investigation and discovery. The *Common Introductory Course* will enable learners to link formal geometrical results to their study of space and shape in primary mathematics. Learners are asked to accept these results as true for the purpose of applying them to various contextualised and abstract problems. They should come to appreciate that certain features of shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features or results can be established in a formal manner through logical proof. Even at the investigative stage, ideas involved in mathematical proof can be developed. Learners should become familiar with the formal proofs of the specified theorems (some of which are examinable at Higher level).

¹ P.D. Barry. *Geometry with Trigonometry*, Horwood, Chichester (2001)

It is envisaged that learners will engage with dynamic geometry software, paper folding and other active investigative methods.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- recall basic facts related to geometry and trigonometry
- construct a variety of geometric shapes and establish their specific properties or characteristics
- solve geometrical problems and in some cases present logical proofs
- interpret information presented in graphical and pictorial form
- analyse and process information presented in unfamiliar contexts
- select appropriate formulae and techniques to solve problems.

Strand 2: Geometry and Trigonometry

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
2.1 Synthetic geometry	<p>Concepts (see <i>Geometry Course</i> section 9.1 for OL and 10.1 for HL)</p> <p>Axioms (see <i>Geometry Course</i> section 9.3 for OL and 10.3 for HL):</p> <ol style="list-style-type: none"> [Two points axiom] There is exactly one line through any two given points. [Ruler axiom] The properties of the distance between points [Protractor Axiom] The properties of the degree measure of an angle Congruent triangles (SAS, ASA and SSS) [Axiom of Parallels] Given any line l and a point P, there is exactly one line through P that is parallel to l. <p>Theorems: [Formal proofs are not examinable at OL. Formal proofs of theorems 4, 6, 9, 14 and 19 are examinable at HL.]</p> <ol style="list-style-type: none"> Vertically opposite angles are equal in measure. In an isosceles triangle the angles opposite the equal sides are equal. Conversely, if two angles are equal, then the triangle is isosceles. If a transversal makes equal alternate angles on two lines then the lines are parallel, (and converse). The angles in any triangle add to 180°. Two lines are parallel if and only if, for any transversal, the corresponding angles are equal. Each exterior angle of a triangle is equal to the sum of the interior opposite angles. In a parallelogram, opposite sides are equal and opposite angles are equal (and converses). The diagonals of a parallelogram bisect each other. If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal. Let ABC be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s:t$, then it also cuts $[AC]$ in the same ratio (and converse). If two triangles are similar, then their sides are proportional, in order (and converse). [Theorem of Pythagoras] In a right-angled triangle the square of the hypotenuse is the sum of the squares of the other two sides. If the square of one side of a triangle is the sum of the squares of the other two sides, then the angle opposite the first side is a right angle. The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc. 	<ul style="list-style-type: none"> – recall the axioms and use them in the solution of problems – use the terms: theorem, proof, axiom, corollary, converse and implies – apply the results of all theorems, converses and corollaries to solve problems – prove the specified theorems

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
	<p>Corollaries:</p> <ol style="list-style-type: none"> 1. A diagonal divides a parallelogram into 2 congruent triangles. 2. All angles at points of a circle, standing on the same arc, are equal, (and converse). 3. Each angle in a semi-circle is a right angle. 4. If the angle standing on a chord [BC] at some point of the circle is a right-angle, then [BC] is a diameter. 5. If ABCD is a cyclic quadrilateral, then opposite angles sum to 180°, (and converse). <p>Constructions:</p> <ol style="list-style-type: none"> 1. Bisector of a given angle, using only compass and straight edge. 2. Perpendicular bisector of a segment, using only compass and straight edge. 3. Line perpendicular to a given line l, passing through a given point not on l. 4. Line perpendicular to a given line l, passing through a given point on l. 5. Line parallel to a given line, through a given point. 6. Division of a line segment into 2 or 3 equal segments, without measuring it. 7. Division of a line segment into any number of equal segments, without measuring it. 8. Line segment of a given length on a given ray. 9. Angle of a given number of degrees with a given ray as one arm. 10. Triangle, given lengths of three sides 11. Triangle, given SAS data 12. Triangle, given ASA data 13. Right-angled triangle, given the length of the hypotenuse and one other side. 14. Right-angled triangle, given one side and one of the acute angles (several cases). 15. Rectangle, given side lengths. 	<p>– complete the constructions specified</p>

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
2.2 Co-ordinate geometry	Co-ordinating the plane. Properties of lines and line segments including midpoint, slope, distance and the equation of a line in the form. $y - y_1 = m(x - x_1)$. $y = mx + c$. $ax + by + c = 0$ where a, b, c , are integers and m is the slope of the line. Intersection of lines. Parallel and perpendicular lines and the relationships between the slopes.	<ul style="list-style-type: none"> – explore the properties of points, lines and line segments including the equation of a line – find the point of intersection of two lines – find the slopes of parallel and perpendicular lines
2.3 Trigonometry	Right-angled triangles. Trigonometric ratios. Working with trigonometric ratios in surd form for angles of 30°, 45° and 60° Right-angled triangles. Decimal and DMS values of angles.	<ul style="list-style-type: none"> – apply the theorem of Pythagoras to solve right-angled triangle problems of a simple nature involving heights and distances – use trigonometric ratios to solve problems involving angles (integer values) between 0° and 90° – solve problems involving surds – solve problems involving right-angled triangles – manipulate measure of angles in both decimal and DMS forms
2.4 Transformation geometry	Translations, central symmetry, axial symmetry and rotations.	<ul style="list-style-type: none"> – locate axes of symmetry in simple shapes – recognise images of points and objects under translation, central symmetry, axial symmetry and rotations
Students learn about Students should be able to		
2.5 Synthesis and problem-solving skills	<ul style="list-style-type: none"> – explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions. 	

Strand 3: Number

This strand builds on the ideas about number that learners developed in primary school and facilitates the transition between arithmetic and algebra; the *Common Introductory Course* provides appropriate continuity with, and progression from, primary school mathematics. Within this strand, in the context of learning about numbers and computation, learners explore and investigate some generalisations that are central to our number system, the properties and relationships of binary operations, and the results of operating on particular kinds of numbers. Learners are introduced to the notion of justification or proof. They extend their work with ratios to develop an understanding of proportionality which can be applied to solve single and multi-step problems in numerous contexts. Learners are expected to be able to use calculators appropriately and accurately, as well as carrying out calculations by hand and mentally.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- revisit previous learning on number and number operations
- develop a meaningful understanding of different number types, their use and properties
- engage in applications of numeracy to solve real life problems
- apply set theory as a strategy for solving problems in arithmetic.

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
<p>3.1 Number systems</p> <p>N: the set of natural numbers, $\mathbf{N} = \{1,2,3,4,\dots\}$</p> <p>Z: the set of integers, including 0</p> <p>Q: the set of rational numbers</p> <p>R: the set of real numbers</p> <p>R/Q: the set of irrational numbers</p>	<p>The binary operations of addition, subtraction, multiplication and division and the relationships between these operations, beginning with whole numbers and integers. They explore some of the laws that govern these operations and use mathematical models to reinforce the algorithms they commonly use. Later, they revisit these operations in the context of rational numbers and irrational numbers (R/Q) and refine, revise and consolidate their ideas.</p> <p>Students learn strategies for computation that can be applied to any numbers; implicit in such computational methods are generalisations about numerical relationships involving the operations being used. Students articulate the generalisation that underlies their strategy, firstly in the vernacular and then in symbolic language.</p> <p>Problems set in context, using diagrams to solve the problems so they can appreciate how the mathematical concepts are related to real life. Algorithms used to solve problems involving fractional amounts.</p>	<ul style="list-style-type: none"> – investigate models such as decomposition, skip counting, arranging items in arrays and accumulating groups of equal size to make sense of the operations of addition, subtraction, multiplication and division in N where the answer is in N, including the inverse operations – investigate the properties of arithmetic: commutative, associative and distributive laws and the relationships between them – appreciate the order of operations, including the use of brackets – investigate models such as the number line to illustrate the operations of addition, subtraction, multiplication and division in Z – use the number line to order numbers in N, Z, Q (and R for HL) – generalise and articulate observations of arithmetic operations – investigate models to help think about the operations of addition, subtraction, multiplication and division of rational numbers – consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value – analyse solution strategies to problems – engage with the idea of mathematical proof – calculate percentages – use the equivalence of fractions, decimals and percentages to compare proportions – consolidate their understanding and their learning of factors, multiples and prime numbers in N – consolidate their understanding of the relationship between ratio and proportion – check a result by considering whether it is of the right order of magnitude – check a result by working the problem backwards – justify approximations and estimates of calculations – present numerical answers to the degree of accuracy specified

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
3.2 Indices	Binary operations of addition, subtraction, multiplication and division in the context of numbers in index form.	<ul style="list-style-type: none"> – use and apply the rules for indices (where $a \in \mathbf{Z}$, $a \neq 0$; $p, q \in \mathbf{N}$): <ul style="list-style-type: none"> • $a^p a^q = a^{p+q}$ • $\frac{a^p}{a^q} = a^{p-q}$ $p > q$ • $(a^p)^q = a^{pq}$ – use and apply rules for indices (where $a, b \in \mathbf{R}$, $a, b \neq 0$; $p, q \in \mathbf{Q}$; $a^p, a^q \in \mathbf{R}$; complex numbers not included): <ul style="list-style-type: none"> • $a^p a^q = a^{p+q}$ • $\frac{a^p}{a^q} = a^{p-q}$ • $a^0 = 1$ • $(a^p)^q = a^{pq}$ • $a^{1/q} = \sqrt[q]{a}$, $q \in \mathbf{Z}$, $q \neq 0$, $a > 0$ • $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$, $p, q \in \mathbf{Z}$, $q \neq 0$, $a > 0$ • $a^{-p} = \frac{1}{a^p}$ • $(ab)^p = a^p b^p$ • $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$ – operate on the set of irrational numbers $\mathbf{R} \setminus \mathbf{Q}$ – use the notation $a^{1/2}$, $a \in \mathbf{N}$ – express rational numbers ≥ 1 in the approximate form $a \times 10^n$, where a is in decimal form correct to a specified number of places and where $n = 0$ or $n \in \mathbf{N}$ – express non-zero positive rational numbers in the approximate form $a \times 10^n$, where $n \in \mathbf{Z}$ and $1 \leq a < 10$ – compute reciprocals
3.3 Applied arithmetic	<p>Solving problems involving, e.g., mobile phone tariffs, currency transactions, shopping, VAT and meter readings.</p> <p>Making value for money calculations and judgments.</p> <p>Using ratio and proportionality.</p>	<ul style="list-style-type: none"> – solve problems that involve finding profit or loss, % profit or loss (on the cost price), discount, % discount, selling price, compound interest for not more than 3 years, income tax (standard rate only), net pay (including other deductions of specified amounts) – solve problems that involve cost price, selling price, loss, discount, mark up (profit as a % of cost price), margin (profit as a % of selling price) compound interest, income tax and net pay (including other deductions)

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
3.4 Applied measure	<p>Measure and time.</p> <p>2D shapes and 3D solids, including nets of solids (two-dimensional representations of three-dimensional objects).</p> <p>Using nets to analyse figures and to distinguish between surface area and volume.</p> <p>Problems involving perimeter, surface area and volume.</p> <p>Modelling real-world situations and solve a variety of problems (including multi-step problems) involving surface areas, and volumes of cylinders and prisms. The circle and develop an understanding of the relationship between its circumference, diameter and π.</p>	<ul style="list-style-type: none"> – calculate, interpret and apply units of measure and time – solve problems that involve calculating average speed, distance and time – investigate the nets of rectangular solids – find the volume of rectangular solids and cylinders – find the surface area of rectangular solids – identify the necessary information to solve a problem – select and use suitable strategies to find length of the perimeter and the area of the following plane figures: disc, triangle, rectangle, square, and figures made from combinations of these – draw and interpret scaled diagrams – investigate nets of prisms (polygonal bases) cylinders and cones – solve problems involving surface area of triangular base prisms (right angle, isosceles, equilateral), cylinders and cones – solve problems involving curved surface area of cylinders, cones and spheres – perform calculations to solve problems involving the volume of rectangular solids, cylinders, cones, triangular base prisms (right angle, isosceles, equilateral), spheres and combinations of these

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
3.5 Sets	Set language as an international symbolic mathematical tool; the concept of a set as being a well-defined collection of objects or elements. They are introduced to the concept of the universal set, null set, subset, cardinal number; the union, intersection, set difference operators, and Venn diagrams. They investigate the properties of arithmetic as related to sets and solve problems involving sets.	<ul style="list-style-type: none"> – use suitable set notation and terminology – list elements of a finite set – describe the rule that defines a set – consolidate the idea that equality of sets is a relationship in which two equal sets have the same elements – perform the operations of intersection, union (for two sets), set difference and complement – investigate the commutative property for intersection, union and difference – explore the operations of intersection, union (for three sets), set difference and complement – investigate the associative property in relation to intersection, union and difference – investigate the distributive property of union over intersection and intersection over union.
Students learn about	Students should be able to	
3.6 Synthesis and problem-solving skills	<ul style="list-style-type: none"> – explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions. 	

Strand 4: Algebra

Algebra builds on the proficiency that learners have been developing in Strand 3. Two aspects of algebra that underlie all others are algebra as a systematic way of expressing generality and abstraction, including algebra as generalised arithmetic, and algebra as syntactically guided transformations of symbols. These two main aspects of algebra have led to the categorisation of three types of activities that learners of school algebra should engage in: representational activities, transformational activities and activities involving generalising and justifying.

In this strand the approaches to teaching and learning should promote inquiry, build on prior knowledge, and enable learners to have a deep understanding of algebra which allows easy movement between equations, graphs, and tables. The *Common Introductory Course* provides the initial engagement with patterns, relationships and expressions, laying the groundwork for progression to symbolic representation, equations and formulae.

Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- make use of letter symbols for numeric quantities
- emphasise relationship-based algebra
- connect graphical and symbolic representations of algebraic concepts
- use real life problems as vehicles to motivate the use of algebra and algebraic thinking
- use appropriate graphing technologies (calculators, computer software) throughout the strand activities.

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
4.1 Generating arithmetic expressions from repeating patterns	Patterns and the rules that govern them; students construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a correspondence from each input to each output.	<ul style="list-style-type: none"> – use tables to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence
4.2 Representing situations with tables, diagrams and graphs	Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks. Students look at various patterns and make predictions about what comes next.	<ul style="list-style-type: none"> – use tables, diagrams and graphs as tools for representing and analysing linear, quadratic and exponential patterns and relations (exponential relations limited to doubling and tripling) – develop and use their own generalising strategies and ideas and consider those of others – present and interpret solutions, explaining and justifying methods, inferences and reasoning
4.3 Finding formulae	Ways to express a general relationship arising from a pattern or context.	<ul style="list-style-type: none"> – find the underlying formula written in words from which the data are derived (linear relations) – find the underlying formula algebraically from which the data are derived (linear, quadratic relations)
4.4 Examining algebraic relationships	Features of a relationship and how these features appear in the different representations. Constant rate of change: linear relationships. Non-constant rate of change: quadratic relationships. Proportional relationships.	<ul style="list-style-type: none"> – show that relations have features that can be represented in a variety of ways – distinguish those features that are especially useful to identify and point out how those features appear in different representations: in tables, graphs, physical models, and formulas expressed in words, and algebraically – use the representations to reason about the situation from which the relationship is derived and communicate their thinking to others – recognise that a distinguishing feature of quadratic relations is the way change varies – discuss rate of change and the y-intercept; consider how these relate to the context from which the relationship is derived, and identify how they can appear in a table, in a graph and in a formula – decide if two linear relations have a common value – investigate relations of the form $y=mx$ and $y=mx+c$ – recognise problems involving direct proportion and identify the necessary information to solve them

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
4.5 Relations without formulae	Using graphs to represent phenomena quantitatively.	<ul style="list-style-type: none"> – explore graphs of motion – make sense of quantitative graphs and draw conclusions from them – make connections between the shape of a graph and the story of a phenomenon – describe both quantity and change of quantity on a graph
4.6 Expressions	Using letters to represent quantities that are variable. Arithmetic operations on expressions; applications to real life contexts. Transformational activities: collecting like terms, simplifying expressions, substituting, expanding and factoring.	<ul style="list-style-type: none"> – evaluate expressions of the form <ul style="list-style-type: none"> • $ax + by$ • $a(x + y)$ • $x^2 + bx + c$ • $\frac{ax + by}{cx + dy}$ • axy where $a, b, c, d, x, y \in \mathbf{Z}$ <ul style="list-style-type: none"> • $ax^2 + bx + c$ • $x^3 + bx^2 + cx + d$ where $a, b, c, d, x, y \in \mathbf{Q}$ – add and subtract simple algebraic expressions of forms such as: <ul style="list-style-type: none"> • $(ax + by + c) \pm (dx + ey + f)$ • $(ax^2 + bx + c) \pm (dx^2 + ex + f)$ • $\frac{ax + b}{c} \pm \frac{dx + e}{f}$ where $a, b, c, d, e, f \in \mathbf{Z}$ <ul style="list-style-type: none"> • $\frac{ax + b}{c} \pm \dots \pm \frac{dx + e}{f}$ • $(ax + by + c) \pm \dots \pm (dx + ey + f)$ • $(ax^2 + bx + c) \pm \dots \pm (dx^2 + ex + f)$ where $a, b, c, d, e, f \in \mathbf{Z}$ <ul style="list-style-type: none"> • $\frac{a}{bx + c} \pm \frac{p}{qx + r}$ where $a, b, c, p, q, r \in \mathbf{Z}$. – use the associative and distributive property to simplify such expressions as: <ul style="list-style-type: none"> • $a(bx + cy + d) + e(fx + gy + h)$ • $a(bx + cy + d) + \dots + e(fx + gy + h)$ • $a(bx^2 + cx + d)$ • $ax(bx^2 + c)$ where $a, b, c, d, e, f, g, h \in \mathbf{Z}$ <ul style="list-style-type: none"> • $(x+y)(x+y); (x-y)(x-y)$ – multiply expressions of the form: <ul style="list-style-type: none"> • $(ax + b)(cx + d)$ • $(ax + b)(cx^2 + dx + e)$ where $a, b, c, d, e \in \mathbf{Z}$ – divide expressions of the form: <ul style="list-style-type: none"> • $ax^2 + bx + c \div dx + e$, where $a, b, c, d, e \in \mathbf{Z}$ • $ax^3 + bx^2 + cx + d \div ex + f$, where $a, b, c, d, e \in \mathbf{Z}$ – factorise expressions such as <ul style="list-style-type: none"> ax, axy where $a \in \mathbf{Z}$ $abxy + ay$, where $a, b \in \mathbf{Z}$ $sx - ty + tx - sy$, where s, t, x, y are variable $ax^2 + bx$, where $a, b, c \in \mathbf{Z}$ $x^2 + bx + c$, where $b, c \in \mathbf{Z}$ $x^2 - a^2$ $ax^2 + bx + c$, $a \in \mathbf{N}$ $b, c \in \mathbf{Z}$ difference of two squares $a^2x^2 - b^2y^2$ where $a, b \in \mathbf{N}$ – rearrange formulae

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
4.7 Equations and inequalities	<p>Selecting and using suitable strategies (graphic, numeric, algebraic, mental) for finding solutions to equations and inequalities.</p> <p>They identify the necessary information, represent problems mathematically, making correct use of symbols, words, diagrams, tables and graphs.</p>	<ul style="list-style-type: none"> – consolidate their understanding of the concept of equality – solve first degree equations in one or two variables, with coefficients elements of \mathbf{Z} and solutions also elements of \mathbf{Z} – solve first degree equations in one or two variables with coefficients elements of \mathbf{Q} and solutions also in \mathbf{Q} – solve quadratic equations of the form $x^2 + bx + c = 0$ where $b, c \in \mathbf{Z}$ and $x^2 + bx + c$ is factorisable $ax^2 + bx + c = 0$ where $a, b, c \in \mathbf{Q}$ $x \in \mathbf{R}$ – form quadratic equations given whole number roots – solve simple problems leading to quadratic equations – solve equations of the form $\frac{ax + b}{c} \pm \frac{dx + e}{f} = \frac{g}{h}$, where $a, b, c, d, e, f, g, h \in \mathbf{Z}$ – solve linear inequalities in one variable of the form $g(x) \leq k$ where $g(x) = ax + b$, $a \in \mathbf{N}$ and $b, k \in \mathbf{Z}$; $k \leq g(x) \leq h$ where $g(x) = ax + b$, and $k, a, b, h, \in \mathbf{Z}$ and $x \in \mathbf{R}$
Students should learn about	Students should be able to	
4.8 Synthesis and problem-solving skills	<ul style="list-style-type: none"> – explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions. 	

Strand 5: Functions

This strand seeks to make explicit the connections and relationships already encountered in strand 3 and strand 4. Learners revisit and consolidate the learning outcomes of the previous strands.

N.B. Topic descriptions and learning outcomes listed in bold text are for Higher Level only.

In the course of studying this strand the learner will

- engage with the concept of a function (that which involves a set of inputs, a set of possible outputs and a rule that assigns one output to each input)
- emphasise the relationship between functions and algebra
- connect graphical and symbolic representations of functions
- use real life problems as motivation for the study and application of functions
- use appropriate graphing technologies.

Topic	Description of topic Students learn about	Learning outcomes Students should be able to
5.1 Functions	The meaning and notation associated with functions.	<ul style="list-style-type: none"> – engage with the concept of a function, domain, co-domain and range – make use of function notation $f(x) =$, $f : x \rightarrow$, and $y =$
5.2 Graphing functions	Interpreting and representing linear, quadratic and exponential functions in graphical form.	<ul style="list-style-type: none"> – interpret simple graphs – plot points and lines – draw graphs of the following functions and interpret equations of the form $f(x) = g(x)$ as a comparison of functions <ul style="list-style-type: none"> • $f(x) = ax + b$, where $a, b \in \mathbf{Z}$ • $f(x) = ax^2 + bx + c$, where $a \in \mathbf{N}$; $b, c \in \mathbf{Z}$; $x \in \mathbf{R}$ • $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{Z}$, $x \in \mathbf{R}$ • $f(x) = a2^x$ and $f(x) = a3^x$, where $a \in \mathbf{N}$, $x \in \mathbf{R}$ – use graphical methods to find approximate solutions where $f(x) = g(x)$ and interpret the results – find maximum and minimum values of quadratic functions from a graph – interpret inequalities of the form $f(x) \leq g(x)$ as a comparison of functions of the above form; use graphical methods to find approximate solution sets of these inequalities and interpret the results – graph solution sets on the number line for linear inequalities in one variable

Students learn about	Students should be able to
5.3 Synthesis and problem-solving skills	<ul style="list-style-type: none"> – explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.

General principles

Assessment in education involves gathering, interpreting and using information about the processes and outcomes of learning. It takes different forms and can be used in a variety of ways, such as to test and certify achievement, to determine the appropriate route for learners to take through a differentiated curriculum, or to identify specific areas of difficulty (or strength) for a given learner. While different techniques may be employed for formative, diagnostic and certification purposes, assessment of any kind can improve learning by exerting a positive influence on the curriculum at all levels. To do this it must reflect the full range of curriculum goals.

Assessment should be used as a continuous part of the teaching-learning process and involve learners, wherever possible, as well as teachers in identifying next steps. In this context, the most valuable assessment takes place at the site of learning. Assessment also provides an effective basis for communication with parents in a way that helps them to support their children's learning. Assessment must be valid, reliable and equitable. These aspects of assessment are particularly relevant for national assessment for certification purposes.

Assessment for certification

Junior Certificate Mathematics is assessed at Foundation, Ordinary and Higher levels. At Foundation level there is one examination paper. There are two assessment components at Ordinary and Higher level

- Mathematics Paper 1
- Mathematics Paper 2

Differentiation at the point of assessment is achieved through the language level in the examination questions, the stimulus material presented, and the amount of structured support given in the questions, especially for candidates at Foundation level.

The learner's understanding of mathematics will be assessed through a focus on *concepts and skills* and *contexts and applications*. Learners will be asked to engage with mathematical and real life problems and to explain and justify conclusions. In this regard some assessment items will differ from those traditionally presented in examination papers.

Learners at Foundation level can expect to engage with a variety of tasks, including word problems, but in language that is appropriate to this level. They will be expected to deal with concepts at a concrete level and will not be expected to engage in more formal abstraction. There will be structured support within tasks to assist in progression through a problem. Learners will be expected to give an opinion and to justify and explain their reasoning in some answers. The assessment will reflect the changed methodology and active nature of teaching and learning in the classroom.

The tasks for learners at Ordinary level will be more challenging than Foundation level tasks and candidates may not receive the same level of structured support in a problem. They will be expected to deal with problem solving in real world contexts and to draw conclusions from answers. The quality of the answering expected will be higher than that at Foundation level.

Learners at Higher level will be expected to deal with more complex and challenging problems than those at Ordinary level. They will be asked to demonstrate a deeper understanding of concepts and an ability to employ a variety of strategies to solve problems as well as to apply mathematical knowledge. Learners at this level can expect to be tested on Ordinary level learning outcomes but their tasks will be, to an appropriate degree, more complex and difficult.

Appendix: Common Introductory Course for Junior Cycle Mathematics

The *Common Introductory Course* is the minimum course to be covered by all learners at the start of junior cycle. It is intended that the experience of this course will lay the foundation for conceptual understanding which learners can build on subsequently. The order in which topics are introduced is left to the discretion of the teacher. The topics and strands should not be treated in isolation; where appropriate, connections should be made between them. Classroom strategies should be adopted which will encourage students to develop their synthesis and problem-solving skills.

Once the introductory course has been completed, teachers can decide which topics to extend or explore to a greater depth, depending on the progress being made by the class group.

The following table, when read in conjunction with the section on the Bridging Framework for Mathematics (see page 8), may help teachers to prepare teaching and learning plans for the *Common Introductory Course* in order to facilitate a smooth transition for learners from their mathematics education in the primary school.

Strand /Topic Title	Learning outcomes Students should be able to
Strand 1: 1.1 Counting	<ul style="list-style-type: none"> – list all possible outcomes of an experiment – apply the fundamental principle of counting
Strand 1: 1.2 Concepts of probability It is expected that the conduct of experiments (including simulations), both individually and in groups, will form the primary vehicle through which the knowledge, understanding and skills in probability are developed.	<ul style="list-style-type: none"> – decide whether an everyday event is likely or unlikely to occur – recognise that probability is a measure on a scale of 0 - 1 of how likely an event is to occur
Strand 1: 1.5 Finding, collecting and organising data	<ul style="list-style-type: none"> – explore different ways of collecting data – plan an investigation involving statistics and conduct the investigation – summarise data in diagrammatic form – reflect on the question(s) posed in light of data collected
Strand 1: 1.6 Representing data graphically and numerically	<ul style="list-style-type: none"> – select appropriate graphical or numerical methods to represent and describe the sample (univariate data only) – use stem and leaf plots, line plots and bar charts to display data
Strand 2: 2.1 Synthetic geometry (<i>see Geometry for Post-primary School Mathematics</i>) The geometrical results should be first encountered through discovery and investigation.	<ul style="list-style-type: none"> – convince themselves through investigation that theorems 1-6 appear to be true – construct <ol style="list-style-type: none"> 1. the bisector of a given angle, using only compass and straight edge 2. the perpendicular bisector of a segment, using only compass and straight edge 4. a line perpendicular to a given line l, passing through a given point on l 5. a line parallel to a given line l, through a given point 6. divide a line segment into 2, 3 equal segments, without measuring it 8. a line segment of given length on a given ray
Strand 2: 2.2 Co-ordinate geometry	<ul style="list-style-type: none"> – coordinate the plane – locate points on the plane using coordinates
Strand 2: 2.4 Transformation geometry	<ul style="list-style-type: none"> – use drawings to show central symmetry, axial symmetry and rotations

Strand /Topic Title	Learning outcomes Students should be able to
<p>Strand 3: 3.1: Number systems</p> <p>Students explore the operations of addition, subtraction, multiplication and division and the relationships between these operations – in the first instance with whole numbers and integers. They will explore some of the laws that govern these operations and use mathematical models to reinforce the algorithms they commonly use. Later, they revisit these operations in the contexts of rational numbers and refine and revise their ideas.</p> <p>Students will devise strategies for computation that can be applied to any number. Implicit in such computational methods are generalisations about numerical relationships involving the operations being used. Students will articulate the generalisation that underlies their strategy, firstly in common language and then in symbolic language.</p>	<ul style="list-style-type: none"> – investigate models such as decomposition, skip counting, arranging items in arrays and accumulating groups of equal size to make sense of the operations of addition, subtraction, multiplication, and division in N where the answer is in N, including the inverse operations – investigate the properties of arithmetic: commutative, associative and distributive laws and the relationships between them – appreciate the order of operations, including use of brackets – investigate models, such as the number line, to illustrate the operations of addition, subtraction, multiplication and division in Z – use the number line to order numbers in N, Z, Q (and R for HL) – generalise and articulate observations of arithmetic operations – investigate models to help think about the operations of addition, subtraction, multiplication and division of rational numbers – consolidate the idea that equality is a relationship in which two mathematical expressions hold the same value – analyse solution strategies to problems – begin to look at the idea of mathematical proof – calculate percentages – use the equivalence of fractions, decimals and percentages to compare proportions – consolidate their understanding and their learning of factors, multiples and prime numbers in N – consolidate their understanding of the relationship between ratio and proportion – check a result by considering whether it is of the right order of magnitude – check a result by working the problem backwards – justify approximations and estimates of calculations – present numerical answers to degree of accuracy specified
<p>Strand 3: 3.5 Sets</p> <p>Students learn the concept of a set as being a collection of well-defined objects or <i>elements</i>. They are introduced to the concept of the universal set, null set, sub-set; the union and intersection operators and to Venn diagrams: simple closed bounded curves that contain the elements of a set.</p> <p>They investigate the properties of arithmetic as related to sets and solve problems involving sets.</p>	<ul style="list-style-type: none"> – list elements of a set – describe the rule that defines a finite set – consolidate the idea that equality of sets is a relationship in which two equal sets have the same elements – use the cardinal number terminology when referring to set membership – perform the operations of intersection, union (for two sets) – investigate the commutative property for intersection and union – illustrate sets using Venn diagrams

Strand /Topic Title	Learning outcomes Students should be able to
<p>Strand 4: 4.1 Generating arithmetic expressions from repeating patterns</p> <p>Students examine patterns and the rules that govern them and so construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a correspondence from each input to each output.</p>	<ul style="list-style-type: none"> – use tables and diagrams to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms in a sequence
<p>Strand 4: 4.2 Representing situations with tables diagrams and graphs</p> <p>Students examine relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks.They look at various patterns and make predictions about what comes next.</p>	<ul style="list-style-type: none"> – use tables, diagrams and graphs as a tool for analysing relations – develop and use their own mathematical strategies and ideas and consider those of others – present and interpret solutions, explaining and justifying methods, inferences and reasoning
<p>All Strands Synthesis and problem-solving skills</p>	<ul style="list-style-type: none"> – explore patterns and formulate conjectures – explain findings – justify conclusions – communicate mathematics verbally and in written form – apply their knowledge and skills to solve problems in familiar and unfamiliar contexts – analyse information presented verbally and translate it into mathematical form – devise, select and use appropriate mathematical models, formulae or techniques to process information and to draw relevant conclusions.



SECTION B

Geometry for Post-primary School Mathematics

This section sets out the course in geometry for both Junior Certificate Mathematics and Leaving Certificate Mathematics. Strand 2 of the relevant syllabus document specifies the learning outcomes at the different syllabus levels.

Geometry for Post-primary School Mathematics

1 Introduction

The Junior Certificate and Leaving Certificate mathematics course committees of the National Council for Curriculum and Assessment (NCCA) accepted the recommendation contained in the paper [4] to base the logical structure of post-primary school geometry on the level 1 account in Professor Barry's book [1].

To quote from [4]: We distinguish three levels:

Level 1: The fully-rigorous level, likely to be intelligible only to professional mathematicians and advanced third- and fourth-level students.

Level 2: The semiformal level, suitable for digestion by many students from (roughly) the age of 14 and upwards.

Level 3: The informal level, suitable for younger children.

This document sets out the agreed geometry for post-primary schools. It was prepared by a working group of the NCCA course committees for mathematics and, following minor amendments, was adopted by both committees for inclusion in the syllabus documents. Readers should refer to Strand 2 of the syllabus documents for Junior Certificate and Leaving Certificate mathematics for the range and depth of material to be studied at the different levels. A summary of these is given in sections 9–13 of this document.

The preparation and presentation of this document was undertaken principally by Anthony O'Farrell, with assistance from Ian Short. Helpful criticism from Stefan Bechluft-Sachs, Ann O'Shea, Richard Watson and Stephen Buckley is also acknowledged.

2 The system of geometry used for the purposes of formal proofs

In the following, Geometry refers to plane geometry.

There are many formal presentations of geometry in existence, each with its own set of axioms and primitive concepts. What constitutes a valid proof in the context of one system might therefore not be valid in the context of another. Given that students will be expected to present formal proofs in the examinations, it is therefore necessary to specify the system of geometry that is to form the context for such proofs.

The formal underpinning for the system of geometry on the Junior and Leaving Certificate courses is that described by Prof. Patrick D. Barry in [1]. A properly formal presentation of such a system has the serious disadvantage that it is not readily accessible to students at this level. Accordingly, what is presented below is a necessarily simplified version that treats many concepts far more loosely than a truly formal presentation would demand. Any readers who wish to rectify this deficiency are referred to [1] for a proper scholarly treatment of the material.

Barry's system has the primitive undefined terms **plane**, **point**, **line**, $<_l$ (**precedes on a line**), **(open) half-plane**, **distance**, and **degree-measure**, and seven axioms: A_1 : about incidence, A_2 : about order on lines, A_3 : about how lines separate the plane, A_4 : about distance, A_5 : about degree measure, A_6 : about congruence of triangles, A_7 : about parallels.

3 Guiding Principles

In constructing a level 2 account, we respect the principles about the relationship between the levels laid down in [4, Section 2].

The choice of material to study should be guided by applications (inside and outside Mathematics proper).

The most important reason to study synthetic geometry is to prepare the ground logically for the development of trigonometry, coordinate geometry, and vectors, which in turn have myriad applications.

We aim to keep the account as simple as possible.

We also take it as desirable that the official Irish syllabus should avoid imposing terminology that is nonstandard in international practice, or is used in a nonstandard way.

No proof should be allowed at level 2 that cannot be expanded to a complete rigorous proof at level 1, or that uses axioms or theorems that come later in the logical sequence. We aim to supply adequate proofs for all the theorems, but do not propose that only those proofs will be acceptable. It should be open to teachers and students to think about other ways to prove the results, provided they are correct and fit within the logical framework. Indeed, such activity is to be encouraged. Naturally, teachers and students will need some assurance that such variant proofs will be acceptable if presented in examination. We suggest that the discoverer of a new proof should discuss it with students and colleagues, and (if in any doubt) should refer it to the National Council for Curriculum and Assessment and/or the State Examinations Commission.

It may be helpful to note the following non-exhaustive list of salient differences between Barry's treatment and our less formal presentation.

- Whereas we may use set notation and we expect students to understand the conceptualisation of geometry in terms of sets, we more often use the language that is common when discussing geometry informally, such as “the point is/lies on the line”, “the line passes through the point”, etc.
- We accept and use a much lesser degree of precision in language and notation (as is apparent from some of the other items on this list).
- We state five explicit axioms, employing more informal language than Barry's, and we do not explicitly state axioms corresponding to Axioms A2 and A3 – instead we make statements without fanfare in the text.
- We accept a much looser understanding of what constitutes an **angle**, making no reference to angle-supports. We do not define the term angle. We mention reflex angles from the beginning (but make no use of them until we come to angles in circles), and quietly assume (when the time comes) that axioms that are presented by Barry in the context of wedge-angles apply also in the naturally corresponding way to reflex angles.
- When naming an angle, it is always assumed that the non-reflex angle is being referred to, unless the word “reflex” precedes or follows.

- We make no reference to results such as Pasch’s property and the “crossbar theorem”. (That is, we do not expect students to consider the necessity to prove such results or to have them given as axioms.)
- We refer to “the number of degrees” in an angle, whereas Barry treats this more correctly as “the degree-measure” of an angle.
- We take it that the definitions of parallelism, perpendicularity and “sidedness” are readily extended from lines to half-lines and line segments. (Hence, for example, we may refer to the opposite sides of a particular quadrilateral as being parallel, meaning that the lines of which they are subsets are parallel).
- We do not refer explicitly to triangles being **congruent** “under the correspondence $(A, B, C) \rightarrow (D, E, F)$ ”, taking it instead that the correspondence is the one implied by the order in which the vertices are listed. That is, when we say “ $\triangle ABC$ is congruent to $\triangle DEF$ ” we mean, using Barry’s terminology, “Triangle $[A,B,C]$ is congruent to triangle $[D,E,F]$ under the correspondence $(A, B, C) \rightarrow (D, E, F)$ ”.
- We do not always retain the distinction in language between an angle and its measure, relying frequently instead on the context to make the meaning clear. However, we continue the practice of distinguishing notationally between the angle $\angle ABC$ and the number $|\angle ABC|$ of degrees in the angle¹. In the same spirit, we may refer to two angles being equal, or one being equal to the sum of two others, (when we should more precisely say that the two are equal in measure, or that the measure of one is equal to the sum of the measures of the other two). Similarly, with length, we may loosely say, for example: “opposite sides of a parallelogram are equal”, or refer to “a circle of radius r ”. Where ambiguity does not arise, we may refer to angles using a single letter. That is, for example, if a diagram includes only two rays or segments from the point A , then the angle concerned may be referred to as $\angle A$.

Having pointed out these differences, it is perhaps worth mentioning some significant structural aspects of Barry’s geometry that are retained in our less formal version:

¹In practice, the examiners do not penalise students who leave out the bars.

- The primitive terms are almost the same, subject to the fact that their properties are conceived less formally. We treat **angle** as an extra undefined term.
- We assume that results are established in the same order as in Barry [1], up to minor local rearrangement. The exception to this is that we state all the axioms as soon as they are useful, and we bring the theorem on the angle-sum in a triangle forward to the earliest possible point (short of making it an axiom). This simplifies the proofs of a few theorems, at the expense of making it easy to see which results are theorems of so-called Neutral Geometry².
- **Area** is not taken to be a primitive term or a given property of regions. Rather, it is defined for triangles following the establishment of the requisite result that the products of the lengths of the sides of a triangle with their corresponding altitudes are equal, and then extended to convex quadrilaterals.
- **Isometries or other transformations** are not taken as primitive. Indeed, in our case, the treatment does not extend as far as defining them. Thus they can play no role in our proofs.

4 Outline of the Level 2 Account

We present the account by outlining:

1. A list (Section 5), of the terminology for the geometrical concepts. Each term in a theory is either undefined or defined, or at least definable. There have to be some undefined terms. (In textbooks, the undefined terms will be introduced by descriptions, and some of the defined terms will be given explicit definitions, in language appropriate to the level. We assume that previous level 3 work will have laid a foundation that will allow students to understand the undefined terms. We do not give the explicit definitions of all the definable terms. Instead we rely on the student's ordinary language, supplemented sometimes by informal remarks. For instance, we do not write out in cold blood the definition of the **side opposite** a given angle in a triangle, or the

² Geometry without the axiom of parallels. This is not a concern in secondary school.

definition (in terms of set membership) of what it means to say that a line **passes through** a given point. The reason why some terms **must** be given explicit definitions is that there are alternatives, and the definition specifies the starting point; the alternative descriptions of the term are then obtained as theorems.

2. A logical account (Section 6) of the synthetic geometry theory. All the material through to LC higher is presented. The individual syllabuses will identify the relevant content by referencing it by number (e.g. Theorems 1,2, 9).
3. The geometrical constructions (Section 7) that will be studied. Again, the individual syllabuses will refer to the items on this list by number when specifying what is to be studied.
4. Some guidance on teaching (Section 8).
5. Syllabus entries for each of JC-OL, JC-HL, LC-FL, LC-OL, LC-HL.

5 Terms

Undefined Terms: angle, degree, length, line, plane, point, ray, real number, set.

Most important Defined Terms: area, parallel lines, parallelogram, right angle, triangle, congruent triangles, similar triangles, tangent to a circle, area.

Other Defined terms: acute angle, alternate angles, angle bisector, arc, area of a disc, base and corresponding apex and height of triangle or parallelogram, chord, circle, circumcentre, circumcircle, circumference of a circle, circumradius, collinear points, concurrent lines, convex quadrilateral, corresponding angles, diameter, disc, distance, equilateral triangle, exterior angles of a triangle, full angle, hypotenuse, in-centre, incircle, inradius, interior opposite angles, isosceles triangle, median lines, midpoint of a segment, null angle, obtuse angle, perpendicular bisector of a segment, perpendicular lines, point of contact of a tangent, polygon, quadrilateral, radius, ratio, rectangle, reflex angle ordinary angle, rhombus, right-angled triangle, scalene triangle,

sector, segment, square, straight angle, subset, supplementary angles, transversal line, vertically-opposite angles.

Definable terms used without explicit definition: angles, adjacent sides, arms or sides of an angle, centre of a circle, endpoints of segment, equal angles, equal segments, line passes through point, opposite sides or angles of a quadrilateral, or vertices of triangles or quadrilaterals, point lies on line, side of a line, side of a polygon, the side opposite an angle of a triangle, vertex, vertices (of angle, triangle, polygon).

6 The Theory

Line³ is short for straight line. Take a fixed **plane**⁴, once and for all, and consider just lines that lie in it. The plane and the lines are **sets**⁵ of **points**⁶. Each line is a **subset** of the plane, i.e. each element of a line is a point of the plane. Each line is endless, extending forever in both directions. Each line has infinitely-many points. The points on a line can be taken to be ordered along the line in a natural way. As a consequence, given any three distinct points on a line, exactly one of them lies **between** the other two. Points that are not on a given line can be said to be on one or other **side** of the line. The sides of a line are sometimes referred to as **half-planes**.

Notation 1. We denote points by roman capital letters A, B, C , etc., and lines by lower-case roman letters l, m, n , etc.

Axioms are statements we will accept as true⁷.

Axiom 1 (Two Points Axiom). *There is exactly one line through any two given points. (We denote the line through A and B by AB .)*

Definition 1. The line **segment** $[AB]$ is the part of the line AB between A and B (including the endpoints). The point A divides the line AB into two pieces, called **rays**. The point A lies between all points of one ray and all

³Line is undefined.

⁴Undefined term

⁵Undefined term

⁶Undefined term

⁷ An **axiom** is a statement accepted without proof, as a basis for argument. A **theorem** is a statement deduced from the axioms by logical argument.

points of the other. We denote the ray that starts at A and passes through B by $[AB$. Rays are sometimes referred to as **half-lines**.

Three points usually determine three different lines.

Definition 2. If three or more points lie on a single line, we say they are **collinear**.

Definition 3. Let A , B and C be points that are not collinear. The **triangle** $\triangle ABC$ is the piece of the plane enclosed by the three line segments $[AB]$, $[BC]$ and $[CA]$. The segments are called its **sides**, and the points are called its **vertices** (singular **vertex**).

6.1 Length and Distance

We denote the set of all **real numbers**⁸ by \mathbb{R} .

Definition 4. We denote the **distance**⁹ between the points A and B by $|AB|$. We define the **length** of the segment $[AB]$ to be $|AB|$.

We often denote the lengths of the three sides of a triangle by a , b , and c . The usual thing for a triangle $\triangle ABC$ is to take $a = |BC|$, i.e. the length of the side opposite the vertex A , and similarly $b = |CA|$ and $c = |AB|$.

Axiom 2 (Ruler Axiom¹⁰). *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*
2. $|AB| = |BA|$;
3. *if C lies on AB , between A and B , then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A , and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k .*

⁸Undefined term

⁹Undefined term

¹⁰ Teachers used to traditional treatments that follow Euclid closely should note that this axiom (and the later Protractor Axiom) guarantees the existence of various points (and lines) without appeal to postulates about constructions using straight-edge and compass. They are powerful axioms.

Definition 5. The **midpoint** of the segment $[AB]$ is the point M of the segment with ¹¹

$$|AM| = |MB| = \frac{|AB|}{2}.$$

6.2 Angles

Definition 6. A subset of the plane is **convex** if it contains the whole segment that connects any two of its points.

For example, one side of any line is a convex set, and triangles are convex sets.

We do not define the term angle formally. Instead we say: There are things called **angles**. To each angle is associated:

1. a unique point A , called its **vertex**;
2. two rays $[AB$ and $[AC$, both starting at the vertex, and called the **arms** of the angle;
3. a piece of the plane called the **inside** of the angle.

An angle is either a null angle, an ordinary angle, a straight angle, a reflex angle or a full angle. Unless otherwise specified, you may take it that any angle we talk about is an ordinary angle.

Definition 7. An angle is a **null angle** if its arms coincide with one another and its inside is the empty set.

Definition 8. An angle is an **ordinary angle** if its arms are not on one line, and its inside is a convex set.

Definition 9. An angle is a **straight angle** if its arms are the two halves of one line, and its inside is one of the sides of that line.

Definition 10. An angle is a **reflex angle** if its arms are not on one line, and its inside is not a convex set.

Definition 11. An angle is a **full angle** if its arms coincide with one another and its inside is the rest of the plane.

¹¹ Students may notice that the first equality implies the second.

Definition 12. Suppose that A , B , and C are three noncollinear points. We denote the (ordinary) angle with arms $[AB$ and $[AC$ by $\angle BAC$ (and also by $\angle CAB$). We shall also use the notation $\angle BAC$ to refer to straight angles, where A , B , C are collinear, and A lies between B and C (either side could be the inside of this angle).

Sometimes we want to refer to an angle without naming points, and in that case we use lower-case Greek letters, α, β, γ , etc.

6.3 Degrees

Notation 2. We denote the number of **degrees** in an angle $\angle BAC$ or α by the symbol $|\angle BAC|$, or $|\angle \alpha|$, as the case may be.

Axiom 3 (Protractor Axiom). *The number of degrees in an angle (also known as its degree-measure) is always a number between 0° and 360° . The number of degrees of an ordinary angle is less than 180° . It has these properties:*

1. *A straight angle has 180° .*
2. *Given a ray $[AB$, and a number d between 0 and 180, there is exactly one ray from A on each side of the line AB that makes an (ordinary) angle having d degrees with the ray $[AB$.*
3. *If D is a point inside an angle $\angle BAC$, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Null angles are assigned 0° , full angles 360° , and reflex angles have more than 180° . To be more exact, if A , B , and C are noncollinear points, then the reflex angle “outside” the angle $\angle BAC$ measures $360^\circ - |\angle BAC|$, in degrees.

Definition 13. The ray $[AD$ is the **bisector** of the angle $\angle BAC$ if

$$|\angle BAD| = |\angle DAC| = \frac{|\angle BAC|}{2}.$$

We say that an angle is ‘an angle of’ (for instance) 45° , if it has 45 degrees in it.

Definition 14. A **right angle** is an angle of exactly 90° .

Definition 15. An angle is **acute** if it has less than 90° , and **obtuse** if it has more than 90° .

Definition 16. If $\angle BAC$ is a straight angle, and D is off the line BC , then $\angle BAD$ and $\angle DAC$ are called **supplementary angles**. They add to 180° .

Definition 17. When two lines AB and AC cross at a point A , they are **perpendicular** if $\angle BAC$ is a right angle.

Definition 18. Let A lie between B and C on the line BC , and also between D and E on the line DE . Then $\angle BAD$ and $\angle CAE$ are called **vertically-opposite angles**.

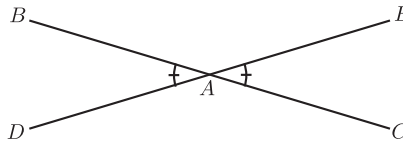


Figure 1.

Theorem 1 (Vertically-opposite Angles).

Vertically opposite angles are equal in measure.

Proof. See Figure 1. The idea is to add the same supplementary angles to both, getting 180° . In detail,

$$\begin{aligned} |\angle BAD| + |\angle BAE| &= 180^\circ, \\ |\angle CAE| + |\angle BAE| &= 180^\circ, \end{aligned}$$

so subtracting gives:

$$\begin{aligned} |\angle BAD| - |\angle CAE| &= 0^\circ, \\ |\angle BAD| &= |\angle CAE|. \end{aligned}$$

□

6.4 Congruent Triangles

Definition 19. Let A, B, C and A', B', C' be triples of non-collinear points. We say that the triangles $\triangle ABC$ and $\triangle A'B'C'$ are **congruent** if all the sides and angles of one are equal to the corresponding sides and angles of the other, i.e. $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $|\angle ABC| = |\angle A'B'C'|$, $|\angle BCA| = |\angle B'C'A'|$, and $|\angle CAB| = |\angle C'A'B'|$. See Figure 2.

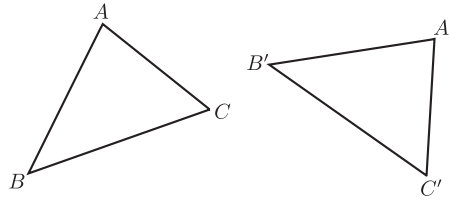


Figure 2.

Notation 3. Usually, we abbreviate the names of the angles in a triangle, by labelling them by the names of the vertices. For instance, we write $\angle A$ for $\angle CAB$.

Axiom 4 (SAS+ASA+SSS¹²).

If (1) $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|\angle A| = |\angle A'|$,

or

(2) $|BC| = |B'C'|$, $|\angle B| = |\angle B'|$, and $|\angle C| = |\angle C'|$,

or

(3) $|AB| = |A'B'|$, $|BC| = |B'C'|$, and $|CA| = |C'A'|$

then the triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent.

Definition 20. A triangle is called **right-angled** if one of its angles is a right angle. The other two angles then add to 90° , by Theorem 4, so are both acute angles. The side opposite the right angle is called the **hypotenuse**.

Definition 21. A triangle is called **isosceles** if two sides are equal¹³. It is **equilateral** if all three sides are equal. It is **scalene** if no two sides are equal.

Theorem 2 (Isosceles Triangles).

(1) In an isosceles triangle the angles opposite the equal sides are equal.

(2) Conversely, If two angles are equal, then the triangle is isosceles.

Proof. (1) Suppose the triangle $\triangle ABC$ has $AB = AC$ (as in Figure 3). Then $\triangle ABC$ is congruent to $\triangle ACB$ [SAS]

$\therefore \angle B = \angle C$.

¹²It would be possible to prove all the theorems using a weaker axiom (just SAS). We use this stronger version to shorten the course.

¹³ The simple “equal” is preferred to “of equal length”

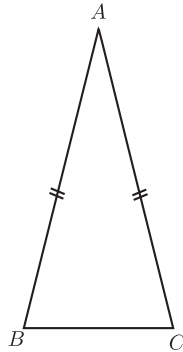


Figure 3.

(2) Suppose now that $\angle B = \angle C$. Then $\triangle ABC$ is congruent to $\triangle ACB$ [ASA]
 $\therefore |AB| = |AC|$, $\triangle ABC$ is isosceles. □

Acceptable Alternative Proof of (1). Let D be the midpoint of $[BC]$, and use SAS to show that the triangles $\triangle ABD$ and $\triangle ACD$ are congruent. (This proof is more complicated, but has the advantage that it yields the extra information that the angles $\angle ADB$ and $\angle ADC$ are equal, and hence both are right angles (since they add to a straight angle)). □

6.5 Parallels

Definition 22. Two lines l and m are **parallel** if they are either identical, or have no common point.

Notation 4. We write $l \parallel m$ for “ l is parallel to m ”.

Axiom 5 (Axiom of Parallels). *Given any line l and a point P , there is exactly one line through P that is parallel to l .*

Definition 23. If l and m are lines, then a line n is called a **transversal** of l and m if it meets them both.

Definition 24. Given two lines AB and CD and a transversal BC of them, as in Figure 4, the angles $\angle ABC$ and $\angle BCD$ are called **alternate** angles.

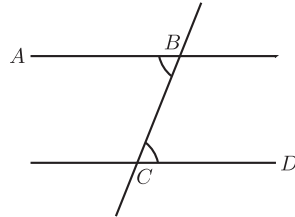


Figure 4.

Theorem 3 (Alternate Angles). *Suppose that A and D are on opposite sides of the line BC .*

(1) *If $|\angle ABC| = |\angle BCD|$, then $AB \parallel CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.*

(2) *Conversely, if $AB \parallel CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.*

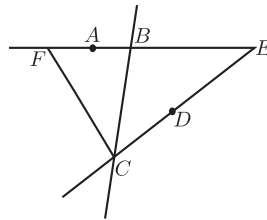


Figure 5.

Proof. (1) Suppose $|\angle ABC| = |\angle BCD|$. If the lines AB and CD do not meet, then they are parallel, by definition, and we are done. Otherwise, they meet at some point, say E . Let us assume that E is on the same side of BC as D .¹⁴ Take F on EB , on the same side of BC as A , with $|BF| = |CE|$ (see Figure 5). [Ruler Axiom]

¹⁴Fuller detail: There are three cases:

1°: E lies on BC . Then (using Axiom 1) we must have $E = B = C$, and $AB = CD$.

2°: E lies on the same side of BC as D . In that case, take F on EB , on the same side of BC as A , with $|BF| = |CE|$. [Ruler Axiom]

Then $\triangle BCE$ is congruent to $\triangle CBF$. [SAS]

Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

Then $\triangle BCE$ is congruent to $\triangle CBF$. [SAS]
 Thus

$$|\angle BCF| = |\angle CBE| = 180^\circ - |\angle ABC| = 180^\circ - |\angle BCD|,$$

so that F lies on DC . [Ruler Axiom]

Thus AB and CD both pass through E and F , and hence coincide, [Axiom 1]

Hence AB and CD are parallel. [Definition of parallel]

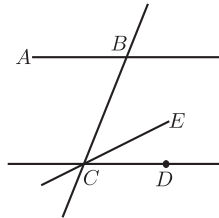


Figure 6.

(2) To prove the converse, suppose $AB \parallel CD$. Pick a point E on the same side of BC as D with $|\angle BCE| = |\angle ABC|$. (See Figure 6.) By Part (1), the line CE is parallel to AB . By Axiom 5, there is only one line through C parallel to AB , so $CE = CD$. Thus $|\angle BCD| = |\angle BCE| = |\angle ABC|$. \square

Theorem 4 (Angle Sum 180). *The angles in any triangle add to 180° .*

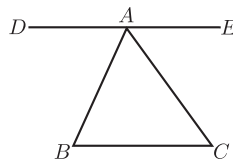


Figure 7.

so that F lies on DC . [Protractor Axiom]
 Thus AB and CD both pass through E and F , and hence coincide. [Axiom 1]
 3°: E lies on the same side of BC as A . Similar to the previous case.
 Thus, in all three cases, $AB = CD$, so the lines are parallel.

Proof. Let $\triangle ABC$ be given. Take a segment $[DE]$ passing through A , parallel to BC , with D on the opposite side of AB from C , and E on the opposite side of AC from B (as in Figure 7). [Axiom of Parallels]

Then AB is a transversal of DE and BC , so by the Alternate Angles Theorem,

$$|\angle ABC| = |\angle DAB|.$$

Similarly, AC is a transversal of DE and BC , so

$$|\angle ACB| = |\angle CAE|.$$

Thus, using the Protractor Axiom to add the angles,

$$\begin{aligned} & |\angle ABC| + |\angle ACB| + |\angle BAC| \\ &= |\angle DAB| + |\angle CAE| + |\angle BAC| \\ &= |\angle DAE| = 180^\circ, \end{aligned}$$

since $\angle DAE$ is a straight angle. □

Definition 25. Given two lines AB and CD , and a transversal AE of them, as in Figure 8(a), the angles $\angle EAB$ and $\angle ACD$ are called **corresponding angles**¹⁵.

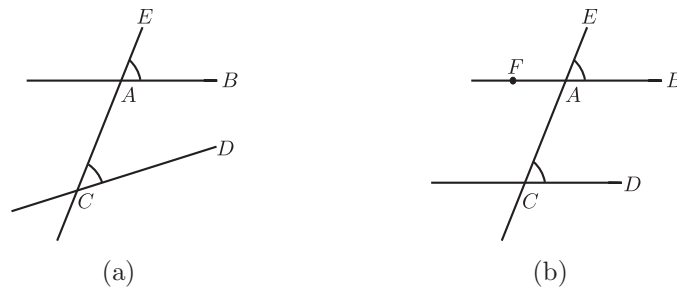


Figure 8.

Theorem 5 (Corresponding Angles). *Two lines are parallel if and only if for any transversal, corresponding angles are equal.*

¹⁵with respect to the two lines and the given transversal.

Proof. See Figure 8(b). We first assume that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. Let F be a point on AB such that F and B are on opposite sides of AE . Then we have

$$|\angle EAB| = |\angle FAC| \quad \text{[Vertically opposite angles]}$$

Hence the alternate angles $\angle FAC$ and $\angle ACD$ are equal and therefore the lines $FA = AB$ and CD are parallel.

For the converse, let us assume that the lines AB and CD are parallel. Then the alternate angles $\angle FAC$ and $\angle ACD$ are equal. Since

$$|\angle EAB| = |\angle FAC| \quad \text{[Vertically opposite angles]}$$

we have that the corresponding angles $\angle EAB$ and $\angle ACD$ are equal. \square

Definition 26. In Figure 9, the angle α is called an **exterior angle** of the triangle, and the angles β and γ are called (corresponding) **interior opposite angles**.¹⁶

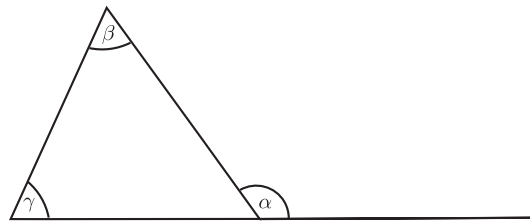


Figure 9.

Theorem 6 (Exterior Angle). *Each exterior angle of a triangle is equal to the sum of the interior opposite angles.*

Proof. See Figure 10. In the triangle $\triangle ABC$ let α be an exterior angle at A . Then

$$|\alpha| + |\angle A| = 180^\circ \quad \text{[Supplementary angles]}$$

and

$$|\angle B| + |\angle C| + |\angle A| = 180^\circ. \quad \text{[Angle sum } 180^\circ]$$

Subtracting the two equations yields $|\alpha| = |\angle B| + |\angle C|$. \square

¹⁶The phrase **interior remote angles** is sometimes used instead of **interior opposite angles**.

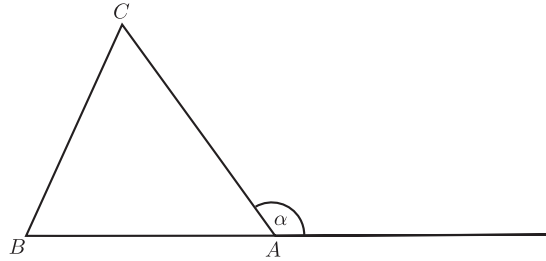


Figure 10.

Theorem 7.

(1) In $\triangle ABC$, suppose that $|AC| > |AB|$. Then $|\angle ABC| > |\angle ACB|$. In other words, the angle opposite the greater of two sides is greater than the angle opposite the lesser side.

(2) Conversely, if $|\angle ABC| > |\angle ACB|$, then $|AC| > |AB|$. In other words, the side opposite the greater of two angles is greater than the side opposite the lesser angle.

Proof.

(1) Suppose that $|AC| > |AB|$. Then take the point D on the segment $[AC]$ with $|AD| = |AB|$. [Ruler Axiom]

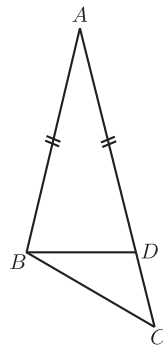


Figure 11.

See Figure 11. Then $\triangle ABD$ is isosceles, so

$$\begin{aligned}
 |\angle ACB| &< |\angle ADB| && \text{[Exterior Angle]} \\
 &= |\angle ABD| && \text{[Isosceles Triangle]} \\
 &< |\angle ABC|.
 \end{aligned}$$

Thus $|\angle ACB| < |\angle ABC|$, as required.

(2)(This is a Proof by Contradiction!)

Suppose that $|\angle ABC| > |\angle ACB|$. See Figure 12.

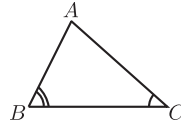


Figure 12.

If it could happen that $|AC| \leq |AB|$, then
either Case 1°: $|AC| = |AB|$, in which case $\triangle ABC$ is isosceles, and then $|\angle ABC| = |\angle ACB|$, which contradicts our assumption,
or Case 2°: $|AC| < |AB|$, in which case Part (1) tells us that $|\angle ABC| < |\angle ACB|$, which also contradicts our assumption. Thus it cannot happen, and we conclude that $|AC| > |AB|$. \square

Theorem 8 (Triangle Inequality).

Two sides of a triangle are together greater than the third.

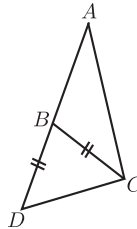


Figure 13.

Proof. Let $\triangle ABC$ be an arbitrary triangle. We choose the point D on AB such that B lies in $[AD]$ and $|BD| = |BC|$ (as in Figure 13). In particular

$$|AD| = |AB| + |BD| = |AB| + |BC|.$$

Since B lies in the angle $\angle ACD$ ¹⁷ we have

$$|\angle BCD| < |\angle ACD|.$$

¹⁷ B lies in a segment whose endpoints are on the arms of $\angle ACD$. Since this angle is $< 180^\circ$ its inside is convex.

Because of $|BD| = |BC|$ and the Theorem about Isosceles Triangles we have $|\angle BCD| = |\angle BDC|$, hence $|\angle ADC| = |\angle BDC| < |\angle ACD|$. By the previous theorem applied to $\triangle ADC$ we have

$$|AC| < |AD| = |AB| + |BC|.$$

□

6.6 Perpendicular Lines

Proposition 1. ¹⁸ *Two lines perpendicular to the same line are parallel to one another.*

Proof. This is a special case of the Alternate Angles Theorem. □

Proposition 2. *There is a unique line perpendicular to a given line and passing through a given point. This applies to a point on or off the line.*

Definition 27. The **perpendicular bisector** of a segment $[AB]$ is the line through the midpoint of $[AB]$, perpendicular to AB .

6.7 Quadrilaterals and Parallelograms

Definition 28. A closed chain of line segments laid end-to-end, not crossing anywhere, and not making a straight angle at any endpoint encloses a piece of the plane called a **polygon**. The segments are called the **sides** or edges of the polygon, and the endpoints where they meet are called its **vertices**. Sides that meet are called **adjacent sides**, and the ends of a side are called **adjacent vertices**. The angles at adjacent vertices are called **adjacent angles**. A polygon is called **convex** if it contains the whole segment connecting any two of its points.

Definition 29. A **quadrilateral** is a polygon with four vertices.

Two sides of a quadrilateral that are not adjacent are called **opposite sides**. Similarly, two angles of a quadrilateral that are not adjacent are called **opposite angles**.

¹⁸In this document, a proposition is a useful or interesting statement that could be proved at this point, but whose proof is not stipulated as an essential part of the programme. Teachers are free to deal with them as they see fit. For instance, they might be just mentioned, or discussed without formal proof, or used to give practice in reasoning for HLC students. It is desirable that they be mentioned, at least.

Definition 30. A **rectangle** is a quadrilateral having right angles at all four vertices.

Definition 31. A **rhombus** is a quadrilateral having all four sides equal.

Definition 32. A **square** is a rectangular rhombus.

Definition 33. A polygon is **equilateral** if all its sides are equal, and **regular** if all its sides and angles are equal.

Definition 34. A **parallelogram** is a quadrilateral for which both pairs of opposite sides are parallel.

Proposition 3. *Each rectangle is a parallelogram.*

Theorem 9. *In a parallelogram, opposite sides are equal, and opposite angles are equal.*

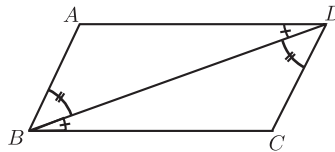


Figure 14.

Proof. See Figure 14. Idea: Use Alternate Angle Theorem, then ASA to show that a diagonal divides the parallelogram into two congruent triangles. This gives opposite sides and (one pair of) opposite angles equal.

In more detail, let $ABCD$ be a given parallelogram, $AB \parallel CD$ and $AD \parallel BC$. Then

$$\begin{aligned} |\angle ABD| &= |\angle BDC| && \text{[Alternate Angle Theorem]} \\ |\angle ADB| &= |\angle DBC| && \text{[Alternate Angle Theorem]} \\ \Delta DAB &\text{ is congruent to } \Delta BCD. && \text{[ASA]} \end{aligned}$$

$$\therefore |AB| = |CD|, |AD| = |CB|, \text{ and } |\angle DAB| = |\angle BCD|.$$

□

Remark 1. Sometimes it happens that the converse of a true statement is false. For example, it is true that if a quadrilateral is a rhombus, then its diagonals are perpendicular. But it is not true that a quadrilateral whose diagonals are perpendicular is always a rhombus.

It may also happen that a statement admits several valid converses. Theorem 9 has two:

Converse 1 to Theorem 9: *If the opposite angles of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. First, one deduces from Theorem 4 that the angle sum in the quadrilateral is 360° . It follows that adjacent angles add to 180° . Theorem 3 then yields the result. \square

Converse 2 to Theorem 9: *If the opposite sides of a convex quadrilateral are equal, then it is a parallelogram.*

Proof. Drawing a diagonal, and using SSS, one sees that opposite angles are equal. \square

Corollary 1. *A diagonal divides a parallelogram into two congruent triangles.*

Remark 2. The converse is false: It may happen that a diagonal divides a convex quadrilateral into two congruent triangles, even though the quadrilateral is not a parallelogram.

Proposition 4. *A quadrilateral in which one pair of opposite sides is equal and parallel, is a parallelogram.*

Proposition 5. *Each rhombus is a parallelogram.*

Theorem 10. *The diagonals of a parallelogram bisect one another.*

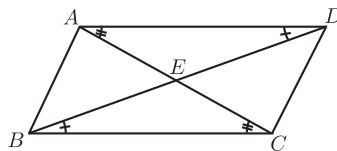


Figure 15.

Proof. See Figure 15. Idea: Use Alternate Angles and ASA to establish congruence of $\triangle ADE$ and $\triangle CBE$.

In detail: Let AC cut BD in E . Then

$$\begin{aligned} |\angle EAD| &= |\angle ECB| \text{ and} \\ |\angle EDA| &= |\angle EBC| && \text{[Alternate Angle Theorem]} \\ |AD| &= |BC|. && \text{[Theorem 9]} \end{aligned}$$

$\therefore \triangle ADE$ is congruent to $\triangle CBE$. [ASA] \square

Proposition 6 (Converse). *If the diagonals of a quadrilateral bisect one another, then the quadrilateral is a parallelogram.*

Proof. Use SAS and Vertically Opposite Angles to establish congruence of $\triangle ABE$ and $\triangle CDE$. Then use Alternate Angles. \square

6.8 Ratios and Similarity

Definition 35. If the three angles of one triangle are equal, respectively, to those of another, then the two triangles are said to be **similar**.

Remark 3. Obviously, two right-angled triangles are similar if they have a common angle other than the right angle.

(The angles sum to 180° , so the third angles must agree as well.)

Theorem 11. *If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.*

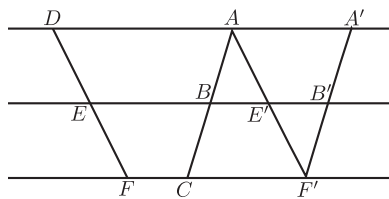


Figure 16.

Proof. Uses opposite sides of a parallelogram, AAS, Axiom of Parallels.

In more detail, suppose $AD \parallel BE \parallel CF$ and $|AB| = |BC|$. We wish to show that $|DE| = |EF|$.

Draw $AE' \parallel DE$, cutting EB at E' and CF at F' .

Draw $F'B' \parallel AB$, cutting EB at B' . See Figure 16.

Then

$$\begin{array}{llll}
 |B'F'| & = & |BC| & \text{[Theorem 9]} \\
 & = & |AB|. & \text{[by Assumption]} \\
 |\angle BAE'| & = & |\angle E'F'B'|. & \text{[Alternate Angle Theorem]} \\
 |\angle AE'B| & = & |\angle F'E'B'|. & \text{[Vertically Opposite Angles]} \\
 \therefore \triangle ABE' & \text{is congruent to} & \triangle F'B'E'. & \text{[ASA]} \\
 \therefore |AE'| & = & |F'E'|. &
 \end{array}$$

But

$$|AE'| = |DE| \text{ and } |F'E'| = |FE|. \quad \text{[Theorem 9]}$$

$$\therefore |DE| = |EF|. \quad \square$$

Definition 36. Let s and t be positive real numbers. We say that a point C divides the segment $[AB]$ in the ratio $s : t$ if C lies on the line AB , and is between A and B , and

$$\frac{|AC|}{|CB|} = \frac{s}{t}.$$

We say that a line l cuts $[AB]$ in the ratio $s : t$ if it meets AB at a point C that divides $[AB]$ in the ratio $s : t$.

Remark 4. It follows from the Ruler Axiom that given two points A and B , and a ratio $s : t$, there is exactly one point that divides the segment $[AB]$ in that exact ratio.

Theorem 12. Let $\triangle ABC$ be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[AC]$ in the same ratio.

Proof. We prove only the commensurable case.

Let l cut $[AB]$ in D in the ratio $m : n$ with natural numbers m, n . Thus there are points (Figure 17)

$$D_0 = A, D_1, D_2, \dots, D_{m-1}, D_m = D, D_{m+1}, \dots, D_{m+n-1}, D_{m+n} = B,$$

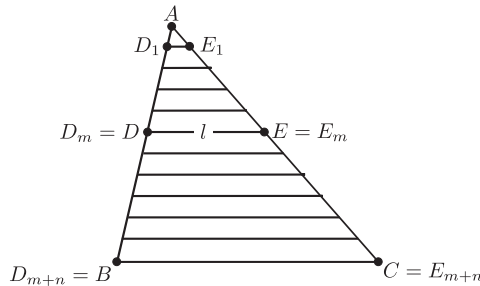


Figure 17.

equally spaced along $[AB]$, i.e. the segments

$$[D_0D_1], [D_1D_2], \dots [D_iD_{i+1}], \dots [D_{m+n-1}D_{m+n}]$$

have equal length.

Draw lines D_1E_1, D_2E_2, \dots parallel to BC with E_1, E_2, \dots on $[AC]$.

Then all the segments

$$[AE_1], [E_1E_2], [E_2E_3], \dots, [E_{m+n-1}C]$$

have the same length,

[Theorem 11]

and $E_m = E$ is the point where l cuts $[AC]$.

[Axiom of Parallels]

Hence E divides $[AC]$ in the ratio $m : n$. \square

Proposition 7. *If two triangles $\triangle ABC$ and $\triangle A'B'C'$ have*

$$|\angle A| = |\angle A'|, \text{ and } \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|},$$

then they are similar.

Proof. Suppose $|A'B'| \leq |AB|$. If equal, use SAS. Otherwise, note that then $|A'B'| < |AB|$ and $|A'C'| < |AC|$. Pick B'' on $[AB]$ and C'' on $[AC]$ with $|A'B'| = |AB''|$ and $|A'C'| = |AC''|$. [Ruler Axiom] Then by SAS, $\triangle A'B'C'$ is congruent to $\triangle AB''C''$.

Draw $[B''D]$ parallel to BC [Axiom of Parallels], and let it cut AC at D . Now the last theorem and the hypothesis tell us that D and C'' divide $[AC]$ in the same ratio, and hence $D = C''$.

Thus

$$\begin{aligned} |\angle B| &= |\angle AB''C''| \text{ [Corresponding Angles]} \\ &= |\angle B'|, \end{aligned}$$

and

$$|\angle C| = |\angle AC''B''| = |\angle C'|,$$

so $\triangle ABC$ is similar to $\triangle A'B'C'$.

[Definition of similar]

□

Remark 5. The **Converse to Theorem 12** is true:

Let $\triangle ABC$ be a triangle. If a line l cuts the sides AB and AC in the same ratio, then it is parallel to BC .

Proof. This is immediate from Proposition 7 and Theorem 5.

□

Theorem 13. If two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar, then their sides are proportional, in order:

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|}.$$

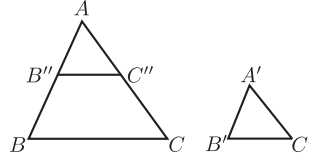


Figure 18.

Proof. We may suppose $|A'B'| \leq |AB|$. Pick B'' on $[AB]$ with $|AB''| = |A'B'|$, and C'' on $[AC]$ with $|AC''| = |A'C'|$. Refer to Figure 18. Then

$$\begin{array}{llll} \triangle AB''C'' & \text{is congruent to} & \triangle A'B'C' & \text{[SAS]} \\ \therefore |\angle AB''C''| & = & |\angle ABC| & \\ \therefore B''C'' & \parallel & BC & \text{[Corresponding Angles]} \\ \therefore \frac{|A'B'|}{|A'C'|} & = & \frac{|AB''|}{|AC''|} & \text{[Choice of } B'', C''] \\ & = & \frac{|AB|}{|AC|} & \text{[Theorem 12]} \\ \frac{|AC|}{|A'C'|} & = & \frac{|AB|}{|A'B'|} & \text{[Re-arrange]} \end{array}$$

Similarly, $\frac{|BC|}{|B'C'|} = \frac{|AB|}{|A'B'|}$

□

Proposition 8 (Converse). *If*

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|CA|}{|C'A'|},$$

then the two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar.

Proof. Refer to Figure 18. If $|A'B'| = |AB|$, then by SSS the two triangles are congruent, and therefore similar. Otherwise, assuming $|A'B'| < |AB|$, choose B'' on AB and C'' on AC with $|AB''| = |A'B'|$ and $|AC''| = |A'C'|$. Then by Proposition 7, $\triangle AB''C''$ is similar to $\triangle ABC$, so

$$|B''C''| = |AB''| \cdot \frac{|BC|}{|AB|} = |A'B'| \cdot \frac{|BC|}{|AB|} = |B'C'|.$$

Thus by SSS, $\triangle A'B'C'$ is congruent to $\triangle AB''C''$, and hence similar to $\triangle ABC$. \square

6.9 Pythagoras

Theorem 14 (Pythagoras). *In a right-angle triangle the square of the hypotenuse is the sum of the squares of the other two sides.*

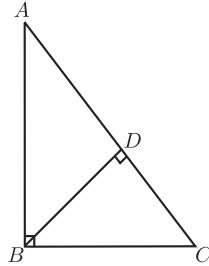


Figure 19.

Proof. Let $\triangle ABC$ have a right angle at B . Draw the perpendicular BD from the vertex B to the hypotenuse AC (shown in Figure 19).

The right-angle triangles $\triangle ABC$ and $\triangle ADB$ have a common angle at A . $\therefore \triangle ABC$ is similar to $\triangle ADB$.

$$\therefore \frac{|AC|}{|AB|} = \frac{|AB|}{|AD|},$$

so

$$|AB|^2 = |AC| \cdot |AD|.$$

Similarly, $\triangle ABC$ is similar to $\triangle BDC$.

$$\therefore \frac{|AC|}{|BC|} = \frac{|BC|}{|DC|},$$

so

$$|BC|^2 = |AC| \cdot |DC|.$$

Thus

$$\begin{aligned} |AB|^2 + |BC|^2 &= |AC| \cdot |AD| + |AC| \cdot |DC| \\ &= |AC| (|AD| + |DC|) \\ &= |AC| \cdot |AC| \\ &= |AC|^2. \end{aligned}$$

□

Theorem 15 (Converse to Pythagoras). *If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.*

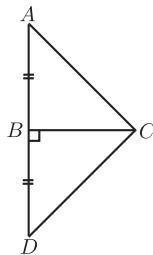


Figure 20.

Proof. (Idea: Construct a second triangle on the other side of $[BC]$, and use Pythagoras and SSS to show it congruent to the original.)

In detail: We wish to show that $|\angle ABC| = 90^\circ$.

Draw $BD \perp BC$ and make $|BD| = |AB|$ (as shown in Figure 20).

Then

$$\begin{aligned}
 |DC| &= \sqrt{|DC|^2} \\
 &= \sqrt{|BD|^2 + |BC|^2} && \text{[Pythagoras]} \\
 &= \sqrt{|AB|^2 + |BC|^2} && [|AB| = |BD|] \\
 &= \sqrt{|AC|^2} && \text{[Hypothesis]} \\
 &= |AC|.
 \end{aligned}$$

$\therefore \triangle ABC$ is congruent to $\triangle DBC$. [SSS]

$\therefore |\angle ABC| = |\angle DBC| = 90^\circ$. □

Proposition 9 (RHS). *If two right angled triangles have hypotenuse and another side equal in length, respectively, then they are congruent.*

Proof. Suppose $\triangle ABC$ and $\triangle A'B'C'$ are right-angle triangles, with the right angles at B and B' , and have hypotenuses of the same length, $|AC| = |A'C'|$, and also have $|AB| = |A'B'|$. Then by using Pythagoras' Theorem, we obtain $|BC| = |B'C'|$, so by SSS, the triangles are congruent. □

Proposition 10. *Each point on the perpendicular bisector of a segment $[AB]$ is equidistant from the ends.*

Proposition 11. *The perpendiculars from a point on an angle bisector to the arms of the angle have equal length.*

6.10 Area

Definition 37. If one side of a triangle is chosen as the base, then the opposite vertex is the **apex** corresponding to that base. The corresponding **height** is the length of the perpendicular from the apex to the base. This perpendicular segment is called an **altitude** of the triangle.

Theorem 16. *For a triangle, base times height does not depend on the choice of base.*

Proof. Let AD and BE be altitudes (shown in Figure 21). Then $\triangle BCE$ and $\triangle ACD$ are right-angled triangles that share the angle C , hence they are similar. Thus

$$\frac{|AD|}{|BE|} = \frac{|AC|}{|BC|}.$$

Re-arrange to yield the result. □

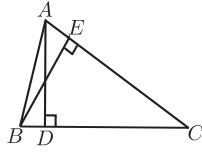


Figure 21.

Definition 38. The **area** of a triangle is half the base by the height.

Notation 5. We denote the area by “area of $\triangle ABC$ ”¹⁹.

Proposition 12. *Congruent triangles have equal areas.*

Remark 6. This is another example of a proposition whose converse is false. It may happen that two triangles have equal area, but are not congruent.

Proposition 13. *If a triangle $\triangle ABC$ is cut into two by a line AD from A to a point D on the segment $[BC]$, then the areas add up properly:*

$$\text{area of } \triangle ABC = \text{area of } \triangle ABD + \text{area of } \triangle ADC.$$

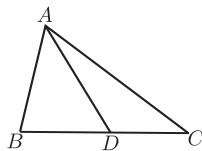


Figure 22.

Proof. See Figure 22. All three triangles have the same height, say h , so it comes down to

$$\frac{|BC| \times h}{2} = \frac{|BD| \times h}{2} + \frac{|DC| \times h}{2},$$

which is obvious, since

$$|BC| = |BD| + |DC|.$$

□

¹⁹ $|\triangle ABC|$ will also be accepted.

If a figure can be cut up into nonoverlapping triangles (i.e. triangles that either don't meet, or meet only along an edge), then its area is taken to be the sum of the area of the triangles²⁰.

If figures of equal areas are added to (or subtracted from) figures of equal areas, then the resulting figures also have equal areas²¹.

Proposition 14. *The area of a rectangle having sides of length a and b is ab .*

Proof. Cut it into two triangles by a diagonal. Each has area $\frac{1}{2}ab$. □

Theorem 17. *A diagonal of a parallelogram bisects the area.*

Proof. A diagonal cuts the parallelogram into two congruent triangles, by Corollary 1. □

Definition 39. Let the side AB of a parallelogram $ABCD$ be chosen as a base (Figure 23). Then the **height** of the parallelogram **corresponding to that base** is the height of the triangle $\triangle ABC$.

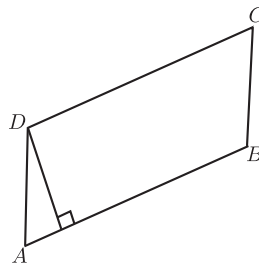


Figure 23.

Proposition 15. *This height is the same as the height of the triangle $\triangle ABD$, and as the length of the perpendicular segment from D onto AB .*

²⁰ If students ask, this does not lead to any ambiguity. In the case of a convex quadrilateral, $ABCD$, one can show that

$$\text{area of } \triangle ABC + \text{area of } \triangle CDA = \text{area of } \triangle ABD + \text{area of } \triangle BCD.$$

In the general case, one proves the result by showing that there is a common refinement of any two given triangulations.

²¹ Follows from the previous footnote.

Theorem 18. *The area of a parallelogram is the base by the height.*

Proof. Let the parallelogram be $ABCD$. The diagonal BD divides it into two triangles, $\triangle ABD$ and $\triangle CDB$. These have equal area, [Theorem 17] and the first triangle shares a base and the corresponding height with the parallelogram. So the areas of the two triangles add to $2 \times \frac{1}{2} \times \text{base} \times \text{height}$, which gives the result. \square

6.11 Circles

Definition 40. A **circle** is the set of points at a given distance (its **radius**) from a fixed point (its **centre**). Each line segment joining the centre to a point of the circle is also called a **radius**. The plural of radius is radii. A **chord** is the segment joining two points of the circle. A **diameter** is a chord through the centre. All diameters have length twice the radius. This number is also called **the diameter** of the circle.

Two points A, B on a circle cut it into two pieces, called **arcs**. You can specify an arc uniquely by giving its endpoints A and B , and one other point C that lies on it. A **sector** of a circle is the piece of the plane enclosed by an arc and the two radii to its endpoints.

The length of the whole circle is called its **circumference**. For every circle, the circumference divided by the diameter is the same. This ratio is called π .

A **semicircle** is an arc of a circle whose ends are the ends of a diameter.

Each circle divides the plane into two pieces, the inside and the outside. The piece inside is called a **disc**.

If B and C are the ends of an arc of a circle, and A is another point, not on the arc, then we say that the angle $\angle BAC$ is the angle at A **standing on the arc**. We also say that it **stands on the chord** $[BC]$.

Theorem 19. *The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.*

Proof. There are several cases for the diagram. It will be sufficient for students to examine one of these. The idea, in all cases, is to draw the line through the centre and the point on the circumference, and use the Isosceles Triangle Theorem, and then the Protractor Axiom (to add or subtract angles, as the case may be).

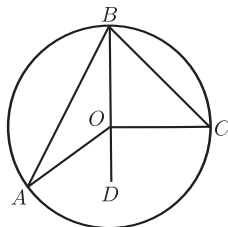


Figure 24.

In detail, for the given figure, Figure 24, we wish to show that $|\angle AOC| = 2|\angle ABC|$.

Join B to O and continue the line to D . Then

$$\begin{aligned}
 |OA| &= |OB|. && \text{[Definition of circle]} \\
 \therefore |\angle BAO| &= |\angle ABO|. && \text{[Isosceles triangle]} \\
 \therefore |\angle AOD| &= |\angle BAO| + |\angle ABO| && \text{[Exterior Angle]} \\
 &= 2 \cdot |\angle ABO|.
 \end{aligned}$$

Similarly,

$$|\angle COD| = 2 \cdot |\angle CBO|.$$

Thus

$$\begin{aligned}
 |\angle AOC| &= |\angle AOD| + |\angle COD| \\
 &= 2 \cdot |\angle ABO| + 2 \cdot |\angle CBO| \\
 &= 2 \cdot |\angle ABC|.
 \end{aligned}$$

□

Corollary 2. *All angles at points of the circle, standing on the same arc, are equal. In symbols, if A, A', B and C lie on a circle, and both A and A' are on the same side of the line BC , then $\angle BAC = \angle BA'C$.*

Proof. Each is half the angle subtended at the centre. □

Remark 7. The converse is true, but one has to be careful about sides of BC :

Converse to Corollary 2: *If points A and A' lie on the same side of the line BC , and if $|\angle BAC| = |\angle BA'C|$, then the four points A, A', B and C lie on a circle.*

Proof. Consider the circle s through A, B and C . If A' lies outside the circle, then take A'' to be the point where the segment $[A'B]$ meets s . We then have

$$|\angle BA'C| = |\angle BAC| = |\angle BA''C|,$$

by Corollary 2. This contradicts Theorem 6.

A similar contradiction arises if A' lies inside the circle. So it lies on the circle. \square

Corollary 3. *Each angle in a semicircle is a right angle. In symbols, if BC is a diameter of a circle, and A is any other point of the circle, then $\angle BAC = 90^\circ$.*

Proof. The angle at the centre is a straight angle, measuring 180° , and half of that is 90° . \square

Corollary 4. *If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.*

Proof. The angle at the centre is 180° , so is straight, and so the line BC passes through the centre. \square

Definition 41. A **cyclic** quadrilateral is one whose vertices lie on some circle.

Corollary 5. *If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to 180° .*

Proof. The two angles at the centre standing on the same arcs add to 360° , so the two halves add to 180° . \square

Remark 8. The converse also holds: *If $ABCD$ is a convex quadrilateral, and opposite angles sum to 180° , then it is cyclic.*

Proof. This follows directly from Corollary 5 and the converse to Corollary 2. \square

It is possible to approximate a disc by larger and smaller equilateral polygons, whose area is as close as you like to πr^2 , where r is its radius. For this reason, we say that the area of the disc is πr^2 .

Proposition 16. *If l is a line and s a circle, then l meets s in zero, one, or two points.*

Proof. We classify by comparing the length p of the perpendicular from the centre to the line, and the radius r of the circle. If $p > r$, there are no points. If $p = r$, there is exactly one, and if $p < r$ there are two. \square

Definition 42. The line l is called a **tangent** to the circle s when $l \cap s$ has exactly one point. The point is called the **point of contact** of the tangent.

Theorem 20.

(1) Each tangent is perpendicular to the radius that goes to the point of contact.

(2) If P lies on the circle s , and a line l through P is perpendicular to the radius to P , then l is tangent to s .

Proof. (1) This proof is a proof by contradiction.

Suppose the point of contact is P and the tangent l is not perpendicular to OP .

Let the perpendicular to the tangent from the centre O meet it at Q . Pick R on PQ , on the other side of Q from P , with $|QR| = |PQ|$ (as in Figure 25).

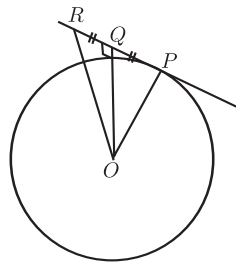


Figure 25.

Then $\triangle OQR$ is congruent to $\triangle OQP$. [SAS]

$$\therefore |OR| = |OP|,$$

so R is a second point where l meets the circle. This contradicts the given fact that l is a tangent.

Thus l must be perpendicular to OP , as required.

(2) (Idea: Use Pythagoras. This shows directly that each other point on l is further from O than P , and hence is not on the circle.)

In detail: Let Q be any point on l , other than P . See Figure 26. Then

$$\begin{aligned} |OQ|^2 &= |OP|^2 + |PQ|^2 && \text{[Pythagoras]} \\ &> |OP|^2. \\ \therefore |OQ| &> |OP|. \end{aligned}$$

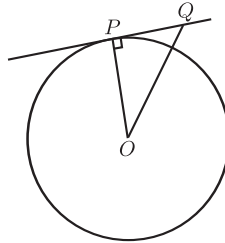


Figure 26.

$\therefore Q$ is not on the circle. [Definition of circle]
 $\therefore P$ is the only point of l on the circle.
 $\therefore l$ is a tangent. [Definition of tangent]

□

Corollary 6. *If two circles share a common tangent line at one point, then the two centres and that point are collinear.*

Proof. By part (1) of the theorem, both centres lie on the line passing through the point and perpendicular to the common tangent. □

The circles described in Corollary 6 are shown in Figure 27.

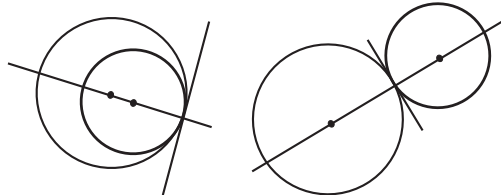


Figure 27.

Remark 9. Any two distinct circles will intersect in 0, 1, or 2 points.

If they have two points in common, then the common chord joining those two points is perpendicular to the line joining the centres.

If they have just one point of intersection, then they are said to be *touching* and this point is referred to as their *point of contact*. The centres and the point of contact are collinear, and the circles have a common tangent at that point.

Theorem 21.

- (1) *The perpendicular from the centre to a chord bisects the chord.*
- (2) *The perpendicular bisector of a chord passes through the centre.*

Proof. (1) (Idea: Two right-angled triangles with two pairs of sides equal.)
See Figure 28.

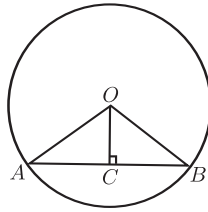


Figure 28.

In detail:

$$\begin{aligned} |OA| &= |OB| && \text{[Definition of circle]} \\ |OC| &= |OC| \end{aligned}$$

$$\begin{aligned} |AC| &= \sqrt{|OA|^2 - |OC|^2} && \text{[Pythagoras]} \\ &= \sqrt{|OB|^2 - |OC|^2} \\ &= |CB|. && \text{[Pythagoras]} \end{aligned}$$

$\therefore \triangle OAC$ is congruent to $\triangle OBC$. [SSS]

$\therefore |AC| = |CB|$.

(2) This uses the Ruler Axiom, which has the consequence that a segment has exactly one midpoint.

Let C be the foot of the perpendicular from O on AB .

By Part (1), $|AC| = |CB|$, so C is the midpoint of $[AB]$.

Thus CO is the perpendicular bisector of AB .

Hence the perpendicular bisector of AB passes through O . □

6.12 Special Triangle Points

Proposition 17. *If a circle passes through three non-collinear points A , B , and C , then its centre lies on the perpendicular bisector of each side of the triangle $\triangle ABC$.*

Definition 43. The **circumcircle** of a triangle $\triangle ABC$ is the circle that passes through its vertices (see Figure 29). Its centre is the **circumcentre** of the triangle, and its radius is the **circumradius**.

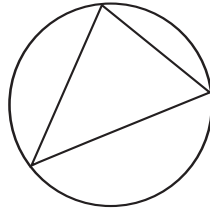


Figure 29.

Proposition 18. *If a circle lies inside the triangle $\triangle ABC$ and is tangent to each of its sides, then its centre lies on the bisector of each of the angles $\angle A$, $\angle B$, and $\angle C$.*

Definition 44. The **incircle** of a triangle is the circle that lies inside the triangle and is tangent to each side (see Figure 30). Its centre is the **incentre**, and its radius is the **inradius**.

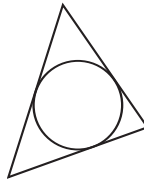


Figure 30.

Proposition 19. *The lines joining the vertices of a triangle to the centre of the opposite sides meet in one point.*

Definition 45. A line joining a vertex of a triangle to the midpoint of the opposite side is called a **median** of the triangle. The point where the three medians meet is called the **centroid**.

Proposition 20. *The perpendiculars from the vertices of a triangle to the opposite sides meet in one point.*

Definition 46. The point where the perpendiculars from the vertices to the opposite sides meet is called the **orthocentre** (see Figure 31).

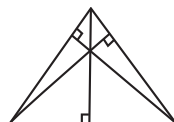


Figure 31.

7 Constructions to Study

The instruments that may be used are:

straight-edge: This may be used (together with a pencil) to draw a straight line passing through two marked points.

compass: This instrument allows you to draw a circle with a given centre, passing through a given point. It also allows you to take a given segment $[AB]$, and draw a circle centred at a given point C having radius $|AB|$.

ruler: This is a straight-edge marked with numbers. It allows you measure the length of segments, and to mark a point B on a given ray with vertex A , such that the length $|AB|$ is a given positive number. It can also be employed by sliding it along a set square, or by other methods of sliding, while keeping one or two points on one or two curves.

protractor: This allows you to measure angles, and mark points C such that the angle $\angle BAC$ made with a given ray $[AB]$ has a given number of degrees. It can also be employed by sliding it along a line until some line on the protractor lies over a given point.

set-squares: You may use these to draw right angles, and angles of 30° , 60° , and 45° . It can also be used by sliding it along a ruler until some coincidence occurs.

The prescribed constructions are:

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line l , passing through a given point not on l .

4. Line perpendicular to a given line l , passing through a given point on l .
5. Line parallel to given line, through given point.
6. Division of a segment into 2, 3 equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of 60° , without using a protractor or set square.
19. Tangent to a given circle at a given point on it.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.

8 Teaching Approaches

8.1 Practical Work

Practical exercises and experiments should be undertaken before the study of theory. These should include:

1. Lessons along the lines suggested in the Guidelines for Teachers [2]. We refer especially to Section 4.6 (7 lessons on Applied Arithmetic and Measure), Section 4.9 (14 lessons on Geometry), and Section 4.10 (4 lessons on Trigonometry).
2. Ideas from Technical Drawing.
3. Material in [3].

8.2 From Discovery to Proof

It is intended that all of the geometrical results on the course would first be encountered by students through investigation and discovery. As a result of various activities undertaken, students should come to appreciate that certain features of certain shapes or diagrams appear to be independent of the particular examples chosen. These apparently constant features therefore seem to be general results that we have reason to believe might always be true. At this stage in the work, we ask students to accept them as true for the purpose of applying them to various contextualised and abstract problems, but we also agree to come back later to revisit this question of their truth. Nonetheless, even at this stage, students should be asked to consider whether investigating a number of examples in this way is sufficient to be convinced that a particular result always holds, or whether a more convincing argument is required. Is a person who refuses to believe that the asserted result will always be true being unreasonable? An investigation of a statement that appears at first to be always true, but in fact is not, may be helpful, (e.g. the assertion that $n^2 + n + 41$ is prime for all $n \in \mathbb{N}$). Reference might be made to other examples of conjectures that were historically believed to be true until counterexamples were found.

Informally, the ideas involved in a mathematical proof can be developed even at this investigative stage. When students engage in activities that lead to closely related results, they may readily come to appreciate the manner

in which these results are connected to each other. That is, they may see for themselves or be led to see that the result they discovered today is an inevitable logical consequence of the one they discovered yesterday. Also, it should be noted that working on problems or “cuts” involves logical deduction from general results.

Later, students at the relevant levels need to proceed beyond accepting a result on the basis of examples towards the idea of a more convincing logical argument. Informal justifications, such as a dissection-based proof of Pythagoras’ theorem, have a role to play here. Such justifications develop an argument more strongly than a set of examples. It is worth discussing what the word “prove” means in various contexts, such as in a criminal trial, or in a civil court, or in everyday language. What mathematicians regard as a “proof” is quite different from these other contexts. The logic involved in the various steps must be unassailable. One might present one or more of the readily available dissection-based “proofs” of fallacies and then probe a dissection-based proof of Pythagoras’ theorem to see what possible gaps might need to be bridged.

As these concepts of argument and proof are developed, students should be led to appreciate the need to formalise our idea of a mathematical proof to lay out the ground rules that we can all agree on. Since a formal proof only allows us to progress logically from existing results to new ones, the need for axioms is readily identified, and the students can be introduced to formal proofs.

9 Syllabus for JCOL

9.1 Concepts

Set, plane, point, line, ray, angle, real number, length, degree, triangle, right-angle, congruent triangles, similar triangles, parallel lines, parallelogram, area, tangent to a circle, subset, segment, collinear points, distance, midpoint of a segment, reflex angle, ordinary angle, straight angle, null angle, full angle, supplementary angles, vertically-opposite angles, acute angle, obtuse angle, angle bisector, perpendicular lines, perpendicular bisector of a segment, ratio, isosceles triangle, equilateral triangle, scalene triangle, right-angled triangle, exterior angles of a triangle, interior opposite angles, hypotenuse, alternate angles, corresponding angles, polygon, quadrilateral, convex quadrilateral,

rectangle, square, rhombus, base and corresponding apex and height of triangle or parallelogram, transversal line, circle, radius, diameter, chord, arc, sector, circumference of a circle, disc, area of a disc, circumcircle, point of contact of a tangent, vertex, vertices (of angle, triangle, polygon), endpoints of segment, arms of an angle, equal segments, equal angles, adjacent sides, angles, or vertices of triangles or quadrilaterals, the side opposite an angle of a triangle, opposite sides or angles of a quadrilateral, centre of a circle.

9.2 Constructions

Students will study constructions 1, 2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15.

9.3 Axioms and Proofs

The students should be exposed to some formal proofs. They will not be examined on these. They will see Axioms 1,2,3,4,5, and study the proofs of Theorems 1, 2, 3, 4, 5, 6, 9, 10, 13 (statement only), 14, 15; and direct proofs of Corollaries 3, 4.

10 Syllabus for JCHL

10.1 Concepts

Those for JCOL, and concurrent lines.

10.2 Constructions

Students will study all the constructions prescribed for JC-OL, and also constructions 3 and 7.

10.3 Logic, Axioms and Theorems

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

They will study Axioms 1, 2, 3, 4, 5. They will study the proofs of Theorems 1, 2, 3, 4*, 5, 6*, 9*, 10, 11, 12, 13, 14*, 15, 19*, Corollaries 1,

2, 3, 4, 5, and their converses. Those marked with a * may be asked in examination.

The formal material on area will not be studied at this level. Students will deal with area only as part of the material on arithmetic and mensuration.

11 Syllabus for LCFL

Students are expected to build on their mathematical experiences to date.

11.1 Constructions

Students revisit constructions 4, 5, 10, 13, 15, and learn how to apply these in real-life contexts.

12 Syllabus for LCOL

12.1 Constructions

A knowledge of the constructions prescribed for JC-OL will be assumed, and may be examined. In addition, students will study constructions 16–21.

12.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-OL will be assumed.

Students will study proofs of Theorems 7, 8, 11, 12, 13, 16, 17, 18, 20, 21, and Corollary 6.

No proofs are examinable. Students will be examined using problems that can be attacked using the theory.

13 Syllabus for LCHL

13.1 Constructions

A knowledge of the constructions prescribed for JC-HL will be assumed, and may be examined. In addition, students will study the constructions prescribed for LC-OL, and construction 22.

13.2 Theorems and Proofs

Students will be expected to understand the meaning of the following terms related to logic and deductive reasoning: **Theorem, proof, axiom, corollary, converse, implies, is equivalent to, if and only if, proof by contradiction.**

A knowledge of the Axioms, concepts, Theorems and Corollaries prescribed for JC-HL will be assumed.

Students will study all the theorems and corollaries prescribed for LC-OL, but will not, in general, be asked to reproduce their proofs in examination.

However, they may be asked to give proofs of the Theorems 11, 12, 13, concerning ratios, which lay the proper foundation for the proof of Pythagoras studied at JC, and for trigonometry.

They will be asked to solve geometrical problems (so-called “cuts”) and write reasoned accounts of the solutions. These problems will be such that they can be attacked using the given theory. The study of the propositions may be a useful way to prepare for such examination questions.

References

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- [2] Junior Cycle Course Committee, NCCA. *Mathematics: Junior Certificate Guidelines for Teachers*. Stationary Office, Dublin. 2002. ISBN 0-7557-1193-9.
- [3] Fiacre O’Cairbre, John McKeon, and Richard O. Watson. *A Resource for Transition Year Mathematics Teachers*. DES. Dublin. 2006.

- [4] Anthony G. O'Farrell. *School Geometry*. IMTA Newsletter 109 (2009) 21-28.