Mathematics for Control Theory

Geometric Concepts in Control
Submanifolds of $\mathbb{R}^n$, vector fields and differential equations on manifolds

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Reading materials

Reference:

- Francesco Bullo and Andrew Lewis [2005], *Geometric Control of Mechanical Systems: Modeling, Analysis and Design for Simple Mechanical Control Systems*, Springer, ISBN: 0-387-22195-6 (chapter 3, Sects. 3.1-3.5)
- Vladimir Igorevich Arnold 1991, [1974], *Mathematical Methods of Classical Mechanics*, Springer-Verlag, ISBN: 0-387-96890-3 (chapter 4, section on manifolds)
- Ralph Abraham, Jerrold E. Marsden and Tudor Ratiu, 1991, [1983], *Manifolds, Tensor Analysis and Applications*, Springer, ISBN 978-1-4612-6990-8 (chapters 3 and 4 for serious study).

Motivation

Many control problems can be stated as follows: given the system with state space $\mathbb{R}^n$:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= g(x) \end{aligned}$$

we seek to find a control function $u(x)$ such that $y(t)$ is regulated to zero.

Differentiating y gives

$$\dot{y} = \frac{\partial g}{\partial x} f(x,u)$$

In principle, we can solve for $u = u(x)$ from here to attain the intended objective. Substituting $u(x)$ into the system equations gives

$$\begin{aligned} \dot{x} &= f(x,u(x)) = f_u(x) \\ g(x) &= 0 \end{aligned}$$

Now we have a differential equation whose states evolve in something more specific than $\mathbb{R}^n$. They are constrained to the surface ("manifold") described by $g(x) = 0$.

Example: simple pendulum

The dynamics of a simple pendulum are given by

$$ml^2\ddot{x}_1 + mgl\sin(x_1) = 0$$

Using angle x_1 and angular velocity x_2 as state variables:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{g}{l}\sin(x_1)\end{aligned}$$

We usually define the state as belonging to $\mathbb{R}^2$, but we know that the motion of the pendulum is restricted to circular trajectories. We can reveal this by involving the Cartesian coordinates of the mass, $x = l\cos x_1$ and $y = -l\sin(x_1)$ and their velocities v_x and v_y.

Example: simple pendulum

We can easily describe the geometric locus of the positions by $x^2 + y^2 = l^2$. The velocities satisfy: $v_x = yx_2$ and $v_y = -xx_2$. We can then write $xv_x + yv_y = 0$. Therefore, the pendulum evolves in a higher-dimensional manifold described by

$$\mathcal{M} = \{(x, y, v_x, v_y) \ : \ x^2 + y^2 = l^2, \ \ xv_x + yv_y = 0\}$$

The manifold is like $g(x)$ in our previous example, it only gives the "shape" of the place where the system evolves. A separate time evolution equation can parameterize actual trajectories for various initial conditions.

The manifold contains key information about system dynamics. For instance, the ratio of Cartesian positions is the negative reciprocal of the ratio of negative Cartesian velocities. Have you thought of pendulum motion like that before?

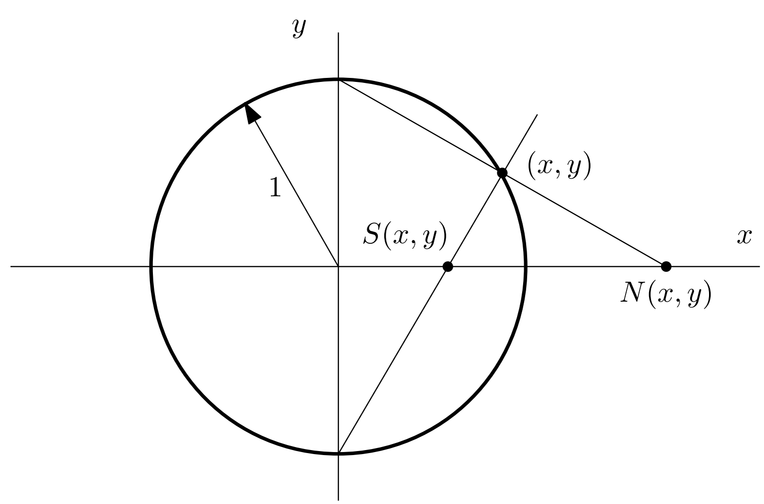
Thus, can have a dynamic description on $\mathbb{R}^2$ (based on the angle and angular velocity), or we can have a description on the manifold (based on two Cartesian positions and two Cartesian velocities).

Intuitive idea of manifolds

A manifold is a set whose points can be mapped to a subset of Euclidean space via a set of continuous functions with continuous inverses (local charts). These charts cover overlapping portions of the manifold.

Example: the unit circle $\mathcal{S} = \{(x, y) \ : \ x^2 + y^2 = 1\}$ can be mapped to a subset of $\mathbb{R}$ with two charts:

- Chart "North" is defined on the arc $\mathcal{S} - \{(0, 1)\}$ by *stereographic projection*.
- Chart "South" is defined on the arc $\mathcal{S} - \{(0, -1)\}$ by stereographic projection.



Unit circle as a manifold...

We can't cover the whole circle using a single chart (this can be proven using topology). Our two charts have overlapping domains, and are defined by:

$$
\begin{aligned}
N(x,y) &= \frac{x}{1-y}, \quad (x,y) \in \mathcal{S}, y \neq 1 \\
S(x,y) &= \frac{x}{1+y}, \quad (x,y) \in \mathcal{S}, y \neq -1
\end{aligned}
$$

The ranges of $N(x,y)$ and $S(x,y)$ are both equal to $\mathbb{R}$. It can be easily verified that each function is continuous and invertible in its domain (homeomorphism).

Transition maps

If we start with one point on Euclidean space, we don't know where it came from on the manifold without selecting a chart. If we select one chart, we can uniquely find a point on the manifold. Then we can map forward with another chart, to find a point on Euclidean space. This composite mapping is called transition map.

The transition map coincides with our idea of *coordinate transformation*. For example, coordinates $(x, y, z) \in \mathbb{R}^3$ can be changed to and from coordinates $(\rho, \theta, \phi) \in \mathbb{R}^3$ using the spherical transformation.

Charts and Atlases

A *diffeomorphism* is a continuously differentiable mapping between two open sets having a continuously differentiable inverse. A C^k diffeomorphism can be differentiated k times.

A *chart* on a set S is a bijection ϕ from $U \in S$ to an open subset of $\mathbb{R}^n$. A C^k atlas is a collection of charts on S such that:

- The union of the charts equals S
- Any two charts $(U_i, \phi_i), (U_j, \phi_j)$ with $U_i \cap U_j \neq \varnothing$ are *compatible*: If $\phi_i(U_i \cap U_j)$ is open, consider the restriction of the transition mapping $\phi_{ji} = \phi_j \circ \phi_i^{-1}$ to $\phi_i(U_i \cap U_j)$. This restricted mapping must be a C^k diffeomorphism.

Differentiable Manifolds

Define an equivalence relation on all C^k atlases on a set S by: $\mathcal{A}_1 \, \rho \, \mathcal{A}_1$ if $\mathcal{A}_1 \cup \mathcal{A}_1$ is still a C^k atlas. It can be shown that ρ is indeed an equivalence relation.

As we saw before, ρ partitions the set of all C^k atlases into equivalence classes. We call such classes C^k *differentiable structures* on S.

A differentiable manifold $\mathcal{M}$ is a pair $(S, \mathcal{D})$, where S is a set and $\mathcal{D}$ is a differentiable structure on S.

For convenience, we identify $\mathcal{M}$ with S in the notation.

We note that $\mathbb{R}^n$ is a trivial example of a manifold, with a single chart that works in the whole set: the identity mapping.

A manifold has dimension n if every chart takes values in an n-dimensional linear space.

Submanifolds in $\mathbb{R}^n$

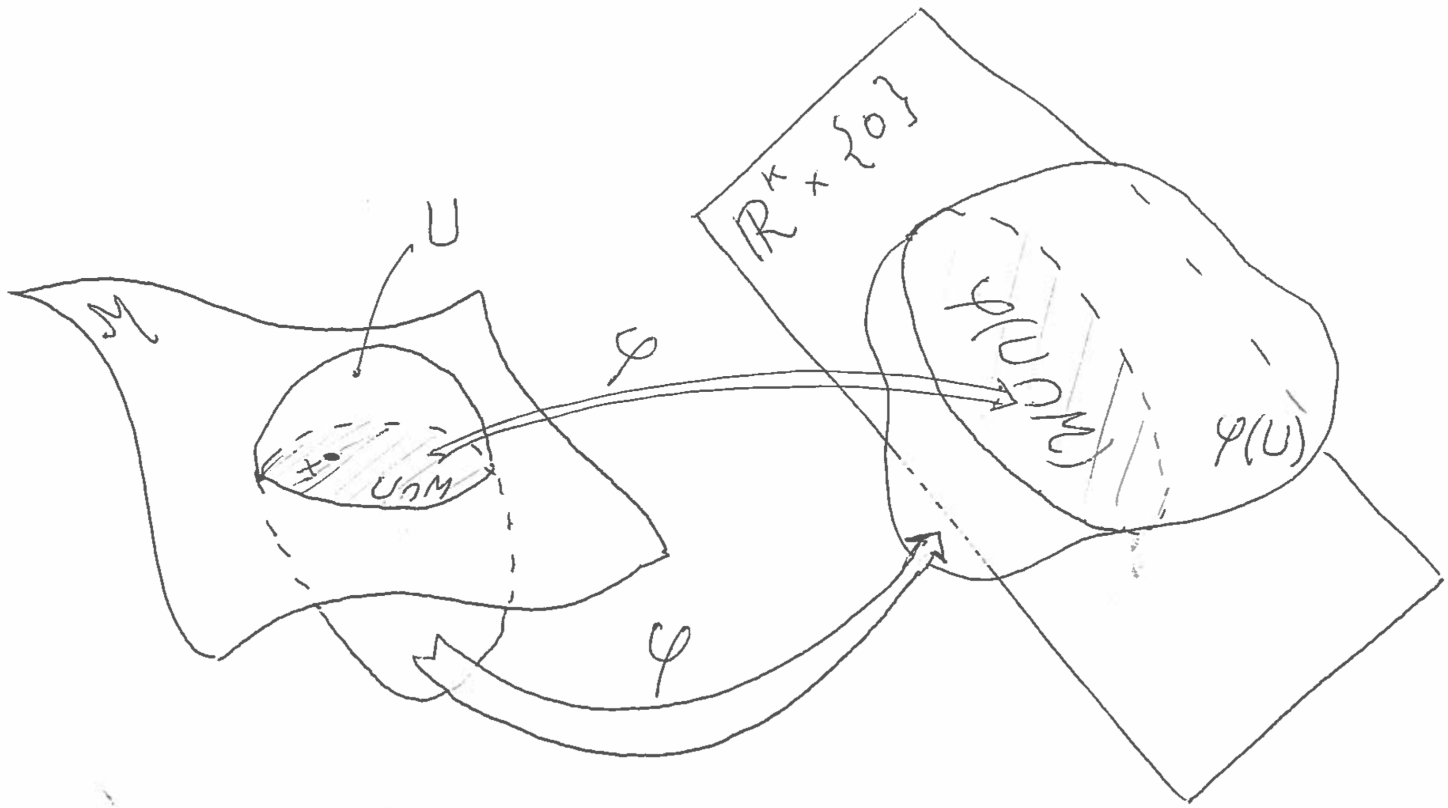
Submanifolds are a generalization of the idea of linear subspaces.

A set $\mathcal{M} \in \mathbb{R}^n$ is a (differentiable) submanifold if for all $a \in \mathcal{M}$, there are open sets $U, V \in \mathbb{R}^n$ with $a \in U$ and a diffeomorphism $\phi : U \mapsto V$ such that

$$\phi(U \cap \mathcal{M}) = \phi(U) \cap \left(\mathbb{R}^k \times \{0\}\right)$$

The dimension of $\mathcal{M}$ is k.

The submanifold property



$$\varphi(U \cap M) = \varphi(U) \cap \{\mathbb{R}^k \times \{0\}\}$$

Example

The unit circle $\mathcal{M} = \{(x, y) \ : \ x^2 + y^2 = 1\}$ is a 1-dimensional submanifold of $\mathbb{R}^2$. As an example, we take the point $(\sqrt{2}/2, \sqrt{2}/2) = a \in \mathcal{M}$ and use the polar-Cartesian coordinate transformation as the diffeomorphism ϕ.

Let U be an open rectangle in $\mathbb{R}^n$ defined by $U = (0.25, 1.5) \times (0.25, 1.5)$ which contains a. The mapping

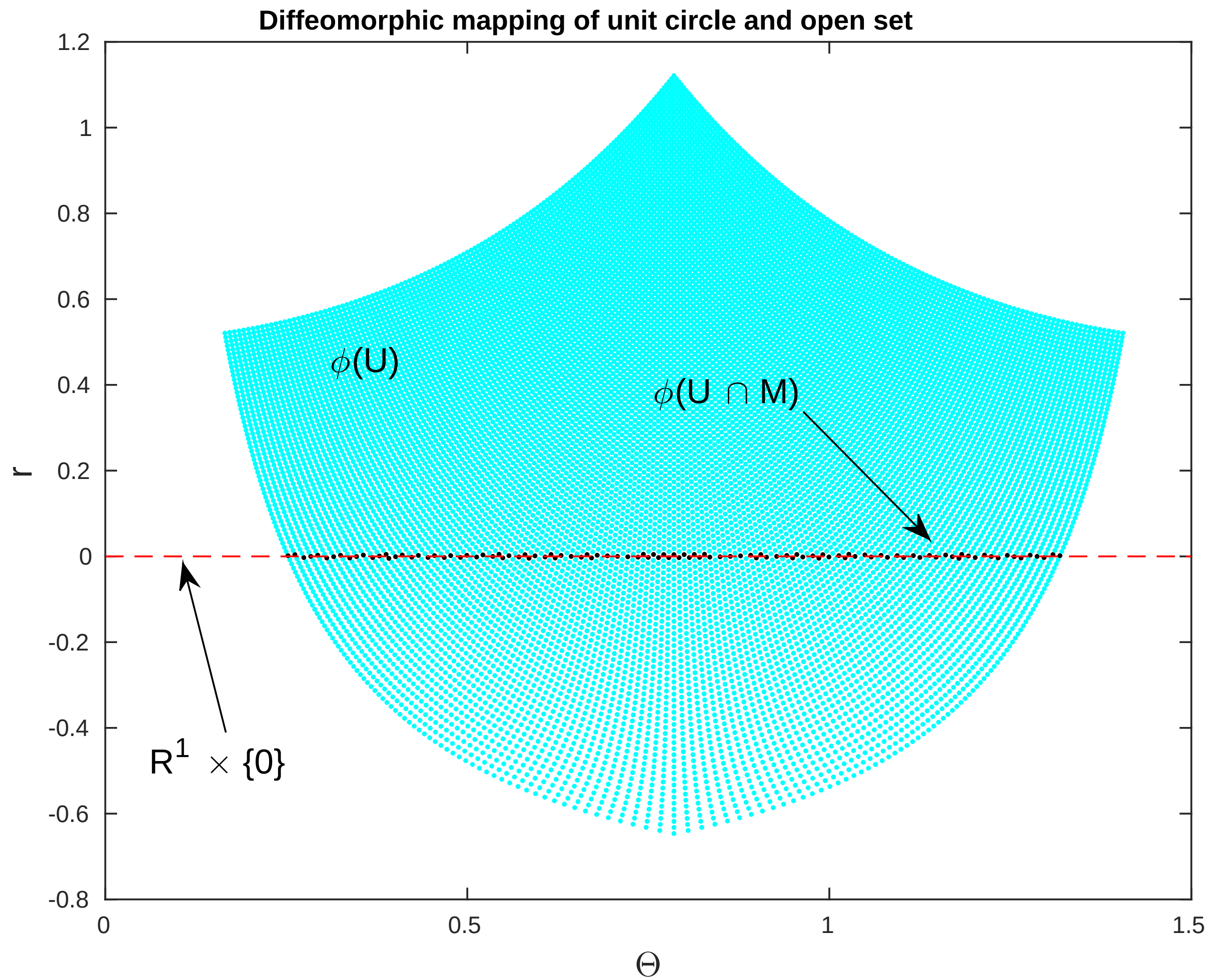
$$\phi(x, y) \mapsto (\tan^{-1}(y/x), \sqrt{x^2 + y^2} - 1)$$

is a diffeomorphism in U mapping it to some region $V \in \mathbb{R}^2$. Its inverse is

$$\phi^{-1}(\theta, r) \mapsto ((r + 1)\cos(\theta), (r + 1)\sin(\theta))$$

Example...

In this case, the set $\mathbb{R}^1 \times \{0\}$ is $\{(\theta, r) \in \mathbb{R}^2 \ : \ r = 0\}$, the horizontal axis. As the figure shows, if $V = \phi(U)$ is intersected with $\mathbb{R}^1 \times \{0\}$, the result is the same as the mapping only $U \cap \mathcal{M}$ through ϕ.



Example...

This code explains the figure:

```
for x=[0.25:0.01:1.5];
for y=[0.25:0.01:1.5];
th=atan(y/x);
r=sqrt(x^2+y^2)-1;
if abs(r)<0.005,
color='.k'; %to plot φ(U ∩ M)
else
color='.c'; %to plot φ(U)
 end
plot(th,r,color);hold on
end
```

The Tangent Space of a Submanifold of $\mathbb{R}^n$

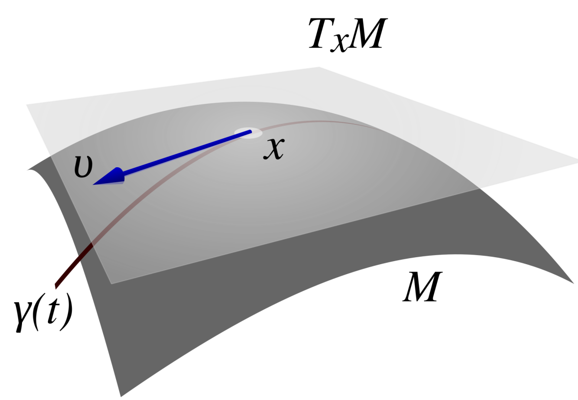
Informally, the tangent space of a submanifold $\mathcal{M}$ at a point a is the set of all vectors $v \in \mathbb{R}^n$ which are tangent to a some continuously differentiable curve contained in $\mathcal{M}$ passing through a.

Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^n$ and let $a \in \mathcal{M}$. The tangent space of $\mathcal{M}$ at a is given by

$$T_a\mathcal{M} = \{v \in \mathbb{R}^n \ : \ \exists \gamma : (-s, s) \mapsto \mathbb{R}^n \ :$$

$$\gamma(t) \in \mathcal{M} \text{ for some } s > 0 \text{ and } t \in (-s, s), \ \gamma(0) = a, \ \dot{\gamma}(0) = v\}$$

with $\gamma \in \mathcal{C}^\infty$
A tangent vector can be identified with $\dot{\gamma}(0)$.



https://en.wikipedia.org/wiki/Tangent_space

Tangent Space as a Linear Space

Vectors in $T_a\mathcal{M}$ form a linear space. To define addition and scalar multiplication, we select an arbitrary chart $\phi : U \mapsto \mathbb{R}^n$ and carry tangent vectors to $\mathbb{R}^n$ with a mapping $(d\phi)_a : T_a\mathcal{M} \mapsto \mathbb{R}^n$ defined by

$$(d\phi)_a(\dot{\gamma}(0)) = \frac{d}{dt}(\phi \circ \gamma)(0)$$

1. The images of $d\phi_a$ can be operated upon using the standard addition and multiplication in $\mathbb{R}^n$.
2. Because $T_a\mathcal{M}$ is a linear space, the linear combination of tangent vectors is some tangent vector w, so there must be a curve δ with the properties required by the definition such that $(\dot{\delta})(0) = w$.

Finding the Tangent Space

If $\mathcal{M}$ can be described by the equation $g(x) = 0$ in some neighborhood of a, then

$$T_a\mathcal{M} = \text{ null } g'(a) = \{v \in \mathbb{R}^n \ : \ g'(a)v = 0\}$$

Example: Find the tangent space of the unit sphere centered at the origin at $(1, 0, 0)$.

The sphere is described everywhere by $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. The derivative of g (gradient) is

$$g' = (2x, 2y, 2z)$$

We need to find the set of $v \in \mathbb{R}^3$ such that $(2, 0, 0).v = 0$. This is the set of all vectors orthogonal to $(1, 0, 0)$, which is just

$$T_a\mathcal{M} = \text{ span } \{(0, 1, 0), (0, 0, 1)\}$$

Vector Fields and Differential Equations on Manifolds

Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^n$. A vector field on $\mathcal{M}$ is a $\mathcal{C}^1$ mapping $f : \mathcal{M} \mapsto \mathbb{R}^n$ such that $f(x) \in T_x\mathcal{M}$ for all $x \in \mathcal{M}$.

A differential equation on $\mathcal{M}$ is written as

$$\dot{x} = f(x)$$

A solution $x(t)$ is a curve defined on an interval $\mathcal{I}$ (time) such that $\dot{x}(t) = f(x(t))$ for all $t \in \mathcal{I}$.

Example: Back to the pendulum equation, the dynamic evolution of Cartesian positions and velocities can be described by:

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= h_x(x, y, v_x, v_y) \\ \dot{v}_y &= h_y(x, y, v_x, v_y) \end{aligned}$$

Exercise for you to find h_x and h_y

Given an initial condition for 2 positions and 2 velocities *on the manifold*, a solution curve lying on the manifold is obtained. Of course, there are smaller representations for the same dynamics. In some cases, reduction to a smaller representation is not so easy.