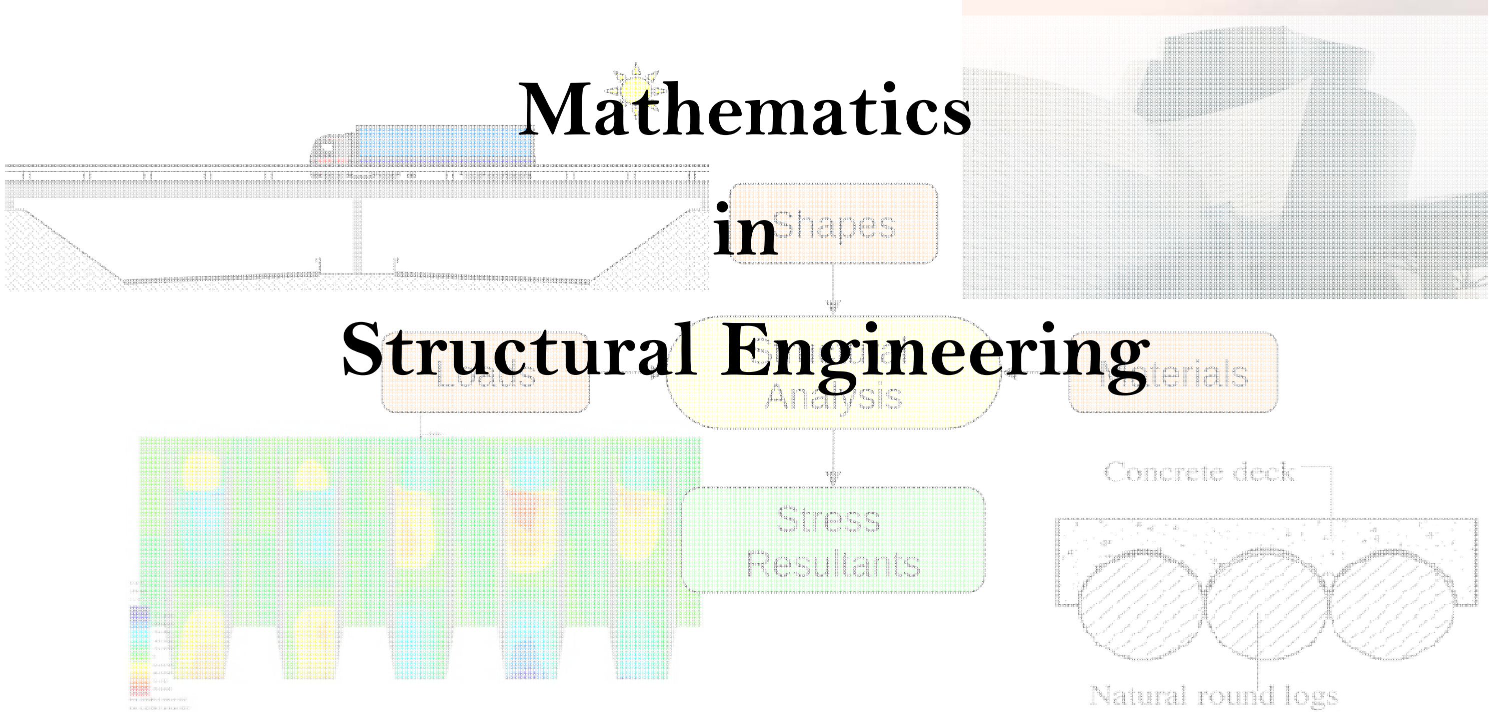
Dr Colin Caprani

PhD, BSc(Eng), DipEng, CEng, MIEI, MIABSE, MIStructE



Mathematics
in
Structural Engineering

Coláiste Cois Life – 5th and 6th Year

About Me

- Degree in Structural Engineering 1999
- Full time consultancy until 2001
- PhD in UCD from 2001 to 2006
- Lecturing in DIT and UCD
- Consultant in buildings & bridges

Guess my Leaving result!

C1 in Honours Maths

You don’t have to be a genius…

Definition of Structural Engineering

Institution of Structural Engineers:

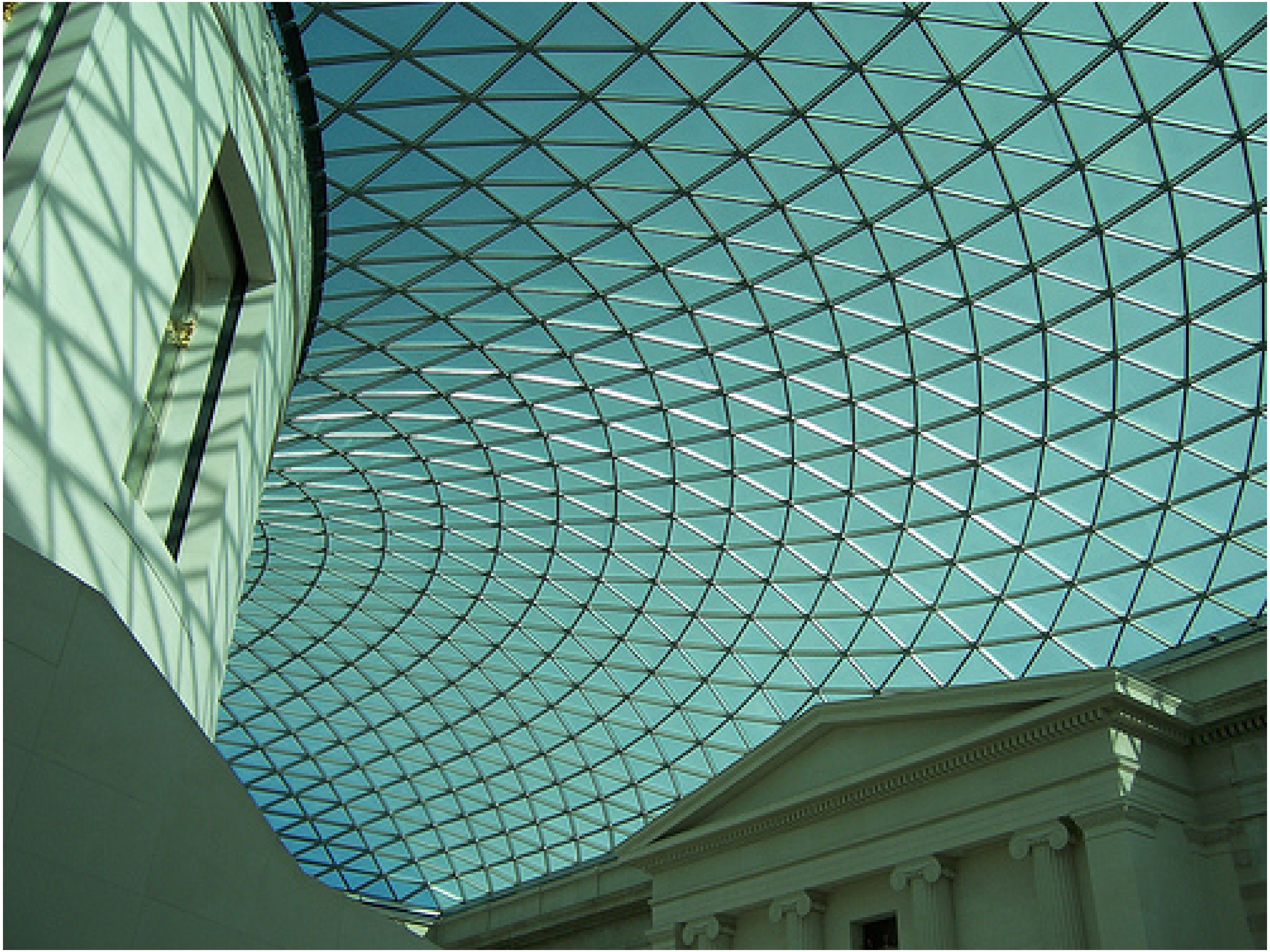
"…the science and art of designing and making with economy and elegance buildings, bridges, frameworks and other similar structures so that they can safely resist the forces to which they may be subjected"

Prof. Tom Collins, University of Toronto:

"…the art of moulding materials we do not really understand into shapes we cannot really analyze so as to withstand forces we cannot really assess in such a way that the public does not really suspect"

Some examples of structural engineering…


British Waterways Scotland
welcome to
The Falkirk Wheel



Important Maths Topics

Essential maths topics are:

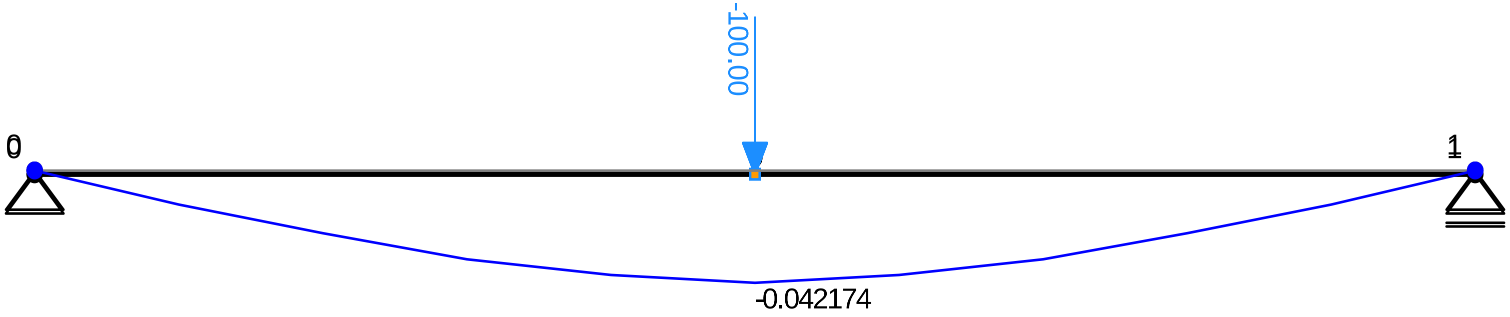
1. Algebra
2. Calculus – differentiation and integration
3. Matrices
4. Complex numbers
5. Statistics and probability

For each of these, I'll give an example of its application…

Algebra

How stiff should a beam be?

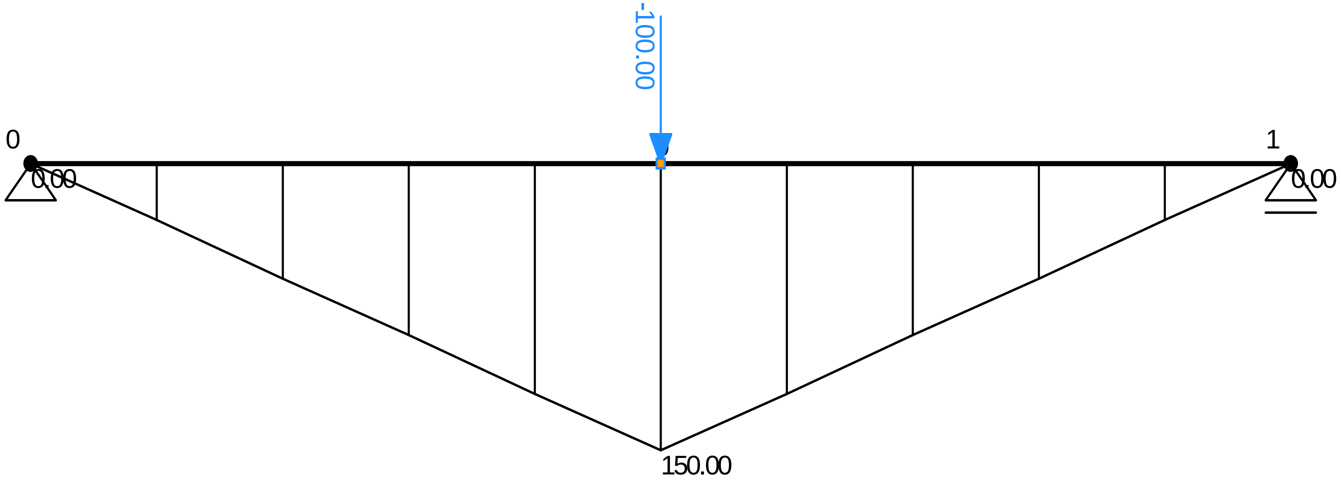
For a point load on the centre of a beam we will work it out…



Calculus I

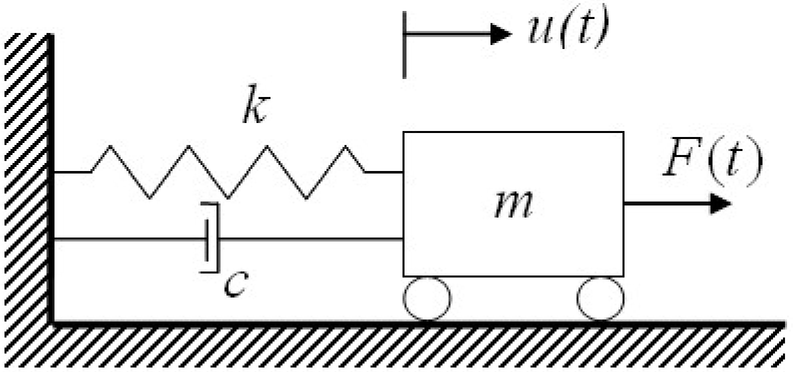
Beam deflection:

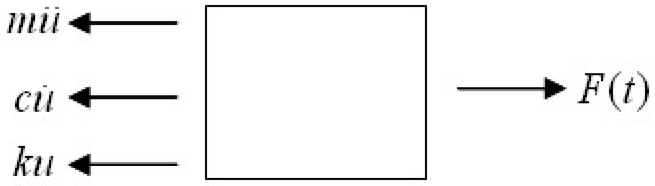
Given the bending in a beam, can we find the deflection?



Calculus II

Vibration of structures





$$\mathrm{F}_{\text{applied}} = \mathrm{F}_{\text{stiffness}} + \mathrm{F}_{\text{damping}} + \mathrm{F}_{\text{inertia}}$$

$$\begin{aligned} \mathrm{F}_{\text{stiffness}} &= ku \\ \mathrm{F}_{\text{damping}} &= c\dot{u} \\ \mathrm{F}_{\text{inertia}} &= m\ddot{u} \end{aligned}$$

Fundamental Equation of Motion:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F(t)$$

Matrices I

In structural frames displacement is related to forces:

$$\mathbf{F} = \mathbf{K} \cdot \boldsymbol{\delta}$$

Force Vector

Stiffness Matrix

Displacement Vector

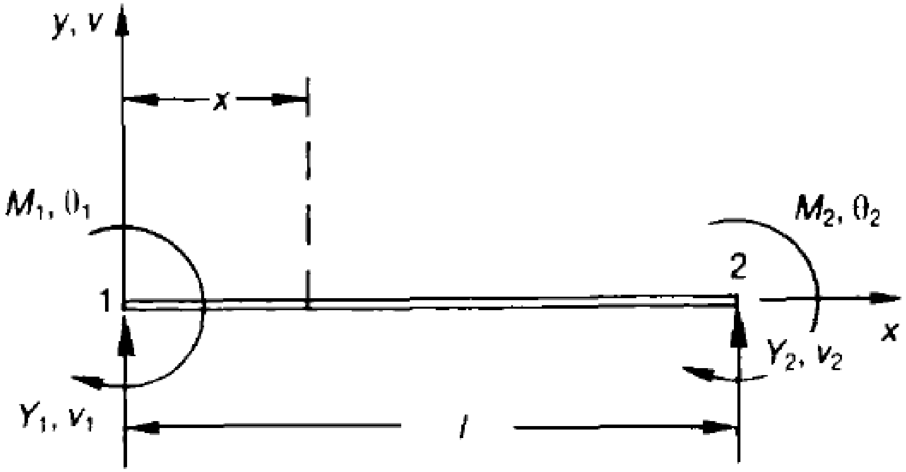
To solve, we pre-multiply each side by the inverse of the stiffness matrix:

$$\mathbf{K}^{-1} \cdot \mathbf{F} = \mathbf{K}^{-1}\mathbf{K} \cdot \boldsymbol{\delta} = \mathbf{I} \cdot \boldsymbol{\delta}$$

$$\therefore \boldsymbol{\delta} = \mathbf{K}^{-1} \cdot \mathbf{F}$$

Matrices II

Each member in a frame has its own stiffness matrix:



$$[\mathbf{k}] = EI \begin{array}{cccc|c} v_1 & \theta_1 & v_2 & \theta_2 & \\ 12/l^3 & -6/l^2 & -12/l^3 & -6/l^2 & v_1 \\ -6/l^2 & 4/l & 6/l^2 & 2/l & \theta_1 \\ -12/l^3 & 6/l^2 & 12/l^3 & 6/l^2 & v_2 \\ -6/l^2 & 2/l & 6/l^2 & 4/l & \theta_2 \end{array}$$

These are assembled to solve for the whole structure displacements

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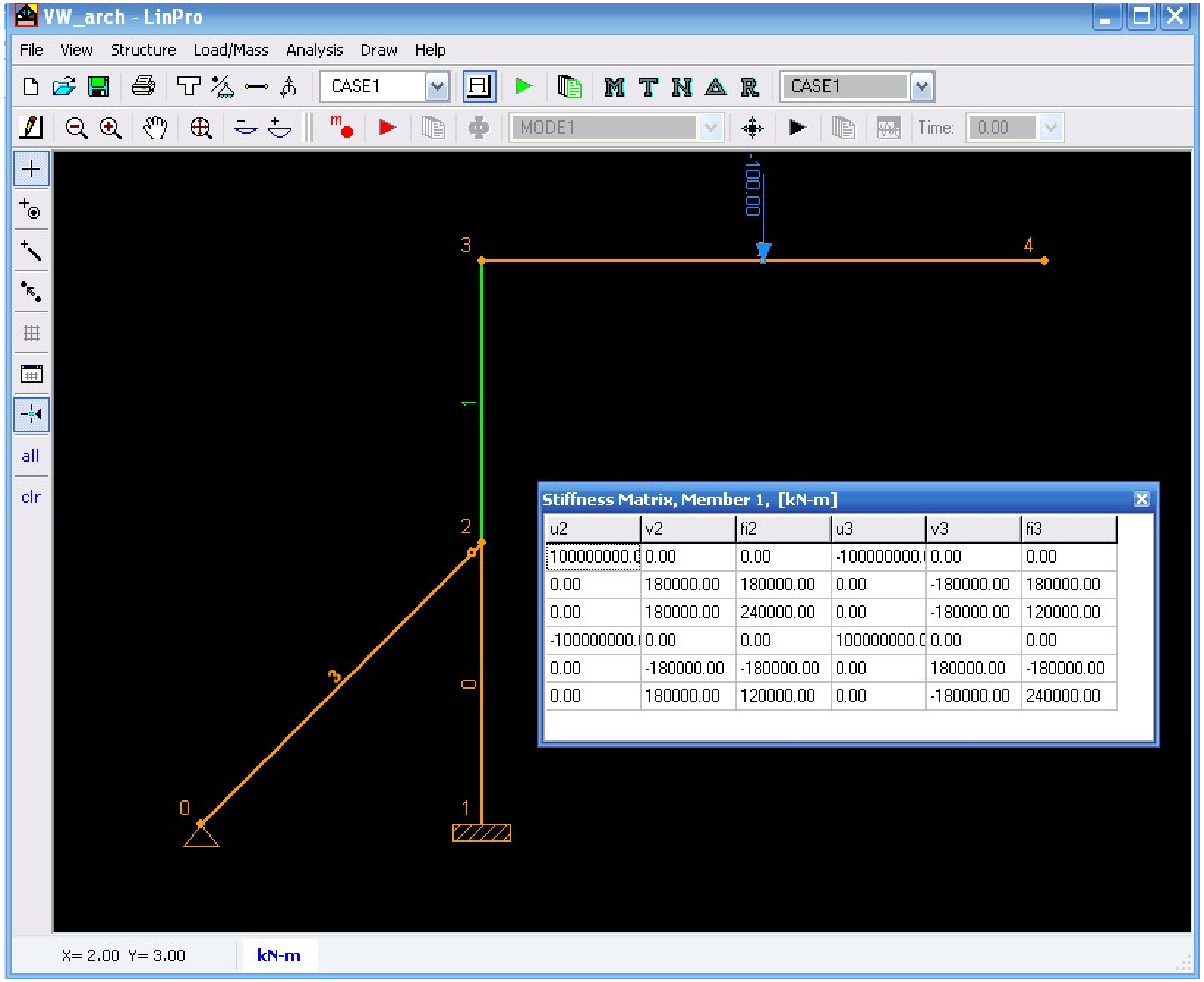
Matrices III

LinPro Software:

Displays the stiffness matrix for a member

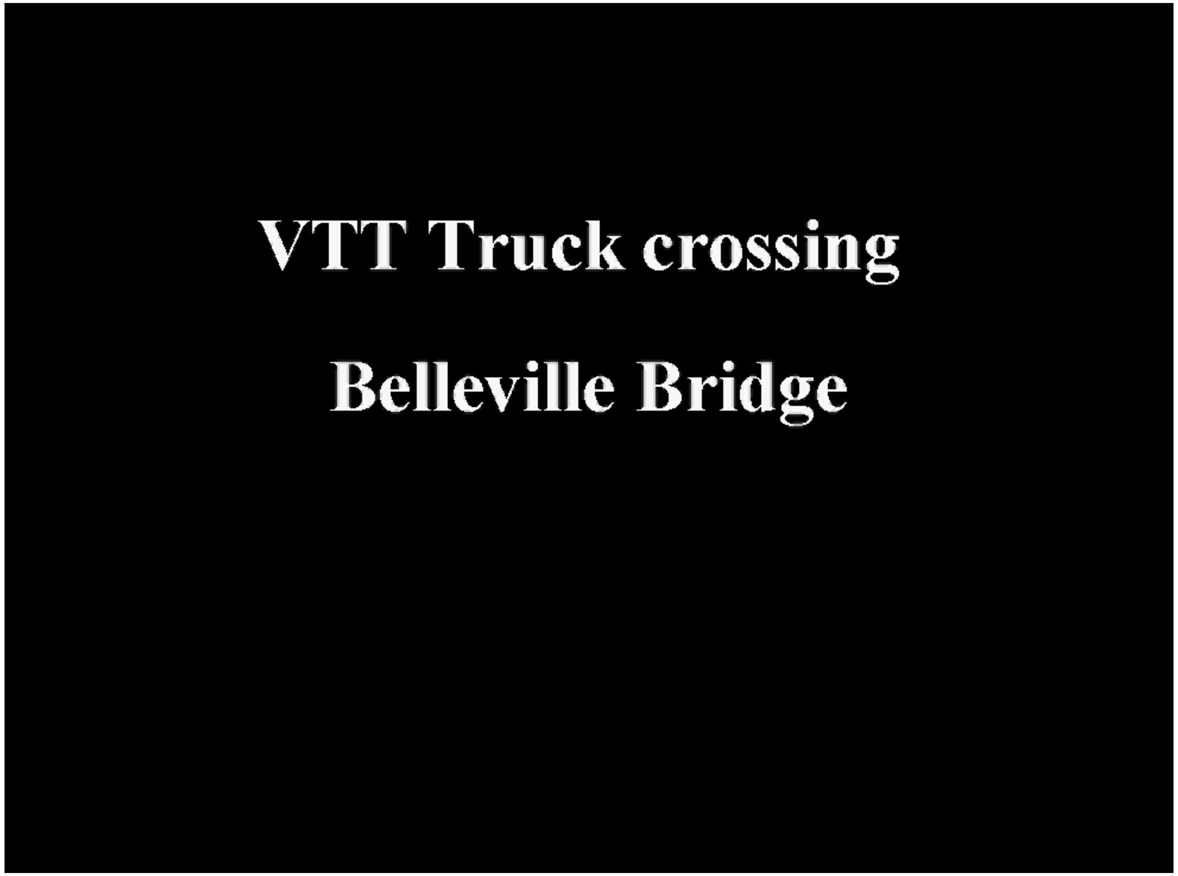


Stiffness Matrix, Member 1, [kN-m]

u2	v2	fi2	u3	v3	fi3
100000000.	0.00	0.00	-100000000.	0.00	0.00
0.00	180000.00	180000.00	0.00	-180000.00	180000.00
0.00	180000.00	240000.00	0.00	-180000.00	120000.00
-100000000.	0.00	0.00	100000000.	0.00	0.00
0.00	-180000.00	-180000.00	0.00	180000.00	-180000.00
0.00	180000.00	120000.00	0.00	-180000.00	240000.00

Matrices IV

Assembling the simple matrices for each member lets us calculate complex structures:



VTT Truck crossing

Belleville Bridge

Complex Numbers I

Free vibration:

$$\ddot{u}(t)+\omega^2 u(t)=0$$

$$\lambda^2+\omega^2=0$$

$$\lambda_{1,2}=\pm i\omega$$

$$u(t)=C_1 e^{\lambda_1 t}+C_2 e^{\lambda_2 t}$$

$$u(t)=C_1 e^{+i\omega t}+C_2 e^{-i\omega t}$$

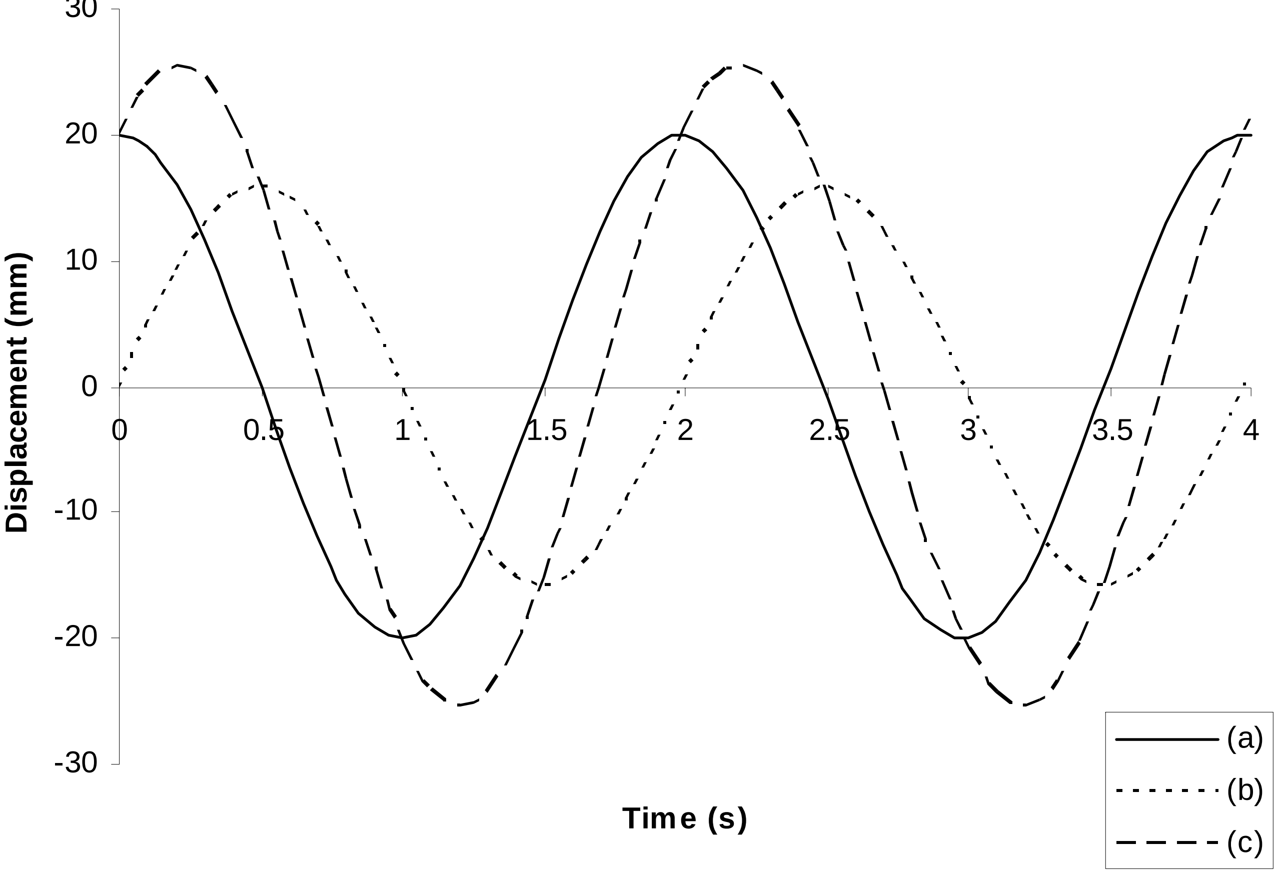
Since $e^{\pm i\theta}=\cos\theta\pm i\sin\theta$

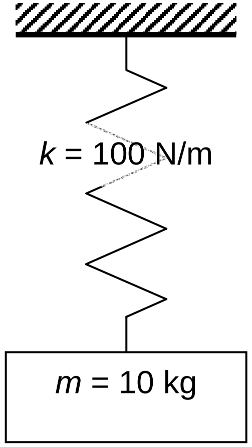
$$u(t)=C_1\left(\cos\omega t+i\sin\omega t\right)+C_2\left(\cos\omega t-i\sin\omega t\right)$$
$$=A\cos\omega t+B\sin\omega t$$

$$u(t)=u_0\cos\omega t+\left(\frac{\dot{u}_0}{\omega}\right)\sin\omega t$$

Complex Numbers II

$$u\left(t\right)=u_0\cos\omega t+\left(\frac{\dot{u}_0}{\omega}\right)\sin\omega t$$





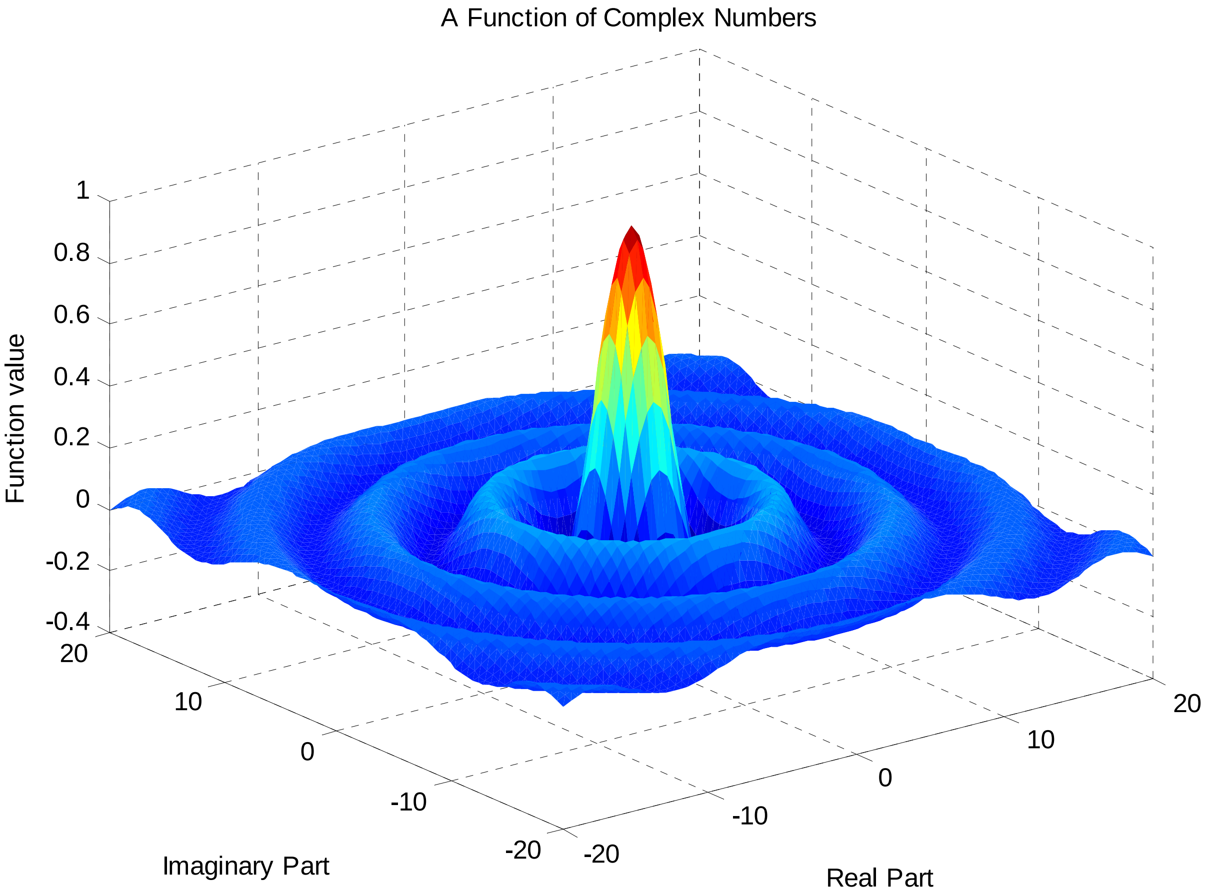
$u_0 = 20\text{mm} \quad \dot{u}_0 = 0$

$u_0 = 0 \quad \dot{u}_0 = 50\text{mm/s}$

$u_0 = 20\text{mm} \quad \dot{u}_0 = 50\text{mm/s}$

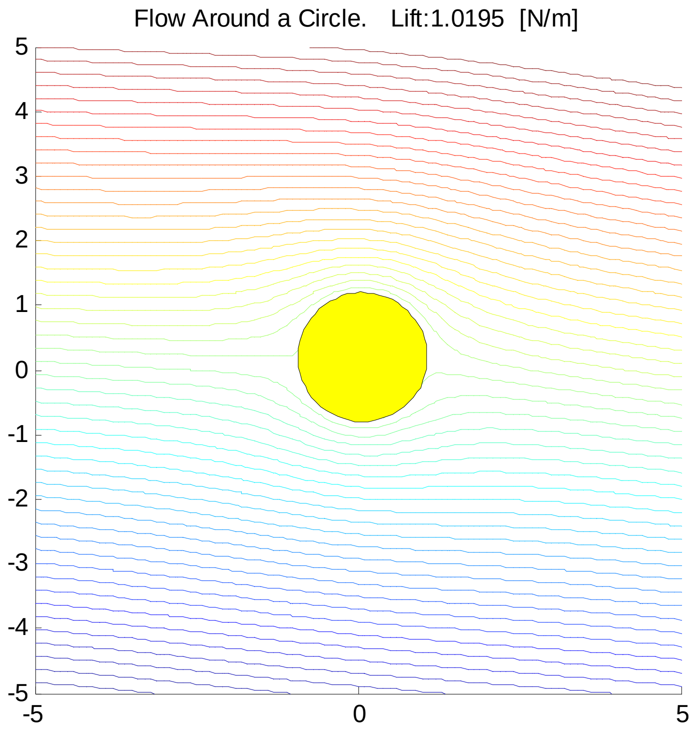
Complex Numbers III

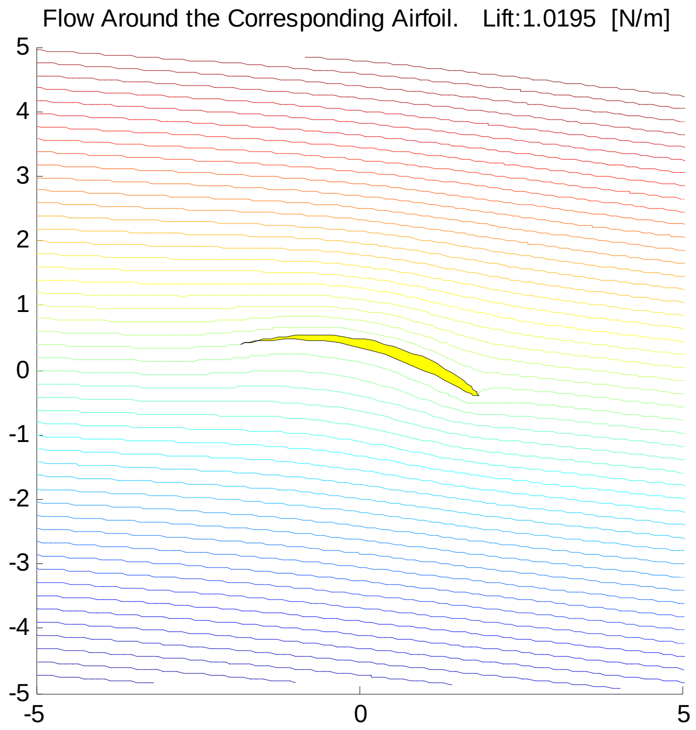
Are used to model complex geometries:



Complex Numbers IV

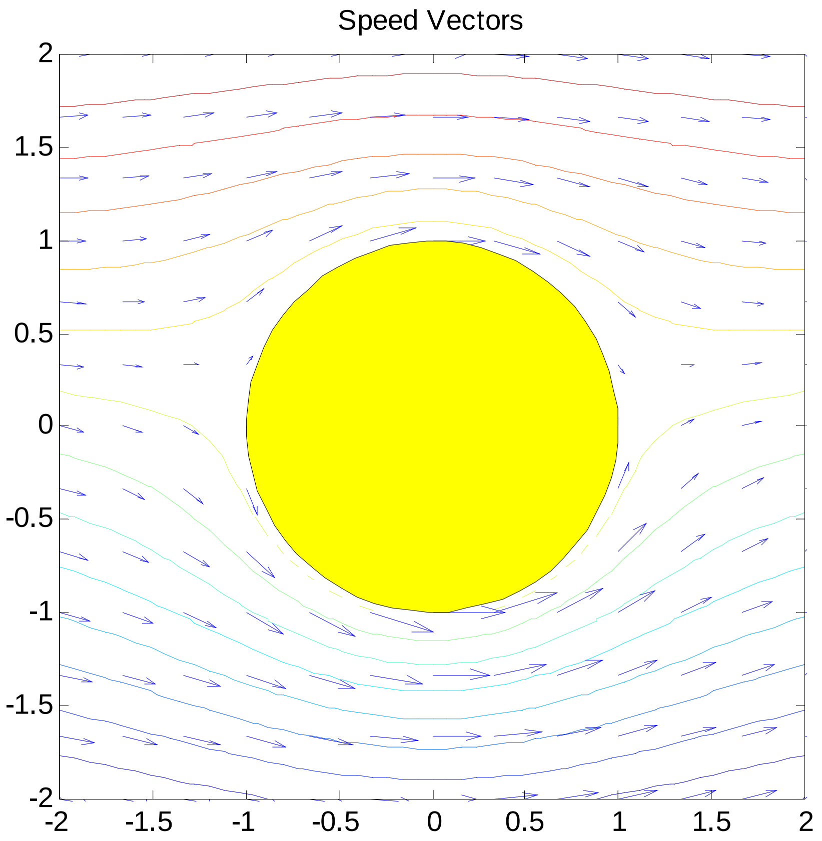
Aerofoil lift


Flow Around a Circle. Lift:1.0195 [N/m]


Flow Around the Corresponding Airfoil. Lift:1.0195 [N/m]

Complex Numbers V

Why does the ball curl?



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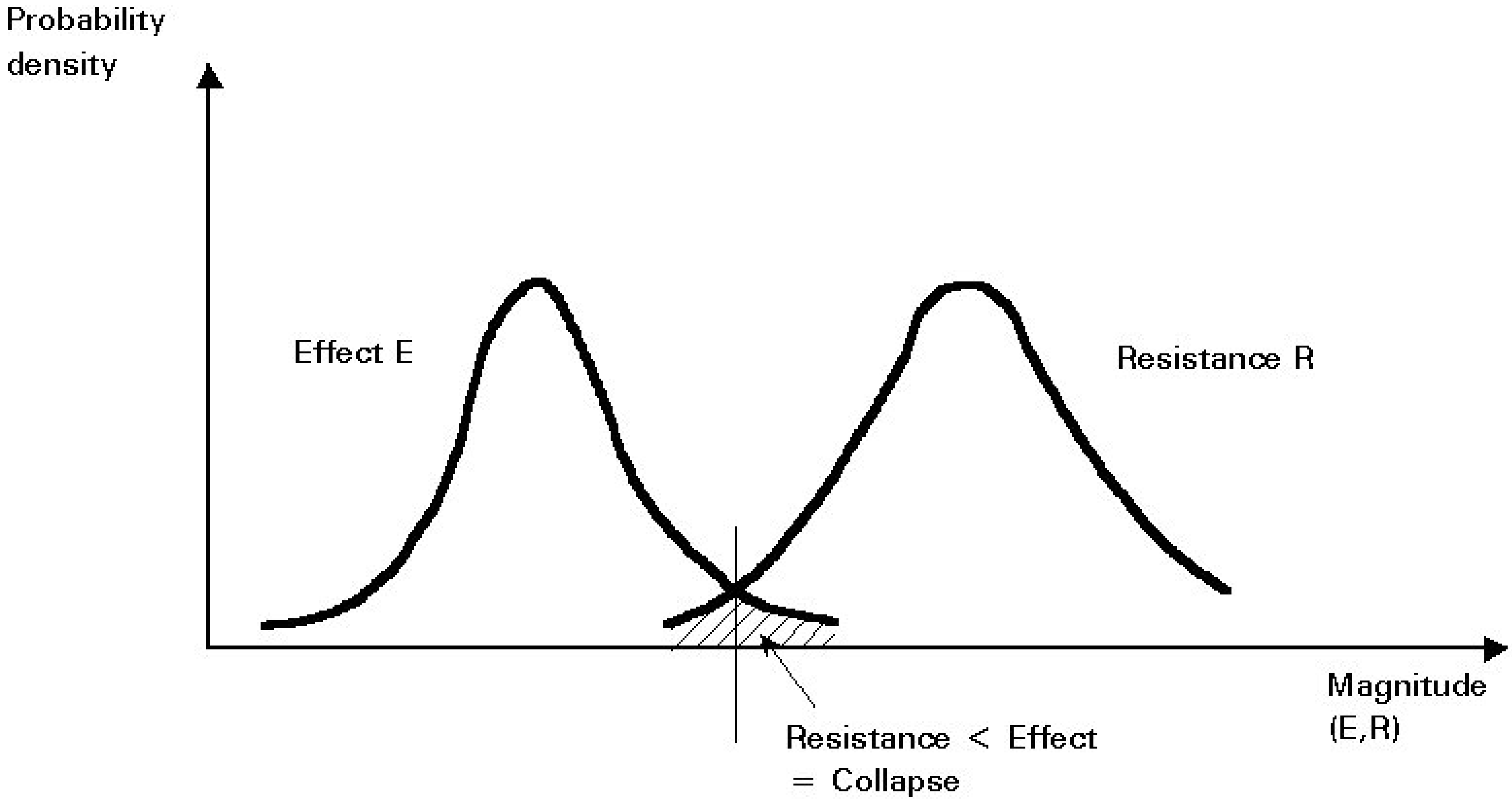
Statistics and Probability I

How strong is a structure?

How much load is on a structure?



Statistics and Probability II

How strong is a beam?

What is the effect of the load on the beam?

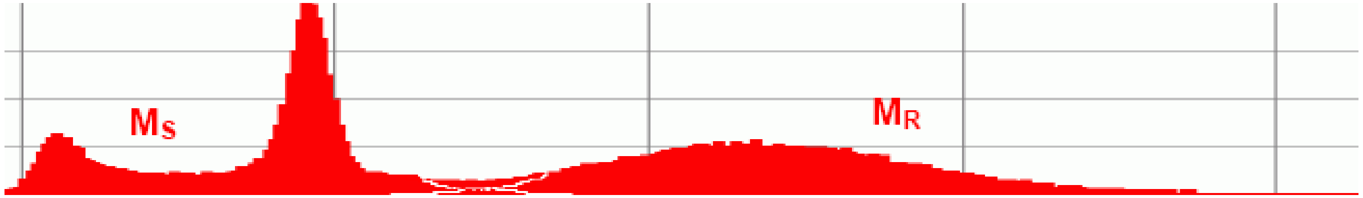


*Fig.6 SFF1, cross-section of Fig.7 in the span center (**x** =2 **m**), P_f is determined by the intersection of the resistance and response graphs*

Profile 1
x = 2 m

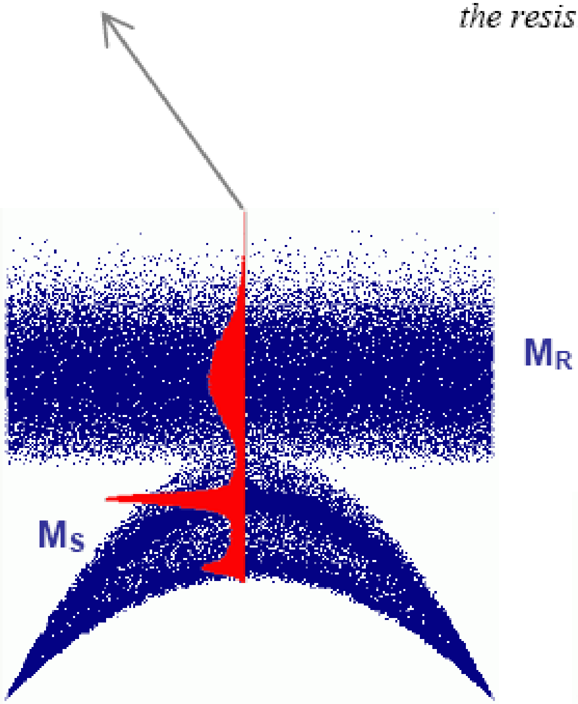


Fig.7 AntHill: response M_x and resistance M_{R1}

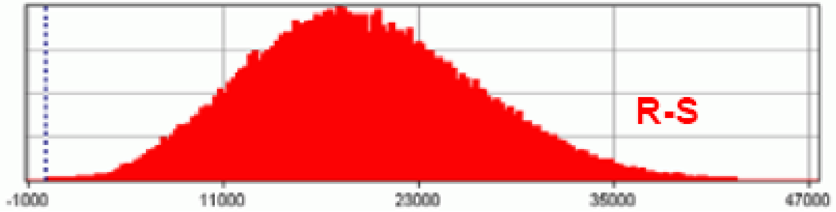
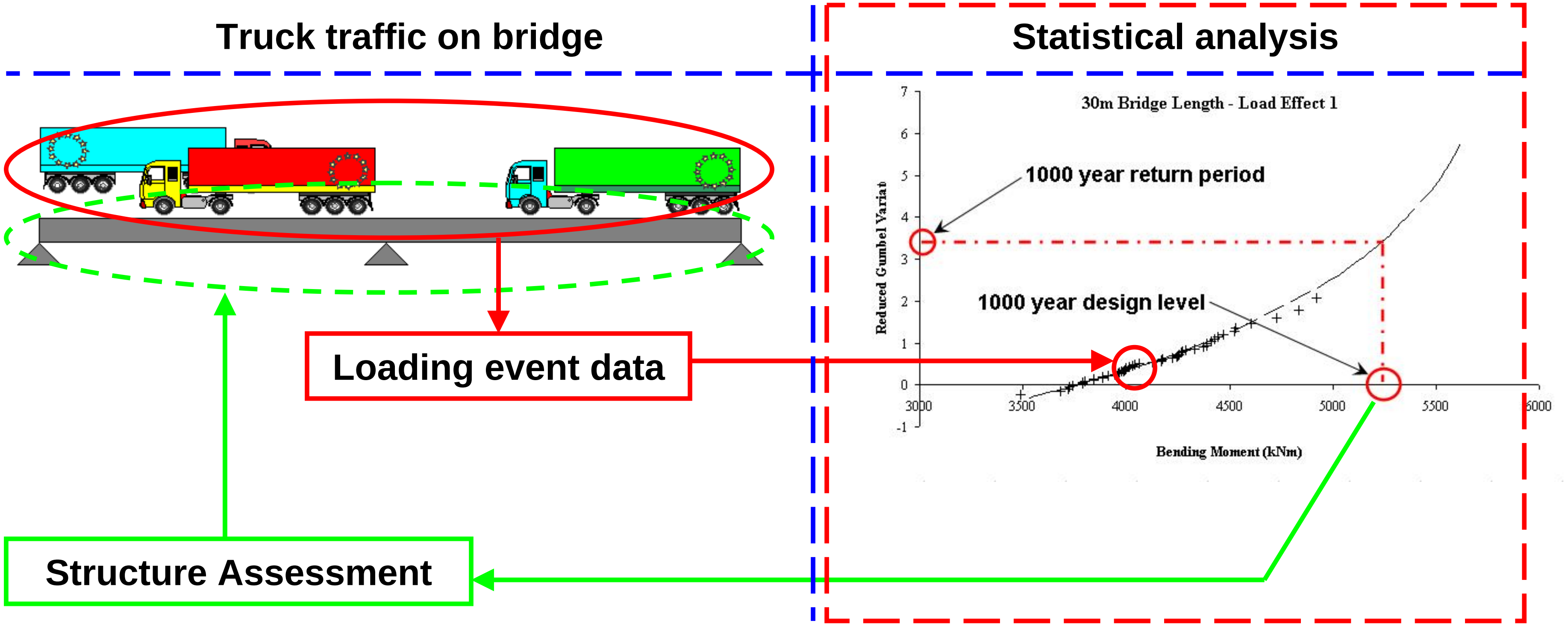


Fig.8 SF1 = R – S

Statistics and Probability III

What about bridges?

▬ Represents my area of interest



Truck traffic on bridge

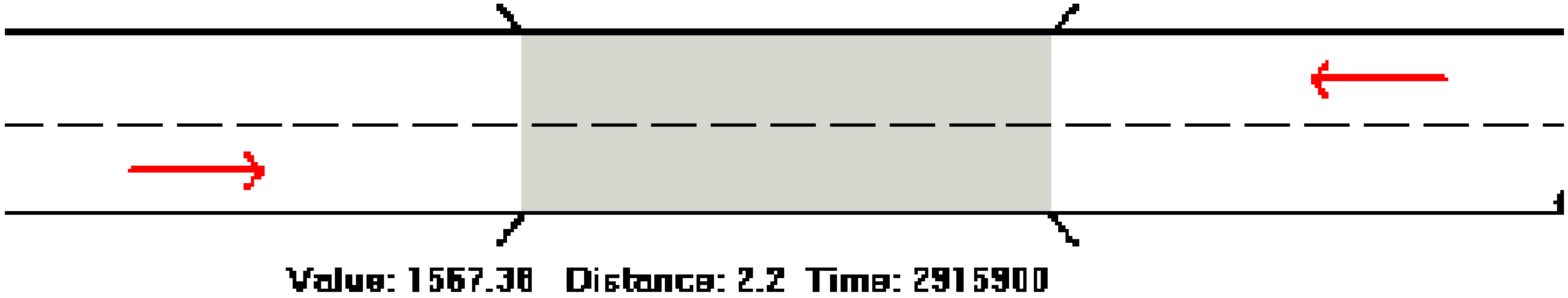
Statistical analysis

Loading event data

Structure Assessment

Statistics and Probability IV

Simulated bridge loading events…



Maths for the sake of it…

Once voted the most beautiful relation in maths:

$$e^{i\pi} + 1 = 0$$

It links the five most important numbers in maths:

$$e = 2.718281...$$

$$\pi = 3.141592...$$

$$i = \sqrt{-1}$$

$$1$$

$$0$$

Of this, a professor once said:

"it is surely true, it is paradoxical, we can't understand it, and we don't know what it means, but we have proved it, and therefore we know it is the truth"

Conclusion

- **All designed objects require mathematics to describe them**
- **I've just shown you my area of structural engineering**
- **Maths is essential for any profession involved in technical design**
- **It can also be enjoyable for its own sake**

Thanks for listening…but one last question

Question

If there are 23 people in a room,

what are the chances two of them share a birthday?

a) **Over 80%**

b) **Over 50%**

c) **Over 20%**

d) **Almost zilch!**