Interactive STEM

RESEARCH BRIEF

Interactive Technologies in STEM Teaching and Learning

Mathematics in the Early Grades: Counting & Cardinality

In the domain of *Counting & Cardinality*, Common Core State Standards indicate that by the time students leave kindergarten, they should know the names and sequence of number words up to 100, be able to count to determine the number of objects in a small set (up to 20), and make comparisons between small sets of objects (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In this brief, we describe the key concepts and skills that are involved in meeting the standards in this domain and provide excerpts from learning trajectories that illustrate how they develop over time and with experience.

Early Numeracy & Counting

Infants have demonstrated the ability to make judgements about quantity within the first months of life (e.g., Antell & Keating, 1983; Starkey & Cooper, 1980). However, research suggests that these earliest abilities may be based on an approximate sense of number (Huttenlocher, Jordan & Levine, 1994) and also reflect a sensitivity to different continuous amounts (e.g., area or length) rather than the precise number of objects in a display. It is not until their toddler years that children develop the ability to perceive, describe, and reason about exact quantities using the process of *subitizing* and later, through *counting* (Clements & Sarama, 2014).

Subitizing is a perceptual process that allows for the instant recognition of the quantity of small sets (Clements, 1999). It is an important skill that young children can build on to recognize slightly larger sets of objects in certain arrangements. However, in order to describe and reason about larger quantities, children need to develop and refine their understanding of number using processes that rely on abstract representation rather than perception.

Counting enables children to determine the quantity of any set by using number words to mark each item in a set. It involves the following big ideas:

» **Number Words and Sequence**
Language plays an important role in developing number sense, where number words provide children with verbal tools to make their thinking about number explicit.

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Learning the number names from 1 to 20 may involve memorization, but children should learn to recognize and use the repeated patterns that occur after 20 as they continue the number sequence (Fosnot & Dolk, 2001; Fuson, 2012; Fuson, Richards, & Briars, 1982).

- **One-to-One Correspondence**
  When counting to determine "how many," each number word should be applied to only one item in the set.

- **Cardinality**
  Cardinality refers to the quantity or total number of items in a set and can be determined by subitizing (for very small sets) or counting. While subitizing allows children to perceive the cardinality of small sets, counting requires them to understand that the last number in the counting sequence represents the quantity of the set.

Gelman and Gallistel (1978) outline five principles for counting that require children to coordinate their knowledge across all three of these big ideas and to generalize them across different counting situations. Proficient counters must be able to do the following:

1. Consistently use the number words in the same order (stable order principle)
2. Count every item in a set only once, using only one number word (one-one principle)
3. Understand that the last number word used represents the cardinality of the set (cardinal principle)
4. Recognize that any collection of like or unlike items can be counted as a set (abstraction principle)
5. Understand that the result is the same no matter the order in which the objects are counted (order irrelevance principle)

Counting Situations and Strategies: When children first begin to use the verbal counting sequence to enumerate sets of objects, they will likely mark every item vocally (using a number name) and physically (using an action such as pointing or touching). The accuracy of their earliest counts will depend on a variety of factors, and the number of errors they produce in more challenging counting situations will reduce over time with sufficient practice, demonstration, and scaffolding (see Counting and Cardinality *Under Construction*). Factors that contribute to the difficulty of a counting situation and the accuracy of children's counts are listed below (ranging from *less difficult/lower error rate to more difficult/higher error rate*; Clements & Sarama, 2014; Cross, Woods, & Schweingruber, 2009; Fuson, 2012):

- Set size (*small sets* → *large sets*)
- Set layout (*rows* → *unorganized arrangements*)
- Item type (*concrete objects* → *more abstract elements, e.g., actions*)
- Student's level of effort (*high effort* → *low effort*)

After success with smaller, easily counted sets, children begin to generalize their knowledge so that they're successful on more difficult counting tasks. To do so, it is critical that children have a variety of opportunities to apply and practice what they're learning, which will support these connections as well as their success in counting in a variety of situations.

As their counting skills and number knowledge become more robust and sophisticated, children may use other strategies, including these:

- Skip counting (counting off by more than one, e.g., 2s, 5s, or 10s)
- Counting backward
- Counting on (starting from a number other than one and continuing the counting sequence)

Counting and Cardinality *Under Construction*	
As children demonstrate what they know about counting and cardinality across a variety of tasks and situations, they may produce errors that indicate they are still developing and generalizing their knowledge and skills.	
Concept/Skill	Typical Errors[1] (Clements & Sarama, 2014; Fuson, 2012)
Number words and sequence	When reciting number words or using them in counting situations, a student: • Omits numbers • Repeats numbers
One-to-one correspondence	When counting a set of items, a student: • Skips an item/does not include it in the counting sequence • Assigns more than one number word to a single item • Points to two or more items while saying one number word
Cardinality	After counting, when asked how many there are in the set, a student: • Gives the wrong number • Recounts to determine the number of items
Comparing number/ Number conservation	When asked to compare two sets and identify which has more, for example, a student: *Set 1* – ••••• *Set 2* – • • • • • Answers "*Set 2*" based purely on perceptual cues • Answers "*Set* 2" even after prompted to count

Written Number

Where number names give children a verbal means for representing quantity, numerals give them written symbols that make it easier to represent large sets.

Before learning to write numbers, children represent quantity in a variety of other ways that evolve as their ideas around number develop. Initially, they may draw pictures of each item in a set, later using abstract symbols (e.g., a tick mark or dot) to replace more literal representations. It is not until children have some understanding of cardinality—that one number can describe the quantity of the entire set—that they can learn to use written numbers instead of depictions that rely on one-to-one correspondence.

Zero: The counting sequence, usually starting at one, deals with numbers that describe quantity. *Zero*, as a number word and written symbol, represents a new concept—the absence of quantity—which is often difficult for children to understand. Young children's understanding of zero develops over several stages. Initially they may learn the name and associate it with the symbol "0," but it is not until later that they recognize its quantitative meaning (i.e., the lack of quantity), and then finally, they can relate it to other small numbers (Wellman & Miller, 1986).

[1] The errors described in this table do not represent an exhaustive list and do not reflect other kinds of errors that may be most common in children with mathematics learning disabilities

Comparing Number

Children's ability to compare quantities begins in infancy as well, when they can perceive differences between two sets depending on the set size and the size ratio between the two sets (Clements & Sarama, 2014). As they learn to count, and as their thinking about number becomes more explicit through physical, verbal, and written representations, they are able to determine whether a set has more than, less than, or an equal number as another set where differences aren't so obvious.

Initially, infants and young children base their comparisons on perceptual cues, such as the length of the arrangement (even if the number of items is the same). They later begin using matching strategies, where they pair items between the two sets until they determine which has more items that can't be paired. Once they have a firm grasp on cardinality, they can use counting strategies to compare the cardinality of two sets. Comparison situations also allow children to begin thinking about *how much* more or less a set has. This is initially difficult, but it builds on their knowledge of part-whole relationships and connects to operations of addition and subtraction (Cross et al., 2009).

Number Conservation and Hierarchical Inclusion

There are two important milestones that children must reach in order to fully understand cardinality and begin to think of numbers as parts and wholes (Fuson, 2012).

Number Conservation: According to Piagetian theory of cognitive development, children do not have a complete understanding of counting as a means for determining cardinality until they demonstrate success on number conservation tasks (Piaget, 2013), that is, they recognize that the number of items in a set remains the same regardless of their physical arrangement. While more recent research suggests that children can actually develop important knowledge about number and counting that can *contribute* to their ability to conserve number (Clements, 1984), it is a critical milestone that children must reach to fully appreciate the cardinal meaning of number and to be successful on comparison tasks where perceptual cues are misleading.

Hierarchical Inclusion: Hierarchical inclusion is the idea that (1) each number in the counting sequence is exactly one greater than the one before it and (2) the number of the resulting set contains the previous within it. For example, when counting *1, 2, 3, 4, 5, 6, 7…8*, the number *8* has a value that is equal to *7 + 1*, and contains the numbers 1 through 7 within it. Children need a firm grasp on hierarchical inclusion to use the strategy of counting on.

Key Vocabulary

Cardinality – The last number word said when counting tells how many objects have been counted.

Hierarchical inclusion – Each number (a) represents one more than the previous number in the counting sequence and (b) includes all previous numbers within it.

Number conservation – Understanding that the quantity of a set doesn't change if the set is rearranged.

Number sequence – The names and the ordered list of number words.

One-to-one correspondence – Saying number words in correspondence with the objects counted.

Subitizing – Visually recognizing the number of items in a small set without counting.

Sample Learning Trajectories

The following table contains samples from two of Clements and Sarama's (2009) learning trajectories: (1) Verbal and Object Counting and (2) Comparing, Ordering, & Estimating Numbers.[2] Note that the ages are approximate and heavily dependent on experience. Complete versions of their trajectories, which include the full age range and sample instructional tasks, are available in *Learning and Teaching Early Math: The Learning Trajectories Approach* (2014; see Resources).

Developmental Progression	Description
Age 4	
Counter (Small Numbers)	Accurately counts objects in a line to 5 and answers the "how many" question with the last number counted. When objects are visible, and especially with small numbers, begins to understand cardinality.
Counter (10)	• Counts arrangements of objects to 10. May be able to write numerals to represent 1– 10. • May be able to tell the number just after or just before another number, but only by counting up from 1. • Verbal counting to 20 is developing.
Producer (Small Numbers)	Counts out objects to 5. Recognizes that counting is relevant to situations in which a certain number must be placed.
Nonverbal Comparer of Dissimilar Items	Matches small, equal collections consisting of different items, showing that they are the same number.
Matching Comparer	Compares groups of 1– 6 by matching.
Counting Comparer (Same Size)	Accurate comparison via counting, but only when objects are about the same size and groups are small (up to 5).
Age 5	
Counter and Producer (10+)	• Counts and counts out objects accurately to 10, then beyond (to about 30). Has explicit understanding of cardinality (how numbers tell how many). Keeps track of objects that have and have not been counted, even in different arrangements. Writes or draws to represent 1 to 10 (then, 20, then 30). • Recognizes errors in others' counting and can eliminate most errors in own counting (point-object) if asked to try hard.
Counter Backward from 10	Counts backward from 10 to 1, verbally, or when removing objects from a group.
Counting Comparer (5)	Compares with counting, even when larger collection's objects are smaller. Later, figures out how many more or less.
Counting Comparer (10)	Compares with counting, even when larger collection's objects are smaller, up to 10.

Age 6	
Counter from *N*	• Counts verbally and with objects from numbers other than 1 (but does not yet keep track of the number of counts). • Immediately determines numbers just after or just before.
Skip Counter by 10s to 100	Skip counts by tens up to 100 or beyond with understanding; e.g., "sees" groups of 10 within a quantity and counts those groups by 10 (this relates to multiplication and algebraic thinking).
Counter to 100	Counts to 100. Makes decade transitions (e.g., from 29 to 30) starting at any number.
Counter On Using Patterns	Keeps track of a few counting acts, but only by using numerical pattern (spatial, auditory, or rhythmic).
Skip Counter	Counts by fives and twos with understanding.
Counter of Imagined Items	Counts mental images of hidden objects.
Counter On Keeping Track	Keeps track of counting acts numerically, first with objects, then by "counting counts." Counts up 1 to 4 more from a given number.

Resources

Children's Counting and Concept of Number
Fuson, K., 2012
Part I: Number Words
Part II: Correspondence Errors in Counting Objects
Part III: Concepts of Cardinality
Part IV: Number Words, Counting, & Cardinality

Early Childhood Mathematics Education Research: Learning Trajectories for Young Children
Sarama, J., & Clements, D., 2009
Part II: Number and Quantitative Thinking

Learning and Teaching Math: The Learning Trajectories Approach (2nd ed.)
Clements, D., & Sarama, J., 2014
Chapter 3: Verbal and Object Counting
Chapter 4: Comparing, Ordering, and Estimating

Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity
National Research Council, 2009
Chapter 5: The Teaching and Learning Paths for Number, Relations, and Operations

Teaching Student-Centered Mathematics: Developmentally Appropriate Instruction for Grades Pre-K–2
Van de Walle, J., Lovin, L.H., Karp, K. S., & Bay-Williams, J. M., 2013
Chapter 8: Developing Early Number Concepts and Number Sense

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