Chapter 1

Mathematics Involved in Electromagnetic Theory

Gradiant of a scalar function:

If $\phi(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$ is called the gradient of the scalar function ϕ. And is denoted by grad ϕ.

Thus, $$grad\ \phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

$$grad\ \phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$grad\ \phi=\vec{\nabla}\phi$$

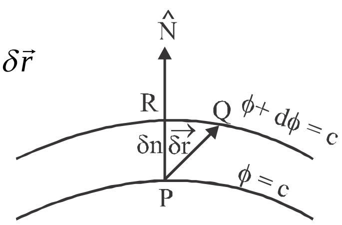
Geometrical Meaning of Gradient, Normal:

If a surface $\phi(x, y, z)=c$ passes through a point P. The value of the function at each point on the surface is the same as at P. Then such a surface is called a level surface through P. For example, if $\phi(x, y, z)$ represents potential at the point P, then equipotential surface $\phi(x, y, z)=c$ is a level surface.

Two level surfaces can not intersect.

Let the level surface pass through the point P at which the value of the function is ϕ. Consider another level surface passing through Q, where the value of the function is $\phi+d\phi$.

Let $\vec{r}$ and $\vec{r}+\delta\vec{r}$ be the position vector of P and Q then $\overrightarrow{PQ}=\delta\vec{r}$



$$\vec{\nabla}\phi\cdot d\vec{r}=\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}dx+\hat{j}dy+\hat{k}dz\right)$$

$$=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi \quad \text{... (1)}$$

If Q lies on the level surface of P, then $d\phi=0$

Equation (1) becomes , $\vec{\nabla}\phi\cdot d\vec{r}=0$, then $\vec{\nabla}\phi$ is perpendicular to $d\vec{r}$ (tangent)

Hence, $\vec{\nabla}\phi$ is normal to the surface $\phi(x, y, z)=c$

The Divergence:

From the defnition of ∇ we construct the divergence:

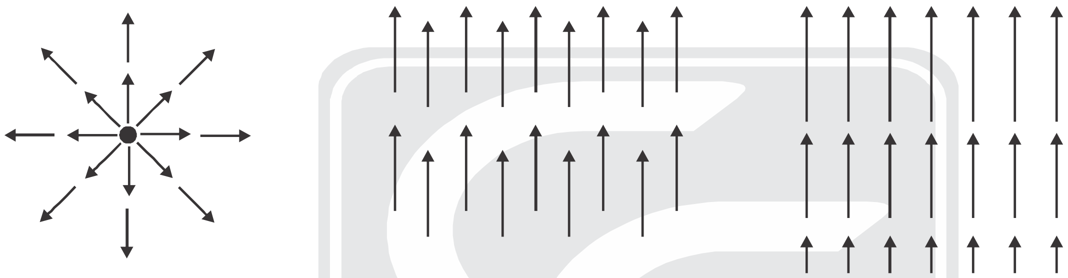
$$\vec{\nabla}\cdot\vec{v}=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(v_x\hat{x}+v_y\hat{y}+v_z\hat{z}\right)$$

$$=\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}$$

Observe that the divergence of a vector function $\vec{v}$ is itself a scalar $\vec{\nabla}\cdot\vec{v}$

Gemetrical interpretation :

The name divergence is well chosen, for $\vec{\nabla}\cdot\vec{v}$ is a measure of how much the vector $\vec{v}$ spreads out (diverges) from the point in question. For example, the vector function in figure (a) has a large (positive) divergence (if the arrows pointed in, it would be a large negative divergence), the function in figure (b) has zero divergence, and the function in figure (c) again has a positive divergence.



(a) (b) (c)

The Curl:

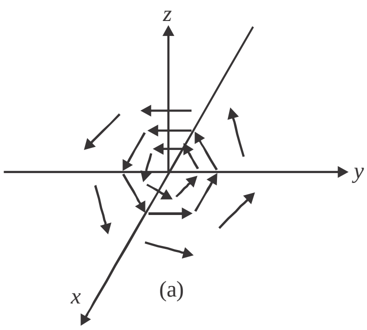
From the defintion ∇ we construct the curl:

$$\vec{\nabla}\times\vec{v}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}=\hat{x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)$$

Notice that the curl of a vector function $\vec{v}$ is, like any cross product, a vector. Carl of a scalar does not exist.

Geometrical interpretation :

The name curl is also well chosen, for $\vec{\nabla}\times\vec{v}$ is a measure of how much the vector $\vec{v}$ 'curls around' the point in question. Thus the three functions in figure (a) have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.



(a)

Product rules:

(i) $\vec{\nabla}(fg) = f\,\vec{\nabla}g + g\vec{\nabla}f$

(ii) $\vec{\nabla}(\vec{A}.\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}.\vec{\nabla})\vec{B} + (\vec{B}.\vec{\nabla})\vec{A}$,

(iii) $\vec{\nabla}.(f\vec{A}) = f(\vec{\nabla}.\vec{A}) + \vec{A}.(\vec{\nabla}f)$

(iv) $\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A}) - \vec{A}.(\vec{\nabla}\times\vec{B})$,

(v) $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f)$

(vi) $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}.\vec{B}) - \vec{B}(\vec{\nabla}.\vec{A})$

Problem-1: Find the gradients of the following functions:

(a) $f(x,y,z) = x^2 + y^3 + z^4$ (b) $f(x,y,z) = x^2y^3z^4$ (c) $f(x,y,z) = e^x \sin(y)\ln(z)$

Problem-2: Let ‘r’ be the separation vector from a fixed point (x', y', z') to the point (x, y, z) and let ‘r’ be its length. Show that

(a) $\vec{\nabla}(r^2) = 2\vec{r}$ (b) $\vec{\nabla}(1/r) = -\hat{r}/r^2$ (c) what is the general formula $\vec{\nabla}(r^n)$?

Problem-3: Calculate the divergence of the following vector functions

(a) $\vec{v}_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$ (b) $\vec{v}_b = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$

(c) $\vec{v}_c = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}$

Problem-4: The vector function, $v = \frac{\hat{r}}{r^2}$ compute its divergence.

The gradient, the divergence, and the curl are the only first derivatives we can make with ∇; by applying ∇ twice we can construct five species of second derivatives. The gradient ∇T is a vector, so we can take the divergence and curl of it.

(1) Divergence of gradient: $\vec{\nabla}.(\vec{\nabla}T) = \nabla^2 T$ (Laplacian of T)

(2) Curl of gradient: $\vec{\nabla}\times(\vec{\nabla}T) = 0$

(3) Gradient of divergence: $\vec{\nabla}(\vec{\nabla}.\vec{v})$ (This is not same as laplacian)

(4) Divergence of curl: $\vec{\nabla}.(\vec{\nabla}\times\vec{v}) = 0$

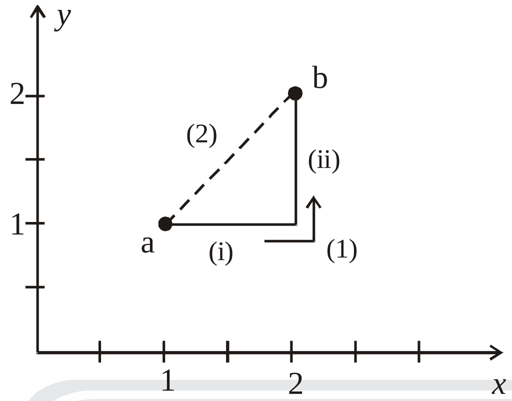
(5) Curl of curl: $\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}.\vec{v}) - \nabla^2\vec{v}$

Problem-5: Calculate the Laplacian of the following functions:

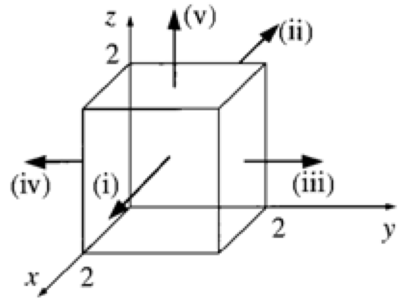
(a) $T_a = x^2 + 2xy + 3z + 4$ (b) $T_b = \sin x \sin y \sin z$

(c) $T_c = e^{-5x}\sin 4y \cos 3z$ (d) $v = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

Problem-6: Calculate the line integral of the function $\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point a = (1, 1, 0) to the point b = (2, 2, 0), along the paths (1) and (2) in the following figure. What is $\oint v.dl$ for the loop that goes from 'a' to 'b' along (1) and returns to 'a' along (2)?



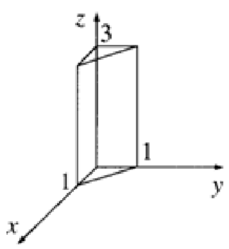
Problem-7: Calculate the surface integral of $v = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) in figure. Let "upward and outward" be the positive direction, as indicated by the arrows.



Problem-8: Calculate the volume integral of $T = xyz^2$ over the prism in figure.

Soln. You can do the three integrals in any order. Let's do 'x' first: it runs from 0 to (1–y); then y(it goes from 0 to 1); and finally z (0 to 3):

$$\int T d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x dx \right] dy \right\} dz = \frac{1}{2}\int_0^3 z^2 dz \int_0^1 (1-y^2)^2 y dy = \frac{1}{2} \times 9 \times \frac{1}{12} = \frac{3}{8}$$



Problem-9: Calculate the line integral of the function $v = x^2\hat{x} + 2yz\,\hat{y} + y^2\hat{z}$ from the origin to the point (1, 1, 1) by three different routes:

(a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

(c) The direct straight line.

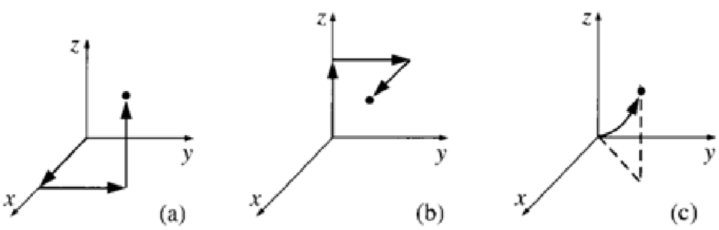
Example-10: Let $T = xy^2$, and take point 'a' to be the origin (0, 0, 0) and 'b' the point (2, 1, 0). Check the fundamental theorem for gradients.

Problem-11: Check the fundamental theorem for the gradients, using $T = x^2 + 4xy + 2yz^3$, the points $a = (0,0,0)$, $b = (1,1,1)$ and the three paths in figures (a), (b) and (c) are

(a) $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$

(b) $(0,0,0) \to (0,0,1) \to (0,1,1) \to (1,1,1)$

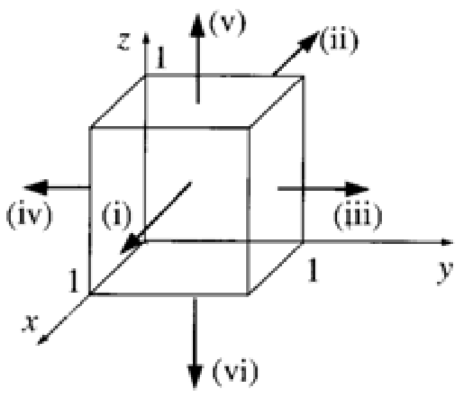
(c) The parabolic path $z = x^2;\ y = x$



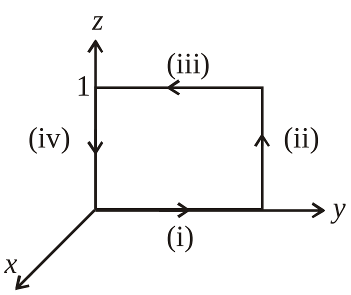
Problem-12: Check the divergence theorem using the function

$$\vec{v} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$$

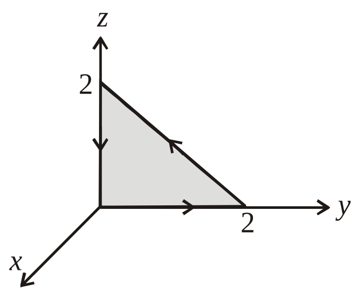
and the unit cube situated at the origin has shown in the following figure.



Problem-13: Suppose $\vec{v} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check Stokes' theorem for the square surface shown in figure.



Problem-14: Test Stokes' theorem for the function $\vec{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$, using the triangular shaded area of figure.



Problem-15: Evaluate the integral

$$\int_0^\infty x\, e^{-x} dx$$

Formulas for gradient divergence and curl for spherical coordinates.

Gradient: $$\vec{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

Divergence: $$\vec{\nabla}.\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

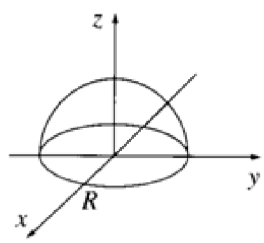
Curl: $$\vec{\nabla}\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta v_\phi\right) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\theta}$$
$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian: $$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

Problem-16: Compute the divergence of the function

$$\vec{v} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy-plane and centered at the origin (Figure)



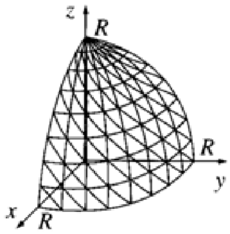
Problem-17: Find the formulas for r, θ, ϕ in terms of $\hat{x}, \hat{y}, \hat{z}$.

Problem-18: Express the unit vector, $\hat{r}, \hat{\theta}, \hat{\phi}$ in terms of $\hat{x}, \hat{y}, \hat{z}$.

Problem-19: Express the cylinderical unit vectors $\hat{s}, \hat{\phi}, \hat{z}$ in terms of $\hat{x}, \hat{y}, \hat{z}$. Invert your formulas to get $\hat{x}, \hat{y}, \hat{z}$ in terms of $\hat{s}, \hat{\phi}, \hat{z}$.

Problem-20: Check the divergence theorem for the function

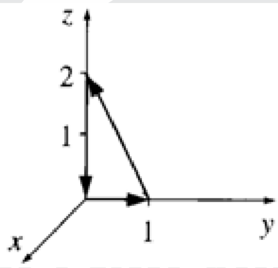
$$\vec{v} = r^2 \cos\theta\, \hat{r} + r^2 \cos\phi\, \hat{\theta} - r^2 \cos\theta \sin\phi\, \hat{\phi}$$



using as your volume one octant of the sphere of radius R. Make sure you include the entire surface.
Ans. $\pi R^4/4$.

Problem-21: Compute the line integral of

$$\vec{v} = 6\hat{x} + yz^2\hat{y} + (3y+z)\hat{z}$$



along the triangular path shown in figure. Check your answer using Stoke's theorem.
Ans. 8/3.

Coordinate System:

Coordinate system	$\hat{u}_1$	$\hat{u}_2$	$\hat{u}_3$	h_1	h_2	h_3	ds	dv
Cartesian	$\hat{x}$	$\hat{y}$	$\hat{z}$	1	1	1	$dxdy$ or $dydz$ or $dzdx$	$dxdydz$
Spherical	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$	1	r	$r\sin\theta$	$rd\theta r\sin\theta d\phi$	$r^2\sin\theta drd\theta d\phi$
Cylindrical	$\hat{\rho}$	$\hat{\theta}$	$\hat{z}$	1	ρ	1	$rd\theta dz$	$\rho d\rho d\theta dz$

(i) $\nabla = \frac{1}{h_1}\frac{\partial}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}\hat{e}_3$

(ii) $\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$

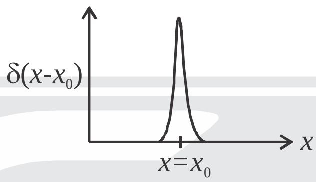
(iii) $\vec{\nabla}\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}^3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$

(iv) $\vec{\nabla}^2\phi=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$

Dirac-delta function:

(A) One dimensional delta function:

$$\delta(x-x_0)=\begin{cases}0 & \text{for } x\neq x_0\\ \infty & \text{for } x=x_0\end{cases}$$



Properties of delta function:

(1) The delta function is even $\delta(-x)=\delta(x)$

(2) $\delta(ax)=\frac{1}{|a|}\delta(x)\quad (a\neq 0)$

(3) $\int_{-\infty}^{\infty}\delta(x)\,dx=1$

(4) $\int_a^b f(x)\delta(x-x_0)\,dx=\begin{cases}f(x_0) & \text{if } a<x_0<b\\ 0 & \text{elsewhere}\end{cases}$

(5) $\int\delta(a-x)\delta(x-b)\,dx=\delta(a-b)$

(6) $\delta[(x-a)(x-b)]=\frac{1}{|a-b|}[\delta(x-a)+\delta(x-b)]\ (a\neq b)$

(7) $\delta(x^2-a^2)=\frac{1}{2|a|}[\delta(x-a)+\delta(x+a)]\ (a\neq 0)$

(8) $\int_a^b f(x)\delta^n(x-x_0)\,dx=(-1)^n f^n(x_0)\quad a<x_0<b$

(B) Three dimensional delta function:

(1) The three dimensional form delta function in cartesian coordinates is

$$\delta(\vec{r}-\vec{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$$

(2) In spherical coordinate

$$\delta(\vec{r}-\vec{r}_0)=\frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$$