Mathematics of Seismic Imaging

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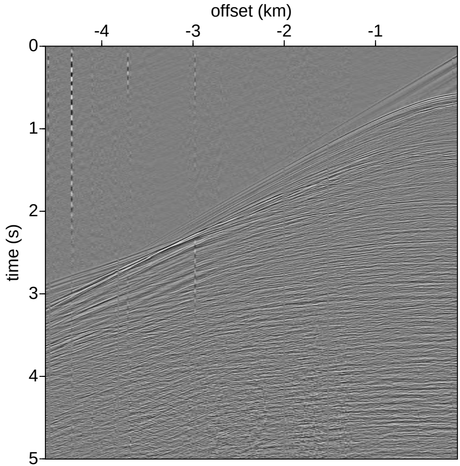
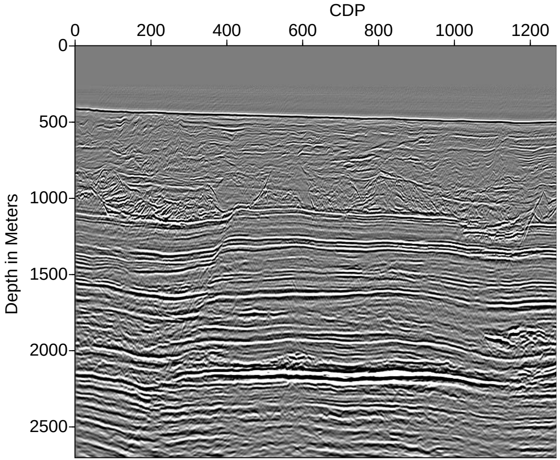
Introduction

How do you turn lots of this...



Introduction

into this?



Introduction

Main goal of these lectures: coherent mathematical view of reflection seismic imaging, as practiced in petroleum industry

- imaging = approximate solution of inverse problem for wave equation
- most practical imaging methods based on linearization ("perturbation theory")
- high frequency asymptotics ("microlocal analysis") key to understanding
- limitations of linearization lead to many open problems

Lots of mathematics - much yet to be created - with practical implications!

Agenda

Seismic inverse problem: the sedimentary Earth, reflection seismic measurements, the acoustic model, linearization, reflectors and reflections idealized via *harmonic analysis of singularities*

High frequency asymptotics: why adjoints of modeling operators are imaging operators ("Kirchhoff migration"). Beylkin-Rakesh-... theory of high frequency asymptotic inversion

Adjoint state imaging with the wave equation: reverse time and reverse depth

Geometric optics, Rakesh's construction, and asymptotic inversion w/ caustics and multipathing, imaging artifacts, and prestack migration après Claerbout.

A step beyond linearization: a mathematical framework for velocity analysis

Reflection seismology

aka active source seismology, seismic sounding/profiling

uses acoustic (sound) waves to probe the Earth's sedimentary crust

main exploration tool of oil & gas industry, also used in environmental and civil engineering (hazard detection, bedrock profiling) and academic geophysics (structure of crust and mantle)

highest resolution imaging technology for deep Earth exploration, in comparison with static (gravimetry, resistivity) or diffusive (active source EM) techniques - works because

waves transfer space-time resolved information from one place to another with (relatively) little loss

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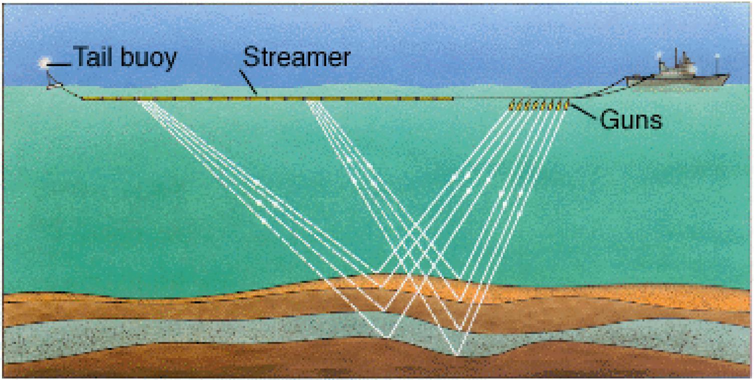
Reflection seismology

Three components:

- energy/sound source - creates wave traveling into subsurface
- receivers - record waves (echoes) reflected from subsurace
- recording and signal processing instrumentation



Reflection seismology

Three components:

- energy/sound source - creates wave traveling into subsurface
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Survey consists of many experiments = shots

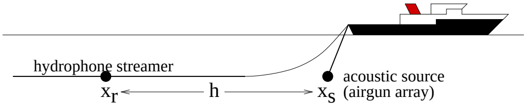
Each shot = use of one source, localized in time and space - position $\mathbf{x}_s$

Simultaneous recording of reflections as many localized receivers, positions $\mathbf{x}_s$, time interval $= 0 - O(10)$s after initiation of source.

Reflection seismology

Marine reflection seismology:

- typical energy source: *airgun array* - releases (array of) supersonically expanding bubbles of compressed air, generates sound pulse in water
- typical receivers: hydrophones (waterproof microphones) in one or more 5-10 km flexible streamer(s) - wired together 500 - 30000 *groups*
- survey ships - lots of recording, processing capacity



Land acquisition similar, but acquisition and processing are more complex. Vast bulk (90%+) of data acquired each year is marine.

Reflection seismology

Marine seismic data parameters:

- time t - $0 \leq t \leq t_{\max}$, $t_{\max} = 5 - 15$s
- source location $\mathbf{x}_s$ - 100 - 100000 distinct values
- receiver location $\mathbf{x}_r$
- typically the same range of *offsets* $= \mathbf{x}_r - \mathbf{x}_s$ for each shot - *half offset* $\mathbf{h} = \frac{\mathbf{x}_r - \mathbf{x}_s}{2}$, $h = |\mathbf{h}|$: 100 - 500000 values (typical: 5000)
- data values: microphone output (volts), filtered version of local pressure (force/area)

Reflection seismology

Idealized marine "streamer" geometry: $\mathbf{x}_s$ and $\mathbf{x}_r$ lie roughly on constant depth plane, source-receiver lines are parallel $\rightarrow$ 3 spatial degrees of freedom (eg. $\mathbf{x}_s, h$): *codimension 1*. [Other geometries are interesting, eg. ocean bottom cables, but streamer surveys still prevalent.]

How much data? Contemporary surveys may feature

- Simultaneous recording by multiple streamers (up to 12!)
- Many (roughly) parallel ship tracks ("lines")
- Recent development: Wide Angle Towed Streamer (WATS) survey - uses multiple survey ships for areal sampling of source and receiver positions
- single line ("2D") $\sim$ Gbytes; multiple lines ("3D") $\sim$ Tbytes; WATS $\sim$ Pbytes

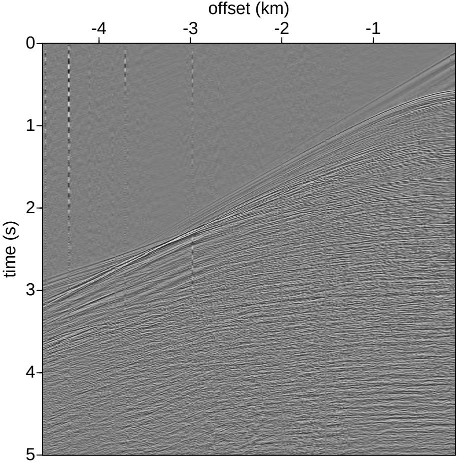
Distinguished data subsets

- *traces* = data for one source, one receiver: $t \mapsto d(x_r, t; x_s)$ - function of t, time series, single channel
- *gathers* or *bins* = subsets of traces, extracted from data after acquisition. Characterized by common value of an acquisition parameter

Examples:

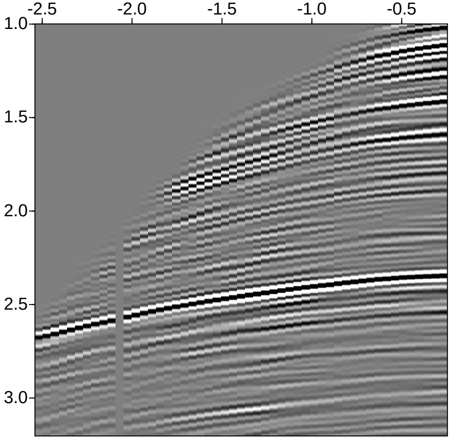
- shot (or common source) gather: traces w/ same shot location $\mathbf{x}_s$ (previous expls)
- offset (or common offset) gather: traces w/ same half offset h
- ...

Shot gather, Mississippi Canyon



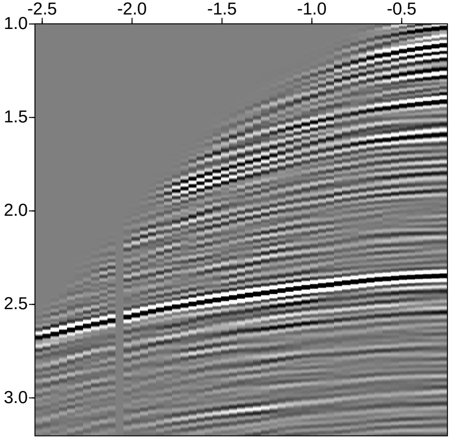
(thanks: Exxon)

Shot gather, Mississippi Canyon



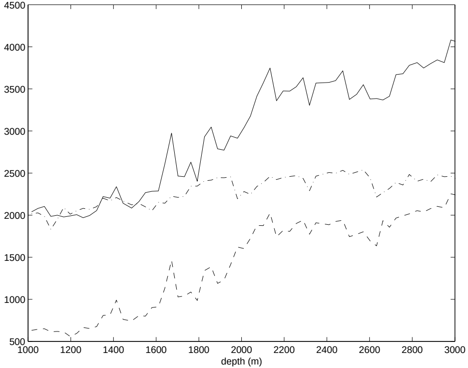
Lightly processed - bandpass filter 4-10-25-40 Hz, mute. Most striking visual characteristic: waves = coherent space-time structures ("reflections")

Shot gather, Mississippi Canyon



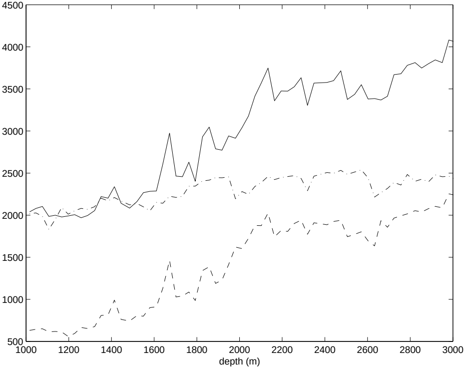
What features in the subsurface structure cause reflections? How to model?

Well logs: a “direct” view of the subsurface



Blocked logs from well in North Sea (thanks: Mobil R & D). Solid: p-wave velocity (m/s), dashed: s-wave velocity (m/s), dash-dot: density ($\mathrm{kg/m^3}$). “Blocked” means “averaged” (over 30 m windows). Original sample rate of log tool < 1 m.

Well logs: a "direct" view of the subsurface



You see:

- **Trends** = slow increase in velocities, density - scale of km
- **Reflectors** = jumps in velocities, density - scale of m or 10s of m

The Modeling Task

A useful model of the reflection seismology experiment must

- predict wave motion
- produce reflections from reflectors
- accomodate significant variation of wave velocity, material density,...

A *really good* model will also accomodate

- multiple wave modes, speeds
- material anisotropy
- attenuation, frequency dispersion of waves
- complex source, receiver characteristics

The Acoustic Model

Not *really good*, but good enough for this week and basis of most contemporary processing.

Relates $\rho(\mathbf{x})$= material density, $\lambda(\mathbf{x})$ = bulk modulus, $p(\mathbf{x},t)$= pressure, $\mathbf{v}(\mathbf{x},t)$ = particle velocity, $\mathbf{f}(\mathbf{x},t)$= force density (sound source):

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{f},$$

$$\frac{\partial p}{\partial t} = -\lambda \nabla \cdot \mathbf{v} \;(+\mathrm{i.c.'s, b.c.'s})$$

(compressional) wave speed $c = \sqrt{\frac{\lambda}{\rho}}$

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The Acoustic Model

acoustic field potential $u(\mathbf{x},t) = \int_{-\infty}^{t} ds\, p(\mathbf{x},s)$:

$$p = \frac{\partial u}{\partial t},\ \mathbf{v} = \frac{1}{\rho}\nabla u$$

Equivalent form: second order wave equation for potential

$$\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla u = \int_{-\infty}^{t} dt\, \nabla \cdot \left(\frac{\mathbf{f}}{\rho}\right) \equiv \frac{f}{\rho}$$

plus initial, boundary conditions.

The Acoustic Model

Further idealizations:

- density ρ is constant,
- source force density is *isotropic point radiator with known time dependence* ("source pulse" $w(t)$, typically of compact support)

$$f(\mathbf{x}, t; \mathbf{x}_s) = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

$\Rightarrow$ acoustic potential, pressure depends on source location $\mathbf{x}_s$ also.

Homogeneous acoustics

Suppose also that

- velocity c is constant

("homogeneous" acoustic medium - same stress-strain relation everywhere)

Explicit *causal* (= vanishing for $t << 0$) solution for 3D:

$$u(\mathbf{x}, t) = \frac{w(t - r/c)}{4\pi r}, \; r = |\mathbf{x} - \mathbf{x}_s|$$

[Proof: exercise!]

Nomenclature: expanding or outgoing spherical wave

Homogeneous acoustics

Also explicit solution (up to quadrature) in 2D - a bit more complicated (Poisson's formula - exercise: find it! eg. in Courant and Hilbert)

Looks like expanding circular wavefront for typical $w(t)$ [SIMULATION]

Observe: no reflections!!! [SIMULATION]

Upshot: if acoustic model is at all appropriate, must use non-constant c to explain observations.

Natural mathematical question: how nonconstant can c be and still permit "reasonable" solutions of wave equation?

Heterogeneous acoustics

Weak solution of Dirichlet problem in $\Omega \subset \mathbf{R}^3$ (similar treatment for other b. c.'s):

$$u \in C^1([0, T]; L^2(\Omega)) \cap C^0([0, T]; H_0^1(\Omega))$$

satisfying for any $\phi \in C_0^\infty((0, T) \times \Omega)$,

$$\int_0^T \int_\Omega dt\, dx \left\{ \frac{1}{\rho c^2} \frac{\partial u}{\partial t} \frac{\partial \phi}{\partial t} - \frac{1}{\rho} \nabla u \cdot \nabla \phi + \frac{1}{\rho} f \phi \right\} = 0$$

Theorem (Lions, 1972) Suppose that $\log \rho, \log c \in L^\infty(\Omega)$, $f \in L^2(\Omega \times \mathbf{R})$. Then weak solutions of Dirichlet problem exist; initial data

$$u(\cdot, 0) \in H_0^1(\Omega),\ \frac{\partial u}{\partial t}(\cdot, 0) \in L^2(\Omega)$$

uniquely determine them.

Key Ideas in Proof

1. Conservation of energy: first assume that $f \equiv 0$, set

$$E[u](t) = \frac{1}{2}\int_{\Omega}\left(\frac{1}{\rho c^2}p(\cdot,t)^2 + \rho|\mathbf{v}(\cdot,t)|^2\right)$$

$=$ elastic strain energy (potential $+$ kinetic)

$$= \frac{1}{2}\int_{\Omega}\left(\frac{1}{\rho c^2}\left(\frac{\partial u}{\partial t}(\cdot,t)\right)^2 + \frac{1}{\rho}|\nabla u(\cdot,t)|^2\right)$$

Then if u is smooth enough that integrations by parts and differentiations under integral sign make sense, easy to see that

$$\frac{dE[u]}{dt} = 0$$

Key Ideas in Proof

General case ($f \neq 0$): with help of Cauchy-Schwarz $\leq$,

$$\frac{dE[u]}{dt}(t) \leq \mathrm{const.} \left(E[u](t) + \int_0^t ds \int_\Omega dx\, f^2(\mathbf{x}, s) \right)$$

whence for $0 \leq t \leq T$,

$$E[u](t) \leq \mathrm{const.} \left(E[u](0) + \int_0^t ds \int_\Omega dx\, f^2(\mathbf{x}, s) \right)$$

(Gronwall's $\leq$)

const on RHS bounded by T, $\|\log \rho\|_{L^\infty(\Omega)}, \|\log c\|_{L^\infty(\Omega)}$

Key Ideas in Proof

Poincaré's $\le\ \Rightarrow$ "*a priori* estimate"

$$\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)^2} + \|u(\cdot,t)\|^2_{H^1(\Omega)}$$

$$\le \mathrm{const.}\left(\left\|\frac{\partial u}{\partial t}(\cdot,0)\right\|_{L^2(\Omega)^2} + \|u(\cdot,0)\|^2_{H^1(\Omega)} + \int_{-\infty}^{t} ds \int_{\Omega} dx\, f^2(\mathbf{x},s)\right)$$

Derivation presumed more smoothness than weak solutions have, ex def. First serious result:

Weak solutions obey same *a priori* estimate

Proof via approximation argument.

Corollary: Weak solutions uniquely determined by $t=0$ data

Key Ideas in Proof

2. Galerkin approximation: Pick increasing sequence of subspaces

$$W^0 \subset W^1 \subset W^2 \subset \ldots \subset H_0^1(\Omega)$$

so that

$$\cup_{n=0}^{\infty} W^n \, \mathrm{dense\,in}\, L^2(\Omega)$$

Typical example: piecewise linear Finite Element subspaces on sequence of meshes, each refinement of preceding.

Galerkin principle: find $u^n \in C^2([0,T], W^n)$ so that for any $\phi^n \in C^1([0,T], W^n)$,

$$\int_0^T \int_\Omega dt\, dx \left\{ \frac{1}{\rho c^2} \frac{\partial u^n}{\partial t} \frac{\partial \phi^n}{\partial t} - \frac{1}{\rho} \nabla u^n \cdot \nabla \phi^n + \frac{1}{\rho} f \phi^n \right\} = 0$$

Key Ideas in Proof

In terms of basis $\{\phi^n_m : m = 0, ..., N^n\}$ of W^n, write

$$u^n(t, \mathbf{x}) = \sum_{m=0}^{N^n} U^n_m(t)\phi^n_m(\mathbf{x})$$

Then integration by parts in $t \Rightarrow$ coefficient vector $U^n(t) = (U^n_0(t), ..., U^n_{N^n})^T$ satisfies ODE

$$M^n \frac{d^2 U^n}{dt^2} + K^n U^n = F^n$$

where

$$M^n_{i,j} = \int_\Omega \frac{1}{\rho c^2}\phi^n_i \phi^n_j,\ K^n_{i,j} = \int_\Omega \frac{1}{\rho}\nabla\phi^n_i \cdot \nabla\phi^n_j$$

and sim for F^n

Key Ideas in Proof

Assume temporarily that $f \in C^0([0,T], L^2(\Omega)) \subset L^2([0,T] \times \Omega)$ - then $F^n \in C^0([0,T], W^n)$, so...

basic theorem on ODEs $\Rightarrow$ existence of Galerkin approximation u^n.

Energy estimate for Galerkin approximation -

$$E[u^n](t) \leq \mathrm{const.} \left(E[u^n](0) + \int_0^t \|f(\cdot,t)\|^2_{L^2(\Omega)} \right)$$

constant independent of n.

Alaoglu Thm $\Rightarrow$ $\{u^n\}$ weakly precompact in $L^2([0,T], H^1_0(\Omega))$, $\{\partial u^n/\partial t\}$ weakly precompact in $L^2([0,T], L^2(\Omega))$, so can select weakly convergent sequence, limit $u \in L^2([0,T], H^1_0(\Omega))$, $\{\partial u/\partial t\} \in L^2([0,T], L^2(\Omega))$.

Key Ideas in Proof

Final cleanup of Galerkin existence argument:

- u is weak solution (necessarily the weak solution!)
- remove regularity assumption on f via density of $C^0([0,T],L^2(\Omega))$ in $L^2([0,T]\times\Omega)$, energy estimate

More *time* regularity of $f \Rightarrow$ more *time* regularity of u. If you want more *space* regularity, then coefficients must be more regular! (examples later)

See Stolk 2000 for details, Blazek et al. 2008 for similar results re symmetric hyperbolic systems

Reflection seismic inverse problem

Forward map S = time history of pressure for each source location $\mathbf{x}_s$ at receiver locations $\mathbf{x}_r$, *as function of c*

Reality: $\mathbf{x}_s$ samples finitely many points near surface of Earth $(z = 0)$, active receiver locations $\mathbf{x}_r$ may depend on source locations and are also discrete

but: sampling is *reasonably fine* (see plots!) so...

Idealization: $(\mathbf{x}_s, \mathbf{x}_r)$ range over 4-diml closed submfd with boundary Σ, source and receiver depths constant.

Reflection seismic inverse problem

(predicted seismic data), depends on velocity field $c(\mathbf{x})$:

$$\mathcal{F}[c] = p|_{\Sigma\times[0,T]}$$

Inverse problem: given *observed seismic data* $d \in L^2(\Sigma \times [0, T])$, find c so that

$$\mathcal{F}[c] \simeq d$$

This inverse problem is

- large scale - Tbytes of data, Pflops to simulate forward map
- nonlinear
- yields to no known direct attack (no “solution formula”)

Linearization

Almost all useful technology to date relies on linearization (aka perturbation theory, Born approximation,...): write $c = v(1+r)$ and treat r as relative first order perturbation about v, resulting in perturbation of presure field $\delta p = \frac{\partial \delta u}{\partial t} = 0, t \leq 0$, where

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u = \frac{2r}{v^2}\frac{\partial^2 u}{\partial t^2}$$

Define **linearized forward map** F by

$$F[v]r = \delta p|_{\Sigma\times[0,T]}$$

Analysis of $F[v]$ is the main content of contemporary reflection seismic theory.

Linearization in theory

Recall Lions-Stolk result: if $\log c \in L^\infty(\Omega)$ ($\rho = 1!$) and $f \in L^2(\Omega \times [0, T])$, then weak solution has *finite energy*, i.e.

$$u = u[c] \in C^1([0, T], L^2(\Omega)) \cap C^0([0, T], H_0^1(\Omega))$$

Suppose $\delta c \in L^\infty(\Omega)$, define δu by solving perturbational problem: set $v = c$, $r = \delta c / c$.

Linearization in theory

Stolk (2000): for small enough $h \in \mathbf{R}$,

$$\|u[c + h\delta c] - u[c] - \delta u\|_{C^0([0,T],L^2(\Omega))} = o(h)$$

Note "loss of derivative": error in Newton quotient is $o(1)$ in weaker norm than that of space of weak solns

Implication for $\mathcal{F}[c]$: under suitable circumstances ($c =$ const. near Σ - "marine" case),

$$\|\mathcal{F}[c]\|_{L^2(\Sigma\times[0,T])} = O(\|w\|_{L^2(\mathbf{R})})$$

but

$$\|\mathcal{F}[v(1+r)] - \mathcal{F}[v] - F[v]r\|_{L^2(\Sigma\times[0,T])} = O(\|w\|_{H^1(\mathbf{R})})$$

and these estimates are both sharp

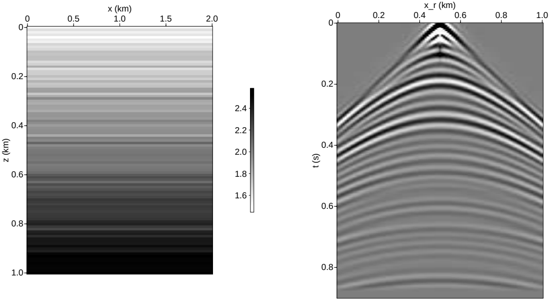
Linearization in practice

Physical intuition, numerical simulation, and not nearly enough mathematics: linearization error

$$\mathcal{F}[v(1+r)] - \mathcal{F}[v] - F[v]r$$

- *small* when v smooth, r rough or oscillatory on wavelength scale - well-separated scales
- *large* when v not smooth and/or r not oscillatory - poorly separated scales

Illustration: 2D finite difference simulation: shot gathers with typical marine seismic geometry. Smooth (linear) $v(x,z)$, oscillatory (random) $r(x,z)$ depending only on z("layered medium"). Source wavelet $w(t) =$ bandpass filter.



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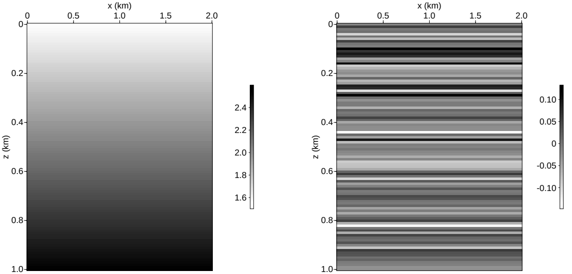
Left: Total velocity $c = v(1 + r)$ with smooth (linear) background $v(x, z)$, oscillatory (random) $r(x, z)$. Std dev of $r = 5\%$.
Right: Simulated seismic response ($\mathcal{F}[v(1 + r)]$), wavelet = bandpass filter 4-10-30-45 Hz. Simulator is (2,4) finite difference scheme.



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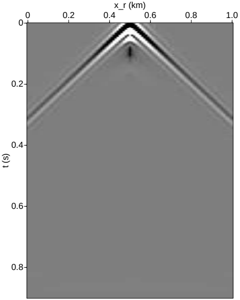
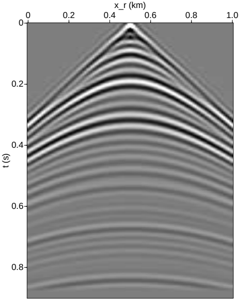
Decomposition of model in previous slide as smooth background (left, $v(x,z)$) plus rough perturbation (right, $r(x,z)$).





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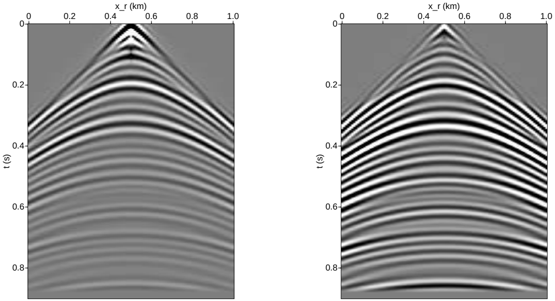
Left: Simulated seismic response of smooth model ($\mathcal{F}[v]$),
Right: Simulated linearized response, rough perturbation of smooth model ($F[v]r$)



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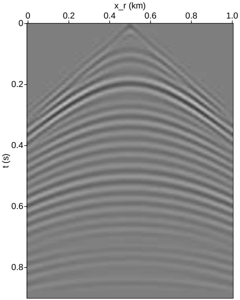
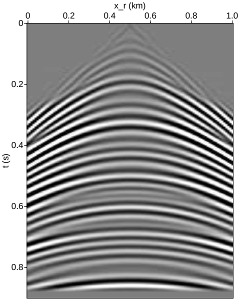
Left: Simulated seismic response of rough model $(\mathcal{F}[0.95v + r])$,
Right: Simulated linearized response, smooth perturbation of rough model $(F[0.95v + r]((0.05v)/(0.95v + r)))$





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Left: linearization error $(\mathcal{F}[v(1+r)] - \mathcal{F}[v] - F[v]r)$, rough perturbation of smooth background
Right: linearization error, smooth perturbation of rough background (plotted with same grey scale).

Summary

For the same pulse w,

- v smooth, r oscillatory $\Rightarrow F[v]r$ approximates **primary reflection** = result of one-time wave-material interaction (single scattering); error = **multiple reflections**, "not too large" if r is "not too big"
- v nonsmooth, r smooth $\Rightarrow$ error = *time shifts* - very large perturbations since waves are oscillatory.

For typical oscillatory w ($\|w\|_{H^1} >> \|w\|_{L^2}$), tends to imply that *in scale-separated case, effectively no loss of derivative!*

Math. justification available only in 1D (Lewis & S., 1991)

Velocity Analysis and Imaging

Velocity analysis problem = partially linearized inverse problem: given d find v, r so that

$$\mathcal{F}[v] + F[v]r \simeq d$$

Linearized inversion problem: given d and v, find r so that

$$F[v]r \simeq d - \mathcal{F}[v]$$

Imaging problem - relaxation of linearized inversion: given d and v, find an *image* r of "reality" = solution of linearized inversion problem

Velocity Analysis and Imaging

Last 20 years:

- much progress on imaging
- lots of progress on linearized inversion
- much less on velocity analysis

Interesting question: what's an image?

> *"...I know it when I see it." - Associate Justice Potter Stewart, 1964*

Aymptotic assumption

Linearization is accurate $\Leftrightarrow$ length scale of $v >> $ length scale of $r \simeq$ wavelength, properties of $F[v]$ dominated by those of $F_\delta[v]$ ($= F[v]$ with $w = \delta$). Implicit in migration concept (eg. Hagedoorn, 1954); explicit use: Cohen & Bleistein, SIAM JAM 1977.

Key idea: **reflectors** (rapid changes in r) emulate *singularities*; **reflections** (rapidly oscillating features in data) also emulate singularities.

NB: "everybody's favorite reflector": the smooth interface across which r jumps. *But* this is an oversimplification - reflectors in the Earth may be complex zones of rapid change, pehaps in all directions. More flexible notion needed!!

Wave Front Sets

Paley-Wiener characterization of local smoothness for distributions: $u \in \mathcal{D}'(\mathbf{R}^n)$ is smooth at $\mathbf{x}_0 \Leftrightarrow$ for some nbhd X of $\mathbf{x}_0$, any $\chi \in C_0^\infty(X)$ and $N \in \mathbf{N}$, any $\boldsymbol{\xi} \in \mathbf{R}^n, |\boldsymbol{\xi}| = 1$,

$$|\widehat{(\chi u)}(\tau\boldsymbol{\xi})| = O(\tau^{-N}),\ \tau \to \infty$$

Proof (sketch): smooth at $\mathbf{x}_0$ means: for some nbhd X, $\chi u \in C_0^\infty(\mathbf{R}^n)$ for any $\chi \in C_0^\infty(X) \Leftrightarrow$.

$$\widehat{\chi u}(\boldsymbol{\xi}) = \int dx\, e^{i\boldsymbol{\xi}\cdot\mathbf{x}} \chi(x) u(x)$$

Wave Front Sets

$$= \int dx\,(1+|\boldsymbol{\xi}|^2)^{-p}[(I-\nabla^2)^p e^{i\boldsymbol{\xi}\cdot\mathbf{x}}]\chi(x)u(x)$$

$$= (1+|\boldsymbol{\xi}|^2)^{-p}\int dx\,e^{i\boldsymbol{\xi}\cdot\mathbf{x}}[(I-\nabla^2)^p\chi(x)u(x)]$$

whence

$$|\widehat{\chi u}(\boldsymbol{\xi})| \le \mathrm{const.}(1+|\boldsymbol{\xi}|^2)^{-p}$$

where the const. depends on p, χ and u. For any N, choose p large enough, replace $\boldsymbol{\xi} \leftarrow \tau\boldsymbol{\xi}$, get desired $\le$.

Wave Front Sets

Harmonic analysis of singularities, *après* Hörmander: the **wave front set** $WF(u) \subset \mathbf{R}^n \times \mathbf{R}^n \setminus 0$ of $u \in \mathcal{D}'(\mathbf{R}^n)$ - captures orientation as well as position of singularities - microlocal smoothness

$(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF(u) \Leftrightarrow$, there is open nbhd $X \times \Xi \subset \mathbf{R}^n \times \mathbf{R}^n \setminus 0$ of $(\mathbf{x}_0, \boldsymbol{\xi}_0)$ so that for any $\chi \in C_0^\infty(\mathbf{R}^n), \mathrm{supp}\chi \subset X$, $N \in \mathbf{N}$, all $\boldsymbol{\xi} \in \Xi$ so that $|\boldsymbol{\xi}| = |\boldsymbol{\xi}_0|$,

$$|\widehat{\chi u}(\tau \boldsymbol{\xi})| = O(\tau^{-N})$$

Housekeeping chores

(i) note that the nbhds Ξ may naturally be taken to be *cones*

(ii) $WF(u)$ is invariant under chg. of coords if it is regarded as a subset of the *cotangent bundle* $T^*(\mathbf{R}^n)$ (i.e. the $\boldsymbol{\xi}$ components transform as covectors).

(iii) The standard example: if u jumps across the interface $\phi(\mathbf{x}) = 0$, otherwise smooth, then $WF(u) \subset \mathcal{N}_\phi = \{(\mathbf{x}, \boldsymbol{\xi}) : \phi(\mathbf{x}) = 0, \boldsymbol{\xi} || \nabla\phi(\mathbf{x})\}$ (*normal bundle* of $\phi = 0$)

[Good refs for basics on WF: Duistermaat, 1996; Taylor, 1981; Hörmander, 1983]

Housekeeping chores

Proof of (ii): follows from

(iv) Basic estimate for oscillatory integrals: suppose that $\psi \in C^\infty(\mathbf{R}^n), \nabla\psi(\mathbf{x}_0) \neq \mathbf{0}$, $(\mathbf{x}_0, \nabla\psi(\mathbf{x}_0)) \notin WF(u)$. Then for any $\chi \in C_0^\infty(\mathbf{R}^n)$ supported in small enough nbhd of $\mathbf{x}_0$, and any $N \in \mathbf{N}$,

$$\int dx\, e^{i\tau\psi(\mathbf{x})}\chi(\mathbf{x})u(\mathbf{x}) = O(\tau^{-N})$$

Housekeeping chores

Proof of (iv): choose nbhd $X \times \Xi$ of $(\mathbf{x}_0, \nabla\psi(\mathbf{x}_0))$ as in definition.

Choose $a \in C^\infty(\mathbf{R}^n \setminus \{0\})$ homogeneous of degree 0 ($a(\boldsymbol{\xi}) = a(\boldsymbol{\xi}/|\boldsymbol{\xi}|)$) for $|\boldsymbol{\xi}| > 1$ so that $a(\boldsymbol{\xi}) = 0$ if $\boldsymbol{\xi} \notin \Xi$ or $|\boldsymbol{\xi}| \le 1/2$, $a(\boldsymbol{\xi}) = 1$ if $|\boldsymbol{\xi}| > 1$ and $\boldsymbol{\xi} \in \Xi_1 \subset \Xi$, another conic nbhd of $\nabla\psi(\mathbf{x}_0)$.

Pick $\chi_1 \in C_0^\infty(\mathbf{R}^n)$ st $\chi_1 \equiv 1$ on $\mathrm{supp}\chi$, and write

$$\chi(x)u(x) = \chi_1(x)(2\pi)^{-n} \int d\boldsymbol{\xi}\, e^{i\mathbf{x}\cdot\boldsymbol{\xi}} \widehat{\chi u}(\boldsymbol{\xi})$$

$$= \chi_1(x)(2\pi)^{-n} \int d\boldsymbol{\xi}\, e^{i\mathbf{x}\cdot\boldsymbol{\xi}} g_1(\boldsymbol{\xi}) + \chi_1(x)(2\pi)^{-n} \int d\boldsymbol{\xi}\, e^{i\mathbf{x}\cdot\boldsymbol{\xi}} g_2(\boldsymbol{\xi})$$

in which $g_1 = a\widehat{\chi u}$, $g_2 = (1-a)\widehat{\chi u}$

Housekeeping chores

So

$$\int dx\, e^{i\tau\psi(\mathbf{x})}\chi(\mathbf{x})u(\mathbf{x}) = \sum_{j=1,2}\int dx \int d\boldsymbol{\xi}\, e^{i(\tau\psi(x)+\mathbf{x}\cdot\boldsymbol{\xi})}\chi_1(x)g_j(\boldsymbol{\xi})$$

For $\boldsymbol{\xi} \in \mathrm{supp}(1-a)$ (excludes a conic nbhd of $\nabla\psi(\mathbf{x}_0)$), can write

$$e^{i(\tau\psi(\mathbf{x})+\mathbf{x}\cdot\boldsymbol{\xi})} = [-i|\tau\nabla\psi(\mathbf{x})+\boldsymbol{\xi}|^{-2}(\tau\nabla\psi(\mathbf{x})+\boldsymbol{\xi})\cdot\nabla]^p e^{i(\tau\psi(\mathbf{x})+\mathbf{x}\cdot\boldsymbol{\xi})}$$

Can guarantee that $|\tau\nabla\psi(\mathbf{x})+\boldsymbol{\xi}| > 0$ by choosing $\mathrm{supp}\chi_1$ suff. small, so that in dom. of integration $\nabla\psi(\mathbf{x})$ is close to $\nabla\psi(\mathbf{x}_0)$. In fact, for $\boldsymbol{\xi} \in \mathrm{supp}(1-a)$, $\mathrm{supp}\chi_1$ small enough, and $\mathbf{x} \in \mathrm{supp}\chi_1$,

$$|\tau\nabla\psi(\mathbf{x})+\boldsymbol{\xi}| > C\tau$$

for some $C > 0$. Exercise: prove this!

Housekeeping chores

Substitute and integrate by parts, use above estimate to get

$$\left| \int dx \int d\boldsymbol{\xi}\, e^{i(\tau\psi(\mathbf{x})+\mathbf{x}\cdot\boldsymbol{\xi})}\chi_1(\mathbf{x})g_2(\boldsymbol{\xi}) \right| \le const.\tau^{-N}$$

for any N.

Note that for $\boldsymbol{\xi} \in \mathrm{supp}a$,

$$|\widehat{\chi u}(\boldsymbol{\xi})| \le \mathrm{const.}|\boldsymbol{\xi}|^{-p}$$

for any p (with p-dep. const, of course!). Follows that

$$h(bx) = \int d\boldsymbol{\xi}\, e^{i\mathbf{x}\cdot\boldsymbol{\xi}} g_1(\boldsymbol{\xi})$$

converges absolutely, also after differentiating any number of times under the integral sign, therefore $h \in C^\infty(\mathbf{R}^n)$, whence

$$\int dx \int d\boldsymbol{\xi}\, e^{i(\tau\psi(\mathbf{x})+\mathbf{x}\cdot\boldsymbol{\xi})}\chi_1(\mathbf{x})g_1(\boldsymbol{\xi}) = \int dx\, e^{i\tau\psi(\mathbf{x})}\chi_1(\mathbf{x})h(\mathbf{x})$$

Housekeeping chores

with integrand supported as near as you like to $\mathbf{x}_0$. Since $\nabla\psi(\mathbf{x}_0) \neq 0$, same is true of $\mathrm{supp}\chi_1$ provided this is chosen small enough; now use

$$e^{i\tau\psi(\mathbf{x})} = \tau^{-p}(-i|\nabla\psi(\mathbf{x})|^{-2}\nabla\psi(\mathbf{x}) \cdot \nabla)^p e^{i\tau\psi(\mathbf{x})}$$

and integration by parts again to show that this term is also $O(\tau^{-N})$ any N.

Housekeeping chores

Proof of (ii), for u integrable (Exercise: formulate and prove similar statement for distributions)

Equivalent statement: suppose that $F : U \rightarrow \mathbf{R}^n$ is a diffeomorphism on an open $U \subset \mathbf{R}^n$, $\mathrm{supp} u \subset F(U)$, $\mathbf{x}_0 \in U$, $\mathbf{y}_0 = F(\mathbf{x}_0)$, and $(\mathbf{y}_0, \eta_0) \notin WF(u)$.

Claim: then $(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF(u \circ F)$, where $\boldsymbol{\xi}_0 = DF(\mathbf{x}_0)^T \eta_0$.

Housekeeping chores

Need to show that if $\chi \in C_0^\infty(\mathbf{R}^n), \mathbf{x}_0 \in \mathrm{supp}\chi$ and small enough, then $\widehat{\chi u \circ F}(\tau\boldsymbol{\xi}) = O(\tau^{-N})$ any N for $\boldsymbol{\xi}$ conically near $\boldsymbol{\xi}_0$. From the change-of-variables formula

$$\widehat{\chi u \circ F}(\tau\boldsymbol{\xi}) = \int dx \chi(\mathbf{x})(u \circ F)(\mathbf{x})e^{i\tau\mathbf{x}\cdot\boldsymbol{\xi}}$$

$$= \int dy\, (\chi \circ F^{-1})(\mathbf{y})u(\mathbf{y})e^{i\tau\boldsymbol{\xi}\cdot F^{-1}(\mathbf{y})} \det D(F^{-1})(\mathbf{y})$$

Set $j = \chi \circ F^{-1} detD(F^{-1})$. Note: $j \in C_0^\infty(\mathbf{R}^n)$ supported in nbhd $\mathcal{V}$ of $\mathbf{y}_0$ if χ supported in $F^{-1}(\mathcal{V})$.

Housekeeping chores

MVT: for $\mathbf{y}$ close enough to $\mathbf{y}_0$,

$$F^{-1}(\mathbf{y}) = \mathbf{x}_0 + \int_0^1 d\sigma DF^{-1}(\mathbf{y}_0 + \sigma(\mathbf{y} - \mathbf{y}_0))(\mathbf{y} - \mathbf{y}_0)$$

Insert in exponent to get

$$\widehat{\chi u \circ F}(\tau \boldsymbol{\xi}) = e^{i\tau \mathbf{x}_0 \cdot \boldsymbol{\xi}} \int dy\, j(\mathbf{y}) u(\mathbf{y}) e^{i\tau\psi_{\boldsymbol{\xi}}(\mathbf{y})}$$

where

$$\psi_{\boldsymbol{\xi}}(\mathbf{y}) = (\mathbf{y} - \mathbf{y}_0) \cdot \int_0^1 d\sigma DF^{-1}(\mathbf{y}_0 + \sigma(\mathbf{y} - \mathbf{y}_0))^T \boldsymbol{\xi}$$

Since

$$\nabla \psi_{\boldsymbol{\xi}}(\mathbf{y}_0) = DF^{-1}(\mathbf{y}_0)\boldsymbol{\xi}$$

claim now follows from basic thm on oscillatory integrals.

Housekeeping chores

Proof of (iii): Function of compact supp, jumping across $\phi = 0$

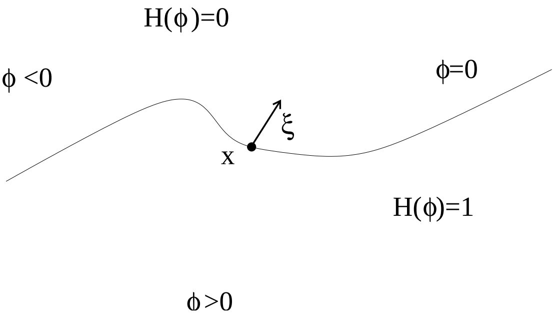
$$u = \chi H(\phi)$$

with χ smooth, $H =$ Heaviside function ($H(t) = 1, t > 0$ and $H(t) = 0, t < 0$).

Pick $\mathbf{x}_0$ with $\phi(\mathbf{x}_0) = 0$. Surface $\phi = 0$ regular near $\mathbf{x}_0$ if $\nabla\phi(\mathbf{x}_0) \neq 0$ - assume this.

Suffices to consider case of $\chi \in C_0^\infty(\mathbf{R}^n)$ of small support cont'g $\mathbf{x}_0$. Inverse Function Thm $\Rightarrow$ exists diffeo F mapping nbhd of $\mathbf{x}_0$ to nbhd of 0 so that $F(\mathbf{x}_0) = 0$ and $F_1(\mathbf{x}) = \phi(\mathbf{x})$. Fact (ii) $\Rightarrow$ reduce to case $\phi(\mathbf{x}) = x_1$ - Exercise: do this!

Wavefront set of a jump discontinuity



$$WF(H(\phi)) = \{(\mathbf{x}, \boldsymbol{\xi}) : \phi(\mathbf{x}) = 0,\ \boldsymbol{\xi} || \nabla\phi(\mathbf{x})\}$$

Formalizing the reflector concept

Key idea, restated: reflectors (or "reflecting elements") will be points in $WF(r)$. Reflections will be points in $WF(d)$.

These ideas lead to a usable definition of *image*: a reflectivity model $\tilde{r}$ is an image of r if $WF(\tilde{r}) \subset WF(r)$ (the closer to equality, the better the image).

Idealized **migration problem**: given d (hence $WF(d)$) deduce somehow a function which has *the right reflectors*, i.e. a function $\tilde{r}$ with $WF(\tilde{r}) \simeq WF(r)$.

NB: you're going to need v! ("It all depends on v(x,y,z)" - J. Claerbout)

Agenda

Seismic inverse problem: the sedimentary Earth, reflection seismic measurements, the acoustic model, linearization, reflectors and reflections idealized via *harmonic analysis of singularities*

High frequency asymptotics: why adjoints of modeling operators are imaging operators ("Kirchhoff migration"). Beylkin-Rakesh-... theory of high frequency asymptotic inversion

Adjoint state imaging with the wave equation: reverse time and reverse depth

Geometric optics, Rakesh's construction, and asymptotic inversion w/ caustics and multipathing, imaging artifacts, and prestack migration après Claerbout.

A step beyond linearization: a mathematical framework for velocity analysis

Microlocal property of differential operators

Suppose $u \in \mathcal{D}'(\mathbf{R}^n)$, $(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF(u)$, and $P(\mathbf{x}, D)$ is a partial differential operator:

$$P(\mathbf{x}, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

$$D = (D_1, ..., D_n),\ \ D_i = -i \frac{\partial}{\partial x_i}$$

$$\alpha = (\alpha_1, ..., \alpha_n),\ \ |\alpha| = \sum_i \alpha_i,$$

$$D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$$

Then $(\mathbf{x}_0, \xi_0) \notin WF(P(\mathbf{x}, D)u)$ [i.e.: $WF(Pu) \subset WF(u)$].

Proof

Choose $X \times \Xi$ as in the definition, $\phi \in \mathcal{D}(X)$ form the required Fourier transform

$$\int dx\, e^{i\mathbf{x}\cdot(\tau\xi)}\phi(\mathbf{x})P(\mathbf{x},D)u(\mathbf{x})$$

and start integrating by parts: eventually

$$= \sum_{|\alpha|\le m} \tau^{|\alpha|}\xi^{\alpha} \int dx\, e^{i\mathbf{x}\cdot(\tau\xi)}\phi_{\alpha}(\mathbf{x})u(\mathbf{x})$$

where $\phi_\alpha \in \mathcal{D}(X)$ is a linear combination of derivatives of ϕ and the a_αs. Since each integral is rapidly decreasing as $\tau \to \infty$ for $\xi \in \Xi$, it remains rapidly decreasing after multiplication by $\tau^{|\alpha|}$, and so does the sum. **Q. E. D.**

Integral representation of linearized operator

With $w = \delta$, acoustic potential u is same as Causal Green's function $G(\mathbf{x}, t; \mathbf{x}_s)$ = retarded fundamental solution:

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\mathbf{x}, t; \mathbf{x}_s) = \delta(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

and $G \equiv 0, t < 0$. Then $(w = \delta!)$ $p = \frac{\partial G}{\partial t}$, $\delta p = \frac{\partial \delta G}{\partial t}$, and

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) \delta G(\mathbf{x}, t; \mathbf{x}_s) = \frac{2}{v^2(\mathbf{x})}\frac{\partial^2 G}{\partial t^2}(\mathbf{x}, t; \mathbf{x}_s) r(\mathbf{x})$$

Simplification: from now on, define $F[v]r = \delta G|_{\mathbf{x}=\mathbf{x}_r}$ - i.e. lose a t-derivative. Duhamel's principle $\Rightarrow$

$$\delta G(\mathbf{x}_r, t; \mathbf{x}_s) = \int dx \, \frac{2r(\mathbf{x})}{v(\mathbf{x})^2} \int ds \, G(\mathbf{x}_r, t - s; \mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x}, s; \mathbf{x}_s)$$

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Add geometric optics...

Geometric optics approximation of G should be good, as v is smooth. Local version: if $\mathbf{x}$ "not too far" from $\mathbf{x}_s$, then

$$G(\mathbf{x}, t; \mathbf{x}_s) = a(\mathbf{x}; \mathbf{x}_s)\delta(t - \tau(\mathbf{x}; \mathbf{x}_s)) + R(\mathbf{x}, t; \mathbf{x}_s)$$

where the traveltime $\tau(\mathbf{x}; \mathbf{x}_s)$ solves the eikonal equation

$$v|\nabla\tau| = 1$$

$$\tau(\mathbf{x}; \mathbf{x}_s) \sim \frac{|\mathbf{x} - \mathbf{x}_s|}{v(\mathbf{x}_s)},\ \mathbf{x} \to \mathbf{x}_s$$

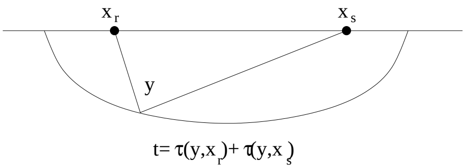
and the amplitude $a(\mathbf{x}; \mathbf{x}_s)$ solves the transport equation

$$\nabla \cdot (a^2 \nabla\tau) = 0$$

Simple Geometric Optics

"Not too far" means: there should be one and only one ray of geometric optics connecting each $\mathbf{x}_s$ or $\mathbf{x}_r$ to each $\mathbf{x} \in \mathrm{supp}\, r$.

Will call this the **simple geometric optics** assumption.



An oft-forgotten detail

All of this is meaningful only if the remainder R is small in a suitable sense: energy estimate (**Exercise!**) $\Rightarrow$

$$\int dx \int_0^T dt \, |R(\mathbf{x}, t; \mathbf{x}_s)|^2 \leq C \|v\|_{\mathrm{C}^4}$$

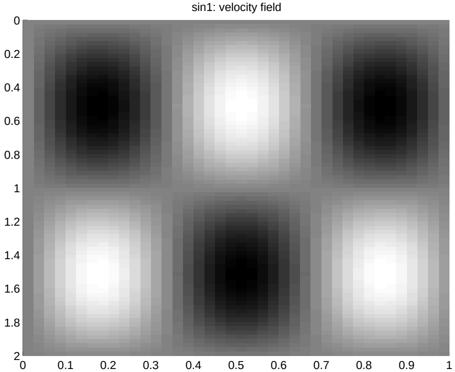
Numerics, and a caution

Numerical solution of eikonal, transport: ray tracing (Lagrangian), various sorts of upwind finite difference (Eulerian) methods. See eg. Sethian book, WWS 1999 MGSS notes (online) for details.

For “random but smooth” $v(\mathbf{x})$ with variance σ, more than one connecting ray occurs as soon as the distance is $O(\sigma^{-2/3})$. Such *multipathing* is invariably accompanied by the formation of a *caustic* (White, 1982).

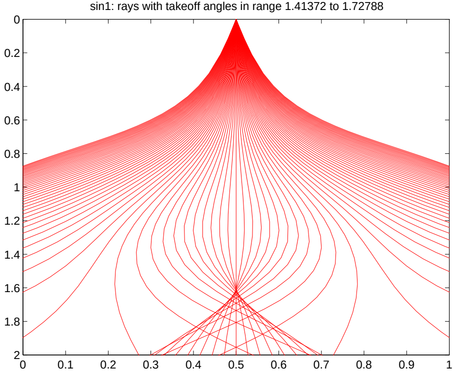
Upon caustic formation, the simple geometric optics field description above is no longer correct (Ludwig, 1966).

A caustic example (1)



2D Example of strong refraction: Sinusoidal velocity field
$v(x, z) = 1 + 0.2 \sin \frac{\pi z}{2} \sin 3\pi x$

A caustic example (2)



Rays in sinusoidal velocity field, source point = origin. Note formation of caustic, multiple rays to source point in lower center.

The linearized operator as Generalized Radon Transform

Assume: $\mathrm{supp}\, r$ contained in simple geometric optics domain (each point reached by unique ray from any source or receiver point).

Then distribution kernel K of $F[v]$ is

$$K(\mathbf{x}_r, t, \mathbf{x}_s; \mathbf{x}) = \int ds\, G(\mathbf{x}_r, t-s; \mathbf{x}) \frac{\partial^2 G}{\partial t^2}(\mathbf{x}, s; \mathbf{x}_s) \frac{2}{v^2(\mathbf{x})}$$

$$\simeq \int ds\, \frac{2a(\mathbf{x}_r, \mathbf{x}) a(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta'(t - s - \tau(\mathbf{x}_r, \mathbf{x})) \delta''(s - \tau(\mathbf{x}, \mathbf{x}_s))$$

$$= \frac{2a(\mathbf{x},\mathbf{x}_r)a(\mathbf{x},\mathbf{x}_s)}{v^2(\mathbf{x})}\delta''(t-\tau(\mathbf{x},\mathbf{x}_r)-\tau(\mathbf{x},\mathbf{x}_s))$$

provided that

$$\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{x}_r)+\nabla_{\mathbf{x}}\tau(\mathbf{x},\mathbf{x}_s) \neq 0$$

$\Leftrightarrow$ velocity at $\mathbf{x}$ of ray from $\mathbf{x}_s$ **not** negative of velocity of ray from $\mathbf{x}_r$ $\Leftrightarrow$ *no forward scattering*. [Gel'fand and Shilov, 1958 - when is pullback of distribution again a distribution?].

Q: What does $\simeq$ mean?

A: It means "differs by something smoother".

In theory, can complete the geometric optics approximation of the Green's function so that the difference is C^∞ - then the two sides have the same singularities, ie. the same wavefront set.

In practice, it's sufficient to make the difference just a bit smoother, so the first term of the geometric optics approximation (displayed above) suffices (can formalize this with modification of wavefront set defn).

These lectures will ignore the distinction.

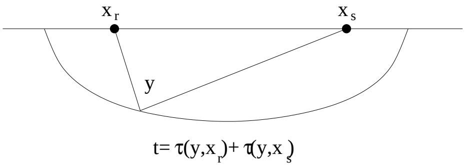
GRT = "Kirchhoff" modeling

So: for r supported in simple geometric optics domain, no forward scattering $\Rightarrow$

$$\delta G(\mathbf{x}_r, t; \mathbf{x}_s) \simeq$$

$$\frac{\partial^2}{\partial t^2} \int dx \, \frac{2r(\mathbf{x})}{v^2(\mathbf{x})} a(\mathbf{x}, \mathbf{x}_r) a(\mathbf{x}, \mathbf{x}_s) \delta(t - \tau(\mathbf{x}, \mathbf{x}_r) - \tau(\mathbf{x}, \mathbf{x}_s))$$

That is: pressure perturbation is sum (integral) of r over *reflection isochron* $\{\mathbf{x} : t = \tau(\mathbf{x}, \mathbf{x}_r) + \tau(\mathbf{x}, \mathbf{x}_s)\}$, w. weighting, filtering. Note: if v =const. then isochron is ellipsoid, as $\tau(\mathbf{x}_s, \mathbf{x}) = |\mathbf{x}_s - \mathbf{x}|/v$!



Zero Offset data and the Exploding Reflector

Zero offset data ($\mathbf{x}_s = \mathbf{x}_r$) is seldom actually measured (contrast radar, sonar!), but routinely *approximated* through *NMO-stack* (to be explained later).

Extracting image from zero offset data, rather than from all (100's) of offsets, is tremendous *data reduction* - when approximation is accurate, leads to excellent images.

Imaging basis: the *exploding reflector* model (Claerbout, 1970's).

For zero-offset data, distribution kernel of $F[v]$ is

$$K(\mathbf{x}_s, t, \mathbf{x}_s; \mathbf{x}) = \frac{\partial^2}{\partial t^2} \int ds \, \frac{2}{v^2(\mathbf{x})} G(\mathbf{x}_s, t - s; \mathbf{x}) G(\mathbf{x}, s; \mathbf{x}_s)$$

Under some circumstances (explained below), K ($= G$ time-convolved with itself) is "similar" (also explained) to $\tilde{G} =$ Green's function for $v/2$. Then

$$\delta G(\mathbf{x}_s, t; \mathbf{x}_s) \sim \frac{\partial^2}{\partial t^2} \int dx \, \tilde{G}(\mathbf{x}_s, t, \mathbf{x}) \frac{2r(\mathbf{x})}{v^2(\mathbf{x})}$$

$\sim$ solution w of

$$\left(\frac{4}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w = \delta(t) \frac{2r}{v^2}$$

Thus reflector "explodes" at time zero, resulting field propagates in "material" with velocity $v/2$.

Explain when the exploding reflector model "works", i.e. when G time-convolved with itself is "similar" to $\tilde{G}$ = Green's function for $v/2$. If supp r lies in simple geometry domain, then

$$K(\mathbf{x}_s, t, \mathbf{x}_s; \mathbf{x}) = \int ds \, \frac{2a^2(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta(t - s - \tau(\mathbf{x}_s, \mathbf{x})) \delta''(s - \tau(\mathbf{x}, \mathbf{x}_s))$$

$$= \frac{2a^2(\mathbf{x}, \mathbf{x}_s)}{v^2(\mathbf{x})} \delta''(t - 2\tau(\mathbf{x}, \mathbf{x}_s))$$

whereas the Green's function $\tilde{G}$ for $v/2$ is

$$\tilde{G}(\mathbf{x}, t; \mathbf{x}_s) = \tilde{a}(\mathbf{x}, \mathbf{x}_s) \delta(t - 2\tau(\mathbf{x}, \mathbf{x}_s))$$

(half velocity = double traveltime, same rays!).

Difference between effects of K, $\tilde{G}$: for each $\mathbf{x}_s$ scale r by smooth fcn - preserves $WF(r)$ hence $WF(F[v]r)$ and relation between them. Also: adjoints have same effect on WF sets.

Upshot: from imaging point of view (i.e. apart from amplitude, derivative (filter)), kernel of $F[v]$ restricted to zero offset is same as Green's function for $v/2$, *provided that simple geometry hypothesis holds:* only one ray connects each source point to each scattering point, ie. *no multipathing*.

See Claerbout, IEI, for examples which demonstrate that multipathing really does invalidate exploding reflector model.

Standard Processing

Inspirational interlude: the sort-of-layered theory = "Standard Processing"

Suppose were v,r functions of $z = x_3$ only, all sources and receivers at $z = 0$. Then the entire system is translation-invariant in $x_1, x_2 \Rightarrow$ Green's function G its perturbation δG, and the idealized data $\delta G|_{z=0}$ are really only functions of t and *half-offset* $h = |\mathbf{x}_s - \mathbf{x}_r|/2$. There would be *only one seismic experiment*, equivalent to any *common midpoint gather* ("CMP").

This isn't really true - *look at the data!!!* However it is *approximately* correct in many places in the world: CMPs change very slowly with midpoint $\mathbf{x}_m = (\mathbf{x}_r + \mathbf{x}_s)/2$.

Standard processing: treat each CMP *as if it were the result of an experiment performed over a layered medium*, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(\mathbf{x}_m, z), r = r(\mathbf{x}_m, z)$.

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s)$$

$$\simeq \int dx \frac{2r(z)}{v^2(z)} a(\mathbf{x}, x_r) a(\mathbf{x}, x_s) \delta''(t - \tau(\mathbf{x}, x_r) - \tau(\mathbf{x}, x_s))$$

$$= \int dz \, \frac{2r(z)}{v^2(z)} \int d\omega \int dx \omega^2 a(\mathbf{x}, x_r) a(\mathbf{x}, x_s) e^{i\omega(t - \tau(\mathbf{x}, x_r) - \tau(\mathbf{x}, x_s))}$$

Since we have already thrown away smoother (lower frequency) terms, do it again using *stationary phase.* Upshot (see 2000 MGSS notes for details): up to smoother (lower frequency) error,

$$F[v]r(h,t) \simeq A(z(h,t),h)R(z(h,t))$$

Here $z(h,t)$ is the inverse of the 2-way traveltime

$$t(h,z) = 2\tau((h,0,z),(0,0,0))$$

i.e. $z(t(h,z'),h) = z'$. R is (yet another version of) "reflectivity"

$$R(z) = \frac{1}{2}\frac{dr}{dz}(z)$$

That is, $F[v]$ *is a a derivative followed by a change of variable followed by multiplication by a smooth function*. Substitute t_0 (vertical travel time) for z (depth) and you get "Inverse NMO" ($t_0 \rightarrow (t,h)$). Will be sloppy and call $z \rightarrow (t,h)$ INMO.

Anatomy of an adjoint

$$\int dt \int dh\, d(t,h) F[v] r(t,h)$$

$$= \int dt \int dh\, d(t,h) A(z(t,h),h) R(z(t,h))$$

$$= \int dz\, R(z) \int dh\, \frac{\partial t}{\partial z}(z,h) A(z,h) d(t(z,h),h)$$

$$= \int dz\, r(z) (F[v]^* d)(z)$$

so $F[v]^* = -\frac{\partial}{\partial z} S M[v] N[v]$, where

- $N[v]$ = **NMO operator** $N[v]d(z,h) = d(t(z,h),h)$
- $M[v]$ = multiplication by $\frac{\partial t}{\partial z} A$
- S = **stacking operator** $Sf(z) = \int dh\, f(z,h)$

RICE

$$F[v]^*F[v]r(z) = -\frac{\partial}{\partial z}\left[\int dh\,\frac{dt}{dz}(z,h)A^2(z,h)\right]\frac{\partial}{\partial z}r(z)$$

Microlocal property of PDOs $\Rightarrow WF(F[v]^*F[v]r) \subset WF(r)$ i.e. $F[v]^*$ is an imaging operator.

If you leave out the amplitude factor ($M[v]$) and the derivatives, as is commonly done, then you get essentially the same expression - so (NMO, stack) is an imaging operator!

It's even easy to get an (asymptotic) inverse out of this - exercise for the reader.

Now make everything dependent on $\mathbf{x}_m$ and you've got standard processing. (end of layered interlude).

Multioffset ("Prestack") Imaging, après Beylkin

If $d = F[v]r$, then

$$F[v]^* d = F[v]^* F[v] r$$

In the layered case, $F[v]^*F[v]$ is an operator which preserves wave front sets. *Whenever $F[v]^*F[v]$ preserves wave front sets, $F[v]^*$ is an imaging operator.*

Beylkin, JMP 1985: for r supported in simple geometric optics domain,

- $WF(F_\delta[v]^* F_\delta[v] r) \subset WF(r)$
- if $S^{\rm obs} = S[v] + F_\delta[v] r$ (data consistent with linearized model), then $F_\delta[v]^*(S^{\rm obs} - S[v])$ is an image of r
- an operator $F_\delta[v]^\dagger$ exists for which $F_\delta[v]^\dagger(S^{\rm obs} - S[v]) - r$ is *smoother* than r, under some constraints on r - an *inverse modulo smoothing operators* or *parametrix*.

Outline of proof

Express $F[v]^*F[v]$ as "Kirchhoff modeling" followed by "Kirchhoff migration"; (ii) introduce Fourier transform; (iii) approximate for large wavenumbers using stationary phase, leads to representation of $F[v]^*F[v]$ modulo smoothing error as *pseudodifferential operator* ("ΨDO"):

$$F[v]^*F[v]r(\mathbf{x}) \simeq p(\mathbf{x},D)r(\mathbf{x}) \equiv \int d\xi\, p(\mathbf{x},\boldsymbol{\xi})e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\hat{r}(\boldsymbol{\xi})$$

in which $p \in C^\infty$, and for some m (the *order* of p), all multiindices α,β, and all compact $K \subset \mathbf{R}^n$, there exist constants $C_{\alpha,\beta,K} \geq 0$ for which

$$|D_{\mathbf{x}}^\alpha D_{\boldsymbol{\xi}}^\beta p(\mathbf{x},\boldsymbol{\xi})| \leq C_{\alpha,\beta,K}(1+|\boldsymbol{\xi}|)^{m-|\beta|},\ \mathbf{x}\in K$$

Explicit computation of **symbol** p - for details, see Notes on Math Foundations.

RICE

Microlocal Propertyof ΨDOs

if $p(x, D)$ is a ΨDO, $u \in \mathcal{E}'(\mathbf{R}^n)$ then
$WF(p(x, D)u) \subset WF(u)$.

Will prove this, from which imaging property of prestack Kirchhoff migration follows. First, a few other properties:

- differential operators are ΨDOs (easy - exercise)
- ΨDOs of order m form a module over $C^\infty(\mathbf{R}^n)$ (also easy)
- product of ΨDO order m, ΨDO order l = ΨDO order $\leq m + l$; adjoint of ΨDO order m is ΨDO order m (much harder)

Complete accounts of theory, many apps: books of Duistermaat, Taylor, Nirenberg, Treves, Hörmander.

Proof of Microlocal Property

Suppose $(\mathbf{x}_0, \boldsymbol{\xi}_0) \notin WF(u)$, choose neighborhoods X, Ξ as in defn, with Ξ conic. Need to choose analogous nbhds for $P(x, D)u$. Pick $\delta > 0$ so that $B_{3\delta}(\mathbf{x}_0) \subset X$, set $X' = B_\delta(\mathbf{x}_0)$.

Similarly pick $0 < \epsilon < 1/3$ so that $B_{3\epsilon}(\boldsymbol{\xi}_0/|\boldsymbol{\xi}_0|) \subset \Xi$, and chose $\Xi' = \{\tau\boldsymbol{\xi} : \boldsymbol{\xi} \in B_\epsilon(\boldsymbol{\xi}_0/|\boldsymbol{\xi}_0|), \tau > 0\}$.

Need to choose $\phi \in \mathcal{E}'(X')$, estimate $\mathcal{F}(\phi P(\mathbf{x}, D)u)$. Choose $\psi \in \mathcal{E}(X)$ so that $\psi \equiv 1$ on $B_{2\delta}(\mathbf{x}_0)$.

NB: this implies that if $\mathbf{x} \in X'$, $\psi(\mathbf{y}) \neq 1$ then $|\mathbf{x} - \mathbf{y}| \geq \delta$.

Write $u = (1-\psi)u + \psi u$. Claim: $\phi P(\mathbf{x}, D)((1-\psi)u)$ is smooth.

$$\phi(\mathbf{x})P(\mathbf{x}, D)((1-\psi)u))(\mathbf{x})$$

$$= \phi(\mathbf{x}) \int d\xi \, P(\mathbf{x}, \boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} \int dy \, (1-\psi(\mathbf{y}))u(\mathbf{y})e^{-i\mathbf{y}\cdot\boldsymbol{\xi}}$$

$$= \int d\xi \int dy \, P(\mathbf{x}, \boldsymbol{\xi})\phi(\mathbf{x})(1-\psi(\mathbf{y}))e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} u(\mathbf{y})$$

$$= \int d\xi \int dy \, (-\nabla_{\xi}^2)^M P(\mathbf{x}, \boldsymbol{\xi})\phi(\mathbf{x})(1-\psi(\mathbf{y}))|\mathbf{x}-\mathbf{y}|^{-2M} e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} u(\mathbf{y})$$

using the identity

$$e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}} = |\mathbf{x}-\mathbf{y}|^{-2}\left[-\nabla_\xi^2 e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi}}\right]$$

and integrating by parts $2M$ times in $\boldsymbol{\xi}$. This is permissible because $\phi(\mathbf{x})(1-\psi(\mathbf{y})) \neq 0 \Rightarrow |\mathbf{x}-\mathbf{y}| > \delta$.

According to the definition of ΨDO,

$$|(-\nabla_\xi^2)^M P(\mathbf{x},\boldsymbol{\xi})| \leq C|\boldsymbol{\xi}|^{m-2M}$$

For any K, the integral thus becomes absolutely convergent after K differentiations of the integrand, provided M is chosen large enough. Q.E.D. Claim.

This leaves us with $\phi P(\mathbf{x}, D)(\psi u)$. Pick $\eta \in \Xi'$ and w.l.o.g. scale $|\eta| = 1$.

Fourier transform:

$$\mathcal{F}(\phi P(\mathbf{x}, D)(\psi u))(\tau\eta) = \int dx \int d\xi \, P(\mathbf{x}, \boldsymbol{\xi})\phi(\mathbf{x})\hat{\psi u}(\xi) e^{i\mathbf{x}\cdot(\boldsymbol{\xi}-\tau\eta)}$$

Introduce $\tau\theta = \xi$, and rewrite this as

$$= \tau^n \int dx \int d\theta \, P(\mathbf{x}, \tau\theta)\phi(\mathbf{x})\hat{\psi u}(\tau\theta) e^{i\tau\mathbf{x}\cdot(\theta-\eta)}$$

Divide the domain of the inner integral into $\{\theta : |\theta - \eta| > \epsilon\}$ and its complement. Use

$$-\nabla_x^2 e^{i\tau\mathbf{x}\cdot(\theta-\eta)} = \tau^2|\theta-\eta|^2 e^{i\tau\mathbf{x}\cdot(\theta-\eta)}$$

Integrate by parts $2M$ times to estimate the first integral:

$$\tau^{n-2M}\left|\int dx \int_{|\theta-\eta|>\epsilon} d\theta\,(-\nabla_x^2)^M[P(\mathbf{x},\tau\theta)\phi(\mathbf{x})]\hat{\psi u}(\tau\theta) \times |\theta-\eta|^{-2M} e^{i\tau\mathbf{x}\cdot(\theta-\eta)}\right|$$

$$\leq C\tau^{n+m-2M}$$

m being the order of P. Thus the first integral is rapidly decreasing in τ.

For the second integral, note that $|\theta - \eta| \le \epsilon \Rightarrow \theta \in \Xi$, per the defn of Ξ'. Since $X \times \Xi$ is disjoint from the wavefront set of u, for a sequence of constants C_N, $|\hat{\psi u}(\tau\theta)| \le C_N \tau^{-N}$ uniformly for θ in the (compact) domain of integration, whence the second integral is also rapidly decreasing in τ. **Q. E. D.**

And that's why Kirchhoff migration works, at least in the simple geometric optics regime.

Asymptotic Prestack Inversion

Recall: in layered case,

$$F[v]r(h,t) \simeq A(z(h,t),h)\frac{1}{2}\frac{dr}{dz}(z(h,t))$$

$$F[v]^*d(z) \simeq -\frac{\partial}{\partial z}\int dh\, A(z,h)\frac{\partial t}{\partial z}(z,h)d(t(z,h),h)$$

$$F[v]^*F[v] = -\frac{\partial}{\partial z}\left[\int dh\, \frac{dt}{dz}(z,h)A^2(z,h)\right]\frac{\partial}{\partial z}$$

In particular, the normal operator $F[v]^*F[v]$ is an elliptic PDO.

Thus normal operator is *asymptotically invertible* and you can construct approximate least-squares solution to $F[v]r = d$:

$$\tilde{r} \simeq (F[v]^* F[v])^{-1} F[v]^* d$$

Relation between r and $\tilde{r}$: difference is *smoother* than either. Thus difference is *small* if r is oscillatory - consistent with conditions under which linearization is accurate.

Analogous construction in simple geometric optics case: due to Beylkin (1985).

Complication: $F[v]^* F[v]$ cannot be invertible - because $WF(F[v]^* F[v] r)$ generally quite a bit "smaller" than $WF(r)$.

Inversion aperture

$\Gamma[v] \subset \mathbf{R}^3 \times \mathbf{R}^3 - 0$:

if $WF(r) \subset \Gamma[v]$, then $WF(F[v]^* F[v] r) = WF(r)$ and $F[v]^* F[v]$ "acts invertible". [construction of $\Gamma[v]$ - later!]

Beylkin: with proper choice of amplitude $b(\mathbf{x}_r, t; \mathbf{x}_s)$, the modified Kirchhoff migration operator

$$F[v]^\dagger d(\mathbf{x}) =$$

$$\int\int\int dx_r\, dx_s\, dt\, b(\mathbf{x}_r, t; \mathbf{x}_s) \delta(t - \tau(\mathbf{x}; \mathbf{x}_s) - \tau(\mathbf{x}; \mathbf{x}_r)) d(\mathbf{x}_r, t; \mathbf{x}_s)$$

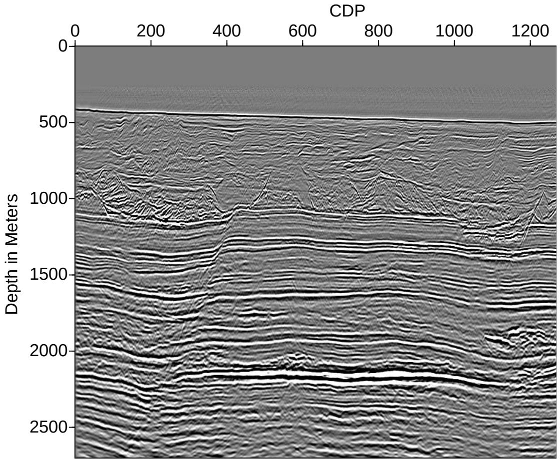
yields $F[v]^\dagger F[v] r \simeq r$ if $WF(r) \subset \Gamma[v]$

For details of Beylkin construction: Beylkin, 1985; Miller et al 1989; Bleistein, Cohen, and Stockwell 2000; WWS Math Foundations, MGSS notes 1998. All components are by-products of eikonal solution.

aka: Generalized Radon Transform ("GRT") inversion, Ray-Born inversion, migration/inversion, true amplitude migration,...

Many extensions, eg. to elasticity: Bleistein, Burridge, deHoop, Lambaré,...

Apparent limitation: construction relies on simple geometric optics (no multipathing) - how much of this can be rescued? cf. Part III.



Example of GRT Inversion (application of $F[v]^\dagger$): K. Araya (1995), "2.5D" inversion of marine streamer data from Gulf of Mexico: 500 source positions, 120 receiver channels, 750 Mb.

Agenda

Seismic inverse problem: the sedimentary Earth, reflection seismic measurements, the acoustic model, linearization, reflectors and reflections idealized via *harmonic analysis of singularities*

High frequency asymptotics: why adjoints of modeling operators are imaging operators ("Kirchhoff migration"). Beylkin-Rakesh-... theory of high frequency asymptotic inversion

Adjoint state imaging with the wave equation: reverse time and reverse depth

Geometric optics, Rakesh's construction, and asymptotic inversion w/ caustics and multipathing, imaging artifacts, and prestack migration après Claerbout.

A step beyond linearization: a mathematical framework for velocity analysis

Wave Equation Migration

Techniques for computing $F[v]^*$:

(i) Reverse time

(ii) Reverse depth

Reverse Time Migration, Zero Offset

Start with the zero-offset case - easier, but only if you replace it with the exploding reflector model, which replaces $F[v]$ by

$$\tilde{F}[v]r(\mathbf{x}_s, t) = w(\mathbf{x}_s, t),\ \mathbf{x}_s \in X_s, 0 \le t \le T$$

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) w = \delta(t)\frac{2r}{v^2},\ w \equiv 0,\ t < 0$$

To compute the adjoint, start with its definition: choose $d \in \mathcal{E}(X_s \times (0, T))$, so that

$$< \tilde{F}[v]^* d, r >=< d, \tilde{F}[v] r >$$

$$= \int_{X_s} dx_s \int_0^T dt\, d(\mathbf{x}_s, t) w(\mathbf{x}_s, t)$$

The only thing you know about w is that it solves a wave equation with r on the RHS. To get this fact into play, (i) rewrite the integral as a space-time integral:

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \int_{X_s} dx_s \, d(\mathbf{x}_s, t)\delta(\mathbf{x} - \mathbf{x}_s) w(\mathbf{x}, t)$$

(ii) write the other factor in the integrand as the image of a field q under the (adjoint of the) wave operator (it's self-adjoint), that is,

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{x}, t) = \int_{X_s} dx_s \, d(\mathbf{x}_s, t)\delta(\mathbf{x} - \mathbf{x}_s)$$

so

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \left[\left(\frac{4}{v^2(\mathbf{x})}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{x}, t)\right] w(\mathbf{x}, t)$$

(iii) integrate by parts

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \left[\left(\frac{4}{v^2(\mathbf{x})} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) w(\mathbf{x}, t) \right] q(\mathbf{x}, t)$$

which works if $q \equiv 0,\ t > T$ (*final value condition*); (iv) use the wave equation for w

$$= \int_{\mathbf{R}^3} dx \int_0^T dt \, \frac{2}{v(\mathbf{x})^2} r(\mathbf{x}) \delta(t) q(\mathbf{x}, t)$$

(v) observe that you have computed the adjoint:

$$= \int_{\mathbf{R}^3} dx \, r(\mathbf{x}) \left[\frac{2}{v(\mathbf{x})^2} q(\mathbf{x}, 0) \right] =< r, \tilde{F}[v]^* d >$$

i.e.

$$\tilde{F}[v]^* d = \frac{2}{v(\mathbf{x})^2} q(\mathbf{x}, 0)$$

Summary of the computation, with the usual description:

- Use that data as sources, backpropagate in time - i.e. solve the final value ("reverse time") problem

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) q(\mathbf{x},t) = \int_{X_s} dx_s\, d(\mathbf{x}_s,t)\delta(\mathbf{x}-\mathbf{x}_s),\ \ q \equiv 0,\ t > T$$

- read out the "image" (= adjoint output) at $t = 0$:

$$\tilde{F}[v]^* d = \frac{2}{v(\mathbf{x})^2} q(\mathbf{x},0)$$

Note: The adjoint (time-reversed) field q is *not* the physical field (δu) run backwards in time, contrary to some imputations in the literature.

Historical Remarks

- Known as “two way reverse time finite difference poststack migration” in geophysical literature (Whitmore, 1982)
- uses full (two way) wave equation, propagates adjoint field backwards in time, generally implemented using finite difference discretization.
- Same as “adjoint state method”, Lions 1968, Chavent 1974 for control and inverse problems for PDEs - much earlier for control of ODEs - Lailly, Tarantola ’80s.
- My buddy Tapia says: all you’re doing is transposing a matrix! True (after discretization), but it’s important that these matrices are triangular, so can be implemented by recursions - forward for simulation, backwards for adjoint.

RICE

Reverse Time Migration, Prestack

A slightly messier computation computes the adjoint of $F[v]$ (i.e. multioffset or *prestack* migration):

$$F[v]^* d(\mathbf{x}) = -\frac{2}{v(\mathbf{x})} \int dx_s \int_0^T dt \left(\frac{\partial q}{\partial t} \nabla^2 u \right) (\mathbf{x}, t; \mathbf{x}_s)$$

where *adjoint field* q satisfies $q \equiv 0,\ t \geq T$ and

$$\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q(\mathbf{x}, t; \mathbf{x}_s) = \int dx_r\, d(\mathbf{x}_r, t; \mathbf{x}_s) \delta(\mathbf{x} - \mathbf{x}_r)$$

Proof

$$< F[v]^* d, r > = < d, F[v] r >$$

$$= \int \int dx_s \, dx_r \int_0^T dt \, d(\mathbf{x}_r, t; \mathbf{x}_s) \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s)$$

$$= \int dx_s \int dx \int_0^T dt \left\{ \int dx_r \, d(\mathbf{x}_r, t; \mathbf{x}_s) \delta(\mathbf{x} - \mathbf{x}_r) \right\} \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

$$= \int dx_s \int dx \int_0^T dt \left[\left(\frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) q \right] \frac{\partial \delta u}{\partial t}(\mathbf{x}, t; \mathbf{x}_s)$$

RICE

$$= -\int dx_s \int dx \int_0^T dt \left[\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u\right]\frac{\partial q}{\partial t}(\mathbf{x},t;\mathbf{x}_s)$$

(boundary terms in integration by parts vanish because (i) $\delta u \equiv 0,\ t << 0$; (ii) $q \equiv 0,\ t >> 0$; (iii) both vanish for large $\mathbf{x}$, at each t)

$$= -\int dx_s \int dx \int_0^T dt \left(\frac{2r}{v^2}\frac{\partial^2 u}{\partial t^2}\frac{\partial q}{\partial t}\right)(\mathbf{x},t;\mathbf{x}_s)$$

$$= -\int dx_s \int dx\, r(\mathbf{x})\frac{2}{v^2(\mathbf{x})}\int_0^T dt \left(\frac{\partial^2 u}{\partial t^2}\frac{\partial q}{\partial t}\right)(\mathbf{x},t;\mathbf{x}_s)$$

$$=< r, F[v]^* d >$$

q.e.d.

Implementation

Algorithm: finite difference or finite element discretization in $\mathbf{x}$, finite difference time stepping.

- For each $\mathbf{x}_s$, solve wave equation for u forward in t, record final (t=T) Cauchy data, also (for example) Dirichlet boundary data.
- Step u and q backwards in time together; at each time step, data serves as source for q ("backpropagate data")
- During backwards time stepping, accumulate (approximations to)

$$Q(\mathbf{x}) += \frac{2}{v^2(\mathbf{x})} \int_0^T dt \left(\frac{\partial^2 u}{\partial t^2} \frac{\partial q}{\partial t} \right) (\mathbf{x}, t; \mathbf{x}_s)$$

  ("crosscorrelate reference and backpropagated field").
- next $\mathbf{x}_s$ - after last $\mathbf{x}_s$, $F[v]^* d = Q$.

Reverse Depth Migration, Zero Offset

aka: depth extrapolation, downward continuation, or simply "wave equation migration".

Introduced by Claerbout, early 70's ("swimming pool equation"). Again, assume exploding reflector model:

$$\tilde{F}[v]r(\mathbf{x}_s,t) = w(\mathbf{x}_s,t),\ \mathbf{x}_s \in X_s, 0 \le t \le T$$

$$\left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) w = \delta(t)\frac{2r}{v^2},\ w \equiv 0,\ t < 0$$

Basic idea: 2nd order wave equation permits waves to move in all directions, but waves carrying reflected energy are (mostly) moving *up*. Should satisfy a 1st order equation for wave motion in one direction.

Coming up...

For the moment use 2D notation $\mathbf{x} = (x, z)$ etc. Write wave equation as evolution equation in z:

$$\frac{\partial^2 w}{\partial z^2} - \left(\frac{4}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) w = -\delta(t)\frac{2r}{v^2}$$

Suppose that you could take the square root of the operator in parentheses - call it B. Then the LHS of the wave equation becomes

$$\left(\frac{\partial}{\partial z} - B\right)\left(\frac{\partial}{\partial z} + B\right) w = -\delta(t)\frac{2r}{v^2}$$

so setting $\tilde{w} = \left(\frac{\partial}{\partial z} + B\right) w$ you get

$$\left(\frac{\partial}{\partial z} - B\right)\tilde{w} = -\delta(t)\frac{2r}{v^2}$$

Some issues

This *might* be the required equation for upcoming waves.

Two major problems: (i) how the h–l do you take the square root of a PDO?

(ii) what guarantees that the equation just written governs upcoming waves?

Answers to be found in the theory of ΨDOs!

Classical ΨDOs

Important *subclass* of *classical* ΨDOs: those whose ("classical") symbols have asymptotic expansions:

$$p(\mathbf{x},\boldsymbol{\xi}) \sim \sum_{j\le m} p_j(\mathbf{x},\boldsymbol{\xi}),\ |\boldsymbol{\xi}| \to \infty$$

in which p_j is *homogeneous in* $\boldsymbol{\xi}$ *of degree* j:

$$p_j(\mathbf{x},\tau\boldsymbol{\xi}) = \tau^j p_j(\mathbf{x},\tau\boldsymbol{\xi}),\ \tau, |\boldsymbol{\xi}| \ge 1$$

The *principal symbol* is the homogeneous term of highest degree, i.e. p_m above.

Products of ΨDOs are ΨDOs.

Classical ΨDOs have more complete *calculus*, including prescriptions for "computing" adjoints, products, and the like. From now on unless otherwise stated, all ΨDOs are classical.

Product rule for ΨDOs: if p^1, p^2 are classical,

$$p^1(\mathbf{x},\boldsymbol{\xi}) = \sum_{j\le m^1} p^1_j(\mathbf{x},\boldsymbol{\xi}),\ p^2(\mathbf{x},\boldsymbol{\xi}) = \sum_{j\le m^2} p^2_j(\mathbf{x},\boldsymbol{\xi})$$

then so is $p^1(\mathbf{x},D)p^2(\mathbf{x},D)$, and its principal symbol is $p^1_{m^1}(\mathbf{x},\xi)p^2_{m^2}(\mathbf{x},\xi)$, and there is an algorithm for computing the rest of the expansion.

In an open neighborhood $X\times\Xi$ of $(\mathbf{x}_0,\boldsymbol{\xi}_0)$, symbol of $p^1(\mathbf{x},D)p^2(\mathbf{x},D)$ depends only on symbols of p^1, p^2 in $X\times\Xi$.

RICE

Consequence: if $a(\mathbf{x}, D)$ has an asymptotic expansion and is of order $m \in \mathbf{R}$, and $a_m(\mathbf{x}_0, \boldsymbol{\xi}_0) > 0$ in $\mathcal{P} \subset \mathbf{R}^n \times \mathbf{R}^n - 0$, then there exists $b(\mathbf{x}, D)$ of order $m/2$ with asymptotic expansion for which

$$(a(\mathbf{x}, D) - b(\mathbf{x}, D)b(\mathbf{x}, D))u \in \mathcal{E}(\mathbf{R}^n)$$

for any $u \in \mathcal{E}'(\mathbf{R}^n)$ with $WF(u) \subset \mathcal{P}$.

Moreover, $b_{m/2}(\mathbf{x}, \boldsymbol{\xi}) = \sqrt{a_m(\mathbf{x}, \boldsymbol{\xi})},\ (\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{P}$. Will call b a *microlocal square root* of a.

Similar construction: if $a(\mathbf{x}, \boldsymbol{\xi}) \neq 0$ in $\mathcal{P}$, then there is $c(\mathbf{x}, D)$ of order $-m$ so that

$$c(\mathbf{x}, D)a(\mathbf{x}, D)u - u,\ a(\mathbf{x}, D)c(\mathbf{x}, D)u - u \in \mathcal{E}(\mathbf{R}^n)$$

for any $u \in \mathcal{E}'(\mathbf{R}^n)$ with $WF(u) \subset \mathcal{P}$.

Moreover, $c_{-m}(\mathbf{x}, \boldsymbol{\xi}) = 1/a_m(\mathbf{x}, \boldsymbol{\xi}),\ (\mathbf{x}, \boldsymbol{\xi}) \in \mathcal{P}$. Will call b a *microlocal inverse* of a.

Application: the Square Root Operator

$$a(x,z,D_t,D_x) = \frac{\partial^2}{\partial x^2} - \frac{4}{v(x,z)^2}\frac{\partial^2}{\partial t^2} = \frac{4}{v(x,z)^2}D_t^2 - D_x^2$$

is

$$a(x,z,\tau,\xi) = \frac{4}{v(x,z)^2}\tau^2 - \xi^2$$

For $\delta > 0$, set

$$\mathcal{P}_\delta(z) = \left\{(x,t,\xi,\tau) : \frac{4}{v(x,z)^2}\tau^2 > (1+\delta)\xi^2\right\}$$

The SSR Operator

Then according to the last slide, there is an order 1 ΨDO-valued function of z, $b(x,z,D_t,D_x)$, with principal symbol

$$b_1(x,z,\tau,\xi) = \sqrt{\frac{4}{v(x,z)^2}\tau^2 - \xi^2} = \tau\sqrt{\frac{4}{v(x,z)^2} - \frac{\xi^2}{\tau^2}},\ (x,t,\xi,\tau) \in \mathcal{P}_\delta($$

for which $a(x,z,D_t,D_x)u \simeq b(x,z,D_t,D_x)b(x,z,D_t,D_x)u$ if $WF(u) \subset \mathcal{P}_\delta(z)$.

b is the world-famous **single square root** ("SSR") operator - see Claerbout, IEI.

The SSR Assumption

To what extent has this construction factored the wave operator:

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t)\right)\left(\frac{\partial}{\partial z} + ib(x,z,D_x,D_t)\right)$$

$$= \frac{\partial^2}{\partial z^2} + b(x,z,D_x,D_t)b(x,z,D_x,D_t) + \frac{\partial b}{\partial z}(x,z,D_x,D_t)$$

SSR Assumption: For some $\delta > 0$, the wavefield w satisfies

$$(x,z,t,\xi,\zeta,\tau) \in WF(w) \;\Rightarrow\; (x,t,\xi,\tau) \in \mathcal{P}_\delta(z) \text{ and } \zeta\tau > 0$$

This statement has a ray-theoretic interpretation (which will eventually make sense): rays carrying significant energy are nowhere horizontal. Along any such ray, z decreases as t increases - *coming up!*

$$\tilde{w}(x,z,t) = \left(\frac{\partial}{\partial z} + ib(x,z,D_x,D_t)\right) w(x,z,t)$$

$$b(x,z,D_x,D_t)b(x,z,D_x,D_t)w \simeq \left(\frac{4}{v(x,z)^2}D_t^2 - D_x^2\right) w$$

with a smooth error, so

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t)\right)\tilde{w}(x,z,t) = -\frac{2r(x,z)}{v(x,z)^2}\delta(t)$$

$$+i\left(\frac{\partial}{\partial z}b(x,z,D_x,D_t)\right)w(x,z,t)$$

(since b depends on z, the z deriv. does not commute with b). So $\tilde{w} = \tilde{w}_0 + \tilde{w}_1$, where

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t)\right)\tilde{w}_0(x,z,t) = -\frac{2r(x,z)}{v(x,z)^2}\delta(t)$$

(this is the **SSR modeling equation**)

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t)\right)\tilde{w}_1(x,z,t) = i\left(\frac{\partial}{\partial z}b(x,z,D_x,D_t)\right)w(x,z,t)$$

Claim: $WF(\tilde{w}_1) \subset WF(w)$. Granted this $\Rightarrow WF(\tilde{w}_0) \subset WF(w)$ also.

Upshot: SSR modeling

$$\tilde{F}_0[v]r(x_s, z_s, t) = \tilde{w}_0(x_s, z_s, t)$$

produces the same singularities (i.e. the same waves) as exploding reflector modeling, so is as good a basis for migration.

SSR migration: assume that sources all lie on $z_s = 0$.

$$< \tilde{F}_0[v]^* d, r >=< d, \tilde{F}_0[v]r >$$

$$= \int dx_s \int dt\, d(x_s, t)\tilde{w}_0(x_s, 0, t)$$

$$= \int dx_s \int dt \int dz \, d(\bar{x}_s, t)\delta(z)\tilde{w}_0(x_s, z, t)$$

Define the adjoint field q by

$$\left(\frac{\partial}{\partial z} - b(x, z, D_x, D_t)\right) q(x, z, t) = d(x, t)\delta(z), \;\; q(x, z, t) \equiv 0, \; z < 0$$

which is equivalent to solving the initial value problem

$$\left(\frac{\partial}{\partial z} - ib(x, z, D_x, D_t)\right) q(x, z, t) = 0, \; z > 0; , \;\; q(x, 0, t) = d(x, t)$$

Insert in expression for inner product, integrate by parts, use self-adjointness of b, get

$$< d, \tilde{F}_0[v]r >= \int dx \int dz \, \frac{2r(x, z)}{v(x, z)^2} q(x, z, 0)$$

whence

$$\tilde{F}_0[v]^* d(x,z) = \frac{2}{v(x,z)^2} q(x,z,0)$$

Standard description of the SSR migration algorithm:

- downward continue data (i.e. solve for q)
- image at $t = 0$.

The art of SSR migration: computable approximations to $b(x, z, D_x, D_t)$ - swimming pool operator, many successors.

Proof of the Claim

Unfinished business: proof of claim

Depends on celebrated **Propagation of Singularities** theorem of Hörmander (1970).

Given symbol $p(\mathbf{x}, \boldsymbol{\xi})$, order m, with asymptotic expansion, define *bicharateristics* as solutions $(\mathbf{x}(t), \boldsymbol{\xi}(t))$ of Hamiltonian system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial p}{\partial \boldsymbol{\xi}}(\mathbf{x}, \boldsymbol{\xi}), \;\; \frac{d\boldsymbol{\xi}}{dt} = -\frac{\partial p}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\xi})$$

with $p(\mathbf{x}(t), \boldsymbol{\xi}(t)) \equiv 0$.

Theorem: Suppose $p(\mathbf{x}, D)u = f$, and suppose that for $t_0 \le t \le t_1$, $(\mathbf{x}(t), \boldsymbol{\xi}(t)) \notin WF(f)$. Then either $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \le t \le t_1\} \subset WF(u)$ or $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \le t \le t_1\} \subset T^*(\mathbf{R}^n) - WF(u)$.

P of S has at least two distinct proofs:

- Nirenberg, 1972
- Hörmander, 1970 (in Taylor, 1981)

Proof of claim: check that bicharacteristics for SSR operator are just upcoming rays of geom. optics for wave equation. These pass into $t < 0$ where RHS is smooth, also initial condn at large z is smooth - so each ray has one "end" outside of $WF(\tilde{w}_1)$. If ray carries singularity, must pass of WF of w, but then it's entirely contained by P of S applied to w. **q. e. d.**

Reverse Depth Migration, Prestack

Nonzero offset ("prestack"): starting point is integral representation of the scattered field

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int dx \, \frac{2r(\mathbf{x})}{v(\mathbf{x})^2} \int ds \, G(\mathbf{x}_r, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

By analogy with zero offset case, would like to view this as "exploding reflectors in both directions": reflectors propagate energy upward to sources and to receivers.

However can't do this because reflection location is *same* for both.

The “survey sinking” idea

Bold stroke: introduce a new space variable $\mathbf{y}$ (a “sunken source”, think of $\mathbf{x}$ as a “sunken receiver”), define

$$\tilde{F}[v]R(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int \int dx\, dy\, R(\mathbf{x}, \mathbf{y}) \int ds\, G(\mathbf{x}_r, t-s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{y})$$

and note that $\tilde{F}[v]R = F[v]r$ if

$$R(\mathbf{x}, \mathbf{y}) = \frac{2r}{v^2}\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)\delta(\mathbf{x}-\mathbf{y})$$

This trick decomposes $F[v]$ into *two* "exploding reflectors":

$$\tilde{F}[v]R(\mathbf{x}_r, t; \mathbf{x}_s) = u(\mathbf{x}, t; \mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) u(\mathbf{x}, t; \mathbf{x}_s) = \int dy\, R(\mathbf{x}, \mathbf{y}) G(\mathbf{x}_s, t; \mathbf{y})$$

$$\equiv w_s(\mathbf{x}_s, t; \mathbf{x})$$

("upward continue the receivers"),

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) w_s(\mathbf{y}, t; \mathbf{x}) = R(\mathbf{x}, \mathbf{y})\delta(t)$$

("upward continue the sources").

This factorization of $F[v]$ ($r \mapsto R \mapsto \tilde{F}[v]R$) leads to a reverse time computation of adjoint $\tilde{F}[v]^*$ - will discuss this later.

It's equally possible to continue the receivers first, then the sources, which leads to

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) u(\mathbf{x}_r, t; \mathbf{y}) = \int dx\, R(\mathbf{x}, \mathbf{y}) G(\mathbf{x}_r, t; \mathbf{x})$$

$$\equiv w_r(\mathbf{x}_r, t; \mathbf{y})$$

("upward continue the sources"),

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) w_r(\mathbf{x}, t; \mathbf{y}) = R(\mathbf{x}, \mathbf{y})\delta(t)$$

("upward continue the receivers").

The DSR Assumption

Apply reverse depth concept: as before, go 2D temporarily, $\mathbf{x} = (x, z_r), \mathbf{y} = (y, z_s)$, all sources and receivers on $z = 0$.

Double Square Root ("DSR") assumption: For some $\delta > 0$, the wavefield u satisfies

$$(x, z_r, t, y, z_s, \xi, \zeta_s, \tau, \eta, \zeta_r) \in WF(u) \Rightarrow$$

$$(x, t, \xi, \tau) \in \mathcal{P}_\delta(z_r),\ (y, t, \eta, \tau) \in \mathcal{P}_\delta(z_s),\ \text{and } \zeta_r \tau > 0,\ \zeta_s \tau > 0,$$

As for SSR, there is a ray-theoretic interpretation: rays from source and receiver to scattering point stay away from the vertical and decrease in z for increasing t, i.e. they are all upcoming.

Since z will be singled out (and eventually $R(\mathbf{x}, \mathbf{y})$ will have a factor of $\delta(\mathbf{x}, \mathbf{y})$), impose the constraint that

$$R(x, z, x, z_s) = \tilde{R}(x, y, z)\delta(z - z_s)$$

Define upcoming projections as for SSR:

$$\tilde{w}_s = \left(\frac{\partial}{\partial z_s} + ib(y, z_s, D_y, D_t)\right) w_s,$$

$$\tilde{w}_r = \left(\frac{\partial}{\partial z_r} + ib(x, z_r, D_x, D_t)\right) w_r,$$

$$\tilde{u} = \left(\frac{\partial}{\partial z_s} + ib(y, z_s, D_y, D_t)\right) \left(\frac{\partial}{\partial z_r} + ib(x, z_r, D_x, D_t)\right) u$$

Except for lower order commutators which we justify throwing away as before,

$$\left(\frac{\partial}{\partial z_s} - ib(y, z_s, D_y, D_t)\right)\tilde{w}_s = \tilde{R}\delta(z_r - z_s)\delta(t),$$

$$\left(\frac{\partial}{\partial z_r} - ib(x, z_r, D_x, D_t)\right)\tilde{w}_r = \tilde{R}\delta(z_r - z_s)\delta(t),$$

$$\left(\frac{\partial}{\partial z_r} - ib(x, z_r, D_x, D_t)\right)\tilde{u} = \tilde{w}_s$$

$$\left(\frac{\partial}{\partial z_s} - ib(y, z_s, D_y, D_t)\right)\tilde{u} = \tilde{w}_r$$

Initial (final) conditions are that $\tilde{w}_r, \tilde{w}_s$, and $\tilde{u}$ all vanish for large z - the equations are to be solve in decreasing z ("upward continuation").

Simultaneous upward continuation:

$$\frac{\partial}{\partial z}\tilde{u}(x,z,t;y,z) = \frac{\partial}{\partial z_r}\tilde{u}(x,z_r,t;y,z)|_{z=z_r} + \frac{\partial}{\partial z_r}\tilde{u}(x,z,t;y,z_s)|_{z=z_s}$$

$$= [ib(x,z_r,D_x,D_t)\tilde{u} + \tilde{w}_s + ib(y,z_s,D_y,D_t)\tilde{u} + \tilde{w}_r]_{z_r=z_s=z}$$

Since $\tilde{w}_s(y,z,t;x,z) = \tilde{w}_r(x,z,t;y,z) = \tilde{R}(x,y,z)\delta(t)$, $\tilde{u}$ is seen to satisfy the

DSR modeling equation:

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t) - ib(y,z,D_y,D_t)\right)\tilde{u}(x,z,t;y,z) = 2\tilde{R}(x,y,z)\delta(t$$

$$\tilde{F}[v]\tilde{R}(x_r,t;x_s) = \tilde{u}(x_r,0,t;x_s,0)$$

DSR Migration

Computation of adjoint follows same pattern as for SSR, and leads to

DSR migration equation: solve

$$\left(\frac{\partial}{\partial z} - ib(x,z,D_x,D_t) - ib(y,z,D_y,D_t)\right)\tilde{q}(x,y,z,t) = 0$$

in *increasing* z with initial condition at $z = 0$:

$$\tilde{q}(x_r, x_s, 0, t) = d(x_r, x_s, t)$$

Then $\tilde{F}[v]^* d(x,y,z) = \tilde{q}(x,y,z,0)$

The physical DSR model has $\tilde{R}(x,y,z) = r(x,z)\delta(x-y)$, so final step in DSR computation of $F[v]^*$ is adjoint of $r \mapsto \tilde{R}$:

$$F[v]^* d(x,z) = \tilde{q}(x,x,z,0)$$

Standard description of DSR migration

(See Claerbout, IEI):

- downward continue sources and receivers (solve DSR migration equation)
- image at $t = 0$ and zero offset ($x = y$)

Another moniker: "survey sinking": DSR field $\tilde{q}$ is (related to) the field that you would get by conducting the survey with sources and receivers at depth z. At any given depth, the zero-offset, time-zero part of the field is the instantaneous response to scatterers on which source = receiver is sitting, therefore constitutes an image.

As for SSR, the art of DSR migration is in the approximation of the DSR operator.

RICE

Remarks

Stolk and deHoop (2001) derived DSR modeling and migration via a more systematic argument than that used here, involving ΨDO matrix factorization of the wave equation written as a first order evolution system in z. This idea goes back to Taylor (1975) who used it to show that singularities propagating along bicharacteristics reflect as expected at boundaries.

Stolk (2003) has also carried out a very careful global construction of a family of SSR ΨDOs which are of non-classical type at near-horizontal directions ("nearly evanescent waves"). This construction should lead to more reliable discretizations.

The last part of the course will present the various apparently ad-hoc "prestack modeling" ideas within a unified framework.

Agenda

Seismic inverse problem: the sedimentary Earth, reflection seismic measurements, the acoustic model, linearization, reflectors and reflections idealized via *harmonic analysis of singularities*

High frequency asymptotics: why adjoints of modeling operators are imaging operators ("Kirchhoff migration"). Beylkin-Rakesh-... theory of high frequency asymptotic inversion

Adjoint state imaging with the wave equation: reverse time and reverse depth

Geometric optics, Rakesh's construction, and asymptotic inversion w/ caustics and multipathing, imaging artifacts, and prestack migration après Claerbout.

A step beyond linearization: a mathematical framework for velocity analysis

Why Beylkin isn't enough

The theory developed by Beylkin and others cannot be the end of the story:

- The "single ray" hypotheses generally fails in the presence of strong refraction.
- B. White, "The Stochastic Caustic" (1982): For "random but smooth" $v(\mathbf{x})$ with variance σ, points at distance $O(\sigma^{-2/3})$ from source have more than one ray connecting to source, with probability 1 - *multipathing* associated with formation of *caustics* = ray envelopes.
- Formation of caustics invalidates asymptotic analysis on which Beylkin result is based.

Why it matters

- Strong refraction leading to multipathing and caustic formation typical of salt (4-5 km/s) intrusion into sedimentary rock (2-3 km/s) (eg. Gulf of Mexico), also chalk tectonics in North Sea and elsewhere - some of the most promising petroleum provinces!

RICE

Escape from simplicity - the Canonical Relation

How do we get away from "simple geometric optics", SSR, DSR,... - all violated in sufficiently complex (and realistic) models? Rakesh *Comm. PDE* 1988, Nolan *Comm. PDE* 1997: global description of $F_\delta[v]$ as mapping reflectors $\mapsto$ reflections.

$Y = \{\mathbf{x}_s, t, \mathbf{x}_r\}$ (time $\times$ set of source-receiver pairs) submfd of $\mathbf{R}^7$ of dim. ≤ 5, $\Pi : T^*(\mathbf{R}^7) \rightarrow T^*Y$ the natural projection

$\mathrm{supp}\, r \subset X \subset \mathbf{R}^3$

Canonical relation $C_{F_\delta[v]} \subset T^*(X) \setminus \{\mathbf{0}\} \times T^*(Y) \setminus \{\mathbf{0}\}$ describes singularity mapping properties of F:

$$(\mathbf{x}, \xi, \mathbf{y}, \eta) \in C_{F_\delta[v]} \Leftrightarrow$$

$$\text{for some } u \in \mathcal{E}'(X),\ (\mathbf{x}, \xi) \in WF(u),\ \text{and } (\mathbf{y}, \eta) \in WF(Fu)$$

Geometry of Reflection

Rays of geometric optics: solutions of Hamiltonian system

$$\frac{d\mathbf{X}}{dt} = \nabla_{\mathbf{\Xi}} H(\mathbf{X}, \mathbf{\Xi}),\ \ \frac{d\mathbf{\Xi}}{dt} = -\nabla_{\mathbf{X}} H(\mathbf{X}, \mathbf{\Xi})$$

with $H(\mathbf{X}, \mathbf{\Xi}) = 1 - v^2(\mathbf{X})|\mathbf{\Xi}|^2 = 0$ (*null bicharacteristics*).

Characterization of C_F:

$((\mathbf{x}, \xi), (\mathbf{x}_s, t, \mathbf{x}_r, \xi_{\mathbf{s}}, \tau, \xi_{\mathbf{r}})) \in C_{F_\delta[v]} \subset T^*(X) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$

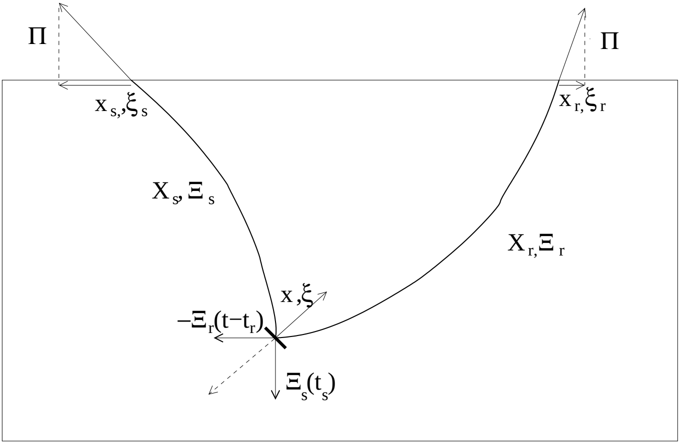
$\Leftrightarrow$ there are *rays of geometric optics* $(\mathbf{X}_s, \mathbf{\Xi}_s)$, $(\mathbf{X}_r, \mathbf{\Xi}_r)$ and times t_s, t_r so that

$$\Pi(\mathbf{X}_s(0), t, \mathbf{X}_r(t), \mathbf{\Xi}_s(0), \tau, \mathbf{\Xi}_r(t)) = (\mathbf{x}_s, t, \mathbf{x}_r, \xi_s, \tau, \xi_r),$$

$$\mathbf{X}_s(t_s) = \mathbf{X}_r(t - t_r) = \mathbf{x},\ t_s + t_r = t,\ \mathbf{\Xi}_s(t_s) - \mathbf{\Xi}_r(t - t_r) || \xi$$

Geometry of Reflection

Since $\mathbf{\Xi}_s(t_s)$, $-\mathbf{\Xi}_r(t-t_r)$ have same length, sum = bisector $\Rightarrow$ *velocity vectors of incident ray from source and reflected ray from receiver (traced backwards in time) make equal angles with reflector at* $\mathbf{x}$ *with normal* ξ.



Geometry of Reflection

Upshot: canonical relation of $F_\delta[v]$ simply enforces the equal-angles law of reflection.

Further, *rays carry high-frequency energy*, in exactly the fashion that seismologists imagine.

Finally, *Rakesh's characterization of C_F is global:* no assumptions about ray geometry, other than no forward scattering and no grazing incidence on the acquisition surface Y, are needed.

Proof: Plan of attack

Recall that

$$F[v]r(\mathbf{x}_r, t; \mathbf{x}_s) = \frac{\partial \delta u}{\partial t}(\mathbf{x}_r, t; \mathbf{x}_s)$$

where

$$\frac{1}{v^2}\frac{\partial^2 \delta u}{\partial t^2} - \nabla^2 \delta u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} r$$

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} - \nabla^2 u = \delta(t)\delta(\mathbf{x} - \mathbf{x}_s)$$

and $u, \delta u \equiv 0, \; t < 0$.

Need to understand (1) $WF(u)$, (2) relation $WF(r) \leftrightarrow WF(ru)$, (3) WF of soln of WE in terms of WF of RHS (this also gives (1)!).

Singularities of the Acoustic Potential Field

Main tool: **Propagation of Singularities** theorem of Hörmander (1970).

Given symbol $p(\mathbf{x}, \boldsymbol{\xi})$, order m, with asymptotic expansion, define *null bicharateristics* (= rays) as solutions $(\mathbf{x}(t), \boldsymbol{\xi}(t))$ of Hamiltonian system

$$\frac{d\mathbf{x}}{dt} = \frac{\partial p}{\partial \boldsymbol{\xi}}(\mathbf{x}, \boldsymbol{\xi}), \;\; \frac{d\boldsymbol{\xi}}{dt} = -\frac{\partial p}{\partial \mathbf{x}}(\mathbf{x}, \boldsymbol{\xi})$$

with $p(\mathbf{x}(t), \boldsymbol{\xi}(t)) \equiv 0$.

Theorem: Suppose $p(\mathbf{x}, D)u = f$, and suppose that for $t_0 \le t \le t_1$, $(\mathbf{x}(t), \boldsymbol{\xi}(t)) \notin WF(f)$. Then either $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \le t \le t_1\} \subset WF(u)$ or $\{(\mathbf{x}(t), \boldsymbol{\xi}(t)) : t_0 \le t \le t_1\} \subset T^*(\mathbf{R}^n) - WF(u)$.

Source to Field

RHS of wave equation for $u = \delta$ function in $\mathbf{x}, t$. WF set $=$ $\{(\mathbf{x}, t, \boldsymbol{\xi}, \tau) : \mathbf{x} = \mathbf{x}_s, t = 0\}$ - i.e. no restriction on covector part.

$\Rightarrow (\mathbf{x}, t, \boldsymbol{\xi}, \tau) \in WF(u)$ iff a ray starting at $(\mathbf{x}_s, 0)$ passes over $(\mathbf{x}, t)$ - i.e. $(\mathbf{x}, t)$ lies on the "light cone" with vertex at $(\mathbf{x}_s, 0)$. Symbol for wave op is $p(\mathbf{x}, t, \boldsymbol{\xi}, \tau) = \frac{1}{2}(\tau^2 - v^2(\mathbf{x})|\boldsymbol{\xi}|^2)$, so Hamilton's equations for null bicharacteristics are

$$\frac{d\mathbf{X}}{dt} = -v^2(\mathbf{X})\boldsymbol{\Xi}, \;\; \frac{d\boldsymbol{\Xi}}{dt} = \nabla \log v(\mathbf{X})$$

Thus $\boldsymbol{\xi}$ is proportional to velocity vector of ray.

[$(\boldsymbol{\xi}, \tau)$ *normal* to light cone.]

RICE

Singularities of Products

To compute $WF(ru)$ from $WF(r)$ and $WF(u)$, use *Gabor calculus* (Duistermaat, Ch. 1)

Here r is really $(r \circ \pi)u$, where $\pi(\mathbf{x}, t) = \mathbf{x}$. Choose bump function ϕ localized near $(\mathbf{x}, t)$

$$\phi\widehat{(r \circ \pi)}u(\boldsymbol{\xi}, \tau) = \int d\boldsymbol{\xi}'\, d\tau' \widehat{\phi r}(\boldsymbol{\xi}')\delta(\tau')\widehat{u}(\boldsymbol{\xi} - \boldsymbol{\xi}', \tau - \tau')$$

$$= \int d\boldsymbol{\xi}' \widehat{\phi r}(\boldsymbol{\xi}')\widehat{u}(\boldsymbol{\xi} - \boldsymbol{\xi}', \tau)$$

This will decay rapidly as $|(\boldsymbol{\xi},\tau)| \to \infty$ unless (i) you can find $(\mathbf{x}',\boldsymbol{\xi}') \in WF(r)$ so that $\mathbf{x},\mathbf{x}' \in \pi(\mathrm{supp}\phi)$, $\boldsymbol{\xi}-\boldsymbol{\xi}' \in WF(u)$, i.e. $(\boldsymbol{\xi},\tau) \in WF(r\circ\pi)+WF(u)$, or (ii) $\boldsymbol{\xi} \in WF(r)$ or $(\boldsymbol{\xi},\tau) \in WF(u)$.

Possibility (ii) will not contribute, so effectively

$$WF((r\circ\pi)u) = \{(\mathbf{x},t_s,\boldsymbol{\xi}+\boldsymbol{\Xi}_s(t_s),\cdot) : (\mathbf{x},\boldsymbol{\xi}) \in WF(r),\, \mathbf{x}=\mathbf{X}_s(t_s)$$

for a ray $(\mathbf{X}_s,\boldsymbol{\Xi}_s)$ with $\mathbf{X}_s(0)=x_s$, some τ.

Wavefront set of Scattered Field

Once again use propagation of singularities: $(\mathbf{x}_r, t, \boldsymbol{\xi}_r, \tau_r) \in WF(\delta u) \Leftrightarrow$ on ray $(\mathbf{X}_r, \boldsymbol{\Xi}_r)$ passing through $WF(ru)$. Can argue that time of intersection is $t - t_r < t$.

That is,

$$\mathbf{X}_r(t) = \mathbf{x}_r, \mathbf{X}_r(t - t_r) = \mathbf{X}_s(t_s) = x,$$

$t = t_r + t_s$, and

$$\boldsymbol{\Xi}_r(t_s) = \boldsymbol{\xi} + \boldsymbol{\Xi}_s(t_s)$$

for some $\boldsymbol{\xi} \in WF(r)$. **Q. E. D.**

Rakesh's Thesis

Rakesh also showed that $F[v]$ is a *Fourier Integral Operator* = class of oscillatory integral operators, introduced by Hörmander and others in the '70s to describe the solutions of nonelliptic PDEs.

Phases and amplitudes of FIOs satisfy certain restrictive conditions. Canonical relations have geometric description similar to that of $F[v]$. Adjoint of FIO is FIO with inverse canonical relation.

ΨDOs are special FIOs.

Composition of FIOs does *not* yield an FIO in general. Beylkin had shown that $F[v]^*F[v]$ is FIO (ΨDO, actually) under simple ray geometry hypothesis - but this is only sufficient. Rakesh noted that this follows from general results of Hörmander: *simple ray geometry* $\Leftrightarrow$ *canonical relation is graph of ext. deriv. of phase function*.

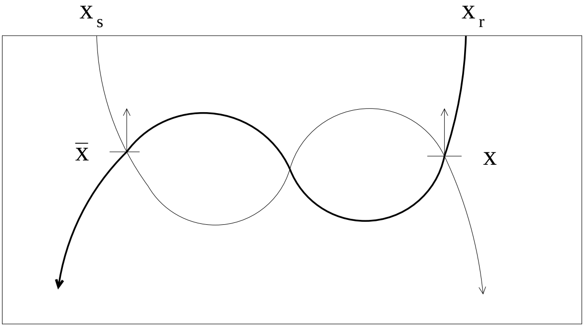
The Shell Guys and TIC

Smit, tenKroode and Verdel (1998): provided that

- source, receiver positions $(\mathbf{x}_s, \mathbf{x}_r)$ form an *open* 4D manifold ("complete coverage" - all source, receiver positions at least locally), and
- the *Traveltime Injectivity Condition* ("TIC") holds: $C_{F[v]}^{-1} \subset T^*Y \setminus \{0\} \times T^*X \setminus \{0\}$ is a *function* - that is, initial data for source and receiver rays and total travel time together determine reflector uniquely.

then $F[v]^*F[v]$ is ΨDO $\Rightarrow$ application of $F[v]^*$ produces image, and $F[v]^*F[v]$ has microlocal parametrix ("asymptotic inversion").

TIC is a nontrivial constraint!



Symmetric waveguide: time ($\mathbf{x}_s \rightarrow \bar{\mathbf{x}} \rightarrow \mathbf{x}_r$) same as time ($\mathbf{x}_s \rightarrow \mathbf{x} \rightarrow \mathbf{x}_r$), so TIC fails.

Stolk's Thesis

Stolk (2000): for dim=2, under "complete coverage" hypothesis, v for which $F[v]^*F[v] = [\Psi\text{DO} + \text{rel. smoothing op}]$ open, dense set in $C^\infty(\mathbf{R}^2)$ (without assuming TIC!). Conjecture: same for dim=3.

Also, for any dim, v for which $F[v]^*F[v]$ is FIO open, dense in $C^\infty(\mathbf{R}^2)$.

Operto's Thesis

Application of $F[v]^*$ involves accounting for *all* rays connecting source and receiver with reflectors.

Standard practice still attempts imaging with single choice of ray pair (shortest time, max energy,...).

Operto et al (2000) give nice illustration that all rays must be included in general to obtain good image.

Nolan's Thesis

Limitation of Smit-tenKroode-Verdel: most idealized data acquisition geometries violate "complete coverage": for example, idealized marine streamer geometry (src-recvr submfd is 3D)

Nolan (1997): result remains true without "complete coverage" condition: requires only TIC plus addl condition so that projection $C_{F[v]} \rightarrow T^*Y$ is embedding - but examples violating TIC are much easier to construct when source-receiver submfd has positive codim.

Sinister Implication: When data is just a single gather - common shot, common offset - image may contain *artifacts*, i.e. spurious reflectors not present in model.

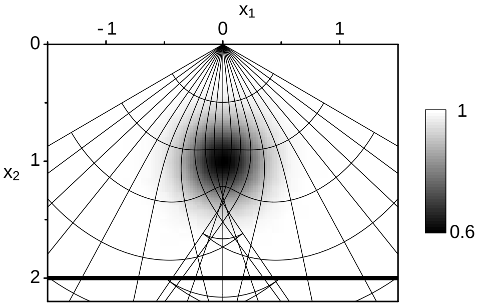
Horrible Example

Synthetic 2D Example (see Stolk and WWS, *Geophysics* 2004 for this and other horrible expls)

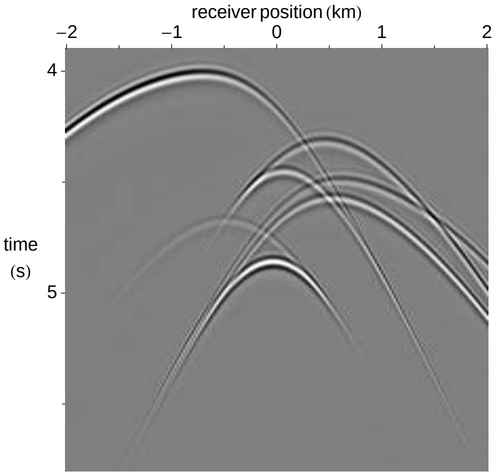
Strongly refracting acoustic lens (v) over horizontal reflector (r), $S^{\rm obs} = F[v]r$.

(i) for open source-receiver set, $F[v]^* S^{\rm obs} =$ good image of reflector - within limits of finite frequency implied by numerical method, $F[v]^* F[v]$ acts like ΨDO;

(ii) for *common offset* submfd (codim 1), TIC is violated and $WF(F[v]^* S^{\rm obs})$ is larger than $WF(r)$.



Gaussian lens velocity model, flat reflector at depth 2 km, overlain with rays and wavefronts (Stolk & S. 2002 SEG).



Typical shot gather - lots of arrivals

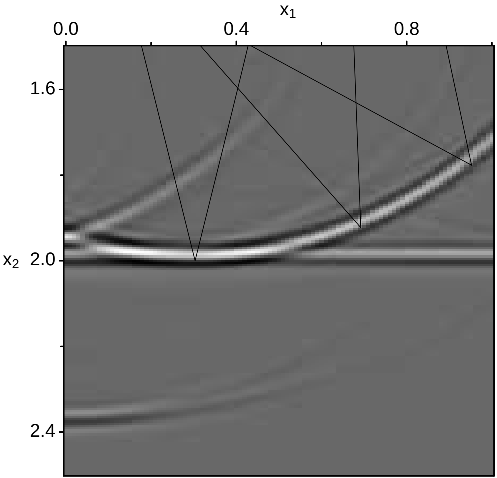


Image from common offset gather at $h = 0.3$ - all three ray pairs belong to the same offset, midpoint, time, midpoint slowness - TIC fails, image has “artifact” WF

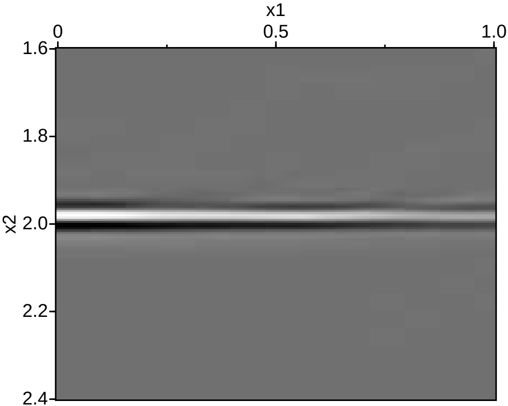


Image from all offsets - TIC holds, “WF” recovered

What it all means

Note that a gather scheme makes the scattering operator block-diagonal: for example with data sorted into common offset gathers $h = (x_r - x_s)/2$,

$$F[v] = [F_{h_1}[v], ..., F_{h_N}[v]]^T, \; d = [d_{h_1}, ..., d_{h_N}]^T$$

Thus $F[v]^* d = \sum_i F_{h_i}[v]^* d_{h_i}$. Otherwise put: to form image, **migrate** ith gather (apply migration operator $F_{h_i}[v]^*$, then **stack** individual migrated images.

Horrible Examples show that individual offset gather images may contain nonphysical apparent reflectors (artifacts).

Smit-tenKroode-Verdel, Nolan, Stolk: if TIC holds, then these artifacts are not stationary with respect to the gather parameter, hence *stack out* (interfere destructively) in final image.

Agenda

Seismic inverse problem: the sedimentary Earth, reflection seismic measurements, the acoustic model, linearization, reflectors and reflections idealized via *harmonic analysis of singularities*

High frequency asymptotics: why adjoints of modeling operators are imaging operators ("Kirchhoff migration"). Beylkin-Rakesh-... theory of high frequency asymptotic inversion

Adjoint state imaging with the wave equation: reverse time and reverse depth

Geometric optics, Rakesh's construction, and asymptotic inversion w/ caustics and multipathing, imaging artifacts, and prestack migration après Claerbout.

A step beyond linearization: a mathematical framework for velocity analysis

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Velocity Analysis

Partially linearized seismic inverse problem ("velocity analysis"): given observed seismic data d, find smooth *velocity* $v \in \mathcal{E}(X), X \subset \mathbf{R}^3$ oscillatory *reflectivity* $r \in \mathcal{E}'(X)$ so that

$$F[v]r \simeq d$$

Acoustic partially linearized model: acoustic potential field u and its perturbation δu solve

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) u = \delta(t)\delta(\mathbf{x}-\mathbf{x}_s),\ \left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta u = 2r\nabla^2 u$$

plus suitable bdry and initial conditions.

$$F[v]r = \left.\frac{\partial \delta u}{\partial t}\right|_Y$$

data acquisition manifold $Y = \{(\mathbf{x}_r, t; \mathbf{x}_s)\} \subset \mathbf{R}^7$, dimn $Y \leq 5$ (many idealizations here!).

$F[v] : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$ is a linear map (FIO of order 1), but dependence on v is quite nonlinear, so this inverse problem is nonlinear.

Agenda:

- reformulation of inverse problem via *extensions*
- "standard processing" extension and standard VA
- the surface oriented extension and standard MVA
- the ΨDO property and why it's important
- global failure of the ΨDO property for the SOE
- Claerbout's depth oriented extension has the ΨDO property
- differential semblance

Extensions

Extension of $F[v]$: manifold $\bar{X}$ and maps $\chi : \mathcal{E}'(X) \to \mathcal{E}'(\bar{X})$, $\bar{F}[v] : \mathcal{E}'(\bar{X}) \to \mathcal{D}'(Y)$ so that

$$\begin{tikzcd}
\mathcal{E}'(\bar{X}) \arrow[r, "{\bar{F}[v]}"] & \mathcal{D}'(Y) \\
\mathcal{E}'(X) \arrow[u, "\chi"] \arrow[r, "{F[v]}"'] & \mathcal{D}'(Y) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

commutes.

Invertible extension: $\bar{F}[v]$ has a *right parametrix* $\bar{G}[v]$, i.e. $I - \bar{F}[v]\bar{G}[v]$ is smoothing. [The trivial extension - $\bar{X} = X, \bar{F} = F$ - is virtually never invertible.] Also χ has a *left inverse* η.

Reformulation of inverse problem: given d, find v so that $\bar{G}[v]d \in \mathcal{R}(\chi)$ (implicitly determines r also!).

Example 1: Standard VA extension

Treat each CMP *as if it were the result of an experiment performed over a layered medium*, but permit the layers to vary with midpoint.

Thus $v = v(z), r = r(z)$ for purposes of analysis, but at the end $v = v(\mathbf{x}_m, z), r = r(\mathbf{x}_m, z)$.

$$F[v]R(\mathbf{x}_m, h, t) \simeq A(\mathbf{x}_m, h, z(\mathbf{x}_m, h, t))R(\mathbf{x}_m, z(\mathbf{x}_m, h, t))$$

Here $z(\mathbf{x}_m, h, t)$ is the inverse of the 2-way traveltime

$$t(\mathbf{x}_m, h, z) = 2\tau(\mathbf{x}_m + (h, 0, z), \mathbf{x}_m)_{v=v(\mathbf{x}_m, z)}$$

computed with the layered velocity $v(\mathbf{x}_m, z)$, i.e. $z(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z')) = z'$.

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That is, $F[v]$ *is a change of variable followed by multiplication by a smooth function*. NB: industry standard practice is to use vertical traveltime t_0 instead of z for depth variable.

Can write this as $F[v] = \bar{F}S^*$, where $\bar{F}[v] = N[v]^{-1}M[v]$ has right parametrix $\bar{G}[v] = M[v]N[v]$:

$N[v]$ = **NMO operator** $N[v]d(\mathbf{x}_m, h, z) = d(\mathbf{x}_m, h, t(\mathbf{x}_m, h, z))$

$M[v]$ = multiplication by A

S = **stacking operator**

$$Sf(\mathbf{x}_m, z) = \int dh\, f(\mathbf{x}_m, h, z),\ \ S^*r(\mathbf{x}_m, h, z) = r(\mathbf{x}, z)$$

Identify as extension: $\bar{F}[v], \bar{G}[v]$ as above,
$X = \{\mathbf{x}_m, z\}, H = \{h\}, \bar{X} = X \times H, \chi = S^*, \eta = S$ - the invertible extension properties are clear.

Standard names for the Standard VA extension objects: $\bar{F}[v] =$ "inverse NMO", $\bar{G}[v] =$ "NMO" [often the multiplication op $M[v]$ is neglected]; $\eta =$ "stack", $\chi =$ "spread"

How this is used for velocity analysis: Look for v that makes $\bar{G}[v]d \in \mathcal{R}(\chi)$

So what is $\mathcal{R}(\chi)$? $\chi[r](\mathbf{x}_m, z, h) = r(\mathbf{x}_m, z)$ Anything in range of χ is *independent of h*. Practical issues $\Rightarrow$ replace "independent of" with "smooth in".

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Flatten them gathers!

Inverse problem reduced to: adjust v to make $\bar{G}[v]d^{\mathrm{obs}}$ smooth in h, i.e. *flat* in z, h display for each $\mathbf{x}_m$ (*NMO-corrected CMP*).

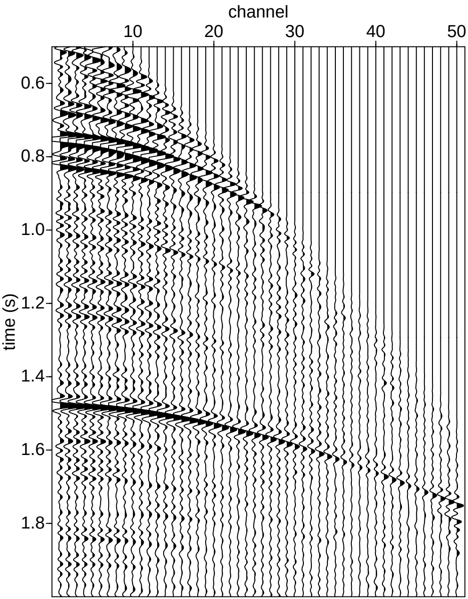
Replace z with t_0, v with v_{RMS} em localizes computation: reflection through $\mathbf{x}_m, t_0, 0$ *flattened* by adjusting $v_{\mathrm{RMS}}(\mathbf{x}_m, t_0) \Rightarrow$ 1D search, do by visual inspection.

Various aids - NMO corrected CMP gathers, velocity spectra, etc.

See: Claerbout: *Imaging the Earth's Interior*

WWS: MGSS 2000 notes

RICE



RICE

Example 2: Surface oriented or standard MVA extension

. This only works where Earth is "nearly layered". Where this fails, replace NMO by prestack migration.

Shot version: Σ_s = set of shot locations, $\bar{X} = X \times \Sigma_s$, $\chi[r](\mathbf{x}, \mathbf{x}_s) = r(\mathbf{x})$.

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t, \mathbf{x}_s) = \frac{\partial^2}{\partial t^2} \int dx\, \bar{r}(\mathbf{x}, \mathbf{x}_s) \int ds\, G(\mathbf{x}_r, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

Offset version (preferred because it minimizes truncation artifacts): Σ_h = set of half-offsets in data, $\bar{X} = X \times \Sigma_h$, $\chi[r](\mathbf{x}, \mathbf{h}) = r(\mathbf{x})$.

$$\bar{F}[v]\bar{r}(\mathbf{x}_s, t, \mathbf{h}) = \frac{\partial^2}{\partial t^2} \int dx\, \bar{r}(\mathbf{x}, \mathbf{h}) \int ds\, G(\mathbf{x}_s + \mathbf{h}, t - s; \mathbf{x}) G(\mathbf{x}_s, s; \mathbf{x})$$

[Parametrize data with source location $\mathbf{x}_s$, time t, offset $\mathbf{h}$.] **NB:** note that both versions are "block diagonal" - family of operators (FIOs) parametrized by $\mathbf{x}_s$ or $\mathbf{h}$.

Properties of SOE

Beylkin (1985), Rakesh (1988): if $\|v\|_{C^2(X)}$ "not too big", then

- $\bar{F}$ has **the ΨDO property**: $\bar{F}\bar{F}^*$ is ΨDO
- singularities of $\bar{F}\bar{F}^*d \subset$ singularities of d
- straightforward construction of right parametrix $\bar{G} = \bar{F}^*Q$, Q $=$ ΨDO, also as generalized Radon Transform - explicitly computable.

Range of χ (offset version): $\bar{r}(\mathbf{x},\mathbf{h})$ independent of $\mathbf{h} \Rightarrow$ "semblance principle": find v so that $\bar{G}[v]d^{\rm obs}$ is independent of $\mathbf{h}$. Practical limitations $\Rightarrow$ replace "independent of $\mathbf{h}$" by "smooth in $\mathbf{h}$".

Industrial MVA

Application of these ideas = industrial practice of migration velocity analysis.

Idea: twiddle v until $\bar{G}[v]d^{\mathrm{obs}}$ is smooth in $\mathbf{h}$.

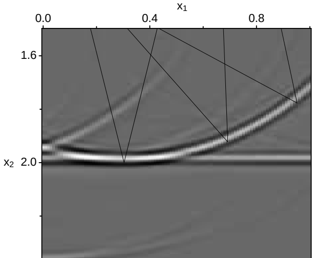
Since it is hard to inspect $\bar{G}[v]d^{\mathrm{obs}}(x, y, z, h)$, pull out subset for constant x, y = **common image gather** ("CIG"): display function of z, h for fixed x, y. These play same role as NMO corrected CMP gathers in layered case.

Try to adjust v so that selected CIGs are *flat* - just as in Standard VA. This is much harder, as there is no RMS velocity trick to localize the computation - each CIG depends globally on v.

Description, some examples: Yilmaz, *Seismic Data Processing*.

Bad news

Nolan (1997): big trouble! In general, standard extension does **not** have the ΨDO property. Geometric optics analysis: for $\|v\|_{C^2(X)}$ "large", multiple rays connect source, receiver to reflecting points in X; block diagonal structure of $\bar{F}[v] \Rightarrow$ info necessary to distinguish multiple rays is *projected out*.



x1
0.0
0.4
0.8
1.6
x2 2.0

Example 3: Claerbout's depth oriented extension

Standard MVA extension only works when Earth has simple ray geometry. Claerbout proposed alternative extension:

$\Sigma_d =$ somewhat arbitrary set of vectors near 0 ("offsets"),
$\bar{X} = X \times \Sigma_d$, $\chi[r](\mathbf{x},\mathbf{h}) = r(\mathbf{x})\delta(\mathbf{h})$, $\eta[\bar{r}](\mathbf{x}) = \bar{r}(\mathbf{x},0)$

$$\bar{F}[v]\bar{r}(\mathbf{x}_s,t,\mathbf{x}_r) = \frac{\partial^2}{\partial t^2}\int dx \int_{\Sigma_d} dh\,\bar{r}(\mathbf{x},\mathbf{h}) \int ds\, G(\mathbf{x}_s, t-s; \mathbf{x}+2\mathbf{h})G(\mathbf{x}_r,$$

$$= \frac{\partial^2}{\partial t^2}\int dx \int_{\mathbf{x}+2\Sigma_d} dy\,\bar{r}(\mathbf{x},\mathbf{y}-\mathbf{x}) \int ds\, G(\mathbf{x}_s, t-s; \mathbf{y})G(\mathbf{x}_r, s; \mathbf{x})$$

NB: in this formulation, there appears to be too many model parameters.

Shot record modeling

for each $\mathbf{x}_s$ solve

$$\bar{F}[v]\bar{r}(\mathbf{x}_r, t; \mathbf{x}_s) = u(\mathbf{x}, t; \mathbf{x}_s)|_{\mathbf{x}=\mathbf{x}_r}$$

where

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{x}}\right) u(\mathbf{x}, t; \mathbf{x}_s) = \int_{\mathbf{x}+2\Sigma_d} dy\, \bar{r}(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, t; \mathbf{x}_s)$$

$$\left(\frac{1}{v(\mathbf{y})^2}\frac{\partial^2}{\partial t^2} - \nabla^2_{\mathbf{y}}\right) G(\mathbf{y}, t; \mathbf{x}_s) = \delta(t)\delta(\mathbf{x}_s - \mathbf{y})$$

Finite difference scheme: form RHS for eqn 1, step u, G forward in t.

Computing $\bar{G}[v]$

Instead of parametrix, be satisfied with adjoint.

Reverse time adjoint computation - specify adjoint field as in standard reverse time prestack migration:

$$\left(\frac{1}{v(\mathbf{x})^2}\frac{\partial^2}{\partial t^2} - \nabla_{\mathbf{x}}^2\right) w(\mathbf{x}, t; \mathbf{x}_s) = \int dx_r \, d(\mathbf{x}_r, t; \mathbf{x}_s)\delta(\mathbf{x} - \mathbf{x}_r)$$

with $w(\mathbf{x}, t; \mathbf{x}_s) = 0, t >> 0$. Then

$$\bar{F}[v]^* d(\mathbf{x}, \mathbf{h}) = \int dx_s \int dt \, G(\mathbf{x} + 2\mathbf{h}, t; \mathbf{x}_s) w(\mathbf{x}, t; \mathbf{x}_s)$$

i.e. exactly the same computation as for reverse time prestack, except that crosscorrelation occurs at an offset 2**h**.

Nomenclature

NB: the "usual computation" of $\bar{G}[v]$ is either DSR or a variant of shot record computation of previous slide using depth extrapolation. $\mathbf{h}$ is usually restricted to be horizontal, i.e. $h_3 = 0$.

Common names: shot-geophone or survey-sinking migration (with DSR), or shot record migration.

"Downward continue sources and receivers, image at $t = 0, h = 0$"

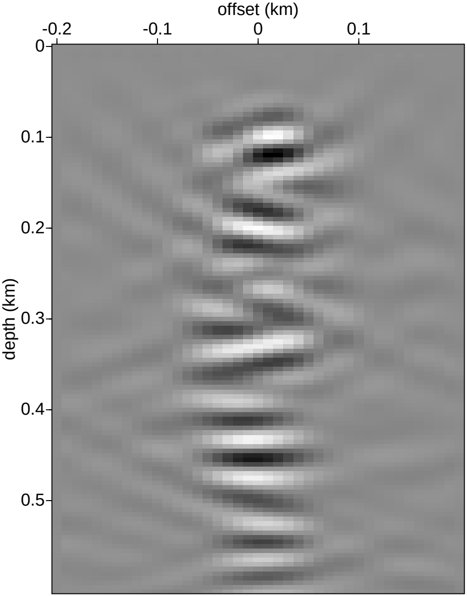
These are what is typically meant by "wave equation migration"!

What should be the character of the image when the velocity is correct?

Hint: for simulation of seismograms, the input reflectivity had the form $r(\mathbf{x})\delta(\mathbf{h})$.

Therefore guess that when velocity is correct, *image is concentrated near* $h = 0$.

Examples: 2D finite difference implementation of reverse time method. Correct velocity $\equiv 1$. Input reflectivity used to generate synthetic data: random! For output reflectivity (image of $\bar{F}[v]^*$), constrain offset to be horizontal: $\bar{r}(\mathbf{x},\mathbf{h}) = \tilde{r}(\mathbf{x},h_1)\delta(h_3)$. Display CIGs (i.e. $x_1 =$const. slices).



Offset Image Gather, x=1 km

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Stolk and deHoop, 2001

Claerbout extension has the ΨDO property, at least when restricted to $\bar{r}$ of the form $\bar{r}(\mathbf{x},\mathbf{h}) = R(\mathbf{x},h_1,h_2)\delta(h_3)$, and under DSR assumption.

Sketch of proof (after Rakesh, 1988):

This will follow from *injectivity* of wavefront or *canonical relation* $C_{\bar{F}} \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$ which describes singularity mapping properties of $\bar{F}$:

$$(\mathbf{x},\mathbf{h},\xi,\nu,\mathbf{y},\eta) \in C_{F_\delta[v]} \Leftrightarrow$$

$$\text{for some } u \in \mathcal{E}'(\bar{X}),\ (\mathbf{x},\mathbf{h},\xi,\nu) \in WF(u),\ \text{and } (\mathbf{y},\eta) \in WF(\bar{F}u)$$

Characterization of $C_{\bar{F}}$

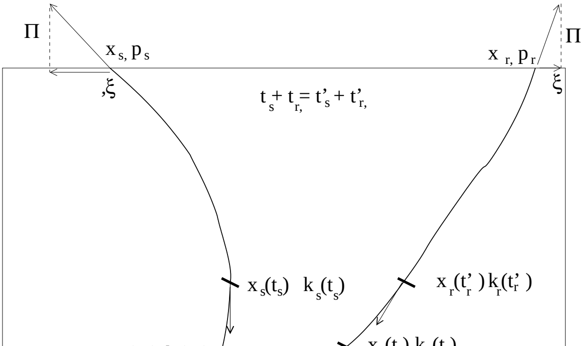
$$((\mathbf{x}, \mathbf{h}, \xi, \nu), (\mathbf{x}_s, t, \mathbf{x}_r, \xi_{\mathbf{s}}, \tau, \xi_{\mathbf{r}})) \in C_{\bar{F}}[v] \subset T^*(\bar{X}) - \{\mathbf{0}\} \times T^*(Y) - \{\mathbf{0}\}$$

$\Leftrightarrow$ there are *rays of geometric optics* $(\mathbf{X}_s, \mathbf{\Xi}_s)$, $(\mathbf{X}_r, \mathbf{\Xi}_r)$ and times t_s, t_r so that

$$\Pi(\mathbf{X}_s(0), t, \mathbf{X}_r(0), \mathbf{\Xi}_s(0), \tau, \mathbf{\Xi}_r(0)) = (\mathbf{x}_s, t, \mathbf{x}_r, \xi_s, \tau, \xi_r),$$

$$\mathbf{X}_s(t_s) = \mathbf{x}, \mathbf{X}_r(t_r) = \mathbf{x} + 2\mathbf{h},\ t_s + t_r = t,$$

$$\mathbf{\Xi}_s(t_s) + \mathbf{\Xi}_r(t_r) || \xi, \mathbf{\Xi}_s(t_s) - \mathbf{\Xi}_r(t_r) || \nu$$



Π
x s, p s
,ξ
t s + t r, = t’s + t’r,
x r, p r
Π
ξ
x s(ts) k s(ts)
x r(t’r) k r(t’r)

Proof

Uses wave equations for u, G and

- Gabor calculus: computes wave front sets of products, pullbacks, integrals, etc. See Duistermaat, Ch. 1.
- Propagation of Singularities Theorem

and that's all! [No integral representations, phase functions,...]

Note intrinsic ambiguity: if you have a ray pair, move times t_s, t_r resp. t_s', t_r', for which $t_s + t_r = t_s' + t_r' = t$ then you can construct two points $(\mathbf{x}, \mathbf{h}, \boldsymbol{\xi}, \nu), (\mathbf{x}', \mathbf{h}', \boldsymbol{\xi}', \nu')$ which are candidates for membership in $WF(\bar{r})$ and which satisfy the above relations with the same point in the cotangent bundle of $T^*(Y)$.

No wonder - there are too many model parameters!

Stolk and deHoop fix this ambiguity by imposing two constraints:

- DSR assumption: all rays carrying significant reflected energy (source or receiver) are upcoming.
- Restrict $\bar{F}$ to the domain $\mathcal{Z} \subset \mathcal{E}'(\bar{X})$

$$\bar{r} \in \mathcal{Z} \Leftrightarrow \bar{r}(\mathbf{x}, \mathbf{h}) = R(\mathbf{x}, h_1, h_2)\delta(h_3)$$

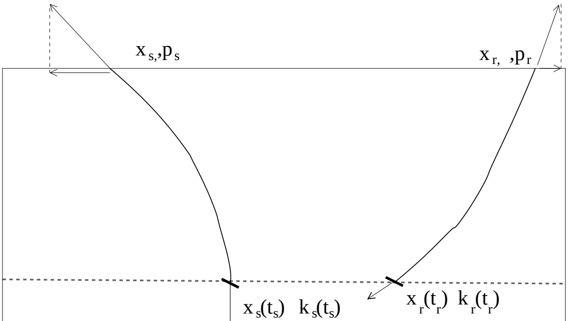
If $\bar{r} \in \mathcal{Z}$, then $(\mathbf{x}, \mathbf{h}, \boldsymbol{\xi}, \nu) \in WF(\bar{r}) \Rightarrow h_3 = 0$. So source and receiver rays in $C_{\bar{F}}$ must terminate at same depth, to hit such a point.

Because of DSR assumption, this fixes the traveltimes t_s, t_r.

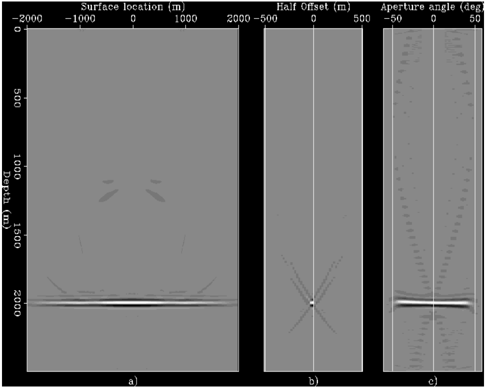
Restricted to $\mathcal{Z}$, $C_{\bar{F}}$ is injective.

$\Rightarrow C_{\bar{F}^*\bar{F}} = I$

$\Rightarrow \bar{F}^*\bar{F}$ is ΨDO when restricted to $\mathcal{Z}$.



$x_{s,}, p_s$
$x_{r,}\ , p_r$
$x_s(t_s)\ \ k_s(t_s)$
$x_r(t_r)\ \ k_r(t_r)$



Surface location (m)
-2000 -1000 0 1000 2000
Half Offset (m)
-500 0 500
Aperture angle (deg)
-50 0 50
Depth (m)
0
500
1000
1500
2000
a)
b)
c)

Quantitative VA

Suppose $W : \mathcal{E}'(\bar{X}) \rightarrow \mathcal{D}'(Z)$ annihilates range of χ:

$$\mathcal{E}'(X) \overset{\chi}{\rightarrow} \mathcal{E}'(\bar{X}) \overset{W}{\rightarrow} \mathcal{D}'(Z) \rightarrow 0$$

and moreover W is bounded on $L^2(\bar{X})$. Then

$$J[v;d] = \frac{1}{2}\|W\bar{G}[v]d\|^2$$

minimized when $[v, \eta\bar{G}[v]d]$ solves partially linearized inverse problem.

Construction of *annihilator* of $\mathcal{R}(F[v])$ (Guillemin, 1985):

$$d \in \mathcal{R}(F[v]) \Leftrightarrow \bar{G}[v]d \in \mathcal{R}(\chi) \Leftrightarrow W\bar{G}[v]d = 0$$

Annihilators, annihilators everywhere...

For Standard Extended Model, several popular choices:

- $W = (I - \Delta)^{-\frac{1}{2}} \nabla_{\mathbf{h}}$ ("differential semblance" - WWS, 1986)
- $W = I - \frac{1}{|H|} \int \, dh$ ("stack power" - Toldi, 1985)
- $W = I - \chi F[v]^{\dagger} \bar{F}[v] \Rightarrow$ minimizing $J[v, d]$ equivalent to least squares.

For Claerbout extension, differential semblance $W = h$.

But not many are good for much...

Since *problem is huge*, only W giving rise to differentiable $v \mapsto J[v, d]$ are useful - must be able to use Newton!!! Once again, idealize $w(t) = \delta(t)$.

Theorem (Stolk & WWS, 2003): $v \mapsto J[v, d]$ smooth $\Leftrightarrow$ W pseudodifferential.

i.e. only *differential semblance* gives rise to smooth optimization problem, *uniformly in source bandwidth*.

NB: Least squares embedded in larger family of optimization formulations, some (others) of which are tractable.

Numerical examples using synthetic and field data: WWS et al., Chauris & Noble 2001, Mulder & tenKroode 2002. deHoop et al. 2004.

Beyond Born

Nonlinear effects not included in linearized model: *multiple reflections*. Conventional approach: treat as *coherent noise*, attempt to eliminate - active area of research going back 40+ years, with recent important developments.

Why not model this "noise"?

Proposal: *nonlinear extensions* with $F[v]r$ replaced by $\mathcal{F}[c]$. Create annihilators in same way (now also nonlinear), optimize differential semblance.

Nonlinear analog of Standard Extended Model appears to be *invertible* - in fact extended nonlinear inverse problem is *underdetermined*.

Open problems: no theory. Also must determine $w(t)$ (Lailly SEG 2003).

And so on...

- Elasticity: theory of asymptotic Born inversion at smooth background in good shape (Beylkin & Burridge 1988, deHoop & Bleistein 1997). Theory of extensions, annihilators, differential semblance partially complete (Brandsberg-Dahl et al 2003).
- Anisotropy - work of deHoop (Brandsberg-Dahl et al 2003).
- Anelasticity - in the sedimentary section, $Q = 100 - 1000$, lower in gassy sediments and near surface. No mathematical results, but some numerics - Minkoff & WWS 1997, Blanch et al 1998.
- Source determination - actually always an issue. Some success in casting as an inverse problem - Minkoff & WWS 1997, Routh et al SEG 2003.
- ...

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