Mathematics, Physics, and the Rainbow

by James D. Nickel, BA, BTh, BMiss, MA

> The heavens are telling of divine glory, and the firmament proclaims his handiwork. Day to day spews forth utterance, and night to night proclaims knowledge. There are no conversations, nor are there words, the articulations of which are not heard. Their sound went out to all the earth, and to the ends of the world their utterances. In the sun he pitched his covert, and he himself, like a bridegroom going forth from his bride's chamber, will rejoice, like a giant, to run his course. From the sky's extremity is his starting point, and his goal is as far as the sky's extremity, and there is no one that will be hid from his heat.
>
> The law of the Lord is faultless, turning souls; the testimony of the Lord is reliable, making infants wise; the statutes of the Lord are upright, making glad the heart; the commandment of the Lord is radiant, enlightening the eyes; the fear of the Lord is pure, enduring forever and ever; the judgments of the Lord are valid, justified altogether, things desired beyond gold and much precious stone and sweeter beyond honey and honeycomb. Indeed, your slave guards them; in guarding them there is great reward. Transgressions--who shall detect them? From my hidden ones clear me. Transgressions–who shall detect them? From my hidden ones clear me. And the sayings of my mouth shall become good pleasure, and the meditation of my heart is before you always, O Lord, my helper and my redeemer.
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> Psalm 19, New English Translation of the Septuagint (NETS)

This essay is extracted from a set of lessons from the forthcoming textbook *Dancing to Infinity*.

This essay is dedicated to Earl Nickel (1918-2008), beloved father, who, on 28 February 2008, left this world for his eternal reward.

Connections with Scripture

The rainbow reveals fascinating properties of mathematical physics. Combined with an analysis of its place in the plot line of the Bible, a diligent study of the rainbow will reap rich rewards, both intellectual and spiritual.

The rainbow first appears in Scripture in Genesis 9:13-17 (NKJV):

Source: iStockPhoto

> I do set my bow in the cloud, and it shall be for a token of a covenant between me and the earth. And it shall come to pass, when I bring a cloud over the earth, that the bow shall be seen in the cloud: And I will remember my covenant, which is between me and you and every living creature of all flesh; and the waters shall no more become a flood to destroy all flesh. And the bow shall be in the cloud; and I will look upon it, that I may remember the everlasting covenant between God and every living creature of all

> flesh that is upon the earth. And God said unto Noah, This is the token of the covenant, which I have established between me and all flesh that is upon the earth.

The Hebrew for bow means "bending," reflecting the nature of the rainbow appearing as a curve in the sky. Mathematically, when seen from the ground, the rainbow forms a portion of a perfect circle. In Scripture, the rainbow is connected to God's covenant, His promise, His faithfulness, His mercy, His throne, and His glory. Let's read Ezekiel's description of the throne of God:

> A voice came from above the firmament that was over their heads; whenever they stood, they let down their wings. And above the firmament over their heads was the likeness of a throne, in appearance like a sapphire stone; on the likeness of the throne was a likeness with the appearance of a man high above it. Also from the appearance of His waist and upward I saw, as it were, the color of amber with the appearance of fire all around within it; and from the appearance of His waist and downward I saw, as it were, the appearance of fire with brightness all around. Like the appearance of a rainbow in a cloud on a rainy day, so was the appearance of the brightness all around it. This was the appearance of the likeness of the glory of the LORD. So when I saw it, I fell on my face, and I heard a voice of One speaking. (Ezekiel 1:25-28, NKJV).

This throne description connects with similar passages in the book of Revelation:

> After these things I looked, and behold, a door standing open in heaven. And the first voice which I heard was like a trumpet speaking with me, saying, "Come up here, and I will show you things which must take place after this" Immediately I was in the Spirit; and behold, a throne set in heaven, and One sat on the throne. And He who sat there was like a jasper and a sardius stone in appearance; and there was a rainbow around the throne, in appearance like an emerald (Revelation 4:1-3, NKJV).

> I saw still another mighty angel coming down from heaven, clothed with a cloud. And a rainbow was on his head, his face was like the sun, and his feet like pillars of fire. He had a little book open in his hand. And he set his right foot on the sea and his left foot on the land, and cried with a loud voice, as when a lion roars. When he cried out, seven thunders uttered their voices (Revelation 10:1-3, NKJV).

In these Scriptures, the rainbow is connected with the throne, the authority, and the power of the glory of God. The throne of God is His authority, justice, judgment, and mercy all revealed completely and perfectly in His Son, the Lion and the Lamb.

Transliterated into English, the Greek word for rainbow is *iris*. The iris is the colored part of the eye. When we use the word iridescent, we mean rainbow colored. Iris was also a Greek god, the messenger of Zeus and an omen of war and storms.

Other names for the rainbow are "flashing arc" (Italy), "the bow of Indra" (India), and "the bride of the rain" (North Africa). In Europe, the rainbow was sometimes called the "bridge of the Holy Spirit," or the "girdle of God." The idea that there is a "pot of gold" at the end of the rainbow, which, of course, no one could ever reach, comes from an old English superstition.

We can also see rainbows in:

- The refracted light of diamonds.
- The parabolic spray of water in a fountain or from a garden hose.
- Waterfalls.
- Fog.
- The morning dew clinging to the web of a spider.
- Ice crystals on trees.
- The reflected image of the Sun in water.

- The encirclement of the moon given specific weather conditions.
- On the bottom side of CD or DVD disks when reflecting light.
- In an airplane, if conditions are right (the Sun, rain, and the plane), you will see a circular rainbow.

GENERAL INTRODUCTION

You see a rainbow in the sky when three conditions are evident:

1. It is raining.
2. The Sun is setting in the west.
3. The observer is between the Sun and the rain and looking east.

There is a coinherence of simplicity and complexity in a rainbow. It is formed by a multitude of raindrops, all nearly spherical[1], in which the light from the Sun is reflected, chromatically refracted, reflected again and dispersed by the gently falling water spheres into a thousand hues that can be unpacked by the human eye into seven distinct colors. This reflection, refraction, and reflection of light conforms to lovely geometric theorems, theorems so simple that anyone can grasp them, but also so complex as to defy logical analysis.

> A rainbow is an arc of intense colors embroidered with needles of sunlight onto the fabric of falling rain.

The description of the physical phenomena of seeing a rainbow in the sky strains the capacity of ordinary language, and some of the most powerful tools of mathematical physics have been devised for the express purpose of explaining how the rainbow works. The rainbow is truly one of the more remarkable and wonderful marvels of God's creation.

THE REVELATION OF THE SKY

The single bright and multicolored semi-circular arc seen after a rain shower is called the **primary rainbow**. Seven sectors of light in a rainbow always follow this sequence:

1. Innermost sector: violet
2. Indigo (dark blue)
3. Blue.
4. Green.
5. Yellow.
6. Orange:
7. Outermost sector: red.

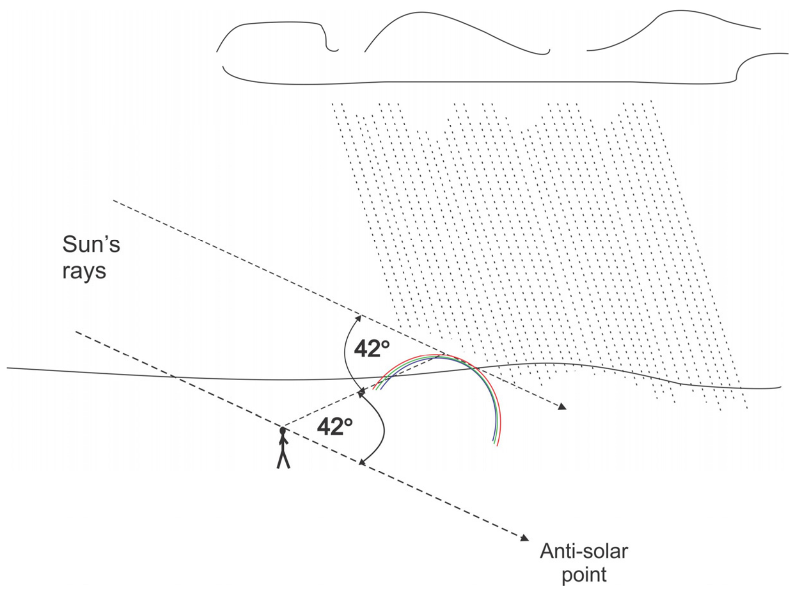
Source: iStockPhoto

On clear days, fainter features of the rainbow are revealed. Higher in the sky, above the primary rainbow, you can see what is called the **secondary rainbow**. The properties of this fainter rainbow are:

1. It is twice the width of the primary rainbow.
2. The colors appear in reverse order of the primary rainbow.

[1] The spherical nature of raindrops is another study, altogether.

3. It is one-tenth as intense, regarding light radiation, as the primary rainbow.



In the section between the primary and secondary rainbows, the sky is considerably darker. It is called **Alexander's Dark Band** in honor of Alexander of Aphrodisias, a Greek philosopher, who first described its characteristics ca. 200 AD.

A series of faint bands, usually pink and green alternatively, can be seen on the inner side of the primary bow. Best seen at the top of the primary box, these bands are called ***supernumerary arcs***. The physics used to describe these arcs is much too advanced for us to consider in this essay.[2]

SOME HISTORY

Aristotle (384-322 BC), Greek philosopher primarily and scientist secondarily, understood the rainbow to be an unusual kind of reflection of sunlight from clouds. This light is reflected at a fixed angle giving rise to a circular cone of rainbow rays.


Source: iStockPhoto

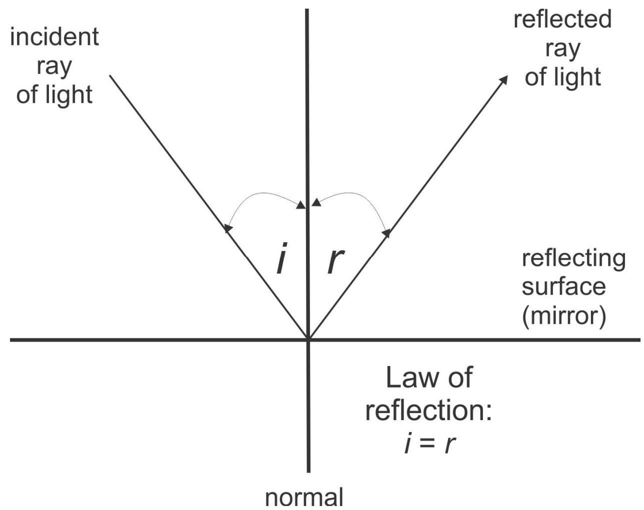
In 1266, Roger Bacon (ca. 1214-1294), English scientist and mathematician, measured this angle and determined it to be 42° for the primary bow and 50° for the secondary bow.

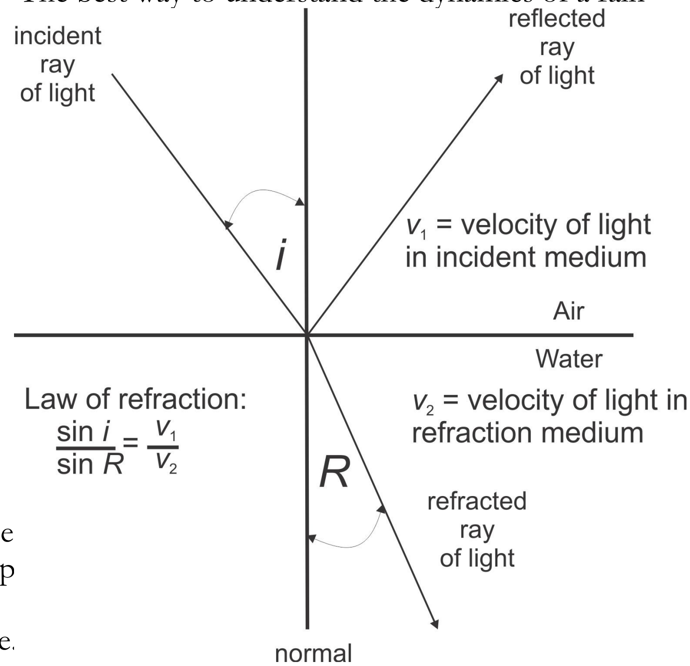
In 1304, the German monk Theodoric of Freiberg (ca. 1250-ca. 1310) rejected Aristotle's interpretation. He stated that each raindrop was individually capable of producing a rainbow. Using a spherical flask filled with water, he was able to confirm his assertion, but his finding remained largely unknown for three centuries.

René Descartes (1596-1650), French philosopher and mathematician rediscovered Theodoric's conclusions. He stated that the primary bow is made up of rays that enter a droplet and are reflected once from the inner surface. The secondary rainbow consists of rays that have undergone two internal reflections and it is fainter because, with each reflection, some of the energy of light is lost. Also, each of the colors in the rainbow comes to the eye from a different set of water droplets.

[2] Just so that the reader is aware of the mathematical complexity, more than 1500 terms in a mathematical formula are needed to describe the distribution in the sky of just one color of the rainbow.

WATER DROP DYNAMICS





The best way to understand the dynamics of a rainbow is to follow Theodoric's approach.[3] Let's conside what happens when light passes through a single drop of water. Physicists have observed that a falling raindrop assumes the shape of a nearly perfect sphere.

To understand what happens when a ray of light strikes a water droplet, we must note two basic laws of optics.

The first law is the **law of reflection**. It states that when a ray of light hits a reflective surface, that angle of incidence equals the angle of reflection.

The second law is the **law of refraction**. It takes into account, quantitatively, the reality that light bends when it enters a new medium. In our example, light is traveling from the medium of air to the medium of water. In air, light travels at about 300,000 km/sec. In water, the speed of light slows to about 225,600 km/sec. The ratio of the speed of light in air to the speed of light in water is $4/3 \approx 1.33$ and it is called the **refractive index** for air to water.

When light strikes the surface of water obliquely, the change in speed results in a change in direction; i.e., light is refracted. The formula governing this creation reality takes into account the trigonometric function called sine (abbreviated sin).[4] If i = the angle of incidence, R = the angle of refraction, v_1 = the speed of light in the incident medium, and v_2 = the speed of light in the refraction medium, then:

$$\frac{\sin i}{\sin R} = \frac{v_1}{v_2}$$

This law is also called Descartes' law, the law of sines, or Snell's law, after the Dutch physicist Willebrord Snell (1591-1626) who derived it but never published it in his lifetime.

From air in a vacuum to the medium:	Refractive index
Vacuum	1.000

[3] The source of some of the physics to follow is from H. Moysés Nussenzveig, "The Theory of the Rainbow," *Scientific American*. 236:4 (April 1977), 116-127.

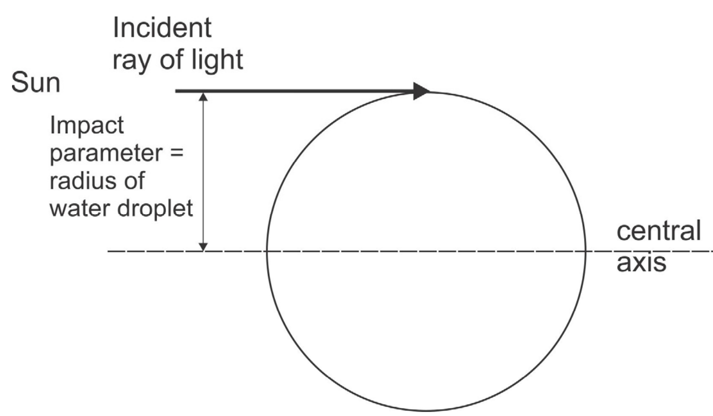
[4] If θ as the measure of one of the acute angles in a right triangle, then the ratio of the side opposite θ to the hypotenuse (the side opposite the right angle) is defined as the sine ratio. In symbols, $\sin\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}}$. That a ratio of a right triangle turns up in the law of the refraction of light is one of the many amazing connections (unity in diversity) in science and mathematics.

From air in a vacuum to the medium:	Refractive index
Atmosphere	1.000277
Ice	1.31
Water	1.33
Ethyl alcohol	1.362
30% sugar solution	1.38
Fused quartz	1.46
Glycerin	1.473
80% sugar solution	1.49
Typical crown glass	1.52
Heavy flint glass	1.65
Sapphire	1.77
Diamond	2.417

THE IMPACT PARAMETER

An important variable in the analysis of the path of light through a water droplet is called the **impact parameter**. It is the distance, or displacement, of the incident ray from an axis passing through the center of the droplet. The values or range of the impact parameter are:

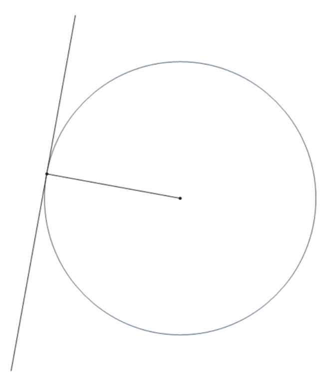
1. From zero (this happens when the incident ray coincides with the central axis).
2. To the radius of the droplet (this happens when the incident ray is tangent to the water droplet).



Impact parameter tangent to the water droplet

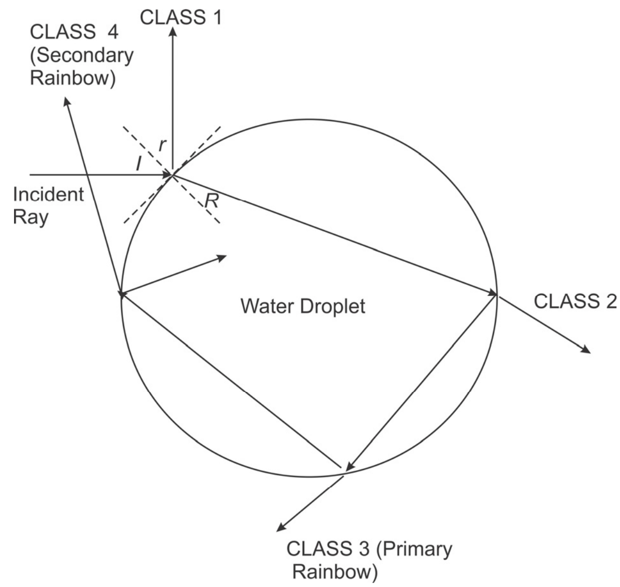
As a side note, in geometry, we can construct a line tangent to a circle at a given point on the circle by drawing the following:

1. Draw a circle.
2. Mark a point on it.
3. Draw the radius.
4. Construct a line perpendicular to the radius at that point.


Line tangent to a point on a circle

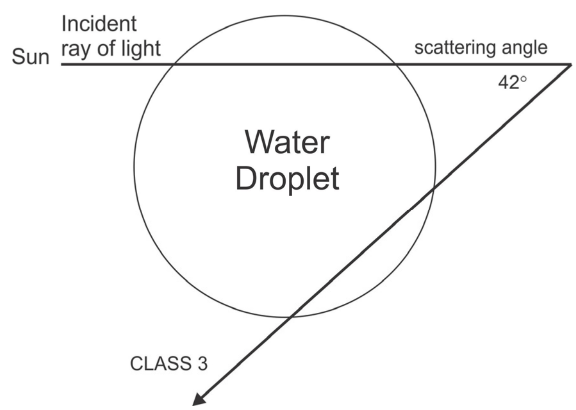
A JOURNEY

Let's now follow the path of a light ray as it makes its journey through a water droplet. First, when the incident ray contacts the surface of the water droplet, it is partially reflected. We denote this reflected ray a CLASS 1 ray. The remaining light is transmitted into the droplet with a change of direction caused by refraction. Second, when the incident ray contacts the next surface, the ray is again partially transmitted out of the water droplet with a change of direction caused by refraction. We denote this exit rays as a CLASS 2 ray. As before, the ray is partially reflected. Third, at the next boundary, the ray is again partially transmitted out of the droplet by the process of refraction. We denote this exit ray as a CLASS 3 ray. *It is the CLASS 3 rays that make up the primary rainbow.* As before, the ray is again partially reflected. This process continues *ad infinitum* and rays of higher classes (4, 5, 6, etc.), each with their associated rainbows, are theoretically formed (and, indeed, fill the sky!). But due to light intensity loss, they cannot be seen with our eyes.



Light's journey through a water droplet

We define the **scattering angle** for the primary rainbow as the measure between the initial incident ray and the CLASS 3 ray. It measures about 42°. The scattering angle for the secondary rainbow is the angle between the initial incident ray and the CLASS 4 ray. It measures about 51°. The scattering angle is a function of the impact parameter. Since, in sunlight, the droplet is illuminated at all impact parameters simultaneously, light is scattered in virtually all directions. An important question to ask, then, is why do we see the rainbow colors enhanced at a particular angle?



Scattering angle

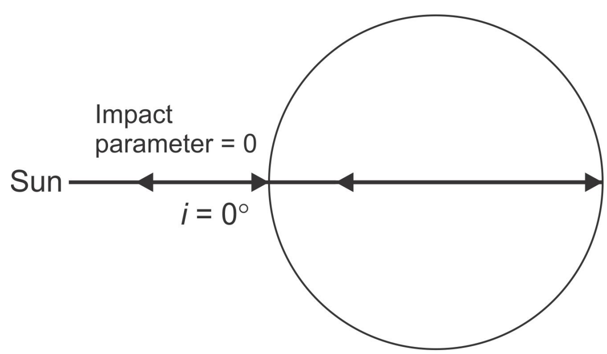
AN IRIDESCENT ANSWER!

Descartes found the answer. He painstakingly applied the laws of reflection and refraction at *each* point where an incident ray strikes the surface of the water droplet. Using these laws, he computed the paths of incident rays based upon various impact parameters.

Before we inspect Descartes' method, let's create a refraction table based upon the index of refraction from air to water of 4/3. Hence, $\frac{\sin i}{\sin R} = \frac{4}{3} \Leftrightarrow 3\sin i = 4\sin R \Leftrightarrow \sin R = \frac{3}{4}\sin i \Leftrightarrow R = \sin^{-1}\left(\frac{3}{4}\sin i\right)$. Using this formula, $R = \sin^{-1}\left(\frac{3}{4}\sin i\right)$, we can compute the angle of refraction, *R*, for incident angles ranging from 0° to 90°:

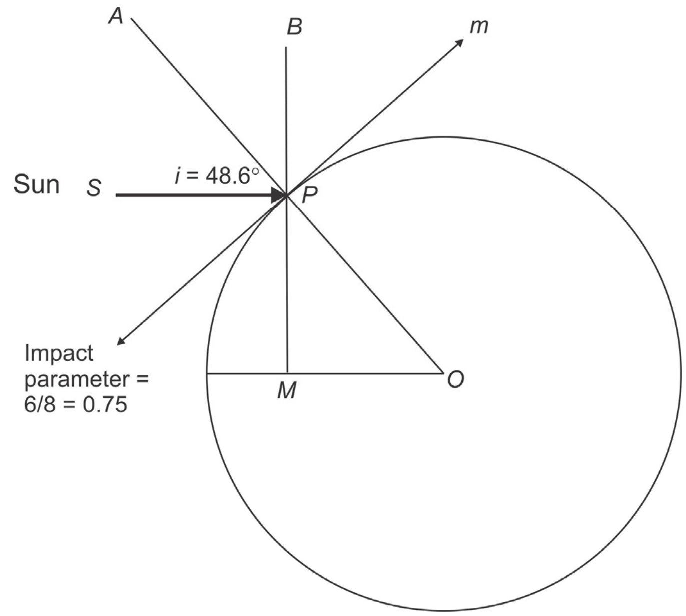
i	*R*	*i*	*R*	*i*	*R*	*i*	*R*	*i*	*R*	*i*	*R*
0°	0°	16°	11.9°	32°	23.4°	48°	33.9°	64°	42.4°	80°	47.6°
1°	0.75°	17°	12.7°	33°	24.1°	49°	34.5°	65°	42.8°	81°	47.8°
2°	1.5°	18°	13.4°	34°	24.8°	50°	35.1°	66°	43.2°	82°	48.0°
3°	2.2°	19°	14.1°	35°	25.5°	51°	35.7°	67°	43.7°	83°	48.1°
4°	3.0°	20°	14.9°	36°	26.2°	52°	36.2°	68°	44.1°	84°	48.2°
5°	3.7°	21°	15.6°	37°	26.8°	53°	36.8°	69°	44.4°	85°	48.3°
6°	4.5°	22°	16.3°	38°	27.5°	54°	37.4°	70°	44.8°	86°	48.4°
7°	5.2°	23°	17.0°	39°	28.2°	55°	37.9°	71°	45.2°	87°	48.5°
8°	6.0°	24°	17.8°	40°	28.8°	56°	38.4°	72°	45.5°	88°	48.6°
9°	6.7°	25°	18.5°	41°	29.5°	57°	39.0°	73°	45.8°	89°	48.6
10°	7.5°	26°	19.2°	42°	30.1°	58°	39.5°	74°	46.1°	90°	48.6°
11°	8.2°	27°	19.9°	43°	30.8°	59°	40°	75°	46.4°		
12°	9.0°	28°	20.6°	44°	31.4°	60°	40.5°	76°	46.7°		
13°	9.7°	29°	21.3°	45°	32.0°	61°	41.0°	77°	47.0°		
14°	10.5°	30°	22.0°	46°	32.6°	62°	41.5°	78°	47.2°		
15°	11.2°	31°	22.7°	47°	33.3°	63°	41.9°	79°	47.4°		

Let's now analyze CLASS 3 rays. When the impact parameter is 0, these rays are scattered through an angle of 0°; i.e., they are backscattered toward the Sun, having passed through the center of the droplet and reflected from the far wall.



CLASS 3 angle of incidence = 0°

Next, Descartes considered impact parameters between 0 and the radius of the droplet. By geometric analysis, we can determine the angle of incidence based upon the ratio of the impact parameter to the radius of the droplet. We consider $\overrightarrow{SP}$, a ray from the sun that strikes the surface of the water droplet at point P. We draw a line tangent to circle *O* and point *P* and label this line with the letter *m*. The normal of this line, or the line perpendicular to it, is $\overline{AO}$. Hence, $\overline{AO} \perp m$. $\overline{MO}$ is a portion of the central axis. We construct $\overline{BM} \perp \overline{MO}$. Note right triangle ΔPMO. Since vertical angles are equal, we know that m$\angle APB$ = m$\angle MPO$. We let $PO = 8$ (the radius of the droplet) and $PM = 6$ (the impact parameter). Therefore, $\frac{PM}{PO} = \frac{6}{8} = 0.75$. Knowing this ratio will give us m$\angle MPO$ = $\cos^{-1}(0.75) = 41.4^\circ$. Therefore, m$\angle APB$ = 41.4° and since m$\angle SPB$ = 90°, then m$\angle APS$ = 90° – 41.4° = 48.6° (the angle of incidence).



Angle of incidence = 48.6°
Impact parameter = 75% of radius

In general, if we know the impact parameter (IP) and the radius (r), then i, the angle of incidence, is calculated as follows:

$$i = 90^\circ - \cos^{-1}\left(\frac{IP}{r}\right)$$

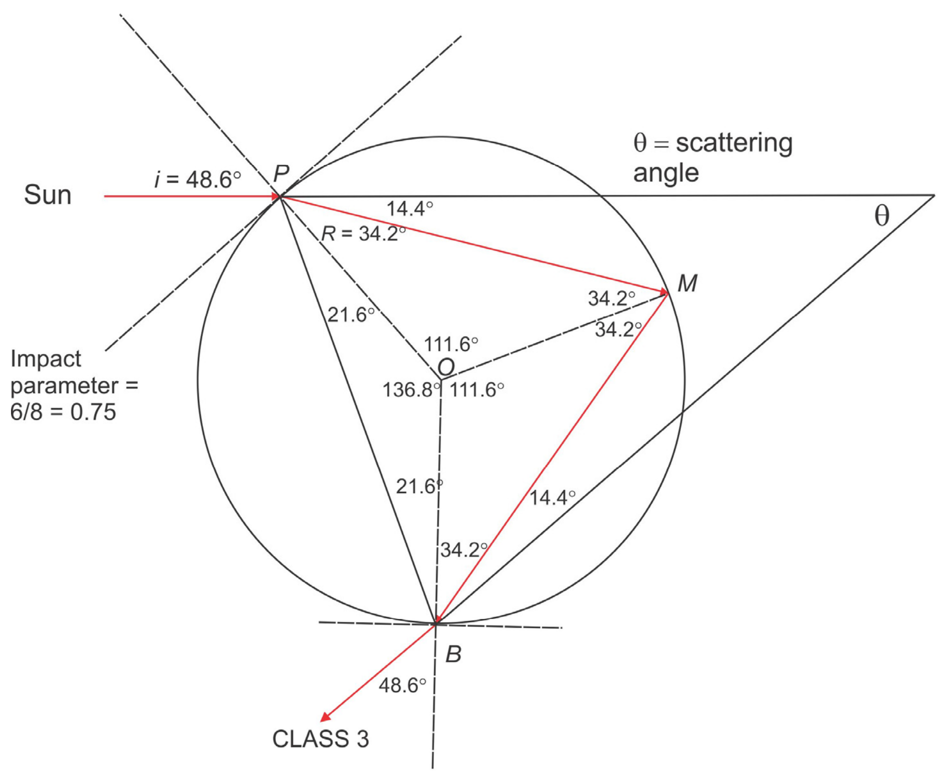
Here is a table for *i*, given particular values of $\frac{IP}{r}$:

$\frac{IP}{r}$	i
0	0
0.0625	3.6°
0.125	7.2°
0.250	14.5°
0.375	22.0°
0.500	30.0°
0.625	38.7°
0.750	48.6°
0.8125	54.3°
0.875	61.0°
0.9375	69.6°
1	90°

Given i, the angle of incidence, we can calculate R, the angle of refraction, based upon our previously derived formula (the angle of refraction from air to water): $R = \sin^{-1}\left(\frac{3}{4}\sin i\right)$. Next on our agenda of geometric analysis is to determine how to find the scattering angle for a CLASS 3 ray. To help us navigate to a conclusion, we again set $\frac{IP}{r} = 0.75$ and we seek to find θ, the scattering angle. Let's first trace the rays inside the droplet. $\overrightarrow{PM}$ is the first refracted ray. $\overrightarrow{MB}$ is the first internally reflected ray and the refracted exit ray at point B is the CLASS 3 ray. What is the measure of θ? By the law of refraction, we know that m∠OPM = 34.2°. $\overline{OM}$ is the normal to the line tangent to the droplet at point M. By the law of reflection m∠PMO = m∠OMB = 34.2°. $\overline{OB}$ is the normal to the line tangent to the droplet at point B, the exit point. m∠OBM = 34.2° and by the law of refraction the exit angle is indicated at 48.6°. The beautiful symmetry of these internal angles and the exit angle exists because the light rays confirm to the reflection and refraction laws.

Now note the triangles thus formed. Since the measure of the angles of a triangle equals 180°, then, in ΔPOM, m∠POM = 180° – (34.2° + 34.2°) = 111.6°. Likewise, m∠MOB = 180°. Since vertical angles are equal, then the initial angle of incidence, 48.6° = m∠APO. Since m∠APO = m∠APM + m∠MPO, then m∠APM = 48.6° – 34.2° = 14.4°. The same reasoning will show that m∠APO = 14.4°.

We now consider ΔAPB. If we can calculate m∠APB and m∠ABP, we can then determine θ. Since the measure of a circle is 360°, then m∠POB = 360° – (111.6° + 111.6°) = 136.8°. Since $\overline{PO}$ and $\overline{OB}$ are radii of a circle, then ΔPOB is isosceles and, by one of the theorems of Euclidean geometry, ∠BPO and ∠PBO are equal angles. With a little bit of arithmetic, we can calculate m∠BPO = m∠PBO = 21.6°. Therefore, m∠APB = m∠ABP = 21.6° + 34.2° + 14.4° = 70.2°. We now have enough information to determine θ = 180° – (70.2° + 70.2°) = 39.6°. QED!



CLASS 3 scattering angle

We can invoke the above reasoning for any impact parameter. To make matters easier, can we determine a formula for finding θ? Note carefully that this formula will involve i, the initial angle of incidence, and R, the initial angle of refraction. We know that θ = 180° – 2i – 2k = 180° – 2(i + k). We can determine k regarding R as follows:

$$k = \frac{180^0 - \left[360^0 - 2(180^0 - 2R)\right]}{2} \Leftrightarrow k = \frac{180^0 - \left(360^0 - 360^0 + 4R\right)}{2} \Leftrightarrow k = 90^0 - 2R$$

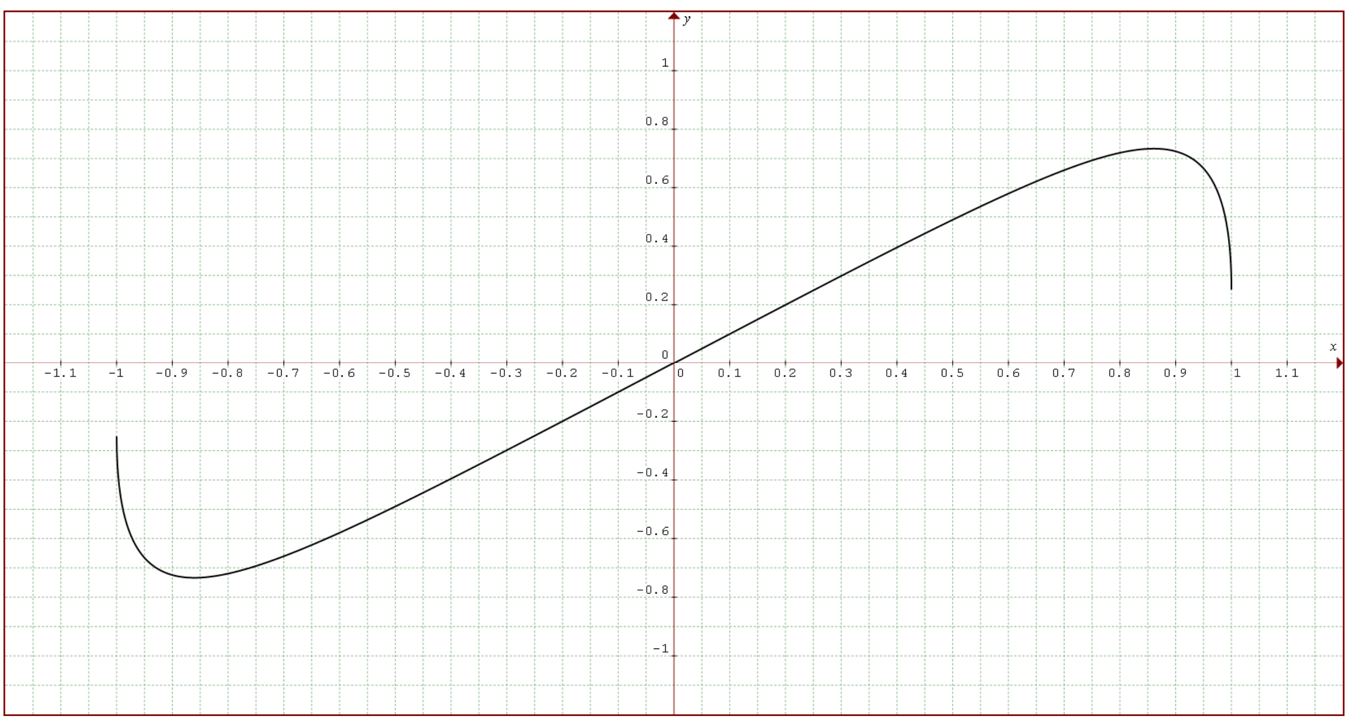
Therefore, $\theta = 180° – 2(i + 90° – 2R) = 180° – 2i – 180° + 4R = -2i + 4R = 4R – 2i$

In summary, our formula for finding θ, the scattering angle for CLASS 3 rays, is:

$\theta = 4R – 2i$

We can now graph the scattering angle verses the impact parameter. We first create a table:

$\frac{IP}{r}$	θ
0	0°
0.0625	3.6°
0.125	7.2°
0.250	14.2°
0.375	21.2°
0.500	28°
0.625	34.6°
0.750	39.6°
0.8125	41.4°
0.875	42°
0.9375	39.6°
1	14.4°



CLASS 3 scattering angle (only images in Quadrant I are relevant)

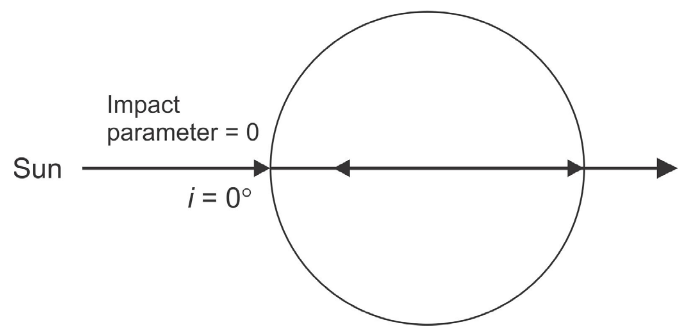
The function that governs this table where $x = \frac{IP}{r}$, $i = \frac{\pi}{2} - \cos^{-1}(x)$, and θ is in radians[5] (i.e., $90° = \frac{\pi}{2}$) is:

$$\theta = f(x) = 4R - 2i \Leftrightarrow \theta = f(x) = 4\left\{\sin^{-1}\left[\frac{3}{4}\sin\left(\frac{\pi}{2} - \cos^{-1}(x)\right)\right]\right\} - 2\left[\frac{\pi}{2} - \cos^{-1}(x)\right] \Leftrightarrow$$

$$\theta = f(x) = 4\sin^{-1}(0.75x) + 2\cos^{-1}(x) - \pi$$

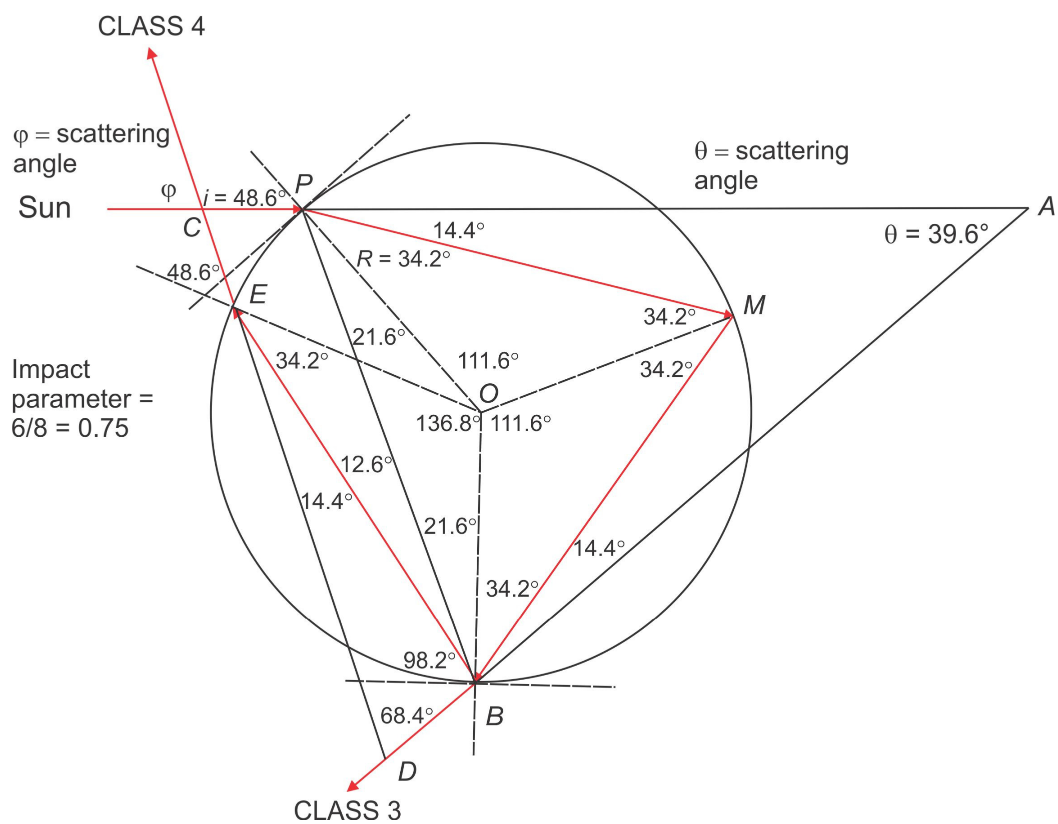
As we can see from the CLASS 3 graph, the scattering angle increases and passes through a maximum (x = 0.73356 radians = 42.03°) when the impact parameter is about 7/8 (to be exact, 0.8607) of the radius of the droplet. After that, the scattering angle decreases.

Next, let's consider the scattering angle, φ, for the CLASS 4 exit ray. When the impact parameter is zero, the scattering angle is 180° or π radians. The reason for this is that the central ray is reflected twice and then continues in its original direction.



CLASS 4 angle of incidence = 0°

We again consider the situation $\frac{IP}{r} = 0.75$ and we seek to find φ, the scattering angle. $\overrightarrow{BE}$ is the ray



CLASS 4 scattering angle

[5] A radian is the measure of a central angle subtending an arc equal in length to the radius and is equal to 57.2958° = $\frac{360^\circ}{2\pi}$

resulting from the second internal reflection. At point E, the CLASS 4 ray exits the droplet. We can find φ by finding m∠*ADC* in Δ*ADC*. Since vertical angles are equal, φ = m∠*ACD* = 180° – (θ + m∠ADC). By the law of reflection, m∠*MBO* = m∠*EBO* = 34.2°. Therefore, m∠*EBD* = 180° – (34.2° + 34.2° + 14.4°) = 98.2°. Since vertical angles are equal, m∠*DEO* = 48.6°. Since m∠*BEO* = 34.2°, then m∠*DEB* = 14.4°. Therefore, m∠*ADC* = 180° – (14.4° + 98.2°) = 68.4°. We can now calculate φ = m∠*ACD* = 180° – (θ + m∠*ADC*) = 180° – (39.6° + 68.4°) = 72°.

Again, we can invoke the same reasoning for any impact parameter. To make matters easier, we can determine a formula for φ just as we did for θ. Note carefully that this formula will again involve *i*, the initial angle of incidence, and *R*, the initial angle of refraction. We know that φ = 180° – (θ + *k*). We know that *k* = 180° – {[180° – (*i* + *R*)] + (*i* – *R*)}. Simplifying this equation, we get:

$k = 180° - (180° - i - R + i - R) \Leftrightarrow k = 180° - 180° + 2R = 2R$

We now have our formula for φ in terms of *R*: φ = 180° – (θ + 2*R*). We can now graph the scattering angle of the CLASS 4 exit ray verses the impact parameter. We first create a table:

$\frac{IP}{r}$	φ
0	180°
0.0625	171°
0.125	162°
0.250	144.2°
0.375	126.2°
0.500	108°
0.625	89.4°
0.750	72°
0.8125	63.6°
0.875	56°
0.9375	51°
1	64.4°

The function that governs this table where $x = \frac{IP}{r}$ and φ is in radians (i.e., 180° = π) is:

$$\varphi = f(x) = \pi - (\theta + 2R) \Leftrightarrow$$
$$\varphi = f(x) = \pi - \left[4\sin^{-1}(0.75x) + 2\cos^{-1}(x) - \pi + 2\left(\sin^{-1}(0.75x)\right)\right] \Leftrightarrow$$
$$\varphi = f(x) = 2\pi - \left(6\sin^{-1}(0.75x) + 2\cos^{-1}(x)\right)$$

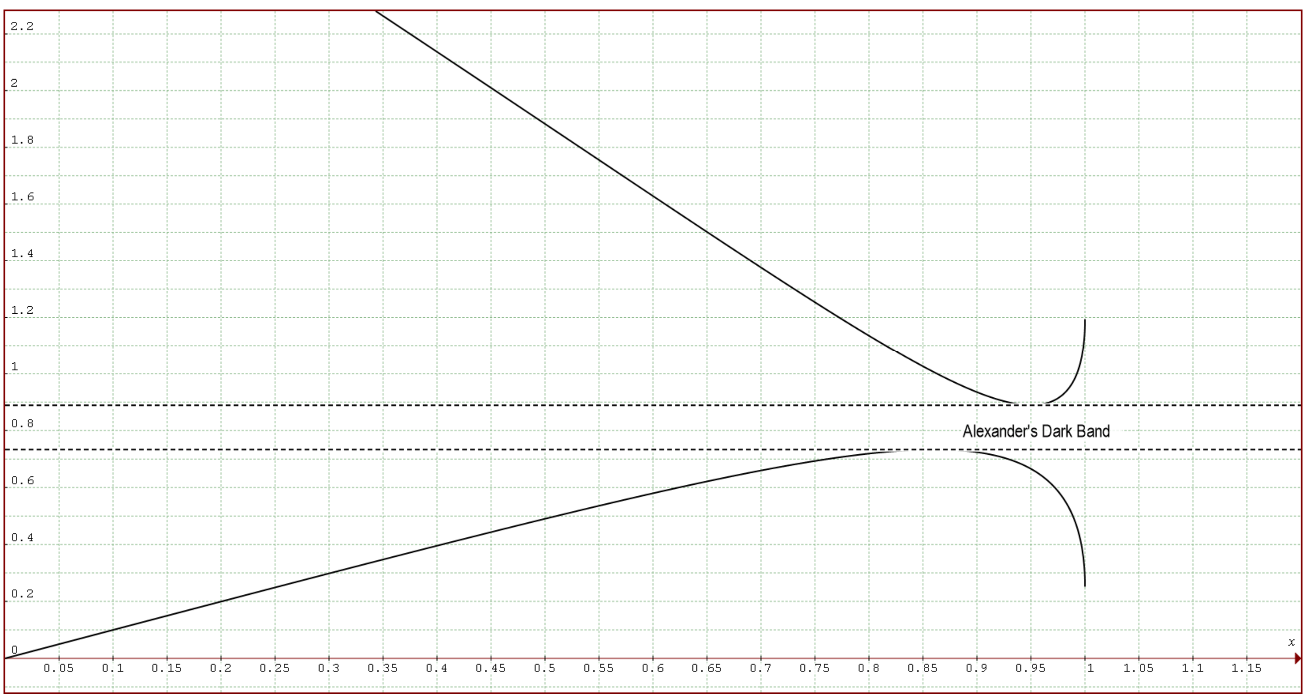
We note that as the impact parameter increases, the scattering angle decreases and reaches a minimum at 0.8897 radians or 50.98°. After that, the scattering angle increases.

CLASS 4 scattering angle

Let's now put all of our tables together to get a sense of the relationships between $\frac{IP}{r}$, i, *R*, θ, and φ.

$\frac{IP}{r}$	*i*	*R*	θ	φ
0	0°	0°	0°	180°
0.0625	3.6°	2.7°	3.6°	171°
0.125	7.2°	5.4°	7.2°	162°
0.250	14.5°	10.8°	14.2°	144.2°
0.375	22°	16.3°	21.2°	126.2°
0.500	30°	22°	28°	108°
0.625	38.7°	28°	34.6°	89.4°
0.750	48.6°	34.2°	39.6°	72°
0.8125	54.3°	37.5°	41.4°	63.6°
0.875	61°	41°	42°	56°
0.9375	69.6°	44.7°	39.6°	51°
1	90°	48.6°	14.4°	64.4°

Plotting the two graphs on the same axes summarizes the possibilities pictorially, along with the revelation of Alexander's Dark Band.



CLASS 3 & 4 scattering angle

Mathematics, Physics, and the Rainbow

by James D. Nickel, BA, BTh, BMiss, MA

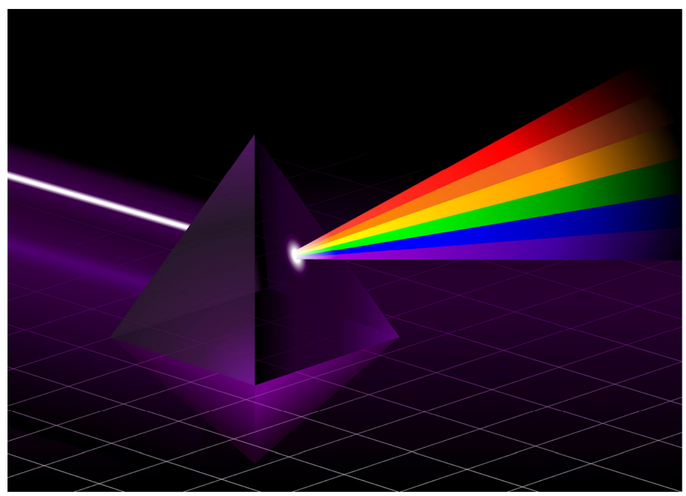
Because a droplet in sunlight is uniformly illuminated, the impact parameters of the incident rays are uniformly distributed. The concentration of scattered light is therefore expected to be greatest where the scattering angle varies most slowly with the changes in the impact parameter. In other words, the scattered light is brightest where it gathers together the incident rays from the largest range of impact parameters.


Alexander's Dark Band. Source: iStockPhoto

From the graphs, we see that the regions of minimum variation are those surrounding the maximum and minimum scattering angles. Thus, the particular status of the primary and secondary rainbow angles is explained. The primary bow is 42.03° and the secondary bow is 50.98°. Since no rays of CLASS 3 and CLASS 4 are scattered into the angular region between 42.03° and 50.98°, Alexander's dark band is also explained, a region measuring 50.98° – 42.03° = 8.95°. It is darker than the rest of the sky because no CLASS 3 or CLASS 4 rays enter this area. This region would be pitch dark if only rays of CLASS 3 and CLASS 4 existed.

The Colors

Why do we see the colors we see? In 1666, Sir Isaac Newton (1642-1727) split white light into its spectrum of colors using a prism. He also showed that the refractive index for each color is different. This difference means that light, as CLASS 3 rays, comes back out of the droplet in different directions (or angles) depending upon its color, or each color goes through a different angle of "bending." Each color has its angle, ranging from a minimum of 40°17′ for violet to 42°2′ for red.


Source: iStockPhoto

How has the Triune God of beauty ordered our brain interpret these particular colors? Color is a sensation of our brains generated when our eyes receive light energy at a particular frequency. What do we mean by frequency? Think of a radio set. You turn the dial to a certain "frequency" to receive a signal (called radio waves or "energy in motion") from a radio station's emitting tower. Like dropping a pebble into a pond, this energy is transmitted in a wavelike (up and down) form in all directions. To help us understand this wavelike motion, tie a rope to a tree or another person. Grab the other end and begin vibrating the rope with your arm in an "up and down" motion. The faster you whip the rope, the higher the frequency (or vibration) and the greater the energy (as the person at the other end of the rope would confirm!).

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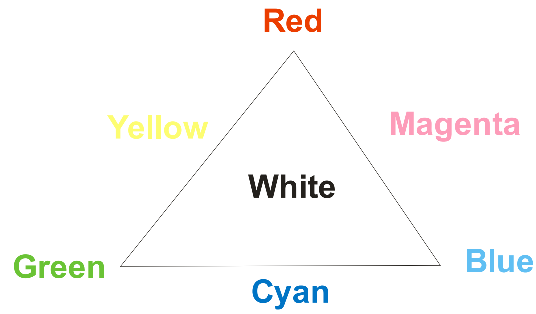
The frequency of the radio waves for AM and shortwave radio range from 10^4 to 10^7 vibrations (or cycles) per second. These cycles are identified by a unit called the hertz (Hz), named after the German physicist Heinrich Rudolph Hertz (1857-1894). These frequencies are on your radio dial in kilohertz units (kHz) where 1 kHz = 1000 Hz: 531 TO 1692 KHz. FM radio and television waves range from 10^7 to 10^8 Hz. On your radio dial, this range is identified in megahertz (MHz) where 1 MHz = 1,000,000 Hz: 88.1 to 107.9 MHz. The frequency of radar waves is 10^{10} Hz. Visible light is also wavelike, and its frequency is between 4×10^{14} (red light) and 7.5×10^{14} Hz (violet light).

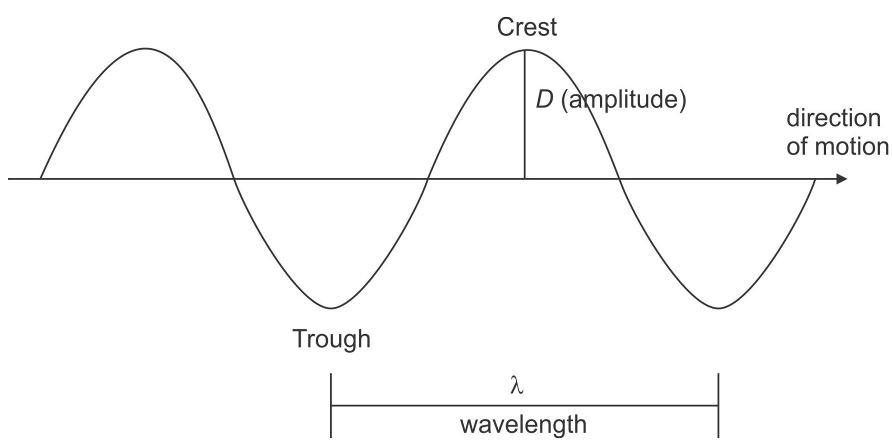


The Father, Son, and Spirit designed the human eye to act as a radio receiver. Light consists of three primary colors: red, green, and blue. The secondary colors are yellow (a mixture of red and green), cyan (a mixture of green and blue), and magenta (a mixture of blue and red). An even mixture of red, green, and blue produces white. Complementary colors, when mixed, also produce white. There are two sets of complementary colors: green and magenta and blue and yellow. Pigments are coloring substances that absorb some colors and reflect others. For example, black paint absorbs all colors and reflects none while white paint reflects all colors and absorbs none. When we mix pigments, they reflect only the colors that neither pigment absorbs. For example, yellow paint reflects red and green light while absorbing blue light. Cyan paint reflects green and blue light while absorbing red light. Magenta paint reflects red and blue light while absorbing green light. The mix of yellow and cyan pigments reflects green light and produces green paint, mixing yellow and magenta reflects red light and produces red paint, and mixing magenta and cyan reflect blue light and produces blue paint.

The human eye contains three sets of nerves (called cones) that respond, by the wisdom design of the Logos, Jesus Christ (John 1:1-3; Colossians 1:15-17), to the three primary colors. If all three cones are equally stimulated, we see white. If the green cones are stimulated, we see green. If the green and red cones are equally stimulated, we see yellow. All the wonderful hues and shades of color are produced by the appropriate stimulation of these color cones. When light is reflected and refracted through water droplets in the sky, our eye receives the precise light wave vibrations, the three cones are subsequently stimulated, and we see the beautiful, semi-circular seven-colored band of the primary rainbow, a visible sign in the heavens reflecting the Triune God's covenants of promise! What a wonder of wonders!

The Electromagnetic Spectrum

We will bring our discussion of the physics of the rainbow to a conclusion with a brief analysis of the electromagnetic spectrum, a complex yet orderly arrangement the flows out of the creative word of God, Jesus Christ, the logos (see Genesis 1:3; John 1:1-3). All energy radiation propagates in waves that oscillate (up and down motion). These waves are also **transverse** meaning that the up and down motion (also called displacements) is perpendicular to the direction in which the waves travel.



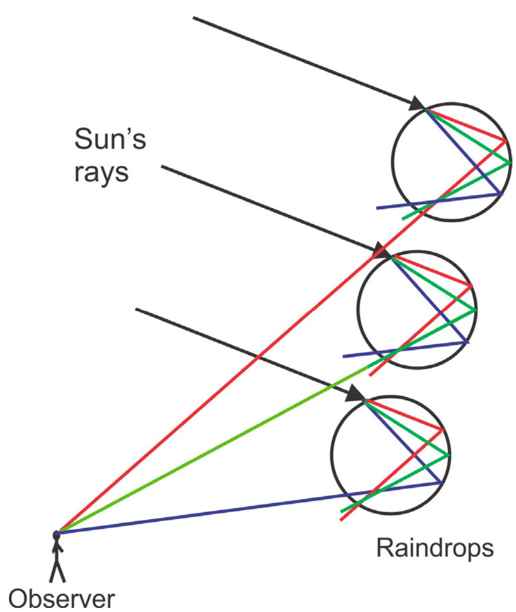
These waves can all travel in a vacuum and their velocity, as discovered by the Scottish physicist James Clerk Maxwell (1831-1879), is equivalent to the speed of light or approximately 3×10^5 km/s = 3×10^8 m/s.

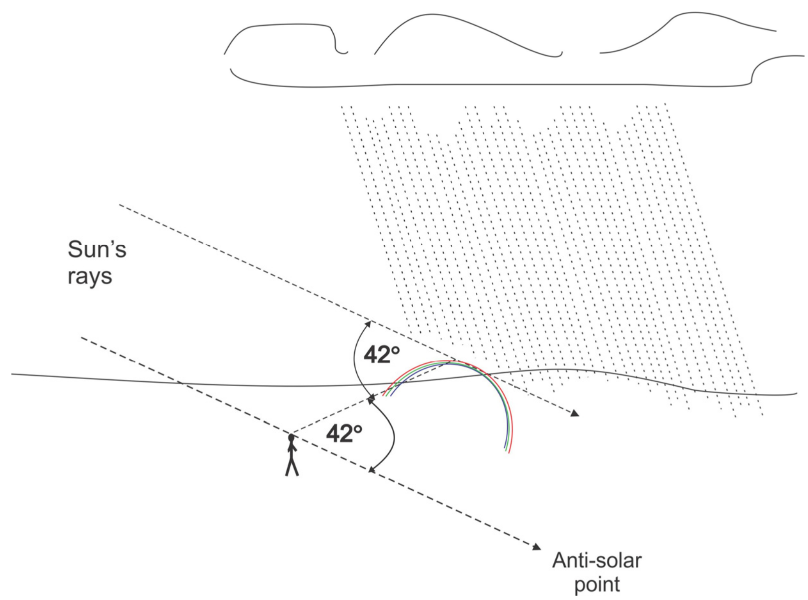
There are three identifying features of each wave:

1. frequency or radiation (f)
2. wavelength (λ)
3. amplitude (*D*)

Frequencies are determined by the wave equation $v = f\lambda$ where v = velocity of light. Scientists commonly measure wavelengths in a unit called Angstrom (Å), named after the Swiss astronomer and physicist Anders Ångström (1814-1874). By definition, 1Å = 10^{-8} cm = 10^{-10} m. The shorter the wavelength, the large the refraction angle. This inverse relationship means that the short wavelengths of violet light are refracted most, and the long wavelengths of red light are refracted least.

When it rains, we see the rainbow colors reflected in an ordered and precise manner from a multiplicity of individual raindrops. Our eyes can always see CLASS 3 rays and, when the sky is clear enough; CLASS 4 rays are visible. From our unique position on the ground our eyes, acting as a radio receiver, gather in these light radiations from raindrops that form a semi-circular arc in the sky. The CLASS 3 scattering angle for red light is 42.03° (the top band of the primary bow) and 40.28° for violet light (the bottom band of the primary bow). When we change positions, we see a different rainbow; i.e., the rainbow we see from our vantage point is uniquely ours.

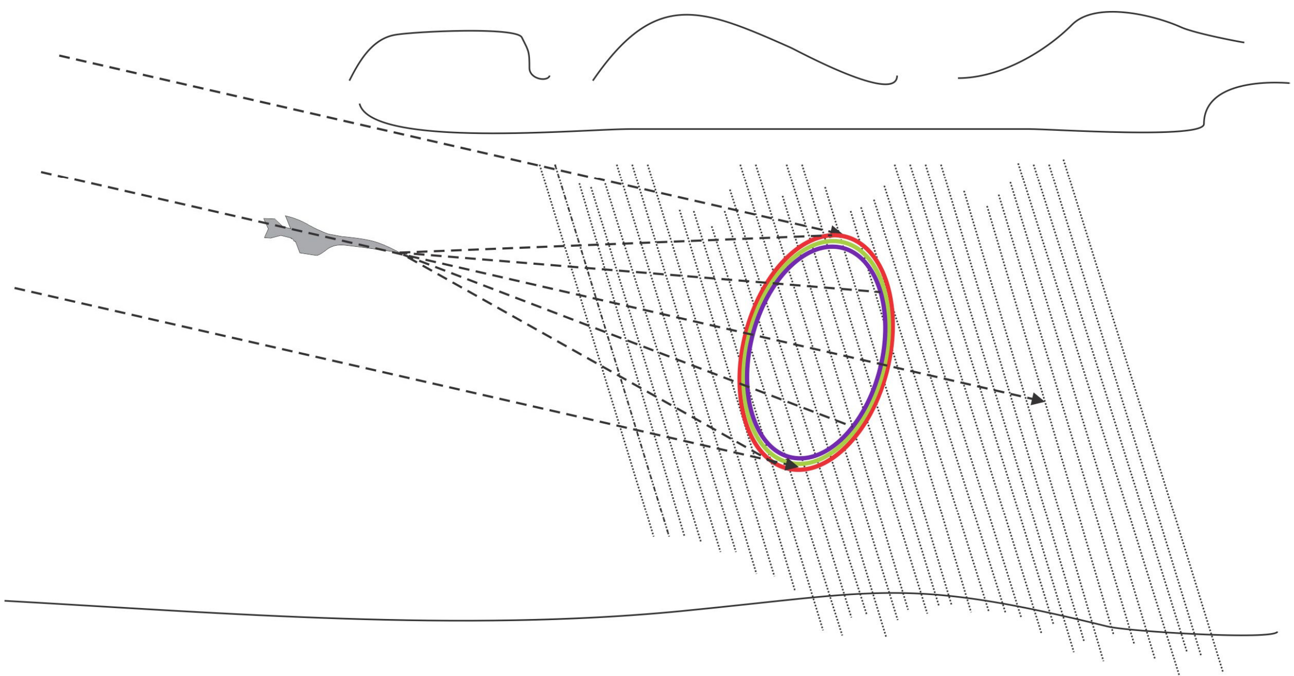




Mathematics, Physics, and the Rainbow

by James D. Nickel, BA, BTh, BMiss, MA

The rainbow in the sky is uniquely ours … a fitting conclusion to our study. The Father, Son, and Spirit, being in loving communion, gave the bow in the sky as a sign of His covenant promise to the world, and this promise is interpreted to us, using the laws of physics that model the revelation of God's created order, *as a very personal message*. This God, the Lord of all, the infinite, eternal, transcendent One in Three, the God who numbers the stars is also the immanent One in Three, the personal God who numbers the hairs on our head. In His ordering of the Sun and the rain, He reveals to each and every human who has eyes to see a very personal rainbow of His manifold mercies.



Circular rainbow as seen from an airplane

The Electromagnetic Spectrum

f (Hz)	λ (meters)						
10^{21}	3×10^{-13}						
10^{20}	3×10^{-12}	↑	Gamma Rays ↓				
10^{19}	3×10^{-11}						
10^{18}	3×10^{-10}			↑			
10^{17}	3×10^{-9}	X-rays					
10^{16}	3×10^{-8}	↓		Ultraviolet radiation ↓			
					Violet	4000Å	4×10^{-7} m
					Indigo	4300Å	
					Blue	4800Å	
				Visible light	**Green**	5300Å	
10^{15}	3×10^{-7}				**Yellow**	5800Å	
					Orange	6100Å	
					Red	7000Å	7×10^{-7} m
10^{14}	3×10^{-6}		↑				
10^{13}	3×10^{-5}		Infrared radiation				↑
10^{12}	3×10^{-4}		↓				
10^{11}	3×10^{-3}				↑		Micro-waves
10^{10}	3×10^{-2}	↑					
10^{9}	3×10^{-1}						↓
10^{8}	3×10^{0}		↑	↑			
10^{7}	3×10^{1}		TV ↓	AM/FM radio	Communication bands: Amateur, police, airplanes, etc.		
10^{6}	3×10^{2}						
10^{5}	3×10^{3}			↓			
10^{4}	3×10^{4}	Radio waves					
10^{3}	3×10^{5}				↓		
10^{2}	3×10^{6}						
10^{1}	3×10^{7}	↓					
10^{0}	3×10^{8}						