Numerical Integration with MATLAB

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Integration

The integral of a function $f(x)$ is denoted as:

$$\int_a^b f(x)dx$$

Integration

Given the function:

$$y = x^2$$

We know that the exact solution is:

$$\int_0^a x^2 dx = \frac{a^3}{3}$$

The integral from 0 to 1 is:

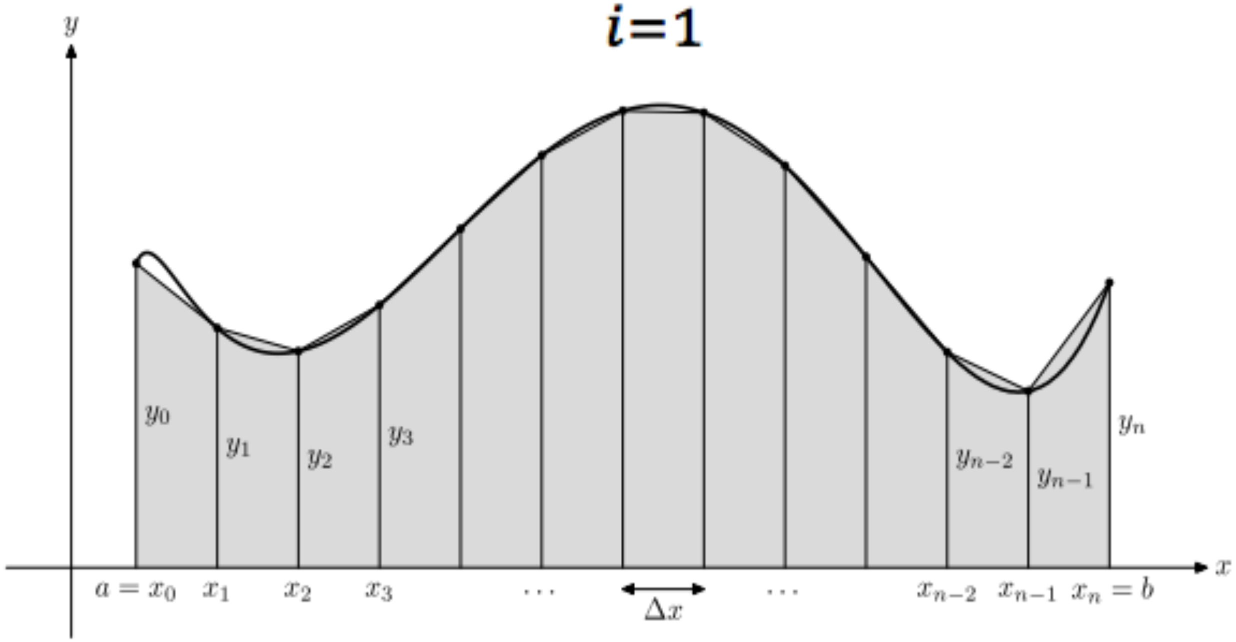
$$\int_0^1 x^2 dx = \frac{1}{3} \approx 0.3333$$

$$\int_a^b f(x)dx$$

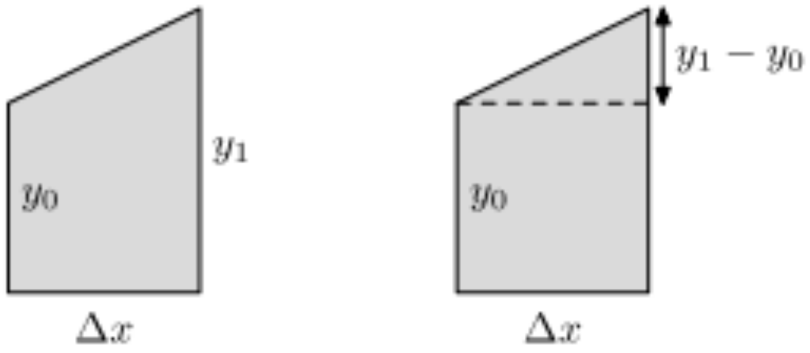
Numerical Integration

An integral can be seen as the area under a curve.
Given $y = f(x)$ the approximation of the Area (A) under the curve can be found dividing the area up into rectangles and then summing the contribution from all the rectangles (trapezoid rule):

$$A = \sum_{i=1}^{n-1} (x_{i+1} - x_i) \cdot (y_{i+1} + y_i)/2$$



$$A = y_0\,\Delta x + \frac{1}{2}(y_1 - y_0)\Delta x = \frac{(y_0 + y_1)\Delta x}{2}.$$



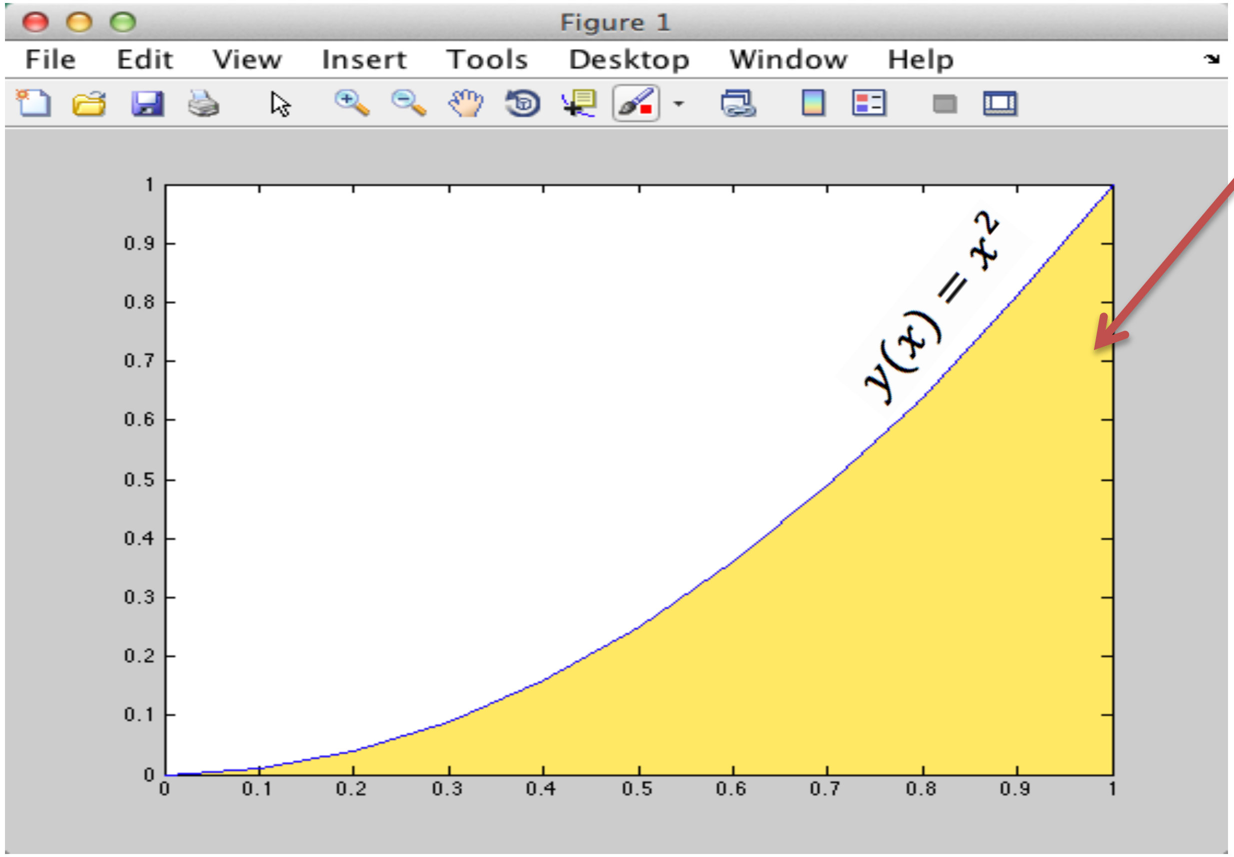
Example:

Numerical Integration

We know that the exact solution is:

$$y(x) = x^2 \rightarrow \int_a^b y(x)\,dx = ? \rightarrow \int_0^a x^2 dx = \frac{a^3}{3}$$

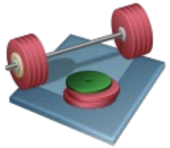
$$\int_0^1 x^2 dx = \frac{1}{3} \approx 0.3333$$



We use MATLAB (trapezoid rule):

```
x=0:0.1:1;
y=x.^2;
plot(x,y)

% Calculate the Integral:
avg_y=y(1:length(x)-1)+diff(y)/2;
A=sum(diff(x).*avg_y)
```

A = 0.3350



Students: Try this example

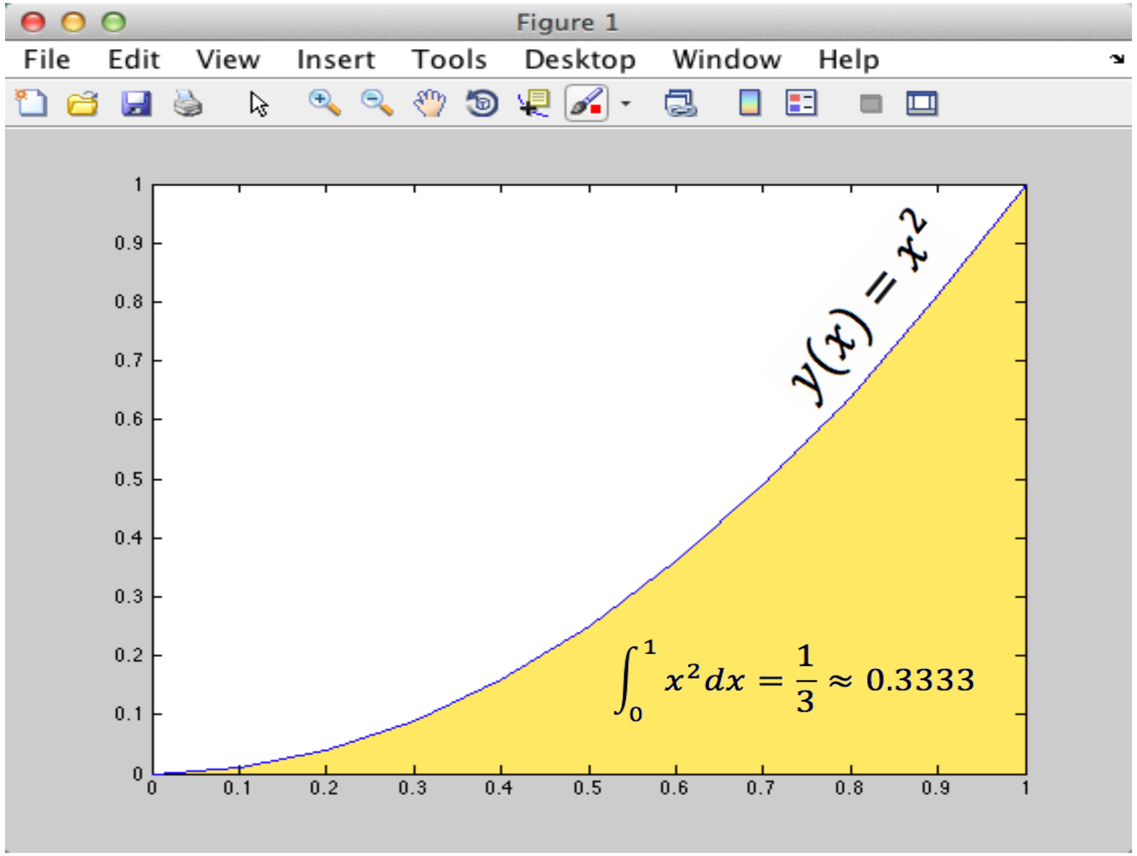
Example:

Numerical Integration

We know that the exact solution is:

$$y(x) = x^2 \rightarrow \int_a^b y(x)\,dx = ? \rightarrow \int_0^a x^2 dx = \frac{a^3}{3}$$

In MATLAB we have several built-in functions we can use for numerical integration:




```
clear
clc
close all

x=0:0.1:1;
y=x.^2;

plot(x,y)

% Calculate the Integral (Trapezoid method):
avg_y = y(1:length(x)-1) + diff(y)/2;
A = sum(diff(x).*avg_y)

% Calculate the Integral (Simpson method):
A = quad('x.^2', 0,1)

% Calculate the Integral (Lobatto method):
A = quadl('x.^2', 0,1)
```

Numerical Integration

Given the following equation:

$$y = x^3 + 2x^2 - x + 3$$

- We will find the integral of y with respect to x, evaluated from -1 to 1
- We will use the built-in MATLAB functions *diff()*, *quad()* and *quadl()*

Numerical Integration – Exact Solution

The exact solution is:

$$I = \int_a^b (x^3 + 2x^2 - x + 3)dx = \left(\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} + 3x \right) \Bigg|_a^b$$

$$= \frac{1}{4}(b^4 - a^4) + \frac{2}{3}(b^3 - a^3) - \frac{1}{2}(b^2 - a^2) + 3(b - a)$$

$a = -1$ and $b = 1$ gives:

$$I = \frac{1}{4}(1 - 1) + \frac{2}{3}(1 + 1) - \frac{1}{2}(1 - 1) + 3(1 + 1) = \frac{22}{3}$$

Symbolic Math Toolbox

We start by finding the Integral using the Symbolic Math Toolbox:

```
clear, clc

syms f(x)
syms x

f(x) = x^3 + 2*x^2 -x +3

I = int(f)
```

This gives: `I(x) = x^4/4 + (2*x^3)/3 - x^2/2 + 3*x`

http://mathworks.com/help/symbolic/getting-started-with-symbolic-math-toolbox.html

Symbolic Math Toolbox

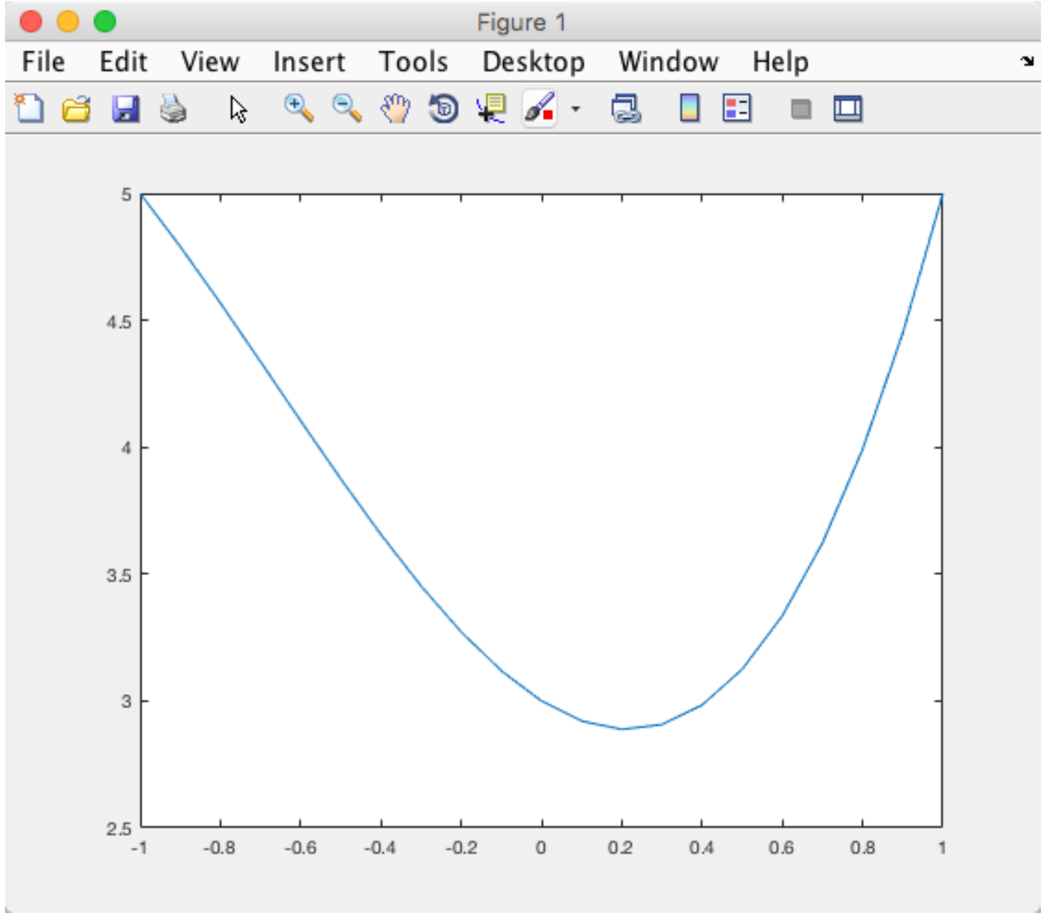
The Integral from a to b:

```
clear, clc
syms f(x)
syms x
f(x) = x^3 + 2*x^2 -x +3

a = -1;
b = 1;
Iab = int(f, a, b)
```

This gives: `Iab = 22/3` ≈ 7.33

http://mathworks.com/help/symbolic/getting-started-with-symbolic-math-toolbox.html

```
clear, clc

x = -1:0.1:1;

y = myfunc(x);

plot(x,y)

% Exact Solution
a = -1;
b = 1;
Iab = 1/4*(b^4-a^4 )+2/3*(b^3-a^3 )-
1/2*(b^2-a^2 )+3*(b-a)

% Method 1
avg_y = y(1:length(x)-1) + diff(y)/2;
A1 = sum(diff(x).*avg_y)

% Method 2
A2 = quad(@myfunc, -1,1)

% Method 3
A3 = quadl(@myfunc, -1,1)
```



MATLAB gives the following results:

```
Iab =    7.3333
A1  =    7.3400
A2  =    7.3333
A3  =    7.3333
```

Integration on Polynomials

Given the following equation:

$$y = x^3 + 2x^2 - x + 3$$

Which is also a polynomial. A polynomial can be written on the following general form: $y(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + \ a_n$

- We will find the integral of y with respect to x, evaluated from -1 to 1
- We will use the *polyint()* function in MATLAB

```
clear
clc

p = [1 2 -1 3];

polyint(p)
```

The solution is:

```
ans =
    0.2500    0.6667   -0.5000    3.0000         0
```

The solution is a new polynomial:

```
[0.25, 0.67, -0.5, 3, 0]
```

Which can be written like this:

$$0.25x^4 + 0.67x^3 - 0.5x^2 + 3x$$

We know from an example that the exact solution is:

$$\int_a^b (x^3 + 2x^2 - x + 3)dx = \left.\underline{\left(\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} + 3x\right)}\right|_a^b$$

→ So wee se the answer is correct (as expected).

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