Matrices and Systems of Linear Equations

Key Definitions

- **Matrix:** Numbers written in a rectangular array that are enclosed by square brackets.
- **Order:** The size of the matrix. The size is written as $m \times n$ (read "m by n") where m is the number of rows the matrix has and n is the number of columns the matrix has.
- **Element:** Each numbers in the matrix denoted a_{ij} where i tells you which row the number can be found and j tells you which column the number can be found. Each element can also be referred to as an entry.
- **Row Matrix:** A matrix with only one row.
- **Coefficient Matrix:** A matrix used to represent an algebraic expression where each column represents a variable. The only value in each entry is the coefficient that corresponds to the given variable.
- **Augmented Matrix:** A matrix used to represent a system of equations. It consist of the coefficient matrix with an extra column on the far right with the solutions. This column is separated from the coefficient matrix with a vertical bar.
- **Row-Echelon Form:** A matrix in a form with the following criteria:
    1. If there is a row of all 0's, it is the bottom row of the matrix.
    2. The first entry in every row is a 1 (any entry to the left is 0). This 1 is called the leading 1.
    3. For two successive rows, the higher row's leading 1 is in a column to the left of the lower row's leading 1.
- **Reduced Row-Echelon Form:** A matrix where the criteria for Row-Echelon Form holds but also consists of 0's in every entry above and below any leading 1. A matrix is usually put in this form when solving systems of equations.
- **Independent System:** A system is independent if there is only one solution.
- **Dependent System:** A system is dependent if there are infinitely many solutions.
- **Inconsistent System:** A system is inconsistent if there is no solution.

Matrix Order and Entries

- The order (or size) of a matrix is written as $m \times n$ (read "m by n") where m is the number of rows and n is the number of columns.
- **Example:** What is the order of the matrix below?

$$\begin{bmatrix} 6 & 3 & 7 & 8 \\ 2 & 1 & 12 & 5 \\ 5 & 8 & 4 & 10 \end{bmatrix}$$

Since the matrix as 3 rows and 4 columns, its size is 3×4

- If a matrix A is size $m \times n$, each element (or entry) a_{ij} can be located in the i^{th} row and the j^{th} column. Below is a diagram to represent the location of each entry.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

- **Example:** Using matrix B, what is the value of b_{34}?

$$B = \begin{bmatrix} -12 & 4 & -22 & 5 \\ 15 & -9 & 18 & 16 \\ 8 & 20 & 9 & -8 \\ 2 & 7 & 28 & 1 \end{bmatrix}$$

Step 1: To find b_{34}, we are going to go to the 3^{rd} row.
Step 2: Once we are at the third row, we will go to the 4^{th} column to find the value. We see that this value is -8.

Row-Echelon and Reduced Row-Echelon Form

- **Row-Echelon Form:** Row Echelon form is a matrix that meets the following criteria:
  - If there is a row of all 0's, it is the bottom row of the matrix.
  - The first entry in every row is a 1 (any entry to the left is 0). This 1 is called the leading 1.
  - For two successive rows, the higher row's leading 1 is in a column to the left of the lower row's leading 1.
- Here are a few examples of what a matrix could look like in Row-Echelon Form:

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 6 & 8 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 10 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 7 & 12 \\ 0 & 1 & 15 & 30 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Reduced Row-Echelon Form:** Reduced Row-Echelon form is a matrix that is still in Row-Echelon Form, but just adds one more item to our list of criteria:
  - There is a 0 in every entry above and below every leading 1 in the matrix.

- Here are a few examples of what a matrix could look like in Reduced Row-Echelon Form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Row Operations of Matrices

- **Row Operations:** Suppose you are working with a matrix where any two of its rows are represented generally by R_A and R_B.
  - **Interchanging rows** ($R_A \leftrightarrow R_B$): One row operation that we may perform is simply switching 2 rows. For example:
    $\begin{bmatrix} 2 & 5 & 7 \\ 1 & 3 & 8 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 3 & 8 \\ 2 & 5 & 7 \end{bmatrix}$ upon the action $R_1 \leftrightarrow R_2$ because we interchanged the two rows.
  - **Multiplying a Row by a Nonzero Constant** ($cR_A \to R_A$): Another row operation is multiplying an *entire* row by a constant. For example:
    $\begin{bmatrix} 2 & 5 & 7 \\ 1 & 3 & 8 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 2 & 5 & 7 \\ 3 & 9 & 24 \end{bmatrix}$ upon the action $3R_2 \to R_2$ because we multiplied every number in the second row by 3 and it became our new second row.
  - **Adding a Multiple of a Row to Another Row** ($cR_A + R_B \to R_B$): The final row operation that we can perform is rewriting a row is by adding that row to another row that is multiplied by a constant. For example:
    $\begin{bmatrix} 2 & 5 & 7 \\ 1 & 3 & 8 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 4 & 11 & 23 \\ 1 & 3 & 8 \end{bmatrix}$ upon the action $2R_2 + R_1 \to R_1$ because we multiplied the second row by 2 and added it to the first row and it became our new first row.

Writing Systems with Matrices

- **Coefficient Matrices:**
  - To find a coefficient matrix for equations, we must organize each column as a variable where each row represents a specific equation.
  - The entries that we will put into this matrix will only be the coefficient of that variable.
    - Recall: If a variable is not represented in an equation, then its coefficient is 0.
- **Example:** Find the coefficient matrix of the following equations:

$$\frac{1}{2}x + y + 2z = 2$$
$$x + 3y + 5z = 3$$
$$2y - 4z = 4$$

Step 1: We want to begin by distinguishing all the variables in the equation. We see that there are 3 unknown variables, $x, y, and\ z$.

***Step 2: Identify the coefficients of each variable in the equations.*.**

	Coefficient of x	***Coefficient of y***	***Coefficient of z***
$\frac{1}{2}x + y + 2z = 2$	$\frac{1}{2}$	1	2
$x + 3y + 5z = 3$	1	3	5
$2y - 4z = 4$	0	2	-4

Step 3: We will set up our coefficient matrix with the coefficients of x in the first column, the coefficients of y in the second column, and the coefficients of z in the third column.

$$\begin{bmatrix} \frac{1}{2} & 1 & 2 \\ 1 & 3 & 5 \\ 0 & 2 & -4 \end{bmatrix}$$

- **Augmented Matrices:**
  - An augmented matrix is a coefficient matrix with an extra column on the end with the constant solutions to the expression.
  - This column is separated from the coefficient matrix with a vertical bar.
  - Knowing how to write an augmented matrix is important because we will eventually need them to solve systems of equations.
- **Example:** Using the equations and coefficient matrix we found in the previous example, create the augmented matrix.

Step 1: find the constant values that the expression equals for each equation.

	Constant the expression is equal to
$\frac{1}{2}x + y + 2z = 2$	2
$x + 3y + 5z = 3$	3
$2y - 4z = 4$	4

Step 2: Use these values to create a fourth column to the matrix, separating it from the coefficient matrix with a vertical bar.

$$\left[\begin{array}{ccc|c} \frac{1}{2} & 1 & 2 & 2 \\ 1 & 3 & 5 & 3 \\ 0 & 2 & -4 & 4 \end{array}\right]$$

Solving Systems with Matrices

- **How to Solve a System with a Matrix:**
    - To solve a system of equations using a matrix, you must:
        1. Create an augmented matrix using the given equations
        2. Perform row operations on the matrix until it is in Reduced Row-Echelon Form.
        3. Rewrite the equations from the Reduced Row-Echelon Form.
        4. Solve for each variable

  This process is called **Gauss-Jordan Elimination**
- **Example:** Solve the following system of equations

$$\frac{1}{2}x + y + 2z = 2$$
$$x + 3y + 5z = 3$$
$$2y - 4z = 4$$

Step 1: Create an augmented matrix for the system. We found the augmented matrix for this specific system in a previous example.

$$\left[\begin{array}{ccc|c} \frac{1}{2} & 1 & 2 & 2 \\ 1 & 3 & 5 & 3 \\ 0 & 2 & -4 & 4 \end{array}\right]$$

Step 2: Perform row operations on the matrix until it is in Reduced Row-Echelon Form.

We will want to begin by obtaining a 1 in the first column of the first row. We can do this by multiplying the first row by 2.

$$\left[\begin{array}{ccc|c} \frac{1}{2} & 1 & 2 & 2 \\ 1 & 3 & 5 & 3 \\ 0 & 2 & -4 & 4 \end{array}\right] \qquad 2R_1 \to R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 1 & 3 & 5 & 3 \\ 0 & 2 & -4 & 4 \end{array}\right]$$

Next, we will want to make every entry below this leading 1 a zero. To get the first entry in the second row to be zero, we will multiply the first row by – $\mathbf{1}$ and add it to the second row.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 1 & 3 & 5 & 3 \\ 0 & 2 & -4 & 4 \end{array}\right] \qquad (-1)R_1 + R_2 \to R_2 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & -4 & 4 \end{array}\right]$$

Since the entire first column starts with a leading $\mathbf{1}$ and has all $\mathbf{0}$'s below it, we can move on to the second column.

We see that the leading entry in the second row is already a 1, so we will work on making the entry below a 0. We can do this by multiplying the second row by -2 and adding it to the third row.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & -4 & 4 \end{array}\right] \qquad (-2)R_2 + R_3 \to R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

Now we will proceed in obtaining our third leading 1.

> ***Note: We do not begin making the entries above leading 1's zero until we have all of our leading 1's.***

To obtain this leading 1 in the third column, we will multiply the third row by $-\frac{\mathbf{1}}{\mathbf{6}}$ because $(-\frac{\mathbf{1}}{\mathbf{6}}) \cdot (-\mathbf{6}) = \mathbf{1}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -6 & 6 \end{array}\right] \qquad (-\tfrac{1}{6})R_3 \to R_3 \qquad \begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

> ****At this step, we have completed Gaussian Elimination and can rewrite the system here and solve like we would in previous chapters. However, we are going to keep going to solve using Gauss-Jordan Elimination.***

Now that we have obtained all of our leading 1's, we may begin making the entries above the leading 1's zero.

We will start by using the leading 1 in the third row to get the "1" above it zero. To do this, we will multiply the third row by -1 and add it to the second column

$$\begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \qquad (-1)R_3 + R_2 \rightarrow R_2 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(Notice this does not mess up any of the values above the zeros in the third row. This is a result of obtaining all the leading 1's first.)

We will continue making the entries above the third leading 1 zero by multiplying the third row by -4 and adding it to the first row.

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad (-4)R_3 + R_1 \rightarrow R_1 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Now we will perform the same task, but now we are looking to get the values above the <u>second</u> leading entry to be zero. We will do this by multiplying the second row by -2 and adding it to the first row.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad (-2)R_2 + R_1 \rightarrow R_1 \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

We now have our matrix in Reduced Row-Echelon Form.

Step 3: Write the equations using the new matrix we obtained. Remember, the first column is used to find our coefficients of x, our second to find the coefficients of y, the third to find the coefficients of z. Since the values to the right of the vertical bar are the values that they are equal to, our new equations are:

$$1x + 0y + 0z = 8$$
$$0x + 1y + 0z = 0$$
$$0x + 0y + 1z = -1$$

Step 4: Solve the equations. We see that the equations are worked out so that there is no arithmetic to further complete, so the solution to the system is:

$$x = 8$$
$$y = 0$$
$$z = -1$$

Different Types of Systems

- **Independent System:**
  - A system is said to be independent if there is only one unique solution to the system of equations. We see that the example above resulted in a unique solution for x, y, and z, therefore it was independent.

  - In a matrix: There would be the same number of leading 1's as there are unknown variables.
- **Dependent System:**
  - A system is said to be dependent if there are infinitely many solutions to the system of equations. For example, if we have the system
    $$x - 3z = 0$$
    $$y + z = 3$$
    we see that x and y can both we written in terms of z:
    $$x = 3z$$
    $$y = 3 - z$$
    So there can be infinitely many solutions to the system because x and y can be infinitely many values depending on what value is chosen for z.
  - In a matrix: There would be fewer leading ones than unknown variables.
- **Inconsistent System:**
  - A system is said to be inconsistent if there is no solution to the system. We can tell that a system has no solution if we get a result:
    $$0 = c$$
    Where c is any nonzero constant.
  - In a matrix: There would be a row of all zeros in the coefficient matrix, but with a corresponding nonzero value in the augmented column.

Matrix Algebra

Matrix Equality

- Any two matrices are said to be equal if the two matrices have the same order ($m \times n$) and if every pair of corresponding entries are equal.
- **Example:** If the following matrices are equivalent, what are the values of a_{11}, a_{42}, and a_{23}?

(Recall: When an element/entry is written as a_{ij}, i tells us what row the element is in and j tells us what column our element is in.)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 5 & 10 & 1 & 8 \\ 6 & 7 & -2 & 0 \\ -6 & 9 & 8 & 4 \\ 2 & 3 & 4 & -5 \end{bmatrix}$$

Since 5 is in the corresponding entry to a_{11}, conclude $a_{11} = 5$.

Since 3 is in the corresponding entry to a_{42}, conclude $a_{42} = 3$.

Since -2 is in the corresponding entry to a_{23}, conclude $a_{23} = -2$.

Matrix Addition/Subtraction

- In order to add or subtract matrices, they must be of the *same* order ($m \times n$). Then, to add/subtract the matrices, you must add/subtract corresponding elements. In general:

$$A \pm B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2m} \pm b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \dots & a_{mn} \pm b_{mn} \end{bmatrix}$$

- **Example:** Compute $A + B$ and $B - C$ given:

$$A = \begin{bmatrix} 2 & 8 & 0 \\ 1 & 5 & 10 \\ 4 & 3 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 15 & 5 \\ 6 & 6 & 13 \\ 12 & 18 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 10 & 2 \\ 1 & 3 & 6 \\ 5 & 9 & 7 \end{bmatrix}.$$

$$A + B = \begin{bmatrix} 2 & 8 & 0 \\ 1 & 5 & 10 \\ 4 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 8 & 15 & 5 \\ 6 & 6 & 13 \\ 12 & 18 & 8 \end{bmatrix} = \begin{bmatrix} 2+8 & 8+15 & 0+5 \\ 1+6 & 5+6 & 10+13 \\ 4+12 & 3+18 & 7+8 \end{bmatrix} = \begin{bmatrix} 10 & 23 & 5 \\ 7 & 11 & 23 \\ 16 & 21 & 15 \end{bmatrix}$$

$$B - C = \begin{bmatrix} 8 & 15 & 5 \\ 6 & 6 & 13 \\ 12 & 18 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 10 & 2 \\ 1 & 3 & 6 \\ 5 & 9 & 7 \end{bmatrix} = \begin{bmatrix} 8-4 & 15-10 & 5-2 \\ 6-1 & 6-3 & 13-6 \\ 12-5 & 18-9 & 8-7 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 3 \\ 5 & 3 & 7 \\ 7 & 9 & 1 \end{bmatrix}$$

- **Properties of Matrix Addition:**
  - If A, B, and C are $m \times n$ matrices and $\mathbf{0}$ is the $m \times n$ matrix of all 0's, then they uphold the following properties.

Communitive Property:	$A + B = B + A$
Associative Property:	$(A + B) + C = A + (B + C)$
Additive Identity Property:	$A + \mathbf{0} = A$
Additive Inverse Property:	$A + (-A) = \mathbf{0}$

Scalar Multiplication

- Scalar multiplication is when a matrix is being multiplied by a constant number. It is computed by multiplying every entry in the matrix by this constant (we call this constant the scalar).
- **Example:** Compute $\frac{1}{3}A$ where $A = \begin{bmatrix} 15 & 9 \\ 1 & 3 \end{bmatrix}$.

$$\tfrac{1}{3}A=\tfrac{1}{3}\begin{bmatrix}15 & 9\\ 1 & 3\end{bmatrix}=\begin{bmatrix}\tfrac{1}{3}(15) & \tfrac{1}{3}(9)\\ \tfrac{1}{3}(1) & \tfrac{1}{3}(3)\end{bmatrix}=\begin{bmatrix}5 & 3\\ \tfrac{1}{3} & 1\end{bmatrix}$$

Matrix Multiplication

- **Matrix Order when Multiplying:** To multiply two matrices, let's say matrix A and matrix B, the order/size of the matrices are very important. The number of columns in the first matrix must equal the number of rows in the second matrix. For instance, say we wanted to compute AB. The number of columns in A bust equal the number of rows in B.

$$\begin{array}{ccc} AB & = & \begin{array}{cc} A & B \\ m\times n & n\times p \end{array} \end{array}$$

$$\textit{the same value}!$$

Multiplication of 2 matrices will result in a matrix with the same number of rows as the first matrix and the same number of columns as the second.

$$\begin{array}{ccc} AB & = & \begin{array}{cc} A & B \\ m\times n & n\times p \end{array} \end{array}$$

$$\textit{order of resulting matrix}!\ (m\times p)$$

- **Example:** If matrix D has order 3×2 and matrix E has order 3×3, is it possible to compute DB? If it is, what is the order of the resulting matrix?
  Step 1: Determine how many columns are in D and how many rows are in E.

$$\begin{array}{ccc} DE = & D & E \\ & 3\times 2 & 3\times 3 \end{array}$$

Since D has 2 rows and E has 3 columns, and since 2≠3, we can conclude that these matrices cannot be multiplied together.

- **Example:** If matrix F has order 3×6 and matrix G has order 6×2, is it possible to compute FG? If it is, what is the order of the resulting matrix?
  Step 1: Determine how many columns are in D and how many rows are in G.

$$\begin{array}{ccc} FG = & F & G \\ & 3\times 6 & 6\times 2 \end{array}$$

Since the values are equal, it is possible to multiply these matrices together.
Step 2: Find the number of rows in F and the number of columns in G.

$$\begin{array}{ccc} FG = & F & G \\ & 3\times 6 & 6\times 2 \end{array}$$

Since there are 3 rows in F and 2 columns in G, the order of the resulting matrix is 3×2.

- **Multiplying Matrices:**

*Before we begin looking into how to multiply matrices, note that any entry $a_{i\,j}$ can be denoted as the (i,j)-entry. So if you look at the (1,3)-entry, it would be the entry in the first row and the third column.

The First Row:

1. When we multiply 2 matrices, let's say AB, of appropriate order together, we compute the (1,1)-entry of the new matrix by multiplying the first entry in the first ***row* of $\boldsymbol{A}$** with the first entry in the **first column of $\boldsymbol{B}$** added to the product found by multiplying the second entry in the first ***row* of $\boldsymbol{A}$** with the second entry in the first **column of $\boldsymbol{B}$**. We will continue this process until every entry in the first row of A is multiplied with an entry of the first column of B and added.
2. To compute the (1,2)-entry of the new matrix, we will repeat the same process still using the first row of A, but then using the *second* column of B. We will then complete this process with every column of B which will result in the first row of the new matrix from matrix multiplication.

The Second Row:

1. Repeat the same process as the first row, but now use the *second* row of A for all computations

Repeat this process until all the rows of A have been used for computing rows of the new matrix.

- **Example:** Compute CD given $C=\begin{bmatrix}5 & 3 & 1\\ 1 & 2 & 4\end{bmatrix}$ and $D=\begin{bmatrix}2 & 4 & -1\\ 1 & 2 & 5\\ 0 & -3 & 4\end{bmatrix}$

Step 1: Compute the (1,1)-entry by multiplying the corresponding entries in the first row of C and the first column of D.

$$CD=\begin{bmatrix}5 & 3 & 1\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}2 & 4 & -1\\ 1 & 2 & 5\\ 0 & -3 & 4\end{bmatrix}=\begin{bmatrix}5\cdot 2+3\cdot 1+1\cdot 0 & \_\_ & \_\_\\ \_\_ & \_\_ & \_\_\end{bmatrix}=\begin{bmatrix}13 & \_\_ & \_\_\\ \_\_ & \_\_ & \_\_\end{bmatrix}$$

Step 2: Compute the (1,2)-entry by multiplying the corresponding entries in the first row of C and the second column of D.

$$CD=\begin{bmatrix}5 & 3 & 1\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}2 & 4 & -1\\ 1 & 2 & 5\\ 0 & -3 & 4\end{bmatrix}=\begin{bmatrix}13 & 5\cdot 4+3\cdot 2+1\cdot -3 & \_\_\\ \_\_ & \_\_ & \_\_\end{bmatrix}=\begin{bmatrix}13 & 23 & \_\_\\ \_\_ & \_\_ & \_\_\end{bmatrix}$$

Step 3: Compute the (1,3)-entry by multiplying the corresponding entries in the first row of C and the third column of D.

$$CD=\begin{bmatrix}5 & 3 & 1\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}2 & 4 & -1\\ 1 & 2 & 5\\ 0 & -3 & 4\end{bmatrix}=\begin{bmatrix}13 & 23 & 5\cdot -1+3\cdot 5+1\cdot 4\\ \_\_ & \_\_ & \_\_\end{bmatrix}=\begin{bmatrix}13 & 23 & 14\\ \_\_ & \_\_ & \_\_\end{bmatrix}$$

Step 4: Compute the (2,1)-entry by multiplying the corresponding entries in the second row of C and the first column of D.

$$CD = \begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 1 & 2 & 5 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 14 \\ 1\cdot 2 + 2\cdot 1 + 4\cdot 0 & \_\_ & \_\_ \end{bmatrix} = \begin{bmatrix} 13 & 23 & 14 \\ 4 & \_\_ & \_\_ \end{bmatrix}$$

Step 5: Compute the (2,2)-entry by multiplying the corresponding entries in the second row of C and the second column of D.

$$CD = \begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 1 & 2 & 5 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 14 \\ 4 & 1\cdot 4 + 2\cdot 2 + 4\cdot -3 & \_\_ \end{bmatrix} = \begin{bmatrix} 13 & 23 & 14 \\ 4 & -4 & \_\_ \end{bmatrix}$$

Step 6: Compute the (2,3)-entry by multiplying the corresponding entries in the second row of C and the third column of D.

$$CD = \begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 1 & 2 & 5 \\ 0 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 14 \\ 4 & -4 & 1\cdot -1 + 2\cdot 5 + 4\cdot 4 \end{bmatrix} = \begin{bmatrix} 13 & 23 & 14 \\ 4 & -4 & 25 \end{bmatrix}$$

Therefore,

$$CD = \begin{bmatrix} 13 & 23 & 14 \\ 4 & -4 & 25 \end{bmatrix}$$