MATRICES

EXAM QUESTIONS

(Part One)

Question 1 (**)

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given below in terms of the scalar constants a, b, c and d, by

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}.$$

Given that $\mathbf{A}+\mathbf{B}=\mathbf{C}$, find the value of a, b, c and d.

$$\boxed{a=8,\ b=3,\ c=2,\ d=3}$$

$$A+B=C \Rightarrow \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix} + \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2+b & -2 \\ 3 & a-4 \end{pmatrix} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}$$

So $a-4=4$, $a=8$; $-2+b=1$, $b=3$; $c=-2$; $d=3$

Question 2 (**)

Find, in terms of k, the inverse of the following 2×2 matrix.

$$\mathbf{M} = \begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}.$$

Verify your answer by multiplication.

$$\boxed{\mathbf{M}^{-1} = \begin{pmatrix} -k-2 & k+1 \\ k+1 & -k \end{pmatrix}}$$

- $\underline{M} = \begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}$
- $\det(\underline{M}) = k(k+2)-(k+1)(k+1) = k^2+2k-(k^2+2k+1)$

  $= k^2+2k-k^2-2k-1 = -1$
- $\underline{M}^{-1} = \frac{1}{-1}\begin{bmatrix} k+2 & -(k+1) \\ -(k+1) & k \end{bmatrix} = -\begin{bmatrix} k+2 & -k-1 \\ -k-1 & k \end{bmatrix} = \begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix}$
- NOW VERIFYING BY MULTIPLICATION

$$\underline{M}\,\underline{M}^{-1} = \begin{bmatrix} k & k+1 \\ k+1 & k+2 \end{bmatrix}\begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix}$$

$$= \begin{bmatrix} k(-k-2)+(k+1)^2 & k(k+1)-k(k+1) \\ (k+1)(-k-2)+(k+1)(k+2) & (k+1)^2-k(k+2) \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{-k^2}\cancel{-2k}\cancel{+k^2}\cancel{+2k}+1 & \cancel{k^2}\cancel{+k}\cancel{-k^2}\cancel{-k} \\ \cancel{-k^2}\cancel{-3k}\cancel{-2}\cancel{+k^2}\cancel{+3k}\cancel{+2} & \cancel{k^2}\cancel{+2k}+1\cancel{-k^2}\cancel{-2k} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underline{I}$$

- INDEED THE INVERSE

Question 3 (**)

The 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given below in terms of the scalar constants a, b and c.

$$\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 4 \\ b & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & c \\ 3 & 2 \end{pmatrix}.$$

Given that

$$2\mathbf{A} - 3\mathbf{B} = 4\mathbf{C},$$

find the value of a, b and c.

$\boxed{a = 1, \ b = -2, \ c = -2}$

$$2A - 3B = 4C \Rightarrow 2\begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix} - 3\begin{pmatrix} 2 & 4 \\ b & 2 \end{pmatrix} = 4\begin{pmatrix} -1 & c \\ 3 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2a & 4 \\ 6 & 14 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 3b & 6 \end{pmatrix} = \begin{pmatrix} -4 & 4c \\ 12 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2a-6 & -8 \\ 6+3b & 8 \end{pmatrix} = \begin{pmatrix} -4 & 4c \\ 12 & 8 \end{pmatrix}$$

So $2a - 6 = -4$, $2a = 2$, $a = 1$ $6 + 3b = 12$, $3b = 6$, $b = 2$ $4c = -8$, $c = -2$

Question 4 (**)

The 2×2 matrix $\mathbf{A}$ represents a rotation by $90°$ anticlockwise about the origin O.

The 2×2 matrix $\mathbf{B}$ represents a reflection in the straight line with equation $y=-x$.

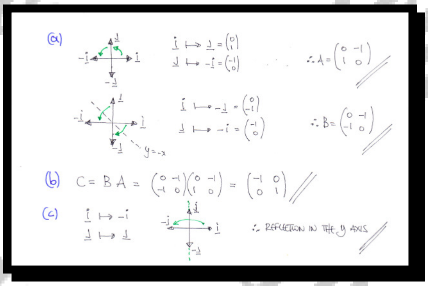
a) Write down the matrices $\mathbf{A}$ and $\mathbf{B}$.

The 2×2 matrix $\mathbf{C}$ represents a rotation by $90°$ anticlockwise about the origin O, followed by a reflection about the straight line with equation $y=-x$.

b) Find the elements of $\mathbf{C}$.

c) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{A}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}}$, $\boxed{\mathbf{B}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}}$, $\boxed{\mathbf{C}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}}$, $\boxed{\text{reflection in the } y \text{ axis}}$



Question 5 (**)

The 2×2 matrix $\mathbf{A}$, is defined as

$$\mathbf{A}=\begin{pmatrix}2 & a\\ b & -2\end{pmatrix}$$

where a and b are constants.

The matrix $\mathbf{A}$, maps the point $P(2,5)$ onto the point $Q(-1,2)$.

a) Find the value of a and the value of b.

A triangle T_1 with an area of 9 square units is transformed by $\mathbf{A}$ into the triangle T_2.

b) Find the area of T_2.

$\boxed{a=-1}$, $\boxed{b=6}$, $\boxed{\text{area}=18}$

(a) $\begin{pmatrix}2 & a\\ b & -2\end{pmatrix}\begin{pmatrix}2\\ 5\end{pmatrix}=\begin{pmatrix}-1\\ 2\end{pmatrix}$

$\begin{pmatrix}4+5a\\ 2b-10\end{pmatrix}=\begin{pmatrix}-1\\ 2\end{pmatrix}$

$\therefore\ 5a=-5$, $2b=12$ $\quad a=-1$, $b=6$

(b) $A=\begin{pmatrix}2 & -1\\ 6 & -2\end{pmatrix}$

$\det A=-4+6=2$

$\therefore\ 9\times2=18$ units^2

Question 6 (**)

The 2×2 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given below in terms of the scalar constants x.

$$\mathbf{A} = \begin{pmatrix} 2 & x \\ 3 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{pmatrix} 3x+2 & 7 \\ 7-x & 7 \end{pmatrix}.$$

a) Find an expression for $\mathbf{AB}$, in terms of x.

b) Determine the value of x, given $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{C}$.

$$\boxed{\mathbf{AB} = \begin{pmatrix} 4+x & 2+4x \\ 7 & 7 \end{pmatrix}}, \quad \boxed{x = 1}$$

(a) $AB = \begin{bmatrix} 2 & x \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{pmatrix} 4+x & 2+4x \\ 7 & 7 \end{pmatrix}$ //

(b) $B^T A^T = (AB)^T = C$

$\therefore \begin{pmatrix} 4+x & 7 \\ 2+4x & 7 \end{pmatrix} = \begin{pmatrix} 3x+2 & 7 \\ 7-x & 7 \end{pmatrix}$ use a_{11}: $4+x = 3x+2$, $2 = 2x$, $x = 1$ //

Question 7 (**)

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}.$$

Find the 2×2 matrix $\mathbf{X}$ that satisfy the equation

$$\mathbf{AX} = \mathbf{B}.$$

$$\boxed{\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$$

$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}$

$\Rightarrow AX = B$

$\Rightarrow A^{-1}AX = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = \frac{1}{5\times5-2\times2}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}$

$\Rightarrow X = 1\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}$

$\Rightarrow X = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ //

Question 8 (**)

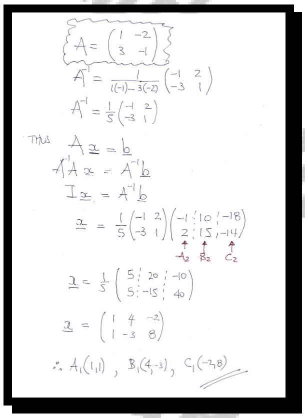
The triangle T_1 is mapped by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}$$

onto another triangle T_2, whose vertices have coordinates $A_2(-1,2)$, $B_2(10,15)$ and $C_2(-18,-14)$.

Find the coordinates of the vertices of T_1.

$\boxed{A_1(1,1)}$, $\boxed{B_1(4,-3)}$, $\boxed{C_1(-2,8)}$



$$\boxed{A = \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}}$$

$$A^{-1} = \frac{1}{1(-1) - 3(-2)} \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{5} \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$$

THUS $A\underline{x} = \underline{b}$

$$A^{-1}A\underline{x} = A^{-1}\underline{b}$$

$$I\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 10 & -18 \\ 2 & 15 & -14 \end{pmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ A_2 & B_2 & C_2 \end{array}$$

$$\underline{x} = \frac{1}{5}\begin{pmatrix} 5 & 20 & -10 \\ 5 & -15 & 40 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} 1 & 4 & -2 \\ 1 & -3 & 8 \end{pmatrix}$$

$\therefore A_1(1,1)$, $B_1(4,-3)$, $C_1(-2,8)$

Question 9 (**)

A plane transformation maps the general point (x, y) onto the general point (X, Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix},$$

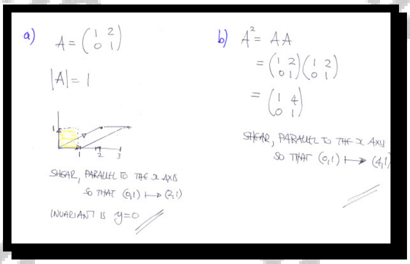
where $\mathbf{A}$ is the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

a) Give a geometrical description for the transformation represented by $\mathbf{A}$, stating the equation of the line of invariant points under this transformation

b) Calculate $\mathbf{A}^2$ and describe geometrically the transformation it represents.

$\boxed{\text{shear parallel to } y=0,\ (0,1) \mapsto (2,1)}$ $\boxed{\text{line of invariant points } y=0}$,

$\boxed{\text{shear parallel to } y=0,\ (0,1) \mapsto (4,1)}$



Question 10 (**)

The triangle T_1 is mapped by the 2×2 matrix

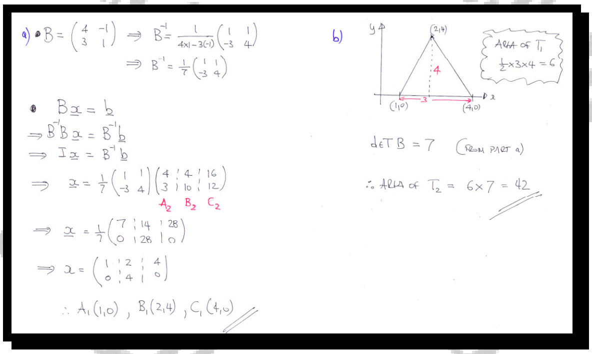
$$\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$$

onto a triangle T_2, whose vertices are the points with coordinates $A_2(4,3)$, $B_2(4,10)$ and $C_2(16,12)$.

a) Find the coordinates of the vertices of T_1.

b) Determine the area of T_2.

$\boxed{A_1(1,0)}$, $\boxed{B_1(2,4)}$, $\boxed{C_1(4,0)}$, $\boxed{\text{area} = 42}$



Question 11 (**)

The 2×2 matrix $\mathbf{C}$ is defined, in terms of a scalar constant a, by

$$\mathbf{C} = \begin{pmatrix} 3 & a \\ 5 & 2 \end{pmatrix}.$$

a) Determine the value of a, given that $\mathbf{C}$ is singular.

The 2×2 matrix $\mathbf{D}$ is given by

$$\mathbf{D} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}.$$

b) Find the inverse of $\mathbf{D}$.

The point P is transformed by $\mathbf{D}$ onto the point $Q(6k+1, 14k+1)$, where k is a scalar constant.

c) Determine, in terms of k, the coordinates of P.

$\boxed{a = \frac{6}{5}}$, $\boxed{\mathbf{D}^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}}$, $\boxed{P(2k+1, 2k-1)}$

(a) $\det C = 0 \Rightarrow (3\times2)-(5\times a)=0$
$\Rightarrow 6-5a=0$
$\Rightarrow 6=5a$
$\Rightarrow a=\frac{6}{5}$

(b) $D^{-1} = \frac{1}{(2\times3)-(4\times1)}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$
$D^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$
$D^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$

(c) $Dp = q$
$\Rightarrow D^{-1}Dp = D^{-1}q$
$\Rightarrow Ip = D^{-1}q$
$\Rightarrow p = D^{-1}q$

$\Rightarrow p = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6k+1 \\ 14k+1 \end{pmatrix} = \begin{bmatrix} \frac{3}{2}(6k+1)-\frac{1}{2}(14k+1) \\ -2(6k+1)+1(14k+1) \end{bmatrix}$
$= \begin{bmatrix} 9k+\frac{3}{2}-7k-\frac{1}{2} \\ -12k-2+14k+1 \end{bmatrix} = \begin{bmatrix} 2k+1 \\ 2k-1 \end{bmatrix}$

$\therefore P(2k+1, 2k-1)$

Question 12 (**)

A plane transformation maps the general point (x, y) to the general point (X, Y) by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the area scale factor of the transformation.

The points on a straight line which passes through the origin remain invariant under this transformation.

b) Determine the equation of this straight line.

$\boxed{\text{SF} = 16}$, $\boxed{y = \frac{3}{4}x}$

(a) AREA SCALE FACTOR $= \begin{vmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{vmatrix} = 67.84 - 51.84 = 16$

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.4x - 7.2y \\ -7.2x + 10.6y \end{pmatrix}$

$\therefore\ 7.2y = 5.4x$ OR $7.2x = 9.6y$ $\quad$ ie $y = \frac{3}{4}x$

Question 13 (**)

The distinct square matrices $\mathbf{A}$ and $\mathbf{B}$ are non singular.

Simplify the expression, showing all steps in the workings.

$$\mathbf{AB}\left(\mathbf{A}^{-1}\mathbf{B}\right)^{-1}.$$

$\boxed{\mathbf{A}^2}$

$AB(A^{-1}B)^{-1} = AB(B^{-1}(A^{-1})^{-1}) = ABB^{-1}A = AIA$

$= AA = A^2$

Question 14 (**)

The distinct square matrices $\mathbf{A}$ and $\mathbf{B}$ have the properties

$$\mathbf{AB} = \mathbf{B}^5\mathbf{A} \quad \text{and} \quad \mathbf{B}^6 = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix.

Prove that

$$\mathbf{BAB} = \mathbf{A}.$$

$\boxed{\text{proof}}$

$$AB = B^5A$$
$$\Rightarrow BAB = BB^5A$$
$$\Rightarrow BAB = B^6A$$
$$\Rightarrow BAB = IA$$
$$\Rightarrow BAB = A$$

Question 15 (**)

The 2×2 matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}.$$

The 2×2 matrix $\mathbf{B}$ satisfies

$$\mathbf{BA}^2 = \mathbf{A}.$$

Find the elements of $\mathbf{B}$.

$$\boxed{\mathbf{B} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}}$$

$A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ $\quad \det A = (1\times4)-(1\times3)=1$

$A^{-1} = \frac{1}{1}\begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$

Now $BA^2 = A$

$\Rightarrow BAA = A$

$\Rightarrow BA^2A^{-1} = AA^{-1}$

$\Rightarrow BA = I$

$\Rightarrow BAA^{-1} = IA^{-1}$

$\Rightarrow B = A^{-1}$

$\therefore B = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$

Question 16 (**)

The triangle T_1 is mapped by the 2×2 matrix

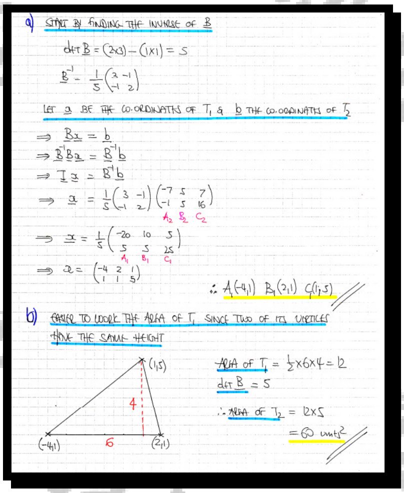
$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

onto the triangle T_2, whose vertices have coordinates $A_2(-7,-1)$, $B_2(5,5)$ and $C_2(7,16)$.

a) Find the coordinates of the vertices of T_1.

b) Determine the area of T_2.

$\boxed{}$, $\boxed{A_1(-4,1)}$, $\boxed{B_1(2,1)}$, $\boxed{C_1(1,5)}$, $\boxed{\text{area} = 60}$



a) START BY FINDING THE INVERSE OF $\underline{B}$

$\det \underline{B} = (2\times3) - (1\times1) = 5$

$\underline{B}^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$

LET $\underline{x}$ BE THE CO-ORDINATES OF T_1 & $\underline{b}$ THE CO-ORDINATES OF T_2

$\Rightarrow \underline{B}\underline{x} = \underline{b}$

$\Rightarrow \underline{B}^{-1}\underline{B}\underline{x} = \underline{B}^{-1}\underline{b}$

$\Rightarrow \underline{I}\underline{x} = \underline{B}^{-1}\underline{b}$

$\Rightarrow \underline{x} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -7 & 5 & 7 \\ -1 & 5 & 16 \end{pmatrix}$

$A_2 \quad B_2 \quad C_2$

$\Rightarrow \underline{x} = \frac{1}{5}\begin{pmatrix} -20 & 10 & 5 \\ 5 & 5 & 25 \end{pmatrix}$

$A_1 \quad B_1 \quad C_1$

$\Rightarrow \underline{x} = \begin{pmatrix} -4 & 2 & 1 \\ 1 & 1 & 5 \end{pmatrix}$

$\therefore A_1(-4,1) \quad B_1(2,1) \quad C_1(1,5)$

b) EASIER TO WORK THE AREA OF T_1 SINCE TWO OF ITS VERTICES HAVE THE SAME HEIGHT

$(1,5)$, $(-4,1)$, $(2,1)$, 4, 6

AREA OF $T_1 = \frac{1}{2} \times 6 \times 4 = 12$

$\det \underline{B} = 5$

$\therefore$ AREA OF $T_2 = 12 \times 5$

$= 60$ units^2

Question 17 (**+)

The transformation represented by the 2×2 matrix $\mathbf{A}$ maps the point $(3,4)$ onto the point $(10,4)$, and the point $(5,-2)$ onto the point $(8,-2)$.

Determine the elements of $\mathbf{A}$.

$$\boxed{\mathbf{A}=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}}$$

LET $A=\begin{pmatrix}a & b\\ b & d\end{pmatrix}$

$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}3\\ 4\end{pmatrix}=\begin{pmatrix}10\\ 4\end{pmatrix} \Rightarrow \begin{matrix}3a+4b=10\\ 3c+4d=4\end{matrix}$

$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}5\\ -2\end{pmatrix}=\begin{pmatrix}8\\ -2\end{pmatrix} \Rightarrow \begin{matrix}5a-2b=8\\ 8c-2d=-2\end{matrix}$

THUS

- $\left.\begin{matrix}3a+4b=10\\ 5a-2b=8\end{matrix}\right\}$; $\left.\begin{matrix}3a+4b=10\\ 10a-4b=16\end{matrix}\right\}$; $13a=26$; $a=2$; $b=1$
- $\left.\begin{matrix}3c+4d=4\\ 8c-2d=-2\end{matrix}\right\}$; $\left.\begin{matrix}3c+4d=4\\ 16c-4d=-4\end{matrix}\right\}$; $19c=0$; $c=0$; $d=1$

$\therefore A=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}$

Question 18 (**+)

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A}=\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}.$$

Determine the elements of $\mathbf{A}^3$ and hence describe geometrically the transformation represented by $\mathbf{A}$.

$\boxed{\mathbf{A}^3=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}}$, $\boxed{\text{rotation of } 120°\text{, anticlockwise \& enlargement of S.F. 2, both about the origin and in any order.}}$

$A^3=\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}=\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$

$\therefore A^3=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}=8\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

THIS IS A ROTATION BY 120°, ABOUT THE ORIGIN & UNIFORM ENLARGEMENT ABOUT THE ORIGIN BY SCALE FACTOR 2 (EITHER ORDER)

TO DETERMINE CLOCKWISE ANTICLOCKWISE

$\underset{\theta}{\circlearrowleft}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ so $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$

$\therefore$ ANTICLOCKWISE

Question 19 (+)**

It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & -2 \\ 4 & -1 \end{pmatrix}.$$

a) Determine the matrix $\mathbf{AB}$.

b) Find the elements of

$$\mathbf{BA} - 2\mathbf{C}^2.$$

$\boxed{\mathbf{AB} = \begin{pmatrix} 5 \end{pmatrix}}$, $\boxed{\mathbf{BA} - 2\mathbf{C}^2 = \begin{pmatrix} 20 & -3 \\ 2 & 13 \end{pmatrix}}$

(a) $AB = \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \end{pmatrix}$

(b) $BA - 2C^2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 4 & -1 \end{pmatrix}$

$= \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} - 2\begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix}$

$= \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} -14 & 0 \\ 0 & -14 \end{pmatrix} = \begin{pmatrix} 20 & -3 \\ 2 & 13 \end{pmatrix}$

Question 20 (**+)

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are given below

$$\mathbf{A}=\begin{pmatrix}2 & 1\\ 2 & 0\end{pmatrix} \quad\text{and}\quad \mathbf{B}=\begin{pmatrix}1 & 0\\ -2 & 2\end{pmatrix}.$$

The matrix $\mathbf{C}$ represents the combined effect of the transformation represented by the $\mathbf{B}$, followed by the transformation represented by $\mathbf{A}$.

a) Determine the elements of $\mathbf{C}$.

b) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{\mathbf{C}=\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}}$, $\boxed{\text{enlargement by scale factor 2, reflection in the line } y=x\text{, in any order}}$

a) $C = AB = \begin{pmatrix}2 & 1\\ 2 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 2\end{pmatrix} = \begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix}$

b) NOW

$C = \begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix} = 2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$= \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

UNIFORM ENLARGEMENT BY SCALE FACTOR 2

REFLECTION IN THE LINE $y=x$

(ORDER DOES NOT MATTER

Question 21 (+)**

The 2×2 matrix $\mathbf{D}$ is given by

$$\mathbf{D}=\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}.$$

a) Given that $\mathbf{I}$ is the 2×2 identity matrix, show clearly that …

i. … $\mathbf{D}^2+5\mathbf{D}=6\mathbf{I}$.

ii. … $\mathbf{D}^{-1}=\frac{1}{6}(\mathbf{D}+5\mathbf{I})$.

The transformation in the x-y plane, which is represented by the matrix $\mathbf{D}$, maps the point P onto the point Q.

The coordinates of Q are $(7-2k, 9-6k)$, where k is a constant.

b) Determine, in terms of k, the coordinates of P.

$$\boxed{P(2k+3, 2k+1)}$$

(a) $D=\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$

- $D^2 = DD = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} = \begin{pmatrix} -14 & 25 \\ -30 & 51 \end{pmatrix}$
- $D^2+5D = \begin{pmatrix} -14 & 25 \\ -30 & 51 \end{pmatrix} + 5\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} = \begin{pmatrix} -14 & 25 \\ -30 & 51 \end{pmatrix} + \begin{pmatrix} 20 & -25 \\ 30 & -45 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$

$= 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 6I$ AS REQUIRED

(b) $D^2+5D=6I$

$\Rightarrow D^{-1}D^2+5D^{-1}D = 6D^{-1}I$

$\Rightarrow D+5I = 6D^{-1}$

$\Rightarrow D^{-1}=\frac{1}{6}(D+5I)$ AS REQUIRED

(c) $Dp = q$

$\Rightarrow D^{-1}Dp = D^{-1}q$

$\Rightarrow Ip = D^{-1}q$

$\Rightarrow p = D^{-1}q$

$\Rightarrow p = \frac{1}{-36-(-30)}\begin{bmatrix} -9 & 5 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 7-2k \\ 9-6k \end{bmatrix}$

$\Rightarrow p = -\frac{1}{6}\begin{bmatrix} -9(7-2k)+5(9-6k) \\ -6(7-2k)+4(9-6k) \end{bmatrix}$

$\Rightarrow p = -\frac{1}{6}\begin{bmatrix} -63+18k+45-30k \\ -42+12k+36-24k \end{bmatrix}$

$\Rightarrow p = -\frac{1}{6}\begin{bmatrix} -18-12k \\ -6-12k \end{bmatrix}$

$\Rightarrow p = \begin{pmatrix} 3+2k \\ 1+2k \end{pmatrix}$

$\therefore P(2k+3, 2k+1)$

Question 22 (***)

The 2×2 matrix $\mathbf{B}$ maps the points with coordinates $(-1,2)$ and $(1,4)$ onto the points with coordinates $(0,1)$ and $(6,-1)$, respectively.

a) Find the elements of $\mathbf{B}$.

b) Determine whether $\mathbf{B}$ has an invariant line, or a line of invariant points, or both.

c) Describe geometrically the transformation represented by $\mathbf{B}$.

$\boxed{\mathbf{B}=\begin{pmatrix}2&1\\-1&0\end{pmatrix}}$, $\boxed{\text{line of invariant points, } y=-x}$, $\boxed{\text{invariant line } \ y=-x+c}$, $\boxed{\text{shear}}$

a) SOLVE SIMULTANEOUS EQUATIONS OR USE INVERSES

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}-1&1\\2&4\end{pmatrix}=\begin{pmatrix}0&6\\1&-1\end{pmatrix}$$

$\Rightarrow \underline{B}\,\underline{A} = \underline{C}$

$\Rightarrow \underline{B}\,\underline{A}\,\underline{A}^{-1} = \underline{C}\,\underline{A}^{-1}$

$\Rightarrow \underline{B}\,\underline{I} = \underline{C}\,\underline{A}^{-1}$

$\Rightarrow \underline{B} = \begin{pmatrix}0&6\\1&-1\end{pmatrix}\frac{1}{-4-2}\begin{pmatrix}4&-1\\-2&-1\end{pmatrix}$

$\Rightarrow \underline{B} = \frac{1}{6}\begin{pmatrix}0&6\\1&-1\end{pmatrix}\begin{pmatrix}-4&1\\2&1\end{pmatrix}$

$\Rightarrow \underline{B} = \frac{1}{6}\begin{pmatrix}12&6\\-6&0\end{pmatrix}$

$\Rightarrow \underline{B} = \begin{pmatrix}2&1\\-1&0\end{pmatrix}$

b) FIRSTLY LOOK FOR LINE OF INVARIANT POINTS

$$\begin{pmatrix}2&1\\-1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix} \Rightarrow \begin{pmatrix}2x+y\\-x\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

$\therefore y=-x$
is LINE OF INVARIANT POINTS

NOW LOOK FOR INVARIANT LINES

$$\begin{pmatrix}2&1\\-1&0\end{pmatrix}\begin{pmatrix}t\\mt+c\end{pmatrix}=\begin{pmatrix}X\\Y\end{pmatrix}$$

$$\begin{pmatrix}2t+mt+c\\-t\end{pmatrix}=\begin{pmatrix}X\\Y\end{pmatrix}$$

$\Rightarrow \begin{matrix}2t+mt+c = X\\ -t = Y\end{matrix}$

$\Rightarrow \begin{matrix}2t+mt = X-c\\ t = -Y\end{matrix}$

$\Rightarrow \dfrac{2+m}{1} = \dfrac{X-c}{-Y}$

$\Rightarrow Y = -\dfrac{1}{2+m}X - \dfrac{1}{2+m}c$

COMPARE WITH $Y=mX+c$

$m = \dfrac{-1}{2+m}$

$2m+m^2=-1$

$m^2+2m+1=0$

$(m+1)^2=0$

$m=-1$

$\therefore Y = -\dfrac{1}{2-1}X + \dfrac{1}{2-1}c$

$Y=-X+c$

$\therefore$ ALSO INVARIANT LINES PARALLEL TO $y=-x$

c) INVESTIGATE $\underline{B}$

$\det\underline{B} = (2\times0)-(-1\times1)=1$ (AREA INVARIANT)

POSITIVE DETERMINANT $\Rightarrow$ ROTATION OR SHEAR (NO REFLECTION)

INVARIANT LINE OR INVARIANT LINE OF POINTS $\Rightarrow$ SHEAR (NO ROTATION)

$\therefore$ $\underline{B}$ REPRESENTS A SHEAR, WHERE $y=-x$ IS INVARIANT LINE OF POINTS

Question 23 (***)

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}; \ \mathbf{B} = \begin{pmatrix} 19 & 36 \\ 8 & 15 \end{pmatrix}.$$

Find the 2×2 matrix $\mathbf{X}$ that satisfy the equation $\mathbf{AX} = \mathbf{B}$

$$\boxed{\mathbf{X} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}}$$

$AX = B$

$A^{-1}AX = A^{-1}B$

$X = A^{-1}B$

$\therefore A^{-1} = \frac{1}{5\times3 - 2\times7}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$

$\therefore X = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 19 & 36 \\ 8 & 15 \end{pmatrix} = \begin{pmatrix} 57-56 & 108-105 \\ -38+40 & -72+75 \end{pmatrix}$

$\therefore X = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$

Question 24 (***)

It is given that $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices that satisfy

$$\det(\mathbf{AB}) = 18 \quad \text{and} \quad \det\left(\mathbf{B}^{-1}\right) = -3.$$

A square S, of area 6 cm^2, is transformed by $\mathbf{A}$ to produce an image S'.

Given that S' is also a square, determine its **perimeter**.

$$\boxed{72 \text{ cm}}$$

$\left.\begin{array}{l} \det(AB) = 18 \\ \det(B^{-1}) = -3 \end{array}\right\} \Rightarrow$ $\det A \times \det B = 18$

$\Downarrow$ $\qquad \det A \times \left(-\frac{1}{3}\right) = 18$

$\det B = -\frac{1}{3}$ $\qquad \det A = -54$

NOW AREA OF IMAGE IS

$6 \times 54 = 324$

SIDE LENGTH IS $\sqrt{324} = 18$

$\therefore$ PERIMETER OF IMAGE IS 72 cm

Question 25 (***)

The 2×2 matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ 3 & b \end{pmatrix},$$

where a and b are scalar constants.

a) If the point with coordinates $(1,1)$ is mapped by $\mathbf{A}$ onto the point with coordinates $(1,3)$, determine the value of a and the value of b.

b) Show that

$$\mathbf{A}^2 = 2\mathbf{A} - 3\mathbf{I}.$$

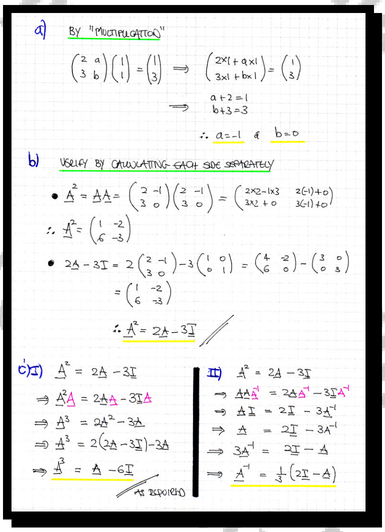
The inverse of $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$ and $\mathbf{I}$ is the 2×2 identity matrix.

c) Use part **(b)** to show further that …

i. … $\mathbf{A}^3 = \mathbf{A} - 6\mathbf{I}$.

ii. … $\mathbf{A}^{-1} = \frac{1}{3}(2\mathbf{I} - \mathbf{A})$

$\boxed{}$, $\boxed{a=-1}$, $\boxed{b=0}$



a) BY "MULTIPLICATION"

$$\begin{pmatrix} 2 & a \\ 3 & b \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2\times1 + a\times1 \\ 3\times1 + b\times1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} a+2=1 \\ b+3=3 \end{array}$$

$\therefore a=-1$ & $b=0$

b) VERIFY BY CALCULATING EACH SIDE SEPARATELY

- $A^2 = AA = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2\times2-1\times3 & 2(-1)+0 \\ 3\times2+0 & 3(-1)+0 \end{pmatrix}$

$\therefore A^2 = \begin{pmatrix} 1 & -2 \\ 6 & -3 \end{pmatrix}$

- $2A - 3I = 2\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 \\ 6 & -3 \end{pmatrix}$

$\therefore A^2 = 2A - 3I$

c)I) $A^2 = 2A - 3I$

$\Rightarrow A^2A = 2AA - 3IA$

$\Rightarrow A^3 = 2A^2 - 3A$

$\Rightarrow A^3 = 2(2A - 3I) - 3A$

$\Rightarrow A^3 = A - 6I$

AS REQUIRED

II) $A^2 = 2A - 3I$

$\Rightarrow AAA^{-1} = 2AA^{-1} - 3IA^{-1}$

$\Rightarrow AI = 2I - 3A^{-1}$

$\Rightarrow A = 2I - 3A^{-1}$

$\Rightarrow 3A^{-1} = 2I - A$

$\Rightarrow A^{-1} = \frac{1}{3}(2I - A)$

Question 26 (***)

A transformation in the x-y plane is represented by the 2×2 matrix

$$\mathbf{T}=\begin{pmatrix}1 & -3\\ -1 & 4\end{pmatrix}.$$

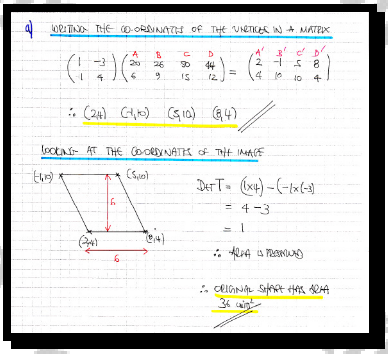
A quadrilateral Q has vertices at the points with coordinates $(20,6)$, $(26,9)$, $(50,15)$ and $(44,12)$. These coordinates are given in cyclic order.

The vertices of Q are transformed by $\mathbf{T}$.

a) Find the positions of the vertices of the image of Q.

b) Determine the area of Q, fully justifying your reasoning.

$\boxed{}$, $\boxed{(2,4),(-1,10),(5,10),(8,4)}$, $\boxed{\text{area}=36}$



Question 27 (***)

The 2×2 matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix},$$

where a and b are scalar constants.

The point with coordinates $(3,1)$ is mapped by $\mathbf{B}$ onto the point with coordinates $(5,13)$.

a) Determine the value of a and the value of b.

The inverse of $\mathbf{B}$ is denoted by $\mathbf{B}^{-1}$ and $\mathbf{I}$ is the 2×2 identity matrix.

b) Show that

$$\mathbf{B}^2 = 5\mathbf{B} + 2\mathbf{I}.$$

c) Show further that …

i. … $\mathbf{B}^3 = 27\mathbf{B} + 10\mathbf{I}$.

ii. … $\mathbf{B}^{-1} = \frac{1}{2}(\mathbf{B} - 5\mathbf{I})$

$\boxed{a = 1}$, $\boxed{b = 4}$

(a) $\begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix} \Rightarrow \begin{matrix} 3a+2=5 \\ 9+b=13 \end{matrix} \Rightarrow \begin{matrix} a=1 \\ b=4 \end{matrix}$

(b) $B^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

$5B + 2I = 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

$\therefore B^2 = 5B + 2I$ AS REQUIRED

(c) (I) $B^2 = 5B + 2I$

$\Rightarrow B^2B = 5BB + 2IB$

$\Rightarrow B^3 = 5B^2 + 2B$

$\Rightarrow B^3 = 5(5B + 2I) + 2B$

$\Rightarrow B^3 = 27B + 10I$ AS REQUIRED

(II) $B^2 = 5B + 2I$

$\Rightarrow B^2B^{-1} = 5BB^{-1} + 2IB^{-1}$

$\Rightarrow B = 5I + 2B^{-1}$

$\Rightarrow B - 5I = 2B^{-1}$

$\Rightarrow B^{-1} = \frac{1}{2}(B - 5I)$ AS REQUIRED

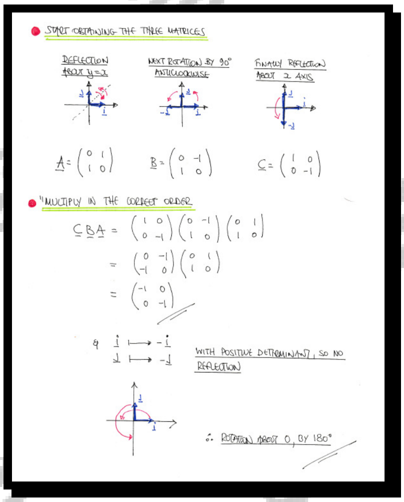
Question 28 (***)

A transformation in the x-y plane consists of ...

- ...a reflection about the line with equation $y = x$
- ... followed by an anticlockwise rotation about the origin by $90°$
- ... followed by a reflection about the x axis.

Use matrices to describe geometrically the resulting combined transformation.

$\boxed{}$, $\boxed{\text{rotation about the origin by } 180°}$



- START OBTAINING THE THREE MATRICES

REFLECTION ABOUT $y=x$ | NEXT ROTATION BY $90°$ ANTICLOCKWISE | FINALLY REFLECTION ABOUT x AXIS

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- "MULTIPLY IN THE CORRECT ORDER

$$CBA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

& $\mathbf{i} \mapsto -\mathbf{i}$, $\mathbf{j} \mapsto -\mathbf{j}$ WITH POSITIVE DETERMINANT, SO NO REFLECTION

$\therefore$ ROTATION ABOUT O, BY $180°$

Question 29 (***+)

The 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A}=\begin{pmatrix}4 & 1\\ 3 & 1\end{pmatrix} \quad \text{and} \quad \mathbf{B}=\begin{pmatrix}2 & -1\\ -3 & 5\end{pmatrix}.$$

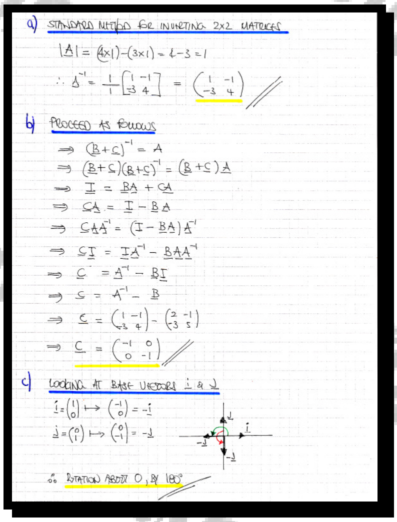
a) Find $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$.

b) Find a matrix $\mathbf{C}$, so that

$$(\mathbf{B}+\mathbf{C})^{-1}=\mathbf{A}.$$

c) Describe geometrically the transformation represented by $\mathbf{C}$.

$\boxed{}$, $\boxed{\mathbf{A}^{-1}=\begin{pmatrix}1 & -1\\ -3 & 4\end{pmatrix}}$, $\boxed{\mathbf{C}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}}$, $\boxed{\text{rotation about } O\text{, by } 180°}$



a) STANDARD METHOD FOR INVERTING 2x2 MATRICES

$|\mathbf{A}| = (4\times1)-(3\times1) = 4-3 = 1$

$\therefore \mathbf{A}^{-1} = \frac{1}{1}\begin{bmatrix}1 & -1\\ -3 & 4\end{bmatrix} = \begin{pmatrix}1 & -1\\ -3 & 4\end{pmatrix}$

b) PROCEED AS FOLLOWS

$\Rightarrow (\mathbf{B}+\mathbf{C})^{-1} = \mathbf{A}$

$\Rightarrow (\mathbf{B}+\mathbf{C})(\mathbf{B}+\mathbf{C})^{-1} = (\mathbf{B}+\mathbf{C})\mathbf{A}$

$\Rightarrow \mathbf{I} = \mathbf{BA} + \mathbf{CA}$

$\Rightarrow \mathbf{CA} = \mathbf{I} - \mathbf{BA}$

$\Rightarrow \mathbf{CAA}^{-1} = (\mathbf{I}-\mathbf{BA})\mathbf{A}^{-1}$

$\Rightarrow \mathbf{CI} = \mathbf{IA}^{-1} - \mathbf{BAA}^{-1}$

$\Rightarrow \mathbf{C} = \mathbf{A}^{-1} - \mathbf{BI}$

$\Rightarrow \mathbf{C} = \mathbf{A}^{-1} - \mathbf{B}$

$\Rightarrow \mathbf{C} = \begin{pmatrix}1 & -1\\ -3 & 4\end{pmatrix} - \begin{pmatrix}2 & -1\\ -3 & 5\end{pmatrix}$

$\Rightarrow \mathbf{C} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

c) LOOKING AT BASE VECTORS $\mathbf{i}$ & $\mathbf{j}$

$\mathbf{i} = \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}-1\\0\end{pmatrix} = -\mathbf{i}$

$\mathbf{j} = \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\-1\end{pmatrix} = -\mathbf{j}$

$\therefore$ ROTATION ABOUT O, BY 180°

Question 30 (***+)

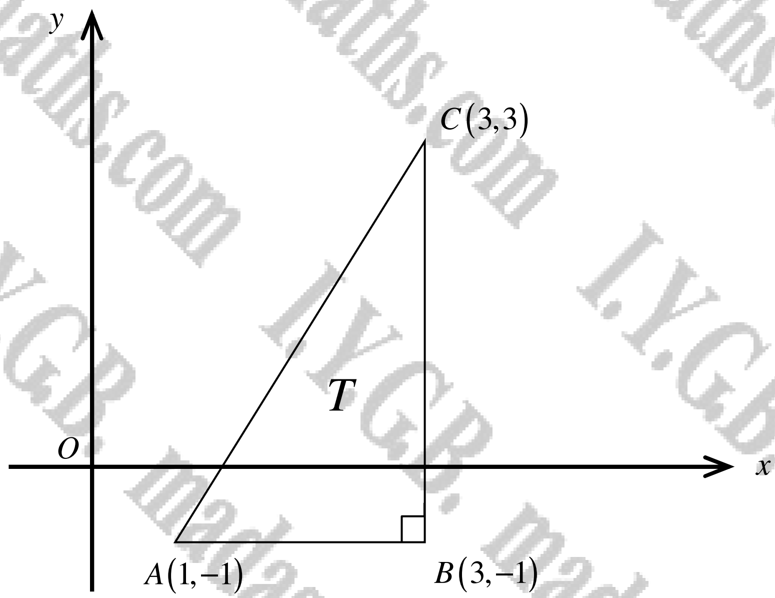
The 2×2 matrices

$$\mathbf{A}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad\text{and}\quad \mathbf{B}=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix},$$

represent linear transformations in the x-y plane.

a) Give full geometrical descriptions for each of the transformations represented by $\mathbf{A}$ and $\mathbf{B}$.

The figure below shows a right angled triangle T, with vertices at the points $A(1,-1)$, $B(3,-1)$ and $C(3,3)$.



The triangle T is first transformed by $\mathbf{A}$ and then by $\mathbf{B}$, producing the triangle T'.

b) Find the single matrix that represents this composite transformation.

c) Determine the coordinates of the vertices of T'.

d) Calculate the area of T'.

[continues overleaf]

[continued from overleaf]

The triangle T' is then reflected in the straight line with equation $y = -x$ to give the triangle T''.

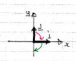
e) Find the single matrix that maps T'' back onto T.

$\boxed{}$, $\boxed{\text{rotation about } O,\ 90°,\ \text{clockwise}}$, $\boxed{\text{enlargement, in } x \text{ only, scale factor } 2}$,

$\boxed{\mathbf{BA} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}}$, $\boxed{A'(-2,-1), B'(-2,-3), C'(6,-3)}$, $\boxed{\text{area} = 8}$, $\boxed{\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}}$

a) $\underline{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\det \underline{A} = +1$ (POSSIBLE ROTATION)



$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\underline{j}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{i}$

$\therefore$ ROTATION ABOUT O, BY 90°, CLOCKWISE

$\underline{B} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$\det \underline{B} = 2$

$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\underline{i}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{j}$

$\therefore$ ENLARGEMENT, PARALLEL TO THE x AXIS, BY SCALE FACTOR OF 2

b) COMPOSING TRANSFORMATIONS, $\underline{A}$ FOLLOWED BY $\underline{B}$, IS $\underline{B}\underline{A}$

$\underline{B}\underline{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$

c) WRITING THE 3 SETS OF CO-ORDINATES AS A SINGLE MATRIX

$\underline{B}\underline{A}\underline{x} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}\begin{array}{c} \begin{pmatrix} 1 & 3 & 3 \\ -1 & -1 & 3 \end{pmatrix} \\ \begin{matrix} A & B & C \end{matrix} \end{array} = \begin{array}{c} \begin{pmatrix} -2 & -2 & 6 \\ -1 & -3 & -3 \end{pmatrix} \\ \begin{matrix} A' & B' & C' \end{matrix} \end{array}$

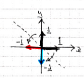
$\therefore A'(-2,-1),\ B'(-2,-3),\ C'(6,-3)$

d) AREA OF T, BY INSPECTION IS $\frac{1}{2} \times 2 \times 4 = 4$ UNITS^2

$\det(\underline{A}\underline{B}) = \det \underline{A} \times \det \underline{B} = 1 \times 2 = 2$

AREA OF T' IS $4 \times 2 = 8$ UNITS^2

e) FIND THE MATRIX WHICH REPRESENT REFLECTION ABOUT $y = -x$



$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto -\underline{j} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto -\underline{i} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\therefore$ REQUIRED MATRIX IS $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

THE MATRIX WHICH DOES THE 3 TRANSFORMATIONS IN THE CORRECT ORDER IS

$\underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{\text{REFLECTION ABOUT } y=-x}\underbrace{\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}}_{\underline{B}\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

FINALLY WE REQUIRE THE INVERSE OF THE ABOVE MATRIX

- $\det \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = -2$
- $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

Question 31 (***+)

The 2×2 matrix $\mathbf{A}$ given below, represents a transformation in the x-y plane.

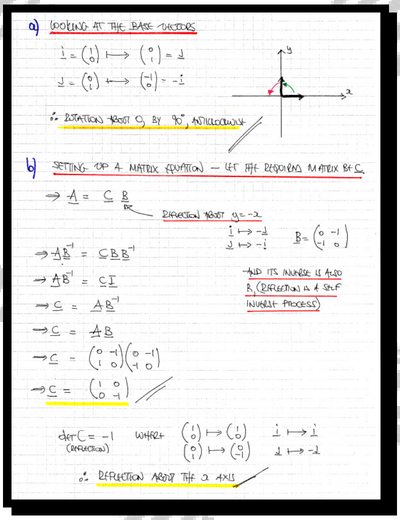
$$\mathbf{A}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

a) Describe geometrically the transformation represented by $\mathbf{A}$.

The transformation described by $\mathbf{A}$ is equivalent to a reflection about the straight line with equation $y=-x$, followed by another transformation described by the matrix $\mathbf{C}$.

b) Find the matrix $\mathbf{C}$, and describe it geometrically.

$\boxed{\text{rotation about } O, \text{by } 90°, \text{ anticlockwise}}$, $\boxed{\mathbf{C}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$, $\boxed{\text{reflection about the } x \text{ axis}}$



a) LOOKING AT THE BASE VECTORS

$\underline{i}=\begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\end{pmatrix}=\underline{j}$

$\underline{j}=\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}-1\\0\end{pmatrix}=-\underline{i}$

∴ ROTATION ABOUT O, BY 90°, ANTICLOCKWISE

b) SETTING UP A MATRIX EQUATION – LET THE REQUIRED MATRIX BE $\underline{C}$.

$\Rightarrow \underline{A}=\underline{C}\,\underline{B}$

REFLECTION ABOUT $y=-x$

$\underline{i}\mapsto -\underline{j}$, $\underline{j}\mapsto -\underline{i}$, $\underline{B}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

AND ITS INVERSE IS ALSO B, (REFLECTION IS A SELF INVERSE PROCESS)

$\Rightarrow \underline{A}\underline{B}^{-1}=\underline{C}\underline{B}\underline{B}^{-1}$

$\Rightarrow \underline{A}\underline{B}^{-1}=\underline{C}\underline{I}$

$\Rightarrow \underline{C}=\underline{A}\underline{B}^{-1}$

$\Rightarrow \underline{C}=\underline{A}\underline{B}$

$\Rightarrow \underline{C}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

$\Rightarrow \underline{C}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$\det C=-1$ (REFLECTION) WHERE $\begin{pmatrix}1\\0\end{pmatrix}\mapsto\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}\mapsto\begin{pmatrix}0\\-1\end{pmatrix}$; $\underline{i}\mapsto\underline{i}$, $\underline{j}\mapsto-\underline{j}$

∴ REFLECTION ABOUT THE x AXIS

Question 32 (***+)

The 2×2 matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

Find, by calculation, the equations of the two lines which pass through the origin, that remain invariant under the transformation represented by $\mathbf{M}$.

$\boxed{}$, $\boxed{y = \pm x}$

METHOD A

- LET A LINE THROUGH THE ORIGIN HAVE EQUATION $y = mx$, WHICH IS THEN MAPPED TO $Y = mX$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} X \\ mX \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} 3mx \\ 3x \end{bmatrix}$$

- HENCE WE OBTAIN THE EQUATIONS

$$\left.\begin{aligned} X &= 3mx \\ mX &= 3x \end{aligned}\right\} \Longrightarrow \text{DIVIDING THE EQUATIONS WE OBTAIN}$$

$$\frac{1}{m} = m$$
$$m^2 = 1$$
$$m = \pm 1$$

$\therefore$ THE REQUIRED LINES ARE $y = x$ & $y = -x$

METHOD B (BY EIGENVECTORS)

- FIND THE CHARACTERISTIC EQUATION OF $\underline{M}$

$$\begin{vmatrix} 0-\lambda & 3 \\ 3 & 0-\lambda \end{vmatrix} = 0 \implies (-\lambda)^2 - 9 = 0$$
$$\implies \lambda^2 - 9 = 0$$
$$\implies (\lambda-3)(\lambda+3) = 0$$
$$\implies \lambda = \begin{cases} -3 \\ 3 \end{cases}$$

- FINDING THE EIGENVECTORS AND HENCE THE LINES

IF $\lambda = 3$

$3y = 3x$

$3x = 3y$

$\therefore y = x$

IF $\lambda = -3$

$3y = -3x$

$3x = -3y$

$\therefore y = -x$

Question 33 (***+)

Find the image of the straight line with equation

$$2x+3y=10,$$

under the transformation represented by the 2×2 matrix

$$\mathbf{A}=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}.$$

$\boxed{}$, $\boxed{11x+y=70}$

METHOD A

BY INSPECTION $A(5,0)$ & $B(2,2)$ LIE ON THE LINE

MAP THESE POINTS ONTO THEIR NEW POSITIONS

$$\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}5 & 2\\ 0 & 2\end{pmatrix}=\begin{pmatrix}5 & 6\\ 15 & 4\end{pmatrix}$$

A B A′ B′

FIND GRADIENT OF $A'(5,15)$ & $B'(6,4)$ $\Rightarrow m=\frac{4-15}{6-5}=\frac{-11}{1}=-11$

$$\therefore\ y-y_0=m(x-x_0)$$
$$y-4=-11(x-6)$$
$$y-4=-11x+66$$
$$y=-11x+70$$

METHOD B

$$2x+3y=10$$
$$3y=-2x+10$$
$$y=-\tfrac{2}{3}x+\tfrac{10}{3}$$

LET A POINT LIE ON ABOVE LINE HAVE COORDINATES $\left(t,-\tfrac{2}{3}t+\tfrac{10}{3}\right)$

THUS

$$\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}\begin{pmatrix}t\\ -\tfrac{2}{3}t+\tfrac{10}{3}\end{pmatrix}=\begin{pmatrix}t-\tfrac{4}{3}t+\tfrac{20}{3}\\ 3t+\tfrac{2}{3}t-\tfrac{10}{3}\end{pmatrix}=\begin{pmatrix}-\tfrac{1}{3}t+\tfrac{20}{3}\\ \tfrac{11}{3}t-\tfrac{10}{3}\end{pmatrix}\begin{matrix}\leftarrow \text{NEW } x\\ \leftarrow \text{NEW } y\end{matrix}$$

$$\left.\begin{aligned}x&=-\tfrac{1}{3}t+\tfrac{20}{3}\\ y&=\tfrac{11}{3}t-\tfrac{10}{3}\end{aligned}\right\}\Rightarrow\left.\begin{aligned}3x&=-t+20\\ 3y&=11t-10\end{aligned}\right\}\Rightarrow\begin{aligned}33x&=-11t+220\\ 3y&=11t-10\end{aligned}$$

Add equations $33x+3y=210$

$$11x+y=70$$

AS BEFORE

Question 34 (***+)

The 2×2 matrix $\mathbf{M}=\begin{pmatrix}-2 & 1\\ -9 & 4\end{pmatrix}$ is given.

Under the transformation represented by $\mathbf{M}$ a straight line passing through the origin remains invariant.

Determine the equation of this line.

$\boxed{}$, $\boxed{y=3x}$

WORKING AS FOLLOWS

- "OBJECT LINE" $y=mx$
- "IMAGE LINE" $Y=mX$

$\Rightarrow \begin{pmatrix}-2 & 1\\ -9 & 4\end{pmatrix}\begin{pmatrix}x\\ mx\end{pmatrix}=\begin{pmatrix}X\\ mX\end{pmatrix}$

$\Rightarrow \begin{array}{l}-2x+mx=X\\ -9x+4mx=mX\end{array}$

DIVIDING THE EQUATIONS

$\Rightarrow \dfrac{-2+m}{-9+4m}=\dfrac{1}{m}$

$\Rightarrow -2m+m^2=4m-9$

$\Rightarrow m^2-6m+9=0$

$\Rightarrow (m-3)^2=0$

$\Rightarrow m=3$

$\therefore$ REQUIRED LINE

$y=3x$

Question 35 (****)

The 2×2 matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}$ is given.

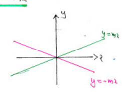
Under the transformation represented by $\mathbf{A}$, a straight line passing through the origin is reflected about the y axis.

Determine the possible equations of this line.

$\boxed{}$, $\boxed{y = x}$, $\boxed{y = -6x}$

LET THE REQUIRED LINE HAVE EQUATION $y = mx$

LOOKING AT THE DIAGRAM THE REFLECTED LINE WILL HAVE EQUATION $y = -mx$



$\Rightarrow \begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ -mX \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2x + mx \\ -6x + 3mx \end{pmatrix} = \begin{pmatrix} X \\ -mX \end{pmatrix}$

$\Rightarrow \left.\begin{array}{l} 2x + mx = X \\ -6x + 3mx = -mX \end{array}\right\}$ DIVIDING $\quad \dfrac{2+m}{-6+3m} = \dfrac{1}{-m}$

$-6 + 3m = -2m - m^2$

$m^2 + 5m - 6 = 0$

$(m-1)(m+6) = 0$

$\therefore m = \begin{cases} 1 \\ -6 \end{cases}$

$\therefore \; y = x \;$ OR $\; y = -6x$

Question 36 (****)

Find the image of the circle with equation

$$x^2 + y^2 = 4,$$

under the transformation represented by the 2×2 matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$.

$\boxed{}$, $\boxed{20x^2 - 32xy + 13y^2 = 16}$

LET THE "BEFORE" CO-ORDINATES BE (x,y) AND THE "AFTER" CO-ORDINATES BE DENOTED BY (X,Y)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

IF $\underline{X} = \underline{A}\underline{x}$

$\underline{A}^{-1}\underline{X} = \underline{A}^{-1}\underline{A}\underline{x}$

$\underline{A}^{-1}\underline{X} = \underline{x}$

$$\underline{A}^{-1} = \frac{1}{(4\times2 - 2\times3)}\begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix}$$

HENCE WE NOW HAVE

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4X - 3Y \\ -2X + 2Y \end{pmatrix} = \begin{pmatrix} 2X - \frac{3}{2}Y \\ -X + Y \end{pmatrix}$$

SUBSTITUTE INTO THE CIRCLE & TIDY

$\Rightarrow x^2 + y^2 = 4$

$\Rightarrow \left(2X - \frac{3}{2}Y\right)^2 + (Y - X)^2 = 4$

$\Rightarrow \left.\begin{array}{l} 4X^2 - 6XY + \frac{9}{4}Y^2 \\ X^2 - 2XY + Y^2 \end{array}\right\} = 4$

$\Rightarrow 5X^2 - 8XY + \frac{13}{4}Y^2 = 4$

$\Rightarrow 20X^2 - 32XY + 13Y^2 = 16$

IE $20x^2 - 32xy + 13y^2 = 16$

Question 37 (****)

The 2×2 matrix $\mathbf{R}$ represents a reflection where the point $(2,1)$ gets mapped onto the point $(6,-5)$, and the line with equation $y=-\frac{1}{2}x$ is a line of invariant points.

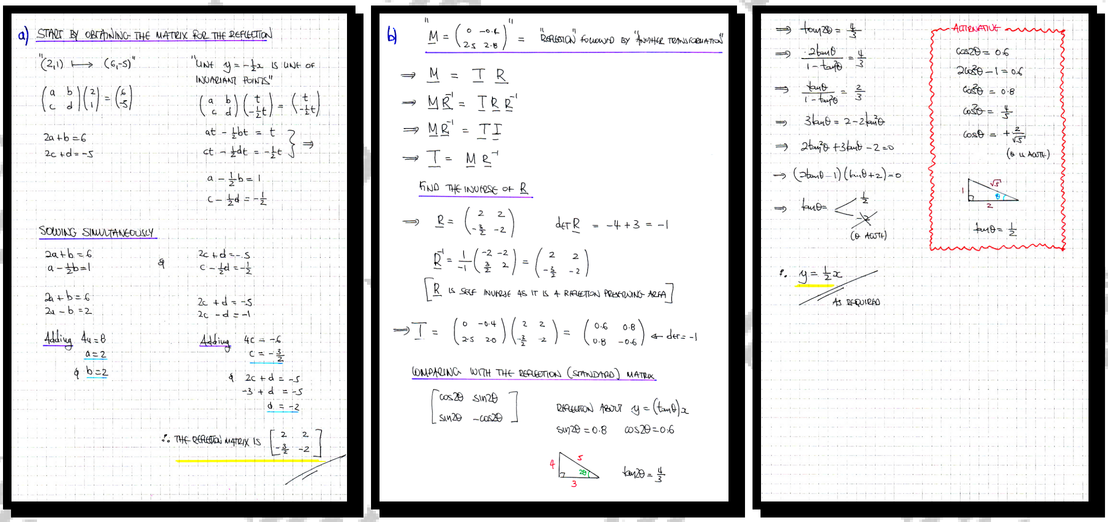
a) Determine the elements of $\mathbf{R}$.

The 2×2 matrix $\mathbf{M}$ represents the combined transformation of the reflection represented by $\mathbf{R}$, followed by another transformation T.

$$\mathbf{M}=\begin{pmatrix}0 & -0.4\\ 2.5 & 2.8\end{pmatrix}.$$

b) Given that T is also a reflection determine, in exact simplified form, the equation of the line of reflection of T.

$\boxed{}$, $\boxed{\mathbf{R}=\begin{pmatrix}2 & 2\\ -\frac{3}{2} & -2\end{pmatrix}}$, $\boxed{\frac{1}{2}x=-\frac{1}{7}y=\frac{1}{2}z}$



Question 38 (****)

Under the transformation represented by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & -7 \end{pmatrix},$$

the straight line with equation $y = mx$ is reflected about the x axis.

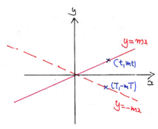
Find the possible values of m.

$\boxed{}$, $\boxed{m = 1, \quad m = 2}$

LOOKING AT A DIAGRAM

under this transformation

$(t, mt) \longmapsto (T, -mT)$



HENCE WE OBTAIN

$$\begin{pmatrix} 1 & 2 \\ 4 & -7 \end{pmatrix}\begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ -mT \end{pmatrix} \Rightarrow \begin{pmatrix} t+2mt \\ 4t-7mt \end{pmatrix} = \begin{pmatrix} T \\ -mT \end{pmatrix}$$

$$\Rightarrow \begin{Bmatrix} t + 2mt = T \\ 4t - 7mt = -mT \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} t(1+2m) = T \\ t(4-7m) = -mT \end{Bmatrix}$$

$$\Rightarrow \frac{t(1+2m)}{t(4-7m)} = \frac{T}{-mT}$$

$$\Rightarrow \frac{1+2m}{4-7m} = \frac{1}{-m}$$

$$\Rightarrow -m - 2m^2 = 4 - 7m$$

$$\Rightarrow 0 = 2m^2 - 6m + 4$$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\Rightarrow m = \begin{cases} 1 \\ 2 \end{cases}$$

Question 39 (****)

A transformation in two dimensional space maps a general point with coordinates (x, y) onto the point with coordinates (X, Y) according to the equation

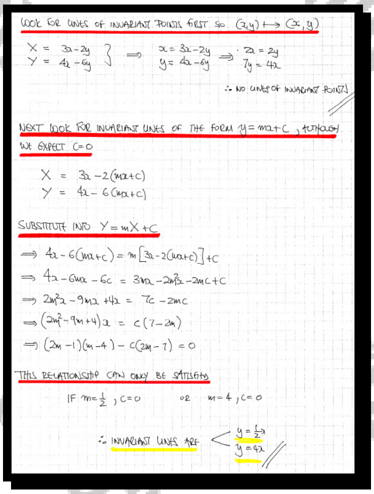
$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix},$$

where $\mathbf{B}$ is the 2×2 matrix $\begin{pmatrix} 3 & -2 \\ 4 & -6 \end{pmatrix}$.

Investigate whether this transformation has any lines of invariant points or any invariant lines, giving any relevant equations of such lines if they exist.

You may not use eigenvalue/eigenvector methods in this question

$\boxed{}$, $\boxed{\text{no lines of invariant points}}$, $\boxed{\text{invariant lines : } y = \tfrac{1}{2}x \ \cup \ y = 4x}$



LOOK FOR LINES OF INVARIANT POINTS FIRST SO $(x,y) \mapsto (x,y)$

$$\left.\begin{aligned} X &= 3x - 2y \\ Y &= 4x - 6y \end{aligned}\right\} \Longrightarrow \begin{aligned} x &= 3x - 2y \\ y &= 4x - 6y \end{aligned} \Longrightarrow \begin{aligned} 2x &= 2y \\ 7y &= 4x \end{aligned}$$

$\therefore$ NO LINES OF INVARIANT POINTS

NEXT LOOK FOR INVARIANT LINES OF THE FORM $y = mx + c$, ALTHOUGH WE EXPECT $c = 0$

$$X = 3x - 2(mx + c)$$
$$Y = 4x - 6(mx + c)$$

SUBSTITUTE INTO $Y = mX + c$

$\Longrightarrow 4x - 6(mx + c) = m[3x - 2(mx + c)] + c$

$\Longrightarrow 4x - 6mx - 6c = 3mx - 2m^2x - 2mc + c$

$\Longrightarrow 2m^2x - 9mx + 4x = 7c - 2mc$

$\Longrightarrow (2m^2 - 9m + 4)x = c(7 - 2m)$

$\Longrightarrow (2m - 1)(m - 4) - c(2m - 7) = 0$

THIS RELATIONSHIP CAN ONLY BE SATISFIED

IF $m = \frac{1}{2}$, $c = 0$ OR $m = 4$, $c = 0$

$\therefore$ INVARIANT LINES ARE $y = \frac{1}{2}x$, $y = 4x$

Question 40 (****)

The 2×2 matrix $\mathbf{A}$ is given below.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}.$$

a) Find scalar constants, k and h, so that

$$\mathbf{A}^2 + k\mathbf{I} = h\mathbf{A}.$$

b) Use part **(a)** to determine $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$.

No credit will be given for finding $\mathbf{A}^{-1}$ by a direct method.

$\boxed{k=1}$, $\boxed{h=8}$, $\boxed{\mathbf{A}^{-1} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}}$

a) BY COMPARING ELEMENTS IN THE MATRIX EQUATION

$\underline{A}^2 + k\underline{I} = h\underline{A}$

$\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} + k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = h\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$

$\begin{pmatrix} 23 & 16 \\ 56 & 39 \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} 3h & 2h \\ 7h & 5h \end{pmatrix}$

LOOKING AT a_{12}

$16 + 0 = 2h$

$h = 8$

LOOKING AT a_{11}

$23 + k = 3h$

$23 + k = 24$

$k = 1$

b) USING THE EQUATION OF PART

$\underline{A}^2 + \underline{I} = 8\underline{A}$

$\underline{A}^2\underline{A}^{-1} + \underline{I}\underline{A}^{-1} = 8\underline{A}\underline{A}^{-1}$

$\underline{A} + \underline{A}^{-1} = 8\underline{I}$

$\underline{A}^{-1} = 8\underline{I} - \underline{A}$

$\underline{A}^{-1} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$

$\underline{A}^{-1} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

or $\underline{A}^{-1}\underline{A}^2 + \underline{A}^{-1}\underline{I} = 8\underline{A}^{-1}\underline{A}$

$\underline{A} + \underline{A}^{-1} = 8\underline{I}$

Question 41 (****)

The 2×2 matrix $\mathbf{A}$ given below represents a transformation in the x-y plane.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

The straight line L with equation

$$y = 2x + 1$$

is transformed by $\mathbf{A}$ into the straight line L'.

a) Find a Cartesian equation of L'.

The straight line M is transformed by $\mathbf{A}$ into the straight line M' with equation

$$11x + 6y = 4.$$

b) Find a Cartesian equation of M.

$\boxed{L' : y = 1 - x}$, $\boxed{M : y = 4 - 3x}$

a) EASIER TO PARAMETERIZE L

$y = 2x+1$ HAS GENERAL POINT $(t, 2t+1)$

THUS

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} t \\ 2t+1 \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 3t-2t-1 \\ -5t+4t+2 \end{pmatrix}$$

$$\left.\begin{aligned} X &= t-1 \\ Y &= 2-t \end{aligned}\right\} \text{Add the parametrics}$$

$$X + Y = 1$$

$$\therefore y = 1 - x$$

b) $\Rightarrow A\underline{x} = \underline{X}$

$\Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{X}$

$\Rightarrow I\underline{x} = A^{-1}\underline{X}$

$\Rightarrow \underline{x} = A^{-1}\underline{X}$

NOW $A^{-1} = \frac{1}{3\times2-(-5)(-1)}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

• EASIER NOT TO PARAMETRIZE BUT TO TRANSFORM BACK TWO "EASY" POINT WHICH EVIDENTLY LIE ON M'.

SAY $(2,-3)$ AND $(8,-14)$ BOTH LIE ON M'

$$\underline{x} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2 & 8 \\ -3 & -14 \end{pmatrix}$$

$$\underline{x} = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$$

• SO $(1,1)$ & $(2,-2)$ LIE ON M

GRADIENT $= \frac{y_2-y_1}{x_2-x_1} = \frac{-2-1}{2-1} = -3$

$$y - y_0 = m(x - x_0)$$
$$y - 1 = -3(x-1)$$
$$y - 1 = -3x + 3$$
$$y = 4 - 3x$$

Question 42 **(****)**

Describe fully the transformation given by the following 2×2 matrix

$$\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

The description must be supported by mathematical calculations.

$\boxed{\text{reflection in } y = 2x}$

$\det\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = -\frac{3}{5}\times\frac{3}{5} - \frac{4}{5}\times\frac{4}{5} = -\frac{9}{25} - \frac{16}{25} = -1$

COMPARE THE MATRIX WITH THE STANDARD REFLECTION MATRIX

$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$\cos 2\theta = -\frac{3}{5}$

$2\theta = 126.87 \pm 360n$
$2\theta = 233.13 \pm 360n$

$\theta = 63.43 \pm 180n$
$\theta = 116.57 \pm 180n$

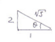
$\theta = 63.43$ PRODUCES ALL THE ENTRIES CORRECTLY

$\therefore$ MATRIX REPRESENTS REFLECTION ABOUT THE LINE $y = (\tan\theta)x$

$\therefore y = \tan(63.43)x$
$y = 2x$

NOTE $\cos 2\theta = -\frac{3}{5}$
$2\cos^2\theta - 1 = -\frac{3}{5}$
$2\cos^2\theta = \frac{2}{5}$
$\cos^2\theta = \frac{1}{5}$

THUS $\cos\theta = +\frac{1}{\sqrt{5}}$ (θ IS ACUTE $63.43°$)



$\therefore \tan\theta = 2$

Question 43 (****)

A composite transformation in the x-y plane consists of …

i. … a uniform enlargement about the origin of scale factor k, $k>0$, denoted by the matrix $\mathbf{E}$.

ii. … a shear parallel to the straight line L, denoted by the matrix $\mathbf{S}$.

It is given that $\mathbf{ES}=\mathbf{SE}=\begin{pmatrix} 12 & 16 \\ -9 & 36 \end{pmatrix}$

a) Show clearly that $k=24$.

b) Find a Cartesian equation of L.

$$\boxed{y=\frac{3}{4}x}$$

a) FIRSTLY $\det\begin{pmatrix} 12 & 16 \\ -9 & 36 \end{pmatrix} = 12\times 36-(-9)\times 16 = 432+144 = 576$ ← AREA SCALE FACTOR

AS THE ENLARGEMENT IS UNIFORM, THE SCALE FACTOR MUST BE $\sqrt{576}=24$ $\therefore k=24$

b) TO FIND L, WE NEED TO FIND AN INVARIANT LINE THROUGH O

THUS

$$\begin{pmatrix} 12 & 16 \\ -9 & 36 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \lambda\begin{pmatrix} x \\ mx \end{pmatrix} \Rightarrow \left.\begin{array}{l} 12x+16mx=\lambda x \\ -9x+36mx=\lambda mx \end{array}\right\} \Rightarrow \text{DIVIDE EQUATIONS}$$

$$\Rightarrow \frac{12x+16mx}{-9x+36mx}=\frac{\lambda x}{\lambda mx}$$

$$\Rightarrow \frac{12+16m}{-9+36m}=\frac{1}{m}$$

$$\Rightarrow 12m+16m^2=-9+36m$$

$$\Rightarrow 16m^2-24m+9=0$$

$$\Rightarrow (4m-3)^2=0$$

$$\therefore m=\frac{3}{4}$$

$$\therefore y=\frac{3}{4}x$$

Question 44 (****)

A plane transformation maps the general point (x, y) onto the general point (X, Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the area scale factor of the transformation.

b) Determine the equation of the straight line of invariant points under this transformation.

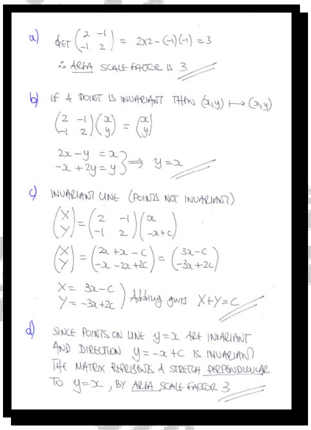
c) Show that all the straight lines with equation of the form

$$x + y = c,$$

where c is a constant, are invariant lines under this transformation.

d) Hence describe the transformation geometrically.

$\boxed{\text{SF} = 3}$, $\boxed{y = x}$, $\boxed{\text{stretch perpendicular to the line } y = x\text{, by area scale factor 3}}$


a) $\det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 2\times2-(-1)(-1) = 3$

$\therefore$ AREA SCALE FACTOR IS 3

b) IF A POINT IS INVARIANT THEN $(x,y) \mapsto (x,y)$

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\left.\begin{array}{l} 2x-y=x \\ -x+2y=y \end{array}\right\} \Rightarrow y=x$

c) INVARIANT LINE (POINTS NOT INVARIANT)

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ -x+c \end{pmatrix}$

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2x+x-c \\ -x-2x+2c \end{pmatrix} = \begin{pmatrix} 3x-c \\ -3x+2c \end{pmatrix}$

$\left.\begin{array}{l} X=3x-c \\ Y=-3x+2c \end{array}\right)$ Adding gives $X+Y=c$

d) SINCE POINTS ON LINE $y=x$ ARE INVARIANT AND DIRECTION $y=-x+c$ IS INVARIANT THE MATRIX REPRESENTS A STRETCH PERPENDICULAR TO $y=x$, BY AREA SCALE FACTOR 3

Question 45 (****)

A transformation $T:\mathbb{R}^2 \mapsto \mathbb{R}^2$ is represented by the following 2×2 matrix .

$$\mathbf{A} = \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}.$$

a) Find the determinant of $\mathbf{A}$ and explain its significance with reference to its sign and its magnitude.

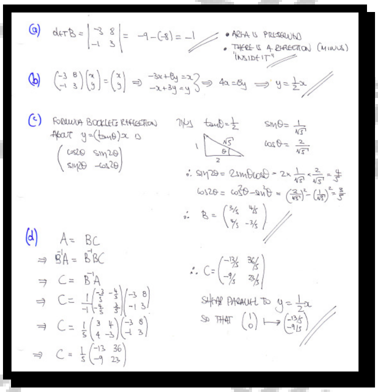
b) Find the equation of the straight line of the invariant points under the transformation represented by $\mathbf{A}$.

c) Determine the entries of the 2×2 matrix $\mathbf{B}$ which represents a reflection about the straight line found in part **(b)**, giving all its entries as simple fractions.

The transformation represented by $\mathbf{A}$, consists of a shear represented by the matrix $\mathbf{C}$, followed by a reflection represented by the matrix $\mathbf{B}$.

d) Determine the matrix $\mathbf{C}$ and describe the shear.

$\boxed{\det \mathbf{A} = -1}$, $\boxed{y = \frac{1}{2}x}$, $\boxed{\mathbf{B} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}}$, $\boxed{\mathbf{C} = \begin{pmatrix} -\frac{13}{5} & \frac{36}{5} \\ -\frac{9}{5} & \frac{23}{5} \end{pmatrix}}$


(a) $\det B = \begin{vmatrix} -3 & 8 \\ -1 & 3 \end{vmatrix} = -9-(-8) = -1$
• AREA IS PRESERVED
• THERE IS A REFLECTION (MINUS) "INSIDE IT"

(b) $\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left.\begin{matrix} -3x+8y = x \\ -x+3y = y \end{matrix}\right\} \Rightarrow 4x = 8y \Rightarrow y = \frac{1}{2}x$

(c) FORMULA BOOKLET: REFLECTION ABOUT $y = (\tan\theta)x$ IS $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

THUS $\tan\theta = \frac{1}{2}$ $\sin\theta = \frac{1}{\sqrt{5}}$ $\cos\theta = \frac{2}{\sqrt{5}}$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{1}{\sqrt{5}}\times\frac{2}{\sqrt{5}} = \frac{4}{5}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{3}{5}$

$\therefore B = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$

(d) $A = BC$

$\Rightarrow B^{-1}A = B^{-1}BC$

$\Rightarrow C = B^{-1}A$

$\Rightarrow C = \frac{1}{-1}\begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$

$\Rightarrow C = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$

$\Rightarrow C = \frac{1}{5}\begin{pmatrix} -13 & 36 \\ -9 & 23 \end{pmatrix}$

$\therefore C = \begin{pmatrix} -13/5 & 36/5 \\ -9/5 & 23/5 \end{pmatrix}$

SHEAR PARALLEL TO $y = \frac{1}{2}x$ SO THAT $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -13/5 \\ -9/5 \end{pmatrix}$

Question 46 (****+)

A transformation in two dimensional space maps a general point with coordinates (x, y) onto the point with coordinates (X, Y) according to the equation

$$\begin{pmatrix} X-4 \\ Y+4 \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix},$$

where $\mathbf{A}$ is the 2×2 matrix $\begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}$.

Investigate whether this transformation has any lines of invariant points or any invariant lines, giving any relevant equations of such lines if they exist.

$\boxed{\text{FP1-N}}$, $\boxed{\text{line of invariant points} : 3x-2y=4}$, $\boxed{\text{invariant line } : y=-x+C}$

START WITH LINES OF INVARIANT POINTS, IF $(x,y) \longmapsto (x,y)$

$$\begin{pmatrix} x-4 \\ y+4 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x-4 \\ y+4 \end{pmatrix} = \begin{pmatrix} -2x+2y \\ 3x-y \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} x-4=-2x+2y \\ y+4=3x-y \end{array}$$

$$\Rightarrow \begin{array}{l} 3x-2y=4 \\ 4=3x-2y \end{array}$$

$\therefore$ $3x-2y=4$ IS A LINE OF INVARIANT POINTS

NEXT INVESTIGATE INVARIANT LINES, SAY $y=mx+C$

$$\left.\begin{array}{l} X-4=-2x+2y \\ Y+4=3x-y \end{array}\right\} \Rightarrow \begin{array}{l} X=4-2x+2(mx+c) \\ Y=-4+3x-(mx+c) \end{array}$$

SUBSTITUTE INTO $Y=mX+C$

$\Rightarrow -4+3x-(mx+c)=[4-2x+2(mx+c)]m+c$

$\Rightarrow -4+3x-mx-c=(4-2x+2mx+2c)m+c$

$\Rightarrow -4+3x-mx-c=4m-2mx+2m^2x+2mc+c$

$\Rightarrow -4-2c-4m-2mc=-2mx+2m^2x+mx-3x$

$\Rightarrow -[2mc+4m+2c+4]=[2m^2-m-3]x$

$\Rightarrow -[2m(c+2)+2(c+2)]=(2m-3)(m+1)x$

$\Rightarrow -(2m+2)(c+2)=(2m-3)(m+1)x$

$\Rightarrow -2(m+1)(c+2)=(2m-3)(m+1)x.$

$\Rightarrow \boxed{(2m-3)(m+1)x+2(m+1)(c+2)=0}$

NOW LOOKING AT THIS EQUATION

- IF $m=-1$ IT IS AUTOMATICALLY SATISFIED FOR ALL C

  $\therefore$ $y=-x+C$ IS AN INVARIANT LINE

- IF $m=\frac{3}{2}$ AND $C=-2$ IT IS ALSO SATISFIED

$$y=\tfrac{3}{2}x-2$$
$$2y=3x-4$$
$$0=3x-2y-4$$
$$3x-2y=4$$

WHICH WAS FOUND TO BE A LINE OF INVARIANT POINTS

Question 47 (****+)

A transformation T, maps the general point (x, y) onto the general point (X, Y), by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

a) Find the area scale factor of the transformation.

b) Determine the equation of the line of invariant points under this transformation.

c) Show that all the straight lines of the form

$$y = x + c,$$

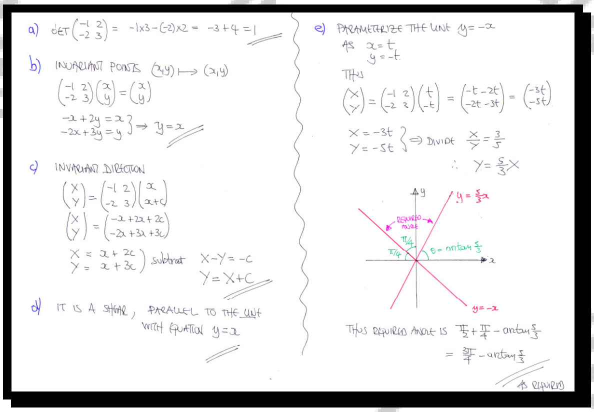
where c is a constant, are invariant lines under T.

d) Hence state the name of T.

e) Show that the acute angle formed by the straight line with equation $y = -x$ and its the image under T is

$$\frac{3\pi}{4} - \arctan\left(\frac{5}{3}\right).$$

$\boxed{\text{SF} = 1}$, $\boxed{y = x}$, $\boxed{\text{shear}}$


a) $\det\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} = -1\times3-(-2)\times2 = -3+4 = 1$

b) INVARIANT POINTS $(x,y) \mapsto (x,y)$

$\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\left.\begin{array}{l} -x+2y = x \\ -2x+3y = y \end{array}\right\} \Rightarrow y = x$

c) INVARIANT DIRECTION

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ x+c \end{pmatrix}$

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -x+2x+2c \\ -2x+3x+3c \end{pmatrix}$

$\left.\begin{array}{l} X = x+2c \\ Y = x+3c \end{array}\right)$ subtract $X-Y = -c$

$Y = X+C$

d) IT IS A SHEAR, PARALLEL TO THE LINE WITH EQUATION $y = x$

e) PARAMETERIZE THE LINE $y = -x$ AS $x = t$, $y = -t$

THUS

$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} -t-2t \\ -2t-3t \end{pmatrix} = \begin{pmatrix} -3t \\ -5t \end{pmatrix}$

$\left.\begin{array}{l} X = -3t \\ Y = -5t \end{array}\right\} \Rightarrow$ DIVIDE $\frac{X}{Y} = \frac{3}{5}$

$\therefore \quad Y = \frac{5}{3}X$

THUS REQUIRED ANGLE IS $\frac{\pi}{2} + \frac{\pi}{4} - \arctan\frac{5}{3}$

$= \frac{3\pi}{4} - \arctan\frac{5}{3}$

AS REQUIRED

Question 48 (****+)

A curve has equation

$$5x^2 - 16xy + 13y^2 = 25.$$

This curve is to be mapped onto another curve C, under the transformation defined by the 2×2 matrix $\mathbf{A}$, given below.

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

Show that the equation of C is the circle with equation

$$x^2 + y^2 = 25.$$

$\boxed{}$, $\boxed{\text{proof}}$

DETERMINE THE TRANSFORMATION EQUATIONS

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-3+4} \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

HENCE WE HAVE

- $x = 3X - 2Y$
- $y = 2X - Y$

SUBSTITUTE INTO THE EQUATION $5x^2 - 16xy + 13y^2 = 25$

$$\Rightarrow 5(3X-2Y)^2 - 16(3X-2Y)(2X-Y) + 13(2X-Y)^2 = 25$$

$$\Rightarrow 5(9X^2 - 12XY + 4Y^2) - 16(6X^2 - 7XY + 2Y^2) + 13(4X^2 - 4XY + Y^2) = 25$$

$$\Rightarrow \left\{ \begin{matrix} 45X^2 - 60XY + 20Y^2 \\ -96X^2 + 112XY - 32Y^2 \\ 52X^2 - 52XY + 13Y^2 \end{matrix} \right\} = 25$$

$$\Rightarrow X^2 + Y^2 = 25$$

OR

$$x^2 + y^2 = 25$$

Question 49 (****+)

The 2×2 matrix $\mathbf{P}$ is given below.

$$\mathbf{P}=\begin{pmatrix}2 & -1\\ -3 & 2\end{pmatrix}$$

The points on the x-y plane which lie on the curve with equation

$$13x^2-16xy+5y^2+8x-6y=4,$$

are transformed by $\mathbf{P}$ onto the points which lie on another curve C.

Determine an equation for C and hence describe it geometrically.

$\boxed{}$, $\boxed{(x-1)^2+(y-2)^2=9}$

START BY OBTAINING THE INVERSE OF $\mathbf{P}$

$$\mathbf{P}^{-1}=\frac{1}{2\times2-3\times1}\begin{pmatrix}2&1\\3&2\end{pmatrix}=\begin{pmatrix}2&1\\3&2\end{pmatrix}$$

THIS CAN BE INTERPRETED BY THE TRANSFORMATION EQUATIONS

$$\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}2&-1\\-3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\iff\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2&1\\3&2\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}$$

I.E $\boxed{\begin{aligned}x&=2X+Y\\ y&=3X+2Y\end{aligned}}$

SUBSTITUTING INTO THE EQUATION WE OBTAIN

$\Rightarrow 13x^2-16xy+5y^2+8x-6y=4$

$\Rightarrow 13(2X+Y)^2-16(2X+Y)(3X+2Y)+5(3X+2Y)^2+8(2X+Y)-6(3X+2Y)=4$

$\Rightarrow 13(4X^2+4XY+Y^2)-16(6X^2+7XY+2Y^2)+5(9X^2+12XY+4Y^2)-2X-4Y=4$

$$\Rightarrow \begin{aligned}&52X^2+52XY+13Y^2\\ &-96X^2-112XY-32Y^2\\ &45X^2+60XY+20Y^2-2X-4Y=4\end{aligned}$$

$\Rightarrow X^2+Y^2-2X-4Y=4$

$\Rightarrow (X-1)^2-1+(Y-2)^2-4=4$

$\Rightarrow (X-1)^2+(Y-2)^2=9$

$\therefore$ A CIRCLE OF RADIUS 3, CENTRE AT $(1,2)$

Question 50 (*****)

A shear is defined by the 2×2 matrix

$$\mathbf{M}=\begin{pmatrix} a & b \\ c & 4 \end{pmatrix},$$

where a, b and c are scalar constants.

Under this transformation the point with coordinates $(1,2)$ is mapped onto the point with coordinates $(-8,11)$.

The shear defined by $\mathbf{M}$ has an invariant line L, which passes through the point with coordinates $(0,1)$.

Determine an equation of L.

$\boxed{}$, $\boxed{L\ :\ y=1-x}$

AS THIS IS A SHEAR THE DETERMINANT MUST BE 1

PRESERVES AREA AND NO REFLECTION

$\Rightarrow 4a-bc=1$

USING THE MAPPING $(1,2)\longmapsto(-8,7)$

$$\begin{pmatrix} a & b \\ c & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}=\begin{pmatrix} -8 \\ 11 \end{pmatrix} \Rightarrow \begin{cases} a+2b=-8 \\ c+8=11 \end{cases}$$

SOLVING TO GET

- $c=3$
- $\left.\begin{array}{l} 4a-3b=1 \\ a+2b=-8 \end{array}\right\} \quad \left.\begin{array}{l} 8a-6b=2 \\ 3a+6b=-24 \end{array}\right\} \Rightarrow 11a=-22 \Rightarrow a=-2$
- $a+2b=-8$
  $-2+2b=-8$
  $2b=-6$
  $b=-3$

$$\therefore \underline{M}=\begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix}$$

NOW WE LOOK FOR INVARIANT LINES OF THE FORM $y=mx+1$

$$\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} X=-2x-3y \\ Y=3x+4y \end{array}$$

$$\Rightarrow \begin{array}{l} X=-2x-3(mx+1) \\ Y=3x+4(mx+1) \end{array}$$

$$\Rightarrow \begin{array}{l} X=-2x-3mx-3 \\ Y=3x+4mx+4 \end{array}$$

NOW SUB INTO $Y=mX+1$

$\Rightarrow 3x+4mx+4=m(-2x-3mx-3)+1$

$\Rightarrow 3x+4mx+4=-2mx-3m^2x-3m+1$

$\Rightarrow 3m^2x+6mx+3x+3m+3=0$

$\Rightarrow 3x(m^2+2m+1)+3(m+1)=0$

$\Rightarrow 3x(m+1)^2+3(m+1)=0$

$\Rightarrow x(m+1)^2+(m+1)=0$

EQUATION IS SATISFIED FOR $m=-1$

$\therefore$ INVARIANT LINE IS $y=1-x$