## Matrix Analysis of Framed Structures

# Matrix Analysis of Framed Structures Third Edition 

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## Preface

Matrix analysis of structures is a vital subject to every structural analyst, whether working in aero-astro, civil, or mechanical engineering. It provides a comprehensive approach to the analysis of a wide variety of structural types, and therefore offers a major advantage over traditional methods which often differ for each type of structure. The matrix approach also provides an efficient means of describing various steps in the analysis and is easily programmed for digital computers. Use of matrices is natural when performing calculations with a digital computer, because matrices permit large groups of numbers to be manipulated in a simple and effective manner.

This book, now in its third edition, was written for both college students and engineers in industry. It serves as a textbook for courses at either the senior or first-year graduate level, and it also provides a permanent reference for practicing engineers. The book explains both the theory and the practical implementation of matrix methods of structural analysis. Emphasis is placed on developing a physical understanding of the theory and the ability to use computer programs for performing structural calculations.

In preparing this new edition, we have tried to maintain the strengths of the earlier editions while also adding new material to allow personal computers to be used in the solution of problems. The direct stiffness method is presented in detail because it is the best and most general approach for the analysis of structures by digital computation. The flexibility method is included as a supplementary method, partly for completeness and partly because it often is necessary to obtain stiffnesses and fixed-end actions by flexibility techniques.

Throughout the book, new examples and problems have been added to aid in teaching the subject. Two new topics, repeated substructures and the omission of axial strains in frames, are now included in Chapter 6. A new chapter, "Finite-Element Method for Framed Structures,'" has been added at the end of the text to show how the analysis of framed structures fits within the scope of the more general finite-element method. This chapter also provides an introduction to the finite-element method, using only onedimensional elements to model the slender members of framed structures.

Prerequisites for the study of matrix analysis of structures are statics, mechanics of materials, algebra, and introductory calculus. In addition, a previous course in elementary structural analysis is certainly beneficial, although not essential. Elementary matrix algebra is used throughout the
book, and the reader must be familiar with basic matrix operations, such as addition, multiplication, and inversion. Because these topics are not difficult, the reader can acquire the necessary knowledge of matrices through self-study during a period of only two or three weeks.

Computer programs for the six basic types of framed structures are given in Chapter 5 in the form of FORTRAN-oriented flow charts. These programs are available on a diskette that also contains the data for all examples shown in that chapter. The diskette may be purchased from Dr. Paul R. Johnston, Manager, Structural Analysis, Failure Analysis Associates, 149 Commonwealth Drive, Menlo Park, CA 94025 [phone (415) 688-7210]. For this purpose, a tear-out order form is included at the back of the book.

The first chapter of the book covers the fundamental concepts of structural analysis that are needed for the remaining chapters. Those who have previously studied structural theory will find that this material is mostly review. However, anyone encountering this subject for the first time will need to become thoroughly familiar with the basic topics presented here.

The flexibility and stiffness methods are introduced in Chapters 2 and 3, respectively. Each of these chapters explains the theory in detail, with examples and problems, and concludes with a section on formalizing the method (Sections 2.7 and 3.6). Although these last sections show the general mathematical approach, they are indirect and are not needed for implementing the methods. In Chapter 4 the direct stiffness method is developed further in a computer-oriented manner, as preparation for programming. Then in Chapter 5 the stiffness method is applied in FORTRAN-oriented flow charts of programs for the analysis of the six basic types of framed structures.

Because the emphasis in the first five chapters is on fundamental concepts, the treatment of many special topics is postponed to Chapter 6. Included in that chapter are such matters as symmetric structures, nonprismatic members, elastic connections, and so on. These topics can be considered as modifications of the basic procedures described in earlier chapters. As already explained, a new Chapter 7 describes how the finite-element method can be applied to framed structures.

Problems for hand solution are given at the ends of the chapters, and they are generally placed in order of increasing difficulty. New problems have been added to all problem sets, and those of Chapter 6 are completely new. The examples and problems in Chapters 1, 2, 3, and 6 (and the Appendixes) are in literal form. Numerical examples and problems of Chapters 4 and 5 are in both US and SI units.

References for further study, lists of notations, five appendixes, and answers to problems are at the end of the book. The appendixes contain tables of useful information, a description of the unit-load method for calculating displacements (Appendix A), and computer routines for solving equations (Appendix D).

The authors wish to thank the graduate students at Stanford and the teachers at other colleges and universities who contributed ideas to this book. Special appreciation is due Paul R. Johnston, who modified the programs for use with personal computers and produced new outputs for Chapter 5. Jeffrey E. Jones solved and organized the problem sets in an excellent manner, using a text editor on his own computer. Finally, Judith C. Clark did an outstanding job of word processing for parts of the revised manuscript.

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