Maximal Regularity in Weighted Spaces, Nonlinear Boundary Conditions, and Global Attractors

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Für meine Eltern

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Introduction

The subject of this thesis is the mathematical analysis of linear and quasilinear parabolic problems with inhomogeneous and nonlinear boundary conditions. We consider static boundary conditions of Dirichlet, Neumann or Robin type, and further boundary conditions of relaxation type, which include dynamic ones as well as boundary conditions that arise in the linearization of free boundary problems.

Evolution equations of this type describe a great variety of physical, chemical and biological phenomena, like reaction-diffusion processes, phase field models, chemotactic behaviour, population dynamics, phase transitions and the behaviour of two phase fluids, for instance. In many cases it is necessary to impose nonlinear boundary conditions into a reaction-diffusion model to capture the dynamics of the phenomenon under investigation. In the context of free boundary problems nonlinear boundary conditions naturally arise after a transformation to a fixed domain.

We focus on maximal regularity results in weighted L_p-spaces for linear nonautonomous parabolic problems with inhomogeneous boundary conditions. Compared to the approach without weights, we are able to reduce the necessary regularity of the initial values, to incorporate an inherent smoothing effect into the solutions and to avoid compatibility conditions at the boundary. These properties serve us as a basis for constructing a local semiflow for the corresponding quasilinear problems in a scale of phase spaces, and for the investigation of the long-time behaviour of solutions in terms of global attractors.

Our approach to quasilinear problems thus relies on linearization and a good understanding of the linear problem. This idea goes back at least to Kato [58], Sobolevskii [77] and Solonnikov [79]. In a semigroup context it was carried out by Grisvard [46], Da Prato & Grisvard [22], Amann [3, 4, 5, 6, 7], Da Prato & Lunardi [23], Lunardi [67] and Prüss [70]. Semilinear problems can be treated in the framework of analytic semigroups, see Henry's monograph [51].

Maximal regularity means that there is an isomorphism between the data and the solution of the linear problem in suitable function spaces. Having established such a sharp regularity result for the linearization, the corresponding quasilinear problem can be treated by quite simple tools, like the contraction principle and the implicit function theorem. There are approaches in spaces of continuous functions (see Angenent [12] and Clément & Simonett [19]), in Hölder spaces (see Lunardi [67]) and in L_p-spaces for $p \in (1, \infty)$ (see Clément

& Li [17] and Prüss [70]). For more details and other approaches to quasilinear parabolic problems we refer to the discussion in [10].

The three mentioned maximal regularity settings have advantages and disadvantages. The continuous setting is quite simple, but strong restrictions on the underlying spaces are necessary. In the Hölder setting the nonlinearities are easy to handle and the approach is also applicable to fully nonlinear problems, but unpleasant compatibility conditions at the initial time are necessary and a priori estimates in high norms are required to show global existence of solutions. In the L_p-setting powerful tools from vector-valued harmonic analysis are available (and needed!), but on the other hand geometric assumptions on the underlying spaces are required and also here one has to work in high norms. For a further discussion we refer again to [10]. In this thesis we entirely work in an L_p-framework.

To decide wether a concrete linear problem enjoys maximal L_p-regularity in a suitable setting is not easy. For linear problems which can be reduced to an abstract equation of the form

$$\partial_t u(t) + Au(t) = f(t), \quad t > 0, \qquad u(0) = u_0, \tag{1}$$

on a Banach space E, where A is a closed and densely defined operator on E, the operator sum method, as developed by Da Prato & Grisvard [21] and extended by Dore & Venni [31] and Kalton & Weis [57], is appropriate in many cases. Weis [85] characterized the maximal L_p-regularity properties of an operator in terms of $\mathcal{R}$-sectoriality. If E is a Hilbert space, then every negative generator of a bounded analytic C_0-semigroup enjoys maximal L_p-regularity. Unfortunately, a Hilbert space setting does often not seem to be suitable for the applications to quasilinear problems.

To treat second order parabolic differential equations with inhomogeneous or nonlinear boundary conditions in a maximal L_p-regularity approach one typically chooses $E = L_p$, $E = W_p^{-1}$ or E as an interpolation space in between as a basic underlying space. If E is close to W_p^{-1} then the boundary conditions are a priori only satisfied in a weak sense, but in this way the problem can be cast in the form (1) and operator sum methods are available, in principle. If E is close to L_p, then the boundary conditions can be understood in a pointwise sense, but a formulation in the abstract form (1) does not seem to be possible in a reasonable way, in general - there is always a 'PDE part' left to deal with. An advantage of choosing E close to L_p is that growth conditions on the nonlinearities can be avoided.

Combining operator sum methods with tools from vector-valued harmonic analysis, Denk, Hieber & Prüss [24, 25] and Denk, Prüss & Zacher [26] showed maximal L_p-regularity with L_p as an underlying space for a large class of vector-valued parabolic problems of even order with inhomogeneous boundary conditions. In [25] problems with boundary conditions of static type are considered, i.e.,

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, && t > 0, \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x), && x \in \Gamma, && t > 0, \qquad j = 1,...,m, \\ u(0,x) &= u_0(x), && x \in \Omega. \end{aligned} \tag{2}$$

This includes the linearization of reaction-diffusion systems and of phase field models with Dirichlet, Neumann and Robin conditions. In [26] the authors study problems with boundary conditions of relaxation type, i.e.,

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, && t>0, \\ \partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x), && x \in \Gamma, && t>0, \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x), && x \in \Gamma, && t>0, && j=1,...,m, \\ u(0,x) &= u_0(x), && x \in \Omega, \\ \rho(0,x) &= \rho_0(x), && x \in \Gamma, \end{aligned} \tag{3}$$

which includes dynamic boundary conditions as well as problems arising as linearizations of free boundary problems that are transformed to a fixed domain. Here $\Omega \subset \mathbb{R}^n$ is a domain with compact smooth boundary $\Gamma = \partial\Omega$. The coefficients of the operators are only assumed to be pointwise multipliers to the underlying spaces, and the top order coefficients are required to be bounded and uniformly continuous. These regularity assumptions allow to apply the linear results to quasilinear problems. Earlier investigations on (2) started at least with Ladyzhenskaya, Solonnikov & Ural'ceva [64] and include also Weidemaier [84].

A principle shortcoming of the maximal L_p-regularity approach to (1), (2) and (3) is that for fixed p one can solve the equation for initial values only in one single space of relatively high regularity, and that one does not have the flexibility to work in a scale of spaces. The L_p-approach to (1) necessarily requires that u_0 belongs to the real interpolation space $(E, D(A))_{1-1/p,p}$. For large p, which is necessary to choose in the L_p-setting to ensure that the nonlinearities are well-defined, this space is close to the domain of A. The situation for (2) and (3) is similar. Thus the long-time behaviour of solutions must be investigated in a phase space of high regularity.

For second order problems $(E, D(A))_{1-1/p,p}$ is usually close to W_p^2 for large p, but often the structure of the problems under consideration does not provide enough information for a priori estimates in such high norms. Such estimates are typically obtained in the energy space H_2^1, in L_∞ or in a Hölder space C^α with small exponent. Thus there is a gap between the regularities inherent to given problems and the regularities which are necessary to apply the nonlinear theory based on maximal L_p-regularity. Due to the lack of a scale of phase spaces it is further not clear how to show relative compactness of bounded orbits and compactness of the solution semiflow without strong a priori bounds. The latter properties are important in the investigation of the ω-limit set of solutions and in the context of global attractors.

The situation is even worse for the maximal Hölder regularity approach. Here it is required that the initial values belong to the domain of the operator under consideration. On the other hand, for semilinear problems the domains of fractional powers of operators serve as a natural scale of phase spaces. The approach to quasilinear problems in interpolation-extrapolation scales developed by Amann also does not have these shortcomings, but requires that the boundary conditions can be absorbed into the domain of an operator on a negative order base space.

To close this regularity gap between theory and applications in the maximal L_p-regularity approach one has introduced temporal weights that vanish at the initial time. In an abstract setting this was done by Clément & Simonett [19] in the context of continuous maximal regularity, and by Prüss & Simonett [71] in the L_p-setting. The latter authors proposed to work in the power weighted spaces

$$L_{p,\mu}(\mathbb{R}_+;E) = \big\{u : \mathbb{R}_+ \to E \ : \ \int_{\mathbb{R}_+} t^{p(1-\mu)} |u(t)|_E^p \,\mathrm{d}t < \infty\big\},$$

where $\mu \in (1/p, 1]$. (Note that the weights $t^{p(1-\mu)}$ belong to the class A_p, see Stein [81].) Functions with worse behaviour at $t = 0$ belong to $L_{p,\mu}$ if one lowers μ. This approach yields the solvability of the abstract equation (1) for initial values in $(E, D(A))_{\mu-1/p,p}$, and thus allows to reduce the initial regularity up to the underlying space E. For fixed p this further gives a useful scale of spaces for the initial values. Since the weights $t^{p(1-\mu)}$ only have an effect at $t = 0$ (on finite time intervals), the maximal regularity approach in the $L_{p,\mu}$-spaces also provides an inherent smoothing effect into solutions, as they regularize instantaneously from $(E_0, E_1)_{\mu-1/p,p}$ to $(E_0, E_1)_{1-1/p,p}$, which corresponds to the unweighted case $\mu = 1$. It was further shown in [71] that the property of maximal $L_{p,\mu}$-regularity for a closed and densely defined operator is independent of $\mu \in (1/p, 1]$. Hence the operator sum methods known from the unweighted case are also available in the weighted approach. The results of [71] were recently used by Köhne, Prüss & Wilke [59] to establish a dynamic theory for abstract quasilinear problems.

It is the main purpose of the present thesis to extend and combine the results of [25, 26, 59, 71] described above and to develop the maximal $L_{p,\mu}$-regularity approach for the problem classes (2) and (3). Here we aim at a systematic and comprehensive treatment of the solution theory as well as of the various prerequisites such as trace and interpolation properties of the underlying anisotropic function spaces on space-time. Besides the reduction of the initial regularity and an inherent smoothing effect of solutions, the approach allows to avoid compatibility conditions at the boundary in linear problems. We apply our linear theory to quasilinear reaction-diffusion systems with nonlinear boundary conditions, of Robin and of reactive-diffusive-convective type, respectively. For such problems we investigate local well-posedness in a scale of phase spaces, global existence and global attractors, employing the flexibility of maximal $L_{p,\mu}$-regularity.

We describe the organization of the thesis, the main results and the methods we have used. In Chapter 1 we investigate the vector-valued $L_{p,\mu}$-spaces and the corresponding anisotropic Sobolev-Slobodetskii spaces in a systematic way, and deduce all the properties required for the applications to parabolic problems. For instance, spaces of type

$$W^\kappa_{p,\mu}\big(\mathbb{R}_+; L_p(\Gamma)\big) \cap L_{p,\mu}\big(\mathbb{R}_+; W_p^{2\kappa}(\Gamma)\big),$$

where $\kappa \in (0,1)$, naturally arise in the L_p-approach to (2) and (3) as the sharp regularity class of the boundary inhomogeneities. For such spaces we establish an intrinsic norm, various embeddings via the Newton polygon, the properties of the temporal and the spatial

traces and mapping properties of pointwise multipliers. Since the multiplication with the weight is not an isomorphism to the unweighted Sobolev-Slobodetskii spaces most of the properties cannot be deduced from known results. We mainly employ interpolation techniques, operator sum methods and the representation of the spaces as domains of operators with a bounded $\mathcal{H}^\infty$-calculus or bounded imaginary powers. Our exposition also gives a comprehensive account of the unweighted case ($\mu = 1$), which has been treated in the literature so far only in a scattered way. It turns out that the weighted spaces enjoy analogous properties as the unweighted spaces, except for the intended reduced regularity of traces at $t = 0$, of course. This makes the weighted setting applicable to parabolic problems without disadvantages.
Certain aspects of weighted fractional order spaces were already investigated by Grisvard [44], Triebel [82] and Prüss & Simonett [71]. Recently Girardi & Weis [42] showed an operator-valued Fourier multiplier theorem for the $L_{p,\mu}$-spaces.

Building on the properties of the weighted spaces, in Chapters 2 and 3 we generalize to the $L_{p,\mu}$-setting the maximal regularity results by Denk, Hieber & Prüss [25] and Denk, Prüss & Zacher [26] on vector-valued linear inhomogeneous, nonautonomous initial-boundary value problems of the form (2) and (3). The unknowns take values in a Banach space of class $\mathcal{HT}$, which is necessary to apply tools from harmonic analysis, and we impose the same ellipticity and Lopatinskii-Shapiro conditions on the operators as in the unweighted case. Again the coefficients of the operators are only required to be pointwise multipliers on the underlying spaces, with continuous top order coefficients, which allows to apply the linear theory to quasilinear problems.
The Chapters 2 and 3 are organized analogously. In Sections 2.1 and 3.1 we give a detailed description of the approach and the involved function spaces, provide examples and formulate the precise assumptions, respectively. The main results are stated in the Theorems 2.1.4 and 3.1.4. Their proofs, which are inspired by the ones in [25, 26], is then carried out in the rest of the chapters. In the Sections 2.2 and 3.2 the case of full- and half-space constant coefficient model problems without lower order terms are considered. Here we employ to a large extent the properties of the weighted spaces derived in Chapter 1. Since these results enter in all points of the reasoning, we give the long and technical proof in detail. The rest of the chapters is then devoted to a perturbation and localization procedure to derive the case of a general domain from the model problems. This procedure is again quite technical, in particular because one has to take care to control the constants in the various perturbation steps. In Proposition 2.5.1 we also show that boundary operators related to (2) are surjective on suitable function spaces and have a bounded linear right-inverse. This result is needed to establish a semiflow for quasilinear problems with Robin boundary conditions in Chapter 4.

In the Chapters 4 and 5 we then apply our linear theory to quasilinear reaction-diffusion systems with nonlinear boundary conditions. Intentionally we do not use the full generality of the linear theory and rather focus from the beginning on some specific problems which also allow for an investigation of their long-time behaviour. On a bounded domain Ω with smooth boundary $\Gamma = \partial\Omega$ and outer unit normal field ν we consider in Chapter 4 systems with Robin boundary conditions, i.e, problems of the form

$$\begin{aligned} \partial_t u - \partial_i(a_{ij}(u)\partial_j u) &= f(u) && \text{in } \Omega, && t > 0, \\ a_{ij}(u)\nu_i\partial_j u &= g(u) && \text{on } \Gamma, && t > 0, \\ u(0,\cdot) &= u_0 && \text{in } \Omega. \end{aligned} \tag{4}$$

It is assumed that (a_{ij}) is elliptic and of separated divergence form, and that the nonlinearities are smooth. A dynamic theory for (4) in a scale of Slobodetskii spaces was established by Amann [6] via extrapolation techniques. Local well-posedness and invariant manifolds near equilibria for (4) based on the unweighted maximal L_p-regularity approach were obtained by Latushkin, Prüss & Schnaubelt [65, 66].

Our focus lies on the global long-time behaviour in strong norms close to W_p^2, where $p < \infty$ is arbitrarily large. We employ maximal $L_{p,\mu}$-regularity together with regularity results on the superposition operators induced by the nonlinearities to construct in Theorem 4.3.6 a compact local semiflow of solutions for (4) in the scale of nonlinear phase spaces

$$\mathcal{M}_p^s := \big\{u_0 \in W_p^s(\Omega, \mathbb{R}^N) \;:\; a_{ij}(u_0)\nu_i\partial_j u_0 = g(u_0) \quad \text{on} \;\; \Gamma\big\},$$

where $p \in (n+2, \infty)$ and $s \in (1+n/p, 2-2/p]$. This high range of regularity is not covered by Amann's theory. In Theorem 4.4.2 we then show that a global attractor of (4) in $\mathcal{M}_p^{2-2/p}$ exists if there is an absorbant set in a Hölder space $C^\alpha(\overline{\Omega}, \mathbb{R}^N)$ for some $\alpha > 0$. The result requires the full strength of the maximal $L_{p,\mu}$-regularity approach for the linearization of (4) and precise estimates for the nonlinear terms, which can be controlled in terms of lower norms of the solution (see Lemma 4.2.3). In particular we obtain from Sobolev's embeddings that the solutions converge to the attractor with respect to the $C^{1+\beta}(\overline{\Omega}, \mathbb{R}^N)$-norm, where $\beta \in (0, 1)$. In important special cases it suffices to have an absorbant set in a weaker norm such as the sup-norm. Improving earlier results, we thus have established that the long-time behaviour of also the spatial gradient of a solution is determined by the dynamics on the attractor with respect to a sup-norm. The convergence in a higher norm can be useful to improve error estimates for numerical algorithms when assuming in a quasi-stationary approximation that parts of a system of partial differential equations are on a fast time scale.

The above statements about convergence in a norm close to W_p^2 are known for semilinear problems with linear boundary conditions, but do not seem to exist for quasilinear problems or nonlinear boundary conditions. Related results rely on the variation of constants formula. The flexibility of the weights builds a bridge from lower to higher regularities, and thus maximal $L_{p,\mu}$-regularity can be seen as a substitute in the case of quasilinear problems.

In Section 4.5 we apply our results to show convergence to an attractor in higher norms for concrete models. We consider semilinear reaction-diffusion systems with nonlinear boundary conditions, as studied by Carvalho, Oliva, Pereira & Rodriguez-Bernal [15], a chemotaxis model with volume filling effect, introduced by Hillen & Painter [53], and the Shigesada-Kawasaki-Teramoto cross-diffusion model for population dynamics, introduced in [76].

In Chapter 5 we turn to systems with nonlinear dynamic boundary conditions, i.e., problems of the form

$$\begin{aligned} \partial_t u &= \partial_i(a_1(u)\partial_i u) + a_2(u)\nabla u + f(u), && \text{in } \Omega, && t > 0,\\ \partial_t u + b(\cdot,u)\partial_\nu u &= \operatorname{div}_\Gamma(c_1(\cdot,u)\nabla_\Gamma u) + c_2(\cdot,u)\nabla_\Gamma u + g(\cdot,u), && \text{on } \Gamma, && t > 0,\\ u(0,\cdot) &= u_0, && \text{in } \overline{\Omega}. \end{aligned} \tag{5}$$

Here Ω and the coefficients are as above, and ∇_Γ and $\operatorname{div}_\Gamma$ denote the surface gradient and the surface divergence on the boundary Γ, respectively. Although they look more nonlinear at a first glance, such boundary conditions are in fact less nonlinear than the ones in (4). In fact, the autonomous version of their linearization may be cast in the abstract form (1) by considering it as an evolution equation on the product space

$$\big\{(v, v_\Gamma) \in L_p(\Omega,\mathbb{R}^N) \times W_p^{1-1/p}(\Gamma,\mathbb{R}^N) \,:\, \operatorname{tr}_\Omega v = v_\Gamma\big\},$$

where tr_Ω denotes the spatial trace on Ω, and one may identify the unknown u with the pair $(u, \operatorname{tr}_\Omega u)$. Consequently one can work in linear phase spaces even for initial regularities close to W_p^2.

The system (5) models the behaviour of the quantities undergoing a reaction-diffusion-convection process in a domain and on its boundary, coupled by the normal flux term $b(\cdot,u)\partial_\nu u$. For $b \equiv 1$, in [43, Section 4] the effect of this coupling is interpreted as sending concentration waves from Γ into an infinitesimal layer near the boundary. Similar dynamical boundary conditions arise in Cahn-Hilliard or Caginalp phase field models if one takes into account the short-ranged interaction with walls [73]. They also arise in two phase flows with soluble surfactant [14]. In the literature these boundary conditions are also called generalized Wentzell boundary conditions [43]. Semilinear versions of (5) with a single equation were investigated by many researchers, for instance by Favini, J. A. Goldstein, G. R. Goldstein & Romanelli [38, 39, 40], Sprekels & Wu [80] and Vazquez & Vitillaro [83]. Results on quasilinear versions do not seem to exist. There are further results on quasilinear systems with dynamic boundary conditions of reactive type, i.e., where tangential derivatives do not occur. A dynamic theory for such problems was established by Escher [36], based on Amann's work. We refer to Constantin & Escher [20] and the references therein for more recent developments.

We first investigate the linear inhomogeneous, nonautonomous version of (5). Under appropriate ellipticity conditions on the coefficients maximal $L_{p,\mu}$-regularity is shown in Theorem 5.2.1. This extends the linear results of [38, 39, 40, 83] to more general problems and to the L_p-case. Next we construct in Theorem 5.3.3 a compact local semiflow of solutions for (5) in the linear phase space

$$\mathcal{M} := \big\{(v, v_\Gamma) \in W_p^{2-2/p}(\Omega,\mathbb{R}^N) \times W_p^{3-3/p}(\Gamma,\mathbb{R}^N) \,:\, \operatorname{tr}_\Omega v = v_\Gamma\big\},$$

using the linear theory and employing the ideas and results of [59] on abstract quasilinear evolution equations in $L_{p,\mu}$-spaces. Assuming an a priori Hölder bound, we are able to show global existence for a solution of (5) in Theorem 5.4.1. Here again maximal regularity, localization techniques in space and time and appropriate nonlinear estimates are the crucial ingredients. In Section 5.5 we specialize to a semilinear version of (5),

$$\begin{aligned} \partial_t u &= \Delta u + f(u) && \text{in } \Omega, && t > 0, \\ \partial_t u + \partial_\nu u &= \Delta_\Gamma u + g(u) && \text{on } \Gamma, && t > 0, \\ u(0,\cdot) &= u_0 && \text{in } \overline{\Omega}, \end{aligned} \tag{6}$$

and investigate the long-time behaviour of solutions in terms of global attractors. Here Δ_Γ denotes the Laplace-Beltrami operator on the boundary. Under appropriate dissipativity assumptions on the reaction terms f and g we show that the system (6) possesses a Lyapunov function, and that solutions are bounded in the energy space $W_2^1(\Omega, \mathbb{R}^N) \times W_2^1(\Gamma, \mathbb{R}^N)$. Employing a Moser-Alikakos iteration procedure, we then deduce global existence. Another a priori estimate on the equilibria of (6) yields the existence of a connected global attractor, and that each solution converges to the set of equilibria (see Theorem 5.5.8).
For a single equation it is shown in [80] that each solution of (6) converges to an equilibrium as time tends to infinity. Our dissipativity conditions differs from the one in [80], and is rather comparable with the one in [15] for Robin boundary conditions. We may allow for one component of a reaction term to have an unfavourable sign, provided the corresponding component of the other reaction term compensates this appropriately in terms of positivity of a Rayleigh quotient related to (6).

Finally, in the appendix we provide facts from interpolation theory, the theory of sectorial operators, differential operators on a boundary and about functions spaces that are used throughout the thesis. We give precise references and also prove some (rather simple) results for which a reference does not seem to exist.

Notations. We write $\mathbb{R}^n_+ := \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ for $n \in \mathbb{N}$, and $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. If it holds $a \le C\, b$ for nonnegative quantities a, b with a generic constant $C > 0$ we write $a \lesssim b$. The Lebesgue, Sobolev and Slobodetskii spaces are denoted by L_p, H_p^s and W_p^s, where $p \in [1, \infty]$ designates integrability and $s \in \mathbb{R}$ designates differentiability. For $\theta \in (0,1)$ and $p \in [1,\infty]$ we denote by $(\cdot,\cdot)_{\theta,p}$ and $[\cdot,\cdot]_\theta$ the real and the complex interpolation functor, respectively. The space of bounded linear operators between two Banach spaces E, F is denoted by $\mathcal{B}(E,F)$, where $\mathcal{B}(E) := \mathcal{B}(E,E)$. If F is densely and continuously embedded into E we write $F \overset{d}{\hookrightarrow} E$, and if E, F coincide as sets and have equivalent norms we write $E = F$. The domain, the spectrum and the resolvent set of a closed operator A on E are denoted by $D(A)$, $\sigma(A)$ and $\rho(A)$, respectively. For $p \in [1, \infty]$ and $s = [s] + s_*$ with $[s] \in \mathbb{N}_0$ and $s_* \in [0,1)$ we set $D_A(s,p) := D(A^s)$ if $s \in \mathbb{N}_0$ and $D_A(s,p) := \{x \in D(A^{[s]}) : A^{[s]}x \in (E, D(A))_{s_*,p}\}$ for $s \notin \mathbb{N}_0$.

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Chapter 1

The Spaces $L_{p,\mu}$ and Weighted Anisotropic Spaces

In this chapter we investigate the vector-valued $L_{p,\mu}$-spaces and the corresponding weighted anisotropic Sobolev-Slobodetskii spaces in a systematic way, where we restrict to spaces of class $\mathcal{HT}$ from the beginning (cf. Appendix A.3). We first consider the Sobolev-Slobodetskii spaces over the half-line and a finite interval, and derive basic properties. Of particular importance is here that the derivative with positive and negative sign admits a bounded $\mathcal{H}^\infty$-calculus on the $L_{p,\mu}$-spaces over the half-line, respectively. Next we briefly review the results from [71] and [42] on abstract maximal $L_{p,\mu}$-regularity and operator-valued Fourier-multipliers in $L_{p,\mu}$. Then we turn to anisotropic spaces, and investigate the Newton polygon, temporal and spatial traces and pointwise multipliers.
Throughout we use the facts on interpolation theory, sectorial operators and function spaces reviewed in the appendix.

1.1 Basic Properties

Let $(E, |\cdot|_E)$ be a complex Banach space of class $\mathcal{HT}$ and let $J = \mathbb{R}_+ = (0,\infty)$ or $J = (0,T)$ for some $T > 0$. Let further

$$p \in (1,\infty), \qquad \mu \in (1/p, 1].$$

For $u : J \to E$ we denote by $t^{1-\mu}u$ the function $t \mapsto t^{1-\mu}u(t)$ on J. We define

$$L_{p,\mu}(J;E) := \big\{u : J \to E \,:\, t^{1-\mu}u \in L_p(J;E)\big\},$$

which becomes a Banach space when equipped with the norm

$$|u|_{L_{p,\mu}(J;E)} := |t^{1-\mu}u|_{L_p(J;E)} = \left(\int_J t^{p(1-\mu)}|u(t)|_E^p \,\mathrm{d}t\right)^{1/p}.$$

Note that $\mu = 1$ corresponds to the unweighted case, $L_{p,1} = L_p$, and that the weight $t^{p(1-\mu)}$ only has an effect at $t = 0$ and $t = \infty$. Thus, for $T > 0$,

$$L_p(0,T;E) \hookrightarrow L_{p,\mu}(0,T;E), \qquad L_{p,\mu}(0,T;E) \hookrightarrow L_p(\tau,T;E), \quad \tau \in (0,T),$$

but $L_p(\mathbb{R}_+;E) \not\subseteq L_{p,\mu}(\mathbb{R}_+;E)$ for $\mu \in (1/p,1)$. For $k \in \mathbb{N}_0$ we define the corresponding *weighted Sobolev space*

$$W^k_{p,\mu}(J;E) = H^k_{p,\mu}(J;E) := \big\{u \in W^k_{1,\mathrm{loc}}(J;E) \ : \ u^{(j)} \in L_{p,\mu}(J;E), \ \ j \in \{0,...,k\}\big\},$$

which becomes a Banach space when equipped with the norm

$$|u|_{W^k_{p,\mu}(J;E)} = |u|_{H^k_{p,\mu}(J;E)} := \Big(\sum_{j=0}^{k} |u^{(j)}|^p_{L_{p,\mu}(J;E)}\Big)^{1/p}.$$

For $s \in \mathbb{R}_+\backslash\mathbb{N}$ with $s = [s]+s_*$, where $[s] \in \mathbb{N}_0$, $s_* \in (0,1)$, we define *weighted Slobodetskii* and *Bessel potential spaces* by real and complex interpolation, respectively, i.e.,

$$W^s_{p,\mu}(J;E) := \big(W^{[s]}_{p,\mu}(J;E), W^{[s]+1}_{p,\mu}(J;E)\big)_{s_*,p},$$

$$H^s_{p,\mu}(J;E) := \big[W^{[s]}_{p,\mu}(J;E), W^{[s]+1}_{p,\mu}(J;E)\big]_{s_*}.$$

By Proposition A.4.2 this definition is consistent with the unweighted case, i.e., we have $W^s_p = W^s_{p,1}$ and $H^s_p = H^s_{p,1}$ for all $s \geq 0$. The general properties of real and complex interpolation spaces (Appendix A.2) imply that for fixed $p \in (1,\infty)$ and $\mu \in (1/p,1]$ one has the scale

$$W^{s_1}_{p,\mu} \overset{d}{\hookrightarrow} H^{s_2}_{p,\mu} \overset{d}{\hookrightarrow} W^{s_3}_{p,\mu} \overset{d}{\hookrightarrow} H^{s_4}_{p,\mu}, \qquad s_1 > s_2 > s_3 > s_4 \geq 0. \tag{1.1.1}$$

In the sequel we will often use that

$$\mathcal{B}(W^k_{p,\mu}(J;E)) \cap \mathcal{B}(W^{k+1}_{p,\mu}(J;E)) \hookrightarrow \mathcal{B}(W^s_{p,\mu}(J;E)) \cap \mathcal{B}(H^s_{p,\mu}(J;E)),$$

where $k \in \mathbb{N}_0$ and $s \in (k,k+1)$, which means that it suffices to consider the spaces of integer order to show that on operator is continuous on the $W^s_{p,\mu}$- and $H^s_{p,\mu}$-scale .

Before continuing with definitions, we derive a first basic property of $L_{p,\mu}$.

Lemma 1.1.1. *Let $J = (0,T)$ be a finite or infinite interval, $p \in (1,\infty)$ and $\mu \in (1/p,1]$. Then*

$$L_{p,\mu}(J;E) \hookrightarrow L_{q,\mathrm{loc}}(\overline{J};E), \qquad 1 \leq q < \frac{1}{1-\mu+1/p}.$$

Consequently, for $k \in \mathbb{N}$ it holds $W^k_{p,\mu}(J;E) \hookrightarrow W^k_{1,\mathrm{loc}}(\overline{J};E)$, and for $u \in W^k_{p,\mu}(J;E)$ the trace $u^{(j)}(0) \in E$ exists for $j \in \{0,...,k-1\}$.

Proof. For a finite interval $J' = (0,T') \subset J$, a function $u \in L_{p,\mu}(J';E)$ and $1 \leq q < p$, Hölder's inequality yields

$$\int_0^{T'} |u(t)|^q_E \,\mathrm{d}t = \int_0^{T'} t^{-q(1-\mu)}(t^{1-\mu}|u(t)|_E)^q \,\mathrm{d}t \leq \Big(\int_0^{T'} t^{-\frac{(1-\mu)pq}{p-q}}\Big)^{\frac{p-q}{p}} |u|^q_{L_{p,\mu}(J';E)},$$

where the integral on the right-hand side is bounded for $\frac{(1-\mu)pq}{p-q} < 1$, i.e., $q < \frac{1}{1-\mu+1/p}$. ■

In view of Lemma 1.1.1 it makes sense to define

$${}_0W^k_{p,\mu}(J;E) = {}_0H^k_{p,\mu}(J;E) := \big\{u \in W^k_{p,\mu}(J;E) \;:\; u^{(j)}(0) = 0, \;\; j \in \{0,...,k-1\}\big\}$$

for $k \in \mathbb{N}$, and for convenience we further set

$${}_0W^0_{p,\mu}(J;E) = {}_0H^0_{p,\mu}(J;E) := L_{p,\mu}(J;E).$$

For a number $s = [s] + s_* \in \mathbb{R}_+\backslash\mathbb{N}$ as above we again define the corresponding fractional order spaces by interpolation, i.e.,

$${}_0W^s_{p,\mu}(J;E) := \big({}_0W^{[s]}_{p,\mu}(J;E), {}_0W^{[s]+1}_{p,\mu}(J;E)\big)_{s_*,p},$$

$${}_0H^s_{p,\mu}(J;E) := \big[{}_0W^{[s]}_{p,\mu}(J;E), {}_0W^{[s]+1}_{p,\mu}(J;E)\big]_{s_*}.$$

This yields as before a scale of function spaces

$${}_0W^{s_1}_{p,\mu} \overset{d}{\hookrightarrow} {}_0H^{s_2}_{p,\mu} \overset{d}{\hookrightarrow} {}_0W^{s_3}_{p,\mu} \overset{d}{\hookrightarrow} {}_0H^{s_4}_{p,\mu}, \qquad s_1 > s_2 > s_3 > s_4 \geq 0, \tag{1.1.2}$$

and we further have that

$${}_0W^s_{p,\mu}(J;E) \hookrightarrow W^s_{p,\mu}(J;E), \qquad {}_0H^s_{p,\mu}(J;E) \hookrightarrow H^s_{p,\mu}(J;E), \qquad s > 0.$$

The following fundamental *Hardy inequalities* are available for the spaces based on vanishing initial values.

Lemma 1.1.2. *Let $J = (0,T)$ be finite or infinite and $p \in (1,\infty)$. Then the following holds true.*

a) *For $\alpha \in (1/p,\infty)$ and a nonnegative function $u \in L_{1,loc}(\mathbb{R}_+;E)$ it holds*

$$\int_0^\infty \left| t^{-\alpha} \int_0^t u(\tau)\,d\tau \right|^p dt \leq \frac{1}{(\alpha - 1/p)^p} \int_0^\infty (t^{1-\alpha}|u(t)|)^p \,dt.$$

b) *For $\mu \in (1/p,1]$ and $k \in \mathbb{N}_0$ it holds*

$$\int_J t^{p(1-\mu-k)} |u(t)|^p_E \,dt \leq C_{p,\mu,k}\, |u^{(k)}|^p_{L_{p,\mu}(J;E)} \qquad \text{if } \; u \in {}_0W^k_{p,\mu}(J;E).$$

c) *For $\mu \in (1/p,1]$ and $s \geq 0$ it holds*

$$\int_J t^{p(1-\mu-s)} |u(t)|^p_E \,dt \leq C_{p,\mu,s}\, |u|^p_{{}_0W^s_{p,\mu}(J;E)} \qquad \text{if } \; u \in {}_0W^s_{p,\mu}(J;E),$$

and further

$$\int_J t^{p(1-\mu-s)} |u(t)|^p_E \,dt \leq C_{p,\mu,s}\, |u|^p_{{}_0H^s_{p,\mu}(J;E)} \qquad \text{if } \; u \in {}_0H^s_{p,\mu}(J;E).$$

Proof. The estimate in a) is shown as in [50, Theorem 330]. For b), in the sequel we identify $u \in {}_0W^k_{p,\mu}(J;E)$ and its derivatives with their trivial extensions to $\mathbb{R}_+$. Then for $j \in \{1, ..., k\}$ we have $u^{(j)} \in L_{1,\mathrm{loc}}([0,\infty);E)$, and, since $u^{(j-1)}(0) = 0$, it holds

$$|u^{(j-1)}(t)|_E \leq \int_0^t |u^{(j)}(\tau)|_E \,\mathrm{d}\tau, \qquad t \in \mathbb{R}_+.$$

For $\alpha > 1/p$ it thus follows from a) that

$$\begin{aligned}\int_J t^{-p\alpha}|u^{(j-1)}(t)|_E^p \,\mathrm{d}t &\leq \int_0^\infty \Big(t^{-\alpha} \int_0^t |u^{(j)}(\tau)|_E \,\mathrm{d}\tau \Big)^p \,\mathrm{d}t \\ &\leq \frac{1}{(\alpha - 1/p)^p} \int_J t^{-p(\alpha-1)} |u^{(j)}(t)|^p \,\mathrm{d}t.\end{aligned}$$

Applying this inequality k times, with $\alpha_j = \mu + k - j > 1/p$ for $j \in \{1, ..., k\}$, we obtain the asserted estimate in b). To prove c), we set for $s \geq 0$

$$L_p(J, t^{p(1-\mu-s)}\,\mathrm{d}t; E) := \big\{ u : J \to E \;:\; t^{1-\mu-s}u \in L_p(J;E) \big\}.$$

It then follows from b) that

$$ {}_0W^k_{p,\mu}(J;E) \hookrightarrow L_p(J, t^{p(1-\mu-k)}\,\mathrm{d}t; E), \qquad k \in \mathbb{N}_0. \tag{1.1.3}$$

In [82, Theorem 1.18.5] the identity

$$(L_p(J, t^{p(1-\mu-k)}\,\mathrm{d}t; E), L_p(J, t^{p(1-\mu-(k+1))}\,\mathrm{d}t; E))_{\theta,p} = L_p(J, t^{p(1-\mu-\theta k)}\,\mathrm{d}t; E) \tag{1.1.4}$$

is shown, where $k \in \mathbb{N}_0$ and $\theta \in (0,1)$, and (1.1.4) remains true if one replaces $(\cdot,\cdot)_{\theta,p}$ by the complex interpolation functor $[\cdot,\cdot]_\theta$. Hence c) follows from (1.1.3) by interpolation. ■

We use the Hardy inequalities to show that the multiplication with the weight is an isomorphism to the unweighted spaces, provided one restricts to vanishing initial values. The following result is shown in [71, Proposition 2.2] for $s = 0$ and $s = 1$.

Lemma 1.1.3. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \geq 0$. Then the map Φ_μ, given by*

$$(\Phi_\mu u)(t) := t^{1-\mu}u(t),$$

induces an isomorphism between ${}_0W^s_{p,\mu}(J;E)$ and ${}_0W^s_p(J;E)$, and between ${}_0H^s_{p,\mu}(J;E)$ and ${}_0H^s_p(J;E)$. The inverse Φ_μ^{-1} is given by $(\Phi_\mu^{-1}u)(t) = t^{-(1-\mu)}u(t)$.

Proof. By interpolation we only have to consider the case $s = k \in \mathbb{N}_0$.
(I) Clearly Φ_μ is an isomorphism in case $k = 0$. For $k \in \mathbb{N}$ we take $u \in {}_0W^k_{p,\mu}(J;E)$ and estimate for $j \in \{1, ..., k\}$, using Lemma 1.1.2,

$$|(t^{1-\mu}u)^{(j)}|^p_{L_p(J;E)} \lesssim \sum_{i=0}^j \int_J t^{-p(\mu+i-1)}|u^{(j-i)}(t)|_E^p \,\mathrm{d}t \lesssim |u|^p_{W^j_{p,\mu}(J;E)}.$$

Thus Φ_μ maps ${}_0W^k_{p,\mu}(J;E)$ continuously into $W^k_p(J;E)$. Since $u \in C^{k-1}(\overline{J};E)$, it holds for $j \in \{0, ..., k-1\}$

$$|(t^{1-\mu}u)^{(j)}(t)|_E \lesssim \sum_{i=0}^{j} t^{1-\mu-i}|u^{(j-i)}(t)|_E \lesssim \sum_{i=0}^{j} t^{-i}|u^{(j-i)}(t)| \to 0, \qquad t \searrow 0,$$

and this shows $(\Phi_\mu u)^{(j)}(0) = 0$ for all $j \in \{0, ..., k-1\}$.
(II) Now take $u \in {}_0W^k_p(J;E)$ and $j \in \{1, ..., k\}$. Then, again by Lemma 1.1.2,

$$|(t^{-(1-\mu)}u)^{(j)}|^p_{L_{p,\mu}(J;E)} \lesssim \sum_{i=0}^{j} \int_J t^{-pi}|u^{(j-i)}|^p_E \,\mathrm{d}t \lesssim |u^{(j)}|^p_{L_p(J;E)},$$

which yields that Φ_μ^{-1} maps ${}_0W^k_p(J;E)$ continuously into $W^k_{p,\mu}(J;E)$. Moreover, for $j \in \{0, ..., k-1\}$ it holds

$$\begin{aligned} |(\Phi_\mu^{-1}u)^{(j)}(t)|_E &\lesssim \sum_{i=0}^{j} t^{\mu-1-i}|u^{(j-i)}(t)|_E \lesssim t^{\mu-1} \sup_{\tau\in(0,t)} |u^{(j)}(\tau)|_E \\ &\le t^{\mu-1}\int_0^t |u^{(j+1)}(\tau)|_E \,\mathrm{d}\tau \lesssim t^{\mu-1/p}|u^{(j+1)}|_{L_p(J;E)}, \end{aligned}$$

which converges to zero as $t \searrow 0$. Hence $(\Phi_\mu^{-1}u)^{(j)}(0) = 0$ for all $j \in \{0, ..., k-1\}$. ■

We next show basic density results for the weighted spaces.

Lemma 1.1.4. *For a finite or infinite interval $J = (0,T)$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \ge 0$ it holds*

$$C_c^\infty(\overline{J}\backslash\{0\};E) \overset{d}{\hookrightarrow} {}_0W^s_{p,\mu}(J;E), {}_0H^s_{p,\mu}(J;E), \qquad C_c^\infty(\overline{J};E) \overset{d}{\hookrightarrow} W^s_{p,\mu}(J;E), H^s_{p,\mu}(J;E).$$

Proof. By the general density results for interpolation spaces (Appendix A.2), we only have to consider the case $s = k \in \mathbb{N}_0$. Throughout, let $\varepsilon > 0$ be given.
(I) For $u \in {}_0W^k_{p,\mu}(J;E)$ it holds $\Phi_\mu u \in {}_0W^k_p(J;E)$ by the preceding lemma. As in [82, Theorems 2.9.1, 4.7.1] for the scalar-valued case, one sees that $C_c^\infty(\overline{J}\backslash\{0\};E)$ is dense in ${}_0W^k_p(J;E)$ for $k \in \mathbb{N}_0$. Thus there is $\psi \in C_c^\infty(\overline{J}\backslash\{0\};E)$ with $|\Phi_\mu u - \psi|_{W^k_p(J;E)} < \varepsilon$. Therefore

$$|u - \Phi_\mu^{-1}\psi|_{W^k_{p,\mu}(J;E)} \lesssim |\Phi_\mu u - \psi|_{W^k_p(J;E)} \lesssim \varepsilon,$$

which yields $C_c^\infty(\overline{J}\backslash\{0\};E) \overset{d}{\hookrightarrow} {}_0W^k_{p,\mu}(J;E)$.
(II) To show the second asserted density, take $u \in W^k_{p,\mu}(J;E)$ and choose $\psi_1 \in C_c^\infty(\overline{J};E)$ with $\psi_1^{(j)}(0) = u^{(j)}(0)$ for $j \in \{0, ..., k-1\}$.[1] By Step I, due to $u - \psi_1 \in {}_0W^k_{p,\mu}(J;E)$, there is $\psi_2 \in C_c^\infty(\overline{J};E)$ which is ε-close to $u - \psi_1$. Hence $\psi_1 + \psi_2$ is ε-close to u. ■

For a finite interval $J = (0,T)$, a linear map $\mathcal{E} : L_{1,\mathrm{loc}}(J;E) \to L_{1,\mathrm{loc}}(\mathbb{R}_+;E)$ is called *extension operator* from J to $\mathbb{R}_+$ if

$$(\mathcal{E}u)|_J = u, \qquad u \in L_{1,\mathrm{loc}}(J;E),$$

[1]For instance, one may take $\psi_1(t) = \varphi(t)\sum_{j=0}^k \frac{1}{j!}u^{(j)}(0)t^j$, where $\varphi \in C_c^\infty([0,\infty)$ with $\varphi \equiv 1$ on $[0,1]$ and $\varphi \equiv 0$ on $[2,\infty)$.

i.e., if it is a right-inverse to the restriction of functions on $\mathbb{R}_+$ to J.
We construct extension operators for the weighted spaces. In the sequel they are frequently employed to deduce properties of the weighted spaces on a finite interval from the half-line case. For $s \in [0, 2]$ we construct one for ${}_0W^s_{p,\mu}(J; E)$ and ${}_0H^s_{p,\mu}(J; E)$ whose norm is independent of the length of J. We do not consider such an extension for $s > 2$, since this case is not needed below for our later applications and the construction would be rather cumbersome.

Lemma 1.1.5. *Let $J = (0, T)$ be a finite interval, $p \in (1, \infty)$, and $\mu \in (1/p, 1]$. Then the following holds true.*

a) *Given $k \in \mathbb{N}$, there is an extension operator $\mathcal{E}_J$ from J to $\mathbb{R}_+$ with*

$$\mathcal{E}_J \in \mathcal{B}(W^s_{p,\mu}(J; E), W^s_{p,\mu}(\mathbb{R}_+; E)) \cap \mathcal{B}(H^s_{p,\mu}(J; E), H^s_{p,\mu}(\mathbb{R}_+; E)), \qquad s \in [0, k].$$

b) *There is an extension operator $\mathcal{E}^0_J$ from J to $\mathbb{R}_+$ with*

$$\mathcal{E}^0_J \in \mathcal{B}({}_0W^s_{p,\mu}(J; E), {}_0W^s_{p,\mu}(\mathbb{R}_+; E)) \cap \mathcal{B}({}_0H^s_{p,\mu}(J; E), {}_0H^s_{p,\mu}(\mathbb{R}_+; E)), \qquad s \in [0, 2],$$

whose operator norm is independent of T.

c) *For the above operators it holds $\mathcal{E}_J, \mathcal{E}^0_J \in \mathcal{B}(L_\infty(J; E), L_\infty(\mathbb{R}_+; E))$, with operator norms independent of T.*

Proof. (I) For $\mathcal{E}_J$, let $k \in \mathbb{N}$ be given. By [1, Theorem 5.19] there is an extension operator $\mathcal{E}$ from $\mathbb{R}_+$ to $\mathbb{R}$ that is continuous from $W^j_p(\mathbb{R}_+; E)$ to $W^j_p(\mathbb{R}; E)$ for all $j \in \{0, ..., k\}$, that satisfies

$$(\mathcal{E}v)^{(j)} = \mathcal{E}_j v^{(j)}, \qquad j \in \{0, ..., k\}, \tag{1.1.5}$$

where $\mathcal{E}_j$ is an extension operator that is continuous from $W^i_p(\mathbb{R}_+; E)$ to $W^i_p(\mathbb{R}; E)$ for all $i \in \{0, ..., k - j\}$. Further $\mathcal{E}$ and $\mathcal{E}_j$ have the property that for $\widetilde{T} > 0$ the function $\mathcal{E}_{(j)} v|_{(-\widetilde{T}/(k+1), 0)}$ is constructed using only $v|_{(0,\widetilde{T})}$. We may thus define

$$(\widetilde{\mathcal{E}}u)(t) := \mathcal{E}\big(u(-\cdot + T)\big)(-t + T), \qquad t \in \big(0, T + T/2(k + 1)\big), \qquad u \in L_{1,\mathrm{loc}}(0, T; E).$$

Then $\widetilde{\mathcal{E}}$ is an extension operator from $(0, T)$ to $\big(0, (1 + \frac{1}{2(k+1)})T\big)$. Due to (1.1.5), and since the weight only has an effect at $t = 0$, for all $j \in \{0, ..., k\}$ it is continuous from $W^j_{p,\mu}(0, T; E)$ to $W^j_{p,\mu}\big(0, (1 + \frac{1}{4(k+1)})T; E\big)$. Choosing a cut-off function $\varphi \in C^\infty_c(\mathbb{R}_+)$ that is equal to 1 on $\big(0, (1 + \frac{1}{4(k+1)})T\big)$ and vanishes on $\big((1 + \frac{1}{3(k+1)})T, \infty\big)$, we define

$$\mathcal{E}_J := \varphi\widetilde{\mathcal{E}}. \tag{1.1.6}$$

Then it holds $\mathcal{E}_J \in \mathcal{B}(W^j_{p,\mu}(J; E), W^j_{p,\mu}(\mathbb{R}_+; E))$ for $j \in \{0, ..., k\}$, which carries over to the fractional order spaces by interpolation. It follows from the representation of $\mathcal{E}$ in [1] that $\mathcal{E}_J$ admits an L_∞-estimate independent of T.
(II) To show b), for $u \in {}_0W^k_{p,\mu}(J; E)$ we define

$$(\mathcal{E}^0_J u)(t) := \begin{cases} u(t), & t \in (0, T), \\ 3(\psi^{1-\mu}u)(2T - t)\mathbb{1}_{[T,2T]}(t) - 2(\psi^{1-\mu}u)(3T - 2t)\mathbb{1}_{[T,\frac{3}{2}T]}(t), & t \geq T, \end{cases}$$

where $\psi(\tau) = \frac{2T\tau-\tau^2}{T^2}$. As above, by interpolation we only have to show that $\mathcal{E}_J^0 \in \mathcal{B}({}_0W^k_{p,\mu}(J;E), {}_0W^k_{p,\mu}(\mathbb{R}_+;E))$ for $k = 0,1,2$. For $k \in \{1,2\}$ we see that the function $\mathcal{E}_J^0 u$ is continuous on $[0,\infty)$. Further, in these cases it holds

$$(\mathcal{E}_J^0 u)'(t) = -3(\psi^{1-\mu}u)'(2T-t)\mathbb{1}_{[T,2T]}(t) + 4(\psi^{1-\mu}u)'(3T-2t)\mathbb{1}_{[T,\frac{3}{2}T]}(t), \qquad t \geq T,$$

and for $\tau \in J$ we have

$$(\psi^{1-\mu}u)'(\tau) = 2(1-\mu)\frac{T^{2(\mu-1)}(T-\tau)}{(2T-\tau)^{\mu}}\tau^{-\mu}u(\tau) + T^{2(\mu-1)}(2T-\tau)^{1-\mu}\tau^{1-\mu}u'(\tau). \quad (1.1.7)$$

For $k=2$ we thus obtain that $\lim_{t\searrow T}(\mathcal{E}_J^0 u)'(t) = u'(T)$, and we infer from $u'(0)=0$ and

$$|u(\tau)|_E\,\tau^{-\mu} \lesssim |u(\tau)|_E\,\tau^{-1} \to |u'(0)|_E = 0, \qquad \tau \searrow 0,$$

that $(\mathcal{E}_J^0 u)'$ is continuous at $t = \frac{3}{2}T$ and $t = 2T$. Therefore $(\mathcal{E}_J^0 u)'$ is continuous on $[0,\infty)$. Moreover, in this case it holds for $\tau \in J$ that

$$\begin{aligned}(\psi^{1-\mu}\,u)''(\tau) &= 4(1-\mu)\Big(-\mu T^{2(\mu-1)}\frac{(T-\tau)^2}{(2T-\tau)^{1+\mu}}\,\tau^{-\mu-1} - \frac{T^{2(\mu-1)}}{(2T-\tau)^{\mu}}\,\tau^{-\mu}\Big)u(\tau)\\ &\quad + 4(1-\mu)T^{2(\mu-1)}\frac{T-\tau}{(2T-\tau)^{\mu}}\tau^{-\mu}u'(\tau) + T^{2(\mu-1)}(2T-\tau)^{1-\mu}\,\tau^{1-\mu}u''(\tau). \quad (1.1.8)\end{aligned}$$

(III) We estimate $\mathcal{E}_J^0 u$ and its derivatives in the weighted norms. Using for $j=1,2$ the substitutions $\tau = (j+1)T - jt$ (i.e., $t = \frac{(j+1)T-\tau}{j}$), we have

$$\begin{aligned}|\mathcal{E}_J^0 u|^p_{L_{p,\mu}(\mathbb{R}_+;E)} &\lesssim |u|^p_{L_{p,\mu}(J;E)} + \sum_{j=1}^{2}\int_T^{\frac{j+1}{j}T} t^{p(1-\mu)}|(\psi^{1-\mu}u)((j+1)T-jt)|_E^p\,\mathrm{d}t\\ &\lesssim |u|^p_{L_{p,\mu}(J;E)} + \sum_{j=1}^{2}\int_0^T\Big(\frac{((j+1)T-\tau)(2T-\tau)}{T^2}\Big)^{p(1-\mu)}\tau^{p(1-\mu)}|u(\tau)|_E^p\,\mathrm{d}\tau\\ &\lesssim |u|^p_{L_{p,\mu}(J;E)},\end{aligned}$$

which yields $\mathcal{E}_J^0 \in \mathcal{B}(L_{p,\mu}(J;E), L_{p,\mu}(\mathbb{R}_+;E))$, with operator norm independent of T. Similarly, for $u \in {}_0W^1_{p,\mu}(J;E)$ we obtain, using (1.1.7) and Hardy's inequality (Lemma 1.1.2),

$$\begin{aligned}|(\mathcal{E}_J^0 u)'|^p_{L_{p,\mu}(\mathbb{R}_+;E)} &\lesssim |u'|^p_{L_{p,\mu}(J;E)} + \sum_{j=1}^{2}\int_0^T\frac{(j+1-\tau/T)^{p(1-\mu)}}{T^{p(1-\mu)}}\Big(\frac{(T-\tau)^p}{(2T-\tau)^{p\mu}}\,\tau^{-p\mu}|u(\tau)|_E^p\\ &\qquad + (2T-\tau)^{p(1-\mu)}\,\tau^{p(1-\mu)}|u'(\tau)|_E^p\Big)\,\mathrm{d}t\\ &\lesssim |u'|^p_{L_{p,\mu}(J;E)}.\end{aligned}$$

Moreover, for $u \in {}_0W^2_{p,\mu}(J;E)$ we use (1.1.8) and Hardy's inequality to estimate

$$\begin{aligned}|(\mathcal{E}_J^0 u)''|^p_{L_{p,\mu}(\mathbb{R}_+;E)} &\lesssim |u''|^p_{L_{p,\mu}(J;E)} + \sum_{j=1}^{2}\int_0^T((j+1)T-\tau)^{p(1-\mu)}T^{2p(\mu-1)}\\ &\qquad\cdot\Big[\Big(\frac{(T-\tau)^{2p}}{(2T-\tau)^{p(1+\mu)}}\,\tau^{-p(\mu+1)} + \frac{T^p}{(2T-\tau)^{p\mu}}\,\tau^{-p(\mu+1)}\Big)|u(\tau)|_E^p\\ &\qquad + \frac{(T-\tau)^p}{(2T-\tau)^{p\mu}}\tau^{-p\mu}|u'(\tau)|_E^p + (2T-\tau)^{p(1-\mu)}\,\tau^{p(1-\mu)}|u''(\tau)|^p\Big]\,\mathrm{d}\tau\\ &\lesssim |u''|^p_{L_{p,\mu}(J;E)},\end{aligned}$$

where the constants in these estimates are independent of T. This shows that $\mathcal{E}_J^0 \in \mathcal{B}({}_0W^k_{p,\mu}(J;E), {}_0W^k_{p,\mu}(\mathbb{R}_+;E))$ for $k = 1, 2$, with operator norm independent of T, respectively. Finally, it follows again from its representation that $\mathcal{E}_J^0$ admits an L_∞-estimate independent of T. ■

We now investigate the realization of the derivative ∂_t and its fractional powers on the weighted spaces. The properties of this operator and its variants are fundamental for all our further considerations. We first show that ∂_t generates the family of left translations.

Lemma 1.1.6. *For $p \in (1,\infty)$, $\mu \in (1/p, 1]$ and $s \geq 0$, the family of left translations $\{\Lambda_t^E\}_{t\geq 0}$, given by*

$$(\Lambda_t^E u)(\tau) := u(\tau + t), \qquad \tau \geq 0,$$

is well-defined and forms a strongly continuous semigroup of contractions on the spaces $W^s_{p,\mu}(\mathbb{R}_+;E)$ and $H^s_{p,\mu}(\mathbb{R}_+;E)$, respectively. Its generator is the derivative ∂_t, with domain $W^{s+1}_{p,\mu}(\mathbb{R}_+;E)$ and $H^{s+1}_{p,\mu}(\mathbb{R}_+;E)$, respectively.

Proof. (I) We write $\Lambda_t = \Lambda_t^E$ for simplicity. For each $t_0 \geq 0$ the operator Λ_{t_0} maps $L_{p,\mu}(\mathbb{R}_+;E)$ into itself and is contractive, due to

$$\begin{aligned}|\Lambda_{t_0} u|^p_{L_{p,\mu}(\mathbb{R}_+;E)} &= \int_0^\infty \tau^{p(1-\mu)} |u(\tau + t_0)|_E^p \,\mathrm{d}\tau \\ &\leq \int_0^\infty (\tau + t_0)^{p(1-\mu)} |u(\tau + t_0)|_E^p \,\mathrm{d}\tau \leq |u|^p_{L_{p,\mu}(\mathbb{R}_+;E)}.\end{aligned}$$

This estimate also shows that Λ_{t_0} maps $W^k_{p,\mu}(\mathbb{R}_+;E)$, $k \in \mathbb{N}$, into itself and is contractive. By interpolation, this carries over to $W^s_{p,\mu}(\mathbb{R}_+;E)$ and $H^s_{p,\mu}(\mathbb{R}_+;E)$, for all $s \geq 0$. It is further clear that $\{\Lambda_t\}_{t\geq 0}$ forms a semigroup of operators on these spaces. Due to Lemma 1.1.4, the set $C_c^\infty([0,\infty);E)$ is dense in all of the spaces above, and the left translations act strongly continuous on this set. By [35, Proposition I.5.3], this yields that the left translations are strongly continuous on $W^s_{p,\mu}(\mathbb{R}_+;E)$ and $H^s_{p,\mu}(\mathbb{R}_+;E)$, respectively.
(II) Now let $k \in \mathbb{N}_0$. Denoting the generator of $\{\Lambda_t\}$ on $W^k_{p,\mu}(\mathbb{R}_+;E)$ by A, we have to show that $\partial_t = A$. To see $A \subseteq \partial_t$, we take $u \in D(A)$. Then $u^{(k)} \in L_{1,\mathrm{loc}}(\mathbb{R}_+;E)$, and for $a, b \in \mathbb{R}_+$ with $a < b$ it holds

$$\int_a^b \frac{1}{h}\big(u^{(k)}(\tau + h) - u^{(k)}(\tau)\big)\,\mathrm{d}\tau = \frac{1}{h}\int_b^{b+h} u^{(k)}(\tau)\,\mathrm{d}\tau - \frac{1}{h}\int_a^{a+h} u^{(k)}(\tau)\,\mathrm{d}\tau.$$

As $h \to 0$, the left-hand side converges to $u^{(k)}(b) - u^{(k)}(a)$ for almost all $a, b \in \mathbb{R}_+$. The integrand on the right converges to $Au^{(k)}$ in $L_{p,\mu}(\mathbb{R}_+;E)$, and thus in $L_1(a,b;E)$. Hence, the right-hand side converges to $\int_a^b Au^{(k)}(\tau)\,\mathrm{d}\tau$. This shows $u \in W^{k+1}_{1,\mathrm{loc}}(\mathbb{R}_+;E)$, with $u^{(k+1)} = Au^{(k)}$. Thus $D(A) \subset W^{k+1}_{p,\mu}(\mathbb{R}_+;E)$ and $\partial_t|_{D(A)} = A$.
The reverse inclusion now follows from abstract arguments. Since A generates a strongly continuous semigroup of contractions, it follows from the Hille-Yosida theorem, [35, Theorem II.3.5], that $1 - A$ is invertible. It is further easy to see that $1 - \partial_t$ is injective on $W^{k+1}_{p,\mu}(\mathbb{R}_+;E)$. From [35, IV.1.21(5)] we thus obtain that $1 - \partial_t = 1 - A$, which yields $\partial_t = A$.

(III) For $s \geq 0$ we concentrate on the W-case, the H-case requires literally the same arguments. By [35, Proposition II.2.3] we have that the generator of the left translations on $W^s_{p,\mu}(\mathbb{R}_+;E)$ is the derivative ∂_t as well, with domain

$$D(\partial_t) = \{u \in W^s_{p,\mu}(\mathbb{R}_+;E) \,:\, \partial_t u \in W^s_{p,\mu}(\mathbb{R}_+;E)\}.$$

It follows from interpolation that $W^{s+1}_{p,\mu}(\mathbb{R}_+;E) \subset D(\partial_t)$. For the converse inclusion, if $u, \partial_t u \in W^s_{p,\mu}(\mathbb{R}_+;E)$ then $(1-\partial_t)u \in W^s_{p,\mu}(\mathbb{R}_+;E)$. Since 1 is contained in the resolvent set of ∂_t, Step II and interpolation yield $u \in W^{s+1}_{p,\mu}(\mathbb{R}_+;E)$. Therefore ∂_t with $D(\partial_t) = W^{s+1}_{p,\mu}(\mathbb{R}_+;E)$ is the generator of the left translations also in the fractional order case. ■

Using a transference principle, we show that the negative generator of the left translations, $-\partial_t$, admits a bounded $\mathcal{H}^\infty$-calculus on $L_{p,\mu}(\mathbb{R}_+;E)$. As shown in [71], the realization of ∂_t with domain ${}_0W^1_{p,\mu}(\mathbb{R}_+;E)$ also admits a bounded $\mathcal{H}^\infty$-calculus, although $-\partial_t$ does not generate a semigroup on $L_{p,\mu}(\mathbb{R}_+;E)$. For a definition and properties of the $\mathcal{H}^\infty$-calculus of a sectorial operator we refer to Appendix A.3.

Theorem 1.1.7. [2] *Let $p \in (1,\infty)$ and $\mu \in (1/p, 1]$. Then on $L_{p,\mu}(\mathbb{R}_+;E)$ the operators*

$$\partial_t, \qquad \textit{with domain} \quad {}_0W^1_{p,\mu}(\mathbb{R}_+;E),$$

and

$$-\partial_t, \qquad \textit{with domain} \quad W^1_{p,\mu}(\mathbb{R}_+;E),$$

admit a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle $\pi/2$, respectively. In particular, both operators are sectorial of angle $\pi/2$.

Proof. (I) The assertion on ∂_t is proved in [71, Theorem 4.5].
(II) For the operator $-\partial_t$ we employ the vector-valued transference principle, which is due to Hieber and Prüss [52]. To this end we introduce vector-valued extensions of operators. Let (Ω,ν) be a measure space, and let S be a bounded, positive operator on $L_p(\Omega,\nu)$.[3] Let further $u_i \in L_p(\Omega,\nu)$ be stepfunctions and $x_i \in E$, $i = 1,...,N$, where $N \in \mathbb{N}$. For simple functions u of the form

$$u = \sum_{i=1}^N u_i x_i, \tag{1.1.9}$$

the vector-valued extension of S, denoted by S^E, is defined as

$$S^E u(\cdot) := \sum_{i=1}^N (Su_i)(\cdot)x_i.$$

Due to [62, Lemma 10.14], the operator S^E extends uniquely to the vector-valued space $L_p(\Omega,\nu;E)$, such that $\|S^E\|_{\mathcal{B}(L_p(\Omega,\nu;E))} = \|S\|_{\mathcal{B}(L_p(\Omega,\nu))}$.
(III) We consider on $L_{p,\mu}(\mathbb{R}_+;E) = L_p(\mathbb{R}_+, t^{p(1-\mu)}\,\mathrm{d}t;E)$ the left translation Λ^E_t, $t \geq 0$. Obviously $\Lambda^{\mathbb{R}}_t$ is a positive operator, and for a simple function u of type (1.1.9) it holds

$$\Lambda^E_t u = (\Lambda^{\mathbb{R}}_t)^E u.$$

[2] Here it is for the first time essential that E is of class $\mathcal{HT}$.
[3] The operator S is called positive if it leaves the positive cone $\{u \in L_p(\Omega,\nu) :\ u \geq 0\ \nu\text{-a.e.}\}$ invariant.

From the density of the simple functions in $L_p(\mathbb{R}_+, t^{p(1-\mu)}\,\mathrm{d}t; E)$ it follows that Λ_t^E is the vector-valued extension of $\Lambda_t^{\mathbb{R}}$, i.e.

$$\Lambda_t^E = (\Lambda_t^{\mathbb{R}})^E.$$

Due to Lemma 1.1.6, the family $\{\Lambda_t^{\mathbb{R}}\}_{t\geq 0}$ forms a strongly continuous semigroup of positive contractions on $L_p(\mathbb{R}_+, t^{p(1-\mu)}\,\mathrm{d}t)$, and ∂_t is the generator of its vector-valued extension $\{\Lambda_t^E\}_{t\geq 0}$ to $L_p(\mathbb{R}_+, t^{p(1-\mu)}; E)$. Moreover, ∂_t is injective on $L_p(\mathbb{R}_+, t^{p(1-\mu)}; E)$. Now [52, Theorem 6] yields that $-\partial_t$ admits a bounded $\mathcal{H}^\infty$-calculus with angle equal to $\pi/2$. ■

The invertibility of $1-\partial_t$ and $1+\partial_t$ yields a useful characterization of the weighted spaces.

Lemma 1.1.8. *For a finite or infinite interval $J = (0,T)$ and $s = [s] + s_*$ with $[s] \in \mathbb{N}_0$, $s_* \in [0,1)$, it holds*

$$W^s_{p,\mu}(J;E) = \{u \in W^{[s]}_{p,\mu}(J;E) \,:\, u^{([s])} \in W^{s_*}_{p,\mu}(J;E)\}, \tag{1.1.10}$$

$$_0W^s_{p,\mu}(J;E) = \{u \in {}_0W^{[s]}_{p,\mu}(J;E) \,:\, u^{([s])} \in {}_0W^{s_*}_{p,\mu}(J;E)\}, \tag{1.1.11}$$

where the spaces on the right-hand side are equipped with their canonical norms.[4] *The norm equivalence constants in (1.1.11) does not depend on the length of J. All these assertions remain true if one replaces the W-spaces by the H-spaces.*

Proof. We only consider the case of W-spaces.
(I) It follows from interpolation that

$$\partial_t^{[s]} \in \mathcal{B}\big(W^s_{p,\mu}(J;E), W^{s_*}_{p,\mu}(J;E)\big) \cap \mathcal{B}\big({}_0W^s_{p,\mu}(J;E), {}_0W^{s_*}_{p,\mu}(J;E)\big),$$

which shows the embeddings from the left to the right in (1.1.10) and (1.1.11), with embedding constants independent of the length of J.
(II) For the converse embedding we first consider the case $J = \mathbb{R}_+$. Since $-\partial_t$ is sectorial we have that $1-\partial_t$ is invertible. Further, interpolation yields that the operator $(1-\partial_t)^{[s]}$ is an isomorphism $W^s_{p,\mu}(\mathbb{R}_+;E) \to W^{s_*}_{p,\mu}(\mathbb{R}_+;E)$. We may therefore estimate

$$|u|_{W^s_{p,\mu}(\mathbb{R}_+;E)} \lesssim |(1-\partial_t)^{[s]}u|_{W^{s_*}_{p,\mu}(\mathbb{R}_+;E)} \lesssim |u|_{W^{[s]}_{p,\mu}(\mathbb{R}_+;E)} + |u^{([s])}|_{W^{s_*}_{p,\mu}(\mathbb{R}_+;E)},$$

and thus obtain (1.1.10). Replacing $1-\partial_t$ by $1+\partial_t$, which is invertible since ∂_t is sectorial, we obtain (1.1.11) in the same way.
(III) Now suppose that J is a finite interval. Using the extension operator $\mathcal{E}_J$ from Lemma 1.1.5 and (1.1.10) for the half-line, we obtain

$$|u|_{W^s_{p,\mu}(J;E)} \lesssim |\mathcal{E}_J u|_{W^{[s]}_{p,\mu}(\mathbb{R}_+;E)} + |(\mathcal{E}_J u)^{([s])}|_{W^{s_*}_{p,\mu}(\mathbb{R}_+;E)} \lesssim |u|_{W^{[s]}_{p,\mu}(J;E)} + |u^{([s])}|_{W^{s_*}_{p,\mu}(J;E)},$$

which shows (1.1.10). Here, the latter inequality follows from the representation (1.1.6) of $\mathcal{E}_J$. For (1.1.11), note that the operator $1+\partial_t$ is also for a finite interval an isomorphism ${}_0W^1_{p,\mu}(J;E) \to L_{p,\mu}(J;E)$. In fact, it is obviously injective. Moreover, the solution of $u' = -u + f$ with $u(0) = 0$ is for $f \in L_{p,\mu}(\mathbb{R}_+;E)$ given by $u = \widetilde{u}|_J$, where $\widetilde{u} \in {}_0W^1_{p,\mu}(\mathbb{R}_+;E)$

[4] For instance, in (1.1.10) the canonical norm on the right-hand side is $\big(|u|^p_{W^s_{p,\mu}(J;E)} + |u^{([s])}|^p_{W^{s_*}_{p,\mu}(J;E)}\big)^{1/p}$.

satisfies $\widetilde{u}' = -\widetilde{u} + \widetilde{f}$, and $\widetilde{f}$ denotes the trivial extension of f to $\mathbb{R}_+$. This shows surjectivity, and further that the operator norm of $(1+\partial_t)^{-1}$ does not depend on the length of J. Now the same arguments as in Step II show (1.1.11) for finite J. ■

We next show general interpolation properties of the weighted spaces.

Lemma 1.1.9. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, $0 \le s_1 < s_2$ and $\theta \in (0,1)$. Then for $s = (1-\theta)s_1 + \theta s_2$ it holds*

$$[H^{s_1}_{p,\mu}(J;E), H^{s_2}_{p,\mu}(J;E)]_\theta = H^s_{p,\mu}(J;E),$$

and if $s \notin \mathbb{N}$ then

$$(H^{s_1}_{p,\mu}(J;E), H^{s_2}_{p,\mu}(J;E))_{\theta,p} = W^s_{p,\mu}(J;E). \tag{1.1.12}$$

Moreover, for $s_1, s_2, s \notin \mathbb{N}_0$ it holds

$$[W^{s_1}_{p,\mu}(J;E), W^{s_2}_{p,\mu}(J;E)]_\theta = W^s_{p,\mu}(J;E),$$

$$(W^{s_1}_{p,\mu}(J;E), W^{s_2}_{p,\mu}(J;E))_{\theta,p} = W^s_{p,\mu}(J;E).$$

If $F \overset{d}{\hookrightarrow} E$ is a further Banach space of class $\mathcal{HT}$, then it holds for $\tau \ge 0$ and $\theta \in (0,1)$

$$(H^\tau_{p,\mu}(\mathbb{R}_+;E), H^\tau_{p,\mu}(\mathbb{R}_+;F))_{\theta,p} \doteq H^\tau_{p,\mu}(\mathbb{R}_+;(E,F)_{\theta,p}),$$

$$[H^\tau_{p,\mu}(\mathbb{R}_+;E), H^\tau_{p,\mu}(\mathbb{R}_+;F)]_\theta \doteq H^\tau_{p,\mu}(\mathbb{R}_+;[E,F]_\theta).$$

All these assertions remain true if one replaces the W- and H-spaces by ${}_0W$- and ${}_0H$-spaces, respectively. Restricting to $s_2 \le 2$ in this case, the norm equivalence constants are independent of the underlying interval J.

Proof. Throughout this proof we set

$$A := 1 - \partial_t, \qquad X := L_{p,\mu}(\mathbb{R}_+;E).$$

(I) We first treat the case $J = \mathbb{R}_+$. Considering A as an operator on X, Theorem 1.1.7, (A.3.1) and (A.3.2) yield that for $\alpha \in (0,1)$ it holds $D(A^\alpha) = H^\alpha_{p,\mu}(\mathbb{R}_+;E)$. Using this, together with the fact that $D(A^k) = H^k_{p,\mu}(\mathbb{R}_+;E)$ for $k \in \mathbb{N}_0$ and Lemma 1.1.8, for $\alpha \ge 1$ we also obtain that

$$\begin{aligned} D(A^\alpha) &= \{u \in D(A^{[\alpha]}) \,:\, A^{[\alpha]}u \in D(A^{\alpha-[\alpha]})\} \\ &= \{u \in H^{[\alpha]}_{p,\mu}(\mathbb{R}_+;E) \,:\, u^{[\alpha]} \in H^{\alpha-[\alpha]}_{p,\mu}(\mathbb{R}_+;E)\} = H^\alpha_{p,\mu}(\mathbb{R}_+;E). \end{aligned}$$

It therefore follows from (A.3.1) that

$$[H^{s_1}_{p,\mu}(\mathbb{R}_+;E), H^{s_2}_{p,\mu}(\mathbb{R}_+;E)]_\theta = [D(A^{s_1}), D(A^{s_2})]_\theta = D(A^s) = H^s_{p,\mu}(\mathbb{R}_+;E),$$

which shows the first asserted equality.

(II) We next show (1.1.12). The operator A^{s_1} induces an isomorphism

$$(H^{s_1}_{p,\mu}(J;E), H^{s_2}_{p,\mu}(J;E))_{\theta,p} = (D(A^{s_1}), D(A^{s_2}))_{\theta,p} \to (X, D(A^\tau))_{\theta,p},$$

where $\tau = s_2 - s_1$. It follows from reiteration that

$$(X, D(A^\tau))_{\theta,p} = (D(A^{[\tau]}), D(A^\tau))_{\sigma,p},$$

with $\sigma = \frac{\tau\theta - [\tau]}{\tau - [\tau]} \in (0,1)$. Further, the operator $A^{[\tau]}$ induces an isomorphism

$$(D(A^{[\tau]}), D(A^\tau))_{\sigma,p} \to (X, D(A^{\tau-[\tau]}))_{\sigma,p} = (X, D(A))_{\sigma(\tau-[\tau]),p} = W^{\sigma(\tau-[\tau])}_{p,\mu}(\mathbb{R}_+; E),$$

where $\sigma(\tau - [\tau]) = \tau\theta - [\tau] \in (0,1)$. Now the operator $A^{-(s_1+[\tau])}$ induces an isomorphism

$$W^{\tau\theta-[\tau]}_{p,\mu}(\mathbb{R}_+; E) \to W^{(s_2-s_1)\theta+s_1}_{p,\mu}(\mathbb{R}_+; E) = W^s_{p,\mu}(\mathbb{R}_+; E),$$

provided $s \notin \mathbb{N}$.

(III) For the third equality we take an integer $k > s_2$, and use the assumption $s, s_1, s_2 \notin \mathbb{N}_0$, (1.1.12), A.2 h), the reflexivity of X and $D(A^k)$ and again (1.1.12), to obtain

$$\begin{aligned}[W^{s_1}_{p,\mu}(\mathbb{R}_+; E), W^{s_2}_{p,\mu}(\mathbb{R}_+; E)]_\theta &= [(X, D(A^k))_{s_1/k,p}, (X, D(A^k))_{s_2/k,p}]_\theta \\ &= (X, D(A^k))_{s/k,p} = W^s_{p,\mu}(\mathbb{R}_+; E).\end{aligned}$$

Similar arguments yield the fourth asserted equality, i.e.,

$$\begin{aligned}(W^{s_1}_{p,\mu}(\mathbb{R}_+; E), W^{s_2}_{p,\mu}(\mathbb{R}_+; E))_{\theta,p} &= ((X, D(A^k))_{s_1/k,p}, (X, D(A^k))_{s_2/k,p})_{\theta,p} \\ &= (X, D(A^k))_{s/k,p} = W^s_{p,\mu}(\mathbb{R}_+; E).\end{aligned}$$

(IV) Now let $F \overset{d}{\hookrightarrow} E$ be a further Banach space of class $\mathcal{HT}$ and $\tau \geq 0$. Then the operator A^τ is an isomorphism

$$(H^\tau_{p,\mu}(\mathbb{R}_+; E), H^\tau_{p,\mu}(\mathbb{R}_+; F))_{\theta,p} \to (L_{p,\mu}(\mathbb{R}_+; E), L_{p,\mu}(\mathbb{R}_+; F))_{\theta,p}.$$

Due to [82, Theorem 1.18.4], the latter space equals $L_{p,\mu}(\mathbb{R}_+; (E,F)_{\theta,p})$, and $A^{-\tau}$ maps this space isomorphically to $H^\tau_{p,\mu}(\mathbb{R}_+; (E,F)_{\theta,p})$. The corresponding assertion on complex interpolation is shown in the same way.

(V) Replacing the operator $A = 1 - \partial_t$ by $A_0 := 1 + \partial_t$, the same arguments as above show the asserted equalities for the ${}_0W$- and the ${}_0H$-spaces. This finishes the case $J = \mathbb{R}_+$. The case of a finite interval can be deduced from the half-line case, using the extension operators $\mathcal{E}_J$ and $\mathcal{E}^0_J$ from Lemma 1.1.5. For instance, one decomposes the identity into $\mathcal{E}_J$ and the restriction $\mathcal{R}_J$ to J and obtains

$$|u|_{[H^{s_1}_{p,\mu}(J;E), H^{s_2}_{p,\mu}(J;E)]_\theta} \leq |\mathcal{E}_J u|_{[H^{s_1}_{p,\mu}(\mathbb{R}_+;E), H^{s_2}_{p,\mu}(\mathbb{R}_+;E)]_\theta} \lesssim |\mathcal{E}_J u|_{H^s_{p,\mu}(\mathbb{R}_+;E)} \lesssim |u|_{H^s_{p,\mu}(J;E)}.$$

The converse embedding is derived in the same way.[5] The dependence of the norm equivalence constants on the length of J carries over from the properties of the extension operators. Note that here it is important that the extension operators act on a whole scale of W- and H-spaces. ■

The following result shows that the good properties of $1 - \partial_t$ and $1 + \partial_t$ carry over to the whole W- and H-scale.

[5] This is nothing but the retraction-coretraction method from [82, Section 1.2.4]

Proposition 1.1.10. *Let $p \in (1,\infty)$, $\mu \in (1/p,1]$, $s \geq 0$, $\alpha \in (0,2)$, and $\omega > 0$. Then the operators*

$$
\begin{array}{llll}
(\omega-\partial_t)^\alpha \text{ on } H^s_{p,\mu}(\mathbb{R}_+;E), & \text{with domain } H^{s+\alpha}_{p,\mu}(\mathbb{R}_+;E), & \\
(\omega-\partial_t)^\alpha \text{ on } W^s_{p,\mu}(\mathbb{R}_+;E), & \text{with domain } W^{s+\alpha}_{p,\mu}(\mathbb{R}_+;E), & s, s+\alpha \notin \mathbb{N}_0, \\
(\omega+\partial_t)^\alpha \text{ on } {}_0H^s_{p,\mu}(\mathbb{R}_+;E), & \text{with domain } {}_0H^{s+\alpha}_{p,\mu}(\mathbb{R}_+;E), & \\
(\omega+\partial_t)^\alpha \text{ on } {}_0W^s_{p,\mu}(\mathbb{R}_+;E), & \text{with domain } {}_0W^{s+\alpha}_{p,\mu}(\mathbb{R}_+;E), & s, s+\alpha \notin \mathbb{N}_0,
\end{array}
$$

are invertible and admit a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle $\alpha\pi/2$, respectively.

Proof. (I) We first consider the case $s = 0$. Theorem 1.1.7 and [24, Proposition 2.11] imply that the realization of $\omega - \partial_t$ on $L_{p,\mu}(\mathbb{R}_+;E)$ with domain $W^1_{p,\mu}(\mathbb{R}_+;E)$ admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to $\pi/2$. Lemma A.3.5 yields that also $(\omega-\partial_t)^\alpha$ admits a bounded $\mathcal{H}^\infty$-calculus, with $\mathcal{H}^\infty$-angle $\alpha\pi/2$, provided $\alpha \in (0,2)$. The same arguments as in Step I of the proof of Lemma 1.1.9 further show that

$$D((\omega-\partial_t)^\alpha) = H^\alpha_{p,\mu}(\mathbb{R}_+;E).$$

(II) Since $\omega - \partial_t$ is invertible, also $(\omega-\partial_t)^s$ is invertible, for all $s \geq 0$. It follows from the definition of the weighted Sobolev spaces that $(\omega-\partial_t)^s$ is an isomorphism $H^{k+s}_{p,\mu}(\mathbb{R}_+;E) \to H^k_{p,\mu}(\mathbb{R}_+;E)$ for $k \in \mathbb{N}_0$, and by interpolation this carries over to an isomorphism $H^{\tau+s}_{p,\mu}(\mathbb{R}_+;E) \to H^\tau_{p,\mu}(\mathbb{R}_+;E)$ for all $\tau \geq 0$. Since $(\omega-\partial_t)^s$ and $(\omega-\partial_t)^\alpha$ commute, it follows from [24, Proposition 2.11] that $(\omega-\partial_t)^\alpha$ has a bounded $\mathcal{H}^\infty$-calculus on $H^s_{p,\mu}(\mathbb{R}_+;E)$, still with angle not larger than $\alpha\pi/2$, and that its domain equals $H^{s+\alpha}_{p,\mu}(\mathbb{R}_+;E)$.
(III) Now let $s, s+\alpha \notin \mathbb{N}_0$. It then follows from interpolation of the H-case and Lemma 1.1.9 that $(\omega-\partial_t)^\alpha$ has a bounded $\mathcal{H}^\infty$-calculus on $W^s_{p,\mu}(\mathbb{R}_+;E)$ with $\mathcal{H}^\infty$-angle $\alpha\pi/2$, and that its domain is $W^{s+\alpha}_{p,\mu}(\mathbb{R}_+;E)$. The same arguments as above show the assertions on the operator $\omega + \partial_t$. ■

We consider the temporal trace on the $W^s_{p,\mu}$- spaces, and characterizations of the ${}_0W^s_{p,\mu}$-spaces in terms of its kernel. These results are mainly due to Grisvard [44]. Observe that the limit number for the existence of a trace is $s = 1-\mu+1/p$. Therefore, if μ runs through the interval $(1/p,1]$ this limit number runs through the interval $[1/p,1)$. Of course, for $\mu = 1$ the limit number $s = 1/p$ for the unweighted case is recovered.

Proposition 1.1.11. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$ and $\mu \in (1/p,1]$. Then the following holds true.*[6]

a) *For $0 < s < 1-\mu+1/p$ it holds $C^\infty_c(\overline{J}\backslash\{0\};E) \overset{d}{\hookrightarrow} W^s_{p,\mu}(J;E)$, and further*

$$W^s_{p,\mu}(J;E) = {}_0W^s_{p,\mu}(J;E).$$

[6] We do not treat the limit cases $s = k+1-\mu+1/p$, $k \in \mathbb{N}_0$, since they are quite complicated and not important for our purposes. For short discussions we refer to [44, Remarque 4.2] and [82, Remark 3.6.3/2]. We also do not consider the corresponding characterizations of the ${}_0H^s_{p,\mu}$-spaces. They should be correct, but it seems that their proofs require a greater effort.

b) *For $k+1-\mu+1/p < s < k+1+(1-\mu+1/p)$ with $k \in \mathbb{N}_0$ it holds*

$$W^s_{p,\mu}(J;E) \hookrightarrow BUC^k(\overline{J};E), \tag{1.1.13}$$

where here one may replace $W^s_{p,\mu}$ by $H^s_{p,\mu}$, and moreover

$${}_0W^s_{p,\mu}(J;E) \doteq \big\{u \in W^s_{p,\mu}(J;E) \ : \ u^{(j)}(0) = 0, \ \ j \in \{0,...,k\}\big\}, \tag{1.1.14}$$

where the latter space is considered as a closed subspace of $W^s_{p,\mu}(J;E)$.

The embedding constants for

$${}_0W^s_{p,\mu}(J;E) \hookrightarrow BUC^k(\overline{J};E), \qquad {}_0H^s_{p,\mu}(J;E) \hookrightarrow BUC^k(\overline{J};E)$$

where $s \in [0,2]$ and $k \in \mathbb{N}_0$ are as in b), are independent of J, respectively.

Proof. The results in [44] for the $W^s_{p,\mu}(\mathbb{R}_+;E)$-spaces are obtained in the scalar-valued case, $E = \mathbb{C}$. An inspection of the proofs there shows that, besides basic facts on vector-valued spaces, they only make use of interpolation theory and the Lemmas 1.1.4 and 1.1.6. Thus the results of [44] carry over to a general E. Moreover, the case of a finite interval is obtained from the half-line case by extension and restriction, as in Step IV of the proof of Lemma 1.1.9. The fact that one may replace W by H as asserted follows from (1.1.1) and (1.1.2).
Assertion a) is shown in [44, Théorème 2.1, Théorème 4.1]. The embedding in b) is for $k = 0$ proved in [44, Théorème 5.2], and the general case $k \in \mathbb{N}$ is an immediate consequence. For $s \leq 1$, (1.1.14) is shown in [44, Théorème 4.1]. For $s > 1$, note that by definition it holds ${}_0W^s_{p,\mu}(\mathbb{R}_+;E) \hookrightarrow W^s_{p,\mu}(\mathbb{R}_+;E)$.
For the converse embedding in (1.1.14), take $u \in W^s_{p,\mu}(\mathbb{R}_+;E)$ with $u^{(j)}(0) = 0$ for $j \in \{0,...,k\}$. Assume first that $[s] = k$. Then $u \in {}_0W^{[s]}_{p,\mu}(\mathbb{R}_+;E)$. From $u^{([s])}(0) = 0$, $1-\mu+1/p < s-[s]$, (1.1.14) for $s-[s] < 1$ and with Lemma 1.1.8 we infer

$$\begin{aligned} |u|_{{}_0W^s_{p,\mu}(\mathbb{R}_+;E)} &\lesssim |u|_{{}_0W^{[s]}_{p,\mu}(\mathbb{R}_+;E)} + |u^{[s]}|_{{}_0W^{s-[s]}_{p,\mu}(\mathbb{R}_+;E)} \\ &\lesssim |u|_{W^{[s]}_{p,\mu}(\mathbb{R}_+;E)} + |u^{[s]}|_{W^{s-[s]}_{p,\mu}(\mathbb{R}_+;E)} \lesssim |u|_{W^s_{p,\mu}(\mathbb{R}_+;E)}. \end{aligned}$$

Now assume that $[s] = k+1$. Then again $u \in {}_0W^{[s]}_{p,\mu}(\mathbb{R}_+;E)$. Since $s-[s] < 1-\mu+1/p$, it follows from a) that $u^{([s])} \in W^{s-[s]}_{p,\mu}(\mathbb{R}_+;E) = {}_0W^{s-[s]}_{p,\mu}(\mathbb{R}_+;E)$, and (1.1.14) follows as above from Lemma 1.1.8. ■

We next consider embeddings of Sobolev type into weighted and unweighted spaces.

Proposition 1.1.12. *Let $J = (0,T)$ be a finite interval, $p \in (1,\infty)$, $\mu \in (1/p,1]$, $s > \tau \geq 0$ and $q \in (p,\infty)$. Then*

$$W^s_{p,\mu}(J;E) \hookrightarrow W^\tau_{q,\mu}(J;E) \qquad \text{if } \ s-(1-\mu+1/p) > \tau - \frac{p(1-\mu+1/p)}{q}, \tag{1.1.15}$$

and moreover it holds

$$W^s_{p,\mu}(J;E) \hookrightarrow W^\tau_q(J;E) \qquad \text{if } \ s-(1-\mu+1/p) > \tau - 1/q. \tag{1.1.16}$$

These embeddings remain true if one replaces the W-spaces by the H-, the ${}_0W$- and the ${}_0H$-spaces, respectively. In two latter cases, restricting to $s \in [0,2]$, for given $T_0 > 0$ the embeddings hold with a uniform constant for all $0 < T \leq T_0$.

Proof. Throughout this proof, let $T_0 > 0$ be given. Since the inequality signs in (1.1.15) and (1.1.16) are strict, we may assume that $s \notin \mathbb{N}$. Again we only have to consider the W-case due to (1.1.1) and (1.1.2).

(I) We show (1.1.15) for $\tau = 0$. For $s > 1 - \mu + 1/p$, the condition is satisfied for all $q \in (p, \infty)$, and, in fact, Proposition 1.1.11 shows

$$W^s_{p,\mu}(J;E) \hookrightarrow L_\infty(J;E) \hookrightarrow L_{q,\mu}(J;E), \qquad q \in (p,\infty),$$

with the asserted behaviour of the embedding constant in the ${}_0W$-case. For $s \leq 1-\mu+1/p$ we take $\eta > 1 - \mu + 1/p$ and use again that $W^\eta_{p,\mu}(J;E) \hookrightarrow L_{r,\mu}(J;E)$ for $r \in (p,\infty)$, A.2 d), Lemma 1.1.9 and [82, Theorem 1.18.5], to obtain

$$W^s_{p,\mu}(J;E) = (L_{p,\mu}(J;E), W^\eta_{p,\mu}(J;E))_{s/\eta,p} \hookrightarrow (L_{p,\mu}(J;E), L_{r,\mu}(J;E))_{s/\eta,p} = L_{q,\mu}(J;E),$$

which is valid for $\frac{1}{q} = \frac{1-s/\eta}{p} + \frac{s/\eta}{r}$. Letting $r \nearrow \infty$ and $\eta \searrow 1 - \mu - 1/p$, we obtain (1.1.15) for q as asserted. In the ${}_0W$-case, the embedding constant is uniform in $T \leq T_0$ for $s \in [0,2]$.

(II) To prove (1.1.15) for $\tau > 0$, we start with

$$W^s_{p,\mu}(J;E) = (W^\kappa_{p,\mu}(J;E), W^\eta_{p,\mu}(J;E))_{\frac{s-\kappa}{\eta-\kappa},p},$$

which holds by Lemma 1.1.9 for noninteger $\kappa < s < \eta$. Let $k \in \mathbb{N}_0$. Using (1.1.15) with $\tau = 0$, we obtain

$$W^\kappa_{p,\mu}(J;E) \hookrightarrow W^k_{q,\mu}(J;E), \qquad \kappa > k + (1 - p/q)(1 - \mu + 1/p),$$

$$W^\eta_{p,\mu}(J;E) \hookrightarrow W^{k+1}_{q,\mu}(J;E), \qquad \eta > (k+1) + (1 - p/q)(1 - \mu + 1/p).$$

Hence for those q, κ, η it holds

$$W^s_{p,\mu}(J;E) \hookrightarrow (W^k_{q,\mu}(J;E), W^{k+1}_{q,\mu}(J;E))_{\frac{s-\kappa}{\eta-\kappa},q} = W^{k+\frac{s-\kappa}{\eta-\kappa}}_{q,\mu}(J;E),$$

using that $(\cdot,\cdot)_{\theta,p} \hookrightarrow (\cdot,\cdot)_{\theta,q}$ for $\theta \in (0,1)$ and $q \in (p,\infty)$. Letting $\kappa \searrow k + (1-\frac{p}{q})(1-\mu+1/p)$ and $\eta \searrow k + 1 + (1 - \frac{p}{q})(1 - \mu + 1/p)$ we obtain (1.1.15) for τ and q as asserted. For ${}_0W$-spaces, the dependence on T for $s \in [0,2]$ carries over from Lemma 1.1.9 and (1.1.15) with $\tau = 0$.

(III) To show (1.1.16), we again first treat the case $\tau = 0$. As above, for $s > 1 - \mu + 1/p$ the embedding is deduced from Proposition 1.1.11. For $s \leq 1 - \mu + 1/p$ we use

$$H^\eta_{p,\mu}(J;E) \hookrightarrow L_\infty(J;E), \qquad \eta > 1 - \mu + 1/p,$$

and further that

$$L_{p,\mu}(J;E) \hookrightarrow L_r(J;E), \qquad 1 - \mu + 1/p < 1/r,$$

which follows from Lemma 1.1.1. For $0 < \sigma < \eta$, Lemma 1.1.9 and A.2 d) and n) thus yield

$$H^\sigma_{p,\mu}(J;E) \hookrightarrow [L_r(J;E), L_\infty(J;E)]_{\frac{\sigma}{\eta}} = L_{\frac{r}{1-\sigma/\eta}}(J;E).$$

Letting $\eta \searrow 1-\mu+1/p$, $r \nearrow \frac{1}{1-\mu+1/p}$, and employing $W^s_{p,\mu}(J;E) \hookrightarrow H^\sigma_{p,\mu}(J;E)$ for $s > \sigma$ we obtain (1.1.16). Replacing W by ${}_0W$, the embedding constant is uniform in T.
(IV) The case $\tau > 0$ may now be obtained from the case $\tau = 0$ as in Step II. We omit the details. ■

We derive an intrinsic norm for the W-spaces, on a finite and an infinite interval.

Proposition 1.1.13. *Let $J = (0,T)$ with $T \in (0,\infty]$, $p \in (1,\infty)$, $\mu \in (1/p,1]$, and $s \in (0,1)$. Then we have*

$$|u|_{W^s_{p,\mu}(J;E)} \sim |u|_{L_{p,\mu}(J;E)} + [u]_{W^s_{p,\mu}(J;E)},$$

where

$$[u]^p_{W^s_{p,\mu}(J;E)} := \int_0^T \int_0^t \tau^{p(1-\mu)} \frac{|u(t)-u(\tau)|^p_E}{(t-\tau)^{1+sp}}\, d\tau\, dt. \tag{1.1.17}$$

In case $J = \mathbb{R}_+$, the semi-norm $[u]_{W^s_{p,\mu}(J;E)}$ may be replaced by

$$[[u]]^p_{W^s_{p,\mu}(\mathbb{R}_+;E)} := \int_0^\infty \int_0^\infty \tau^{p(1-\mu)} \frac{|u(t+\tau)-u(\tau)|^p_E}{t^{1+sp}}\, d\tau\, dt. \tag{1.1.18}$$

Proof. (I) For $J = \mathbb{R}_+$, it follows immediately from Lemma 1.1.6 and A.2 k) that

$$|u|_{W^s_{p,\mu}(J;E)} \sim |u|_{L_{p,\mu}(J;E)} + [[u]]_{W^s_{p,\mu}(J;E)},$$

and a simple substitution shows that $[[u]]_{W^s_{p,\mu}(J;E)}$ may be replaced by $[u]_{W^s_{p,\mu}(J;E)}$.
(I) Now let $J = (0,T)$ be finite. We deduce this case from the half-line case by localization and extension. We fix a smooth partition of unity $\{\psi_1,\psi_2\}$ for $[0,T]$, such that $\psi_1(t) = 0$ for $t \geq \frac{2}{3}T$ and $\psi_2(t) = 0$ for $t \leq \frac{T}{3}$. The multiplication with ψ_i, $i = 1,2$, is continuous on $L_{p,\mu}(J;E)$ and $W^1_{p,\mu}(J;E)$, respectively, hence it is continuous on $W^s_{p,\mu}(J;E)$ by A.2 i). This implies

$$|u|_{W^s_{p,\mu}(J;E)} \lesssim |\psi_1 u|_{W^s_{p,\mu}(J;E)} + |\psi_2 u|_{W^s_{p,\mu}(J;E)}.$$

(II) Since the restriction to J is continuous on the whole $W^s_{p,\mu}$-scale, we may estimate

$$|\psi_1 u|_{W^s_{p,\mu}(J;E)} \lesssim |\psi_1 u|_{W^s_{p,\mu}(\mathbb{R}_+;E)} \lesssim |\psi_1 u|_{L_{p,\mu}(\mathbb{R}_+;E)} + [\psi_1 u]_{W^s_{p,\mu}(\mathbb{R}_+;E)},$$

identifying $\psi_1 u$ with its trivial extension to $\mathbb{R}_+$. We split the outer t-integral in $[\psi_1 u]^p_{W^s_{p,\mu}(\mathbb{R}_+;E)}$ at $t = T$ into two summands. For the first summand we estimate, using the mean value theorem for ψ_1,

$$\begin{aligned}
&\int_0^T \int_0^t \tau^{p(1-\mu)} \frac{|\psi_1(t)u(t)-\psi_1(\tau)u(\tau)|^p}{(t-\tau)^{1+sp}}\, \mathrm{d}\tau\, \mathrm{d}t \\
&\qquad \lesssim [u]^p_{W^s_{p,\mu}(J;E)} + \int_0^T \int_\tau^T \tau^{p(1-\mu)} |u(t)|^p (t-\tau)^{p(1-s)-1}\, \mathrm{d}t\, \mathrm{d}\tau \\
&\qquad \lesssim [u]^p_{W^s_{p,\mu}(J;E)} + |u|^p_{L_{p,\mu}(J;E)}.
\end{aligned}$$

For the second summand we have

$$\begin{aligned}
&\int_T^\infty \int_0^t \tau^{p(1-\mu)} \frac{|\psi_1(t)u(t) - \psi_1(\tau)u(\tau)|^p}{(t-\tau)^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\qquad = \int_0^{\frac{2}{3}T} \tau^{p(1-\mu)} |u(\tau)|^p \left(\int_T^\infty \frac{|\psi_1(\tau)|^p}{(t-\tau)^{1+sp}} \,\mathrm{d}t \right) \mathrm{d}\tau \lesssim |u|^p_{L_{p,\mu}(J;E)},
\end{aligned}$$

since the integral in brackets is bounded independent of $\tau \in (0, \frac{2}{3}T)$.
(III) It follows from A.2 d) that $W_p^s(J;E) \hookrightarrow W_{p,\mu}^s(J;E)$, from which we obtain

$$|\psi_2 u|_{W^s_{p,\mu}(J;E)} \lesssim |\psi_2 u|_{W^s_p(J;E)} \lesssim |\psi_2 u|_{L_p(J;E)} + [\psi_2 u]_{W^s_p(J;E)},$$

where $[\cdot]_{W^s_p(J;E)}$ denotes the intrinsic semi-norm in the unweighted case (cf. (A.4.2)), i.e.,

$$[\psi_2 u]^p_{W^s_p(J;E)} = \int_0^T \int_0^T \frac{|\psi_2(t)u(t) - \psi_2(\tau)u(\tau)|^p_E}{|t-\tau|^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t.$$

We split the inner τ-integral of $[\psi_2 u]^p_{W^s_p(J;E)}$ at $\tau = t$ into two summands. For the first summand we have

$$\begin{aligned}
&\int_0^T \int_0^t \frac{|\psi_2(t)u(t) - \psi_2(\tau)u(\tau)|^p}{(t-\tau)^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\quad \lesssim \int_{T/3}^T \int_{T/3}^t \tau^{p(1-\mu)} |\psi_2(\tau)|^p \frac{|u(t) - u(\tau)|^p}{(t-\tau)^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\qquad + \int_{T/3}^T \int_{T/3}^t |u(t)|^p \frac{|\psi_2(t) - \psi_2(\tau)|^p}{(t-\tau)^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t \lesssim [u]^p_{W^s_{p,\mu}(J;E)} + |u|^p_{L_{p,\mu}(J;E)}.
\end{aligned}$$

For the second summand we estimate in a similar fashion

$$\begin{aligned}
&\int_0^T \int_t^T \frac{|\psi_2(t)u(t) - \psi_2(\tau)u(\tau)|^p}{(\tau-t)^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t \\
&\quad \lesssim \int_{T/3}^T \int_{T/3}^\tau t^{p(1-\mu)} |\psi_2(t)|^p \frac{|u(\tau) - u(t)|^p}{(\tau-t)^{1+sp}} \,\mathrm{d}t \,\mathrm{d}\tau \\
&\qquad + \int_{T/3}^T \int_{T/3}^\tau |u(\tau)|^p \frac{|\psi_2(t) - \psi_2(\tau)|^p}{(\tau-t)^{1+sp}} \,\mathrm{d}t \,\mathrm{d}\tau \lesssim [u]^p_{W^s_{p,\mu}(J;E)} + |u|^p_{L_{p,\mu}(J;E)}.
\end{aligned}$$

These estimates show $|u|_{W^s_{p,\mu}(J;E)} \lesssim |u|_{L_{p,\mu}(J;E)} + [u]_{W^s_{p,\mu}(J;E)}$.
(IV) For the converse estimate, note that it trivially holds

$$[u]^p_{W^s_{p,\mu}(J;E)} \leq [u]^p_{W^s_{p,\mu}(J;E)} + \int_T^\infty \int_0^t \tau^{p(1-\mu)} \frac{|\mathcal{E}_J u(t) - \mathcal{E}_J u(x)|^p}{(t-\tau)^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}t = [\mathcal{E}_J u]^p_{W^s_{p,\mu}(\mathbb{R}_+;E)},$$

where $\mathcal{E}_J$ is the extension operator from Lemma 1.1.5. We thus obtain

$$|u|_{L_{p,\mu}(J;E)} + [u]_{W^s_{p,\mu}(J;E)} \leq |\mathcal{E}_J u|_{W^s_{p,\mu}(\mathbb{R}_+;E)} \lesssim |u|_{W^s_{p,\mu}(J;E)},$$

which finishes the proof. ■

We next prove Poincaré's inequality in the weighted spaces. It will be used in later chapters to obtain smallness of Lipschitz constants by choosing short time intervals.

Lemma 1.1.14. *Let $J = (0,T)$ be finite, $p \in (1,\infty)$, and $\mu \in (1/p, 1]$. Then it holds*

$$|u|_{L_{p,\mu}(J;E)} \lesssim T\, |u'|_{L_{p,\mu}(J;E)} \qquad \text{if } u \in {}_0W^1_{p,\mu}(J;E),$$

and consequently, for $s \in [0,1)$,

$$|u|_{{}_0W^s_{p,\mu}(J;E)} + |u|_{{}_0H^s_{p,\mu}(J;E)} \lesssim T^{1-s}\, |u|_{W^1_{p,\mu}(J;E)} \qquad \text{if } u \in {}_0W^1_{p,\mu}(J;E).$$

Proof. For $t \in J$ we estimate, using Hölder's inequality,

$$\begin{aligned} t^{p(1-\mu)}|u(t)|_E^p &\le t^{p(1-\mu)} \left(\int_0^T s^{-(1-\mu)} s^{1-\mu} |u'(s)|_E \,\mathrm{d}s \right)^p \\ &\lesssim t^{p(1-\mu)}\, T^{(1-p'(1-\mu))p/p'} |u'|^p_{L_{p,\mu}(J;E)}. \end{aligned}$$

Now the first asserted inequality follows after integration over J. For $s \in [0,1)$ the interpolation inequality A.2 j) yields

$$|u|_{{}_0W^s_{p,\mu}(J;E)} + |u|_{{}_0H^s_{p,\mu}(J;E)} \lesssim |u|^s_{{}_0W^1_{p,\mu}(J;E)} |u|^{1-s}_{L_{p,\mu}(J;E)}, \qquad u \in {}_0W^1_{p,\mu}(J;E),$$

from which the second asserted inequality follows. ■

Remark 1.1.15. In applications one deals with superposition and multiplication operators on the spaces $W^s_{p,\mu}(0,T;E)$ and ${}_0W^s_{p,\mu}(0,T;E)$, equipped with the interpolation norm from their definition in the beginning of this section. Of course, one would rather like to work with the intrinsic norms derived in Proposition 1.1.13, since these much more convenient to work with. At the same time one often assumes that T is small, for instance to make lower order terms small, with Poincaré's inequality (see Lemma 1.3.13). Such a scenario arises, for instance, in the proofs of our main Theorems 2.1.4 and 3.1.4 on linear problems, and also in the proof of Proposition 4.3.2 on local existence for nonlinear problems.
In Proposition 1.1.13 we have shown the equivalence of the interpolation norm and the intrinsic norm for $W^s_{p,\mu}(0,T;E)$ using the extension operator $\mathcal{E}_J$ from Lemma 1.1.5. Thus the equivalence constants for these norms depend on T, and typically become large as T becomes small. This might have the effect that lower order terms, for instance, are not small anymore for small T after having used the intrinsic norm. The situation is the same if one works in ${}_0W^s_{p,\mu}(0,T;E)$ equipped with the intrinsic norm.
To overcome this obstacle for short time intervals, in a situation as above one has to work in ${}_0W^s_{p,\mu}(0,T;E)$ equipped with the interpolation norm from the beginning. Via the extension operator $\mathcal{E}^0_J$ and restriction, this space is T-independently connected to ${}_0W^s_{p,\mu}(\mathbb{R}_+;E)$. In this way one can perform the required estimates with the intrinsic norms (1.1.17) or (1.1.18) on ${}_0W^s_{p,\mu}(\mathbb{R}_+;E)$, without receiving unpleasant T-dependent factors. For examples we refer to the proofs of the Lemmas 1.3.22, 1.3.23 and 4.2.3, for instance. ■

1.2 Abstract Properties

1.2.1 Abstract Maximal $L_{p,\mu}$-Regularity

We briefly review the results of Prüss & Simonett [71] on abstract maximal $L_{p,\mu}$-regularity, and add a few remarks on finite intervals.

Let A be a closed and densely defined operator on a Banach space E with domain $D(A)$. Endowed with its graph norm, $D(A)$ becomes a Banach space. Let further $p \in (1,\infty)$ and $\mu \in (1/p, 1]$. For a finite or infinite interval $J = (0,T)$ we set

$$\mathbb{E}_{0,\mu}(J) := L_{p,\mu}(J;E), \qquad \mathbb{E}_{u,\mu}(J) := W^1_{p,\mu}(J;E) \cap L_{p,\mu}(J;D(A)).$$

Due to Lemma 1.1.1, functions in $\mathbb{E}_{u,\mu}(J)$ have a well-defined trace in E at $t = 0$. We say that A enjoys maximal $L_{p,\mu}$-regularity on J,

$$A \in \mathcal{MR}_{p,\mu}(J;E),$$

if for each $f \in \mathbb{E}_{0,\mu}(J)$ there is a unique solution $u \in \mathbb{E}_{u,\mu}(J)$ of

$$u' + Au = f(t), \quad \text{a.e. } t \in J, \qquad u(0) = 0. \tag{1.2.1}$$

In other words, it holds $A \in \mathcal{MR}_{p,\mu}(J;E)$ if and only if the operator $\partial_t + A$ on $\mathbb{E}_{0,\mu}(J)$, with domain

$${}_0\mathbb{E}_{u,\mu}(J) := {}_0W^1_{p,\mu}(J;E) \cap L_{p,\mu}(J;D(A)),$$

is invertible. For convenience we further set

$$\mathcal{MR}_p(J;E) := \mathcal{MR}_{p,1}(J;E)$$

in the unweighted case. If $A \in \mathcal{MR}_{p,\mu}(J;E)$, then the open mapping theorem implies that the solution u of (1.2.1) depends continuously on the right-hand side f, i.e., there is a constant $C > 0$, which does not depend on f, such that

$$|u|_{\mathbb{E}_{u,\mu}(J)} \le C\,|f|_{\mathbb{E}_{0,\mu}(J)}. \tag{1.2.2}$$

The following lemma shows that for negative generators of analytic semigroups, maximal $L_{p,\mu}$-regularity is only a matter of regularity, since the solution of (1.2.1) is given by the convolution with the semigroup.

Lemma 1.2.1. *Let $J = (0,T)$ be finite or infinite, $p \in (1,\infty)$, $\mu \in (1/p,1]$, and let $-A$ be the generator of an analytic semigroup on E. If $u \in \mathbb{E}_{u,\mu}(J)$ solves (1.2.1) for $f \in \mathbb{E}_{0,\mu}(J)$, then u is given by*

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\,ds, \qquad t \in J.$$

In particular, $\mathbb{E}_{u,\mu}(J)$-solutions of (1.2.1) are unique.

Proof. By Lemma 1.1.1 it holds $L_{p,\mu}(J;E) \hookrightarrow L_{1,\mathrm{loc}}(\overline{J};E)$, and thus the assertion follows immediately from [30, Theorem 2.1]. ■

The following fundamental result due to [71] shows that the maximal regularity properties of A on the half-line are independent of the weight.

Theorem 1.2.2. *For $p \in (1,\infty)$ and $\mu \in (1/p,1]$ it holds $A \in \mathcal{MR}_{p,\mu}(\mathbb{R}_+;E)$ if and only if $A \in \mathcal{MR}_p(\mathbb{R}_+;E)$. Moreover, if $A \in \mathcal{MR}_p(J_0;E)$ for some finite or infinite interval $J_0 = (0,T_0)$ then $A \in \mathcal{MR}_{p,\mu}(J;E)$ for all $\mu \in (1/p,1]$ and all finite intervals $J = (0,T)$ as well, and if $A \in \mathcal{MR}_p(\mathbb{R}_+;E)$ then the constant in (1.2.2) is independent of J.*

Proof. (I) The independence of the class $\mathcal{MR}_{p,\mu}(\mathbb{R}_+;E)$ of $\mu \in (1/p,1]$ is shown in [71, Theorem 2.4].
(II) Assume that $A \in \mathcal{MR}_p(J_0;E)$. It then follows from [30, Corollary 5.3] that there is $\omega > 0$ such that $A-\omega \in \mathcal{MR}_p(\mathbb{R}_+;E)$, and thus $A-\omega \in \mathcal{MR}_{p,\mu}(\mathbb{R}_+;E)$ for all $\mu \in (1/p,1]$ by the result of [71]. It can now be shown as in the proof of [30, Theorem 3.3] that $A \in \mathcal{MR}_{p,\mu}(J;E)$ for each finite interval $J=(0,T)$.
(III) Finally, suppose that $A \in \mathcal{MR}_p(\mathbb{R}_+;E)$, and let J be finite. For $f \in L_{p,\mu}(J;E)$ a solution $u \in \mathbb{E}_{u,\mu}(J)$ of

$$u' + Au = f(t), \qquad \text{a.e. } t \in J, \qquad u(0) = 0,$$

is given by $u = \widetilde{u}|_J$, where $\widetilde{u}$ is the solution of the above problem on $\mathbb{R}_+$ with trivially extended right-hand side f. Since $-A$ is the generator of an analytic C_0-semigroup on E by [30, Corollary 4.2], it follows from Lemma 1.2.1 that this is the only solution. This yields the estimate

$$|u|_{\mathbb{E}_{u,\mu}(J)} \le |\widetilde{u}|_{\mathbb{E}_{u,\mu}(\mathbb{R}_+)} \le C\,|\widetilde{f}|_{\mathbb{E}_{0,\mu}(\mathbb{R}_+)} = C\,|f|_{\mathbb{E}_{0,\mu}(J)},$$

where C is the maximal regularity constant of A on $\mathbb{R}_+$, which is independent of J. ■

We describe some consequences of Theorem 1.2.2 for maximal $L_{p,\mu}$-regularity.

If E is of class $\mathcal{HT}$, then well known sufficient conditions for maximal L_p-regularity are also available for maximal $L_{p,\mu}$-regularity, such as that A admits a bounded $\mathcal{H}^\infty$-calculus or admits bounded imaginary powers, with angles strictly smaller than $\pi/2$, respectively. Moreover, combining Theorem 1.2.2 with a result of Weis [85, Theorem 4.2], it holds $A \in \mathcal{MR}_{p,\mu}(\mathbb{R}_+;E)$ if and only if A is $\mathcal{R}$-sectorial.
From [30, Theorem 7.1] it follows that maximal $L_{p,\mu}$-regularity is independent of the exponent $p \in (1,\infty)$, and together with [30, Corollary 4.2] we further obtain that if $A \in \mathcal{MR}_{p,\mu}(\mathbb{R}_+;E)$, then $-A$ is the generator of an exponentially stable analytic C_0-semigroup on E.

Now let us consider (1.2.1) with nontrivial initial values, i.e.,

$$u' + Au = f(t), \qquad \text{a.e. } t \in J, \qquad u(0) = u_0. \tag{1.2.3}$$

The following result is proved in [71, Theorem 3.2] for $J=\mathbb{R}_+$. The case of finite interval may be deduced from this as in the proof of Theorem 1.2.2.

Theorem 1.2.3. *Let $p \in (1,\infty)$, $\mu \in (1/p,1]$, and let $J=(0,T)$ be finite or infinite. If $A \in \mathcal{MR}_p(\mathbb{R}_+;E)$ then (1.2.3) has a unique solution $u \in \mathbb{E}_{u,\mu}(J)$ if and only if $f \in \mathbb{E}_{0,\mu}(J)$ and $u_0 \in D_A(\mu-1/p,p)$.[7] There is a constant C, which is independent of J, f, and u_0, such that*

$$|u|_{\mathbb{E}_{u,\mu}(J)} \le C\big(|f|_{\mathbb{E}_{0,\mu}(J)} + |u_0|_{D_A(\mu-1/p,p)}\big).$$

[7] Recall the notation $D_A(\mu-1/p,p) = (E, D(A))_{\mu-1/p,p}$.

1.2.2 Operator-Valued Fourier Multipliers

We now turn our attention to operator-valued Fourier multipliers on $L_{p,\mu}$. For Banach spaces E, F and an operator-valued function $m \in L_{1,\mathrm{loc}}(\mathbb{R};\mathcal{B}(E,F))$ one obtains an operator T_m by setting

$$T_m f := \mathcal{F}^{-1} m \mathcal{F} f, \qquad f \in \mathcal{F}^{-1} C_c^\infty(\mathbb{R};E),$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}$. It is not hard to show that T_m is densely defined on $L_{p,\mu}(\mathbb{R}_+;E)$. Now m is called a Fourier-multiplier on $L_{p,\mu}$, if the operator T_m admits an estimate

$$|T_m f|_{L_{p,\mu}(\mathbb{R}_+;F)} \lesssim |f|_{L_{p,\mu}(\mathbb{R}_+;E)}, \qquad f \in \mathcal{F}^{-1} C_c^\infty(\mathbb{R};E),$$

i.e., if it extends to a continuous operator from $L_{p,\mu}(\mathbb{R}_+;E)$ to $L_{p,\mu}(\mathbb{R}_+;F)$.
The following result on $L_{p,\mu}$-multipliers is available. It is due to Girardi and Weis [42], and is an extension of Weis' multiplier theorem [85, Theorem 3.4] in the unweighted case.

Theorem 1.2.4. *Let $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and let E and F be Banach spaces of class $\mathcal{HT}$. Assume that $m \in C^1\big(\mathbb{R}\backslash\{0\};\mathcal{B}(E,F)\big)$ satisfies*

$$\mathcal{R}\{m(\lambda), \lambda m'(\lambda) \,:\, \lambda \neq 0\} \leq \kappa.$$

Then $T_m \in \mathcal{B}\big(L_{p,\mu}(\mathbb{R}_+;E), L_{p,\mu}(\mathbb{R}_+;F)\big)$, with norm not exceeding $C(p,\mu,X,Y)\kappa$. ∎

We remark that a corresponding theorem holds true in arbitrary dimensions, and for more general weights from the class A_p.

Under more restrictive assumptions on m we can give a short proof of Theorem 1.2.4, using a result of Kreé [60] which is also the basis for the theorem of [42].

Proposition 1.2.5. *Under the assumptions of Theorem 1.2.4, let m satisfy in addition $m \in C^2\big(\mathbb{R}\backslash\{0\};\mathcal{B}(E,F)\big)$, such that*

$$|m''(\lambda)|_{\mathcal{B}(E,F)} \lesssim |\lambda|^{-2}, \qquad \lambda \neq 0.$$

Then T_m extends to a continuous operator from $L_{p,\mu}(\mathbb{R}_+;E)$ to $L_{p,\mu}(\mathbb{R}_+;F)$.

Proof. It follows from the operator-valued multiplier theorem in the unweighted case that T_m extends to a bounded operator from $L_p(\mathbb{R}_+;E)$ to $L_p(\mathbb{R}_+;F)$.
Moreover, following the lines of the proof of [81, Lemma VI.4.4.2], the assumptions on m yield that T_m may be represented as a convolution operator, with a kernel $k \in C\big(\mathbb{R}\backslash\{0\};\mathcal{B}(E,F)\big)$, satisfying $|k(t)|_{\mathcal{B}(E,F)} \lesssim \frac{1}{|t|}$.
It now follows from [60, Théorème 2] that T_m is also bounded from $L_{p,\mu}(\mathbb{R}_+;E)$ to $L_{p,\mu}(\mathbb{R}_+;F)$, for all $\mu \in (1/p,1]$. ∎

1.3 Weighted Anisotropic Spaces

Let E be a Banach space of class $\mathcal{HT}$, let $J=(0,T)$ be finite or infinite, and let further $\Omega\subset\mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega\in\{\mathbb{R}^n,\mathbb{R}^n_+\}$. In what follows we refer to $t\in\overline{J}$ as time variables, and to $x\in\overline{\Omega}$ as space variables. For $p,q\in[1,\infty]$ and $r>0$ we denote by

$$H^r_p(\Omega;E),\qquad W^r_p(\Omega;E),\qquad B^r_{p,q}(\Omega;E),$$

the E-valued Bessel potential, Slobodetskii and Besov spaces. Recall that $B^r_{p,p}(\Omega;E)=W^r_p(\Omega;E)$ for $p\in[1,\infty)$ and $r\notin\mathbb{N}_0$. The corresponding spaces over the boundary $\partial\Omega$ are defined via local charts. We refer to Appendix A.4 for definitions and properties of these function spaces.

In this section we investigate weighted anisotropic spaces, i.e., intersections of spaces of the form

$$H^s_{p,\mu}\big(J;H^r_p(\Omega;E)\big),\quad W^s_{p,\mu}\big(J;W^r_p(\Omega;E)\big),\quad H^s_{p,\mu}\big(J;W^r_p(\Omega;E)\big),\quad W^s_{p,\mu}\big(J;W^r_p(\Omega;E)\big), \tag{1.3.1}$$

where $s,r\geq 0$. We are further concerned with the corresponding spaces over $J\times\partial\Omega$, and with intersections of spaces where in (1.3.1) $H^s_{p,\mu}$ and $W^s_{p,\mu}$ are replaced by ${}_0H^s_{p,\mu}$ and ${}_0W^s_{p,\mu}$, respectively. We consider the Newton polygon, temporal and spatial trace theorems, and sufficient conditions for pointwise multipliers for these spaces.

We start with two fundamental tools for anisotropic spaces.

The first is a spatial extension operator. Given $k\in\mathbb{N}$, there is an extension operator $\mathcal{E}_\Omega$ to $\mathbb{R}^n$ for functions defined on Ω, i.e., $(\mathcal{E}_\Omega u)|_{\overline{\Omega}}=u$, such that for all $p,q\in(1,\infty)$ and $r\in[0,k]$ it holds

$$\mathcal{E}_\Omega\in\mathcal{B}\big(B^r_{p,q}(\Omega;E),B^r_{p,q}(\mathbb{R}^n;E)\big)\cap\mathcal{B}\big(H^r_p(\Omega;E),H^r_p(\mathbb{R}^n;E)\big). \tag{1.3.2}$$

For integer $r\in[0,k]$, the proof of [1, Theorems 5.21, 5.22] for the scalar-valued spaces literally carries over to the vector-valued case. The general case $r\in[0,k]$ follows from interpolation. Applying $\mathcal{E}_\Omega$ pointwise almost everywhere in time, we obtain a spatial extension operator for the anisotropic spaces, which we denote by $\mathcal{E}_\Omega$ again,

$$\mathcal{E}_\Omega\in\mathcal{B}\big(H^s_{p,\mu}(J;H^r_p(\Omega;E)),H^s_{p,\mu}(J;H^r_p(\mathbb{R}^n;E))\big),\qquad s\geq 0,\quad r\in[0,k]. \tag{1.3.3}$$

Of course, here a H-space may be replaced by a W-space at the first or the second or at both positions, and this remains true for the ${}_0H^s_{p,\mu}$- and the ${}_0W^s_{p,\mu}$-spaces with respect to time.

Second, we consider operators with bounded imaginary powers (cf. Appendix A.3) on the weighted anisotropic spaces for the case $J\times\Omega=\mathbb{R}_+\times\mathbb{R}^n$. This class of operators is crucial for our purposes, in view of the Dore-Venni Theorem A.3.2 and Yagi's theorem (A.3.1).

Lemma 1.3.1. *Let E be of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, $s, r \geq 0$, $\alpha \in (0,2)$ and $\beta > 0$, let $\omega, \omega' \geq 0$ satisfy $\omega + \omega' \neq 0$ and set*

$$H^s_{p,\mu}(H^r_p) := H^s_{p,\mu}\big(\mathbb{R}_+; H^r_p(\mathbb{R}^n; E)\big), \qquad {}_0H^s_{p,\mu}(H^r_p) := {}_0H^s_{p,\mu}\big(\mathbb{R}_+; H^r_p(\mathbb{R}^n; E)\big),$$

and analogously for the other types of spaces in (1.3.1). Let Δ_n be the Laplacian on $\mathbb{R}^n$. Then the following holds true.

a) *The pointwise realization of $(\omega - \Delta_n)^{\beta/2}$ on the spaces*

$$\begin{aligned}
&H^s_{p,\mu}(H^r_p), && \text{with domain } H^s_{p,\mu}(H^{r+\beta}_p), \\
&H^s_{p,\mu}(W^r_p), && \text{with domain } H^s_{p,\mu}(W^{r+\beta}_p), \quad r, r+\beta \notin \mathbb{N}_0, \\
&W^s_{p,\mu}(H^r_p), && \text{with domain } W^s_{p,\mu}(H^{r+\beta}_p), \\
&W^s_{p,\mu}(W^r_p), && \text{with domain } W^s_{p,\mu}(W^{r+\beta}_p), \quad r, r+\beta \notin \mathbb{N}_0,
\end{aligned}$$

is invertible and admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to zero. This remains true if one replaces the $H^s_{p,\mu}, W^s_{p,\mu}$-spaces by the ${}_0H^s_{p,\mu}, {}_0W^s_{p,\mu}$-spaces.

b) *On $L_{p,\mu}(L_p)$, the operators $(\omega' - \partial_t)^\alpha$ and $(\omega' + \partial_t)^\alpha$ commute with $(\omega - \Delta_n)^{\beta/2}$ in the resolvent sense, respectively.*

c) *The operator $L := (\omega' - \partial_t)^\alpha + (\omega - \Delta_n)^{\beta/2}$, considered on the spaces*

$$\begin{aligned}
&H^s_{p,\mu}(H^r_p), && \text{with domain } H^{s+\alpha}_{p,\mu}(H^\beta_p) \cap H^s_{p,\mu}(H^{r+\beta}_p), \\
&H^s_{p,\mu}(W^r_p), && \text{with domain } H^{s+\alpha}_{p,\mu}(W^\beta_p) \cap H^s_{p,\mu}(W^{r+\beta}_p), \quad r, r+\beta \notin \mathbb{N}_0, \\
&W^s_{p,\mu}(H^r_p), && \text{with domain } W^{s+\alpha}_{p,\mu}(H^\beta_p) \cap W^s_{p,\mu}(H^{r+\beta}_p), \quad s, s+\alpha \notin \mathbb{N}_0, \\
&W^s_{p,\mu}(W^r_p), && \text{with domain } W^{s+\alpha}_{p,\mu}(W^\beta_p) \cap W^s_{p,\mu}(W^{r+\beta}_p), \quad s, s+\alpha, r, r+\beta \notin \mathbb{N}_0,
\end{aligned}$$

is invertible and admits bounded imaginary powers, with power angle not larger than $\alpha\pi/2$, respectively. This remains true for the operator $L_0 := (\omega' + \partial_t)^\alpha + (\omega - \Delta_n)^{\beta/2}$ if one replaces the $H^s_{p,\mu}, W^s_{p,\mu}$-spaces by the ${}_0H^s_{p,\mu}, {}_0W^s_{p,\mu}$-spaces.

d) *For $\tau \in (0,1]$ it holds*

$$D(L^\tau) = D((\omega' - \partial_t)^{\alpha\tau}) \cap D((\omega' - \Delta_n)^{\beta\tau/2}),$$

$$D_L(\tau, p) = D_{(\omega' - \partial_t)^\alpha}(\tau, p) \cap D_{(\omega' - \Delta_n)^{\beta/2}}(\tau, p),$$

and this remains true if one replaces L by L_0 and $\omega' - \partial_t$ by $\omega' + \partial_t$.

Proof. (I) Since E is of class $\mathcal{HT}$, the operator $-\Delta_n$ admits on $L_p(\mathbb{R}^n; E)$ with domain $H^2_p(\mathbb{R}^n; E)$ a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to zero, due to [24, Theorem 5.5], for instance. This remains valid for $(\omega - \Delta_n)^{\beta/2}$ with domain $H^\beta_p(\mathbb{R}^n; E)$, due to Lemma A.3.5 and (A.3.1), and further this operator is invertible.
Using $(\omega - \Delta_n)^{r/2}$ as an isomorphism between $H^r_p(\mathbb{R}^n; E)$ and $L_p(\mathbb{R}^n; E)$, it follows from [24, Proposition 2.11] that $(\omega - \Delta_n)^{\beta/2}$ has the same properties on $H^r_p(\mathbb{R}^n; E)$, with domain $H^{r+\beta}_p(\mathbb{R}^n; E)$, $r \geq 0$. By interpolation, these facts remain true if one considers $(\omega - \Delta_n)^{\beta/2}$

on $W_p^r(\mathbb{R}^n;E)$, with domain $W_p^{r+\beta}(\mathbb{R}^n;E)$, provided $r, r+\beta \notin \mathbb{N}_0$. Due to Lemma A.3.6, these properties carry over to its pointwise realizations.
(II) The explicit representation of the resolvents of $\omega'-\partial_t$ and $\omega'+\partial_t$ (see, for instance, [48, Proposition 8.4.1]) yields that on $L_{p,\mu}(L_p)$ these operators are resolvent commuting with $\omega-\Delta_n$, respectively. By [7, Lemma III.4.9.2], this property carries over to the fractional power case.
(III) Since all the spaces under consideration are of class $\mathcal{HT}$, it follows from Proposition 1.1.10 that $(\omega'-\partial_t)^\alpha$ admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to $\alpha\pi/2$ on $H^s_{p,\mu}(H^r_p)$ with domain $H^{s+\alpha}_{p,\mu}(H^r_p)$, and on the corresponding spaces where H is replaced by W, with the asserted exceptions. Using this fact, together with a) and b), the assertions on L are a consequence of the Dore-Venni Theorem A.3.2. The same arguments show the assertion on L_0. Finally, d) is a consequence of the Lemmas A.3.1 and A.3.4. ■

1.3.1 The Newton Polygon

With the help of the operators from Lemma 1.3.1 we establish fundamental embeddings for the anisotropic spaces. The corresponding results for exponentially weighted spaces are obtained in [27, Lemma 4.3].

Proposition 1.3.2. *Let E be of class $\mathcal{HT}$, let $J=(0,T)$ be finite or infinite, $p\in(1,\infty)$, $\mu\in(1/p,1]$, and let $\Omega\subset\mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega\in\{\mathbb{R}^n,\mathbb{R}^n_+\}$. Let further*

$$s,r\geq 0,\qquad \alpha\in(0,2),\qquad \beta>0,\qquad \sigma\in[0,1],$$

and set $H^s_{p,\mu}(H^r_p):=H^s_{p,\mu}\big(J;H^r_p(\Omega;E)\big)$, and analogously for the other anisotropic spaces. Then it holds

$$H^{s+\alpha}_{p,\mu}(H^r_p)\cap H^s_{p,\mu}(H^{r+\beta}_p)\hookrightarrow H^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p), \tag{1.3.4}$$

and moreover each of the spaces

$$H^{s+\alpha}_{p,\mu}(W^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p),\quad W^{s+\alpha}_{p,\mu}(H^r_p)\cap W^s_{p,\mu}(H^{r+\beta}_p),\quad W^{s+\alpha}_{p,\mu}(H^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p),$$

is continuously embedded in

$$W^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p)\cap H^{s+\sigma\alpha}_{p,\mu}(W^{r+(1-\sigma)\beta}_p),$$

provided all the occurring $W_{p,\mu}$- and W-spaces have a noninteger order of differentiability. Finally, assuming all orders of differentiability to be noninteger, it holds

$$W^{s+\alpha}_{p,\mu}(W^r_p)\cap W^s_{p,\mu}(W^{r+\beta}_p)\hookrightarrow W^{s+\sigma\alpha}_{p,\mu}(W^{r+(1-\sigma)\beta}_p). \tag{1.3.5}$$

These embeddings remain true if one replaces Ω by its boundary $\partial\Omega$. They also remain true if one replaces all the $H_{p,\mu}$-, $W_{p,\mu}$- spaces by the ${}_0H_{p,\mu}$-, ${}_0W_{p,\mu}$-spaces. Restricting in the latter case to $s+\alpha\leq 2$, the embedding constants do not depend on the length of J.

Proof. (I) Using extensions and restrictions, and employing that the spaces over $\partial\Omega$ are defined via local charts, it suffices to consider the case $J\times\Omega = \mathbb{R}_+\times\mathbb{R}^n$. The dependence of the embedding constants on J carries over from the properties of the extension operators.
(II) For (1.3.4) we consider the operators $(1-\partial_t)^\alpha$ and $(1-\Delta_n)^{\beta/2}$ on $H^s_{p,\mu}(H^r_p)$, which were treated in Proposition 1.1.10 and Lemma 1.3.1. Note that to obtain sectoriality of $(1-\partial_t)^\alpha$ we have to restrict to $\alpha\in(0,2)$. Due to the invertibility of these operators, for $\sigma\in(0,1)$ it holds that

$$|(1-\partial_t)^{\alpha\sigma}(1-\Delta_n)^{\beta(1-\sigma)/2}\cdot|_{H^s_{p,\mu}(H^r_p)}$$

is an equivalent norm on $H^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p)$. Since the sum of these operators is invertible by Lemma 1.3.1, it further holds that

$$|((1-\partial_t)^{\alpha}+(1-\Delta_n)^{\beta(1-\sigma)/2})\cdot|_{H^s_{p,\mu}(H^r_p)}$$

is an equivalent norm on $H^{s+\alpha}_{p,\mu}(H^r_p)\cap H^s_{p,\mu}(H^{r+\beta}_p)$. Now (1.3.4) follows from the mixed derivative theorem, Lemma A.3.3. The same arguments show

$$H^{s+\alpha}_{p,\mu}(W^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p)\hookrightarrow H^{s+\sigma\alpha}_{p,\mu}(W^{r+(1-\sigma)\beta}_p),$$

$$W^{s+\alpha}_{p,\mu}(H^r_p)\cap W^s_{p,\mu}(H^{r+\beta}_p)\hookrightarrow W^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p),$$

and (1.3.5), with the indicated exceptions. In the following we derive the remaining embeddings from (1.3.4) by suitable interpolation arguments, which were indicated in [37, Remark 5.3] in a more special situation.
(III) For $H^{s+\alpha}_{p,\mu}(W^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p)$ we suppose that $r, r+\beta\notin\mathbb{N}_0$. We apply the real interpolation functor $(\cdot,\cdot)_{1/2,p}$ to the embedding

$$H^{s+\alpha}_{p,\mu}(H^{r\pm\varepsilon}_p)\cap H^s_{p,\mu}(H^{r\pm\varepsilon+\beta}_p)\hookrightarrow H^{s+\alpha(\sigma\pm\varepsilon/\beta)}_{p,\mu}(H^{r+(1-\sigma)\beta}_p), \tag{1.3.6}$$

where $\varepsilon>0$ is sufficiently small. By Lemma 1.1.9 the right-hand sides interpolate to $W^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p)$. To interpolate the left-hand sides above we consider the operator

$$L=(1-\partial_t)^\alpha+(1-\Delta_n)^{\beta/2},$$

which, due to Lemma 1.3.1, is an isomorphism

$$H^{s+\alpha}_{p,\mu}(H^{r\pm\varepsilon}_p)\cap H^s_{p,\mu}(H^{r\pm\varepsilon+\beta}_p)\to H^s_{p,\mu}(H^{r\pm\varepsilon}_p).$$

Hence L is an isomorphism between

$$\big(H^{s+\alpha}_{p,\mu}(H^{r-\varepsilon}_p)\cap H^s_{p,\mu}(H^{r-\varepsilon+\beta}_p), H^{s+\alpha}_{p,\mu}(H^{r+\varepsilon}_p)\cap H^s_{p,\mu}(H^{r+\varepsilon+\beta}_p)\big)_{\frac12,p}$$

and $\big(H^s_{p,\mu}(H^{r-\varepsilon}_p), H^s_{p,\mu}(H^{r+\varepsilon}_p)\big)_{\frac12,p}$, and the latter space equals $H^s_{p,\mu}(W^r_p)$, due to Lemma 1.1.9 and Proposition A.4.2. By Lemma 1.3.1, the operator L^{-1} maps $H^s_{p,\mu}(W^r_p)$ isomorphically to

$$H^{s+\alpha}_{p,\mu}(W^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p).$$

Thus we have shown that the left hand side in (1.3.6) interpolates to $H^{s+\alpha}_{p,\mu}(W^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p)$. For $W^{s+\alpha}_{p,\mu}(H^r_p) \cap W^s_{p,\mu}(H^{r+\beta}_p)$ we have $s, s+\alpha \notin \mathbb{N}_0$. Here we apply $(\cdot,\cdot)_{1/2,p}$ to

$$H^{s\pm\varepsilon+\alpha}_{p,\mu}(H^r_p) \cap H^{s\pm\varepsilon}_{p,\mu}(H^{r+\beta}_p) \hookrightarrow H^{s+\sigma\alpha}_{p,\mu}(H^{r+\beta(1-\sigma\pm\varepsilon/\alpha)}_p).$$

Using the operator L as above yields the asserted embedding in this case.
(IV) For $W^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p)$ we have $s+\alpha, r+\beta \notin \mathbb{N}$. This time we apply $(\cdot,\cdot)_{1/2,p}$ to the embeddings

$$H^{s+\alpha(1\pm\varepsilon/\beta)}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta\pm\varepsilon}_p) \hookrightarrow H^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta\pm\varepsilon}_p),$$

$$H^{s+\alpha(1\pm\varepsilon/\beta)}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta\pm\varepsilon}_p) \hookrightarrow H^{s+\alpha(\sigma\pm\varepsilon/\beta)}_{p,\mu}(H^{r+(1-\sigma)\beta}_p).$$

As above it follows that the right-hand sides interpolate to $H^{s+\sigma\alpha}_{p,\mu}(W^{r+(1-\sigma)\beta}_p)$ and $W^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_p)$, respectively. To interpolate the left-hand side, we consider on $H^s_{p,\mu}(H^r_p)$ the operator

$$L = (1-\partial_t)^{\alpha(1+\varepsilon/\beta)} + (1-\Delta)^{(\beta+\varepsilon)/2},$$

with domain $D(L) = H^{s+\alpha(1+\varepsilon/\beta)}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta+\varepsilon}_p)$. Due to the Lemmas 1.1.9, 1.3.1 and Proposition A.4.2 it holds

$$D(L^{(\beta-\varepsilon)/(\beta+\varepsilon)}) = H^{s+\alpha(1-\varepsilon/\beta)}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta-\varepsilon}_p),$$

and the reiteration theorem yields

$$(D(L^{(\beta-\varepsilon)/(\beta+\varepsilon)}), D(L))_{1/2,p} = D_L((1+(\beta-\varepsilon)/(\beta+\varepsilon))/2, p).$$

Finally, the Lemmas 1.1.9, 1.3.1 and Proposition A.4.2 imply that the latter space equals $W^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(W^{r+\beta}_p)$.
(V) Starting in Step II with $(1+\partial_t)^\alpha$ instead of $(1-\partial_t)^\alpha$, the same arguments as above show that the asserted embeddings are also true for the ${}_0H_{p,\mu}$- and ${}_0W_{p,\mu}$-spaces. ■

Remark 1.3.3. The proof shows that for the embeddings where only the mixed derivative theorem was used the orders of integrability in space and time do not have to coincide. In fact, considering the Laplacian on $H^r_q(\mathbb{R}^n; E)$ for $q \in (1,\infty)$, and realizing it on $H^s_{p,\mu}(J; H^r_q(\mathbb{R}^n; E))$ for $p \in (1,\infty)$ and $\mu \in (1/p, 1]$, the assertions of Lemma 1.3.1 remain true. Then as in Step II of the above proof we obtain, for instance,

$$H^{s+\alpha}_{p,\mu}(H^r_q) \cap H^s_{p,\mu}(H^{r+\beta}_q) \hookrightarrow H^{s+\sigma\alpha}_{p,\mu}(H^{r+(1-\sigma)\beta}_q),$$

$$W^{s+\alpha}_{p,\mu}(W^r_q) \cap W^s_{p,\mu}(W^{r+\beta}_q) \hookrightarrow W^{s+\sigma\alpha}_{p,\mu}(W^{r+(1-\sigma)\beta}_q),$$

with noninteger orders of differentiability in the W-case and uniform embeddings in the ${}_0H_{p,\mu}$- and ${}_0W_{p,\mu}$-case. ■

The above embeddings turn out to be extremely useful in the sequel. They can be visualized by the Newton polygon. Suppose that an anisotropic space $\mathbb{X}$ of the form

$$\mathbb{X} = \bigcap_{j=1}^{m} H_p^{s_j}\big(J; H_p^{r_j}(\Omega; E)\big),$$

where $0 \le r_1 < ... < r_m$ and $s_j \ge 0$, is given. Consider each space $H_p^{s_j}(J; H_p^{r_j}(\Omega; E))$ as a point (r_j, s_j) in a space-time-regularity diagram, and draw the convex hull of these points with respect to the boundary of the positive cone.

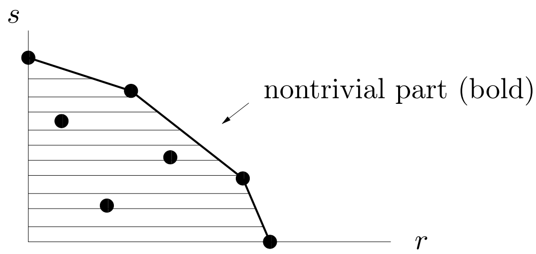

Figure 1.1: The Newton polygon

This hull is called the Newton polygon $\mathcal{NP}$ for $\mathbb{X}$, and the lines on the hull connecting points (r_j, s_j) (including these points) is called the nontrivial part of $\mathcal{NP}$. Proposition 1.3.2 and trivial embeddings in space and time yield that $\mathbb{X}$ embeds into each space $H_p^s(J; H_p^r(\Omega; E))$ for which (r, s) lies inside the Newton polygon. Of course, here one may replace the H-spaces by the W-spaces according to the above result.

A typical application of Proposition 1.3.2 is the following proof of the mapping behaviour of the spatial derivative on anisotropic H-spaces. See [24, Lemma 3.8] for the unweighted case.

Lemma 1.3.4. *Let E be a Banach space of class $\mathcal{HT}$, let $J = (0, T)$ be finite or finite, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or $\Omega \in \{\mathbb{R}^n_+, \mathbb{R}^n\}$. Let further*

$$s \ge 0, \qquad r \in [0, 1), \qquad \alpha \in (0, 2), \qquad \beta \ge 1.$$

Then the pointwise realization of ∂_{x_i}, $i \in \{1, ..., n\}$, is a continuous map

$$H^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta}_p) \to H^{s+\alpha-\alpha/\beta}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta-1}_p).$$

Restricting to $s + \alpha \le 2$, and further to ${}_0H^{s+\alpha}_{p,\mu}$- and ${}_0H^s_{p,\mu}$-spaces in time, its operator norm is independent of the length of J.

Proof. By extension and restriction it suffices to consider the case $J \times \Omega = \mathbb{R}_+ \times \mathbb{R}^n$. Clearly the operator ∂_{x_i} maps continuously

$$H^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta}_p) \to H^s_{p,\mu}(H^{r+\beta-1}_p).$$

It further follows from Proposition 1.3.2 that the embedding

$$H^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta}_p) \hookrightarrow H^{s+\alpha-\alpha/\beta}_{p,\mu}(H^{r+1}_p)$$

is valid, and thus ∂_{x_i} also maps

$$H^{s+\alpha}_{p,\mu}(H^r_p) \cap H^s_{p,\mu}(H^{r+\beta}_p) \to H^{s+\alpha-\alpha/\beta}_{p,\mu}(H^r_p)$$

in a continuous way. ■

1.3.2 Temporal Traces

We now consider the temporal trace for anisotropic spaces. Using integration by parts, it is not hard to see that for a Banach space X and $u \in W^1_{1,\mathrm{loc}}([0,\infty); X)$ the representation

$$u(0) = (2-\mu)\Big(\sigma^{-(2-\mu)} \int_0^\sigma \tau^{1-\mu} u(\tau)\,\mathrm{d}\tau - (2-\mu)\int_0^\sigma t^{-(3-\mu)} \int_0^t \tau^{1-\mu}(u(t)-u(\tau))\,\mathrm{d}\tau\,\mathrm{d}t\Big) \tag{1.3.7}$$

holds true for all $\sigma > 0$. By A.2 l), if $-A$ is the generator of an exponentially stable analytic C_0-semigroup then for $\theta \in (0,1)$ the norm in $D_A(\theta,p)$ is equivalent to $|\cdot|_{D_A(\theta,p),*}$, where

$$|x|^p_{D_A(\theta,p),*} = \int_0^\infty \sigma^{p(1-\theta)} |Ae^{-\sigma A}x|^p_X \frac{\mathrm{d}\sigma}{\sigma}. \tag{1.3.8}$$

The representation (1.3.7) is the key to the following abstract trace theorem, whose proof follows Di Blasio [29].

Lemma 1.3.5. *Let X be a Banach space, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and let the operator A on X with domain $D(A)$ be invertible and admit bounded imaginary powers with power angle strictly smaller than $\pi/2$. Let $s \in (0, 1-\mu+1/p)$ and $\alpha > 0$ satisfy $s+\alpha \in (1-\mu+1/p, 1)$. Then the temporal trace* tr_0, *i.e.,* $\mathrm{tr}_0 u = u(0)$, *maps continuously*

$$W^{s+\alpha}_{p,\mu}\big(\mathbb{R}_+; D(A^s)\big) \cap W^s_{p,\mu}\big(\mathbb{R}_+; D(A^{s+\alpha})\big) \to D_A\big(2s+\alpha-(1-\mu+1/p), p\big). \tag{1.3.9}$$

Moreover, tr_0 *is for $\alpha \in (1-\mu+1/p, 1]$ continuous*

$$W^\alpha_{p,\mu}\big(\mathbb{R}_+; X\big) \cap L_{p,\mu}\big(\mathbb{R}_+; D_A(\alpha,p)\big) \to D_A\big(\alpha-(1-\mu+1/p), p\big), \tag{1.3.10}$$

and for $s \in (0, 1-\mu+1/p)$ it is continuous

$$W^1_{p,\mu}\big(\mathbb{R}_+; D_A(s,p)\big) \cap W^s_{p,\mu}\big(\mathbb{R}_+; D(A)\big) \to D_A\big(1+s-(1-\mu+1/p), p\big).^8 \tag{1.3.11}$$

Proof. The proofs of (1.3.10) and (1.3.11) are very similar to the Lemmas 11 and 12 of [29], starting with (1.3.7) and using the representation (1.1.17) of the weighted Slobodetskii seminorm and Hardy's inequality (Lemma 1.1.2). We therefore concentrate on (1.3.9).
By assumption and Proposition 1.1.11 it holds

$$|u(0)|_X \lesssim |u|_{W^{s+\alpha}_{p,\mu}(\mathbb{R}_+;D(A^s))}, \qquad u \in W^{s+\alpha}_{p,\mu}(\mathbb{R}_+; D(A^s)).$$

[8] The proofs of (1.3.10) and (1.3.11) only require that $-A$ generates an exponentially stable analytic C_0-semigroup.

We further use (1.3.7) and (1.3.8) to obtain

$$
\begin{aligned}
|u(0)|^p_{D_A(2s+\alpha-(1-\mu+1/p),p),*} &= \int_0^\infty \sigma^{p(1-(2s+\alpha-(1-\mu+1/p))}|Ae^{-\sigma A}u(0)|^p_X \sigma^{-1}\,\mathrm{d}\sigma \\
&\lesssim \int_0^\infty \sigma^{-p(2s+\alpha)}\Big(\int_0^\sigma \tau^{1-\mu}|Ae^{-\sigma A}u(\tau)|\,\mathrm{d}\tau\Big)^p\,\mathrm{d}\sigma \qquad (1.3.12)\\
&\quad + \int_0^\infty \Big(\sigma^{2-\mu-(2s+\alpha)}\int_0^\sigma t^{-(3-\mu)}\int_0^t \tau^{1-\mu}|Ae^{-\sigma A}(u(t)-u(\tau))|\,\mathrm{d}\tau\,\mathrm{d}t\Big)^p\,\mathrm{d}\sigma.
\end{aligned}
$$

It follows from A.2 i) that for $\theta \in (0,1)$ we have

$$
|Ae^{-\sigma A}x|_{\mathcal{B}(X)} \lesssim \sigma^{-1+\theta}|A^\theta x|_{\mathcal{B}(X)}, \qquad x \in X. \qquad (1.3.13)
$$

Using Hölder's inequality, (1.3.13), (1.1.17), Hardy's inequality (Lemma 1.1.2) and Proposition 1.1.11, we estimate the first summand in (1.3.12) by

$$
\begin{aligned}
&\int_0^\infty \sigma^{-p(2s+\alpha)}\Big(\int_0^\sigma \tau^{1-\mu}|Ae^{-\sigma A}u(\tau)|\,\mathrm{d}\tau\Big)^p\,\mathrm{d}\sigma \\
&\quad \le \int_0^\infty\int_0^\sigma \tau^{p(1-\mu)}|Ae^{-\sigma A}u(\tau)|^p\sigma^{p-1}\sigma^{-p(2s+\alpha)}\,\mathrm{d}\tau\,\mathrm{d}\sigma \\
&\quad \lesssim \int_0^\infty\int_0^\sigma \tau^{p(1-\mu)}|A^{s+\alpha}u(\tau)|^p\sigma^{-(1+ps)}\,\mathrm{d}\tau\,\mathrm{d}\sigma \\
&\quad \lesssim \int_0^\infty \Big(\int_0^\sigma \tau^{p(1-\mu)}|(u(\sigma)-u(\tau))|^p_{D(A^{s+\alpha})}\sigma^{-(1+ps)}\,\mathrm{d}\tau + \sigma^{p(1-\mu-s)}|u(\sigma)|^p_{D(A^{s+\alpha})}\Big)\,\mathrm{d}\sigma \\
&\quad \lesssim |u|^p_{W^s_{p,\mu}(\mathbb{R}_+;D(A^{s+\alpha}))}.
\end{aligned}
$$

We further use (1.3.13), the Hardy-Young inequality (A.2.1), Hölder's inequality and (1.1.17) to estimate the second summand in (1.3.12),

$$
\begin{aligned}
&\int_0^\infty \Big(\sigma^{2-\mu-(2s+\alpha)}\int_0^\sigma t^{-(3-\mu)}\int_0^t \tau^{1-\mu}|Ae^{-\sigma A}(u(t)-u(\tau))|\,\mathrm{d}\tau\,\mathrm{d}t\Big)^p\,\mathrm{d}\sigma \\
&\quad \lesssim \int_0^\infty \Big(\sigma^{-(s+\alpha-(1-\mu+1/p))}\int_0^\sigma t^{-1}\Big(t^{-(2-\mu)}\int_0^t \tau^{1-\mu}|u(t)-u(\tau)|_{D(A^s)}\,\mathrm{d}\tau\Big)\,\mathrm{d}t\Big)^p\sigma^{-1}\,\mathrm{d}\sigma \\
&\quad \lesssim \int_0^\infty \sigma^{-p(s+\alpha-(1-\mu+1/p))}\sigma^{-p(2-\mu)}\Big(\int_0^\sigma \tau^{1-\mu}|u(\sigma)-u(\tau)|_{D(A^s)}\,\mathrm{d}\tau\Big)^p\sigma^{-1}\,\mathrm{d}\sigma \\
&\quad \le \int_0^\infty\int_0^\sigma \tau^{p(1-\mu)}|u(\sigma)-u(\tau)|^p_{D(A^s)}\sigma^{-(1+p(s+\alpha))}\,\mathrm{d}\tau\,\mathrm{d}\sigma \le [u]^p_{W^{s+\alpha}_{p,\mu}(\mathbb{R}_+;D(A^s))},
\end{aligned}
$$

which shows (1.3.9). ■

From the above lemma we deduce a general trace theorem for the weighted anisotropic spaces. We refer to [89, Theorem 3.2.1] for the unweighted case, and to [27, Lemma 4.4] for anisotropic spaces with exponential weights.

Theorem 1.3.6. *Let E be a Banach space of class $\mathcal{HT}$, let $J=(0,T)$ be finite or infinite, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Assume $r \ge 0$, $\beta > 0$, and suppose that $k \in \mathbb{N}_0$, $s \ge 0$, and $\alpha \in (0,2)$ satisfy*

$$
k-\mu+1/p < s < k+1-\mu+1/p < s+\alpha.
$$

Set $H^s_{p,\mu}(W^r_p) := H^s_{p,\mu}(J; W^r_p(\Omega;E))$, and analogously for the other anisotropic spaces. Throughout, assume that the orders of differentiability of all occurring $W_{p,\mu}$- and W-spaces are noninteger. Then each of the spaces

$$H^{s+\alpha}_{p,\mu}(W^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p),\quad W^{s+\alpha}_{p,\mu}(H^r_p)\cap W^s_{p,\mu}(H^{r+\beta}_p),\quad W^{s+\alpha}_{p,\mu}(H^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p), \tag{1.3.14}$$

is continuously embedded into

$$BUC^k\big(\overline{J}, B^{r+\beta(1+(s-(k+1-\mu+1/p))/\alpha)}_{p,p}(\Omega;E)\big). \tag{1.3.15}$$

Moreover, for $\alpha \le 1$ it holds

$$W^\alpha_{p,\mu}(W^r_p)\cap L_{p,\mu}(W^{r+\beta}_p)\hookrightarrow BUC\big(\overline{J}, B^{r+\beta(1-(1-\mu+1/p)/\alpha)}_{p,p}(\Omega;E)\big), \tag{1.3.16}$$

$$W^1_{p,\mu}(W^r_p)\cap W^s_{p,\mu}(W^{r+\beta}_p)\hookrightarrow BUC\big(\overline{J}, B^{r+\beta(\mu-1/p)/(1-s)}_{p,p}(\Omega;E)\big). \tag{1.3.17}$$

All these embeddings remain true if one replaces Ω by its boundary $\partial\Omega$. Restricting to $s+\alpha\le 2$ and ${}_0H_{p,\mu}$- resp. ${}_0W_{p,\mu}$-spaces in time, the embedding constants are independent of the length of J.

Proof. (I) Using extensions and restrictions, it again suffices to treat the case $J\times\Omega = \mathbb{R}_+\times\mathbb{R}^n$. We only have to consider the case $k=0$, since for $k\ge 1$ it holds, due to $s > k-\mu+1/p$, (1.1.1) and Proposition 1.1.11,

$$H^s_{p,\mu}(H^{r+\beta}_p)\cap H^s_{p,\mu}(W^{r+\beta}_p)\cap W^s_{p,\mu}(H^{r+\beta}_p)\hookrightarrow BUC^{k-1}(\overline{J}; H^{r+\beta}_p),$$

and the latter space embeds into (1.3.15). We further claim that the proof of the asserted embedding reduces to show that the temporal trace operator $\mathrm{tr}_0 u = u(0)$ maps each of the five spaces under consideration continuously into

$$Y := B^{r+\beta(1+(s-(k+1-\mu+1/p))/\alpha)}_{p,p}(\mathbb{R}^n;E),$$

where one has to set $s=k=0$ for (1.3.16) and $k=0$, $\alpha = 1-s$, for (1.3.17). To see this, note that for a function u we have

$$u(t) = \mathrm{tr}_0\Lambda_t u,\qquad t\ge 0,$$

where Λ_t denotes the left translation by t. Due to Lemma 1.1.6, the family of left translations forms on each space $W^\kappa_{p,\mu}$, $H^\kappa_{p,\mu}$, $\kappa\ge 0$, a strongly continuous semigroup of contractions. We thus have for $t>\tau\ge 0$, assuming that tr_0 is continuous,

$$|u(t)-u(\tau)|_Y \lesssim |\Lambda_{t-\tau}u - u|_{\mathbb{Y}},$$

where $\mathbb{Y}$ stands for any of the spaces under consideration. This shows uniform continuity and boundedness of u with values in Y.

(II) We show the asserted continuity of tr_0 on the space spaces in (1.3.14). It follows from Proposition 1.3.2 that

$$H^{s+\alpha}_{p,\mu}(W^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p)\hookrightarrow W^{s+(1-\varepsilon)\alpha}_{p,\mu}(H^{r+\varepsilon\beta}_p)\cap W^{s+\varepsilon\alpha}_{p,\mu}(H^{r+(1-\varepsilon)\beta}_p).$$

$$W^{s+\alpha}_{p,\mu}(H^r_p)\cap H^s_{p,\mu}(W^{r+\beta}_p)\hookrightarrow W^{s+(1-\varepsilon)\alpha}_{p,\mu}(H^{r+\varepsilon\beta}_p)\cap W^{s+\varepsilon\alpha}_{p,\mu}(H^{r+(1-\varepsilon)\beta}_p),$$

where $\varepsilon > 0$ is sufficiently small. Since it is asserted that the spaces on the left- and the right-hand side above have the same trace spaces, it suffices to consider tr_0 on

$$W^{s+\alpha}_{p,\mu}(H^r_p)\cap W^s_{p,\mu}(H^{r+\beta}_p).$$

Moreover, using again Proposition 1.3.2, the same argument shows that it suffices to consider the case $s+\alpha<1$. We apply (1.3.9) with

$$X = H^{r-s\beta/\alpha}_p,\qquad A=(1-\Delta_n)^{\beta/2\alpha},\qquad D(A)=H^{r+(1-s)\beta/\alpha}_p.$$

Since the operator $1-\Delta_n$ admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to zero on the whole H-scale, it follows from (A.3.1), Lemma A.3.5 and Proposition A.4.2 that

$$D(A^s)=H^r_p,\qquad D(A^{s+\alpha})=H^{r+\beta}_p.$$

Thus Lemma 1.3.5 implies that tr_0 maps continuously

$$W^{s+\alpha}_{p,\mu}(H^r_p)\cap W^s_{p,\mu}(H^{r+\beta}_p)\to D_A\big(2s+\alpha-(1-\mu+1/p),p\big)=B^{r+\beta(1+(s-(k+1-\mu+1/p))/\alpha)}_{p,p}.$$

(III) It follows from real interpolation that the operator $1-\Delta_n$ has the same properties on the B-scale as on the H-scale. We may therefore use (1.3.10), applied to

$$X=B^r_{p,p},\qquad A=(1-\Delta_n)^{\beta/2\alpha},\qquad D(A)=B^{r+\beta/\alpha}_{p,p},$$

giving $D_A(\alpha-(1-\mu+1/p),p)=B^{r+\beta(1-(1-\mu+1/p)/\alpha)}_{p,p}$, to obtain (1.3.16). Similarly, applying (1.3.11) with

$$X=B^{r-s\beta/(1-s)}_{p,p},\qquad A=(1-\Delta_n)^{\beta/2(1-s)},\qquad D(A)=W^{r+\beta}_p,$$

giving $D_A(1+s-(1-\mu+1/p),p)=B^{r+\beta(\mu-1/p)/(1-s)}_{p,p}$, yields (1.3.17). ■

The above theorem can again be visualized by the Newton polygon, cf. Figure 1.3.2. Consider, for instance, the space $\mathbb{X}=W^{s+\alpha}_{p,\mu}(H^r_p)\cap W^s_{p,\mu}(H^{r+\beta}_p)$, where s, r, α and β are as above. Then the temporal trace space of $\mathbb{X}$ is obtained by intersecting the horizontal line $(\tau,1-\mu+1/p)$, $\tau\in\mathbb{R}$, with the nontrivial part of the Newton polygon $\mathcal{NP}$ corresponding to $\mathbb{X}$.

Remark 1.3.7. In the situation of Theorem 1.3.6, one can also consider the case

$$k_1-\mu+1/p<s<k_2+1-\mu+1/p<s+\alpha,\qquad 0\le k_1\le k_2,\quad k_1,k_2\in\mathbb{N}_0.$$

This case can be reduced to $k_1=k_2$, where the theorem is applicable, using Proposition 1.3.2. Here one has the choice between high temporal and low spatial regularity and vice versa.

Using arguments as in the proof of [27, Theorem 4.5], one should be able to show that the temporal trace is surjective, for all of the spaces under consideration in the Theorem 1.3.6. At this point we only consider a right-inverse in a special case. We also refer to Lemma 3.2.2, where we consider a right-inverse for the boundary spaces from Chapter 3.

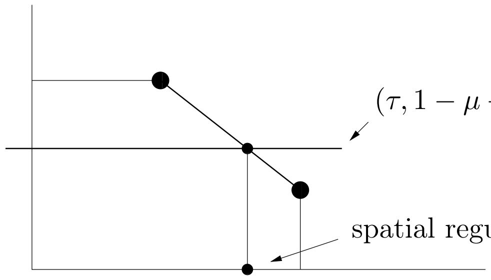

Figure 1.2: The trace space and the Newton polygon

Lemma 1.3.8. *Let E be a Banach space, let $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and let $-A$ be the generator of an exponentially stable analytic C_0-semigroup on E, with domain $D(A)$. Let further $\alpha > 1-\mu+1/p$ with $\alpha - (1-\mu+1/p) \notin \mathbb{N}$. Then it holds*

$$|e^{-\cdot A}x|_{W^{\alpha}_{p,\mu}(\mathbb{R}_+;E)\cap L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))} \lesssim |x|_{D_A(\alpha-(1-\mu+1/p),p)}.^{9}$$

Proof. (I) First let $\alpha \in \mathbb{N}$. Using (1.3.8), for $x \in D_A(\alpha-(1-\mu+1/p),p)$ we obtain

$$\begin{aligned} |e^{-\cdot A}x|_{L_{p,\mu}(\mathbb{R}_+;D(A^{\alpha}))} &= |Ae^{\cdot A}A^{\alpha-1}x|_{L_{p,\mu}(\mathbb{R}_+;E)} \\ &\lesssim |A^{\alpha-1}x|_{D_A(\mu-1/p,p)} = |x|_{D_A(\alpha-(1-\mu+1/p),p)}. \end{aligned}$$

Since $\partial_t^k e^{-\cdot A}x = (-A)^k e^{-\cdot A}x$ for $k \le \alpha$, we further have

$$|e^{-\cdot A}x|_{W^{\alpha}_{p,\mu}(\mathbb{R}_+;E)} \lesssim |e^{-\cdot A}x|_{L_{p,\mu}(\mathbb{R}_+;D(A^{\alpha}))},$$

which shows the assertion for integer α.
(II) We now consider the case $1-\mu+1/p < \alpha < 1$, and show $e^{-\cdot A}x \in L_{p,\mu}(\mathbb{R}_+; D_A(\alpha,p))$. Take $x \in D_A\big(\alpha-(1-\mu+1/p),p\big)$. Then it holds $|e^{-\cdot A}x|_{L_{p,\mu}(\mathbb{R}_+;E)} \lesssim |x|_E$, due to the exponential stability of the semigroup. Moreover we have

$$|e^{-\cdot A}x|^p_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))} = \int_0^\infty \int_0^\infty s^{p(1-\mu)} t^{p(1-\alpha)} |Ae^{-(t+s)A}x|_E^p \,\mathrm{d}s\, \frac{\mathrm{d}t}{t}.$$

We split the inner integral at $s=t$ and estimate the first summand with some small $\varepsilon > 0$ by

$$\begin{aligned} &\int_0^\infty \int_0^t s^{p(1-\mu)} t^{p(1-\alpha)} |\, Ae^{-(t+s)A}x|_E^p \,\mathrm{d}s \frac{\mathrm{d}t}{t} \\ &\qquad\lesssim \int_0^\infty t^{p(1-\alpha+(1-\mu+1/p))} |Ae^{-tA}x|_E^p \left(\frac{1}{t}\int_0^t e^{-\varepsilon p s}\,\mathrm{d}s\right) \frac{\mathrm{d}t}{t} \\ &\qquad\lesssim [x]^p_{D_A(\alpha-(1-\mu+1/p),p)}. \end{aligned}$$

[9] Recall that $D_A(\alpha,p) = D(A^{\alpha})$ for $\alpha \in \mathbb{N}_0$.

For the second summand we use the Hardy-Young inequality (A.2.2), to estimate

$$\begin{aligned}\int_0^\infty \int_t^\infty s^{p(1-\mu)} t^{p(1-\alpha)} | Ae^{-(t+s)A} x|_E^p \, \mathrm{d}s \, \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty t^{p(1-\alpha)} \left(\int_t^\infty s^{p(1-\mu+1)} |Ae^{-sA}x|_E^p \frac{\mathrm{d}s}{s} \right) \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty s^{p(1-\alpha+(1-\mu+1/p))} |Ae^{-sA}x|_E^p \, \frac{\mathrm{d}s}{s} \\ &= [x]^p_{D_A(\alpha-(1-\mu+1/p),p)}.\end{aligned}$$

We thus obtain for $1-\mu+1/p<\alpha<1$ that

$$|e^{-\cdot A}x|_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))} \lesssim |x|_{D_A(\alpha-(1-\mu+1/p),p)}. \tag{1.3.18}$$

Now let $\alpha>1$ with $\alpha\notin\mathbb{N}$. Then we choose $\beta>0$ such that $1-\mu+1/p<\alpha-\beta<1$, which yields $A^\beta x\in D_A(\alpha-\beta-(1-\mu+1/p),p)$ by the reiteration theorem. It now follows from (1.3.18) that

$$\begin{aligned}|e^{-\cdot A}x|_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))} &\lesssim |e^{-\cdot A}A^\beta x|_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha-\beta,p))} \\ &\lesssim |A^\beta x|_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha-\beta-(1-\mu+1/p),p))} \lesssim |x|_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha-(1-\mu+1/p),p))},\end{aligned}$$

and therefore (1.3.18) holds for all $\alpha>1-\mu+1/p$.

(III) For the temporal regularity, observe that

$$\partial_t^k e^{-\cdot A}x=(-A)^k e^{-\cdot A}x, \qquad k\in\{0,...,[\alpha]\},$$

where $x\in D_A(\alpha-(1-\mu+1/p),p)$. In view of the exponential stability of the semigroup, the representations (1.1.18) for the norm of $W^\theta_{p,\mu}(\mathbb{R}_+;E)$ and A.2 l) for the norm of $D_A(\alpha-(1-\mu+1/p),p)$ we have

$$\begin{aligned}|A^k e^{-\cdot A}x|^p_{W^\theta_{p,\mu}(\mathbb{R}_+;E)} &\sim |A^k e^{-\cdot A}x|_E^p + \int_0^\infty\int_0^\infty s^{p(1-\mu)}t^{-\theta p}|A^k e^{-(t+s)A}x - A^k e^{sA}x|_E^p \, \mathrm{d}s \frac{\mathrm{d}t}{t} \\ &\sim |A^k e^{-\cdot A}x|^p_{L_{p,\mu}(\mathbb{R}_+;D_A(\theta,p))}\end{aligned}$$

for $k=[\alpha]$ and $\theta\in(0,1)$. This yields that for $\alpha>1-\mu+1/p$ with $\alpha\notin\mathbb{N}$, using (1.1.8) and the estimates of Step II,

$$\begin{aligned}|e^{-\cdot A}x|_{W^\alpha_{p,\mu}(\mathbb{R}_+;E)} &\lesssim \sum_{k=0}^{[\alpha]} |\partial_t^k e^{-\cdot A}x|_{L_{p,\mu}(\mathbb{R}_+;E)} + |\partial_t^{[\alpha]} e^{-\cdot A}x|_{W^{\alpha-[\alpha]}_{p,\mu}(\mathbb{R}_+;E)} \\ &\lesssim |e^{-\cdot A}x|_{L_{p,\mu}(\mathbb{R}_+;D_A(\alpha,p))} \lesssim |x|_{D_A(\alpha-(1-\mu+1/p),p)},\end{aligned}$$

which finishes the proof. ■

An immediate application of the above result yields a continuous right-inverse of the temporal trace for weighted anisotropic spaces arising in the context of maximal $L_{p,\mu}$-regularity.

Lemma 1.3.9. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, $m \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$. Then a right-inverse for* tr_0 *that is continuous*

$$B_{p,p}^{2m(\mu-1/p)}(\Omega; E) \to W_{p,\mu}^1(\mathbb{R}_+; L_p(\Omega; E)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{2m}(\Omega; E))$$

is given by

$$t \mapsto \mathcal{R}_{\mathbb{R}^n} e^{-t(1-\Delta_n)^m} \mathcal{E}_\Omega u_0, \qquad u_0 \in B_{p,p}^{2m(\mu-1/p)}(\Omega; E).$$

Here $\mathcal{E}_\Omega$ is the extension operator to $\mathbb{R}^n$ from (1.3.2), and $\mathcal{R}_{\mathbb{R}^n}$ denotes the restriction from $\mathbb{R}^n$ to Ω.

1.3.3 Spatial Traces

We now specialize to weighted anisotropic spaces of the form

$$H_{p,\mu}^{s,2ms}(J \times \Omega; E) := H_{p,\mu}^s(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; H_p^{2ms}(\Omega; E)), \tag{1.3.19}$$

where $m \in \mathbb{N}$ and $s \in (0, 1]$, and to the corresponding spaces where H is replaced by W. Our motivation is to investigate the mapping properties of a boundary differential operator $\mathrm{tr}_\Omega \nabla^\beta$ with $\beta \in \mathbb{N}_0^n$ and $|\beta| \le 2m-1$, where tr_Ω and ∇ denote the spatial trace, i.e., $\mathrm{tr}_\Omega u = u|_{\partial\Omega}$, and the euclidian gradient on $\mathbb{R}^n$, respectively. The iterative application of Lemma 1.3.4 implies that ∇^β maps the maximal regularity space

$$W_{p,\mu}^1(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W_p^{2m}(\Omega; E))$$

continuously into

$$H_{p,\mu}^{1-|\beta|/2m}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; H_p^{2m-|\beta|}(\Omega; E)),$$

which is a space as in (1.3.19) with $s = 1 - |\beta|/2m$. We are therefore led to investigate the properties of tr_Ω on a space like (1.3.19). We follow the proof of [25, Lemma 3.5]. For the spatial trace on unweighted anisotropic spaces we also refer to [11, Chapter 4].

For our further considerations we assume that

$$2ms \in \mathbb{N}.$$

It is known that the trace operator tr_Ω, which is originally only defined on $C_c^\infty(\mathbb{R}^n; E)$, extends uniquely to a continuous map

$$H_p^{2ms}(\Omega; E) \to W_p^{2ms-1/p}(\partial\Omega; E). \tag{1.3.20}$$

This can be seen as in [82, Theorems 2.9.3, 4.7.1] for the scalar-valued case. Applied pointwise almost everywhere in time, tr_Ω extends further to a continuous map

$$L_{p,\mu}(J; H_p^{2ms}(\Omega; E)) \to L_{p,\mu}(J; W_p^{2ms-1/p}(\partial\Omega; E)).$$

Observe that Proposition 1.3.2 yields the embedding

$$H_{p,\mu}^{s,2ms}(J \times \Omega; E) \hookrightarrow H_{p,\mu}^{s-1/2mp}(J; H_p^{1/p}(\Omega; E)).$$

Although tr_Ω is not continuous from $H_p^{1/p}(\Omega; E)$ to $L_p(\partial\Omega; E)$, this fact suggests that tr_Ω maps $H_{p,\mu}^{s,2ms}(J \times \Omega; E)$ into $H_{p,\mu}^{s-1/2mp}(J; L_p(\partial\Omega; E))$.

To give a rigorous proof, the following simple density result is useful.

Lemma 1.3.10. *Let $J = (0,T)$ be a finite or infinite interval, let E be a Banach space, and let D be a dense subset of E. Then the set $\mathrm{Step}(J; D)$, consisting step functions of the form*

$$\phi = \sum_{i=1}^{l} \alpha_i(\cdot)\phi_i, \qquad \alpha_i \in C_c(J), \quad \phi_i \in D, \quad l \in \mathbb{N},$$

is dense in $L_{p,\mu}(J; E)$.

Proof. Since $C_c(J; E)$ is dense in $L_{p,\mu}(J; E)$, it suffices to approximate functions from this set. Let $\varepsilon > 0$ be given and take $u \in C_c(J; E)$, such that $\mathrm{supp}\, u \subset (a, b)$ for some $a, b \in J$. Choose numbers $a = t_1 < ... < t_{l-1} < t_l = b$, $l \in \mathbb{N}$, with

$$|u(t) - u(t_i)|_E < \varepsilon \qquad \text{for } t \in [t_i, t_{i+1}], \quad i = 1, ..., l.$$

By assumption, for each i there is $\varphi_i \in D$ such that $|u(t_i) - \varphi_i|_E < \varepsilon$, where we can take $\varphi_1 = \varphi_l = 0$. Now define $\phi \in \mathrm{Step}(J; D)$ by

$$\phi(t) = \sum_{i=1}^{l} \mathbb{1}_{[t_i, t_{i+1})}(t) \frac{(t_{i+1} - t)\varphi_i + (t - t_i)\varphi_{i+1}}{t_{i+1} - t_i}, \qquad t \in J.$$

Then $|u - \phi|_{L_\infty(J;E)} < 2\varepsilon$, and thus

$$|u - \phi|_{L_{p,\mu}(J;E)} < 2b^{p(1-\mu)}(b - a)\varepsilon.$$

Since a and b only depend on u, the assertion follows. ■

Let us now assume that

$$J \times \Omega = \mathbb{R}_+ \times \mathbb{R}^n_+.$$

For this case we describe an alternative representation of $\mathrm{tr}_{\mathbb{R}^n_+}$. In the sequel we write

$$x = (x', y) \in \mathbb{R}^n_+, \qquad x' \in \mathbb{R}^{n-1}, \quad y \in \mathbb{R}_+.$$

Considering a function $u = u(t, x', y)$ on $\mathbb{R}_+ \times \mathbb{R}^n_+$ as a function of $y \in \mathbb{R}_+$ with values in the functions of $(t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$, Fubini's theorem yields the embedding

$$\iota_1 : L_{p,\mu}\big(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}^n_+; E)\big) \hookrightarrow H_p^{2ms}\big(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))\big).$$

Thus, since $2ms \geq 1$, the trace $\mathrm{tr}_0 := \mathrm{tr}_{y=0}$ acts on $L_{p,\mu}(\mathbb{R}_+; H_p^{2m\tau}(\mathbb{R}^n_+; E))$ via $\mathrm{tr}_0 \circ \iota_1$, and maps this space continuously into $L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$. For $\phi \in \mathrm{Step}(\mathbb{R}_+; C_c^\infty(\mathbb{R}^n; E))$ it trivially holds $\mathrm{tr}_{\mathbb{R}^n_+}\phi = (\mathrm{tr}_0 \circ \iota_1)\phi$. Due to the density of $\mathrm{Step}(\mathbb{R}_+; C_c^\infty(\mathbb{R}^n; E))$, proved in Lemma 1.3.10, we obtain that

$$\mathrm{tr}_{\mathbb{R}^n_+} = \mathrm{tr}_0 \circ \iota_1 \qquad \text{on } L_{p,\mu}(\mathbb{R}_+; H_p^{2ms}(\mathbb{R}^n_+; E)). \tag{1.3.21}$$

This representation allows to prove the temporal regularity for spatial traces of functions in $H_{p,\mu}^{s,2ms}(\mathbb{R}_+ \times \mathbb{R}^n_+; E)$ as suggested above.

Lemma 1.3.11. *Let E be a Banach space of class $\mathcal{HT}$, and let $m \in \mathbb{N}$ and $s \in (0,1]$ satisfy $2ms \in \mathbb{N}$. Then the trace $\mathrm{tr}_{\mathbb{R}^n_+}$ maps continuously*

$$H^{s,2ms}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n_+; E) \to W^{s-1/(2mp),2ms-1/p}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E).$$

Proof. Throughout this proof we set $\mathbb{X} := L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$.
(I) Considering a function in $L_{p,\mu}(\mathbb{R}_+; H^{2ms}_p(\mathbb{R}^n_+; E))$ as a function of $y \in \mathbb{R}_+$ taking values in the functions of $(t, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$, we obtain that

$$L_{p,\mu}(\mathbb{R}_+; H^{2ms}_p(\mathbb{R}^n_+; E)) \hookrightarrow H^{2ms}_p(\mathbb{R}_+; \mathbb{X}).$$

Moreover, it follows from $H^{2ms}_p(\mathbb{R}^n_+; E) \hookrightarrow L_p(\mathbb{R}_+; H^{2ms}_p(\mathbb{R}^{n-1}; E))$ and Fubini's theorem that

$$L_{p,\mu}(\mathbb{R}_+; H^{2ms}_p(\mathbb{R}^n_+; E)) \hookrightarrow L_p(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; H^{2ms}_p(\mathbb{R}^{n-1}; E))).$$

Fubini's theorem and interpolation further yield

$$H^s_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E)) = L_p(\mathbb{R}_+; H^s_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))).$$

By Lemma 1.3.1, the realization of the operator $L = 1 - \partial_t + (-\Delta_{n-1})^m$ on $\mathbb{X}$ is invertible and admits bounded imaginary powers with power angle not exceeding $\pi/2$. Hence, by Lemma A.3.5, for $\tau \in (0,1]$ its power L^τ has bounded imaginary powers with angle not larger than $\tau\pi/2$, and it holds

$$D(L^\tau) = H^\tau_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; H^{2m\tau}_p(\mathbb{R}^{n-1}; E)). \tag{1.3.22}$$

Therefore

$$H^{s,2ms}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n_+; E) \hookrightarrow H^{2ms}_p(\mathbb{R}_+; \mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^s)).$$

Denoting the above embedding by $\widetilde{\iota}_1$, equation (1.3.21) implies $\mathrm{tr}_{\mathbb{R}^n_+} = \mathrm{tr}_0 \circ \widetilde{\iota}_1$.
(II) We now claim that the space $H^{2ms}_p(\mathbb{R}; \mathbb{X}) \cap L_p(\mathbb{R}; D(L^s))$ embeds continuously into $H^1_p(\mathbb{R}; D(L^{s-1/2m}))$. To see this, we consider the realization of the operators $A = 1 + (-\partial_y^2)^{sm}$ and $B = L^s$ on $L_p(\mathbb{R}; \mathbb{X})$ with domains

$$D(A) = H^{2ms}_p(\mathbb{R}; \mathbb{X}) \qquad \text{and} \qquad D(B) = L_p(\mathbb{R}; D(L^s)),$$

respectively. These operators are invertible, and admit bounded imaginary powers with power angles equal to zero and $s\pi/2$, respectively. Moreover, A and B are resolvent commuting on step functions in $L_p(\mathbb{R}; \mathbb{X})$, which carries over to $L_p(\mathbb{R}; \mathbb{X})$ by density. Thus the Dore-Venni Theorem A.3.2 shows that the operator $A + B$ is invertible on $L_p(\mathbb{R}; X)$ with domain

$$D(A + B) = H^{2ms}_p(\mathbb{R}; \mathbb{X}) \cap L_p(\mathbb{R}; D(L^s)).$$

Since it holds that $|A^{1/2ms}B^{1-1/2ms} \cdot |_{L_p(\mathbb{R};\mathbb{X})}$ and $|(A + B) \cdot |_{L_p(\mathbb{R};\mathbb{X})}$ are equivalent norms on $H^1_p(\mathbb{R}; D(L^{s-1/2m}))$ and $D(A+B)$, respectively, the mixed derivative theorem (Lemma A.3.3) implies the asserted embedding.
(III) It follows from restriction and extension that also

$$H^{2ms}_p(\mathbb{R}_+; \mathbb{X}) \cap L_p(\mathbb{R}_+; D(L^s)) \hookrightarrow H^1_p(\mathbb{R}_+; D(L^{s-1/2m})),$$

which implies that the operator $L^{s-1/2m}$ maps continuously

$$H_p^{2ms}(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+;D(L^s)) \to H_p^1(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+;D(L^{1/2m})).$$

Note that $L^{1/2m}$ is sectorial of angle at most $\pi/4m < \pi/2$, and thus $-L^{1/2m}$ is the generator of an exponentially stable analytic C_0-semigroup on $\mathbb{X}$. Due to [7, Theorem III.4.10.2] we have

$$H_p^1(\mathbb{R}_+;\mathbb{X}) \cap L_p(\mathbb{R}_+;D(L^{1/2m})) \hookrightarrow BUC([0,\infty);D_{L^{1/2m}}(1-1/p,p)),$$

and from the reiteration theorem we infer

$$D_{L^{1/2m}}(1-1/p,p) = D_L((1-1/p)/2m,p).$$

(IV) We now write

$$\mathrm{tr}_{\mathbb{R}^n_+} = \mathrm{tr}_0\, L^{-(s-1/2m)}\, L^{s-1/2m}\,\widetilde{\iota}_1,$$

where $L^{s-1/2m}$ and its inverse are applied pointwise. By the above considerations, the operator $L^{s-1/2m}\widetilde{\iota}_1$ maps continuously

$$H^{s,2ms}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^n_+;E) \to BUC([0,\infty);D_L((1-1/p)/2m,p)).$$

Clearly, tr_0 and $L^{-(s-1/2m)}$ commute on $BUC([0,\infty);D_L((1-1/p)/2m,p))$, and by reiteration and Lemma A.3.1, $L^{-(s-1/2m)}$ maps $D_L((1-1/p)/2m,p)$ continuously into

$$D_L(s-1/2mp,p) = W^{s-1/(2mp),2ms-1/p}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^{n-1};E).$$

This shows that the trace $\mathrm{tr}_{\mathbb{R}^n_+}$ maps continuously as asserted. ■

Via localization we extend the above result to general domains and finite intervals.

Proposition 1.3.12. *Let E be a Banach space of class $\mathcal{HT}$, $p\in(1,\infty)$, $\mu\in(1/p,1]$, let $m\in\mathbb{N}$ and $s\in(0,1]$ be such that $2ms\in\mathbb{N}$, let $J=(0,T)$ be a finite or infinite interval, and let $\Omega\subset\mathbb{R}^n$ be a domain with compact smooth boundary, or $\Omega\in\{\mathbb{R}^n,\mathbb{R}^n_+\}$. Then the spatial trace tr_Ω maps continuously*

$$H^{s,2ms}_{p,\mu}(J\times\Omega;E) \to W^{s-1/2mp,2ms-1/p}_{p,\mu}(J\times\partial\Omega;E).$$

The operator norm of tr_Ω on ${}_0H^{\tau,2m\tau}_{p,\mu}(J\times\Omega;E)$ is independent of the length of J.

Proof. (I) Using the extension operators $\mathcal{E}_J$ and $\mathcal{E}_J^0$ from Lemma 1.1.5, it suffices to consider the case $J=\mathbb{R}_+$. We describe $\partial\Omega$ by a finite number of charts (U_i,φ_i) and a partition of unity $\{\psi_i\}$ subordinate to the cover $\bigcup_i U_i$. We further denote by Φ_i the push-forward with respect to φ_i, i.e., $\Phi_i u = u\circ\varphi_i^{-1}$. For a function $\phi\in\mathrm{Step}(\mathbb{R}_+;C_c^\infty(\mathbb{R}^n;E))$ it holds

$$\mathrm{tr}_\Omega\phi = \sum\nolimits_i \Phi_i^{-1}\big(\mathrm{tr}_{\mathbb{R}^n_+}\Phi_i(\psi_i\phi)\big) \qquad \text{on } \partial\Omega. \tag{1.3.23}$$

(II) By restriction to $\Omega\cap U_i$, Lemma A.4.1 and trivial extension from $\mathbb{R}^n_+\cap\varphi_i(U_i)$ to $\mathbb{R}^n_+$, for each i we obtain that the $\Phi_i(\psi_i\cdot)$ maps continuously

$$H^{s,2ms}_{p,\mu}(\mathbb{R}_+\times\Omega;E) \to H^{s,2ms}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^n_+;E).$$

Applying Lemma 1.3.11, restricting back to $\mathbb{R}^n_+ \cap \varphi_i(U_i)$ and using again Lemma A.4.1 yields that $\Phi_i^{-1}\mathrm{tr}_{\mathbb{R}^n_+}$ maps the latter space continuously into

$$W^{s-1/2mp,2ms-1/p}_{p,\mu}(\mathbb{R}_+ \times \partial\Omega; E).$$

Thus the operator $\sum_i \Phi_i^{-1}(\mathrm{tr}_{\mathbb{R}^n_+}\Phi_i(\psi_i \cdot))$ maps continuously

$$H^{s,2ms}_{p,\mu}(\mathbb{R}_+ \times \Omega; E) \to W^{s-1/2mp,2ms-1/p}_{p,\mu}(\mathbb{R}_+ \times \partial\Omega; E).$$

Since $\mathrm{Step}(\mathbb{R}_+; C_c^\infty(\mathbb{R}^n; E))$ is dense in $L_{p,\mu}(\mathbb{R}_+; H^{2ms}_p(\Omega; E))$ by Lemma 1.3.10, the representation (1.3.23) holds for all elements of this space, and in particular for all functions from $H^{s,2ms}_{p,\mu}(J \times \Omega; E)$. This shows that tr_Ω is continuous as asserted. ■

Arguing as in [25, Lemma 3.5] one can show that in the situation of the above proposition the spatial trace is surjective.

We use the results derived so far to estimate differential operators of lower order on spaces of type (1.3.19).

Lemma 1.3.13. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$ and $\mu \in (1/p, 1]$. Let $J = (0,T)$ be a finite interval, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Let further the numbers $m \in \mathbb{N}$ and $s \in [0,1)$ be given. Then for every $\eta > 0$ there is $T_0 > 0$ such that for $T \le T_0$ the following holds true.*

- **a)** *For $\alpha \in \mathbb{N}_0^n$ with $s + |\alpha|/2m < 1$ it holds*

$$|\nabla^\alpha u|_{{}_0H^{s,2ms}_{p,\mu}(J\times\Omega;E)} \le \eta\, |u|_{W^{1,2m}_{p,\mu}(J\times\Omega;E)} \qquad \text{for } u \in {}_0W^{1,2m}_{p,\mu}(J \times \Omega; E).$$

- **b)** *For $\beta \in \mathbb{N}_0^n$ with $s + |\beta|/2m + 1/2mp < 1$ it holds*

$$|\mathrm{tr}_\Omega \nabla^\beta u|_{{}_0W^{s,2ms}_{p,\mu}(J\times\partial\Omega;E)} \le \eta\, |u|_{W^{1,2m}_{p,\mu}(J\times\partial\Omega;E)} \qquad \text{for } u \in {}_0W^{1,2m}_{p,\mu}(J \times \Omega; E).$$

Proof. (I) It follows from Lemma 1.3.4 that there is constant C_0, which is independent of J, such that

$$|\nabla^\alpha u|_{{}_0H^{s,2ms}_{p,\mu}(J\times\Omega;E)} \le C_0\, |u|_{{}_0H^{s+|\alpha|/2m,2ms+|\alpha|}_{p,\mu}(J\times\Omega;E)}.$$

From the interpolation inequality A.2 j), the assumption $s + |\alpha|/2m < 1$ and Young's inequality we infer

$$\begin{aligned} |u|_{{}_0H^{s+|\alpha|/2m}_{p,\mu}(J;L_p(\Omega;E))} &\le |u|^{s+|\alpha|/2m}_{{}_0W^1_{p,\mu}(J;L_p(\Omega;E))}\, |u|^{1-s-|\alpha|/2m}_{L_{p,\mu}(J;L_p(\Omega;E))} \\ &\le \frac{\eta}{4C_0}\, |u|_{{}_0W^1_{p,\mu}(J;L_p(\Omega;E))} + C_\eta |u|_{L_{p,\mu}(J;L_p(\Omega;E))}, \end{aligned}$$

where C_η is a constant that depends on η. Here it is important that for complex interpolation the constant in the interpolation inequality is equal to 1 and thus independent of the underlying spaces. It further follows from Poincaré's inequality (Lemma 1.1.14) that

$$|u|_{L_{p,\mu}(J;L_p(\Omega;E))} \le \frac{\eta}{4C_0 C_\eta}\, |u|_{W^1_{p,\mu}(J;L_p(\Omega;E))},$$

provided that $T \leq T_0$ with sufficiently small T_0. This shows

$$|u|_{{}_0H^{s+|\alpha|/2m}_{p,\mu}(J;L_p(\Omega;E))} \leq \frac{\eta}{2C_0}\,|u|_{W^1_{p,\mu}(J;L_p(\Omega;E))}.$$

In a similar way we estimate

$$\begin{aligned}|u|_{L_{p,\mu}(J;H^{2ms+|\alpha|}_p(\Omega;E))} &\leq \frac{\eta}{4C_0}\,|u|_{L_{p,\mu}(J;W^{2m}_p(\Omega;E))} + C_\eta |u|_{L_{p,\mu}(J;L_p(\Omega;E))}\\ &\leq \frac{\eta}{2C_0}\,\big(|u|_{L_{p,\mu}(J;W^{2m}_p(\Omega;E))} + |u|_{W^1_{p,\mu}(J;L_p(\Omega;E))}\big).\end{aligned}$$

This shows a).

(II) For b) we obtain as above that for given $\widetilde{\eta} > 0$ it holds

$$|\mathrm{tr}_\Omega \nabla^\beta u|_{{}_0W^{s,2ms}_{p,\mu}(J\times\partial\Omega;E)} \leq \widetilde{\eta}\,|\mathrm{tr}_\Omega \nabla^\beta u|_{{}_0W^{1-|\beta|/2m-1/2mp,2m-|\beta|-1/p}_{p,\mu}(J\times\partial\Omega;E)},$$

provided that $T \leq T_0$ is sufficiently small. By Proposition 1.3.12 and Lemma 1.3.4 there is constant $\widetilde{C}_0$, which does not depend on J, such that

$$|\mathrm{tr}_\Omega \nabla^\beta u|_{{}_0W^{1-|\beta|/2m-1/2mp,2m-|\beta|-1/p}_{p,\mu}(J\times\partial\Omega;E)} \leq \widetilde{C}_0\,|u|_{W^{1,2m}_{p,\mu}(J\times\Omega;E)}.$$

Setting $\widetilde{\eta} = \eta/\widetilde{C_0}$, we obtain the asserted estimate. ■

We end this section with a useful density result for anisotropic spaces.

Lemma 1.3.14. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and let $s = 1$ or $s \in (0,1)$ with $2ms \notin \mathbb{N}$ and $s \neq 1-\mu+1/p$. Then*

$$C^\infty_c(\mathbb{R}_+; W^{2m}_p(\mathbb{R}^n;E)) \overset{d}{\hookrightarrow} {}_0W^{s,2ms}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^n;E).$$

Proof. Throughout we set $Y_0 := L_p(\mathbb{R}^n, E)$, and $Y_1 := W^{2m}_p(\mathbb{R}^n, E)$.

(I) We first consider the case $s = 1$, i.e., we show that $C^\infty_c(\mathbb{R}_+; Y_1)$ is dense in ${}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^n;E)$. To this end we first show that the set of functions in ${}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^n;E)$ which are compactly supported in $\mathbb{R}_+$ are dense in ${}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^n;E)$. Let $\varepsilon > 0$ and $u \in {}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+\times\mathbb{R}^n;E)$ be given. Choose $T_\varepsilon > t_\varepsilon > 0$ such that the numbers

$$|\mathbb{1}_{\mathbb{R}_+\setminus(t_\varepsilon,T_\varepsilon)}u|_{L_{p,\mu}(\mathbb{R}_+;Y_0)},\quad |\mathbb{1}_{\mathbb{R}_+\setminus(t_\varepsilon,T_\varepsilon)}u'|_{L_{p,\mu}(\mathbb{R}_+;Y_0)},\quad |\mathbb{1}_{\mathbb{R}_+\setminus(t_\varepsilon,T_\varepsilon)}u|_{L_{p,\mu}(\mathbb{R}_+;Y_1)},$$

are smaller than ε, respectively. Choose further a smooth nonnegative cut-off function α_ε on $\mathbb{R}_+$ with $\alpha_\varepsilon \leq 1$ and

$$\alpha_\varepsilon(t) = \begin{cases} 1, & t\in(t_\varepsilon,T_\varepsilon),\\ 0, & t\in(0,t_\varepsilon/2),\\ 0, & (T_\varepsilon+1,\infty),\end{cases}\qquad |\alpha'_\varepsilon||_{(t_\varepsilon/2,t_\varepsilon)} \lesssim \frac{1}{t_\varepsilon},\quad |\alpha'_\varepsilon||_{(T_\varepsilon,T_\varepsilon+1)} \lesssim 1.$$

Then it holds

$$|u-\alpha_\varepsilon u|_{L_{p,\mu}(\mathbb{R}_+;Y_0)} \lesssim \varepsilon,\qquad |u-\alpha_\varepsilon u|_{L_{p,\mu}(\mathbb{R}_+;Y_1)} \lesssim \varepsilon,$$

and further that

$$\begin{aligned}|u'-(\alpha_\varepsilon u)'|_{L_{p,\mu}(\mathbb{R}_+;Y_0)} &\leq |u'-\alpha_\varepsilon u'|_{L_{p,\mu}(\mathbb{R}_+;Y_0)} + |\alpha'_\varepsilon u|_{L_{p,\mu}(\mathbb{R}_+;Y_0)}\\ &\lesssim \varepsilon + |\alpha'_\varepsilon u|_{L_{p,\mu}(\mathbb{R}_+;Y_0)}.\end{aligned}$$

The properties of α_ε yield

$$|\alpha_\varepsilon' u|_{L_{p,\mu}(\mathbb{R}_+;Y_0)} \lesssim |\alpha_\varepsilon' u|_{L_{p,\mu}(t_\varepsilon/2,t_\varepsilon;Y_0)} + |\alpha_\varepsilon' u|_{L_{p,\mu}(T_\varepsilon,T_\varepsilon+1;Y_0)}. \tag{1.3.24}$$

To estimate the first summand we use that $1/t_\varepsilon^p \leq 1/t^p$ for $t \leq t_\varepsilon$, to obtain

$$|\alpha_\varepsilon' u|^p_{L_{p,\mu}(t_\varepsilon/2,t_\varepsilon;Y_0)} \leq \frac{2^p}{t_\varepsilon^p}\, |u|^p_{L_{p,\mu}(t_\varepsilon/2,t_\varepsilon;Y_0)} \lesssim \int_{t_\varepsilon/2}^{t_\varepsilon} t^{-p\mu}|u(t)|^p_{Y_0}\,\mathrm{d}t \lesssim \varepsilon$$

for t_ε sufficiently small, since the function $t^{-p\mu}|u|^p_{Y_0}$ belongs to $L_1(\mathbb{R}_+)$ by Hardy's inequality (Lemma 1.1.2). The assumption for α_ε' on $(T_\varepsilon, T_\varepsilon+1)$ implies that the second summand of (1.3.24) is smaller than a constant multiple of ε. Hence $|\alpha_\varepsilon' u|_{L_{p,\mu}(\mathbb{R}_+;Y_0)} \lesssim \varepsilon$, and thus

$$|u - \alpha_\varepsilon u|_{W^1_{p,\mu}(\mathbb{R}_+;Y_0)} + |u - \alpha_\varepsilon u|_{L_{p,\mu}(\mathbb{R}_+;Y_1)} \lesssim \varepsilon.$$

Therefore the functions $\alpha_\varepsilon u$, which belong to $W^{1,2m}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n; E)$ and are supported in $(t_\varepsilon/2, T_\varepsilon + 1)$, approximate u in $W^{1,2m}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n; E)$ as $\varepsilon \searrow 0$.

(II) To approximate a function $u \in {}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n, E)$ with compact support in $\mathbb{R}_+$ by functions in $C_c^\infty(\mathbb{R}_+; Y_1)$ we have to approximate u in $W^1_p(\mathbb{R}_+; Y_0)$ and $L_p(\mathbb{R}_+; Y_1)$ simultaneously. This can be achieved using a standard mollification method, as in the proofs of [1, Theorem 2.29, Lemma 3.16]. We omit the details. The assertion of this lemma for $s = 1$ follows.

(III) Now let $s \in (0,1)$. By A.2 a) the dense embedding

$${}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n; E) \overset{d}{\hookrightarrow} (L_{p,\mu}(\mathbb{R}_+; Y_0), {}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n; E))_{s,p}$$

is valid, and the Lemmas 1.1.9, 1.3.1 and Proposition A.4.2 yield

$$(L_{p,\mu}(\mathbb{R}_+; Y_0), {}_0W^{1,2m}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n; E))_{s,p} = {}_0W^{s,2ms}_{p,\mu}(\mathbb{R}_+ \times \mathbb{R}^n; E),$$

provided $2ms \notin \mathbb{N}$ and $s \neq 1 - \mu + 1/p$. ■

1.3.4 Pointwise Multipliers

If E, J, and Ω are as in Lemma 1.3.4, then the operator ∇^α, where $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq 2m$ and $m \in \mathbb{N}$, maps continuously

$$W^{1,2m}_{p,\mu}(J \times \Omega; E) \to H^{1-|\alpha|/2m,2m-|\alpha|}_{p,\mu}(J \times \Omega; E).$$ [10]

Motivated by linear differential operators with variable coefficients, we are looking for sufficient conditions on a function $a = a(t,x) \in \mathcal{B}(E)$ to be a pointwise multiplier to $L_{p,\mu}(J; L_p(\Omega; E))$, i.e., such that the multiplication with it is a continuous map

$$H^{\tau,2m\tau}_{p,\mu}(J \times \Omega; E) \to L_{p,\mu}(J; L_p(\Omega; E)), \qquad \tau \in (0,1].$$

We have the following result for coefficients a which belong to an unweighted space.

[10] Recall for $\tau > 0$ the notation $H^{\tau,2m\tau}_{p,\mu}(J \times \Omega; E) = H^\tau_{p,\mu}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; H^{2m\tau}_p(\Omega; E))$, and analogously for the W-spaces.

Lemma 1.3.15. *Let E be a Banach space of class $\mathcal{HT}$, let $J = (0,T)$ be finite, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or let $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Let further $p \in (1,\infty)$, $\mu \in (1/p, 1]$, $r, s \in [p,\infty)$, and $\tau \in (0,1]$ satisfy*

$$\frac{p(1-\mu)+1}{s} + \frac{n}{2mr} < \tau.$$

Then there is $C > 0$ such that

$$|au|_{L_{p,\mu}(J;L_p(\Omega;E))} \leq C\,|a|_{L_s(J;L_r(\Omega;\mathcal{B}(E)))}\,|u|_{H^{\tau,2m\tau}_{p,\mu}(J\times\Omega;E)}$$

is valid for all $a \in L_s(J; L_r(\Omega; \mathcal{B}(E)))$ and $u \in H^{\tau,2m\tau}_{p,\mu}(J \times \Omega; E)$. Restricting to $u \in {}_0H^{\tau,2m\tau}_{p,\mu}(J \times \Omega; E)$, for given $T_0 > 0$ the constant C may be chosen uniformly for all $T \leq T_0$.

Proof. Applying Hölder's inequality twice yields

$$\begin{aligned} |au|^p_{L_{p,\mu}(J;L_p(\Omega;E))} &= \int_J t^{p(1-\mu)} |a(t,\cdot)u(t,\cdot)|^p_{L_p(\Omega;E)}\,\mathrm{d}t \\ &\leq \int_J |a(t,\cdot)|^p_{L_r(\Omega;\mathcal{B}(E))} |t^{1-\mu}u(t,\cdot)|^p_{L_{r'}(\Omega;E)}\,\mathrm{d}t \\ &\leq |a|^p_{L_s(J;L_r(\Omega;\mathcal{B}(E)))} |u|^p_{L_{s',\mu}(L_{r'}(\Omega;E))}, \end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{p}$.[11] Due to Proposition 1.3.2, for $\sigma \in (0,1)$ the embedding

$$H^{\tau,2m\tau}_{p,\mu}(J\times\Omega;E) \hookrightarrow H^{\tau(1-\sigma)}_{p,\mu}(J; H^{2m\tau\sigma}_p(\Omega;E)),$$

is valid, and the embedding constant is independent of J if one restricts to ${}_0H_{p,\mu}$-spaces in time. Sobolev's embedding yields

$$H^{2m\tau\sigma}_p(\Omega;E) \hookrightarrow L_{r'}(\Omega;E) \qquad \text{for } \sigma = \frac{n}{2m\tau r} < 1.$$

It follows from Proposition 1.1.12 that

$$H^{\tau-\frac{n}{2mr}}_{p,\mu}(J;L_{r'}(\Omega;E)) \hookrightarrow L_{s'}(J;L_{r'}(\Omega;E)) \quad \text{for } \tau - \frac{n}{2mr} - \Big(1-\mu+\frac{1}{p}\Big) > -\frac{p(1-\mu+1/p)}{s'},$$

with an embedding constant as asserted in the ${}_0H_{p,\mu}$-case. Since the latter condition is equivalent to $\frac{p(1-\mu+1/p)}{s} + \frac{n}{2mr} < \tau$, this finishes the proof. ■

We are also interested in the case where the coefficients belong to a temporally weighted space. If $2m(\mu - 1/p) > 2m - 1 + n/p$, then Theorem 1.3.6 and Sobolev's embedding yield

$$W^{1,2m}_{p,\mu}(J\times\Omega;E) \hookrightarrow C(\overline{J}; BUC^{2m-1}(\overline{\Omega};E)).$$

Thus ∇^α maps for $|\alpha| < 2m$ continuously

$$W^{1,2m}_{p,\mu}(J\times\Omega;E) \to BUC(\overline{J}\times\overline{\Omega};E),$$

and the multiplication with a is continuous from $BUC(\overline{J}\times\overline{\Omega};E)$ to $L_{p,\mu}(J;L_p(\Omega;E))$ if $a \in L_{p,\mu}(J;L_p(\Omega;\mathcal{B}(E)))$. These considerations together with Lemma 1.3.15 yield the following result for differential operators with variable coefficients.

[11] Note that r' is not the standard dual exponent of r.

Proposition 1.3.16. *Let E be of class $\mathcal{HT}$, $p \in (1,\infty)$ and $\mu \in (1/p,1]$, let $J=(0,T)$ be a finite interval, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or let $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Assume that for the $\mathcal{B}(E)$-valued coefficient $a = a(t,x)$ of the operator $a\nabla^\alpha$, where $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2m$, $m \in \mathbb{N}$, it holds $a \in BC(\overline{J} \times \overline{\Omega}; \mathcal{B}(E))$ in case $|\alpha| = 2m$, and that in case $|\alpha| < 2m$ one of the following conditions is valid: either*

$$2m(\mu - 1/p) > 2m - 1 + n/p \quad \text{and} \quad a \in L_{p,\mu}\big(J; L_p(\Omega; \mathcal{B}(E))\big),$$

or $a \in L_{s_\alpha}\big(J; (L_{r_\alpha} + L_\infty)(\Omega; \mathcal{B}(E))\big)$ for some numbers $s_\alpha, r_\alpha \in [p,\infty)$ with

$$\frac{p(1-\mu)+1}{s_\alpha} + \frac{n}{2mr_\alpha} < 1 - \frac{|\alpha|}{2m}.$$

Then we have

$$a\nabla^\alpha \in \mathcal{B}\big(W^{1,2m}_{p,\mu}(J \times \Omega; E), L_{p,\mu}(J; L_p(\Omega; E))\big). \qquad \blacksquare$$

In the same setting as above, we now consider pointwise multipliers for anisotropic spaces related to boundary differential operators. Lemma 1.3.4, together with Proposition 1.3.11, yields that $\mathrm{tr}_\Omega \nabla^\beta$, where, $\beta \in \mathbb{N}_0^n$, $|\alpha| \leq 2m-1$, maps continuously

$$W^{1,2m}_{p,\mu}(J \times \Omega; E) \to W^{1-|\beta|/2m - 1/2mp, 2m-|\beta|-1/p}_{p,\mu}(J \times \partial\Omega; E).$$

Thus our aim is to provide sufficient conditions on a $\mathcal{B}(E)$-valued function b such that the multiplication with it is a bounded linear map

$$W^{\tau,2m\tau}_{p,\mu}(J \times \partial\Omega; E) \to W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E),$$

where $0 < \kappa \leq \tau < 2$. Moreover, the estimates should be suitable for the localization procedures in the next chapters. To obtain rather sharp results we use the paraproduct techniques presented in [74, Section 4.4].

Choose a function $\psi \in C_c^\infty(\mathbb{R}^n)$ with the property

$$\psi(\xi) = 1, \quad |\xi| \leq 1, \qquad \psi(\xi) = 0, \quad |\xi| \geq 3/2,$$

and define the family φ_j, $j \in \mathbb{N}_0$, by

$$\varphi_0(\xi) = \psi(\xi), \qquad \varphi_1(\xi) = \psi(\xi/2) - \psi(\xi), \qquad \varphi_j(\xi) = \varphi_1(2^{-j+1}\xi), \qquad j \geq 2.$$

Then it holds $\sum_{j=0}^\infty \varphi_j \equiv 1$ on $\mathbb{R}^n$, and further

$$\mathrm{supp}\, \varphi_j \subset \{2^{j-1} \leq |\xi| \leq 3\, 2^{j-1}\}, \quad j \in \mathbb{N}, \qquad \sum_{j=0}^k \varphi_j(\xi) = \psi(2^{-k}\xi), \quad k \in \mathbb{N}_0.$$

Denoting by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^n$, we use this dyadic partition of unity to define operators S_j and S^k which cut off dyadic frequencies in the Fourier image,

$$S_j := \mathcal{F}^{-1}\varphi_j\mathcal{F}, \quad j \in \mathbb{N}_0, \qquad S^k := \sum_{j=0}^k S_j, \quad k \in \mathbb{N}_0, \qquad S^{-l} = S_{-l} := 0, \quad l \in \mathbb{N}.$$

Observe that for $u \in \mathcal{S}'(\mathbb{R}^n; E) = \mathcal{B}\big(\mathcal{S}(\mathbb{R}^n), E\big)$ it holds $u = \lim_{k\to\infty} S^k u$ in the sense of distributions.[12] The Besov spaces may be characterized with the help of the operators S_j. By [75, Definition 4.3], for $\sigma > 0$, $p \in [1,\infty)$ and $q \in [1,\infty]$ the *Littlewood-Paley representation*

$$B^\sigma_{p,q}(\mathbb{R}^n; E) = \big\{u \in \mathcal{S}'(\mathbb{R}^n; E) \,:\, |u|_{B^\sigma_{p,q}(\mathbb{R}^n;E)} = |(2^{\sigma j}|S_j u|_{L_p(\mathbb{R}^n;E)})_{j\in\mathbb{N}_0}|_{l_q} < \infty\big\}$$

is valid. Here l_q denote the standard sequence spaces, $q \in [1,\infty]$. We observe the following.

Lemma 1.3.17. *For $q \in [1,\infty]$, the operator families $(S_j)_{j\in\mathbb{N}_0}$ and $(S^k)_{k\in\mathbb{N}_0}$ are uniformly bounded on $L_q(\mathbb{R}^n; E)$.*

Proof. Since $\varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi)$ for $j \ge 1$ and $\sum_{j=0}^k \varphi_j(\xi) = \psi(2^{-k}\xi)$, we only have to show that the operator norm of the convolution operator $\mathcal{F}^{-1}\psi(2^{-j}\cdot)\mathcal{F} = (\mathcal{F}^{-1}\psi(2^{-j}\cdot))*$ is bounded independent of $j \in \mathbb{N}_0$. The convolution inequality shows that for $q \in [1,\infty]$ we have

$$|(\mathcal{F}^{-1}\psi(2^{-j}\cdot)) * |_{\mathcal{B}(L_q(\mathbb{R}^n;E))} \le |(\mathcal{F}^{-1}\psi(2^{-j}\cdot))|_{L_1(\mathbb{R}^n;E)}.$$

Now it is easy to see that $\big(\mathcal{F}^{-1}\psi(2^{-j}\cdot)\big)(x) = 2^{jn}(\mathcal{F}^{-1}\psi)(2^j x)$ for $x \in \mathbb{R}^n$, and further that

$$|(\mathcal{F}^{-1}\psi)(2^j\cdot)|_{L_1(\mathbb{R}^n;E)} \le 2^{-jn}|\mathcal{F}^{-1}\psi|_{L_1(\mathbb{R}^n;E)},$$

which yields an estimate independent of j. ■

For $f \in \mathcal{S}'(\mathbb{R}^n; \mathcal{B}(E)) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n; \mathcal{B}(E))$ and $g \in \mathcal{S}'(\mathbb{R}^n; E) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n; E)$ we formally decompose the product fg into the paraproducts

$$\Pi_1(f,g) := \sum_{k=2}^\infty S^{k-2} f S_k g, \qquad \Pi_2(f,g) := \sum_{k=0}^\infty (S_{k-1}f + S_k f + S_{k+1} f) S_k g$$

$$\Pi_3(f,g) := \sum_{k=2}^\infty S_k f S^{k-2} g,$$

so that it holds

$$fg = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_3(f,g), \tag{1.3.25}$$

whenever the paraproducts exist in the sense of distributions. Observe that for $k \in \mathbb{N}_0$ it holds

$$\operatorname{supp}\mathcal{F}(S^{k-2} f S_k g) \,\cup\, \operatorname{supp}\mathcal{F}\sum_{l=k-1}^{k+1} S^l f S_k g \,\cup\, \operatorname{supp}\mathcal{F}(S_k f S^{k-2} g) \,\subset \{|\xi| \le 2^{k+3}\}.$$

The following lemma is the vector-valued version of [74, Proposition 2.3.2/2], and gives a criterion for the existence of a paraproduct in a Besov space.

[12]We refer to [75] and [7] for details on vector-valued distributions.

Lemma 1.3.18. *Let E be a Banach space of class $\mathcal{HT}$, $\sigma > 0$, $p,q \in (1,\infty)$, and let $h_k \in L_p(\mathbb{R}^n;E)$, $k \in \mathbb{N}_0$, satisfy*

$$\text{supp}\,\mathcal{F}h_k \subset \{|\xi| \leq 2^{k+3}\}, \qquad k \geq 0.$$

If $(2^{k\sigma}|h_k|_{L_p(\mathbb{R}^n;E)})_{k\in\mathbb{N}_0} \in l_q$ then $\sum_{k=0}^\infty h_k$ converges to some $h \in B^\sigma_{p,q}(\mathbb{R}^n;E)$ in the sense of distributions, and it holds

$$|h|_{B^\sigma_{p,q}(\mathbb{R}^n;E)} \lesssim |(2^{k\sigma}|h_k|_{L_p(\mathbb{R}^n;E)})_{k\in\mathbb{N}_0}|_{l_q}.$$

Proof. The support condition implies that $S_j h_k = 0$ for $j \geq k+4$. Thus

$$S_j \sum_{k=0}^N h_k = S_j \sum_{k=j-3}^N h_k \mathbb{1}_{[0,\infty)}(k), \qquad j, N \in \mathbb{N}_0.$$

(I) We first show that $\big(\sum_{k=0}^N h_k\big)_{N\in\mathbb{N}}$ is convergent in the sense of distributions as $N\to\infty$. For integer $N_1 < N_2$ it holds, using the uniform boundedness of $(S_j)_{j\in\mathbb{N}_0}$,

$$\begin{aligned}
|\sum_{k=N_1}^{N_2} h_k|_{B^{\sigma/2}_{p,\infty}(\mathbb{R}^n;E)} &= \sup_{j\in\mathbb{N}_0} 2^{j\sigma/2}|S_j \sum_{k=N_1}^{N_2} h_k|_{L_p(\mathbb{R}^n;E)} \\
&\lesssim \sup_{j\in\mathbb{N}_0} \sum_{k=\max\{N_1,j-3\}}^{N_2} 2^{(j/2-k)\sigma} 2^{k\sigma}|h_k|_{L_p(\mathbb{R}^n;E)} \\
&\lesssim \sup_{j\in\mathbb{N}_0} \sum_{k=\max\{N_1,j-3\}}^{N_2} 2^{(j/2-k)\sigma},
\end{aligned}$$

which is smaller than any given number if $N_1 < N_2$ are sufficiently large. Therefore $\big(\sum_{k=0}^N h_k\big)_{N\in\mathbb{N}}$ is a Cauchy sequence in $B^{\sigma/2}_{p,\infty}(\mathbb{R}^n;E)$, and thus converges in the sense of distributions to a function h.

(II) We show that in fact $h \in B^\sigma_{p,q}(\mathbb{R}^n;E)$. To this end we estimate for $N\in\mathbb{N}$

$$\begin{aligned}
|\sum_{k=0}^N h_k|_{B^\sigma_{p,q}(\mathbb{R}^n;E)} &= |(2^{j\sigma}|S_j \sum_{k=j-3}^N h_k\mathbb{1}_{[0,\infty)}(k)|_{L_p(\mathbb{R}^n;E)})_{j\in\mathbb{N}_0}|_{l_q} \\
&\lesssim |(2^{j\sigma} \sum_{k=j-3}^N |h_k \mathbb{1}_{[0,\infty)}(k)|_{L_p(\mathbb{R}^n;E)})_{j\in\mathbb{N}_0}|_{l_q} \\
&\leq \sum_{l=-3}^\infty |(2^{j\sigma}|h_{j+l}\mathbb{1}_{[0,\infty)}(j+l)|_{L_p(\mathbb{R}^n;E)})_{j\in\mathbb{N}_0}|_{l_q} \\
&\leq \sum_{l=-3}^\infty 2^{-l\sigma}|(2^{k\sigma}|h_k|_{L_p(\mathbb{R}^n;E)})_{k\in\mathbb{N}_0}|_{l_q} \lesssim |(2^{k\sigma}|h_k|_{L_p(\mathbb{R}^n;E)})_{k\in\mathbb{N}_0}|_{l_q},
\end{aligned}$$

which yields that $\big(\sum_{k=0}^N h_k\big)_{N\in\mathbb{N}}$ is uniformly bounded in $B^\sigma_{p,q}(\mathbb{R}^n;E)$. Since E is assumed to be reflexive the sequence has a weakly convergent subsequence in $B^\sigma_{p,q}(\mathbb{R}^n;E)$, and in particular, this convergence is in the distributional sense. From the uniqueness of distributional limits we obtain $h \in B^\sigma_{p,q}(\mathbb{R}^n;E)$. ■

After these preparations we can estimate the Besov norm of a product in a way that is suitable for our purposes.

Lemma 1.3.19. *Let E be a Banach space of class $\mathcal{HT}$, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Let further $\sigma > 0$, $p, q \in (1, \infty)$, and assume that $r, r', \rho, \rho' \in [p, \infty]$, $r, \rho' \neq \infty$, satisfy $\frac{1}{r} + \frac{1}{r'} = \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{p}$. Then it holds*

$$|fg|_{B^\sigma_{p,q}(\Omega;E)} \lesssim |f|_{L_\rho(\Omega;\mathcal{B}(E))}|g|_{B^\sigma_{\rho',q}(\Omega;E)} + |f|_{B^\sigma_{r,q}(\Omega;\mathcal{B}(E))}|g|_{L_{r'}(\Omega;E)}.$$

In this estimate $\Omega \neq \mathbb{R}^n$ may be replaced by its boundary $\partial\Omega$.

We remark that of particular interest is here the case $\rho = r' = \infty$.

Proof. (I) We first consider the case $\Omega = \mathbb{R}^n$, and estimate the paraproducts for fg given by (1.3.25). Using Lemma 1.3.18, Hölder's inequality in $L_p(\mathbb{R}^n; E)$ and Lemma 1.3.17 we estimate for $\Pi_1(f, g)$

$$\begin{aligned}
|\sum_{k=2}^\infty S^{k-2} f S_k g|_{B^\sigma_{p,q}(\mathbb{R}^n;E)} &\lesssim |(2^{k\sigma}|S^{k-2} f S_k g|_{L_p(\mathbb{R}^n;E)})_{k\geq 2}|_{l_q} \\
&\leq |(2^{k\sigma}|S^{k-2} f|_{L_\rho(\mathbb{R}^n;\mathcal{B}(E))}|S_k g|_{L_{\rho'}(\mathbb{R}^n;E)})_{k\geq 2}|_{l_q} \\
&\leq \sup_{j\in\mathbb{N}_0} |S^j f|_{L_\rho(\mathbb{R}^n;\mathcal{B}(E))}\ |(2^{k\sigma}|S_k g|_{L_{\rho'}(\mathbb{R}^n;E)})_{k\geq 2}|_{l_q} \\
&\lesssim |f|_{L_\rho(\mathbb{R}^n;\mathcal{B}(E))}|g|_{B^\sigma_{\rho',q}(\mathbb{R}^n;E)}.
\end{aligned}$$

In a similar way we obtain for $\Pi_2(f, g)$, with $l \in \{-1, 0, 1\}$,

$$\begin{aligned}
|\sum_{k=0}^\infty S_{k+l} f S_k g|_{B^\sigma_{p,q}(\mathbb{R}^n;E)} &\lesssim |(2^{k\sigma}|S_{k+l} f|_{L_r(\mathbb{R}^n;\mathcal{B}(E))}|S_k g|_{L_{r'}(\mathbb{R}^n;E)})_{k\in\mathbb{N}_0}|_{l_q} \\
&\leq |(2^{k\sigma}|S_{k+l} f|_{L_r(\mathbb{R}^n;\mathcal{B}(E))})_{k\in\mathbb{N}_0}|_{l_q}\ \sup_{j\in\mathbb{N}_0} |S_j g|_{L_{r'}(\mathbb{R}^n;E)} \\
&\lesssim |f|_{B^\sigma_{r,q}(\mathbb{R}^n;\mathcal{B}(E))}|g|_{L_{r'}(\mathbb{R}^n;E)},
\end{aligned}$$

and for $\Pi_3(f, g)$

$$\begin{aligned}
|\sum_{k=2}^\infty S_k f S^{k-2} g|_{B^\sigma_{p,q}(\mathbb{R}^n;E)} &\lesssim |(2^{k\sigma}|S_k f|_{L_r(\mathbb{R}^n;\mathcal{B}(E))}|S^{k-2} g|_{L_{r'}(\mathbb{R}^n;E)})_{k\geq 2}|_{l_q} \\
&\lesssim |f|_{B^\sigma_{r,q}(\mathbb{R}^n;\mathcal{B}(E))}|g|_{L_{r'}(\mathbb{R}^n;E)}.
\end{aligned}$$

Thus the paraproducts exist in the sense of distributions, with the given estimates. This yields the assertion for $\Omega = \mathbb{R}^n$.

(II) The estimate for general Ω may be obtained from the full-space case using the extension operator $\mathcal{E}_\Omega$ from (1.3.2). It is left to show the estimate for $\Omega \neq \mathbb{R}^n$ replaced by its boundary. We describe $\partial\Omega$ by a finite collection of charts (U_i, φ_i) and a partition of unity $\{\psi_i\}$ subordinate to the cover $\bigcup_i U_i$. For each i we choose an open set $W_i \subset \mathbb{R}^n$ with

$\operatorname{supp}\psi_i \subset W_i \subset \overline{W_i} \subset U_i$ and estimate, using Lemma A.4.1,

$$\begin{aligned}
|fg|_{B^\sigma_{p,q}(\partial\Omega;E)} &\lesssim \sum\nolimits_i |f(\varphi_i^{-1})g(\varphi_i^{-1})|_{B^\sigma_{p,q}(\varphi_i(W_i)\cap\mathbb{R}^{n-1};E)}\\
&\lesssim \sum\nolimits_i |f(\varphi_i^{-1})|_{L_r(\varphi_i(W_i)\cap\mathbb{R}^{n-1};\mathcal{B}(E))}|g(\varphi_i^{-1})|_{B^\sigma_{r',q}(\varphi_i(W_i)\cap\mathbb{R}^{n-1};E)}\\
&\qquad + |f(\varphi_i^{-1})|_{B^\sigma_{r,q}(\varphi_i(W_i)\cap\mathbb{R}^{n-1};\mathcal{B}(E))}|g(\varphi_i^{-1})|_{L_{r'}(\varphi_i(W_i)\cap\mathbb{R}^{n-1};E)}\\
&\lesssim \sum\nolimits_i |f|_{L_r(\partial\Omega;\mathcal{B}(E))}|g(\varphi_i^{-1})|_{B^\sigma_{r',q}(\varphi_i(W_i)\cap\mathbb{R}^{n-1};E)}\\
&\qquad + |f(\varphi_i^{-1})|_{B^\sigma_{r,q}(\varphi_i(W_i)\cap\mathbb{R}^{n-1};\mathcal{B}(E))}|g|_{L_{r'}(\partial\Omega;E)}.
\end{aligned}$$

Now take a function $\psi_i^* \in C_c^\infty(U_i)$ with $\psi_i^* \equiv 1$ on W_i, to deduce again from Lemma A.4.1 that

$$\begin{aligned}
|fg|_{B^\sigma_{p,q}(\partial\Omega;E)} &\lesssim \sum\nolimits_i |f|_{L_r(\partial\Omega;\mathcal{B}(E))}|\psi_i^*(\varphi_i^{-1})g(\varphi_i^{-1})|_{B^\sigma_{r',q}(\varphi_i(U_i)\cap\mathbb{R}^{n-1};E)}\\
&\qquad + |\psi_i^*(\varphi_i^{-1})f(\varphi_i^{-1})|_{B^\sigma_{r,q}(\varphi_i(U_i)\cap\mathbb{R}^{n-1};\mathcal{B}(E))}|g|_{L_{r'}(\partial\Omega;E)}\\
&\lesssim |f|_{L_r(\partial\Omega;\mathcal{B}(E))}|g|_{B^\sigma_{r',q}(\partial\Omega;E)} + |f|_{B^\sigma_{r,q}(\partial\Omega;\mathcal{B}(E))}|g|_{L_{r'}(\partial\Omega;E)}.
\end{aligned}$$ ∎

We derive a similar result for certain vector-valued Besov spaces on the half-line.

Lemma 1.3.20. *Let E be a Banach space of class $\mathcal{HT}$, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary, or $\Omega \in \{\mathbb{R}^n, \mathbb{R}^n_+\}$. Let further $\sigma > 0$, $p, q \in (1,\infty)$, and assume that $r, r', \rho, \rho', s, s', \sigma, \sigma' \in [p,\infty]$, $s, \sigma' \neq \infty$, satisfy $\frac{1}{r} + \frac{1}{r'} = \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{s} + \frac{1}{s'} = \frac{1}{\sigma} + \frac{1}{\sigma'} = \frac{1}{p}$. Then it holds*

$$\begin{aligned}
|fg|_{B^\sigma_{p,q}(\mathbb{R}_+;L_p(\Omega;E))} &\lesssim |f|_{L_\sigma(\mathbb{R}_+;L_\rho(\Omega;\mathcal{B}(E)))}|g|_{B^\sigma_{\sigma',q}(\mathbb{R}_+;L_{\rho'}(\Omega;E))}\\
&\qquad + |f|_{B^\sigma_{s,q}(\mathbb{R}_+;L_r(\Omega;\mathcal{B}(E)))}|g|_{L_{s'}(\mathbb{R}_+;L_{r'}(\Omega;E))}.
\end{aligned}$$

In this estimate $\Omega \neq \mathbb{R}^n$ may be replaced by its boundary $\partial\Omega$.

Proof. Using extensions and restrictions, we may consider the estimate on $\mathbb{R}$ instead of $\mathbb{R}_+$. We proceed as in the previous lemma. For $\Pi_1(f,g)$ we obtain, using Hölder's inequality twice,

$$\begin{aligned}
\Big|\sum_{k=2}^\infty S^{k-2}fS_kg\Big|_{B^\sigma_{p,q}(\mathbb{R};L_p(\Omega;E))} &\lesssim |(2^{k\sigma}|S^{k-2}fS_kg|_{L_p(\mathbb{R};L_p(\Omega;E))})_{k\geq 2}|_{l_q}\\
&\leq |(2^{k\sigma}|S^{k-2}f|_{L_s(\mathbb{R};L_r(\Omega;\mathcal{B}(E)))}|S_kg|_{L_{s'}(\mathbb{R};L_{r'}(\Omega;E))})_{k\geq 2}|_{l_q}\\
&\leq \sup_{j\in\mathbb{N}_0}|S^jf|_{L_s(\mathbb{R};L_r(\Omega;\mathcal{B}(E)))}\ |(2^{k\sigma}|S_kg|_{L_{s'}(\mathbb{R};L_{r'}(\Omega;E))})_{k\geq 2}|_{l_q}\\
&\lesssim |f|_{L_s(\mathbb{R};L_r(\Omega;\mathcal{B}(E)))}|g|_{B^\sigma_{s',q}(\mathbb{R};L_r(\Omega;E))}.
\end{aligned}$$

In a similar way one treats the terms $\Pi_2(f,g)$ and $\Pi_3(f,g)$. As in the proof of Lemma 1.3.19 this implies the asserted estimate. Since for the spatial variables only Hölder's inequality was used, one may replace $\Omega \neq \mathbb{R}^n$ by its boundary in the above arguments. ∎

We can now prove the desired sufficient conditions for pointwise multipliers on boundaries. We start with spatial Besov regularity.

Lemma 1.3.21. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p,1]$, let $J=(0,T)$ be finite, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega = \mathbb{R}^n_+$. Let further $s,r \in [p,\infty)$, $m \in \mathbb{N}$, $\kappa \in (0,1)$, $\tau \in (0,2)$ and $\vartheta > 0$ satisfy*

$$\tau > \kappa, \qquad \vartheta > 2m\kappa, \qquad \frac{p(1-\mu+1/p)}{s} < \Big(1 - \frac{n-1}{\vartheta r}\Big)\tau.$$

Then it holds

$$|bu|_{L_{p,\mu}(J;B^{2m\kappa}_{p,p}(\partial\Omega;E))} \lesssim |b|_{L_s(J;B^{2m\kappa}_{r,p}(\partial\Omega;\mathcal{B}(E))}|u|_{W^{\tau}_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^{\vartheta}_{p,p}(\partial\Omega;E))}. \quad (1.3.26)$$

Moreover, for $\frac{p(1-\mu+1/p)}{s} + \frac{n-1}{2mr} < \kappa$ there is $\delta \in (0,\kappa)$ such that

$$\begin{aligned}|bu|_{L_{p,\mu}(J;B^{2m\kappa}_{p,p}(\partial\Omega;E))} &\lesssim |b|_{L_\infty(J;L_\infty(\partial\Omega;\mathcal{B}(E)))}|u|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^{2m\kappa}_{p,p}(\partial\Omega;E))} \quad (1.3.27)\\ &\quad + |b|_{L_s(J;B^{2m\kappa}_{r,p}(\partial\Omega;\mathcal{B}(E))}|u|_{W^{\kappa-\delta}_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^{2m(\kappa-\delta)}_{p,p}(\partial\Omega;E))}.\end{aligned}$$

Restricting to $u \in {}_0W_{p,\mu}$, for given $T_0 > 0$ these estimates hold with a uniform constant for all $T \le T_0$.

Proof. (I) It holds that

$$|bu|^p_{L_{p,\mu}(J;B^{2m\kappa}_{p,p}(\partial\Omega;E))} = \int_J t^{p(1-\mu)}|b(t,\cdot)u(t,\cdot)|^p_{B^{2m\kappa}_{p,p}(\partial\Omega;E))}\,\mathrm{d}t,$$

and for almost every $t \in J$ we use Lemma 1.3.19 to estimate

$$\begin{aligned}|b(t,\cdot)u(t,\cdot)|_{B^{2m\kappa}_{p,p}(\partial\Omega;E))} &\lesssim |b(t,\cdot)|_{L_\rho(\partial\Omega;\mathcal{B}(E))}|u(t,\cdot)|_{B^{2m\kappa}_{\rho',p}(\partial\Omega;E)}\\ &\quad + |b(t,\cdot)|_{B^{2m\kappa}_{r,p}(\partial\Omega;\mathcal{B}(E))}|u(t,\cdot)|_{L_{r'}(\partial\Omega;E)},\end{aligned}$$

where $\frac{1}{r} + \frac{1}{r'} = \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{p}$. Hölder's inequality now yields

$$\begin{aligned}|bu|_{L_{p,\mu}(J,B^{2m\kappa}_{p,p}(\partial\Omega;E))} &\lesssim |b|_{L_\sigma(J;L_\rho(\partial\Omega;\mathcal{B}(E)))}|u|_{L_{\sigma',\mu}(J;B^{2m\kappa}_{\rho',p}(\partial\Omega;E))}\\ &\quad + |b|_{L_s(J;B^{2m\kappa}_{r,p}(\partial\Omega;\mathcal{B}(E)))}|u|_{L_{s',\mu}(J;L_{r'}(\partial\Omega;E))}, \quad (1.3.28)\end{aligned}$$

where $\frac{1}{s} + \frac{1}{s'} = \frac{1}{\sigma} + \frac{1}{\sigma'} = \frac{1}{p}$. To obtain the desired estimates we have to choose these numbers appropriately.

(II) We start with the first summand in (1.3.28). If $\tau = \kappa$ we take $\rho = \sigma = \infty$, $\rho' = \sigma' = p$, and obtain the first summand on the right-hand side of (1.3.27).

Now suppose that $\tau > \kappa$. Then we take $\sigma = s$ and $\sigma' = s'$. The embedding

$$L_s(J;B^{2m\kappa}_{r,p}(\partial\Omega;\mathcal{B}(E))) \hookrightarrow L_s(J;L_\rho(\partial\Omega;\mathcal{B}(E)))$$

is valid for

$$\frac{1}{\rho} > \frac{1}{r} - \frac{2m\kappa}{n-1}, \qquad \text{i.e.,} \qquad \frac{1}{\rho'} < \frac{1}{r'} + \frac{2m\kappa}{n-1}. \quad (1.3.29)$$

We thus need the embedding

$$W^{\tau}_{p,\mu}(J;L_p(\Gamma;E)) \cap L_{p,\mu}(J;B^{\vartheta}_{p,p}(\Gamma;E)) \hookrightarrow L_{s',\mu}(J;B^{2m\kappa}_{\rho',p}(\partial\Omega;E)) \quad (1.3.30)$$

for some ρ that satisfies (1.3.29). Due to Proposition 1.3.2, for given $\theta \in (0,1)$ it holds

$$W^{\tau}_{p,\mu}(J; L_p(\partial\Omega; E)) \cap L_{p,\mu}(J; B^{\vartheta}_{p,p}(\partial\Omega; E)) \hookrightarrow W^{(1-\theta)\tau}_{p,\mu}(J; H^{\vartheta\theta}_p(\partial\Omega; E)),$$

where here the embedding constant is independent of J if one restricts to ${}_0W_{p,\mu}$, and further the Sobolev embedding

$$H^{\vartheta\theta}_p(\partial\Omega; E) \hookrightarrow B^{2m\kappa}_{\rho',p}(\partial\Omega; E) \qquad \text{for} \quad \theta > \frac{2m\kappa}{\vartheta} + \frac{n-1}{\vartheta\rho} \tag{1.3.31}$$

is valid. Therefore, if we choose $\theta > \frac{n-1}{\vartheta r}$, then (1.3.31) holds with some ρ that satisfies (1.3.29). From Proposition 1.1.12 we infer

$$W^{(1-\theta)\tau}_{p,\mu}(J; B^{2m\kappa}_{\rho',p}(\partial\Omega; E)) \hookrightarrow L_{s',\mu}(J; B^{2m\kappa}_{\rho',p}(\partial\Omega; E))$$

for $(1-\theta)\tau - (1-\mu+\frac{1}{p}) > -\frac{p(1-\mu+1/p)}{s'}$, with a uniform embedding constant in the ${}_0W_{p,\mu}$-case. Since it is assumed that $\frac{p(1-\mu+1/p)}{s} < \big(1-\frac{n-1}{\vartheta r}\big)\tau$, the above inequality holds for all $\theta > \frac{n-1}{\vartheta r}$ which are sufficiently close to $\frac{n-1}{\vartheta r}$. This yields (1.3.30) with some ρ that satisfies (1.3.29), and we obtain an estimate of the first summand in (1.3.28) appropriate for (1.3.26).

(III) To estimate the second summand in (1.3.28) we have to show that

$$W^{\tau}_{p,\mu}(J; L_p(\partial\Omega; E)) \cap L_{p,\mu}(J; B^{\vartheta}_{p,p}(\partial\Omega; E)) \hookrightarrow L_{s',\mu}(J; L_{r'}(\partial\Omega; E)),$$

with τ replaced by $\kappa-\delta$ and ϑ replaced by $2m(\kappa-\delta)$ for (1.3.27), where $\delta \in (0,\kappa)$. Using Proposition 1.3.2, it can be seen as above that this embedding holds if $\frac{p(1-\mu+1/p)}{s} < \big(1-\frac{n-1}{\vartheta r}\big)\tau$, with the respective replacements for (1.3.27) and the dependence on J in the ${}_0W_{p,\mu}$-case as asserted. This yields (1.3.26) and (1.3.27), respectively. ■

We next consider temporally weighted Slobodetskii regularity.

Lemma 1.3.22. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, let $J = (0,T)$ be finite, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega = \mathbb{R}^n_+$. Let further $s, r \in [p,\infty)$, $m \in \mathbb{N}$, $\kappa \in (0,1)$, $\kappa \neq 1-\mu+1/p$, $\tau \in (0,2)$ and $\vartheta > 0$ satisfy*

$$\tau > \kappa, \qquad \vartheta > 2m\kappa, \qquad \frac{p(1-\mu+1/p)}{s} < \left(1-\frac{n-1}{\vartheta r}\right)\tau,$$

and suppose further that

$$1 - \frac{1-\mu+1/p}{\tau} > \frac{n-1}{\vartheta r} \qquad \text{if } \kappa > 1-\mu+1/p. \tag{1.3.32}$$

Then it holds

$$|bu|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} \lesssim |b|_{B^{\kappa}_{s,p}(J;L_r(\partial\Omega;\mathcal{B}(E)))} |u|_{W^{\tau}_{p,\mu}(J;L_p(\partial\Omega;E)) \cap L_{p,\mu}(J;B^{\vartheta}_{p,p}(\partial\Omega;E))}. \tag{1.3.33}$$

Moreover, if $\frac{p(1-\mu+1/p)}{s} + \frac{n-1}{2mr} < \kappa$ and

$$\kappa - (1-\mu+1/p) \notin \left(0, \frac{n-1}{2mr}\right), \tag{1.3.34}$$

then there is $\delta \in (0,\kappa)$ *such that*

$$|bu|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} \lesssim |b|_{L_\infty(J;L_\infty(\partial\Omega;\mathcal{B}(E)))}|u|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^{2m\kappa}_{p,p}(\partial\Omega;E))} \qquad (1.3.35)$$
$$+ |b|_{B^{\kappa}_{s,p}(J;L_r(\partial\Omega;\mathcal{B}(E)))}|u|_{W^{\kappa-\delta}_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^{2m(\kappa-\delta)}_{p,p}(\partial\Omega;E))}.$$

Restricting to $u \in {}_0W_{p,\mu}$, *and assuming that* b *is defined on a larger interval* $J_0 = (0,T_0)$, $T_0 > 0$, *these estimates, with* J *replaced by* J_0 *in the norms for* b, *hold with a uniform constant for all* $T \leq T_0$. *In this case, and further in the unweighted case* $\mu = 1$, *one can ignore (1.3.32) and (1.3.34).*

Proof. (I) We intend to use Lemma 1.3.20, and thus have to reduce the estimate to the unweighted case on $\mathbb{R}_+$. If $\kappa > 1-\mu+1/p$ then it is assumed that $1 - \frac{1-\mu+1/p}{\tau} > \frac{n-1}{\vartheta r}$. In this case it follows from the Propositions 1.3.2 and 1.1.11 that

$$W^{\tau}_{p,\mu}(J;L_p(\partial\Omega;E)) \cap L_{p,\mu}(J;B^{\vartheta}_{p,p}(\partial\Omega;E)) \hookrightarrow BUC(\overline{J};L_{r'}(\partial\Omega;E)), \qquad (1.3.36)$$

such that $u_0 := u(0,\cdot) \in L_{r'}(\partial\Omega;E)$ is well defined. Moreover, if $\kappa < 1-\mu+1/p$ and in the ${}_0W_{p,\mu}$-case we set $u_0 := 0$. In both cases we have

$$|bu|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} \leq |b(u-u_0)|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} + |bu_0|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))}. \qquad (1.3.37)$$

In case $\kappa > 1-\mu+1/p$ we use interpolation and (1.3.36) to obtain that for the second summand in (1.3.37) it holds

$$|bu_0|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} \lesssim |b|_{B^{\kappa}_{p,p}(J;L_r(\partial\Omega;\mathcal{B}(E)))}|u_0|_{L_{r'}(\partial\Omega;E)}$$
$$\lesssim |b|_{B^{\kappa}_{s,p}(J;L_r(\partial\Omega;\mathcal{B}(E)))}|u|_{W^{\tau}_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^{\vartheta}_{p,p}(\partial\Omega;E))}, \qquad (1.3.38)$$

as desired for (1.3.33). Replacing τ by κ and ϑ by $2m\kappa$, and noting that the condition on κ is strict, (1.3.38) yields a term as in the second summand of (1.3.35), with some $\delta \in (0,\kappa)$.
(II) To estimate the first summand in (1.3.37) we set $v := u - u_0$, such that

$$v \in {}_0W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))$$

in any case, due to the assumption $\kappa \neq 1-\mu+1/p$ and Proposition 1.1.11. It further follows from Proposition 1.1.11, Lemma 1.1.3, with the help of the extension operators $\mathcal{E}_J$ and $\mathcal{E}^0_J$ from Lemma 1.1.5, and from Lemma 1.3.20, that

$$|bv|_{W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} \lesssim |bv|_{{}_0W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} \lesssim |b(t^{1-\mu}v)|_{{}_0W^{\kappa}_{p}(J;L_p(\partial\Omega;E))}$$
$$\leq |(\mathcal{E}_Jb)(\mathcal{E}^0_Jt^{1-\mu}v)|_{{}_0W^{\kappa}_{p}(\mathbb{R}_+;L_p(\partial\Omega;E))} \lesssim |(\mathcal{E}_Jb)(\mathcal{E}^0_Jt^{1-\mu}v)|_{W^{\kappa}_{p}(\mathbb{R}_+;L_p(\partial\Omega;E))}$$
$$\lesssim |\mathcal{E}_Jb|_{L_\sigma(\mathbb{R}_+;L_\rho(\partial\Omega;\mathcal{B}(E)))}|(\mathcal{E}^0_Jt^{1-\mu}v)|_{B^{\kappa}_{\sigma',p}(\mathbb{R}_+;L_{\rho'}(\partial\Omega;E))}$$
$$+ |\mathcal{E}_Jb|_{B^{\kappa}_{s,p}(\mathbb{R}_+;L_r(\partial\Omega;\mathcal{B}(E)))}|(\mathcal{E}^0_Jt^{1-\mu}v)|_{L_{s'}(\mathbb{R}_+;L_{r'}(\partial\Omega;E))}$$
$$\lesssim |\mathcal{E}_Jb|_{L_\sigma(\mathbb{R}_+;L_\rho(\partial\Omega,\mathcal{B}(E)))}|t^{1-\mu}v|_{{}_0B^{\kappa}_{\sigma',p}(J;L_{\rho'}(\partial\Omega;E))}$$
$$+ |\mathcal{E}_Jb|_{B^{\kappa}_{s,p}(\mathbb{R}_+;L_r(\partial\Omega;\mathcal{B}(E)))}|v|_{L_{s',\mu}(J;L_{r'}(\partial\Omega;E))}, \qquad (1.3.39)$$

where $\frac{1}{r} + \frac{1}{r'} = \frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{p}$ and $\frac{1}{s} + \frac{1}{s'} = \frac{1}{\sigma} + \frac{1}{\sigma'} = \frac{1}{p}$, so that $s', \sigma' \geq p$, have to be chosen appropriately. In the ${}_0W_{p,\mu}$-case, assuming that b is defined on $J_0 = (0,T_0)$ with $T \leq T_0$, in (1.3.39) we can replace J by J_0 and $\mathcal{E}_J$ by $\mathcal{E}_{J_0}$, which leads to constants in (1.3.33) and (1.3.35) as desired.

(III) We consider the first summand in (1.3.39). For $\tau = \kappa$ we take $\rho = \sigma = \infty$, $\rho' = \sigma' = p$, and deduce from Lemma 1.1.3 and (1.3.36) that

$$|t^{1-\mu}v|_{{}_0W^\kappa_p(J;L_p(\partial\Omega;E))} \lesssim |u|_{W^\kappa_{p,\mu}(J;L_p(\partial\Omega;E))} + |u_0|_{W^\kappa_{p,\mu}(J;L_p(\partial\Omega;E))} \lesssim |u|_{W^\kappa_{p,\mu}(J;L_p(\partial\Omega;E))},$$

where the constant for this estimate is of course independent of J if $u_0 = 0$. This yields the first summand in (1.3.35) as desired. Further, for $\tau > \kappa$ we take $\rho = r$, $\rho' = r'$, and estimate for sufficiently small $\varepsilon > 0$

$$\begin{aligned} |t^{1-\mu}v|_{{}_0B^\kappa_{\sigma',p}(J;L_{r'}(\partial\Omega;E))} &\lesssim |v|_{W^{\kappa+\varepsilon}_{\sigma',\mu}(J;L_{r'}(\partial\Omega;E))} \\ &\leq |u|_{W^{\kappa+\varepsilon}_{\sigma',\mu}(J;L_{r'}(\partial\Omega;E))} + |u_0|_{W^{\kappa+\varepsilon}_{\sigma',\mu}(J;L_{r'}(\partial\Omega;E))} \\ &\lesssim |u|_{W^{\kappa+\varepsilon}_{\sigma',\mu}(J;L_{r'}(\partial\Omega;E))} + |u|_{W^\tau_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^\vartheta_{p,p}(\partial\Omega;E))}, \end{aligned}$$

with a uniform constant in the ${}_0W_{p,\mu}$-case. Observe that

$$B^\kappa_{s,p}(\mathbb{R}_+;L_r(\partial\Omega;\mathcal{B}(E))) \hookrightarrow L_\sigma(\mathbb{R}_+;L_r(\partial\Omega;\mathcal{B}(E))) \qquad \text{if} \quad \frac{1}{\sigma'} < \frac{1}{s'} + \kappa. \tag{1.3.40}$$

Since all occurring relations are strict, it thus suffices to show that there exists a $\sigma \geq p$, satisfying (1.3.40), such that

$$W^\tau_{p,\mu}(J;L_p(\partial\Omega;E)) \cap L_{p,\mu}(J;B^{2m\tau}_{p,p}(\partial\Omega;E)) \hookrightarrow W^\kappa_{\sigma',\mu}(J;L_{r'}(\partial\Omega;E))$$

holds true. For $\theta \in (0,1)$, Proposition 1.3.2 yields

$$W^\tau_{p,\mu}(J;L_p(\partial\Omega;E)) \cap L_{p,\mu}(J;B^\vartheta_{p,p}(\partial\Omega;E)) \hookrightarrow W^{(1-\theta)\tau}_{p,\mu}(J;H^{\vartheta\theta}_p(\partial\Omega;E)),$$

with a uniform embedding constant in the ${}_0W_{p,\mu}$-case. It holds

$$H^{\vartheta\theta}_p(\partial\Omega;E) \hookrightarrow L_{r'}(\partial\Omega;E) \qquad \text{if} \quad \theta > \frac{n-1}{\vartheta r}, \tag{1.3.41}$$

and moreover it follows from Proposition 1.1.12 that

$$W^{(1-\theta)\tau}_{p,\mu}(J;L_{r'}(\partial\Omega;E)) \hookrightarrow W^\kappa_{\sigma',\mu}(J;L_{r'}(\partial\Omega;E))$$

if

$$(1-\theta)\tau - (1-\mu+1/p) > \kappa - \frac{p(1-\mu+1/p)}{\sigma'}, \tag{1.3.42}$$

again with a uniform constant in the ${}_0W_{p,\mu}$-case. We can now choose θ and $\sigma \geq p$ satisfying (1.3.40) and (1.3.36) such that (1.3.42) holds, using $\frac{p(1-\mu+1/p)}{s} < \big(1 - \frac{n-1}{\vartheta r}\big)\tau$. So we have shown the asserted estimates for the first summand in (1.3.39).

(IV) For the second summand in (1.3.39) it holds, as above,

$$|v|_{L_{s',\mu}(J;L_{r'}(\partial\Omega;E))} \lesssim |u|_{L_{s',\mu}(J;L_{r'}(\partial\Omega;E))} + |u|_{W^\tau_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;B^\vartheta_{p,p}(\partial\Omega;E))}.$$

Using the Propositions 1.3.2 and 1.1.12, it can be seen as in the previous step that

$$W^\tau_{p,\mu}(J;L_p(\partial\Omega;E)) \cap L_{p,\mu}(J;B^\vartheta_{p,p}(\partial\Omega;E)) \hookrightarrow L_{s',\mu}(J;L_{r'}(\partial\Omega;E))$$

is valid if $\frac{p(1-\mu+1/p)}{s} < \left(1 - \frac{n-1}{\vartheta r}\right)\tau$, with a uniform constant in the ${}_0W_{p,\mu}$-case. This shows (1.3.33). For (1.3.35) the same arguments are valid with τ replaced by $\kappa-\delta$ and ϑ replaced by $2m(\kappa-\delta)$ with $\delta \in (0,\kappa)$. ■

It seems that the exceptions (1.3.32) and (1.3.34) are not essential and only due to our proof. If one had a Littlewood-Paley representation of the spaces $W^s_{p,\mu}$, as for the unweighted Slobodetskii spaces, then one could argue as in Lemma 1.3.20 and also cover the exceptional values.

The above results on pointwise multiplication are rather sharp, and valueable for low values of $p \in (1,\infty)$, compared to κ. It turns out that if p is sufficiently large then the spaces $W^{\kappa,2m\kappa}_{p,\mu}(J\times\partial\Omega)$[13] are closed under pointwise multiplications, and then also b may belong to a temporally weighted space.

Lemma 1.3.23. *Let E be of class $\mathcal{HT}$, let $J=(0,T)$ be finite or infinite and let $p\in(1,\infty)$, $\mu\in(1/p,1]$ and $\kappa\in(0,1)$, $\vartheta\in\mathbb{R}_+\backslash\mathbb{N}$. Then it holds*

$$|bu|_{W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;E)} \lesssim |b|_{L_\infty(J\times\partial\Omega,\mathcal{B}(E))}|u|_{W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;E)} + |b|_{W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;\mathcal{B}(E))}|u|_{L_\infty(J\times\partial\Omega,E)}.$$

Moreover, if

$$\Big(1-\frac{1-\mu+1/p}{\kappa}\Big)\vartheta > \frac{n-1}{p}, \tag{1.3.43}$$

then there is an estimate

$$|bu|_{W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;E)} \lesssim |b|_{W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;\mathcal{B}(E))}|u|_{W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;E)}.$$

Replacing $W^{\kappa,\vartheta}_{p,\mu}$ by ${}_0W^{\kappa,\vartheta}_{p,\mu}$, these estimates are independent of the length of J. If b is defined on a larger interval $J_0=(0,T_0)$, $T\le T_0$, and one restricts to a ${}_0W^{\kappa,\vartheta}_{p,\mu}$-space for u, then the estimates are uniform in T.

Proof. Throughout we denote any occurring sup-norm by $|\cdot|_\infty$.
(I) By Lemma 1.3.19 we have for almost all $t\in J$ that

$$|b(t,\cdot)u(t,\cdot)|_{W^\vartheta_p(\partial\Omega;E)} \lesssim |b(t,\cdot)|_\infty|u(t,\cdot)|_{W^\vartheta_p(\partial\Omega;E)} + |b(t,\cdot)|_{W^\vartheta_p(\partial\Omega;\mathcal{B}(E))}|u(t,\cdot)|_\infty,$$

and we obtain the asserted estimate for $|bu|_{L_{p,\mu}(J;W^\vartheta_p(\partial\Omega;E))}$ by taking the $L_{p,\mu}$-norm. This estimate is always independent of the length of J. For $W^\kappa_{p,\mu}(J;L_p(\partial\Omega;E))$ we use the intrinsic norm given by Proposition 1.1.13 to obtain

$$\begin{aligned}|bu|_{W^\kappa_{p,\mu}(J;L_p(\partial\Omega;E))} \lesssim{}& |b|_\infty|u|_{L_{p,\mu}(J;L_p(\partial\Omega;E))} \\ &+ |b|_\infty[u]_{W^\kappa_{p,\mu}(J;L_p(\partial\Omega;E))} + [b]_{W^\kappa_{p,\mu}(J;L_p(\partial\Omega;\mathcal{B}(E)))}|u|_\infty.\end{aligned} \tag{1.3.44}$$

Note that this estimate also holds true for $J=\mathbb{R}_+$.
(II) Now let $b,u\in{}_0W^{\kappa,\vartheta}_{p,\mu}$. To get an estimate independent of J we cannot use the intrinsic norm for ${}_0W^\kappa_{p,\mu}$ directly (see the discussion in Remark 1.1.15). We therefore take the

[13] Recall the notation $W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;E) = W^\kappa_{p,\mu}(J;L_p(\partial\Omega;E))\cap L_{p,\mu}(J;W^\vartheta_p(\partial\Omega;E))$.

extension operator $\mathcal{E}_J^0$ from Proposition 1.1.5, whose norm is independent of the length of J, and estimate, using Proposition 1.1.11 and (1.3.44) on the half-line,

$$\begin{aligned}|bu|_{{}_0W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} &\le |\mathcal{E}_J^0 b\, \mathcal{E}_J^0 u|_{W^{\kappa}_{p,\mu}(\mathbb{R}_+;L_p(\partial\Omega;E))}\\ &\le |\mathcal{E}_J^0 b|_\infty |\mathcal{E}_J^0 u|_{W^{\kappa}_{p,\mu}(\mathbb{R}_+;L_p(\partial\Omega;E))} + |\mathcal{E}_J^0 b|_{W^{\kappa}_{p,\mu}(\mathbb{R}_+;L_(\partial\Omega;\mathcal{B}(E)))}|\mathcal{E}_J^0 u|_\infty\\ &\lesssim |b|_\infty |u|_{{}_0W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;E))} + |b|_{{}_0W^{\kappa}_{p,\mu}(J;L_p(\partial\Omega;\mathcal{B}(E)))}|u|_\infty.\end{aligned}$$

These estimates are independent of the length of J. If b is defined on a larger interval $J_0 = (0, T_0)$, then one may replace $\mathcal{E}_J^0$ by $\mathcal{E}_{J_0}$ in the above arguments to obtain an estimate uniformly in T in this case.

(III) Finally, if (1.3.43) is valid then the asserted estimate follows from

$$W^{\kappa,\vartheta}_{p,\mu}(J\times\partial\Omega;E) \hookrightarrow C(\overline{J}\times\partial\Omega;E), \tag{1.3.45}$$

which is due to Proposition 1.3.2 and Sobolev's embeddings, and is independent of the length of J in the ${}_0W^{\kappa,\vartheta}_{p,\mu}$-case and independent of $T \le T_0$ if b is defined on J_0. ∎

We emphasize that for $\vartheta = 2m\kappa$ the condition (1.3.43) is equivalent to $\kappa > 1-\mu+1/p+\frac{n-1}{2mp}$.

We summarize the above pointwise multiplication results for the coefficients of boundary differential operators as follows.

Proposition 1.3.24. *Let E be a Banach space of class $\mathcal{HT}$, $p\in(1,\infty)$, $\mu\in(1/p,1]$, let $J=(0,T)$ be a finite interval, and let $\Omega\subset\mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, or $\Omega=\mathbb{R}^n_+$. Let further $m\in\mathbb{N}$ and $k\in\mathbb{N}_0$, $k\le 2m-1$, define the number*

$$\kappa := 1-\frac{k}{2m}-\frac{1}{2mp} \qquad \textit{and suppose that } \ \kappa\ne 1-\mu+1/p.$$

Assume that for the $\mathcal{B}(E)$-valued coefficient $b=b(t,x)$ of the operator $b\,\mathrm{tr}_\Omega\nabla^\beta$, where $\beta\in\mathbb{N}_0^n$ with $|\beta|\le k$, one of the following two conditions is valid: either

$$\kappa > 1-\mu+1/p+\frac{n-1}{2mp} \quad \textit{and} \quad b\in W^{\kappa,2m\kappa}_{p,\mu}(J\times\partial\Omega;\mathcal{B}(E)), \tag{1.3.46}$$

or it holds

$$b\in B^{\kappa}_{s_\beta,p}(J;L_{r_\beta}(\partial\Omega;\mathcal{B}(E)))\cap L_{s_\beta}(J;B^{2m\kappa}_{r_\beta,p}(\partial\Omega;\mathcal{B}(E))), \tag{1.3.47}$$

with numbers $s_\beta, r_\beta\in[p,\infty)$ so that

$$\frac{p(1-\mu)+1}{s_\beta}+\frac{n-1}{2mr_\beta} < \kappa+\frac{k-|\beta|}{2m}, \quad \kappa+\frac{k-|\beta|}{2m}-(1-\mu+1/p)\notin\Big(0,\frac{n-1}{2mr_\beta}\Big).$$

Then in both cases we have

$$b\,\mathrm{tr}_\Omega\nabla^\beta \in \mathcal{B}\big(W^{1,2m}_{p,\mu}(J\times\Omega;E), W^{\kappa,2m\kappa}_{p,\mu}(J\times\partial\Omega;E)\big),$$

and if $|\beta|=k$ then

$$b\in BUC(\overline{J}\times\partial\Omega;\mathcal{B}(E)), \qquad |\beta|=m_j, \qquad j=1,...,m.$$

Proof. It follows from Lemma 1.3.4 and Proposition 1.3.12 that $\mathrm{tr}_\Omega \nabla^\beta$ maps

$$W^{1,2m}_{p,\mu}(J \times \Omega; E) \to W^{1-|\beta|/2m-1/2mp, 2m-|\beta|-1/p}_{p,\mu}(J \times \partial\Omega; E)$$

in a continuous way. Observe that the latter space embeds into $W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E)$, since $|\beta| \leq k$. Assume that (1.3.46) holds. Then Lemma 1.3.23 implies that $b\,\mathrm{tr}_\Omega \nabla^\beta$ maps continuously as asserted. The continuity of b follows from (1.3.45).
Next assume (1.3.47). Then we can apply the Lemmas 1.3.21 and 1.3.22 with $\tau = 1 - |\beta|/2m - 1/2mp$ and $\vartheta = 2m\tau$ to obtain the asserted mapping property of $b\,\mathrm{tr}_\Omega \nabla^\beta$. In case $|\beta| = k$ it holds

$$B^{\kappa}_{s_\beta,p}(J; L_{r_\beta}(\partial\Omega; \mathcal{B}(E))) \cap L_{s_\beta}(J; B^{2m\kappa}_{r_\beta,p}(\partial\Omega; \mathcal{B}(E))) \hookrightarrow C(\overline{J} \times \partial\Omega; \mathcal{B}(E)),$$

which follows from Proposition 1.3.2, the remark thereafter, Proposition 1.1.11 and Sobolev's embeddings, and shows that b is a continuous function. ■

We finish this section with a technical result on compatible data on the boundary.

Lemma 1.3.25. *In the situation of Proposition 1.3.24, for $\kappa > 1 - \mu + 1/p$ the sets*

$$\mathcal{D} := \big\{(g, u_0) \in W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E) \times B^{2m(\mu-1/p)}_{p,p}(\Omega; E) \ :\ b(0,\cdot)\,\mathrm{tr}_\Omega \nabla^\beta u_0 = g(0,\cdot) \ \text{ on } \Gamma\big\},$$

$$\mathcal{D}_0 := \big\{(g, u_0) \in \mathcal{D} \ :\ g \in {}_0W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E)\big\},$$

are well-defined and closed subspaces of $W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E) \times B^{2m(\mu-1/p)}_{p,p}(\Omega; E)$ and ${}_0W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E) \times B^{2m(\mu-1/p)}_{p,p}(\Omega; E)$, respectively.

Proof. If (1.3.46) is valid, then $b(0,\cdot)$ always exists. To obtain this in case (1.3.47), note that $\kappa > 1 - \mu + 1/p$ in particular yields $\kappa > 1/s$. Hence $\mathcal{D}$ and $\mathcal{D}_0$ are well-defined in both cases.
To show that $\mathcal{D}$ is closed in $W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E) \times B^{2m(\mu-1/p)}_{p,p}(\Omega; E)$ take a sequence $(g^k, u_0^k)_{k\in\mathbb{N}} \in \mathcal{D}$, and assume that (g^k, u_0^k) converges to (g, u_0) as $k \to \infty$ with respect to the norm of $W^{\kappa,2m\kappa}_{p,\mu}(J \times \partial\Omega; E) \times B^{2m(\mu-1/p)}_{p,p}(\Omega; E)$. It is then a consequence of (1.3.20) and Theorem 1.3.6 that $(g^k(0,x), \mathrm{tr}_\Omega \nabla^\beta u_0^k(x))_{k\in\mathbb{N}}$ converges (up to a subsequence) to $(g(0,x), \mathrm{tr}_\Omega \nabla^\beta u_0(x))$ as $k \to \infty$, for almost every $x \in \partial\Omega$. Moreover, for all k the identity

$$b(0,x)\mathrm{tr}_\Omega \nabla^\beta u_0^k(x) = g^k(0,x)$$

is valid for almost every $x \in \partial\Omega$. Taking the limit, we obtain that $b(0,x)\mathrm{tr}_\Omega \nabla^\beta u_0(x) = g(0,x)$ holds true for all $x \in \partial\Omega$ which are not contained in a countable union of subsets of surface measure zero of $\partial\Omega$. This yields $(g, u_0) \in \mathcal{D}$. The closedness of $\mathcal{D}_0$ follows from the same arguments. ■

Chapter 2

Maximal $L_{p,\mu}$-Regularity for Static Boundary Conditions

In this chapter we develop the maximal $L_{p,\mu}$-regularity approach for a general class of parabolic initial-boundary value problems with inhomogeneous static boundary conditions, generalizing the results of Denk, Hieber & Prüss [25]. In Section 2.1 we describe the approach and the involved function spaces in detail, provide examples, describe the advantages compared to the unweighted approach, and give an outline of the strategy how to obtain the main result of the present chapter, Theorem 2.1.4. The proof of the theorem is carried out in detail in the Sections 2.2, 2.3, and 2.4, and follows [25]. In Section 2.5 we show that related boundary operators admit a continuous right-inverse.

2.1 The Problem and the Approach in Weighted Spaces

The Problem

For the unknown $u = u(t,x) \in E$ we consider the linear inhomogeneous, nonautonomous parabolic initial-boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, && t \in J, \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x), && x \in \Gamma, && t \in J, && j = 1,...,m, \\ u(0,x) &= u_0(x), && x \in \Omega. \end{aligned} \tag{2.1.1}$$

We assume that $\Omega \subset \mathbb{R}^n$ is a domain with compact smooth boundary $\Gamma = \partial\Omega$, that $J = (0,T)$ is a finite interval, $T > 0$, and that E is a complex Banach space of class $\mathcal{HT}$. The differential operator $\mathcal{A}$ of order $2m$, where $m \in \mathbb{N}$, is given by

$$\mathcal{A}(t,x,D) = \sum_{|\alpha| \leq 2m} a_\alpha(t,x) D^\alpha, \qquad x \in \Omega, \qquad t \in J,$$

where $D = -\mathrm{i}\nabla$, and $\nabla = (\partial_{x_1}, ..., \partial_{x_n})$ denotes the euclidian gradient on $\mathbb{R}^n$. The dynamic equation in the domain is complemented by m boundary conditions of order at most $2m-1$.

The boundary operators $\mathcal{B}_j$ are of the form

$$\mathcal{B}_j(t,x,D) = \sum_{|\beta|\le m_j} b_{j\beta}(t,x)\mathrm{tr}_\Omega D^\beta, \qquad x\in\Gamma,\quad t\in J, \qquad j=1,...,m,$$

where tr_Ω denotes the trace on Ω, and where the integer $m_j \in \{0,...,2m-1\}$ is the order of $\mathcal{B}_j$. Observe how $\mathcal{B}_j$ acts on a function u: one first applies the components of the euclidian gradient and then the spatial trace. We assume that each of the operators $\mathcal{B}_j$ is nontrivial, $\mathcal{B}_j \neq 0$, and write

$$\mathcal{B} := (\mathcal{B}_1, ..., \mathcal{B}_m).$$

The coefficients of the operators take values in the bounded linear operators on E, i.e.,

$$a_\alpha(t,x) \in \mathcal{B}(E), \qquad x\in\Omega,\quad t\in J, \qquad |\alpha|\le 2m,$$

$$b_{j\beta}\in\mathcal{B}(E), \qquad x\in\Gamma,\quad t\in J, \qquad |\beta|\le m_j, \qquad j=1,...,m.$$

Finally, the data on the right-hand side is E-valued, and is assumed to be given.

Example 2.1.1. We consider two problems that fit into the above framework. The first is a linearized reaction-diffusion system, given by

$$\begin{aligned} \partial_t u - \Delta u &= f(t,x), && x\in\Omega, && t\in J,\\ \partial_\nu u &= g(t,x), && x\in\Gamma, && t\in J,\\ u(0,x) &= u_0(x), && x\in\Omega, \end{aligned}$$

where $\partial_\nu = \nu\cdot\mathrm{tr}_\Omega\nabla$ denotes the derivative with respect to the outer unit normal field ν of Γ. Here, the order of $\mathcal{A}(D) = -\Delta$ is 2, thus we have $m=1$, and the order of the boundary operator $\mathcal{B}_1(x,D) = \partial_\nu$ is $m_1 = 1$.
A further problem that fits into our framework is a linearized Cahn-Hilliard phase field model, given by

$$\begin{aligned} \partial_t u + \Delta^2 u - \Delta u &= f(t,x), && x\in\Omega, && t\in J,\\ -\partial_\nu\Delta u + \partial_\nu u &= g_1(t,x), && x\in\Gamma, && t\in J,\\ \partial_\nu u &= g_2(t,x), && x\in\Gamma, && t\in J,\\ u(0,x) &= u_0(x), && x\in\Omega. \end{aligned}$$

Here $\mathcal{A}(D) = \Delta^2 - \Delta$ is of order 4, which means $m=2$, and the dynamic equation in Ω is complemented by two boundary conditions, with $\mathcal{B}_1(x,D) = -\partial_\nu\Delta + \partial_\nu$, $m_1 = 3$, and $\mathcal{B}_2(x,D) = \partial_\nu$, $m_2 = 1$. ■

The Approach in the $L_{p,\mu}$-spaces

We describe the maximal $L_{p,\mu}$-regularity approach for (2.1.1). Let

$$p\in(1,\infty), \qquad \mu\in(1/p,1].$$

The basis of the approach is that the domain inhomogeneity f and the solution u shall satisfy

$$f,\ u,\ \partial_t u,\ \mathcal{A}u \in \mathbb{E}_{0,\mu} := L_{p,\mu}(J; L_p(\Omega; E)).$$

These assumptions determine the regularity of u and the other data as follows. Since $\mathcal{A}$ is of order $2m$, it should hold

$$u \in \mathbb{E}_{u,\mu} := W^1_{p,\mu}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W^{2m}_p(\Omega; E))$$

for the solution of (2.1.1). For the initial value, Theorem 1.3.6 on temporal traces yields

$$u_0 \in X_{u,\mu} := B^{2m(\mu-1/p)}_{p,p}(\Omega; E).$$

For the boundary inhomogeneities, since the operator $\mathcal{B}_j$ is of order m_j, a successive application of Lemma 1.3.4 on spatial derivatives, together with Proposition 1.3.12 on the spatial trace on anisotropic spaces yields

$$g_j \in \mathbb{F}_{j,\mu} := W^{\kappa_j}_{p,\mu}(J; L_p(\Gamma; E)) \cap L_{p,\mu}(J; W^{2m\kappa_j}_p(\Gamma; E)), \qquad j = 1, ..., m,$$

where the number $\kappa_j \in (0,1)$ is given by

$$\kappa_j := 1 - \frac{m_j}{2m} - \frac{1}{2mp}.$$

In the sequel we also write

$$\mathbb{F}_\mu := \mathbb{F}_{1,\mu} \times ... \times \mathbb{F}_{m,\mu}, \qquad g = (g_1, ..., g_m) \in \mathbb{F}_\mu,$$

and we further put

$${}_0\mathbb{E}_{u,\mu} := {}_0W^1_{p,\mu}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W^{2m}_p(\Omega; E)),$$

$${}_0\mathbb{F}_{j,\mu} := {}_0W^{\kappa_j}_{p,\mu}(J; L_p(\Gamma; E)) \cap L_{p,\mu}(J; W^{2m\kappa_j}_p(\Gamma; E)), \qquad {}_0\mathbb{F}_\mu := {}_0\mathbb{F}_{1,\mu} \times ... \times {}_0\mathbb{F}_{m,\mu}.$$

We also write $\mathbb{E}_{0,\mu}(J)$ and $\mathbb{E}_{0,\mu}(J \times \Omega)$, and similar for the other spaces above, if the dependence on the underlying interval and domain might not be clear from the context.

As a consequence of the above regularity assumption, (2.1.1) might a priori not be solveable for all data (f, g, u_0). In fact, for $\kappa_j > 1 - \mu + 1/p$, which is equivalent to $2m(\mu - 1/p) > m_j + 1/p$, it holds

$$\mathbb{F}_{j,\mu} \hookrightarrow BUC(\overline{J}; B^{2m(\mu-1/p)-m_j-1/p}_{p,p}(\Gamma; E)),$$

due to Theorem 1.3.6. In this case, if the boundary equation in (2.1.1) holds for $t > 0$, by continuity it necessarily also holds for $t = 0$, and this yields

$$\mathcal{B}_j(0, x, D)u_0(x) = g_j(0, x), \qquad x \in \Gamma, \qquad \text{if } \kappa_j > 1 - \mu + 1/p. \tag{2.1.2}$$

Here $\mathcal{B}_j u_0$ is well-defined for $u_0 \in X_{u,\mu}$ and $2m(\mu - 1/p) > m_j + 1/p$ provided the coefficients of $\mathcal{B}_j$ are sufficiently smooth.

Thus if $\kappa_j > 1 - \mu + 1/p$ for some j, then with the above approach (2.1.1) it is not solvable in $\mathbb{E}_{u,\mu}$ for arbitrary data $g \in \mathbb{F}_\mu$ and $u_0 \in X_{u,\mu}$. In this case the compatibility condition (2.1.2) on g and u_0 is necessary. For short, the boundary equation has to hold up to $t = 0$ if the involved expressions are well defined.

Example 2.1.2. We reconsider the problems from Example 2.1.1. For the linearized reaction-diffusion system, the weighted maximal regularity class and the regularity classes of the data are given by

$$\mathbb{E}_{u,\mu} = W^1_{p,\mu}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W^2_p(\Omega; E)),$$

$$X_{u,\mu} = B^{2(\mu-1/p)}_{pp}(\Omega; E), \qquad \mathbb{F}_{1,\mu} = W^{1/2-1/2p}_{p,\mu}(J; L_p(\Gamma; E)) \cap L_{p,\mu}(J; W^{1-1/p}_p(\Gamma; E)),$$

i.e., $\kappa_1 = 1/2 - 1/2p$. Compatibility conditions are necessary if $2(\mu - 1/p) > 1 + 1/p$.

For the linearized Cahn-Hilliard model we have

$$\mathbb{E}_{u,\mu} = W^1_{p,\mu}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W^4_p(\Omega; E)), \qquad X_{u,\mu} = B^{4(\mu-1/p)}_{pp}(\Omega; E),$$

as well as $\kappa_1 = 1/4 - 1/4p$, so that

$$\mathbb{F}_{1,\mu} = W^{1/4-1/4p}_{p,\mu}(J; L_p(\Gamma; E)) \cap L_{p,\mu}(J; W^{1-1/p}_p(\Gamma; E)),$$

and further $\kappa_2 = 3/4 - 1/4p$, so that

$$\mathbb{F}_{2,\mu} = W^{3/4-1/4p}_{p,\mu}(J; L_p(\Gamma; E)) \cap L_{p,\mu}(J; W^{3-1/p}_p(\Gamma; E)).$$

Here compatibility conditions in the first and the second boundary equation are necessary if $4(\mu - 1/p) > 3 + 1/p$ and $4(\mu - 1/p) > 1 + 1/p$, respectively. ■

We intend to solve (2.1.1) in the following sense.

Definition 2.1.3. *We say that the problem (2.1.1) enjoys the property of maximal $L_{p,\mu}$-regularity on the interval J, if the regularity assumptions on the data, i.e.,*

$$f \in \mathbb{E}_{0,\mu}, \qquad g \in \mathbb{F}_\mu, \qquad u_0 \in X_{u,\mu},$$

together with the compatibility conditions (2.1.2), are not only necessary for a unique solution $u \in \mathbb{E}_{u,\mu}$ of (2.1.1), but also sufficient.

The Assumptions on the Operators

Let $\mathcal{P}(D) = \sum_{|\gamma|\le k} p_\gamma D^\gamma$ be a differential operator of order $k \in \mathbb{N}_0$, with coefficients p_γ. By the subscript $\sharp$ we denote the principal part of $\mathcal{P}$, i.e.,

$$\mathcal{P}_\sharp(D) = \sum_{|\gamma|=k} p_\gamma D^\gamma.$$

The symbol of $\mathcal{P}$ is given by the polynomial expression $\mathcal{P}(\xi) = \sum_{|\gamma|\le k} p_\gamma \xi^\gamma$, where $\xi \in \mathbb{R}^n$.

We describe the assumptions on the coefficients of the operators. It is required that each summand occurring in $\mathcal{A}$ and $\mathcal{B}_j$ is a continuous operator on the respective underlying spaces, i.e.,

$$a_\alpha D^\alpha \in \mathcal{B}(\mathbb{E}_{u,\mu}, \mathbb{E}_{0,\mu}), \qquad |\alpha| \le 2m, \tag{2.1.3}$$

and further

$$b_{j\beta} \mathrm{tr}_\Omega D^\beta \in \mathcal{B}(\mathbb{E}_{u,\mu}, \mathbb{F}_{j,\mu}), \qquad |\beta| \le m_j, \qquad j = 1, ..., m. \tag{2.1.4}$$

Moreover, the top order coefficients are required to be continuous on $\overline{J} \times \overline{\Omega}$. The Propositions 1.3.16 and 1.3.24 show that the following assumptions are sufficient for these purposes.

(SD) For $|\alpha| < 2m$ one of the following two conditions is valid: either

$$2m(\mu - 1/p) > 2m - 1 + n/p \quad \text{and} \quad a_\alpha \in \mathbb{E}_{0,\mu}(J \times \Omega; \mathcal{B}(E)),$$

or there are $r_\alpha, s_\alpha \in [p, \infty)$ with $\frac{p(1-\mu)+1}{s_\alpha} + \frac{n}{2mr_\alpha} < 1 - \frac{|\alpha|}{2m}$ such that

$$a_\alpha \in L_{s_\alpha}\big(J; (L_{r_\alpha} + L_\infty)(\Omega; \mathcal{B}(E))\big).$$

For $|\alpha| = 2m$ it holds $a_\alpha \in BUC(\overline{J} \times \overline{\Omega}; \mathcal{B}(E))$, and if Ω is unbounded then in addition the limits $a_\alpha(t, \infty) := \lim_{|x|\to\infty} a_\alpha(t, x)$ exist uniformly in $t \in \overline{J}$.

(SB) For $j = 1, ..., m$ and $|\beta| \leq m_j$ one of the following two conditions is valid: either

$$\kappa_j > 1 - \mu + 1/p + \frac{n-1}{2mp} \qquad \text{and} \qquad b_{j\beta} \in \mathbb{F}_{j,\mu}(J \times \Gamma; \mathcal{B}(E)),$$

or there are $r_{j\beta}, s_{j\beta} \in [p, \infty)$ with

$$\frac{p(1-\mu)+1}{s_{j\beta}} + \frac{n-1}{2mr_{j\beta}} < \kappa_j + \frac{m_j - |\beta|}{2m}, \quad \kappa_j + \frac{m_j - |\beta|}{2m} - (1-\mu+1/p) \notin \left(0, \frac{n-1}{2mr_{j\beta}}\right),$$

such that

$$b_{j\beta} \in B^{\kappa_j}_{s_{j\beta},p}\big(J; L_{r_{j\beta}}(\Gamma; \mathcal{B}(E))\big) \cap L_{s_{j\beta}}\big(J; B^{2m\kappa_j}_{r_{j\beta},p}(\Gamma; \mathcal{B}(E))\big).$$

Assuming (SB), Proposition 1.3.24 show that for the top order coefficients of $\mathcal{B}$ it holds

$$b_{j\beta} \in BUC(\overline{J} \times \Gamma; \mathcal{B}(E)), \qquad |\beta| = m_j, \qquad j = 1, ..., m.$$

Observe that the first conditions in (SD) , where the coefficients belong to the weighted space $\mathbb{F}_{j,\mu}$, is made for large p, and will be needed in the applications to quasilinear linear problems. The second condition, where b belongs to an unweighted space, is made for lower values of p, and can be useful in the context of a priori estimates for the underlying problem.

We impose two structural assumptions on the operators. The first is normal ellipticity.

(E) For all $t \in \overline{J}$, $x \in \overline{\Omega}$ and $|\xi| = 1$ it holds $\sigma(\mathcal{A}_\sharp(t, x, \xi)) \subset \mathbb{C}_+ := \{\operatorname{Re} \lambda > 0\}$. If Ω is unbounded then it holds in addition $\sigma(\mathcal{A}_\sharp(t, \infty, \xi)) \subset \mathbb{C}_+$ for all $t \in \overline{J}$ and $|\xi| = 1$.

The second is a condition of Lopatinskii-Shapiro type. For each $x \in \Gamma$ we fix an orthogonal matrix $\mathcal{O}_{\nu(x)}$ that rotates the outer unit normal $\nu(x)$ of Γ at x to $(0, ..., 0, -1) \in \mathbb{R}^n$, and define the rotated operators $(\mathcal{A}^\nu, \mathcal{B}^\nu)$ by

$$\mathcal{A}^\nu(t, x, D) := \mathcal{A}(t, x, \mathcal{O}^T_{\nu(x)} D), \qquad \mathcal{B}^\nu(t, x, D) := \mathcal{B}(t, x, \mathcal{O}^T_{\nu(x)} D).$$

We assume the following.

(LS) For each fixed $t \in \overline{J}$ and $x \in \Gamma$, for all $\lambda \in \overline{\mathbb{C}_+}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$ and all $h \in E^m$ the ordinary initial value problem

$$\begin{aligned} \lambda v(y) + \mathcal{A}^{\nu}_{\sharp}(t, \xi', D_y) v(y) &= 0, \qquad y > 0, \\ \mathcal{B}^{\nu}_{j\sharp}(t, \xi', D_y) v|_{y=0} &= h_j, \qquad j = 1, ..., m, \end{aligned}$$

has a unique solution $v \in C_0([0,\infty); E)$.[1]

If E is finite dimensional, then it is necessary and sufficient for (LS) that the above initial value problem has for $h = 0$ only the trivial solution.

The Main Theorem and the Advantages of the Approach

The main result of this chapter reads as follows.

Theorem 2.1.4. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$ and $\mu \in (1, p, 1]$. Let $J = (0, T)$ be a finite interval, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$. Assume that* (E), (LS), (SD) *and* (SB) *hold true, and that $\kappa_j \neq 1 - \mu + 1/p$ for $j = 1, ..., m$. Then the problem*

$$\begin{aligned} \partial_t u + \mathcal{A}(t, x, D) u &= f(t, x), \qquad x \in \Omega, \qquad t \in J, \\ \mathcal{B}_j(t, x, D) u &= g_j(t, x), \qquad x \in \Gamma, \qquad t \in J, \qquad j = 1, ..., m, \\ u(0, x) &= u_0(x), \qquad x \in \Omega, \end{aligned}$$

enjoys maximal $L_{p,\mu}$-regularity, i.e., it has a unique solution $u = \mathcal{L}(f, g, u_0) \in \mathbb{E}_{u,\mu}$ if and only if

$$\begin{aligned} (f, g, u_0) \in \mathcal{D} := \big\{ (f, g, u_0) \in \mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu} \;:\; &\text{ for } j = 1, ..., m \text{ it holds} \\ &\mathcal{B}_j(0, \cdot, D) u_0 = g_j(0, \cdot) \text{ on } \Gamma \text{ if } \kappa_j > 1 - \mu - 1/p \big\}. \end{aligned}$$

The corresponding solution operator $\mathcal{L} : \mathcal{D} \to \mathbb{E}_{u,\mu}$ is continuous. If $\mathcal{L}$ is restricted to

$$\mathcal{D}_0 := \big\{ (f, g, u_0) \in \mathcal{D} \;:\; g \in {}_0\mathbb{F}_\mu \big\},$$

for given $T_0 > 0$ its operator norm is uniform for all $T \leq T_0$. Finally, if the coefficients

$$(-\mathrm{i})^{|\alpha|} a_\alpha, \quad |\alpha| \leq 2m, \qquad (-\mathrm{i})^{|\beta|} b_{j\beta}, \quad |\beta| \leq m_j, \quad j = 1, ..., m, \tag{2.1.5}$$

and the data are real-valued, then also the solution u is real-valued.

Due to Lemma 1.3.25, the spaces of compatible data $\mathcal{D}$ and $\mathcal{D}_0$ are well-defined and Banach spaces when equipped with the norms of $\mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu}$ and $\mathbb{E}_{0,\mu} \times {}_0\mathbb{F}_\mu \times X_{u,\mu}$, respectively. It is important to distinguish between the norms of $\mathbb{F}_\mu$ and ${}_0\mathbb{F}_\mu$. These are equivalent for $\kappa_j \neq 1 - \mu + 1/p$, but the norm equivalent constants depend on the length of the underlying interval J. Our motivation to introduce the space $\mathcal{D}_0$ is to obtain estimates uniform in time for problems with vanishing initial values, as they typically occur in the

[1] The space $C_0([0, \infty); E)$ consists of the continuous E-valued functions on $[0, \infty)$ vanishing at ∞.

context of fixed point arguments (see the discussion in Remark 1.1.15). Observe that for $(f, g, u_0) \in \mathcal{D}_0$ it necessarily holds $\mathcal{B}_j(0, \cdot, D)u_0 = 0$ on Γ if this expression makes sense, i.e., if $\kappa_j > 1 - \mu + 1/p$.

Compared to the unweighted case, the maximal regularity approach in weighted spaces has the following advantages.

- *Flexible initial regularity:* We obtain solutions for initial values in $B^s_{p,p}(\Omega; E)$, where $s \in (0, 2m(1 - 1/p)]$.
- *Inherent smoothing effect:* Away from the initial time, $\tau \in (0, T)$, the solutions belong to the unweighted space

$$\mathbb{E}_{u,1}(\tau, T) = W^1_p(\tau, T; L_p(\Omega; E)) \cap L_p(\tau, T; W^{2m}_p(\Omega; E)) \hookrightarrow C(\overline{J}; B^{2m(1-1/p)}_{p,p}(\Omega; E)).$$

- *Control solutions in a strong norm at a later time by a weaker norm at an earlier time and the data*: For $s = 2m(\mu - 1/p) \in (0, 2m(1 - 1/p)]$ it holds

$$|u(T)|_{B^{2m(1-1/p)}_{pp}(\Omega;E)} \leq C(T)(|f|_{\mathbb{E}_{0,\mu}} + |g|_{\mathbb{F}_\mu} + |u_0|_{B^s_{p,p}(\Omega;E)}).$$

- *Avoid compatibility conditions:* Given $p \in (1, \infty)$, if μ is sufficiently close to $1/p$ then $\kappa_j < 1 - \mu + 1/p$ for all j, such that there is a unique solution $u \in \mathbb{E}_{u,\mu}$ for arbitrary data in $\mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu}$.

Outline of the Proof

The proof of Theorem 2.1.4 is inspired by the one of Denk, Hieber & Prüss [24, 25] in the unweighted case. The strategy for a bounded domain Ω is as follows, for unbounded domains it has to be slightly modified.

One describes the boundary Γ of Ω by a finite collection of charts (U_i, φ_i), $i = 1, ..., N_H$, and further takes open sets U_i, $i = N_H + 1, ..., N_F$, such that $U_i \cap \Gamma = \emptyset$ and $\overline{\Omega} \subset \bigcup_{i=1}^{N_F} U_i$. This yields local problems, with boundary conditions for $i = 1, ..., N_H$ and without boundary conditions for $i = N_H + 1, ..., N_F$. The problems without boundary conditions are extended to a full-space problem and the problems with boundary conditions are transformed and extended to a half-space problem, using the push-forward corresponding to charts ϕ_i. This is done in Section 2.4. If the diameter of the U_i are sufficiently small then by continuity the top order coefficients of the resulting operators are of small oscillation, such that, by a perturbation argument which is based on the contraction principle, one can neglect lower order terms and assume that the coefficients are constant, see Section 2.3. The resulting full- and half-space problems are solved in Section 2.2. At the end of Section 2.4, these solutions are put together to a solution of the original problem, using a partition of unity for $\overline{\Omega}$ subordinate to the cover $\bigcup_{i=1}^{N_F} U_i$.

2.2 Top Order Constant Coefficient Operators on $\mathbb{R}^n$ and $\mathbb{R}^n_+$

2.2.1 The Full-Space Case without Boundary Conditions

For constant coefficients $a_\alpha \in \mathcal{B}(E)$ we consider the differential operator

$$\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha.$$

Observe that there are no lower order terms so that $\mathcal{A}(D)$ is homogeneous of degree $2m$. We show that (E) implies parameter-ellipticity, in the sense of [24, Definition 5.1], with angle of ellipticity strictly smaller than $\pi/2$.

Lemma 2.2.1. *Assume that $\mathcal{A}$ satisfies* (E). *Then there is $\phi \in (0, \pi/2)$ such that*

$$\sigma(A(\xi)) \subset \Sigma_\phi = \{\lambda \in \mathbb{C}\backslash\{0\} \ : \ |\arg \lambda| < \phi\}, \qquad |\xi| = 1, \qquad \xi \in \mathbb{R}^n.$$

Proof. For $|\xi| = 1$ it holds $|\xi^\alpha| \leq 1$ for all $|\alpha| = 2m$, and therefore

$$|A(\xi)|_{\mathcal{B}(E)} \leq \sum_{|\alpha|=2m} |a_\alpha|_{\mathcal{B}(E)},$$

which yields that the spectral radius of $A(\xi)$ is uniformly bounded in $|\xi| = 1$. Thus there is $R > 0$, independent of $|\xi| = 1$, such that $\lambda \in \rho(A(\xi))$ for all λ with $|\lambda| > R$ or, by assumption, $\lambda \in \overline{\mathbb{C}_-}$. Since the resolvent set is open, it follows from continuity and compactness that for all $\lambda = \mathrm{i}\theta$ with $\theta \in [-R, R]$ there is a neighbourhood $U_{\mathrm{i}\theta} \subset \mathbb{C}$ of $\mathrm{i}\theta$ such that $U_{\mathrm{i}\theta} \subset \rho(A(\xi))$ for all $|\xi| = 1$. Again compactness yields a radius $r > 0$, which does not depend on $\theta \in [-R, R]$, such that $B_r(\mathrm{i}\theta) \subset \rho(A(\xi))$ for all $|\xi| = 1$. We thus obtain an angle $\phi \in (0, \pi/2)$ with $\rho(A(\xi)) \supset \mathbb{C}\backslash\overline{\Sigma_\phi}$. ■

We have the following maximal $L_{p,\mu}$-regularity result for $\mathcal{A}$ on the half-line.

Proposition 2.2.2. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, $\mu \in (1/p, 1]$, and assume that $\mathcal{A}$ satisfies* (E). *Then there is a unique solution $u = \mathcal{S}_F(f, u_0) \in \mathbb{E}_{u,\mu}(\mathbb{R}_+ \times \mathbb{R}^n)$ of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= f(t, x), \qquad x \in \mathbb{R}^n, \qquad t > 0, \\ u(0, x) &= u_0(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{2.2.1}$$

if and only if

$$f \in \mathbb{E}_{0,\mu}(\mathbb{R}_+ \times \mathbb{R}^n), \qquad u_0 \in X_{u,\mu}(\mathbb{R}^n).$$

The corresponding solution operator $\mathcal{S}_F : \mathbb{E}_{0,\mu}(\mathbb{R}_+ \times \mathbb{R}^n) \times X_{u,\mu}(\mathbb{R}^n) \to \mathbb{E}_{u,\mu}$ is continuous.

Proof. It follows from Lemma 2.2.1 that $\mathcal{A}$ is parameter-elliptic, with angle of ellipticity strictly smaller than $\pi/2$. Thus by [24, Theorem 5.5] and the perturbation result [24, Proposition 2.11], the realization of $1 + \mathcal{A}$ on $L_p(\mathbb{R}^n; E)$ with domain $D(1 + \mathcal{A}) = W_p^{2m}(\mathbb{R}^n; E)$ is invertible and admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle strictly smaller than $\pi/2$.

It is now a consequence of (A.3.2) and [85, Theorem 4.2] that $1+\mathcal{A}$ enjoys maximal L_p-regularity on the half-line, i.e., $1+\mathcal{A}\in\mathcal{MR}_p\big(\mathbb{R}_+;L_p(\mathbb{R}^n;E)\big)$. Since

$$X_{u,\mu}(\mathbb{R}^n)=B^{2m(\mu-1/p)}_{p,p}(\mathbb{R}^n;E)=\big(L_p(\mathbb{R}^n;E),W^{2m}_p(\mathbb{R}^n;E)\big)_{\mu-1/p,p}$$

by Proposition A.4.2, the assertion follows from Theorem 1.2.3 by Prüss & Simonett. ■

2.2.2 The Half-Space Case with Boundary Conditions

For constant coefficients $a_\alpha, b_{j\beta}\in\mathcal{B}(E)$ we now consider the operators

$$\mathcal{A}(D)=\sum_{|\alpha|=2m}a_\alpha D^\alpha,\qquad \mathcal{B}_j(D)=\sum_{|\beta|=m_j}b_{j\beta}\mathrm{tr}_{\mathbb{R}^n_+}D^\beta,\quad j=1,...,m.$$

Observe that again there are no lower order terms. We identify the boundary of $\mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$. Now all spaces must be understood over $\mathbb{R}_+\times\mathbb{R}^n_+$ and $\mathbb{R}_+\times\mathbb{R}^{n-1}$, respectively, i.e.,

$$\mathbb{E}_{u,\mu}=W^1_{p,\mu}(\mathbb{R}_+;L_p(\mathbb{R}^n_+;E))\cap L_{p,\mu}(\mathbb{R}_+;W^{2m}_p(\mathbb{R}^n_+;E)),$$

$$\mathbb{F}_{j,\mu}=W^{\kappa_j}_{p,\mu}(\mathbb{R}_+;L_p(\mathbb{R}^n_+;E))\cap L_{p,\mu}(\mathbb{R}_+;W^{2m\kappa_j}_p(\mathbb{R}^n_+;E)),$$

$$\mathbb{E}_{0,\mu}=L_{p,\mu}(\mathbb{R}_+;L_p(\mathbb{R}^n_+;E)),\qquad X_{u,\mu}=B^{2m(\mu-1/p)}_{p,p}(\mathbb{R}^n_+;E).$$

The Banach space of compatible data is given by

$$\mathcal{D}=\big\{(f,g,u_0)\in\mathbb{E}_{0,\mu}\times\mathbb{F}_\mu\times X_{u,\mu}\ :\ \text{for } j=1,...,m \text{ it holds}\\ \mathcal{B}_j(D)u_0=g_j(0,\cdot)\ \text{ on }\ \Gamma\ \text{ if } \kappa_j>1-\mu+1/p\big\}.$$

The main result of this subsection is the following.

Proposition 2.2.3. *Let E be a Banach space of class $\mathcal{HT}$, $p\in(1,\infty)$, $\mu\in(1/p,1]$, and assume that $(\mathcal{A},\mathcal{B})$ satisfies* (E) *and* (LS). *Suppose further that $\kappa_j\neq 1-\mu+1/p$ for all $j=1,...,m$. There is a unique solution $u=\mathcal{S}_H(f,g,u_0)\in\mathbb{E}_{u,\mu}$ for the problem*

$$\begin{aligned} u+\partial_t u+\mathcal{A}(D)u&=f(t,x), && x\in\mathbb{R}^n_+, && t>0,\\ \mathcal{B}_j(D)u&=g_j(t,x), && x\in\mathbb{R}^{n-1}, && t>0, \quad j=1,...,m,\\ u(0,x)&=u_0(x), && x\in\mathbb{R}^n_+, \end{aligned}\tag{2.2.2}$$

if and only if $(f,g,u_0)\in\mathcal{D}$. The solution operator $\mathcal{S}_H:\mathcal{D}\to\mathbb{E}_{u,\mu}$ is continuous.

As explained in Section 2.1, the necessary conditions on the data are a consequence of the mapping behaviour of spatial derivatives, the spatial trace and the temporal trace on the weighted anisotropic spaces, derived in Lemma 1.3.4, Proposition 1.3.12 and Theorem 1.3.6. If a solution operator exists, then its continuity follows from

$$1+\partial_t+\mathcal{A}(D)\in\mathcal{B}(\mathbb{E}_{u,\mu},\mathbb{E}_{0,\mu}),\qquad \mathcal{B}(D)\in\mathcal{B}(\mathbb{E}_{u,\mu},\mathbb{F}_\mu),\qquad \mathrm{tr}_{t=0}\in\mathcal{B}(\mathbb{E}_{u,\mu},X_{u,\mu})$$

and the open mapping theorem.

Our task is thus to show that for any given $(f, g, u_0) \in \mathcal{D}$ the problem (2.2.2) has a unique solution $u \in \mathbb{E}_{u,\mu}$. For this we follow the strategy presented in [25, Section 4]. We first consider (2.2.2) in Lemma 2.2.5 with homogeneous boundary conditions, $g = 0$, and then we consider (2.2.2) in Lemma 2.2.6 with $f = 0$ and $u_0 = 0$. The general case follows from a combination of these lemmas and will be shown at the end of this subsection.

As for normal ellipticity, we first show that also (LS) holds in fact on a larger sector than originally assumed.

Lemma 2.2.4. *Let* $(\mathcal{A}, \mathcal{B})$ *satisfy* (E) *and* (LS). *Then there is* $\phi \in (0, \pi/2)$ *such that for all* $\lambda \in \overline{\Sigma}_{\pi-\phi}$ *and* $\xi' \in \mathbb{R}^{n-1}$ *with* $|\lambda| + |\xi'| \neq 0$ *and all* $h = (h_1, ..., h_m) \in E^m$ *the ordinary initial value problem*

$$\begin{aligned} \lambda v(y) + \mathcal{A}(\xi', D_y)v(y) &= 0, \qquad y > 0, \\ \mathcal{B}_j(\xi', D_y)v|_{y=0} &= h_j, \qquad j = 1, ..., m, \end{aligned} \tag{2.2.3}$$

has a unique solution $v \in C_0([0, \infty); E)$, *i.e., the condition* (LS) *for* $(\mathcal{A}, \mathcal{B})$ *is even valid for* $\lambda \in \overline{\Sigma}_{\pi-\phi}$.

Proof. (I) It follows from Lemma 2.2.1 that $\mathcal{A}$ has angle of ellipticity $\phi_{\mathcal{A}} \in (0, \pi/2)$. For $\lambda \in \overline{\Sigma}_{\pi-\phi_{\mathcal{A}}}$ and $\xi' \in \mathbb{R}^{n-1}$ we rewrite the ordinary differential equation $\lambda v + \mathcal{A}(\xi', D_y)v = 0$ of order $2m$ to a system of $2m$ first order equations,

$$\partial_y \underline{v}(y) = \mathrm{i}A_0(\lambda, \xi')\underline{v}(y), \quad y > 0, \qquad \underline{v} = (v, \partial_y v, ..., \partial_y^{2m-1} v),$$

where $A_0(\lambda, \xi')$ is a $\mathcal{B}(E)$-valued $2m \times 2m$-matrix. The solutions of the above equation are of the form $\underline{v}(y) = e^{y\mathrm{i}A_0(\lambda,\xi')}\underline{v}_0$, where $\underline{v}_0 \in E^{2m}$.
By [24, Proposition 6.1], the matrix $\mathrm{i}A_0(\lambda, \xi')$ has a spectral gap at the imaginary axis. We denote the projection onto the stable part of the spectrum by $P_s(\lambda, \xi') \in \mathcal{B}(E^{2m})$. Denoting further by $\pi_1 : E^{2m} \to E$ the canonical projection onto the first component, we define the operator pencil $T : \overline{\Sigma}_{\pi-\phi_{\mathcal{A}}} \times \mathbb{R}^{n-1} \to \mathcal{B}\big(P_s(\lambda, \xi')E^{2m}, E^m\big)$ by

$$T(\lambda, \xi')\underline{v}_0 := \big(\mathcal{B}(\xi', D_y)\pi_1 e^{y\mathrm{i}A_0(\lambda,\xi')}\underline{v}_0\big)\big|_{y=0}, \qquad \underline{v}_0 \in P_s(\lambda, \xi')E^{2m}.$$

For λ and ξ' from a compact set, the spectral gap for $\mathrm{i}A(\lambda, \xi')$ is uniform, and P_s is continuous in its arguments. By construction, (2.2.3) is uniquely solveable for $\lambda \in \overline{\Sigma}_{\pi-\phi_{\mathcal{A}}}$ and $\xi' \in \mathbb{R}^{n-1}$ if and only if $T(\lambda, \xi')$ is invertible.
(II) Let v be the unique solution of (2.2.3) in $C_0([0, \infty); E)$ for $h \in E^m$. Then for $r > 0$ the function v also satisfies

$$\begin{aligned} \lambda v(r\widetilde{y}) + \mathcal{A}(\xi', D_y)v(r\widetilde{y}) &= 0, \qquad \widetilde{y} > 0, \\ \mathcal{B}_j(\xi', D_y)v|_{y=0} &= h_j, \qquad j = 1, ..., m. \end{aligned}$$

Since $(D_y v)(r\cdot) = r^{-1}D_y(v(r\cdot))$, it follows from homogeneity that $w := v(r\cdot)$ is the unique solution of

$$\begin{aligned} r^{2m}\lambda w(y) + \mathcal{A}(r\xi', D_y)w(y) &= 0, \qquad y > 0, \\ \mathcal{B}_j(r\xi', D_y)w|_{y=0} &= r^{m_j}h_j, \qquad j = 1, ..., m. \end{aligned}$$

Therefore $T(\lambda, \xi')$ is invertible if and only if $T(r^{2m}\lambda, r\xi')$ is invertible.
(III) By (LS), continuity and compactness there is an angle $\phi \in (\phi_{\mathcal{A}}, \pi/2)$ such that $T(\lambda, \xi')$ is invertible for all

$$(\lambda, \xi') \in \{se^{\pm \mathrm{i}\theta} \;:\; s \in [0,1],\; \theta \in [\pi/2, \pi - \phi]\} \times \{|\xi'| = 1\},$$

and further for all

$$(\lambda, \xi') \in \{e^{\pm \mathrm{i}\theta} \;:\; \theta \in [\pi/2, \pi - \phi]\} \times \{|\xi'| \leq 1\}.$$

We use this fact and the scaling property from Step II to show that $T(\lambda, \xi')$ is invertible for all $\lambda \in \overline{\Sigma}_{\pi-\phi}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$. We distinguish four cases.
For $1 \leq |\lambda| \leq |\xi'| =: r_1$ the operator $T(\lambda/r_1^{2m}, \xi'/r_1)$ is invertible because $|\lambda|/r_1^{2m} \leq 1$ and $|\xi'|/r_1 = 1$. The scaling property thus shows that $T(\lambda, \xi')$ is invertible in this case, and hence it is invertible whenever $|\lambda| = 1$. For $1 \leq |\xi'| \leq |\lambda| =: r_2^{2m}$ the operator $T(\lambda/r_2^{2m}, \xi'/r_2)$ is invertible due to $|\lambda|/r_2^{2m} = 1$. So $T(\lambda, \xi')$ is invertible if $|\lambda|, |\xi'| \geq 1$.
Now for $0 < r_3^{2m} := |\lambda| \leq 1$ and arbitrary ξ' we have that $T(\lambda/r_3^{2m}, \xi'/r_3)$ is invertible because $|\lambda|/r_3^{2m} = 1$. Finally, for $0 < r_4 := |\xi'| \leq 1$ and arbitrary λ the operator $T(\lambda/r_4^{2m}, \xi'/r_4)$ is invertible because $|\xi'|/r_4 = 1$. ■

For homogeneous boundary conditions, weighted maximal regularity follows again from the unweighted case, since the abstract result of [71] is applicable.

Lemma 2.2.5. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, $\mu \in (1/p, 1]$, and assume that $(\mathcal{A}, \mathcal{B})$ satisfies* (E) *and* (LS). *Then for all $f \in \mathbb{E}_{0,\mu}$ and $u_0 \in X_{u,\mu}$ there is a unique solution $u \in \mathbb{E}_{u,\mu}$ of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= f(t,x), && x \in \mathbb{R}^n_+, && t > 0, && \\ \mathcal{B}_j(D)u &= 0, && x \in \mathbb{R}^{n-1}, && t > 0, && j = 1, ..., m, \\ u(0,x) &= u_0(x), && x \in \mathbb{R}^n_+. && && \end{aligned} \tag{2.2.4}$$

Denoting by $\mathcal{A}_{\mathcal{B}}$ the realization of the operator $\mathcal{A}$ on $L_p(\mathbb{R}^n_+; E)$, with domain

$$D(\mathcal{A}_{\mathcal{B}}) = \{u \in W^{2m}_p(\mathbb{R}^n_+; E) \;:\; \mathcal{B}u = 0\},$$

the operator $1 + \mathcal{A}_{\mathcal{B}}$ generates an exponentially stable analytic C_0-semigroup, and $1 + \mathcal{A}_{\mathcal{B}} \in \mathcal{MR}_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E))$.

Proof. Due to the Lemmas 2.2.1 and 2.2.4, the operator $\mathcal{A}$ is parameter elliptic with angle of ellipticity $\phi_{\mathcal{A}} < \pi/2$, and for $\phi \in (\phi_{\mathcal{A}}, \pi)$ it holds that $(\mathcal{A}, \mathcal{B})$ satisfies (LS) for all $\lambda \in \overline{\Sigma}_{\pi-\phi}$. Thus, by [24, Theorem 7.4] and the perturbation result [24, Proposition 2.11], $1 + \mathcal{A}_{\mathcal{B}}$ is invertible, and admits a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle strictly smaller than $\pi/2$. It follows from (A.3.2), [85, Theorem 4.2] and Theorem 1.2.3 that $1 + \mathcal{A}_{\mathcal{B}} \in \mathcal{MR}_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E))$. Since

$$X_{u,\mu} = \big(L_p(\mathbb{R}^n_+; E), W^{2m}_p(\mathbb{R}^n; E)\big)_{\mu - 1/p, p}$$

by Proposition A.4.2 we obtain the unique solvability of (2.2.4) in $\mathbb{E}_{u,\mu}$, for $f \in \mathbb{E}_{0,\mu}$ and $u_0 \in X_{u,\mu}$. In particular, $1 + \mathcal{A}_\mathcal{B}$ is the generator of an analytic semigroup. ■

It seems not possible to absorb inhomogeneous boundary conditions, $g \neq 0$, into the domain of a reasonable operator on $L_{p,\mu}(\mathbb{R}_+; E)$. Hence in this case we cannot reduce maximal $L_{p,\mu}$-regularity to the unweighted problem via the abstract result of Theorem 1.2.3.
To treat the inhomogeneous boundary conditions, we first consider an elliptic problem corresponding to (2.2.2). The following result is a combination of the Lemmas 4.3 and 4.4 in [25].

Lemma 2.2.6. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$ and assume that $(\mathcal{A}, \mathcal{B})$ satisfies* (E) *and* (LS). *Then for $\lambda \in \overline{\mathbb{C}_+}\backslash\{0\}$ and $g_j \in W_p^{2m\kappa_j}(\mathbb{R}^{n-1}; E)$, $j = 1, ..., m$, the problem*

$$\begin{aligned} \lambda v + \mathcal{A}(D)v &= 0, && x \in \mathbb{R}^n_+, \\ \mathcal{B}_j(D)v &= g_j(x), && x \in \mathbb{R}^{n-1}, \qquad j = 1, ..., m, \end{aligned} \tag{2.2.5}$$

has a unique solution $v(\lambda) \in W_p^{2m}(\mathbb{R}^n_+; E)$. This solution may be represented in the form

$$v(\lambda) = \sum_{j=1}^m \mathcal{S}_j(\lambda) g_j,$$

for operators $\mathcal{S}_j(\lambda) \in \mathcal{B}(W_p^{2m\kappa_j}(\mathbb{R}^{n-1}; E), W_p^{2m}(\mathbb{R}^n_+; E))$ given by

$$\mathcal{S}_j(\lambda) = \mathcal{T}_j(\lambda) L_\lambda^{1-m_j/2m} \mathcal{E}_\lambda.$$

Here $L_\lambda := \lambda + (-\Delta_{n-1})^m$, and the extension operator $\mathcal{E}_\lambda = e^{-\cdot L_\lambda^{1/2m}}$ maps $g_j \in W_p^{2m\kappa_j}(\mathbb{R}^{n-1}; E)$ to the function $(x', y) \mapsto e^{-yL_\lambda^{1/2m}} g_j(x')$, with $x' \in \mathbb{R}^{n-1}$ and $y > 0$. Moreover, for $\sigma \geq 0$ and $|\alpha| \leq 2m$ it holds $D^\alpha \mathcal{T}_j(\sigma + i\cdot) \in C^1\big(\mathbb{R}\backslash\{0\}; \mathcal{B}(L_p(\mathbb{R}^n_+; E))\big)$, and

$$\Big\{\lambda^{1-\frac{|\alpha|}{2m}} D^\alpha \mathcal{T}_j(\lambda),\ \lambda^{2-\frac{|\alpha|}{2m}} \frac{\partial}{\partial\theta} D^\alpha \mathcal{T}_j(\lambda) \ :\ \lambda = \sigma + i\theta \in \overline{\mathbb{C}_+}\backslash\{0\},\ \ |\alpha| \leq 2m,\ \ j = 1, ..., m\Big\}$$

is an $\mathcal{R}$-bounded set of operators in $\mathcal{B}(L_p(\mathbb{R}^n_+; E))$.

With the above result the time-dependent problem with inhomogeneous boundary conditions can now be solved via Fourier transform with respect to time, using the above representation of the solutions of the corresponding stationary problems. Recall the notation

$$\begin{aligned} {}_0\mathbb{E}_{1,\mu} &= {}_0W^1_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^n_+; E)), \\ {}_0\mathbb{F}_{j,\mu} &= {}_0W^{\kappa_j}_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{2m\kappa_j}(\mathbb{R}^{n-1}; E)), \\ {}_0\mathbb{F}_\mu &= {}_0\mathbb{F}_{1,\mu} \times ... \times {}_0\mathbb{F}_{m,\mu}. \end{aligned}$$

Lemma 2.2.7. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, $\mu \in (1/p, 1]$, and assume that $(\mathcal{A}, \mathcal{B})$ satisfies* (E), (LS). *Then for $g \in {}_0\mathbb{F}_\mu$ there is a unique solution $u \in {}_0\mathbb{E}_{u,\mu}$ of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= 0, && x \in \mathbb{R}^n_+, && t > 0, \\ \mathcal{B}_j(D)u &= g_j(t, x), && x \in \mathbb{R}^{n-1}, && t > 0, \qquad j = 1, ..., m, \\ u(0, x) &= 0, && x \in \mathbb{R}^n_+. \end{aligned} \tag{2.2.6}$$

Proof. Throughout we write $x = (x', y) \in \mathbb{R}^n_+$ with $x' \in \mathbb{R}^{n-1}$ and $y > 0$.
(I) It follows from Lemma 2.2.5 that solutions $u \in {}_0\mathbb{E}_{u,\mu}$ of (2.2.6) are unique. For the existence of a solution we are going to construct a solution operator

$$\mathcal{L} : C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; E))^m \to {}_0\mathbb{E}_{u,\mu},$$

and show that it admits the estimate

$$|\mathcal{L}g|_{\mathbb{E}_{u,\mu}} \lesssim |g|_{{}_0\mathbb{F}_\mu}, \qquad g \in C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; E))^m. \tag{2.2.7}$$

By Lemma 1.3.14, the set $C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; E))^m$ is dense in ${}_0\mathbb{F}_\mu$ (note that $2m\kappa_j \notin \mathbb{N}_0$). Hence, if (2.2.7) holds, then $\mathcal{L}$ extends to a continuous operator ${}_0\mathbb{F}_\mu \to {}_0\mathbb{E}_{u,\mu}$. Since

$$1 + \partial_t + \mathcal{A}(D) \in \mathcal{B}({}_0\mathbb{E}_{u,\mu}, \mathbb{E}_{0,\mu}), \qquad \mathcal{B}(D) \in \mathcal{B}({}_0\mathbb{E}_{u,\mu}, {}_0\mathbb{F}_\mu),$$

the function $u = \mathcal{L}g$ is then the unique solution of (2.2.6) for $g \in {}_0\mathbb{F}_\mu$.
(II) To construct the solution operator $\mathcal{L}$, let $g \in C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; E))^m$. In the sequel we identify such a function with its trivial temporal extension to $\mathbb{R}$. Applying the Fourier transform $\mathcal{F}_t$ with respect to $t \in \mathbb{R}$ to (2.2.6), and denoting the covariable by $\theta \in \mathbb{R}$, we arrive for each θ at the stationary problem

$$\begin{aligned} (1 + \mathrm{i}\theta)v + \mathcal{A}(D)v &= 0, && x \in \mathbb{R}^n_+, \\ \mathcal{B}_j(D)v &= (\mathcal{F}_t g_j)(\theta, x'), && x' \in \mathbb{R}^{n-1}, \qquad j = 1, ..., m. \end{aligned} \tag{2.2.8}$$

By Lemma 2.2.6, the unique solution $v(\theta) \in W_p^{2m}(\mathbb{R}^n_+; E)$ of (2.2.8) is given by

$$v(\theta) = \sum_{j=1}^m \mathcal{T}_j(1 + \mathrm{i}\theta) L_{1+\mathrm{i}\theta}^{1-m_j/2m} \mathcal{E}_{1+\mathrm{i}\cdot} \mathcal{F}_t(g_j)(\theta),$$

where $L_{1+\mathrm{i}\theta} = 1 + \mathrm{i}\theta + (-\Delta_{n-1})^m$, and where for $\theta \in \mathbb{R}$ the extension operator $\mathcal{E}_{1+\mathrm{i}\cdot}$ is for $h \in L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$ defined by

$$(\mathcal{E}_{1+\mathrm{i}\theta} h)(t, x', y) := e^{-yL_{1+\mathrm{i}\theta}^{1/2m}} h(t, x'), \qquad t \in \mathbb{R}, \qquad (x', y) \in \mathbb{R}^n_+.$$

Due to [24, Corollary 1.9], for $\theta \in \mathbb{R}$ and $y > 0$ we have the representation

$$L_{1+\mathrm{i}\theta}^{1-m_j/2m} e^{-yL_{1+\mathrm{i}\theta}^{1/2m}} = \frac{1}{2\pi\mathrm{i}} \int_\Xi z^{1-m_j/2m} e^{-yz^{1/2m}} (z - L_{1+\mathrm{i}\theta})^{-1} \,\mathrm{d}z,$$

where $\Xi = (\infty, \delta]e^{\mathrm{i}3\pi/2} \cup \delta e^{\mathrm{i}[3\pi/2, -3\pi/2]} \cup [\delta, \infty)e^{-\mathrm{i}3\pi/2}$ for some sufficiently small $\delta > 0$. Thus for each $y > 0$ the $\mathcal{B}(L_p(\mathbb{R}^{n-1}; E))$-valued function

$$\theta \mapsto L_{1+\mathrm{i}\theta}^{1-m_j/2m} e^{-yL_{1+\mathrm{i}\theta}^{1/2m}}, \qquad \theta \in \mathbb{R},$$

is smooth and all of its derivatives are bounded. Since $\theta \mapsto \mathcal{F}_t(g_j)(\theta)$ is rapidly decreasing and $\mathcal{T}_j(1 + \mathrm{i}\cdot)$ is by Lemma 2.2.6 a uniformly bounded family of operators, it holds that the solution v of (2.2.8) is rapidly decreasing in θ. We may therefore apply the inverse Fourier transform to v, and obtain that

$$u = \mathcal{L}g := \sum_{j=1}^m \mathcal{F}_t^{-1} \mathcal{T}_j(1 + \mathrm{i}\cdot) L_{1+\mathrm{i}\cdot}^{1-m_j/2m} \mathcal{E}_{1+\mathrm{i}\cdot} \mathcal{F}_t g_j$$

solves the differential equations in (2.2.6). To show $u(0)=0$, we first observe that $u(0) \in D(\mathcal{A}_\mathcal{B})$ holds since u is smooth in t with values in $W_p^{2m}(\mathbb{R}^n_+;E)$ and satisfies the equations. Hence the function

$$\widetilde{u} = u - e^{-\cdot(1+\mathcal{A}_\mathcal{B})}u(0)$$

satisfies $\widetilde{u}(0) = \widetilde{u}'(0) = 0$, which yields

$$\widetilde{u}_\mathbb{R} \in C^1\big(\mathbb{R}; L_p(\mathbb{R}^n_+;E)\big) \cap C\big(\mathbb{R}; W_p^{2m}(\mathbb{R}^n_+;E)\big)$$

for the trivial extension $\widetilde{u}_\mathbb{R}$ of $\widetilde{u}$ to $\mathbb{R}$. Further, as the semigroup generated by $1+\mathcal{A}_\mathcal{B}$ is exponentially stable, the functions $\widetilde{u}$ and $\widetilde{u}'$ are rapidly decreasing on $\mathbb{R}_+$. Thus $(\mathcal{F}_t\widetilde{u}_\mathbb{R})(\theta)$ solves (2.2.8) for each $\theta \in \mathbb{R}$. By uniqueness it holds $\mathcal{F}_t u = \mathcal{F}_t\widetilde{u}$, and therefore $u = \widetilde{u}$, which yields $u(0)=0$.

(III) To show the estimate (2.2.7) we derive another representation for $\mathcal{L}$. We have seen above that for $g_j \in C_c^\infty(\mathbb{R}; W_p^{2m}(\mathbb{R}^{n-1};E))$ the function $\theta \mapsto L_{1+\mathrm{i}\theta}^{1-m_j/2m} e^{-yL_{1+\mathrm{i}\theta}^{1/2m}} \mathcal{F}_t g_j$ belongs to the Schwartz class. Hence Fourier inversion holds, and we may write

$$\mathcal{L}g := \sum_{j=1}^m \big(\mathcal{F}_t^{-1}\mathcal{T}_j(1+\mathrm{i}\cdot)\mathcal{F}_t\big)\big(\mathcal{F}_t^{-1}L_{1+\mathrm{i}\cdot}^{1-m_j/2m}\mathcal{E}_{1+\mathrm{i}\cdot}\mathcal{F}_t\big)g_j.$$

On $\mathcal{S}(\mathbb{R}; W_p^{2m}(\mathbb{R}^{n-1};E))$ it holds $\mathcal{F}_t^{-1}L_{1+\mathrm{i}\theta} = L\mathcal{F}_t^{-1}$, with

$$L := 1 + \partial_t + (-\Delta_{n-1})^m.$$

Moreover, by Lemma 1.3.1 the realization of L on $L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E))$ with domain

$$D(L) = {}_0W^1_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1};E))$$

is invertible and sectorial of angle not larger than $\pi/2$. Using [24, Corollary 1.9] for $L_{1+\mathrm{i}\theta}$ and L, we obtain for $y>0$ and $g_j \in C_c^\infty(\mathbb{R}; W_p^{2m}(\mathbb{R}^{n-1};E))$

$$\begin{aligned}\mathcal{F}_t^{-1}L_{1+\mathrm{i}\cdot}^{1-m_j/2m}e^{-yL_{1+\mathrm{i}\cdot}^{1/2m}}\mathcal{F}_t g_j &= \frac{1}{2\pi\mathrm{i}}\int_\Xi z^{1-m_j/2m}e^{-yz^{1/2m}}(z-L)^{-1}\mathcal{F}_t^{-1}\mathcal{F}_t g_j\,\mathrm{d}z\\ &= L^{1-m_j/2m}e^{-yL^{1/2m}}g_j.\end{aligned}$$

Denoting by $\mathcal{E}$ the extension operator

$$\big(\mathcal{E}h\big)(t,x',y) := e^{-yL^{1/2m}}h(t,x'), \quad t>0, \quad (x',y)\in\mathbb{R}^n_+, \qquad h \in L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E)),$$

we arrive at the representation

$$\mathcal{L}g = \sum_{j=1}^m \big(\mathcal{F}_t^{-1}\mathcal{T}_j(1+\mathrm{i}\cdot)\mathcal{F}_t\big)L^{1-m_j/2m}\mathcal{E}g_j \qquad \text{for } g \in C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1};E))^m.$$

(IV) Due to Lemma 1.3.8, the operator $\mathcal{E}$ maps continuously

$${}_0\mathbb{F}_{j,\mu} = D_{L^{1/2m}}(2m-m_j-1/p,p) \to L_p(\mathbb{R}_+; D_{L^{1/2m}}(2m-m_j,p)),$$

and $L^{1-m_j/2m} = \left(L^{1/2m}\right)^{2m-m_j}$ is continuous

$$L_p(\mathbb{R}_+; D_{L^{1/2m}}(2m-m_j,p)) \to L_p(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E))) = \mathbb{E}_{0,\mu}.$$

Further, due to Lemma 2.2.6 and Theorem 1.2.4, for each $j = 1,...,m$ and $|\alpha| \leq 2m$ the $\mathcal{B}(L_p(\mathbb{R}^n_+;E))$-valued symbol $D^\alpha \mathcal{T}_j(1+\mathrm{i}\cdot)$ is a Fourier multiplier on $L_{p,\mu}$. Therefore the operator $\mathcal{F}_t^{-1}\mathcal{T}_j(1+\mathrm{i}\cdot)\mathcal{F}_t$ is continuous[2]

$$\mathbb{E}_{0,\mu} \to L_{p,\mu}(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^n_+;E)).$$

Finally, it follows from the equation $\partial_t u = -(1+\mathcal{A}(D))u$ that the $\mathbb{E}_{u,\mu}$-norm of u can be controlled by its $L_{p,\mu}(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^n_+;E))$-norm, which shows (2.2.7). ■

The existence of a unique solution of (2.2.2) for given $(f,g,u_0) \in \mathcal{D}$ is now a consequence of the Lemmas 2.2.5 and 2.2.7, as follows. Denote by $u_1 \in \mathbb{E}_{u,\mu}$ the solution of

$$\begin{aligned} w + \partial_t w + \mathcal{A}(D)w &= f(t,x), && x \in \mathbb{R}^n_+, && t > 0, \\ \mathcal{B}_j(D)w &= 0, && x \in \mathbb{R}^{n-1}, && t > 0, \quad j = 1,...,m, \\ w(0,x) &= u_0(x), && x \in \mathbb{R}^n_+, \end{aligned}$$

which exists by Lemma 2.2.5. Since $\mathcal{B}_j(D)u_0 = g_j|_{t=0}$ for $\kappa_j > 1-\mu+1/p$, it follows from Proposition 1.1.11 and $\kappa_j \neq 1-\mu+1/p$ that[3]

$$g_j - \mathcal{B}_j(D)u_1 \in {}_0\mathbb{F}_{j,\mu}, \qquad j = 1,...,m.$$

If we denote by $u_2 \in \mathbb{E}_{u,\mu}$ the solution of

$$\begin{aligned} w + \partial_t w + \mathcal{A}(D)w &= 0, && x \in \mathbb{R}^n_+, && t > 0, \\ \mathcal{B}_j(D)w &= g_j(t,x) - \mathcal{B}_j(D)u_1(t,x), && x \in \mathbb{R}^{n-1}, && t > 0, \quad j = 1,...,m, \\ w(0,x) &= 0, && x \in \mathbb{R}^n_+, \end{aligned}$$

which exists by Lemma 2.2.7, then $u = u_1 + u_2$ solves (2.2.2). The uniqueness of this solution follows from the uniqueness of solutions of (2.2.4). Finally, the continuity of the solution operator $\mathcal{S}_H$ of (2.2.2) is a consequence of the fact that $\mathcal{D}$ is a Banach space and the open mapping theorem. Thus Proposition 2.2.3 is established. ■

2.3 Top Order Coefficients having Small Oscillation

From now on we restrict our considerations to a finite time interval

$$J = (0,T), \qquad T > 0.$$

[2]Proceeding as in the proof of [25, Lemma 4.4], one can show that for $j = 1,...,m$ and $|\alpha| \leq 2m$ it holds $D^\alpha \mathcal{T}_j(1+\mathrm{i}\cdot) \in C^2(\mathbb{R}; \mathcal{B}(L_p(\mathbb{R}^n_+;E)))$, and that $|\partial_\theta^2 D^\alpha \mathcal{T}_j(1+\mathrm{i}\theta)| \lesssim \frac{1}{\theta^2}$. Hence also Proposition 1.2.5 applies.

[3]At this point we have to exclude the value $\kappa_j = 1-\mu+1/p$.

We first consider the half-space case, and write

$$\mathbb{E}_{u,\mu}(J) = \mathbb{E}_{u,\mu}(J \times \mathbb{R}^n_+), \qquad \mathbb{F}_\mu(J) = \mathbb{F}_\mu(J \times \mathbb{R}^{n-1}),$$

and so on. Let the operators $\mathcal{A}$ and $\mathcal{B}_j$, $j = 1, ..., m$, be given by

$$\mathcal{A}(t, x, D) = \sum_{|\alpha| \le 2m} a_\alpha(t, x) D^\alpha, \qquad t \in J, \qquad x \in \mathbb{R}^n_+,$$

and

$$\mathcal{B}_j(t, x, D) = \sum_{|\beta| \le m_j} b_{j\beta}(t, x) \mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad t \in J, \qquad x \in \mathbb{R}^{n-1}.$$

Observe that, in contrast to the previous section, the operators may have lower order terms, and the $\mathcal{B}(E)$-valued coefficients a_α and $b_{j\beta}$ are allowed to depend on (t, x).
The top order coefficients of the boundary operators are assumed to be of the form

$$a_\alpha(t, x) = a_\alpha^0 + \widetilde{a}_\alpha(t, x), \qquad |\alpha| = 2m, \tag{2.3.1}$$

$$b_{j\beta}(t, x) = b_{j\beta}^0 + \widetilde{b}_{j\beta}(t, x), \qquad |\beta| = m_j, \qquad j = 1, ..., m, \tag{2.3.2}$$

where $a_\alpha^0, b_{j\beta}^0 \in \mathcal{B}(E)$ do not depend on (t, x). Using them we define auxiliary top order constant coefficient operators $(\mathcal{A}^0, \mathcal{B}^0)$ by

$$\mathcal{A}^0(D) := \sum_{|\alpha| = 2m} a_\alpha^0 D^\alpha, \qquad \mathcal{B}_j^0(D) := \sum_{|\beta| = m_j} b_{j\beta}^0 \mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad j = 1, ..., m. \tag{2.3.3}$$

Assuming (SD) and (SB) for the coefficients of $\mathcal{A} - \mathcal{A}^0$ and $\mathcal{B} - \mathcal{B}^0$, the Propositions 1.3.16 and 1.3.24 ensure that

$$\mathcal{A} \in \mathcal{B}(\mathbb{E}_{u,\mu}(J), \mathbb{E}_{0,\mu}(J)), \qquad \mathcal{B} \in \mathcal{B}(\mathbb{E}_{u,\mu}(J), \mathbb{F}_\mu(J)). \tag{2.3.4}$$

Moreover, (SD) and (SB) imply

$$\widetilde{a}_\alpha \in BUC(\overline{J} \times \overline{\mathbb{R}^n_+}; \mathcal{B}(E)), \qquad |\alpha| = 2m,$$

$$\widetilde{b}_{j\beta} \in BUC(\overline{J} \times \mathbb{R}^{n-1}; \mathcal{B}(E)), \qquad |\beta| = m_j, \qquad j = 1, ..., m.$$

For an interval $J' = (0, T')$ with $T' > 0$ the set of compatible data is given by

$$\mathcal{D}(J') = \big\{(f, g, u_0) \in \mathbb{E}_{0,\mu}(J') \times \mathbb{F}_\mu(J') \times X_{u,\mu} \;:\; \text{for } j = 1, ..., m \text{ it holds} \\ \mathcal{B}_j(0, \cdot, D) u_0 = g_j(0, \cdot) \text{ on } \mathbb{R}^{n-1} \text{ if } \kappa_j > 1 - \mu + 1/p\big\},$$

and we also consider

$$\mathcal{D}_0(J') = \big\{(f, g, u_0) \in \mathcal{D}(J') \;:\; g \in {}_0\mathbb{F}_\mu(J')\big\}.$$

Due to Lemma 1.3.25, these are Banach spaces as closed subspaces of $\mathbb{E}_{0,\mu}(J') \times \mathbb{F}_\mu(J') \times X_{u,\mu}$ and $\mathbb{E}_{0,\mu}(J') \times {}_0\mathbb{F}_\mu(J') \times X_{u,\mu}$, respectively. We have the following result for the half-space.

Proposition 2.3.1. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, and $\mu \in (1/p, 1]$. Assume that $(\mathcal{A}^0, \mathcal{B}^0)$ satisfies* (E) *and* (LS), *and that the coefficients of $(\mathcal{A}-\mathcal{A}^0, \mathcal{B}-\mathcal{B}^0)$ satisfy* (SD) *and* (SB). *Suppose further that $\kappa_j \neq 1-\mu+1/p$ for $j = 1, ..., m$. Then there are a time $T_0 \in (0,T]$ and a number $\varepsilon > 0$ such that if*

$$\sup_{(t,x)\in[0,T_0]\times\overline{\mathbb{R}^n_+}} |\widetilde{a}_\alpha(t,x)|_{\mathcal{B}(E)} < \varepsilon, \qquad |\alpha| = 2m, \tag{2.3.5}$$

and

$$\sup_{(t,x)\in[0,T_0]\times\mathbb{R}^{n-1}} |\widetilde{b}_{j\beta}(t,x)|_{\mathcal{B}(E)} < \varepsilon, \qquad |\beta| = m_j, \qquad j = 1, ..., m, \tag{2.3.6}$$

then for each interval $J' = (0,T')$ with $T' \in (0,T_0]$ there is a unique solution $u = \mathcal{S}_H^{\mathrm{sm}}(f,g,u_0) \in \mathbb{E}_{u,\mu}(J')$ of

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \mathbb{R}^n_+, && t \in J', \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x), && x \in \mathbb{R}^{n-1}, && t \in J', \qquad j = 1, ..., m, \\ u(0,x) &= u_0(x), && x \in \mathbb{R}^n_+, \end{aligned} \tag{2.3.7}$$

if and only if $(f,g,u_0) \in \mathcal{D}(J')$. The solution operator

$$\mathcal{S}_H^{\mathrm{sm}} : \mathcal{D}(J') \to \mathbb{E}_{u,\mu}(J')$$

is continuous. Restricted to $\mathcal{D}_0(J')$, its operator norm is independent of $T' \in (0,T_0]$.

Proof. Throughout this proof, let $0 < T' \leq T_0 \leq T$, and set $J_0 = (0,T_0)$.
(I) We first consider the necessity part. Let $u \in \mathbb{E}_{u,\mu}(J')$ be a solution of (2.3.7). Then (2.3.4) yields $f \in \mathbb{E}_{0,\mu}(J')$ and $g \in \mathbb{F}_\mu(J')$, and Theorem 1.3.6 implies $u_0 \in X_{u,\mu}$. Hence $(f,g,u_0) \in \mathcal{D}(J')$ is necessary to obtain a solution $u \in \mathbb{E}_{u,\mu}(J')$.
(II) Now suppose that for each $T' \in (0,T_0]$ it holds that for all $(f,g,u_0) \in \mathcal{D}(J')$ there is a unique solution $u \in \mathbb{E}_{u,\mu}(J')$ of (2.3.7), i.e., there is a solution operator $\mathcal{S}_H^{\mathrm{sm}}$ for (2.3.7). Then $\mathcal{S}_H^{\mathrm{sm}}$ is continuous due to (2.3.4) and the open mapping theorem. From this abstract argument its operator norm depends on $T' \in (0,T_0]$ (our construction below does not remove this dependence, see (2.3.10)).
For $(f,g,u_0) \in \mathcal{D}_0(J')$ we may extend $f \in \mathbb{E}_{0,\mu}(J')$ and $g \in {}_0\mathbb{F}_\mu(J')$ to $\mathcal{E}^0_{J'}f \in \mathbb{E}_{0,\mu}(\mathbb{R}_+)$ and $\mathcal{E}^0_{J'}g \in {}_0\mathbb{F}_\mu(\mathbb{R}_+)$, respectively, using the extension operator $\mathcal{E}^0_{J'}$ from Lemma 1.1.5, whose norm is independent of T'. Of course, then it holds $(\mathcal{E}^0_{J'}f|_{J_0}, \mathcal{E}^0_{J'}g|_{J_0}, u_0) \in \mathcal{D}(J_0)$, and it follows from the assumed uniqueness of solutions of (2.3.7) that

$$\mathcal{S}_H^{\mathrm{sm}}(f,g,u_0) = \mathcal{S}_H^{\mathrm{sm}}(\mathcal{E}^0_{J'}f|_{J_0}, \mathcal{E}^0_{J'}g|_{J_0}, u_0)|_{J'}.$$

We therefore obtain

$$\begin{aligned} |\mathcal{S}_H^{\mathrm{sm}}(f,g,u_0)|_{\mathbb{E}_{u,\mu}(J')} &\lesssim |\mathcal{E}^0_{J'}f|_{\mathbb{E}_{0,\mu}(\mathbb{R}_+)} + |\mathcal{E}^0_{J'}g|_{{}_0\mathbb{F}_\mu(\mathbb{R}_+)} + |u_0|_{X_{u,\mu}} \\ &\lesssim |f|_{\mathbb{E}_{0,\mu}(J')} + |g|_{{}_0\mathbb{F}_\mu(J')} + |u_0|_{X_{u,\mu}}, \end{aligned}$$

where the constants in this estimate only depend on T_0, but not on $T' \in (0,T_0]$.

(III) It remains to find a unique solution $u \in \mathbb{E}_{u,\mu}(J')$ of (2.3.7) for given $(f, g, u_0) \in \mathcal{D}(J')$. We define

$$Z_{u_0}(J') := \big\{ v \in \mathbb{E}_{u,\mu}(J') \ : \ v(0,\cdot) = u_0 \big\},$$

which is a nonempty closed subspace of $\mathbb{E}_{u,\mu}(J')$ due to Lemma 1.3.9. For given $v \in Z_{u_0}(J')$ we consider the problem

$$\begin{aligned} w + \partial_t w + \mathcal{A}^0 w &= f + (\mathcal{A}^0 - \mathcal{A} + 1)v && \text{in } J' \times \mathbb{R}^n_+, \\ \mathcal{B}^0 w &= g + (\mathcal{B}^0 - \mathcal{B})v && \text{on } J' \times \mathbb{R}^{n-1}, \\ w(0,\cdot) &= u_0 && \text{in } \mathbb{R}^n_+, \end{aligned} \tag{2.3.8}$$

where the top order constant coefficient operators $\mathcal{A}^0$ and $\mathcal{B}^0$ are given by (2.3.3). Due to Lemma 1.2.1, solutions of (2.3.8) are unique in $\mathbb{E}_{u,\mu}(J')$ for $v \in Z_{u_0}(J')$, since by Lemma 2.2.5 the realization of $1 + \mathcal{A}^0_{\mathcal{B}^0}$ on $L_p(\mathbb{R}^n_+; E)$ is the generator of an analytic C_0-semigroup. To find a solution $w = \mathcal{S}(v) \in \mathbb{E}_{u,\mu}(J')$ of (2.3.8) we consider the problem

$$\begin{aligned} \widetilde{w} + \partial_t \widetilde{w} + \mathcal{A}^0 \widetilde{w} &= \widetilde{f} && \text{on } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ \mathcal{B}^0 \widetilde{w} &= \widetilde{g} && \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ \widetilde{w}(0,\cdot) &= \widetilde{w}_0 && \text{on } \mathbb{R}^n_+. \end{aligned} \tag{2.3.9}$$

Since $(\mathcal{A}^0, \mathcal{B}^0)$ are assumed to satisfy (E) and (LS), Proposition 2.2.3 yields a continuous solution operator

$$\mathcal{S}_H : \mathcal{D}_{\mathcal{B}^0}(\mathbb{R}_+) \to \mathbb{E}_{u,\mu}(\mathbb{R}_+)$$

for (2.3.9), where $\mathcal{D}_{\mathcal{B}^0}(\mathbb{R}_+)$ denotes the space of compatible data with respect to $\mathcal{B}^0$. Since g and u_0 are compatible with respect to $\mathcal{B}$, it follows that

$$(\mathcal{E}_{J'}(f + (\mathcal{A}^0 - \mathcal{A} + 1)v), \mathcal{E}_{J'}(g + (\mathcal{B}^0 - \mathcal{B})v), u_0) \in \mathcal{D}_{\mathcal{B}^0}(\mathbb{R}_+),$$

where $\mathcal{E}_{J'}$ is the extension operator from J' to $\mathbb{R}_+$, see Lemma 1.1.5. Therefore

$$w = \mathcal{S}(v) := \mathcal{S}_H\big(\mathcal{E}_{J'}(f + (\mathcal{A}^0 - \mathcal{A} + 1)v), \mathcal{E}_{J'}(g + (\mathcal{B}^0 - \mathcal{B})v), u_0\big)|_{J'} \tag{2.3.10}$$

is the unique solution of (2.3.8). Observe that a function $u \in \mathbb{E}_{u,\mu}(J')$ solves (2.3.7) if and only if it is a fixed point of $\mathcal{S}$ in $Z_{u_0}(J')$.

(IV) We show that $\mathcal{S}$ has a unique fixed point in $Z_{u_0}(J')$ via the contraction principle, provided T_0 and thus the length of J' are sufficiently small. Clearly $\mathcal{S}$ maps $Z_{u_0}(J')$ into itself. For $v_1, v_2 \in Z_{u_0}(J')$, the difference $\mathcal{S}(v_1) - \mathcal{S}(v_2)$ solves

$$\begin{aligned} w + \partial_t w + \mathcal{A}^0 w &= (\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2) && \text{on } J' \times \mathbb{R}^n_+, \\ \mathcal{B}^0 w &= (\mathcal{B}^0 - \mathcal{B})(v_1 - v_2) && \text{on } J' \times \mathbb{R}^{n-1}, \\ w(0,\cdot) &= 0 && \text{on } \mathbb{R}^n_+. \end{aligned} \tag{2.3.11}$$

From $(v_1 - v_2)(0,\cdot) = 0$ we infer that $(\mathcal{B}^0 - \mathcal{B})(v_1 - v_2) \in {}_0\mathbb{F}_\mu(J')$. We thus may extend the data T'-independently to $\mathbb{R}_+$, using $\mathcal{E}^0_{J'}$ from Lemma 1.1.5. Since the restriction of the solution of the half-line problem

$$\begin{aligned} \widetilde{w} + \partial_t \widetilde{w} + \mathcal{A}^0 \widetilde{w} &= \mathcal{E}^0_{J'}(\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2) && \text{on } \mathbb{R}_+ \times \mathbb{R}^n_+, \\ \mathcal{B}^0 \widetilde{w} &= \mathcal{E}^0_{J'}(\mathcal{B}^0 - \mathcal{B})(v_1 - v_2) && \text{on } \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ \widetilde{w}(0, \cdot) &= 0 && \text{on } \mathbb{R}^n_+, \end{aligned}$$

to J' solves (2.3.11), it follows from the uniqueness of solutions of (2.3.11) that

$$\mathcal{S}(v_1) - \mathcal{S}(v_2) = \mathcal{S}_H\big(\mathcal{E}^0_{J'}(\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2), \mathcal{E}^0_{J'}(\mathcal{B}^0 - \mathcal{B})(v_1 - v_2), 0\big)|_{J'}.$$

The continuity of $\mathcal{S}_H$ and $\mathcal{E}^0_{J'}$ now yield

$$\begin{aligned} |\mathcal{S}(v_1) - \mathcal{S}(v_2)|_{\mathbb{E}_{u,\mu}(J')} &\leq |\mathcal{S}_H\,(\mathcal{E}^0_{J'}(\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2), \mathcal{E}^0_{J'}(\mathcal{B}^0 - \mathcal{B})(v_1 - v_2), 0)|_{\mathbb{E}_{u,\mu}(\mathbb{R}_+)} \\ &\lesssim |\mathcal{E}^0_{J'}(\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2)|_{\mathbb{E}_{0,\mu}(\mathbb{R}_+)} + |\mathcal{E}^0_{J'}(\mathcal{B}^0 - \mathcal{B})(v_1 - v_2)|_{{}_0\mathbb{F}_{u,\mu}(\mathbb{R}_+)} \\ &\lesssim |(\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')} + |(\mathcal{B}^0 - \mathcal{B})(v_1 - v_2)|_{{}_0\mathbb{F}_{\mu}(J')}, \qquad (2.3.12) \end{aligned}$$

where the constant in this estimate is independent of T_0.

(V) It holds

$$\begin{aligned} |(\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')} \leq &\sum_{|\alpha|=2m} |\widetilde{a}_\alpha D^\alpha (v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')} \\ &+ \sum_{|\alpha|<2m} |a_\alpha D^\alpha (v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')} + |v_1 - v_2|_{\mathbb{E}_{0,\mu}(J')}. \end{aligned}$$

For the first summand assumption (2.3.5) yields

$$\sum_{|\alpha|=2m} |\widetilde{a}_\alpha D^\alpha (v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')} \lesssim \varepsilon\,|v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')}.$$

For the second summand and $|\alpha| < 2m$, suppose that the second condition in (SD) holds. Then we take $\delta \in \left(\frac{p(1-\mu)+1}{s_\alpha} + \frac{n}{2mr_\alpha}, 1 - \frac{|\alpha|}{2m}\right)$ and apply Lemma 1.3.15 on pointwise multipliers, to obtain

$$\begin{aligned} &\sum_{|\alpha|<2m} |a_\alpha D^\alpha (v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')} \\ &\quad\lesssim \sum_{|\alpha|<2m} |a_\alpha|_{L_{s_\alpha}(J; L_{r_\alpha}(\mathbb{R}^n_+; \mathcal{B}(E)))} |D^\alpha (v_1 - v_2)|_{{}_0H^\delta_{p,\mu}(J'; L_p(\mathbb{R}^n_+; E)) \cap L_{p,\mu}(J'; H^{2m\delta}_p(\mathbb{R}^n_+; E))}. \end{aligned}$$

It follows from $(v_1 - v_2)(0, \cdot) = 0$ and Lemma 1.3.13 that for given $\eta > 0$ we have

$$|D^\alpha (v_1 - v_2)|_{{}_0H^\delta_{p,\mu}(J'; L_p(\mathbb{R}^n_+; E)) \cap L_{p,\mu}(J'; H^{2m\delta}_p(\mathbb{R}^n_+; E))} \leq \eta\,|v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')},$$

provided T_0 is sufficiently small. If the lower order coefficients satisfy the first condition in (SD) one obtains this estimate for $\sum_{|\alpha|<2m} |a_\alpha D^\alpha (v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')}$ in a similar way. For the third summand we have

$$|v_1 - v_2|_{\mathbb{E}_{0,\mu}(J')} \leq \eta\,|v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')}.$$

Combining these inequalities, we arrive at

$$|(\mathcal{A}^0 - \mathcal{A} + 1)(v_1 - v_2)|_{\mathbb{E}_{0,\mu}(J')} \lesssim (\varepsilon + \eta)|v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')}.$$

(VI) We now estimate the boundary terms in (2.3.12). For $j = 1, ..., m$ it holds

$$|(\mathcal{B}_j^0 - \mathcal{B}_j)(v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} \leq \sum_{|\beta|=m_j} |\widetilde{b}_{j\beta}\, \mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} + \sum_{|\beta|<m_j} |b_{j\beta}\, \mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')}. \tag{2.3.13}$$

For $|\beta| = m_j$ we use (SB), the Lemmas 1.3.21, 1.3.22, 1.3.23 and (2.3.6), to estimate with $\delta \in (0, \kappa_j)$

$$|\widetilde{b}_{j\beta}\, \mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} \lesssim \varepsilon\, |\mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} + |\widetilde{b}_{j\beta}|_{\mathbb{Y}(J)} |\mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0W^{\delta,2m\delta}_{p,\mu}(J'\times\mathbb{R}^n_+;E))}. \tag{2.3.14}$$

Here $\mathbb{Y}(J) = \mathbb{F}_{j,\mu}(J \times \mathbb{R}^{n-1}; \mathcal{B}(E))$ or $\mathbb{Y}(J) = B^{\kappa_j}_{s_{j\beta},p}(L_{r_{j\beta}}) \cap L_{s_{j\beta}}(B^{2m\kappa_j}_{r_{j\beta},p})$, according to the two conditions in (SB). If the first condition in (SB) is valid one further has to use the embedding

$${}_0W^{\delta,2m\delta}_{p,\mu}(J' \times \mathbb{R}^n_+; E)) \hookrightarrow BUC(J' \times \mathbb{R}^n_+; E)$$

to deduce (2.3.14) from Lemma 1.3.23, which is valid for some $\delta \in (0, \kappa_j)$ if $\kappa_j > 1 - \mu + 1/p + \frac{n-1}{2mp}$. Note also that (2.3.14) is uniform in $T \leq T_0$ due to $(v_1 - v_2)(0, \cdot) = 0$. For the first summand, we infer from Proposition 1.3.12, Lemma 1.3.4 and that $|\widetilde{b}_{j\beta}|_{\mathbb{Y}(J)}$ are fixed numbers

$$|\mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} \lesssim |v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')},$$

where this estimate is again uniform in $|J'| \leq T_0$. For the second summand we use Lemma 1.3.13 to obtain for given η

$$|\widetilde{b}_{j\beta}|_{\mathbb{Y}(J)} |\mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0W^{\delta,2m\delta}_{p,\mu}(J\times\mathbb{R}^n_+;E))} \leq \eta\, |v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')},$$

provided T_0 is sufficiently small. This yields

$$\sum_{|\beta|=m_j} |\widetilde{b}_{j\beta}\, \mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} \lesssim (\varepsilon + \eta)|v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')}$$

for the first summand in (2.3.13). For the second summand in (2.3.13) and $|\beta| < m_j$ we use in a similar way (SB) and the Lemmas 1.3.13, 1.3.21, 1.3.23 and 1.3.22, to obtain

$$\sum_{|\beta|<m_j} |b_{j\beta}\, \mathrm{tr}_{\mathbb{R}^n_+} D^\beta (v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} \leq \eta\, |v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')}$$

for sufficiently small T_0. It is thus shown that

$$|(\mathcal{B}_j^0 - \mathcal{B}_j)(v_1 - v_2)|_{{}_0\mathbb{F}_{j,\mu}(J')} \lesssim (\varepsilon + \eta)\, |v_1 - v_2|_{\mathbb{E}_{u,\mu}(J')}.$$

(VII) Comparing with (2.3.12) and choosing ε and η, i.e., T_0, sufficiently small, we obtain that $\mathcal{S}$ is a strict contraction, and therefore has a unique fixed point in $Z_{u_0}(J')$. ■

We now turn to the full space problem, and write

$$\mathbb{E}_{u,\mu}(J) = \mathbb{E}_{u,\mu}(J \times \mathbb{R}^n), \qquad \mathbb{E}_{0,\mu}(J) = \mathbb{E}_{0,\mu}(J \times \mathbb{R}^n), \qquad X_{u,\mu} = X_{u,\mu}(\mathbb{R}^n).$$

We consider the operator $\mathcal{A}$ on $\mathbb{R}^n$ with $\mathcal{B}(E)$-valued variable coefficients a_α, given by

$$\mathcal{A}(t,x,D) = \sum_{|\alpha|\le 2m} a_\alpha(t,x)D^\alpha, \qquad (t,x) \in J \times \mathbb{R}^n.$$

We have the following result.

Proposition 2.3.2. *Let E be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, and $\mu \in (1/p, 1]$. Assume that $\mathcal{A}$ satisfies* (E) *and* (SD). *Then there are a time $T_0 \in (0,T]$ and number $\varepsilon > 0$ such that if*

$$\sup_{(t,x)\in[0,T_0]\times\mathbb{R}^n} |a_\alpha(t,x) - a_\alpha(0,0)|_{\mathcal{B}(E)} < \varepsilon, \qquad |\alpha| = 2m, \tag{2.3.15}$$

then for each interval $J' = (0,T')$ with $T' \in (0,T_0]$ there is a unique solution $u = \mathcal{S}_F^{sm}(f,u_0) \in \mathbb{E}_{u,\mu}(J')$ of

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), \qquad x \in \mathbb{R}^n, \qquad t \in J', \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{2.3.16}$$

if and only if $(f,u_0) \in \mathbb{E}_{0,\mu}(J') \times X_{u,\mu}$. The solution operator

$$\mathcal{S}_F^{sm} : \mathbb{E}_{0,\mu}(J') \times X_{u,\mu} \to \mathbb{E}_{u,\mu}(J')$$

is continuous, and its operator norm is independent of $T' \in (0,T_0]$.

Proof. The proof is completely analogous to the half-space case. We let $0 < T' \le T_0 \le T$. As in the proof of Proposition 2.3.1 we obtain the necessary conditions on the data. To show that for $(f,u_0) \in \mathbb{E}_{0,\mu}(J') \times X_{u,\mu}$ a unique solution of (2.3.16) exists, we consider the space

$$Z_{u_0}(J') = \big\{v \in \mathbb{E}_{u,\mu}(J') \ : \ v(0,\cdot) = u_0\big\},$$

and for $v \in Z_{u_0}(J')$ the problem

$$\begin{aligned} w + \partial_t w + \mathcal{A}^0 w &= f + (\mathcal{A}^0 - \mathcal{A} + 1)v \qquad \text{on } J' \times \mathbb{R}^n, \\ w(0,\cdot) &= u_0 \qquad \text{on } \mathbb{R}^n. \end{aligned} \tag{2.3.17}$$

Here the operator $\mathcal{A}^0$ is given by $\mathcal{A}^0 := \sum_{|\alpha|=2m} a_\alpha(0,0)D^\alpha$. The unique solution of (2.3.17) is given by

$$w = \mathcal{S}(v) := \mathcal{S}_F\big(\mathcal{E}^0_{J'}(f + (\mathcal{A}^0 - \mathcal{A} + 1)v), u_0\big)|_{J'},$$

where $\mathcal{S}_F : \mathbb{E}_{0,\mu}(\mathbb{R}_+) \times X_{u,\mu} \to \mathbb{E}_{u,\mu}(\mathbb{R}_+)$ is the continuous solution operator for (2.3.17) on $\mathbb{R}_+ \times \mathbb{R}^n$ from Proposition 2.2.2. As in the proof of Proposition 2.3.1 one can show that $\mathcal{S}$ is a strict contraction on $Z_{u_0}(J')$, provided ε and T_0 are sufficiently small. The resulting unique fixed point $u \in \mathbb{E}_{u,\mu}(J')$ of $\mathcal{S}$ is the unique solution of (2.3.16). Using that $u = \mathcal{S}_F\big(\mathcal{E}^0_{J'}(f + (\mathcal{A}^0 - \mathcal{A} + 1)u), u_0\big)|_{J'}$ and employing the continuity of $\mathcal{S}_F$ we obtain that the norm of the solution operator $\mathcal{S}_F^{\mathrm{sm}}$ is independent of T', since the norm of $\mathcal{E}^0_{J'}$ is independent of it. ■

2.4 The General Case on a Domain

In this section we finally prove Theorem 2.1.4. Let E be a Banach space of class $\mathcal{HT}$, let $J = (0, T)$ be a finite interval, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$. Now we write

$$\mathbb{E}_{u,\mu} = \mathbb{E}_{u,\mu}(J \times \Omega), \qquad \mathbb{F}_\mu = \mathbb{F}_\mu(J \times \Gamma),$$

and so on. We consider the problem

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, && t \in J, \\ \mathcal{B}_j(t,x,D)u &= g_j(t,x), && x \in \Gamma, && t \in J, \qquad j = 1, ..., m, \\ u(0,x) &= u_0(x), && x \in \Omega, \end{aligned} \tag{2.4.1}$$

where the differential operators $\mathcal{A}$ and $\mathcal{B}_j$, $j = 1, ..., m$, are given by

$$\mathcal{A}(t,x,D) = \sum_{|\alpha| \le 2m} a_\alpha(t,x) D^\alpha, \qquad t \in J, \qquad x \in \Omega,$$

$$\mathcal{B}_j(t,x,D) = \sum_{|\beta| \le m_j} b_{j\beta}(t,x) \operatorname{tr}_\Omega D^\beta, \qquad t \in J, \qquad x \in \Gamma, \qquad m_j \in \{0, ..., 2m-1\}.$$

The $\mathcal{B}(E)$-valued coefficients a_α and $b_{j\beta}$ are assumed to satisfy (SD) and (SB). In this case the Propositions 1.3.16 and 1.3.24 ensure that

$$\mathcal{A} \in \mathcal{B}(\mathbb{E}_{u,\mu}, \mathbb{E}_{0,\mu}), \qquad \mathcal{B} \in \mathcal{B}(\mathbb{E}_{u,\mu}, \mathbb{F}_\mu). \tag{2.4.2}$$

Moreover, it is included in resp. follows from these assumptions that the top order coefficients satisfy

$$\begin{aligned} a_\alpha &\in BUC(\overline{J} \times \overline{\Omega}; \mathcal{B}(E)), \qquad |\alpha| = 2m, \\ b_{j\beta} &\in BUC(\overline{J} \times \Gamma; \mathcal{B}(E)), \qquad |\beta| = m_j, \qquad j = 1, ..., m. \end{aligned}$$

The set of compatible data is given by

$$\mathcal{D} = \big\{(f, g, u_0) \in \mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu} \;:\; \text{for } j = 1, ..., m \text{ it holds} \\ \mathcal{B}_j(0,\cdot,D)u_0 = g_j(0,\cdot) \;\text{ on }\; \Gamma \;\text{ if } \kappa_j > 1 - \mu + 1/p\big\},$$

and further

$$\mathcal{D}_0 = \big\{(f, g, u_0) \in \mathcal{D} \;:\; g \in {}_0\mathbb{F}_\mu\big\}.$$

The following localization procedure is very long, elaborate, and looks sophisticated, but after all it is nothing but a sequence of simple principles, and a lot of notation. For an outline we refer to the end of Section 2.1.

Proof of Theorem 2.1.4.

(I) The necessary conditions on the data follow as in Section 2.1 from (2.4.2) and Theorem 1.3.6. If the solution operator $\mathcal{L}$ exists, then its continuity and its dependence on the length of J can be shown as in Step II of the proof of Proposition 2.3.1.
If the coefficients of the operators are as in (2.1.5) and if the data is real-valued, then we have that if $u \in \mathbb{E}_{u,\mu}$ solves (2.4.1), then also $\operatorname{Re} u \in \mathbb{E}_{u,\mu}$ solves (2.4.1). Hence $u = \operatorname{Re} u$ if one assumes uniqueness, i.e., the solution is real-valued in this case.
(II) Given $(f, g, u_0) \in \mathcal{D}$, we have to show that there exists a unique solution $u \in \mathbb{E}_{u,\mu}$ of (2.4.1). We first show that it suffices to obtain this under the assumption that $T = |J|$ is sufficiently small.
Using the extension operator $\mathcal{E}_J$ from Lemma 1.1.5 we may assume that the coefficients of $\mathcal{A}$ and $\mathcal{B}$ are defined on $[0, 2T]$. Suppose that for each $T_* \in [0, 2T)$ we can find a (small) time $\tau_{T_*} \in (0, 2T - T_*)$ such that the problem

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= \widetilde{f}(t,x), \qquad x \in \Omega, \qquad t \in (T_*, T_* + \tau_{T_*}), \\ \mathcal{B}(t,x,D)u &= \widetilde{g}(t,x), \qquad x \in \Gamma, \qquad t \in (T_*, T_* + \tau_{T_*}), \\ u(T_*, x) &= \widetilde{u}_0(x), \qquad x \in \Omega, \end{aligned} \tag{2.4.3}$$

has a unique solution $u \in \mathbb{E}_{u,\mu}(T_*, T_* + \tau_{T_*})$ for all

$$\widetilde{f} \in \mathbb{E}_{0,\mu}(T_*, T_* + \tau_{T_*}), \qquad \widetilde{g} \in \mathbb{F}_\mu(T_*, T_* + \tau_{T_*}), \qquad \widetilde{u}_0 \in X_{u,\mu},$$

which satisfy the compatibility condition

$$\mathcal{B}_j(T_*, \cdot, D)\widetilde{u}_0 = \widetilde{g}_j(0, \cdot), \qquad \text{on } \Gamma, \qquad \text{if } \kappa_j > 1 - \mu + 1/p, \qquad j = 1, ..., m.$$

In this case we can solve (2.4.1) uniquely for given $(f, g, u_0) \in \mathcal{D}(J)$ as follows. Using $\mathcal{E}_J$ we may assume that also f and g are defined on $(0, 2T)$. The solution intervals for (2.4.3) yield an open cover of $[\tau_0, T]$, from which we choose a finite subcover $\bigcup_{k=1}^K (T_k, T_k + \tau_k)$ with

$$T_1 < \tau_0, \qquad T_k < T_{k-1} + \tau_{k-1} \quad \text{for } 1 < k \leq K, \qquad T < T_k + \tau_k.$$

Let $u^0 \in \mathbb{E}_{u,\mu}(0, \tau_0)$ be the unique solution of (2.4.3) on $(0, \tau_0)$ with data

$$\widetilde{f} = f|_{(0,\tau_0)}, \qquad \widetilde{g} = g|_{(0,\tau_0)}, \qquad \widetilde{u}_0 = u_0.$$

Since $T_1 < \tau_0$, $u^0(T_1) \in X_{u,\mu}$ [4] and the compatibility condition holds, there is a unique solution $u^1 \in \mathbb{E}_{u,\mu}(T_1, T_1 + \tau_1)$ of (2.4.3) on $(T_1, T_1 + \tau_1)$ with data

$$\widetilde{f} = f|_{(T_1, T_1+\tau_1)}, \qquad \widetilde{g} = g|_{(T_1, T_1+\tau_1)}, \qquad \widetilde{u}_0 = u^0(T_1, \cdot).$$

Since we assume that solutions of (2.4.3) are unique for all initial times $T_* \in [0, 2T)$, it follows that u^0 and u^1 coincide on (T_1, τ_0). Iterating this argument yields functions $u^k \in \mathbb{E}_{u,\mu}(T_k, T_k + \tau_k)$, $k = 1, ..., K$, such that u^k satisfies (2.4.3) on $(T_k, T_k + \tau_k)$ with data

$$\widetilde{f} = f|_{(T_k, T_k+\tau_k)}, \qquad \widetilde{g} = g|_{(T_k, T_k+\tau_k)}, \qquad \widetilde{u}_0 = u^{k-1}(T_k, \cdot),$$

[4] In fact, due to the inherent smoothing effect of the weighted spaces it even holds $u^0(T_1) \in X_{u,1}$.

and such that u^k and u^{k+1} coincide on $(T_{k+1}, T_k + \tau_k)$. Since the weight only has an effect at the initial times T_k, it holds

$$u^k|_{(T_{k+1},T_k+\tau_k)} = u^{k+1}|_{(T_{k+1},T_k+\tau_k)} \in \mathbb{E}_{u,1}(T_{k+1}, T_k + \tau_k).$$

Hence we may put together the functions u^k, $k = 0, ..., K$, to a function $u \in \mathbb{E}_{u,\mu}(0, T_K + \tau_K)$, that solves (2.4.1) on $J = (0, T)$. Our assumption also implies that this solution is unique.

Observe that the restriction of $(\mathcal{A}, \mathcal{B})$ to any subinterval of $J = (0, T)$ is still subject to (E), (LS), (SD) and (SB). Therefore, due to the above considerations, our objective is to show the unique solvability of (2.4.1) for all $(f, g, u_0) \in \mathcal{D}(J)$, under the assumption that $T = |J|$ is sufficiently small.

(III) We intend to use the Propositions 2.3.1 and 2.3.2 to show unique solvability. To this end we have to localize (2.4.1) also in space. If Ω is unbounded we choose a large number $R > 0$ with $\Gamma \subset B_R(0)$ and set

$$x_0 := \infty, \qquad U_0 := \Omega \backslash \overline{B_R(0)}.$$

We define on $\overline{J} \times \mathbb{R}^n$ extended top order coefficients $a^0_\alpha = a^0_\alpha(t, x)$, $|\alpha| = 2m$, by

$$a^0_\alpha(t,x) := \begin{cases} a_\alpha(t,x), & x \in \overline{U_0}, \\ a_\alpha\big(t, R^2 \frac{x}{|x|^2}\big), & x \in B_R(0)\backslash\{0\}, \\ a_\alpha(t, x_0), & x = 0, \end{cases} \tag{2.4.4}$$

and further on $J \times \mathbb{R}^n$ extended lower order coefficients $a^0_\alpha = a^0_\alpha(t, x)$, $|\alpha| < 2m$, by

$$a^0_\alpha(t,x) := \begin{cases} a_\alpha(t,x), & x \in \overline{U_0}, \\ 0, & x \in B_R(0). \end{cases}$$

Using these coefficients we define the differential operator

$$\mathcal{A}^0(t, x, D) := \sum_{|\alpha| \le 2m} a^0_\alpha(t, x) D^\alpha.$$

Observe that for $|\alpha| = 2m$ the functions a^0_α are continuous extensions of the a_α to $\mathbb{R}^n$, which only use values of $a_\alpha|_{U_0}$. Therefore $\mathcal{A}^0$ satisfies (E), since this is a pointwise condition. Moreover, by assumption (SD), the limit $a_\alpha(t, \infty) = \lim_{|x|\to\infty} a_\alpha(t, x)$ exists uniformly in $t \in \overline{J}$, $|\alpha| = 2m$. Thus, given $\varepsilon > 0$, if R is sufficiently large and T is sufficiently small then by continuity it holds

$$\sup_{t \in \overline{J},\, x \in \overline{U_0}} |a_\alpha(t, x) - a_\alpha(0, x_0)|_{\mathcal{B}(E)} < \varepsilon, \qquad |\alpha| = 2m.$$

By construction, this carries over to the extended top order coefficients,

$$\sup_{t \in \overline{J},\, x \in \mathbb{R}^n} |a^0_\alpha(t, x) - a^0_\alpha(0, 0)|_{\mathcal{B}(E)} < \varepsilon, \qquad |\alpha| = 2m.$$

Hence, due to Proposition 2.3.2, if R is large and T is small then for all $J' = (0, T')$ with $T' \le T$ there is a continuous solution operator

$$\mathcal{S}^{\mathrm{sm},0}_F : \mathbb{E}_{0,\mu}(J' \times \mathbb{R}^n) \times X_{u,\mu}(\mathbb{R}^n) \to \mathbb{E}_{u,\mu}(J' \times \mathbb{R}^n),$$

for the full-space problem

$$\begin{aligned} \partial_t v + \mathcal{A}^0(t,x,D)v &= f^0(t,x), \qquad x \in \mathbb{R}^n \qquad t \in J', \\ v(0,x) &= u_0^0(x), \qquad x \in \mathbb{R}^n. \end{aligned} \tag{2.4.5}$$

(IV) Now, if Ω is unbounded, take a point $x_* \in \Omega \backslash U_0 = \Omega \cap \overline{B_R(0)}$, and if Ω is bounded, take a point $x_* \in \Omega$. In both cases we construct a differential operator $\mathcal{A}^{x_*}$ on $\mathbb{R}^n$ as above. Choose a radius $r_{x_*} > 0$ with

$$B_{r_{x_*}}(x_*) \cap \Gamma = \emptyset,$$

and put

$$U_{x_*} := B_{r_{x_*}}(x_*).$$

We define extended top order coefficients $a_\alpha^{x_*} = a_\alpha^{x_*}(t,x)$, $|\alpha| = 2m$, by

$$a_\alpha^{x_*}(t,x) := \begin{cases} a_\alpha(t,x), & x \in U_{x_*}, \\ a_\alpha\big(t, x_* + r_{x_*}^2 \frac{x-x_*}{|x-x_*|^2}\big), & x \notin U_{x_*}, \end{cases} \tag{2.4.6}$$

extended lower order coefficients $a_\alpha^{x_*} = a_\alpha^{x_*}(t,x)$, $|\alpha| < 2m$, by

$$a_\alpha^{x_*}(t,x) := \begin{cases} a_\alpha(t,x), & x \in U_{x_*}, \\ 0, & x \notin U_{x_*}, \end{cases} \tag{2.4.7}$$

and we finally set

$$\mathcal{A}^{x_*}(t,x,D) := \sum_{|\alpha| \le 2m} a_\alpha^{x_*}(t,x) D^\alpha.$$

As above, the operator $\mathcal{A}^{x_*}$ satisfies (E), and if r_{x_*} and T are sufficiently small, then Proposition 2.3.2 yields that for all $T' \in \le T$ there is a continuous solution operator

$$\mathcal{S}_F^{\mathrm{sm},x_*} : \mathbb{E}_{0,\mu}(J' \times \mathbb{R}^n) \times X_{u,\mu}(\mathbb{R}^n) \to \mathbb{E}_{u,\mu}(J' \times \mathbb{R}^n),$$

where $J' = (0, T')$, for the full-space problem

$$\begin{aligned} \partial_t v + \mathcal{A}^{x_*}(t,x,D)v &= f^*(t,x), \qquad x \in \mathbb{R}^n \qquad t \in J', \\ v(0,x) &= u_0^*(x), \qquad x \in \mathbb{R}^n. \end{aligned} \tag{2.4.8}$$

(V) For a point $x_* \in \Gamma = \partial\Omega$ we choose an open neighbourhood $\widetilde{U}_{x_*}$ of x_* in $\mathbb{R}^n$ such that there are smooth diffeomorphisms $\varphi_{x_*} : \widetilde{U}_{x_*} \to \mathbb{R}^n$ and a radius $r_{x_*} > 0$ with the properties

$$\varphi_{x_*}(x_*) = 0, \qquad \varphi_{x_*}(\widetilde{U}_{x_*}) = B_{2r_{x_*}}(0), \qquad \varphi'_{x_*}(x_*) = \mathcal{O}_{\nu(x_*)},$$

$$\varphi_{x_*}(\widetilde{U}_{x_*} \cap \Omega) \subset \mathbb{R}^n_+, \qquad \varphi_{x_*}(\widetilde{U}_{x_*} \cap \Gamma) \subset \mathbb{R}^{n-1}. \tag{2.4.9}$$

Note that we identify $\mathbb{R}^{n-1}$ with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. Further $\mathcal{O}_{\nu(x_*)}$ is the orthogonal matrix fixed in assumption (LS) that rotates the outer unit normal $\nu(x_*)$ to $(0, ..., 0, -1) \in \mathbb{R}^n$. By Lemma A.1.1, a chart $(\widetilde{U}_{x_*}, \varphi_{x_*})$ with the above properties always exists. We may assume that the sup-norms of any derivative of φ_{x_*} and $\varphi_{x_*}^{-1}$ are uniformly bounded for $r_{x_*} \le 1$. We further set

$$U_{x_*} := \varphi_{x_*}^{-1}\big(B_{r_{x_*}}(0)\big),$$

and denote by Φ_{x_*} the push-forward operator corresponding to φ_{x_*}, i.e., $\Phi_{x_*}v = v \circ \varphi_{x_*}^{-1}$. Now we define the differential operator $\mathcal{A}^{\Phi_{x_*}}$ for functions $v : \mathbb{R}^n_+ \cap B_{r_{x_*}}(0) \to E$ by

$$\mathcal{A}^{\Phi_{x_*}}(t,x,D)v := \big(\Phi_{x_*}\mathcal{A}(t,\cdot,D)\Phi_{x_*}^{-1}v\big)(x), \qquad t \in J, \qquad x \in \mathbb{R}^n_+ \cap B_{r_{x_*}}(0),$$

and the boundary operators $\mathcal{B}_j^{\Phi_{x_*}}$, $j = 1, ..., m$, by

$$\mathcal{B}_j^{\Phi_{x_*}}(t,x,D)v := \big(\Phi_{x_*}\mathcal{B}_j(t,\cdot,D)\Phi_{x_*}^{-1}v\big)(x), \quad t \in J, \quad x \in \mathbb{R}^{n-1} \cap B_{r_{x_*}}(0), \quad j = 1, ..., m.$$

For $v \in \mathbb{E}_{u,\mu}\big(\mathbb{R}^n_+ \cap B_{r_{x_*}}(0)\big)$ and $1 \le |\alpha| \le 2m$ it holds

$$D^\alpha(\Phi_{x_*}^{-1}v)(t,x) = \sum_{1\le|\gamma|\le|\alpha|} q_{\alpha\gamma}(x)(D^\gamma v)\left(t, \varphi_{x_*}(x)\right), \qquad t \in J, \qquad x \in \Omega \cap U_{x_*},$$

where $q_{\alpha\gamma}$ are real-valued bounded smooth functions in x, depending on the partial derivatives of the components of φ_{x_*} (see [69, Section 1.1.7]). Thus the pushed operators are again of the form

$$\mathcal{A}^{\Phi_{x_*}}(t,x,D) = \sum_{|\alpha|\le 2m} a_\alpha^{\Phi_{x_*}}(t,x)D^\alpha,$$

$$\mathcal{B}_j^{\Phi_{x_*}}(t,x,D) = \sum_{|\beta|\le m_j} b_{j\beta}^{\Phi_{x_*}}(t,x)\mathrm{tr}_{\mathbb{R}^n_+}D^\beta, \qquad j = 1, ..., m,$$

where $\mathrm{tr}_{\mathbb{R}^n_+}$ denotes the spatial trace operator for $\mathbb{R}^n_+$. Lemma A.1.2 implies that the principal parts of $(\mathcal{A}^{\Phi_i}, \mathcal{B}^{\Phi_i})$ are given by

$$\mathcal{A}_\sharp^{\Phi_i}(t,x,D) = \mathcal{A}_\sharp(t,x,\mathcal{O}_{\nu(x_*)}^T D), \qquad \mathcal{B}_\sharp^{\Phi_i}(t,x,D) = \mathcal{B}_\sharp(t,x,\mathcal{O}_{\nu(x_*)}^T D). \tag{2.4.10}$$

Due to Lemma A.4.1, and since the functions $q_{\alpha\gamma}$ are smooth and bounded, the coefficients $a_\alpha^{\Phi_{x_*}}$ satisfy (SD), formulated for $J \times \big(\mathbb{R}^n_+ \cap B_{r_{x_*}}(0)\big)$, and the coefficients $b_{j\beta}^{\Phi_{x_*}}$ satisfy (SB) on $J \times \big(\mathbb{R}^{n-1} \cap B_{r_{x_*}}(0)\big)$.

(VI) We now extend the top order coefficients of $\mathcal{A}^{\Phi_{x_*}}$ from $\overline{\mathbb{R}^n_+} \cap B_{r_{x_*}}(0)$ to $\overline{\mathbb{R}^n_+}$ by reflection as in (2.4.6), and the lower order coefficients of $\mathcal{A}^{\Phi_{x_*}}$ trivially from $\overline{\mathbb{R}^n_+} \cap B_{r_{x_*}}(0)$ to $\overline{\mathbb{R}^n_+}$ as in (2.4.7). Denoting the extended coefficients by $a_\alpha^{x_*}$, this yields an operator

$$\mathcal{A}^{x_*}(t,x,D) := \sum_{|\alpha|\le 2m} a_\alpha^{x_*}(t,x)D^\alpha, \qquad t \in J, \qquad x \in \mathbb{R}^n_+.$$

We further define the top order constant coefficient operator $\mathcal{A}^{x_*,0}$ by

$$\mathcal{A}^{x_*,0}(D) := \mathcal{A}^{x_*}(0,0,D) = \sum_{|\alpha|=2m} a_\alpha^{x_*,0}D^\alpha, \qquad a_\alpha^{x_*,0} := a_\alpha^{x_*}(0,0) = a_\alpha(0,x_*).$$

It follows from (2.4.10) that for $\xi \in \mathbb{R}^n$ it holds

$$\mathcal{A}^{x_*,0}(\xi) = \mathcal{A}_\sharp(0,x_*,\mathcal{O}_{\nu(x_*)}^T\xi).$$

Since $\mathcal{A}$ satisfies (E) we thus obtain that $\mathcal{A}^{x_*,0}$ satisfies (E) as well. We write the top order coefficients of $\mathcal{A}^{x_*}$ in the form

$$a_\alpha^{x_*}(t,x) = a_\alpha^{x_*,0} + \widetilde{a}_\alpha^{x_*}(t,x), \qquad |\alpha| = 2m,$$

where $\widetilde{a}_\alpha^{x_*}(t,x) = a_\alpha^{x_*}(t,x) - a_\alpha^{x_*,0}$, as required for (2.3.1). By construction, the coefficients of $\mathcal{A}^{x_*} - \mathcal{A}^{x_*,0}$ satisfy (SD). Given $\varepsilon > 0$, if T, r_{x_*} and the diameter of U_{x_*} are sufficiently small, then the coefficients $a_\alpha^{\Phi_{x_*}}$ have oscillation less than ε around $a_\alpha^{\Phi_{x_*}}(0,0)$ on $\overline{J} \times \big(\mathbb{R}^n_+ \cap B_{r_{x_*}}(0)\big)$ for all $|\alpha| = 2m$. By construction we have

$$\sup_{\overline{J}\times\overline{\mathbb{R}^n_+}} |\widetilde{a}_\alpha^{x_*}|_{\mathcal{B}(E)} < \varepsilon, \qquad |\alpha| = 2m,$$

for the top order coefficients of $\mathcal{A}^{x_*}$ as well. To extend the top order coefficients of $\mathcal{B}_j^{\Phi_i}$ from $\mathbb{R}^{n-1} \cap B_{r_{x_*}}(0)$ to $\mathbb{R}^{n-1}$ to coefficients $b_{j\beta}^{x_*}$ we fix a nonnegative cut-off function $\chi \in C_c^\infty(\mathbb{R}^{n-1})$ with

$$\chi(x) = 1, \quad |x| \le 1, \qquad \chi(x) = 0, \quad |x| \ge 2, \qquad \chi(x) \in [0,1], \quad x \in \mathbb{R}^{n-1},$$

and set for $|\beta| = m_j$ and $j = 1, ..., m$

$$b_{j\beta}^{x_*}(t,x) := b_{j\beta}^{\Phi_{x_*}}(0,0) + \chi(x/2r_{x_*})\big(b_{j\beta}^{\Phi_{x_*}}(t, \chi(x/r_{x_*})x) - b_{j\beta}^{\Phi_{x_*}}(0,0)\big), \quad t \in \overline{J}, \quad x \in \mathbb{R}^{n-1}. \tag{2.4.11}$$

The lower order coefficients of $\mathcal{B}_j^{\Phi_i}$ are extended on $\mathbb{R}^{n-1}$ to coefficients $b_{j\beta}^{x_*}$ by setting

$$b_{j\beta}^{x_*} := \mathcal{E}_{\mathbb{R}^{n-1}\cap B_{r_{x_*}}(0)} b_{j\beta}^{\Phi_{x_*}}, \quad |\beta| < m_j, \quad j = 1, ..., m, \tag{2.4.12}$$

where $\mathcal{E}_{\mathbb{R}^{n-1}\cap B_{r_{x_*}}(0)}$ denotes the spatial extension operator from $\mathbb{R}^{n-1} \cap B_{r_{x_*}}(0)$ to $\mathbb{R}^{n-1}$, given by (1.3.3). These extended coefficients yield boundary operators

$$\mathcal{B}_j^{x_*}(t,x,D) := \sum_{|\beta|\le m_j} b_{j\beta}^{x_*}(t,x)\mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad t \in J, \qquad x \in \mathbb{R}^{n-1}, \qquad j = 1, ..., m.$$

We define the top order constant coefficient operator $\mathcal{B}^{x_*,0} = (\mathcal{B}_1^{x_*,0}, ..., \mathcal{B}_m^{x_*,0})$ by

$$\mathcal{B}_j^{x_*,0}(D) := \sum_{|\beta|=m_j} b_{j\beta}^{x_*,0}\mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad b_{j\beta}^{x_*,0} := b_{j\beta}^{x_*}(0,0) = b_{j\beta}(0,x_*), \qquad j = 1, ..., m.$$

Due to (2.4.10), for $\xi' \in \mathbb{R}^{n-1}$ we have that

$$\mathcal{A}^{x_*,0}(\xi', D_y) = \mathcal{A}_\sharp\big(0, x_*, \mathcal{O}^T_{\nu(x_*)}(\xi', D_y)\big), \qquad \mathcal{B}^{x_*,0}(\xi', D_y) = \mathcal{B}_\sharp\big(0, x_*, \mathcal{O}^T_{\nu(x_*)}(\xi', D_y)\big).$$

Now we see that the assumption (LS) for $(\mathcal{A}, \mathcal{B})$ on Ω is just made that $(\mathcal{A}^{x_*,0}, \mathcal{B}^{x_*,0})$ satisfies (LS) on $\mathbb{R}^n_+$. We write the top order coefficients of $\mathcal{B}^{x_*}$ in the form

$$b_{j\beta}^{x_*} = b_{j\beta}^{x_*,0} + \widetilde{b}_{j\beta}^{x_*}, \qquad |\beta| = m_j, \qquad j = 1, ..., m,$$

as required for (2.3.2). By construction we have that the coefficients of $\mathcal{B}^{x_*} - \mathcal{B}^{x_*,0}$ satisfy (SB). As for the top order coefficients of $\mathcal{A}^{x_*}$, for given ε, if T, r_{x_*} and the diameter of U_{x_*} are sufficiently small then it follows from continuity that

$$\sup_{\overline{J}\times\mathbb{R}^{n-1}} |\widetilde{b}_{j\beta}^{x_*}|_{\mathcal{B}(E)} < \varepsilon, \qquad |\beta| = m_j, \qquad j = 1, ..., m.$$

Therefore $(\mathcal{A}^{x_*}, \mathcal{B}^{x_*})$ satisfies all the assumptions of Proposition 2.3.1, and if ε and T are small, then for all $J' = (0, T')$ with $T' \leq T$ there is a continuous solution operator

$$\mathcal{S}_H^{\mathrm{sm},x_*} : \mathcal{D}_{\mathcal{B}^{x_*}}(J') \to \mathbb{E}_{u,\mu}(J')$$

for the half-space problem

$$\begin{aligned} \partial_t v + \mathcal{A}^{x_*}(t,x,D)v &= f^*(t,x), && x \in \mathbb{R}^n_+, && t \in J', \\ \mathcal{B}_j^{x_*} v &= g_j^*(t,x), && x \in \mathbb{R}^{n-1}, && t \in J', && j = 1,...,m, \\ v(0,x) &= u_0^*(x), && x \in \mathbb{R}^n_+. \end{aligned} \tag{2.4.13}$$

Here, $\mathcal{D}_{\mathcal{B}^{x_*}}(J')$ denotes the set of compatible data (f^*, g^*, u_0^*) with respect to $\mathcal{B}^{x_*}$.

(VII) The sets U_{x_*}, together with U_0 if Ω is unbounded, yield an open cover of $\overline{\Omega}$. If Ω is bounded it follows from compactness that there are finitely many points $x_i \in \Omega$, $i = 1, ..., N_F$ for some $N_F \in \mathbb{N}$, and finitely many points $x_i \in \Gamma$, $i = N_F + 1, ..., N_H$ for some $N_H > N_F$, such that the union of the corresponding sets

$$U_i := U_{x_i}, \qquad i = 1, ..., N_H,$$

covers $\overline{\Omega}$. If Ω is unbounded, we obtain in the same way a finite cover for the compact set $\Omega \backslash U_0$. Setting $U_0 := \emptyset$ if Ω is bounded, we thus obtain in any case a finite cover

$$\overline{\Omega} \subset \bigcup_{i=0}^{N_F} U_i \cup \bigcup_{i=N_F+1}^{N_H} U_i, \tag{2.4.14}$$

together with corresponding points x_i, operators $\mathcal{A}^i$ for $i = 0, ..., N_F$ and $(\mathcal{A}^i, \mathcal{B}^i)$ for $i = N_F + 1, ..., N_H$. If ε and T are small, then there are solution operators

$$\mathcal{S}_F^{\mathrm{sm},i}, \qquad i = 0, ..., N_F, \qquad \text{and } \mathcal{S}_H^{\mathrm{sm},i}, \qquad i = N_F + 1, ..., N_H,$$

for the finitely many full- and half-space problems (2.4.8) and (2.4.13) on $J = (0, T)$, corresponding to $\mathcal{A}^i$ and $(\mathcal{A}^i, \mathcal{B}^i)$, respectively.

(VIII) If Ω is bounded there exists a partition of unity $\{\psi_i\}_{i=1,...,N_H}$ for $\overline{\Omega}$, subordinate to the cover (2.4.14). In the unbounded case there is such a partition for the compact set $\Omega \backslash U_0$. Thus we set in addition $\psi_0 := 0$ in the bounded case, and

$$\psi_0 := 1 - \sum_{i=1}^{N_H} \psi_i$$

in the unbounded case, such that $\{\psi_i\}_{i=0,...,N_H}$ is in any case a partition of unity for $\overline{\Omega}$, subordinate to (2.4.14).

Now take compatible data $(f, g, u_0) \in \mathcal{D}(J)$ for $(\mathcal{A}, \mathcal{B})$, and consider the problem (2.4.1), for which we have to show unique solvability. Suppose that $u \in \mathbb{E}_{u,\mu}(J \times \Omega)$ solves (2.4.1). Then u solves the localized problems

$$\begin{aligned} \partial_t(\psi_i u) + \mathcal{A}(\psi_i u) &= \psi_i f + [\mathcal{A}, \psi_i]u && \text{in } \Omega \cap U_i, && t \in J, \\ \mathcal{B}(\psi_i u) &= \psi_i g + [\mathcal{B}, \psi_i]u && \text{on } \Gamma \cap U_i, && t \in J, \\ (\psi_i u)(0, \cdot) &= \psi_i u_0 && \text{in } \Omega \cap U_i, \end{aligned} \tag{2.4.15}$$

for each $i = 0, ..., N_H$. Here $[\cdot,\cdot]$ denotes the commutator bracket, e.g.,

$$[\mathcal{A},\psi_i]u = \mathcal{A}(\psi_i u) - \psi_i \mathcal{A}u.$$

Observe that $[\mathcal{A},\psi_i]$ and $[\mathcal{B}_j,\psi_i]$ are differential operators of lower order, i.e., less or equal than $2m-1$ and m_j-1, respectively. For $i = 0, ..., N_F$ it holds $\Omega \cap U_i = U_i$, so that there are no boundary conditions involved in (2.4.15) in this case. By the considerations in Step IV, the function $\psi_i u$ is the unique solution of the initial-value problem

$$\begin{aligned} \partial_t v + \mathcal{A}^i(t,x,D)v &= f^i(t,x), && x \in \mathbb{R}^n, && t \in J,\\ v(0,x) &= u_0^i(x), && x \in \mathbb{R}^n, \end{aligned}$$

where we have set

$$f^i := \psi_i f + [\mathcal{A},\psi_i]u, \qquad u_0^i := \psi_i u_0, \qquad i = 0, ..., N_F,$$

and where we identify functions with compact support with their trivial extension to $\mathbb{R}^n$. It therefore holds

$$\psi_i u = \mathcal{S}_F^{\mathrm{sm},i}(f^i, u_0^i; u)|_{U_i}, \qquad i = 0, ..., N_F.$$

Here the notation $\mathcal{S}_F^{\mathrm{sm},i}(f^i, u_0^i; u)$ indicates that f^i is defined with respect to u.
For $i = N_F+1, ..., N_H$ we have $\Gamma \cap U_i \neq \emptyset$, so that boundary conditions are involved in (2.4.15) in this case. We transform (2.4.15) to a flat boundary, using the push forward Φ_i corresponding to φ_i. Then $v = \Phi_i(\psi_i u)$ satisfies

$$\begin{aligned} \partial_t v + \mathcal{A}^i(t,x,D)v &= f^i(t,x), && x \in \mathbb{R}^n_+, && t \in J,\\ \mathcal{B}^i(t,x,D)v &= g^i(t,x), && x \in \mathbb{R}^{n-1}, && t \in J,\\ v(0,x) &= u_0^i(x), && x \in \mathbb{R}^n_+, \end{aligned}$$

where this time we have set, for $i = N_F+1, ..., N_H$,

$$f^i := \Phi_i(\psi_i f + [\mathcal{A},\psi_i]u), \qquad g^i(t,x) = \Phi_i(\psi_i g + [\mathcal{B},\psi_i]u), \qquad u_0^i := \Phi_i(\psi_i u_0),$$

identifying functions with their trivial extension to $\mathbb{R}^n_+$ as above. By the considerations in Step V and uniqueness it holds

$$\psi_i u = \Phi_i^{-1}\big(\mathcal{S}_H^{\mathrm{sm},i}(f^i, g^i, u_0^i; u)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)}\big), \qquad i = N_F+1, ..., N_H,$$

where again the notation $\mathcal{S}_H^{\mathrm{sm},i}(f^i, g^i, u_0^i; u)$ indicates that f^i, g^i are defined with respect to u. Note here that (f^i, g^i, u_0^i) is compatible with respect to $\mathcal{B}^i$, since (f, g, u_0) is compatible with respect to the original boundary operator $\mathcal{B}$.
(IX) We choose scalar-valued functions $\phi_i \in C_c^\infty(\mathbb{R}^n)$, $i = 0, ..., N_H$, such that

$$\phi_i \equiv 1 \text{ on } \operatorname{supp}\psi_i, \qquad \operatorname{supp}\phi_i \subset U_i.$$

Then $\sum_{i=0}^{N_H} \phi_i\psi_i \equiv 1$ on $\overline{\Omega}$. For $(f, g, u_0) \in \mathcal{D}(J)$ we consider the Banach space

$$Z_{u_0}(J) := \big\{u \in \mathbb{E}_{u,\mu}(J \times \Omega) \,:\, u(0,\cdot) = u_0\big\},$$

which is nonempty by Lemma 1.3.9, and define on $Z_{u_0}(J)$ the map $\mathcal{G}$ by

$$\mathcal{G}_{f,g,u_0}(u) := \sum_{i=0}^{N_F} \phi_i \mathcal{S}_F^{\mathrm{sm},i}(f^i, u_0^i; u)|_{U_i} + \sum_{i=N_F+1}^{N_H} \phi_i \Phi_i^{-1}(\mathcal{S}_H^{\mathrm{sm},i}(f^i, g^i, u_0^i; u)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)}),$$

where f^i, g^i and u_0^i are defined as above with respect to u, respectively. By the considerations in the last step, for a solution $u \in \mathbb{E}_{u,\mu}(J)$ of (2.4.1) it holds

$$\mathcal{G}_{f,g,u_0}(u) = \sum_{i=0}^{N_F} \phi_i \psi_i u = u.$$

Thus a solution of (2.4.1) has to be a fixed point of $\mathcal{G}_{f,g,u_0}$ in $Z_{u_0}(J)$.
Using the contraction principle we show that $\mathcal{G}_{f,g,u_0}$ has a unique fixed point in $Z_{u_0}(J)$ for all compatible data $(f, g, u_0) \in \mathcal{D}(J)$, provided T is sufficiently small. By construction, $\mathcal{G}_{f,g,u_0}$ is a self mapping on $Z_{u_0}(J)$. For $i = 0, ..., N_F$ and $u_1, u_2 \in Z_{u_0}(J)$, the function

$$v = \mathcal{S}_F^{\mathrm{sm},i}(f^{i,1}, u_0^i; u_1) - \mathcal{S}_F^{\mathrm{sm},i}(f^{i,2}, u_0^i; u_2)$$

is the unique solution of

$$\begin{aligned} \partial_t v + \mathcal{A}^i(t,x,D)v &= [\mathcal{A}, \psi_i](u_1 - u_2) && \text{in } \mathbb{R}^n, && t \in J, \\ v(0,\cdot) &= 0 && \text{in } \mathbb{R}^n, \end{aligned}$$

where $[\mathcal{A}, \psi_i](u_1 - u_2)$ is identified with its trivial spatial extension to $\mathbb{R}^n$. It therefore holds

$$v = \mathcal{S}_F^{\mathrm{sm},i}\big([\mathcal{A}, \psi_i](u_1 - u_2), 0\big).$$

By Proposition 2.3.2, the operator norm of $\mathcal{S}_F^{\mathrm{sm},i}$ is independent of T. Given $\eta > 0$, we use this fact, that $[\mathcal{A}, \psi_i]$ is of lower order, that the coefficients of $\mathcal{A}$ are subject to (SD), and that $(u_1 - u_2)(0, \cdot) = 0$, to deduce from Lemma 1.3.13 the estimate

$$\begin{aligned} |\phi_i \mathcal{S}_F^{\mathrm{sm},i}(f^{i,1}, u_0^i; u_1) - \phi_i \mathcal{S}_F^{\mathrm{sm},i}(f^{i,2}, u_0^i; u_2)|_{\mathbb{E}_{u,\mu}(J\times\Omega)} \\ &\lesssim |\mathcal{S}_F^{\mathrm{sm},i}([\mathcal{A}, \psi_i](u_1 - u_2), 0)|_{\mathbb{E}_{u,\mu}(J\times\mathbb{R}^n)} \\ &\lesssim |[\mathcal{A}, \psi_i](u_1 - u_2)|_{\mathbb{E}_{0,\mu}(J\times\Omega)} \\ &\le \eta\, |u_1 - u_2|_{\mathbb{E}_{1,\mu}(J\times\Omega)}, \end{aligned}$$

provided T is sufficiently small. Similarly, for $i = N_F + 1, ..., N_H$ the function

$$v = \mathcal{S}_H^{\mathrm{sm},i}(f^{i,1}, g^{i,1}, u_0^i; u_1) - \mathcal{S}_H^{\mathrm{sm},i}(f^{i,2}, g^{i,2}, u_0^i; u_2)$$

is the unique solution of

$$\begin{aligned} \partial_t v + \mathcal{A}^i(t,x,D)v &= \Phi_i([\mathcal{A}, \psi_i](u_1 - u_2)) && \text{in } \mathbb{R}^n_+, && t \in J, \\ \mathcal{B}^i(t,x,D)v &= \Phi_i([\mathcal{B}, \psi_i](u_1 - u_2)) && \text{on } \mathbb{R}^{n-1}, && t \in J, \\ v(0,\cdot) &= 0 && \text{in } \mathbb{R}^n_+, \end{aligned}$$

where we again identify the right-hand sides with their trivial extensions to $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$, respectively. Note here that the required compatibility condition at $t = 0$ holds, due to $(u_1 - u_2)(0, \cdot) = 0$. Therefore

$$v = \mathcal{S}_H^{\mathrm{sm},i}\big(\Phi_i([\mathcal{A}, \psi_i](u_1 - u_2)), \Phi_i([\mathcal{B}, \psi_i](u_1 - u_2)), 0\big).$$

By Proposition 2.3.1, the operator norm of $\mathcal{S}_H^{\mathrm{sm},i}$ restricted to vanishing initial values is uniform in T smaller than a given length. Using the same tools as above, together with the Lemmas 1.3.21, 1.3.22 and 1.3.23 about pointwise multiplication on the boundary, we obtain for given η

$$\begin{aligned}
|\phi_i \Phi_i^{-1}\big(\mathcal{S}_H^{\mathrm{sm},i}(f^{i,1}, g^{i,1}, u_0^i; u_1)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)} - \mathcal{S}_H^{\mathrm{sm},i}(f^{i,2}, g^{i,2}, u_0^i; u_2)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)}\big)|_{\mathbb{E}_{u,\mu}(J\times\Omega)}& \\
\lesssim |\mathcal{S}_H^{\mathrm{sm},i}(\Phi_i([\mathcal{A}, \psi_i](u_1 - u_2)), \Phi_i([\mathcal{B}, \psi_i](u_1 - u_2)), 0)|_{\mathbb{E}_{u,\mu}(J\times\mathbb{R}^n_+)}& \\
\lesssim |[\mathcal{A}, \psi_i](u_1 - u_2)|_{\mathbb{E}_{0,\mu}(J\times\Omega)} + |[\mathcal{B}, \psi_i](u_1 - u_2)|_{{}_0\mathbb{F}_\mu(J\times\Gamma)}& \\
\leq \eta\, |u_1 - u_2|_{\mathbb{E}_{u,\mu}(J\times\Omega)},&
\end{aligned}$$

provided T is sufficiently small. Hence for small T the map $\mathcal{G}_{f,g,u_0}$ is a strict contraction on $Z_{u_0}(J)$ and has a unique fixed point in there. Since this holds true for all $(f, g, u_0) \in \mathcal{D}(J)$, this fact already implies that solutions of (2.4.1) are unique. We further obtain a linear fixed point map

$$\mathcal{Q} : \mathcal{D}(J) \to Z_{u_0}(J), \qquad \mathcal{Q}(f, g, u_0) = \mathcal{G}_{f,g,u_0}(\mathcal{Q}(f, g, u_0)).$$

We define the space

$$\mathcal{D}_{00}(J) := \{(f, g, 0) \in \mathcal{D}_0(J)\},$$

and use the above estimates and the continuity of the solution operators $\mathcal{S}_F^{\mathrm{sm},i}$ and $\mathcal{S}_H^{\mathrm{sm},i}$ to obtain

$$\begin{aligned}
|\mathcal{Q}(f, g, 0)|_{\mathbb{E}_{u,\mu}(J)} &\leq |\mathcal{G}_{f,g,0}(\mathcal{Q}(f, g, 0)) - \mathcal{G}_{f,g,0}(0)|_{\mathbb{E}_{u,\mu}(J)} + |\mathcal{G}_{f,g,0}(0)|_{\mathbb{E}_{u,\mu}(J)} \\
&\lesssim \eta |\mathcal{Q}(f, g, 0)|_{\mathbb{E}_{u,\mu}(J)} + |(f, g, 0)|_{\mathcal{D}_0(J)}
\end{aligned}$$

for $(f, g, 0) \in \mathcal{D}_{00}(J)$, where η is small. Hence the operator norm of $\mathcal{Q} : \mathcal{D}_{00}(J) \to Z_0(J)$ is uniform in T smaller than a given length. Note that, due to the nonempty intersections of the U_i, the function $\mathcal{Q}(f, g, u_0)$ does not solve (2.4.1) with right-hand side $(f, g, u_0) \in \mathcal{D}(J)$, in general.

(X) We construct a solution for (2.4.1) by finding for given $(f, g, u_0) \in \mathcal{D}(J)$ the appropriate $(f^\star, g^\star, u_0^\star) \in \mathcal{D}(J)$ for which $\mathcal{Q}(f^\star, g^\star, u_0^\star)$ solves (2.4.1). In other words, overlappings in the sum in the definition of $\mathcal{G}_{f^\star,g^\star,u_0^\star}$ coming from nonempty intersections of the U_i have to cancel in the right way. As $\mathcal{Q}$ maps into $Z_{u_0}(J)$ it is clear that we must have $u_0^\star = u_0$. So let $(f, g, u_0) \in \mathcal{D}(J)$ be given. For $(f^\star, g^\star, u_0) \in \mathcal{D}(J)$ we use that $\mathcal{Q}(f^\star, g^\star, u_0)$ is the fixed point of $\mathcal{G}_{f^\star,g^\star,u_0}$, to obtain

$$\begin{aligned}(\partial_t+\mathcal{A})\mathcal{Q}(f^\star,g^\star,u_0) &= \sum_{i=0}^{N_F}(\partial_t+\mathcal{A})\phi_i\mathcal{S}_F^{\mathrm{sm},i}(f^{\star,i},u_0^i;\mathcal{Q}(f^\star,g^\star,u_0))|_{U_i}\\ &\quad+\sum_{i=N_F+1}^{N_H}(\partial_t+\mathcal{A})\phi_i\Phi_i^{-1}\big(\mathcal{S}_H^{\mathrm{sm},i}(f^{\star,i},g^{\star,i},u_0^i;\mathcal{Q}(f^\star,g^\star,u_0))|_{\mathbb{R}^n_+\cap B_{r_i}(0)}\big)\\ &= f^\star+\mathcal{K}_1(f^\star,g^\star)+\sum_{i=0}^{N_H}\phi_i[\mathcal{A},\psi_i]\mathcal{Q}(f^\star,g^\star,u_0),\end{aligned}$$

with the correction term

$$\begin{aligned}\mathcal{K}_1(f^\star,g^\star) := &\sum_{i=0}^{N_F}[\mathcal{A},\phi_i]\mathcal{S}_F^{\mathrm{sm},i}(f^{\star,i},u_0^i;\mathcal{Q}(f^\star,g^\star,u_0))|_{U_i}\\ &+\sum_{i=N_F+1}^{N_H}[\mathcal{A},\phi_i]\Phi_i^{-1}\big(\mathcal{S}_H^{\mathrm{sm},i}(f^{\star,i},g^{\star,i},u_0^i;\mathcal{Q}(f^\star,g^\star,u_0))|_{\mathbb{R}^n_+\cap B_{r_i}(0)}\big).\end{aligned}$$

Note that, since $\{\psi_i\}$ is a partition of unity for $\overline{\Omega}$ and $\phi_i\equiv 1$ on $\operatorname{supp}\psi_i$, it holds

$$\sum_{i=0}^{N_H}\phi_i[\mathcal{A},\psi_i]\mathcal{Q}(f^\star,g^\star,u_0)=[\mathcal{A},1]\mathcal{Q}(f^\star,g^\star,u_0)=0.$$

Similarly, on the boundary we have

$$\mathcal{B}\mathcal{Q}(f^\star,g^\star,u_0)=g^\star+\mathcal{K}_2(f^\star,g^\star),$$

with the correction term

$$\mathcal{K}_2(f^\star,g^\star):=\sum_{i=N_F+1}^{N_H}[\mathcal{B},\phi_i]\Phi_i^{-1}\big(\mathcal{S}_H^{\mathrm{sm},i}(f^{\star,i},g^{\star,i},u_0^i;\mathcal{Q}(f^\star,g^\star,u_0))|_{\mathbb{R}^n_+\cap B_{r_i}(0)}\big).$$

Here the terms involving $\mathcal{S}_F^{\mathrm{sm},i}$ do not appear since the functions ϕ_i vanish on Γ for $i=0,...,N_F$. It follows that the desired $(f^\star,g^\star)$ is a solution of the equation

$$(f^\star,g^\star)+(\mathcal{K}_1,\mathcal{K}_2)(f^\star,g^\star)=(f,g). \tag{2.4.16}$$

Since $\mathcal{Q}(f^\star,g^\star,u_0)|_{t=0}=u_0$ for $\kappa_j>1-\mu+1/p$ it holds

$$\mathcal{K}_2(f^\star,g^\star)_j|_{t=0}=\sum_{i=N_F+1}^{N_H}[\mathcal{B}_j(0,\cdot,D),\phi_i]\psi_iu_0=\sum_{i=N_F+1}^{N_H}[\mathcal{B}_j(0,\cdot,D),1]\psi_iu_0=0.$$

Therefore $\mathcal{K}_2$ maps into ${}_0\mathbb{F}_\mu(J)$. In order to solve (2.4.16) we thus consider the equation

$$(f^\flat,g^\flat)+(\mathcal{K}_1,\mathcal{K}_2)(f^\flat,g^\flat)=-(\mathcal{K}_1,\mathcal{K}_2)(f,g). \tag{2.4.17}$$

for $(f^\flat,g^\flat)\in\mathbb{E}_{0,\mu}(J)\times{}_0\mathbb{F}_\mu(J)$. If we can find a solution $(f^\flat,g^\flat)$ of (2.4.17) then

$$(f^\star,g^\star)=(f^\flat+f,g^\flat+g)$$

solves (2.4.16) and satisfies $(f^\star, g^\star, u_0) \in \mathcal{D}(J)$, which finishes the proof.
(XI) We show that (2.4.17) has a unique solution $(f^\flat, g^\flat) \in \mathbb{E}_{0,\mu}(J) \times {}_0\mathbb{F}_\mu(J)$, using once more the contraction principle. We have already seen that

$$(f^\flat, g^\flat) \mapsto -(\mathcal{K}_1, \mathcal{K}_2)(f, g) - (\mathcal{K}_1, \mathcal{K}_2)(f^\flat, g^\flat) \tag{2.4.18}$$

is a self map on $\mathbb{E}_{0,\mu}(J) \times {}_0\mathbb{F}_\mu(J)$. To show that it is a contraction, take $(f_1^\flat, g_1^\flat), (f_2^\flat, g_2^\flat) \in \mathbb{E}_{0,\mu}(J) \times {}_0\mathbb{F}_\mu(J)$, put

$$u_1 := \mathcal{Q}(f_1^\flat, g_1^\flat, u_0), \qquad u_2 := \mathcal{Q}(f_2^\flat, g_2^\flat, u_0),$$

and consider

$$\begin{aligned} |(\mathcal{K}_1, \mathcal{K}_2)(f_1^\flat, g_1^\flat) - & (\mathcal{K}_1, \mathcal{K}_2)(f_2^\flat, g_2^\flat)|_{\mathbb{E}_{0,\mu}(J)\times {}_0\mathbb{F}_\mu(J)} \\ &= |\mathcal{K}_1(f_1^\flat, g_1^\flat) - \mathcal{K}_1(f_2^\flat, g_2^\flat)|_{\mathbb{E}_{0,\mu}(J)} + |\mathcal{K}_2(f_1^\flat, g_1^\flat) - \mathcal{K}_2(f_2^\flat, g_2^\flat)|_{{}_0\mathbb{F}_\mu(J)}. \end{aligned} \tag{2.4.19}$$

For the first summand we use that $[\mathcal{A}, \phi_i]$ is of lower order, to obtain with Lemma 1.3.13 for given $\eta > 0$

$$\begin{aligned} |\mathcal{K}_1(f_1^\flat, g_1^\flat) - \mathcal{K}_1(f_2^\flat, g_2^\flat)|_{\mathbb{E}_{0,\mu}(J\times\Omega)} &\leq \eta \sum_{i=0}^{N_F} |\mathcal{S}_F^{\mathrm{sm},i}(f_1^{\flat,i}, u_0^i; u_1) - \mathcal{S}_F^{\mathrm{sm},i}(f_2^{\flat,i}, u_0^i; u_2)|_{\mathbb{E}_{u,\mu}(J\times\mathbb{R}^n)} \\ &+ \eta \sum_{i=N_F+1}^{N_H} |\mathcal{S}_H^{\mathrm{sm},i}(f_1^{\flat,i}, g_1^{\flat,i}, u_0^i; u_1) - \mathcal{S}_H^{\mathrm{sm},i}(f_2^{\flat,i}, g_2^{\flat,i}, u_0^i; u_2)|_{\mathbb{E}_{u,\mu}(J\times\mathbb{R}^n_+)}, \end{aligned} \tag{2.4.20}$$

provided T is sufficiently small. We concentrate on the second sum in (2.4.20). For $i = N_F + 1, ..., N_H$ the function $v = \mathcal{S}_H^{\mathrm{sm},i}(f_1^{\flat,i}, g_1^{\flat,i}, u_0^i; u_1) - \mathcal{S}_H^{\mathrm{sm},i}(f_2^{\flat,i}, g_2^{\flat,i}, u_0^i; u_2)$ solves

$$\begin{aligned} \partial_t v + \mathcal{A}^i v &= \Phi_i\big(\psi_i(f_1^\flat - f_2^\flat) + [\mathcal{A}, \psi_i](u_1 - u_2)\big) && \text{in } \mathbb{R}^n_+, && t \in J, \\ \mathcal{B}^i v &= \Phi_i\big(\psi_i(g_1^\flat - g_2^\flat) + [\mathcal{B}, \psi_i](u_1 - u_2)\big) && \text{on } \mathbb{R}^{n-1}, && t \in J, \\ v(0, \cdot) &= 0 && \text{in } \mathbb{R}^n_+. \end{aligned}$$

We may thus estimate

$$\begin{aligned} |\mathcal{S}_H^{\mathrm{sm},i}(f_1^{\flat,i}, & g_1^{\flat,i}, u_0^i; u_1) - \mathcal{S}_H^{\mathrm{sm},i}(f_2^{\flat,i}, g_2^{\flat,i}, u_0^i; u_2)|_{\mathbb{E}_{u,\mu}(J\times\mathbb{R}^n_+)} \\ &\lesssim |f_1^\flat - f_2^\flat|_{\mathbb{E}_{0,\mu}(J\times\Omega)} + |[\mathcal{A}, \psi_i](u_1 - u_2)|_{\mathbb{E}_{0,\mu}(J\times\Omega)} \\ &\qquad + |g_1^\flat - g_2^\flat|_{{}_0\mathbb{F}_\mu(J\times\Gamma)} + |[\mathcal{B}, \psi_i](u_1 - u_2)|_{{}_0\mathbb{F}_\mu(J\times\Gamma)} \\ &\lesssim |f_1^\flat - f_2^\flat|_{\mathbb{E}_{0,\mu}(J\times\Omega)} + |g_1^\flat - g_2^\flat|_{{}_0\mathbb{F}_\mu(J\times\Gamma)} + |\mathcal{Q}(f_1^\flat - f_2^\flat, g_1^\flat - g_2^\flat, 0)|_{\mathbb{E}_{u,\mu}(J\times\mathbb{R}^n_+)} \\ &\lesssim |f_1^\flat - f_2^\flat|_{\mathbb{E}_{0,\mu}(J\times\Omega)} + |g_1^\flat - g_2^\flat|_{{}_0\mathbb{F}_\mu(J\times\Gamma)} \end{aligned}$$

uniformly in T, using that the operator norm of $\mathcal{S}_H^{\mathrm{sm},i}$ on the ${}_0\mathbb{F}_\mu$-spaces, of $\mathcal{Q} : \mathcal{D}_{00}(J) \to Z_0(J)$, of the spatial trace and the spatial derivatives are uniform in T, respectively. Similarly one estimates the first sum in (2.4.20) uniformly in T, to obtain for given η

$$|\mathcal{K}_1(f_1^\flat, g_1^\flat) - \mathcal{K}_1(f_2^\flat, g_2^\flat)|_{\mathbb{E}_{0,\mu}(J\times\Omega)} \leq \eta\, |(f_1^\flat, g_1^\flat) - (f_2^\flat, g_2^\flat)|_{\mathbb{E}_{0,\mu}(J\times\Omega)\times\mathbb{F}_\mu(J\times\Gamma)},$$

provided T is sufficiently small. In the same way one shows the corresponding estimate for the term in (2.4.19) where $\mathcal{K}_2$ is involved. Thus the map (2.4.18) is a strict contraction on $\mathbb{E}_{0,\mu}(J) \times {}_0\mathbb{F}_\mu(J)$, and the resulting fixed point is the unique solution of (2.4.17). This finally proves Theorem 2.1.4. ■

2.5 A Right-Inverse for the Boundary Operator

In this section we construct a right-inverse for a class of autonomous boundary operators related to (2.4.1). Let us explain the setting. We specialize to the finite dimensional case

$$E = \mathbb{C}^N, \qquad N \in \mathbb{N}.$$

Let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$, or $\Omega = \mathbb{R}^n_+$, and let

$$p \in (n, \infty), \qquad \mu \in (1/p, 1].$$

For $m \in \mathbb{N}$ we consider the linear boundary operator $\mathcal{B} = (\mathcal{B}_1, ..., \mathcal{B}_m)$, given by

$$\mathcal{B}_j(x, D) = \sum_{|\beta| \le m_j} b_{j\beta}(x) \mathrm{tr}_\Omega D^\beta, \qquad x \in \Gamma, \qquad m_j \in \{0, ..., 2m-1\}, \qquad j = 1, ..., m.$$

As in Section 2.1 we define the corresponding numbers $\kappa_j \in (0, 1)$ by

$$\kappa_j := 1 - m_j/2m - 1/2mp, \qquad j = 1, ..., m.$$

Throughout this section it is assumed that

$$2m\big(\kappa_j - (1 - \mu + 1/p)\big) > (n-1)/p, \qquad j = 1, ..., m. \tag{2.5.1}$$

The coefficients $b_{j\beta}$ of the boundary operator $\mathcal{B}_j$ are supposed to satisfy

$$b_{j\beta} \in B^{2m(\kappa_j - (1-\mu+1/p))}_{p,p}(\Gamma, \mathcal{B}(\mathbb{C}^N)), \qquad j = 1, ..., m. \tag{2.5.2}$$

Due to (2.5.1) and Sobolev's embedding it holds

$$B^{2m(\kappa_j - (1-\mu+1/p))}_{p,p}(\Gamma, \mathcal{B}(\mathbb{C}^N)) \hookrightarrow C(\Gamma, \mathcal{B}(\mathbb{C}^N)). \tag{2.5.3}$$

Thus Lemma 1.3.19 and the continuity properties of the spatial trace tr_Ω (1.3.20) guarantee that $\mathcal{B}$ maps continuously

$$B^{2m(\mu - 1/p)}_{p,p}(\Omega, \mathbb{C}^N) \to \prod_{j=1}^m B^{2m(\kappa_j - (1-\mu+1/p))}_{p,p}(\Gamma, \mathbb{C}^N).$$

We further assume that there is a linear autonomous differential operator $\mathcal{A}$ of the form

$$\mathcal{A}(x, D) = \sum_{|\alpha| = 2m} a_\alpha(x) D^\alpha, \qquad x \in \Omega,$$

with coefficients

$$a_\alpha \in BUC(\overline{\Omega}, \mathcal{B}(\mathbb{C}^N)), \tag{2.5.4}$$

such that $(\mathcal{A}, \mathcal{B})$ satisfies the ellipticity conditions (E) and (LS) from Section 2.1.

Such a situation arises in Chapter 4, where $(\mathcal{A}, \mathcal{B})$ is the linearization of a quasilinear problem at some $u_0 \in B^{2m(\mu-1/p)}_{p,p}(\Omega, \mathcal{B}(\mathbb{C}^N))$. There terms of the form $\mathrm{tr}_\Omega D^\beta u_0$, $|\beta| \le m_j$ enter into the linearization, which leads to coefficients as in (2.5.2).

In Proposition 4.3.4 we use maximal $L_{p,\mu}$-regularity and the implicit function theorem to show continuous dependence on the initial data for quasilinear problems. Due to a nonlinear phase space it is there required that for all $\mu \in (1/p, 1]$ satisfying (2.5.1) an operator $\mathcal{B}$ as above has a bounded linear right-inverse $\mathcal{N}_\mu$, i.e., it holds $\mathcal{B}\mathcal{N}_\mu = \mathrm{id}$ and $\mathcal{N}_\mu$ maps continuously

$$\prod_{j=1}^m B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\Gamma, \mathbb{C}^N) \to B_{p,p}^{2m(\mu-1/p)}(\Omega, \mathbb{C}^N).$$

For the unweighted case, $\mu = 1$, the existence of a right-inverse is shown in [65, Proposition 5]. The proof there makes use of the corresponding parabolic problem and the fact the L_p-spaces on the half-line are invariant under right translations. Since this is not the case for the $L_{p,\mu}$-spaces, $\mu \in (1/p, 1)$, the proof from [65] does not carry over to the weighted case.

It is the purpose of this section to construct a right-inverse $\mathcal{N}_\mu$ also for $\mu \in (1/p, 1)$. The difficulty is that for $v \in W_p^{2m}(\Omega; \mathbb{C}^N)$ it always holds $\mathcal{B}_j v \in W_p^{2m\kappa_j}(\Gamma, \mathbb{C}^N)$, which is a smaller space than $W_p^{2m(\kappa_j-(1-\mu+1/p))}(\Gamma, \mathbb{C}^N)$. Thus the right-inverse cannot directly be constructed as the solution of the elliptic problem

$$\mathcal{A}(x, D)v = 0, \quad x \in \Omega, \qquad \mathcal{B}(x, D)v = (g_1, ..., g_m), \qquad x \in \Gamma,$$

if $\mathcal{A}$ is realized on $L_p(\Omega, \mathbb{C}^N)$. The idea is now to shift the functions $(g_1, ..., g_m)$ to a higher regularity class with an appropriate isomorphism, to solve the above problem and finally to shift the solution back. Since a suitable isomorphism only seems to be available on spaces over $\mathbb{R}^{n-1}$, we have to localize the problem of finding the right-inverse to $\mathbb{R}^{n-1}$, analogously to the proof of Theorem 2.1.4 in the last section.

For clarity reasons, several assertions from the following proof are postponed in a series of lemmas afterwards. Throughout the rest of this section we set

$$X_\mu(\Omega) := B_{p,p}^{2m(\mu-1/p)}(\Omega, \mathbb{C}^N), \qquad Y_\mu(\Gamma) := \prod_{j=1}^m B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\Gamma, \mathbb{C}^N).$$

Proposition 2.5.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$, or $\Omega = \mathbb{R}^n_+$, let $p \in (n, \infty)$ and $\mu \in (1/p, 1]$ satisfy (2.5.1) and assume that $(\mathcal{A}, \mathcal{B})$ are subject to* (E), (LS), *(2.5.2) and (2.5.4). Then $\mathcal{B}$ has a bounded linear right-inverse*

$$\mathcal{N}_\mu : \prod_{j=1}^m B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\Gamma, \mathbb{C}^N) \to B_{p,p}^{2m(\mu-1/p)}(\Omega, \mathbb{C}^N).$$

Proof. **(I)** For each $x_* \in \Gamma$ the constant coefficient operator $(\mathcal{A}(x_*, D), \mathcal{B}_\sharp(x_*, D))$[5] is subject to the pointwise conditions (E) and (LS). We construct a continuous right-inverse $\mathcal{N}_\mu^{0,x_*} : Y_\mu(\mathbb{R}^{n-1}) \to X_\mu(\mathbb{R}^n_+)$ for $\mathcal{B}_\sharp(x_*, D)$ as follows. Let $g = (g_1, ..., g_m) \in Y_\mu(\mathbb{R}^{n-1})$ be given. Then we have

$$h_j := S^{-1} g_j \in W_p^{2m\kappa_j}(\mathbb{R}^{n-1}, \mathbb{C}^N), \qquad j = 1, ..., m,$$

[5] Recall that $\mathcal{B}_{j\sharp}(x_*, D) = \sum_{|\beta|=m_j} b_{j\beta}(x)\mathrm{tr}_\Omega D^\beta$ denotes the principal part of $\mathcal{B}_j$ for $j = 1, ..., m$, and that $\mathcal{B}_\sharp := (\mathcal{B}_{1\sharp}, ..., \mathcal{B}_{m\sharp})$.

where the operator S on $L_p(\mathbb{R}^{n-1}, \mathbb{C}^N)$ is given by

$$S := (1 + (-\Delta_{n-1})^m)^{1-\mu+1/p}.$$

It follows from Lemma 2.2.6 that for all $\lambda > 0$ the unique solution $v \in W_p^{2m}(\mathbb{R}^n_+, \mathbb{C}^N)$ of the elliptic problem

$$\begin{aligned} \lambda v + \mathcal{A}(x_*, D)v &= 0 \qquad \text{on } \mathbb{R}^n_+, \\ \mathcal{B}_{j\sharp}(x_*, D)v &= h_j \qquad \text{on } \mathbb{R}^{n-1}, \qquad j = 1, ..., m, \end{aligned}$$

is of the form $v = \sum_{j=1}^m \mathcal{S}_j(\lambda) h_j$, with operators

$$\mathcal{S}_j(\lambda) \in \mathcal{B}\big(W_p^{2m\kappa_j}(\mathbb{R}^{n-1}, \mathbb{C}^N), W_p^{2m}(\mathbb{R}^n_+, \mathbb{C}^N)\big), \qquad j = 1, ..., m.$$

We now define the operator $\mathcal{N}_\mu^{0,x_*}(\lambda)$ by

$$\mathcal{N}_\mu^{0,x_*}(\lambda) g := S \sum_{j=1}^m \mathcal{S}_j(\lambda) S^{-1} g_j. \tag{2.5.5}$$

Here S acts on the first $n-1$ variables as a pointwise realization on $L_p(\mathbb{R}^n_+, \mathbb{C}^N) = L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}, \mathbb{C}^N))$. It is shown in Lemma 2.5.4 below that $\mathcal{N}_\mu^{0,x_*}(\lambda)$ maps continuously $Y_\mu(\mathbb{R}^{n-1}) \to X_\mu(\mathbb{R}^n_+)$. Further $\mathcal{N}_\mu^{0,x_*}(\lambda)$ is in fact a right-inverse for $\mathcal{B}_\sharp(x_*, D)$, since the realization of S on $L_p(\mathbb{R}^n_+, \mathbb{C}^N)$ commutes with $\mathcal{B}_\sharp(x_*, D)$.

(II) For all $x_* \in \Gamma$ we choose a neighbourhood $\widetilde{U}_{x_*} \subset \mathbb{R}^n$ of x_*, a smooth diffeomorphism $\varphi_{x_*} : \widetilde{U}_{x_*} \to \mathbb{R}^n$ and a radius $r_{x_*} > 0$ with

$$\varphi_{x_*}(x_*) = 0, \qquad \varphi_{x_*}(\widetilde{U}_{x_*}) = B_{2r_{x_*}}(0), \qquad \varphi_{x_*}(\widetilde{U}_{x_*} \cap \Omega) \subset \mathbb{R}^n_+, \qquad \varphi_{x_*}(\widetilde{U}_{x_*} \cap \Gamma) \subset \mathbb{R}^{n-1}.$$

For given $\varepsilon > 0$, if the diameter of $\widetilde{U}_{x_*}$ is sufficiently small, then by continuity the top order coefficients of $\mathcal{B}_j$ satisfy

$$\sup_{x \in \Gamma \cap \widetilde{U}_{x_*}} |b_{j\beta}(x_*) - b_{j\beta}(x)| < \varepsilon, \qquad |\beta| = m_j, \qquad j = 1, ..., m.$$

Setting

$$U_{x_*} := \varphi_{x_*}^{-1}\big(B_{r_{x_*}}(0)\big), \qquad x_* \in \Gamma,$$

we obtain an open cover $\bigcup_{x_* \in \Gamma} U_{x_*}$ for Γ, from which we may choose a finite subcover $\bigcup_i U_i$ corresponding to points $x_i \in \Gamma$ and chart maps φ_i. There further exists a smooth partition of unity $\{\psi_i\}$ of Γ, subordinate to $\bigcup_i U_i$.

(III) Now let $g = (g_1, ..., g_m) \in Y_\mu(\Gamma)$ be given. If $u \in X_\mu(\Omega)$ solves

$$\mathcal{B} u = g \qquad \text{on } \Gamma, \tag{2.5.6}$$

then for each i the function $v = \psi_i u$ solves[6]

$$\mathcal{B} v = \psi_i g + [\mathcal{B}, \psi_i] u \qquad \text{on } \Gamma \cap U_i. \tag{2.5.7}$$

[6]Recall that $[\cdot, \cdot]$ denotes the commutator bracket, i.e., $[\mathcal{B}, \psi_i]u = \mathcal{B}(\psi_i u) - \psi_i \mathcal{B} u$.

Denoting by Φ_i the push forward operator corresponding to φ_i, i.e., $\Phi_i u = u \circ \varphi_i^{-1}$, we have that $\psi_i u$ solves (2.5.7) if and only if $w = \Phi_i(\psi_i u)$ satisfies

$$\mathcal{B}^{\Phi_i} w = \Phi_i\big(\psi_i g + [\mathcal{B}, \psi_i]u\big) =: h^i \qquad \text{on } \mathbb{R}^{n-1} \cap B_{r_i}(0). \tag{2.5.8}$$

Here $\mathcal{B}^{\Phi_i}(x, D) := \big(\Phi_i \mathcal{B}(\cdot, D)\Phi_i^{-1}\big)(x)$ denotes the transformed boundary operator,

$$\mathcal{B}^{\Phi_i}(x, D)w = \big(\mathcal{B}(\cdot, D)(w \circ \varphi_i)\big) \circ \varphi_i^{-1}(x), \qquad x \in \mathbb{R}^{n-1} \cap B_{r_i}(0),$$

and the transformed data $h^i = (h_1^i, ..., h_m^i)$ is identified with its trivial extension to $\mathbb{R}^{n-1}$ so that it belongs to $Y_\mu(\mathbb{R}^{n-1})$.
As in Step V of the proof of Theorem 2.1.4 we obtain that the coefficients of $\mathcal{B}^{\Phi_i}$, which are denoted by $b_{j\beta}^{\Phi_i}$, satisfy

$$b_{j\beta}^{\Phi_i} \in B_{p,p}^{2m(\kappa_j - (1-\mu+1/p))}(\mathbb{R}^{n-1} \cap B_{r_i}(0), \mathbb{C}^N), \qquad |\beta| \le m_j, \qquad j = 1, ..., m.$$

We denote by $\mathcal{E}_{\mathbb{R}^{n-1} \cap B_{r_i}(0)}$ the continuous extension operator from $\mathbb{R}^{n-1} \cap B_{r_i}(0)$ to $\mathbb{R}^{n-1}$, given by (1.3.3), and extend the lower order coefficients of $\mathcal{B}^{\Phi_i}$ to $\mathbb{R}^{n-1}$ by setting

$$b_{j\beta}^i := \mathcal{E}_{\mathbb{R}^{n-1} \cap B_{r_i}(0)} b_{j\beta}^{\Phi_i}, \qquad |\beta| < m_j, \qquad j = 1, ..., m.$$

The top order coefficients are extended to

$$b_{j\beta}^i(x) := b_{j\beta}^{\Phi_i}(0) + \chi(x/2r_i)\big(b_{j\beta}^{\Phi_i}(\chi(x/r_i)x) - b_{j\beta}^{\Phi_i}(0)\big), \quad x \in \mathbb{R}^{n-1}, \quad |\beta| = m_j, \quad j = 1, ..., m,$$

where $\chi \in C_c^\infty(\mathbb{R}^{n-1})$ is an appropriate cut-off function. We denote the operator with extended coefficients $b_{j\beta}^i$ by $\mathcal{B}^i$. Now, if a function $w \in X_\mu(\mathbb{R}_+^n)$ solves

$$\mathcal{B}^i w = h^i \qquad \text{on } \mathbb{R}^{n-1}, \tag{2.5.9}$$

then $w|_{\mathbb{R}^{n-1} \cap B_{r_i}(0)}$ solves (2.5.8).
(IV) To solve (2.5.9) we consider the top order constant coefficient operators

$$\mathcal{A}^{i,0}(D) := \big(\Phi_i \mathcal{A}_\sharp(\cdot, D)\Phi_i^{-1}\big)|_{x=0}, \qquad \mathcal{B}^{i,0}(D) := \mathcal{B}_\sharp^i(0, D) = \sum_{|\beta| = m_j} b_{j\beta}^{\Phi_i}(0) \mathrm{tr}_{\mathbb{R}_+^n} D^\beta.$$

Since $(\mathcal{A}, \mathcal{B})$ satisfies (E) and (LS), it follows that $(\mathcal{A}^{i,0}, \mathcal{B}^{i,0})$ satisfies these conditions as well. A function w solves (2.5.9) if and only if it satisfies

$$\mathcal{B}^{i,0}(D)w = h^i(x) + \mathcal{B}^{i,sm}(x, D)w, \qquad x \in \mathbb{R}^{n-1}. \tag{2.5.10}$$

Here the operator $\mathcal{B}^{i,sm}$ is given by

$$\mathcal{B}^{i,sm}(x, D) = \mathcal{B}^{i,0}(D) - \mathcal{B}^i(x, D), \qquad x \in \mathbb{R}^{n-1},$$

and the coefficients of $\mathcal{B}^{i,sm}$ are denoted by $b_{j\beta}^{i,sm}$. By construction it holds

$$b_{j\beta}^{i,sm} \in B_{p,p}^{2m(\kappa_j - (1-\mu+1/p))}(\mathbb{R}^{n-1}, \mathbb{C}^N), \qquad |\beta| \le m_j, \qquad j = 1, ..., m,$$

and we further have that for given ε

$$\sup_{x\in\mathbb{R}^{n-1}} |b^{i,sm}_{j\beta}(x)| < \varepsilon, \qquad |\beta| = m_j, \qquad j = 1, ..., m,$$

provided the diameter of the neighbourhoods U_i is chosen sufficiently small from the beginning. Due to the considerations in Step I, a continuous right-inverse $\mathcal{N}^{0,i}_\mu(\lambda)$ for $\mathcal{B}^{i,0}$ may be constructed as in (2.5.5), for all $\lambda > 0$. Hence for a function $w \in X_\mu(\mathbb{R}^n_+)$ to solve (2.5.10) it suffices that w satisfies

$$\big(\mathrm{id} - \mathcal{N}^{0,i}_\mu(\lambda)\mathcal{B}^{i,sm}\big)w = \mathcal{N}^{0,i}_\mu(\lambda)h^i \qquad \text{in } \mathbb{R}^n_+.$$

Lemma 2.5.5 shows that if ε is sufficiently small and λ is sufficiently large then this equation is solvable by means of a Neumann series, i.e.,

$$\mathcal{N}^i_\mu(\lambda)h^i := \sum_{k=0}^\infty \mathcal{N}^{0,i}_\mu(\lambda)\big(\mathcal{B}^{i,sm}\mathcal{N}^{0,i}_\mu(\lambda)\big)^k h^i. \tag{2.5.11}$$

This yields for each i a continuous solution operator

$$\mathcal{N}^i_\mu(\lambda) : Y_\mu(\mathbb{R}^{n-1}) \to X_\mu(\mathbb{R}^n_+)$$

for (2.5.9). Therefore $\psi_i u$ solves (2.5.7) if $\psi_i u$ satisfies

$$\psi_i u = \Phi_i^{-1}\big(\mathcal{N}^i_\mu(\lambda)\Phi_i(\psi_i g + [\mathcal{B}, \psi_i]u)\big)|_{\mathbb{R}^n_+\cap B_{r_i}(0)}. \tag{2.5.12}$$

(V) For each i we choose a function $\phi_i \in C^\infty_c(U_i)$ with

$$\phi_i \equiv 1 \qquad \text{on } \operatorname{supp}\psi_i.$$

Using them, we define the operator $\mathcal{K}_1(\lambda) : Y_\mu(\Gamma) \to X_\mu(\Omega)$ by

$$\mathcal{K}_1(\lambda)g := \sum\nolimits_i \phi_i\Phi_i^{-1}(\mathcal{N}^i_\mu(\lambda)\Phi_i\psi_i g)|_{\mathbb{R}^n_+\cap B_{r_i}(0)}, \qquad g \in Y_\mu(\Gamma),$$

and we further define the operator $\mathcal{K}_2 : X_\mu(\Omega) \to X_\mu(\Omega)$ by

$$\mathcal{K}_2(\lambda)u := \sum\nolimits_i \phi_i\Phi_i^{-1}(\mathcal{N}^i_\mu(\lambda)\Phi_i[\mathcal{B}, \psi_i]u)|_{\mathbb{R}^n_+\cap B_{r_i}(0)}, \qquad u \in X_\mu(\Omega).$$

Due to the considerations in the last step, a solution u of (2.5.6) is a solution of the equation

$$\big(\mathrm{id} - \mathcal{K}_2(\lambda)\big)u = \mathcal{K}_1(\lambda)g^\star, \tag{2.5.13}$$

where $g^\star \in Y_\mu(\Gamma)$ must be appropriately chosen such that error terms from nonempty intersections of the U_i cancel when summing up in $\mathcal{K}_1$ and $\mathcal{K}_2$. Lemma 2.5.6 shows that there is a bounded linear solution operator $\mathcal{Q}(\lambda)$ for (2.5.13) if λ is sufficiently large, which is again constructed by means of a Neumann series.
To find the appropriate $g^\star$ for which $\mathcal{Q}(\lambda)g^\star$ solves (2.5.6), note that due to (2.5.13), (2.5.12) and (2.5.7) it holds

$$\begin{aligned}\mathcal{B}\mathcal{Q}(\lambda)g^\star &= \mathcal{B}\big(\mathcal{K}_1(\lambda)g^\star + \mathcal{K}_2(\lambda)\mathcal{Q}(\lambda)g^\star\big)\\ &= \sum\nolimits_i \phi_i\big(\psi_i g^\star + [\mathcal{B}, \psi_i]\mathcal{Q}(\lambda)g^\star\big) - \mathcal{K}_3(\lambda)g^\star\\ &= \big(\mathrm{id} - \mathcal{K}_3(\lambda)\big)g^\star,\end{aligned}$$

where the correction operator $\mathcal{K}_3(\lambda) : Y_\mu(\Gamma) \to Y_\mu(\Gamma)$ comes from commuting $\mathcal{B}$ with ϕ_i in (2.5.12), i.e.,

$$\mathcal{K}_3(\lambda)h := \sum\nolimits_i [\phi_i, \mathcal{B}]\Phi_i^{-1}\big(\mathcal{N}_\mu^i(\lambda)\Phi_i[\psi_i h + [\mathcal{B}, \psi_i]\mathcal{Q}(\lambda)h]\big)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)}, \qquad h \in Y_\mu(\Gamma).$$

Lemma 2.5.7 shows that for sufficiently large λ there is a continuous solution operator $\mathcal{R}(\lambda) : Y_\mu(\Gamma) \to X_\mu(\Gamma)$ for the equation

$$\big(\mathrm{id} - \mathcal{K}_3(\lambda)\big)g^\star = g, \qquad g \in Y_\mu(\Gamma), \tag{2.5.14}$$

It then follows that the continuous operator $\mathcal{N}_\mu : Y_\mu(\Gamma) \to X_\mu(\Omega)$, defined by

$$\mathcal{N}_\mu g := \mathcal{Q}(\lambda)\mathcal{R}(\lambda)g, \qquad g \in Y_\mu(\Gamma),$$

for some sufficiently large λ, is a right-inverse for $\mathcal{B}$. ∎

We still have to prove several assertions claimed in the proof above.

Lemma 2.5.2. *Let $p \in (1, \infty)$, and assume that $\alpha \in [0, 1]$ and $s \geq 0$ satisfy $s - 2m\alpha \geq 0$. Then the pointwise realization of $(1 + (-\Delta_{n-1})^m)^\alpha$ on $L_p(\mathbb{R}^n_+, \mathbb{C}^N)$ maps continuously*

$$H_p^s(\mathbb{R}^n_+, \mathbb{C}^N) \to H_p^{s-2m\alpha}(\mathbb{R}^n_+, \mathbb{C}^N). \tag{2.5.15}$$

Restricting to $p \in [2, \infty)$, for $\sigma \in [0, 2m(\mu - 1/p)]$ the pointwise realization of the operator $S = (1 + (-\Delta_{n-1})^m)^{1-\mu+1/p}$ on $L_p(\mathbb{R}^n_+, \mathbb{C}^N)$ maps continuously

$$W_p^{2m-\sigma}(\mathbb{R}^n_+, \mathbb{C}^N) \to B_{p,p}^{2m(\mu-1/p)-\sigma}(\mathbb{R}^n_+, \mathbb{C}^N).$$

Proof. Using extensions and restrictions, it suffices to show the assertion for $\mathbb{R}^n$ instead of $\mathbb{R}^n_+$. For $k \in \mathbb{N}_0$ it follows from Fubini's theorem that

$$H_p^k(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow H_p^k(\mathbb{R}; L_p(\mathbb{R}^{n-1}, \mathbb{C}^N)) \cap L_p(\mathbb{R}; H_p^k(\mathbb{R}^{n-1}, \mathbb{C}^N)). \tag{2.5.16}$$

The operators $(1 - \partial_y^2)^{k/2}$ and $(1 + (-\Delta_{n-1})^m)^{k/2m}$ on $L_p(\mathbb{R}; L_p(\mathbb{R}^{n-1}, \mathbb{C}^N))$ with domains

$$H_p^k(\mathbb{R}; L_p(\mathbb{R}^{n-1}, \mathbb{C}^N)) \qquad \text{and} \qquad L_p(\mathbb{R}; H_p^k(\mathbb{R}^{n-1}, \mathbb{C}^N)),$$

commute in the resolvent sense and admit bounded imaginary powers with power angle equal to zero, respectively, Therefore, interpolating the embedding (2.5.16) with $L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}, \mathbb{C}^N))$ by the complex method, using Lemma A.3.4 and A.2 m) we obtain

$$H_p^s(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow H_p^s(\mathbb{R}; L_p(\mathbb{R}^{n-1}, \mathbb{C}^N)) \cap L_p(\mathbb{R}; H_p^s(\mathbb{R}^{n-1}, \mathbb{C}^N)), \qquad s > 0.$$

Using the mixed derivative theorem (Lemma A.3.3) with the operators from above we thus have for $s \geq 0$ that

$$H_p^s(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow H_p^{s(1-\theta)}(\mathbb{R}; H_p^{s\theta}(\mathbb{R}^{n-1}, \mathbb{C}^N)), \qquad \theta \in [0, 1].$$

Hence for $s \geq 2m\alpha$ the operator $(1+(-\Delta_{n-1})^m))^\alpha$ maps $H_p^s(\mathbb{R}^n,\mathbb{C}^N)$ continuously into

$$H_p^{s(1-\theta)}(\mathbb{R}; H_p^{s\theta-2m\alpha}(\mathbb{R}^{n-1},\mathbb{C}^N)), \qquad \theta \in [2m\alpha/s, 1].$$

Now suppose that $s-2m\alpha \in \mathbb{N}_0$. In this case, if θ is such that $s\theta - 2m\alpha = k \in \mathbb{N}_0$ then it also holds $s(1-\theta) = s-2m\alpha - k \in \mathbb{N}_0$. Thus, by Fubini's theorem,

$$\bigcap_{\theta\in[2m\alpha/s,1], s\theta-2m\alpha\in\mathbb{N}_0} H_p^{s(1-\theta)}(\mathbb{R}; H_p^{s\theta-2m\alpha}(\mathbb{R}^{n-1},\mathbb{C}^N)) = H_p^{s-2m\alpha}(\mathbb{R}^n,\mathbb{C}^N),$$

which yields that (2.5.15) holds for $s-2m\alpha\in\mathbb{N}_0$. The general case follows from the integer case by complex interpolation.
The asserted mapping property for S follows from (2.5.15) by real interpolation in case $2m-\sigma \notin \mathbb{N}$. For $2m-\sigma\in\mathbb{N}$ it follows from complex interpolation and the embedding

$$H_p^s(\mathbb{R}^n_+,\mathbb{C}^N) \hookrightarrow B_{p,p}^s(\mathbb{R}^n_+,\mathbb{C}^N), \qquad s\geq 0, \tag{2.5.17}$$

which is valid for $p\geq 2$ due to [82, Theorem 2.3.2]. ∎

We next consider the mapping properties of an extension operator to $\mathbb{R}^n_+$.

Lemma 2.5.3. *Let $p\in(1,\infty)$. Consider for $\operatorname{Re}\lambda \geq 1$ the operator $L_\lambda^{1/2m} = (\lambda + (-\Delta_{n-1})^m)^{1/2m}$, and the corresponding extension operator $\mathcal{E}_\lambda = e^{-\cdot L_\lambda^{1/2m}}$ from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$. There is a constant $C>0$, which does not depend on λ, such that*

$$|\mathcal{E}_\lambda S^{-1} h|_{L_p(\mathbb{R}_+; W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N))} \leq C\,|h|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)}, \tag{2.5.18}$$

where S is defined in Lemma 2.5.2, and further

$$|\mathcal{E}_\lambda h|_{L_p(\mathbb{R}^n_+,\mathbb{C}^N)} \leq C\,\lambda^{-1/2mp}\,|h|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}.$$

Proof. (I) First observe that the function

$$z\mapsto \frac{\vartheta+\sqrt{z}}{\vartheta+(\lambda+z^m)^{1/2m}}, \qquad z\in\Sigma_{\pi/4m} = \{w\in\mathbb{C}\backslash\{0\} \;:\; |\arg w| < \pi/4m\},$$

is bounded independent of $\operatorname{Re}\lambda\geq 1$ and, say, $\vartheta\in\Sigma_{2\pi/3}$. Using that $-\Delta_{n-1}$ admits on $L_p(\mathbb{R}^{n-1},\mathbb{C}^N)$ a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle equal to zero, this yields that the operator family

$$\big(\vartheta+(-\Delta_{n-1})^{1/2}\big)\big(\vartheta+(\lambda+(-\Delta_{n-1})^m)^{1/2m}\big)^{-1}, \qquad \operatorname{Re}\lambda\geq 1, \quad \vartheta\in\Sigma_{2\pi/3},$$

is uniformly bounded on $L_p(\mathbb{R}^{n-1},\mathbb{C}^N)$. Since further the operator $(-\Delta_{n-1})^{1/2}$ is sectorial with angle of sectoriality equal to zero, it follows that the resolvent estimate

$$|(\vartheta+L_\lambda^{1/2m})^{-1}|_{\mathcal{B}(L_p(\mathbb{R}^{n-1},\mathbb{C}^N))} \leq C\,|\vartheta|^{-1}, \qquad \vartheta\in\Sigma_{2\pi/3},$$

is valid with a constant C independent of λ and ϑ. This fact implies

$$|e^{-yL_\lambda^{1/2m}}|_{\mathcal{B}(L_p(\mathbb{R}^{n-1},\mathbb{C}^N))} \leq C, \qquad |L_\lambda e^{-yL_\lambda^{1/2m}}|_{\mathcal{B}(L_p(\mathbb{R}^{n-1},\mathbb{C}^N))} \leq C\,y^{-1}, \tag{2.5.19}$$

for $y > 0$, with a constant C independent of $\operatorname{Re}\lambda \geq 1$, since for a generator of an analytic semigroup the constants in these estimates only depend on the sector contained in the resolvent set and on the resolvent estimate for the generator (see the proof of [67, Proposition 2.1.1 (iii)], for instance).
Using $(1-\Delta_{n-1})^{s/2}$ as an isomorphism between $H_p^s(\mathbb{R}^{n-1},\mathbb{C}^N)$ and $L_p(\mathbb{R}^{n-1},\mathbb{C}^N))$ that commutes with $L_\lambda^{1/2m}$, we obtain from interpolation that (2.5.19) remains valid if one replaces $\mathcal{B}(L_p(\mathbb{R}^{n-1},\mathbb{C}^N))$ by $\mathcal{B}(W_p^s(\mathbb{R}^{n-1},\mathbb{C}^N))$ for $s\geq 0$.
(II) In this step we follow the proof of [68, Proposition 6.2]. Take $v \in W_p^{2m\kappa_j}(\mathbb{R}^{n-1},\mathbb{C}^N)$ and let $v = a+b$ with $a\in W_p^{2m-m_j-1}(\mathbb{R}^{n-1},\mathbb{C}^N)$ and $b\in W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N)$. Using that the operator $(1-\Delta_{n-1})^{1/2}L_\lambda^{-1/2m}$ is uniformly bounded in $\operatorname{Re}\lambda\geq 1$, and using further (2.5.19), we obtain for $y>0$ that

$$\begin{aligned}|e^{-yL_\lambda}v|_{W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N)} &\leq C\,|L_\lambda e^{-yL_\lambda}a|_{W_p^{2m-m_j-1}(\mathbb{R}^{n-1},\mathbb{C}^N)} + |e^{-yL_\lambda}b|_{W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N)}\\ &\leq C\,y^{-1}|a|_{W_p^{2m-m_j-1}(\mathbb{R}^{n-1},\mathbb{C}^N)} + C\,|b|_{W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N)}.\end{aligned}$$

Taking the infimum over a and b on the right-hand side leads to

$$|e^{-yL_\lambda}v|_{W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N)} \leq C\,y^{-1}K\big(y,v,W_p^{2m-m_j-1}(\mathbb{R}^{n-1},\mathbb{C}^N),W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N)\big)$$

for $y>0$, where K denotes the K-functional from real interpolation theory (see [68]). It now follows from the definition of the real interpolation functor $(\cdot,\cdot)_{1-1/p,p}$ that

$$|e^{-\cdot L_\lambda}v|_{L_p(\mathbb{R}_+,W_p^{2m-m_j}(\mathbb{R}^{n-1},\mathbb{C}^N))} \leq C\,|v|_{B_{p,p}^{2m\kappa_j}(\mathbb{R}^{n-1},\mathbb{C}^N)},$$

with a constant C independent of λ. The estimate (2.5.18) is now a consequence of the fact that $S=(1+(-\Delta_{n-1})^m)^{1-\mu+1/p}$ is an isomorphism between $B_{p,p}^{2m\kappa_j}(\mathbb{R}^{n-1},\mathbb{C}^N)$ and $B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)$.
(III) The function $z\mapsto \exp\big(y(\lambda^{1/2m}-(\lambda+z^m)^{1/2m})\big)$ is holomorphic and bounded on $\Sigma_{\pi/4m}$, independent of $y>0$ and $\operatorname{Re}\lambda\geq 1$. Using again the bounded $\mathcal{H}^\infty$-calculus of $-\Delta_{n-1}$ we obtain that there is a constant $C>0$, independent of y and λ, such that

$$|e^{-yL_\lambda^{1/2m}}h|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} \leq C\,e^{-y\lambda^{1/2m}}|h|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)},\qquad h\in L_p(\mathbb{R}^{n-1},\mathbb{C}^N).$$

Taking the $L_p(\mathbb{R}_+)$ norm with respect to y shows the second asserted estimate. ■

In dependence on λ we consider the continuity properties of the right-inverses $\mathcal{N}_\mu^{0,x_*}(\lambda)$ on the half-space, defined in (2.5.5).

Lemma 2.5.4. *In the setting of the proof of Proposition 2.5.1, consider for $x_*\in\Gamma$ the operator*

$$\mathcal{N}_\mu^{0,x_*}(\lambda)g := S\sum_{j=1}^m \mathcal{S}_j(\lambda)S^{-1}g_j,\qquad g=(g_1,...,g_m)\in Y_\mu(\mathbb{R}^{n-1}).$$

Then for $\sigma_1\in[0,\mu-1/p]$ we have

$$\begin{aligned}|\mathcal{N}_\mu^{0,x_*}(\lambda)g|_{B_{p,p}^{2m(\mu-1/p-\sigma_1)}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-\sigma_1}\max_{j=1,\dots,m}\big(&|g_j|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)}\\ &+\lambda^{-(1-\mu+1/p)-1/2mp}\lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big),\end{aligned}\qquad(2.5.20)$$

and, for $\sigma_2 \in [0, 2m - 1/p)$,

$$|\mathcal{N}_\mu^{0,x_*}(\lambda)g|_{B^{1/p+\sigma_2}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-(1-\sigma_2/2m)} \max_{j=1,\dots,m} \Big(\lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} + \lambda^{1-\mu+1/p+1/2mp}|g_j|_{B^{2m(\kappa_j-(1-\mu+1/p))}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}^N)}\Big) \tag{2.5.21}$$

Moreover, for $\sigma_3 \in [0, \kappa_j - (1-\mu+1/p))$ *and* $\beta \in \mathbb{N}_0^n$ *with* $|\beta| \le m_j$ *it holds*

$$|\mathrm{tr}_{\mathbb{R}^n_+} D^\beta \mathcal{N}_\mu^{0,x_*}(\lambda)g|_{B^{2m(\kappa_j-(1-\mu+1/p)-\sigma_3)}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}^N)} \lesssim \lambda^{-\sigma_3}\lambda^{-\frac{m_j-|\beta|}{2m}} \max_{l=1,\dots,m} \big(|g_l|_{B^{2m(\kappa_l-(1-\mu+1/p))}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}^N)} + \lambda^{-(1-\mu+1/p)-1/2mp}\lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big), \tag{2.5.22}$$

and further

$$|\mathrm{tr}_{\mathbb{R}^n_+} D^\beta \mathcal{N}_\mu^{0,x_*}(\lambda)g|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} \lesssim \lambda^{-\left(1-\frac{|\beta|}{2m}\right)} \max_{l=1,\dots,m} \big(\lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} + \lambda^{1-\mu+1/p}\lambda^{1/2mp}|g_l|_{B^{2m(\kappa_l-(1-\mu+1/p))}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}^N)}\big). \tag{2.5.23}$$

Proof. (I) Due to Lemma 2.2.6, for $j = 1, ..., m$ the operator $\mathcal{S}_j(\lambda)$ is of the form

$$\mathcal{S}_j(\lambda) = \mathcal{T}_j(\lambda) L_\lambda^{1-m_j/2m}\mathcal{E}_\lambda,$$

where $\mathcal{T}_j(\lambda) \in \mathcal{B}\big(L_p(\mathbb{R}^n_+,\mathbb{C}^N), W^{2m}_p(\mathbb{R}^n_+,\mathbb{C}^N)\big)$, $L_\lambda = \lambda + (-\Delta_{n-1})^m$ and $\mathcal{E}_\lambda = e^{-\cdot L_\lambda^{1/2m}}$. Using that the function $z \mapsto \frac{(\lambda+z^m)^{1-m_j/2m}}{\lambda^{1-m_j/2m}+z^{(2m-m_j)/2}}$ is uniformly bounded for, say, $z \in \Sigma_{\pi/4m}$, and the properties of $\mathcal{T}_j(\lambda)$ stated in Lemma 2.2.6 we may rewrite $\mathcal{S}_j(\lambda)$ to

$$\mathcal{S}_j(\lambda) = \widetilde{\mathcal{T}}_j(\lambda)\left((-\Delta_{n-1})^{\frac{2m-m_j}{2}} + \lambda^{1-\frac{m_j}{2m}}\right)\mathcal{E}_\lambda,$$

where $\widetilde{\mathcal{T}}_j(\lambda)$ has for $\lambda > 0$ and $j = 1, ..., m$ the property

$$|\widetilde{\mathcal{T}}_j(\lambda)v|_{B^{2m-s}_{p,q}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-s/2m}|v|_{L_p(\mathbb{R}^n_+,\mathbb{C}^N)}, \qquad s \in [0, 2m], \quad q \in [1, \infty]. \tag{2.5.24}$$

The proof of [25, Lemma 4.3] shows that $\mathcal{T}_j(\lambda)$ is a convolution operator with respect to $x \in \mathbb{R}^{n-1}$. Therefore $-\Delta_{n-1}$ commutes with $\mathcal{T}_j(\lambda)$, and thus also with $\widetilde{\mathcal{T}}_j(\lambda)$. Now it follows from [7, Lemma III.4.9.2] that S commutes with $\widetilde{\mathcal{T}}_j(\lambda)$. Together with (2.5.24) we obtain

$$|S\widetilde{\mathcal{T}}_j(\lambda)v|_{W^{2m(\mu-1/p-\sigma_1)}_p(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-\sigma_1}\lambda^{-(1-\mu+1/p)}|Sv|_{L_p(\mathbb{R}^n_+,\mathbb{C}^N)}. \tag{2.5.25}$$

(II) To show (2.5.20), first observe that

$$\begin{aligned} |\mathcal{N}_\mu^{0,x_*}(\lambda)g|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)} &\lesssim \max_{j=1,\dots m} |S\widetilde{\mathcal{T}}_j(\lambda)(-\Delta_{n-1})^{\frac{2m-m_j}{2}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)} \\ &\quad + \max_{j=1,\dots,m} |S\widetilde{\mathcal{T}}_j(\lambda)\lambda^{1-\frac{m_j}{2m}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)}. \end{aligned}$$

Hence for each j we have to estimate these two summands. For the first summand we use (2.5.24) and the Lemmas 2.5.2 and 2.5.3 to obtain

$$|S\widetilde{\mathcal{T}}_j(\lambda)(-\Delta_{n-1})^{\frac{2m-m_j}{2}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-\sigma_1}|g_j|_{B^{2m(\kappa_j-(1-\mu+1/p))}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}^N)}.$$

Using in addition (2.5.25), we have for the second summand

$$|S\widetilde{\mathcal{T}}_j(\lambda)\lambda^{1-\frac{m_j}{2m}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-\sigma_1}\lambda^{-(1-\mu+1/p)-1/2mp}\lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}.$$

(III) For (2.5.21) we first estimate, as in the last step,

$$|\mathcal{N}^{0,x_*}_\mu(\lambda)g|_{B^{1/p+\sigma_2}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \max_{j=1,\ldots,m}|S\widetilde{\mathcal{T}}_j(\lambda)(-\Delta_{n-1})^{\frac{2m-m_j}{2}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{1/p+\sigma_2}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)} + \max_{j=1,\ldots,m}|S\widetilde{\mathcal{T}}_j(\lambda)\lambda^{1-\frac{m_j}{2m}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{1/p+\sigma_2}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)}.$$

Using (2.5.24), we obtain for the first summand

$$|S\widetilde{\mathcal{T}}_j(\lambda)(-\Delta_{n-1})^{\frac{2m-m_j}{2}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{1/p+\sigma_2}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-(1-\sigma_2/2m)}\lambda^{1-\mu+1/p+1/2mp}|g_j|_{B^{2m(\kappa_j-(1-\mu+1/p))}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}^N)},$$

and for the second summand

$$|S\widetilde{\mathcal{T}}_j(\lambda)\lambda^{1-\frac{m_j}{2m}}\mathcal{E}_\lambda S^{-1}g_j|_{B^{1/p+\sigma_2}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-(1-\sigma_2/2m)}\lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}.$$

(IV) For (2.5.22) we estimate

$$|\mathrm{tr}_{\mathbb{R}^n_+}D^\beta\mathcal{N}^{0,x_*}_\mu(\lambda)g|_{B^{2m(\kappa_j-(1-\mu+1/p)-\sigma_3)}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}^N)} \lesssim |\mathcal{N}^{0,x_*}_\mu(\lambda)g|_{B^{2m(\mu-1/p-\sigma_3-\frac{m_j-|\beta|}{2m})}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)},$$

and thus (2.5.22) follows from (2.5.20). In (2.5.23), the trace operator meets the L_p-norm. For this we use that $\mathrm{tr}_{\mathbb{R}^n_+}$ is continuous

$$B^{1/p}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)\to L_p(\mathbb{R}^{n-1},\mathbb{C}^N),$$

see [82, Theorem 2.9.3]. Then

$$|\mathrm{tr}_{\mathbb{R}^n_+}D^\beta\mathcal{N}^{0,x_*}_\mu(\lambda)g|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} \lesssim |\mathcal{N}^{0,x_*}_\mu(\lambda)g|_{B^{1/p+|\beta|}_{p,1}(\mathbb{R}^n_+,\mathbb{C}^N)},$$

and (2.5.23) follows from (2.5.21). ■

We next prove the convergence of the Neumann series in (2.5.11).

Lemma 2.5.5. *In the setting of the proof of Proposition 2.5.1, for each i the series*

$$\mathcal{N}^i_\mu(\lambda) = \sum_{k=0}^\infty \mathcal{N}^{0,i}_\mu(\lambda)\big(\mathcal{B}^{i,sm}\mathcal{N}^{0,i}_\mu(\lambda)\big)^k$$

exists in $\mathcal{B}\big(Y_\mu(\mathbb{R}^{n-1}), B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)\big)$, provided ε is sufficiently small and λ is sufficiently large. For $\sigma_1\in[0,\mu-1/p]$ it holds

$$|\mathcal{N}^i_\mu(\lambda)g|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \lambda^{-\sigma_1}\Big(|g|_{Y_\mu(\mathbb{R}^{n-1})} + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{j=1}^m\lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\Big). \tag{2.5.26}$$

Moreover, for $\sigma_2 \in [0, 2m-1/p)$ *we have*

$$|\mathcal{N}_\mu^i(\lambda)g|_{B_{p,1}^{1/p+\sigma_2}(\mathbb{R}^n_+,\mathbb{C}^N)} \leq C\,\lambda^{-(1-\sigma_2/2m)}\big(\lambda^{1-\mu+1/p+1/2mp}|g|_{Y_\mu(\mathbb{R}^{n-1})} + \sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big). \tag{2.5.27}$$

Proof. (I) It follows from (2.5.20) that for $k \in \mathbb{N}_0$ it holds

$$\begin{aligned}&|\mathcal{N}_\mu^{0,i}(\lambda)(\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g|_{B_{p,p}^{2m(\mu-1/p-\sigma_1)}(\mathbb{R}^n_+,\mathbb{C}^N)} \\ &\quad\lesssim \lambda^{-\sigma_1}\max_{j=1,\dots,m}\big(|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g)_j|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad + \lambda^{-(1-\mu+1/p)-1/2mp}\lambda^{1-\frac{m_j}{2m}}|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g)_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big),\end{aligned} \tag{2.5.28}$$

and further, due to (2.5.21),

$$\begin{aligned}&|\mathcal{N}_\mu^{0,i}(\lambda)(\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g|_{B_{p,1}^{1/p+\sigma_2}(\mathbb{R}^n_+,\mathbb{C}^N)} \\ &\quad\lesssim \lambda^{-(1-\sigma_2/2m)}\max_{j=1,\dots,m}\big(\lambda^{1-\mu+1/p+1/2mp}|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g)_j|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad + \lambda^{1-\frac{m_j}{2m}}|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g)_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big).\end{aligned} \tag{2.5.29}$$

(II) For each j we consider the summands in (2.5.28). For the first summand we have

$$\begin{aligned}&|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda)g)_j|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad\lesssim \max_{|\beta|\leq m_j}|b_{j\beta}^{i,sm}\mathrm{tr}_{\mathbb{R}^n_+}D^\beta\mathcal{N}_\mu^{0,i}(\lambda)g|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)}.\end{aligned} \tag{2.5.30}$$

Applying Lemma 1.3.19 and using (2.5.3) for each β we obtain for small $\delta > 0$ that

$$\begin{aligned}&|b_{j\beta}^{i,sm}\mathrm{tr}_{\mathbb{R}^n_+}D^\beta\mathcal{N}_\mu^{0,i}(\lambda)g|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad\leq C\,|b_{j\beta}^{i,sm}|_{L_\infty(\mathbb{R}^{n-1},\mathbb{C}^N)}|\mathrm{tr}_{\mathbb{R}^n_+}D^\beta\mathcal{N}_\mu^{0,i}(\lambda)g|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad\quad + C_\varepsilon|\mathrm{tr}_{\mathbb{R}^n_+}D^\beta\mathcal{N}_\mu^{0,i}(\lambda)g|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p)-\delta)}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad =: I_1 + I_2.\end{aligned}$$

The estimate (2.5.22) yields

$$\begin{aligned}I_1 &\leq (C\varepsilon + C_\varepsilon\lambda^{-1/2m})\max_{l=1,\dots,m}\big(|g_l|_{B_{p,p}^{2m(\kappa_l-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad + \lambda^{-(1-\mu+1/p)-1/2mp}\lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big),\end{aligned}$$

where C is independent of ε and C_ε is independent of λ. In the same way we obtain

$$\begin{aligned}I_2 &\leq C_\varepsilon\,\lambda^{-\delta}\max_{l=1,\dots,m}\big(|g_l|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\ &\qquad + \lambda^{-(1-\mu+1/p)-1/2mp}\lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big).\end{aligned}$$

Combining these estimates with (2.5.30) leads to

$$
\begin{aligned}
&|(\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda)g)_j|_{B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\
&\quad \le (C\varepsilon_i + C_{\varepsilon_i}\lambda^{-\delta}) \max_{l=1,\dots,m} \Big(|g_l|_{W_p^{2m(\kappa_l-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\
&\qquad\qquad + \lambda^{-(1-\mu+1/p)}\lambda^{-1/2mp}\lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\Big).
\end{aligned}
$$

Note that for $g = (\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^{k-1}\widetilde{g}$ the right-hand side above is of the same type as the right-hand side in (2.5.28) with $k-1$ instead of k and the additional factor $(C\varepsilon + C_\varepsilon\lambda^{-\delta})$. For the second summand in (2.5.28) we have, using (2.5.23),

$$
\begin{aligned}
&|(\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda)g)_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} \le C\,|b_{j\beta}^{i,sm}|_{L_\infty(\mathbb{R}^{n-1},\mathbb{C}^N)}\ \lambda^{-\left(1-\frac{|\beta|}{2m}\right)} \\
&\quad \max_{l=1,\dots,m}\Big(\lambda^{1-\mu+1/p+1/2mp}|g_l|_{B_{p,p}^{2m(\kappa_l-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} + \lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\Big),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\lambda^{-(1-\mu+1/p)}\lambda^{-1/2mp}\lambda^{1-\frac{m_j}{2m}}|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g)_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} \\
&\quad \le (C\varepsilon + C_\varepsilon\lambda^{-1/2m}) \max_{l=1,\dots,m}\Big(|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^{k-1}g)_l|_{B_{p,p}^{2m(\kappa_l-(1-\mu+1/p))}(\mathbb{R}^{n-1},\mathbb{C}^N)} \\
&\qquad + \lambda^{-(1-\mu+1/p)-1/2mp}\lambda^{1-\frac{m_l}{2m}}|((\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda)g)^{k-1})_l|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\Big).
\end{aligned}
$$

Again, the right-hand side is of the same type as the right-hand side in (2.5.28), with k replaced by $k-1$ and the additional factor $C\varepsilon + C_\varepsilon\lambda^{-1/2m}$.
(III) Iterating the above estimates yields for $k \in \mathbb{N}_0$ that

$$
\begin{aligned}
&|\mathcal{N}_\mu^{0,i}(\lambda)(\mathcal{B}^{i,sm}\mathcal{N}_\mu^{0,i}(\lambda))^k g|_{B_{p,p}^{2m(\mu-1/p-\sigma_1)}(\mathbb{R}_+^n,\mathbb{C}^N)} \lesssim (C\varepsilon + C_\varepsilon\lambda^{-\tau})^k \\
&\qquad \lambda^{-\sigma_1}\Big(|g|_{Y_\mu(\mathbb{R}^{n-1})} + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\Big),
\end{aligned}
$$

with some $\tau > 0$. This implies that $\mathcal{N}_\mu^i(\lambda)$ exists in $\mathcal{B}(Y_\mu(\mathbb{R}^{n-1}), B_{p,p}^{2m(\mu-1/p-\sigma_1)}(\mathbb{R}_+^n,\mathbb{C}^N))$ and admits the asserted estimate (2.5.26), provided we first choose ε sufficiently small and then λ sufficiently large.
(IV) Starting in Step II with (2.5.29) instead of (2.5.28) one obtains (2.5.27) in a similar fashion, using the estimates from Lemma 2.5.4. ■

The next lemma shows the unique solvability equation (2.5.13).

Lemma 2.5.6. *In the setting of Proposition 2.5.1, consider the operators*

$$
\begin{aligned}
\mathcal{K}_1(\lambda)g &= \sum\nolimits_i \phi_i\Phi_i^{-1}(\mathcal{N}_\mu^i(\lambda)\Phi_i\psi_i g)|_{\mathbb{R}_+^n\cap B_{r_i}(0)}, \qquad g \in Y_\mu(\Gamma), \\
\mathcal{K}_2(\lambda)u &= \sum\nolimits_i \phi_i\Phi_i^{-1}(\mathcal{N}_\mu^i(\lambda)\Phi_i[\mathcal{B},\psi_i]u)|_{\mathbb{R}_+^n\cap B_{r_i}(0)}, \qquad u \in X_\mu(\Omega).
\end{aligned}
$$

For each $g \in Y_\mu(\Gamma)$ the equation $\big(\mathrm{id} - \mathcal{K}_2(\lambda)\big)u = \mathcal{K}_1(\lambda)g$ has a unique solution $u := \mathcal{Q}(\lambda)g \in X_\mu(\Omega)$, provided λ is sufficiently large. For $\sigma_1 \in [0, \mu-1/p]$ the solution operator *satisfies*

$$
\begin{aligned}
&|\mathcal{Q}(\lambda)g|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\Omega,\mathbb{C}^N)} \\
&\qquad \lesssim \lambda^{-\sigma_1}\big(|g|_{Y_\mu(\Gamma)} + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\Gamma,\mathbb{C}^N)}\big),
\end{aligned} \tag{2.5.31}
$$

and for $\sigma_2 \in [0, 2m-1/p)$ *we further have*

$$
\begin{aligned}
&|\mathcal{Q}(\lambda)g|_{B^{1/p+\sigma_2}_{p,1}(\Omega,\mathbb{C}^N)} \\
&\qquad \lesssim \lambda^{-(1-\sigma_2/2m)}\big(\lambda^{1-\mu+1/p+1/2mp}|g|_{Y_\mu(\Gamma)} + \sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\Gamma,\mathbb{C}^N)}\big).
\end{aligned} \tag{2.5.32}
$$

Proof. (I) We concentrate on (2.5.31), similar arguments lead to (2.5.32). We intend to show the absolute convergence of the Neumann series

$$
\mathcal{Q}(\lambda) := \sum_{k=0}^\infty \mathcal{K}_2(\lambda)^k\,\mathcal{K}_1(\lambda)
$$

in $\mathcal{B}\big(Y_\mu(\Gamma), B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\Omega,\mathbb{C}^N)\big)$. It follows from (2.5.26) that

$$
\begin{aligned}
|\mathcal{K}_2(\lambda)u|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\Omega,\mathbb{C}^N)} &\le \max_i |\mathcal{N}_i\Phi_i[\mathcal{B},\psi_i]u|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)} \\
&\lesssim \lambda^{-\sigma_1}\max_i\Big(|\Phi_i[\mathcal{B},\psi_i]u|_{Y_\mu(\mathbb{R}^{n-1})} \\
&\qquad + \lambda^{-(1-\mu-1/p)-1/2mp}\sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|\Phi_i[\mathcal{B}_j,\psi_i]u|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\Big).
\end{aligned}
$$

If $m_j = 0$ for some j then $[\mathcal{B}_j,\psi_i] = 0$, thus we assume that $m_j \ge 1$ for each $j = 1,\dots,m$ in the sequel. As $[\mathcal{B}_j,\psi_i]$ is of order at most $m_j - 1$ we have for each i that

$$
|\Phi_i[\mathcal{B},\psi_i]u|_{Y_\mu(\mathbb{R}^{n-1})} \lesssim |[\mathcal{B},\psi_i]u|_{Y_\mu(\Gamma\cap U_i)} \lesssim |u|_{B^{2m(\mu-1/p)-1}_{p,p}(\Omega,\mathbb{C}^N)},
$$

and further, for each $j = 1,\dots,m$,

$$
|\Phi_i[\mathcal{B}_j,\psi_i]u|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)} \lesssim |u|_{B^{1/p+m_j-1}_{p,1}(\Omega,\mathbb{C}^N)}.
$$

This yields

$$
\begin{aligned}
|\mathcal{K}_2(\lambda)u|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\Omega,\mathbb{C}^N)} &\lesssim \lambda^{-\sigma_1}\big(|u|_{B^{2m(\mu-1/p)-1}_{p,p}(\Omega,\mathbb{C}^N)} \\
&\qquad + \lambda^{-(1-\mu-1/p)-1/2mp}\sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|u|_{B^{1/p+m_j-1}_{p,1}(\Omega,\mathbb{C}^N)}\big).
\end{aligned} \tag{2.5.33}
$$

Moreover, (2.5.27) implies

$$
\begin{aligned}
|\mathcal{K}_2(\lambda)u|_{B^{1/p+m_j-1}_{p,1}(\Omega,\mathbb{C}^N)} &\lesssim \lambda^{-1/2m}\lambda^{-(1-\frac{m_j}{2m})}\big(\lambda^{1-\mu+1/p+1/2mp}|u|_{B^{2m(\mu-1/p)-1}_{p,p}(\Omega,\mathbb{C}^N)} \\
&\qquad + \sum_{l=1}^m \lambda^{1-\frac{m_l}{2m}}|u|_{B^{1/p+m_l-1}_{p,1}(\Omega,\mathbb{C}^N)}\big).
\end{aligned}
$$

Iterating the above estimates we obtain for $k \in \mathbb{N}$

$$|\mathcal{K}_2(\lambda)^k u|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\Omega,\mathbb{C}^N)} \lesssim (C\lambda)^{-\frac{k-1}{2m}} \lambda^{-\sigma_1} \Big(|u|_{B^{2m(\mu-1/p)-1}_{p,p}(\Omega,\mathbb{C}^N)} + \lambda^{-(1-\mu-1/p)-1/2mp} \sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}} |u|_{B^{1/p+m_j-1}_{p,1}(\Omega,\mathbb{C}^N)}\Big), \tag{2.5.34}$$

where the constant C is independent of λ.

(II) We now estimate $\mathcal{K}_1(\lambda)$. From (2.5.26) we infer

$$|\mathcal{K}_1(\lambda)g|_{B^{2m(\mu-1/p)-1}_{p,p}(\Omega,\mathbb{C}^N)} \lesssim \max_i |\mathcal{N}_i \Phi_i \psi_i g|_{B^{2m(\mu-1/p)-1}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)}$$
$$\lesssim \lambda^{-1/2m} \Big(|g|_{Y_\mu(\Gamma)} + \lambda^{-(1-\mu+1/p)-1/2mp} \sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}} |g_j|_{L_p(\Gamma,\mathbb{C}^N)}\Big),$$

and further, from (2.5.27),

$$|\mathcal{K}_1(\lambda)g|_{B^{1/p+m_j-1}_{p,1}(\Omega,\mathbb{C}^N)} \lesssim \lambda^{-1/2m} \lambda^{-\left(1-\frac{m_j}{2m}\right)} \Big(\lambda^{1-\mu+1/p+1/2mp} |g|_{Y_\mu(\Gamma)} + \sum_{l=1}^m \lambda^{1-\frac{m_l}{2m}} |g_l|_{L_p(\Gamma,\mathbb{C}^N)}\Big)$$

for $j = 1, ..., m$. Using these estimates for $u = \mathcal{K}_1(\lambda)g$ in (2.5.34) we obtain for $k \in \mathbb{N}_0$ that

$$|\mathcal{K}_2(\lambda)^k \mathcal{K}_1(\lambda)g|_{B^{2m(\mu-1/p-\sigma_1)}_{p,p}(\Omega,\mathbb{C}^N)} \leq (C\lambda)^{-k/2m} \lambda^{-\sigma_1} \Big(|g|_{Y_\mu(\Gamma)} + \lambda^{-(1-\mu+1/p)-1/2mp} \sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}} |g_j|_{L_p(\Gamma,\mathbb{C}^N)}\Big).$$

This yields the absolute convergence of the Neumann series and the estimate for $\mathcal{Q}(\lambda)$ as asserted, provided λ is sufficiently large. ■

Last but not least we consider the equation (2.5.14).

Lemma 2.5.7. *In the setting of Proposition 2.5.1, consider the operator*

$$\mathcal{K}_3(\lambda)h = \sum\nolimits_i [\phi_i, \mathcal{B}] \Phi_i^{-1} \big(\mathcal{N}^i_\mu(\lambda) \Phi_i [\psi_i h + [\mathcal{B}, \psi_i] \mathcal{Q}(\lambda) h]\big)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)}, \qquad h \in Y_\mu(\Gamma).$$

If λ is sufficiently large then for each $g \in Y_\mu(\Gamma)$ there is a unique solution $h = \mathcal{R}(\lambda)g \in Y_\mu(\Gamma)$ of $(\mathrm{id} - \mathcal{K}_3)h = g$. The solution operator $\mathcal{R}(\lambda)$ is continuous on $Y_\mu(\Gamma)$.

Proof. We show the absolute convergence of the Neumann series $\sum_{k=0}^\infty \mathcal{K}_3(\lambda)^k$ in $\mathcal{B}(Y_\mu(\Gamma))$. We assume that $m_j \geq 1$ for $j = 1, ..., m$ in the sequel, otherwise the corresponding commutators vanish. Using (2.5.26) and that the commutators $[\phi_i, \mathcal{B}_j]$ are of lower order we obtain

$$\begin{aligned}
|\mathcal{K}_3(\lambda)g|_{Y_\mu(\Gamma)} &\lesssim \max_i |\mathcal{N}^i_\mu(\lambda)\Phi_i[\psi_i g + [\mathcal{B},\psi_i]\mathcal{Q}(\lambda)g]|_{B^{2m(\mu-1/p)-1}_{p,p}(\mathbb{R}^n_+,\mathbb{C}^N)} \\
&\lesssim \lambda^{-1/2m}\max_i \big(|\Phi_i[\psi_i g + [\mathcal{B},\psi_i]\mathcal{Q}(\lambda)g]|_{Y_\mu(\mathbb{R}^{n-1})} \\
&\qquad + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|\Phi_i[\psi_i g_j + [\mathcal{B}_j,\psi_i]\mathcal{Q}(\lambda)g]|_{L_p(\mathbb{R}^{n-1},\mathbb{C}^N)}\big) \\
&\lesssim \lambda^{-1/2m}\max_i \big(|g|_{Y_\mu(\Gamma)} + |[\mathcal{B},\psi_i]\mathcal{Q}(\lambda)g|_{Y_\mu(\Gamma)} \\
&\quad + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}(|g_j|_{L_p(\Gamma,\mathbb{C}^N)} + |[\mathcal{B}_j,\psi_i]\mathcal{Q}(\lambda)g|_{L_p(\Gamma,\mathbb{C}^N)})\big).
\end{aligned}$$

We further infer from (2.5.31) that

$$|[\mathcal{B},\psi_i]\mathcal{Q}g|_{Y_\mu(\Gamma)} \lesssim \lambda^{-1/2m}\big(|g|_{Y_\mu(\Gamma)} + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}|g_j|_{L_p(\Gamma,\mathbb{C}^N)}\big),$$

and from (2.5.32) that

$$\begin{aligned}
|[\mathcal{B}_j,\psi_i]\mathcal{Q}(\lambda)g|_{L_p(\Gamma,\mathbb{C}^N)} &\lesssim |\mathcal{Q}(\lambda)g|_{B^{1/p+m_j-1}_{p,1}(\Omega,\mathbb{C}^N)} \\
&\lesssim \lambda^{-1/2m}\lambda^{-(1-\frac{m_j}{2m})}\big(\lambda^{1-\mu+1/p+1/2mp}|g|_{Y_\mu(\Gamma)} + \sum_{l=1}^m \lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\Gamma,\mathbb{C}^N)}\big),
\end{aligned}$$

which yields

$$|\mathcal{K}_3(\lambda)g|_{Y_\mu(\Gamma)} \lesssim \lambda^{-1/2m}\big(|g|_{Y_\mu(\Gamma)} + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{l=1}^m \lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\Gamma,\mathbb{C}^N)}\big).$$

Using (2.5.27), we also obtain for $l = 1, ..., m$ that

$$\begin{aligned}
&|(\mathcal{K}_3(\lambda)g)_l|_{L_p(\Gamma,\mathbb{C}^N)} \\
&\quad\lesssim \lambda^{-1/2m}\lambda^{-\left(1-\frac{m_l}{2m}\right)}\Big(\lambda^{1-\mu+1/p+1/2mp}\big(|g|_{Y_\mu(\Gamma)} + |[\mathcal{B},\psi_i]\mathcal{Q}(\lambda)g|_{Y_\mu(\Gamma)}\big) \\
&\qquad\qquad + \sum_{j=1}^m \lambda^{1-\frac{m_j}{2m}}\big(|g_j|_{L_p(\Gamma,\mathbb{C}^N)} + |[\mathcal{B}_j,\psi_i]\mathcal{Q}(\lambda)g|_{L_p(\Gamma,\mathbb{C}^N)}\big)\Big).
\end{aligned}$$

Hence we have for $k \in \mathbb{N}_0$, with a constant C that is independent of λ,

$$|\mathcal{K}_3(\lambda)^k g|_{Y_\mu(\Gamma)} \leq (C\lambda)^{-k/2m}\big(|g|_{Y_\mu(\Gamma)} + \lambda^{-(1-\mu+1/p)-1/2mp}\sum_{l=1}^m \lambda^{1-\frac{m_l}{2m}}|g_l|_{L_p(\Gamma,\mathbb{C}^N)}\big).$$

This yields the convergence of $\sum_{k=0}^\infty \mathcal{K}_3(\lambda)^k$ in $\mathcal{B}(Y_\mu(\Gamma))$ and the continuity of the solution operator $\mathcal{R}(\lambda)$, provided λ is sufficiently large. ■

Chapter 3

Maximal $L_{p,\mu}$-Regularity for Boundary Conditions of Relaxation Type

In this chapter we show maximal $L_{p,\mu}$-regularity for vector-valued parabolic initial-boundary value problems of relaxation type, generalizing the results of by Denk, Prüss, & Zacher [26]. The approach is analogous to that in Chapter 2 for the case of static boundary conditions. Thus sometimes we are brief, but also repeat some arguments from the last chapter for transparency. We first describe the approach and the involved anisotropic function spaces in detail, and then prove the main result, Theorem 3.1.4, by solving the half-space problem and performing a perturbation and localization procedure. For the geometry of the boundary of a domain and differential operators defined on them we refer to the Appendecies A.1 and A.5.

3.1 The Problem and the Approach in Weighted Spaces

The Problem

For the unknown vector-valued functions

$$u = u(t,x) \in E, \qquad \rho = \rho(t,x) \in F,$$

we consider linear inhomogeneous, non-autonomous, parabolic initial-boundary value problems of relaxation type, i.e.,

$$\begin{aligned}
\partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, \quad t \in J, \\
\partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x), && x \in \Gamma, \quad t \in J, \\
\mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x), && x \in \Gamma, \quad t \in J, \qquad j = 1,...,m, \\
u(0,x) &= u_0(x), && x \in \Omega, \\
\rho(0,x) &= \rho_0(x), && x \in \Gamma.
\end{aligned} \tag{3.1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is assumed to be a domain with compact smooth boundary $\Gamma = \partial\Omega$, $J = (0,T)$ is a finite interval, $T > 0$, and E, F are Banach spaces of class $\mathcal{HT}$. The unknown u lives on $J \times \overline{\Omega}$, while the unknown ρ lives on $J \times \Gamma$, i.e., it is only present on the boundary Γ. It is assumed that the dynamic equation for u and the static boundary equations take place in E, and that the dynamic equation for ρ takes place in F. Consequently, the right-hand sides f, g_1, ..., g_m, and the initial value u_0 take values in E, while g_0 and ρ_0 take values in F.

Formally one obtains the problem (2.1.1) with static boundary conditions by setting $\rho \equiv 0$ and dropping the second dynamic equation.

The differential operator $\mathcal{A}$ is given by

$$\mathcal{A}(t,x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(t,x)D^\alpha, \qquad x \in \Omega, \qquad t \in J,$$

where $m \in \mathbb{N}$ and $D = -\mathrm{i}\nabla$, with the euclidian gradient $\nabla = (\partial_{x_1}, ..., \partial_{x_n})$ on $\mathbb{R}^n$, and coefficients $a_\alpha(t,x) \in \mathcal{B}(E)$. Hence the order of $\mathcal{A}$ is $2m$. The boundary operators $\mathcal{B}_j$ are of the form

$$\mathcal{B}_j(t,x,D) = \sum_{|\beta|\leq m_j} b_{j\beta}(t,x)\mathrm{tr}_\Omega D^\beta, \qquad x \in \Gamma, \qquad t \in J, \qquad j = 0, ..., m,$$

where $m_j \in \{0, ..., 2m-1\}$ is the order of $\mathcal{B}_j$, and the coefficients satisfy

$$b_{0\beta}(t,x) \in \mathcal{B}(E,F), \qquad b_{j\beta}(t,x) \in \mathcal{B}(E), \qquad j = 1, ..., m.$$

Observe that $\mathcal{B} = (\mathcal{B}_0, ..., \mathcal{B}_m)$ only acts on u, in a way that first the euclidian derivatives, and then the spatial trace tr_Ω is applied.

The operators $\mathcal{C} = (\mathcal{C}_0, ..., \mathcal{C}_m)$ only act on ρ, in the following way. For (almost every) $t \in J$ it is assumed that $\mathcal{C}_j(t, \cdot, D_\Gamma)$ is a linear map

$$C^\infty(\Gamma; F) \to L_1(\Gamma; F),$$

such that for all $j = 0, ..., m$, all local coordinates g for Γ and all $\rho \in C^\infty(\Gamma; F)$ it holds

$$\big(\mathcal{C}_j(t,\cdot,D_\Gamma)\rho\big) \circ \mathrm{g}(x) = \sum_{|\gamma|\leq k_j} c^{\mathrm{g}}_{j\gamma}(t,x)D^\gamma_{n-1}(\rho \circ \mathrm{g})(x), \qquad x \in \mathrm{g}^{-1}(\Gamma \cap U), \qquad t \in J,$$

where $U \subset \mathbb{R}^n$ is the domain of the chart corresponding to g. Here we have $D_{n-1} = -\mathrm{i}\nabla_{n-1}$, with the euclidian gradient $\nabla_{n-1} = (\partial_{x_1}, ..., \partial_{x_{n-1}})$ on $\mathbb{R}^{n-1}$, and $k_j \in \mathbb{N}_0$ is the order of $\mathcal{C}_j$. The local coefficients $c^{\mathrm{g}}_{j\gamma}$, that may depend on the coordinates g, are assumed to satisfy

$$c^{\mathrm{g}}_{0\gamma}(t,x) \in \mathcal{B}(F), \qquad c^{\mathrm{g}}_{j\gamma}(t,x) \in \mathcal{B}(F,E), \quad j = 1, ..., m.$$

We do not assume that an operator $\mathcal{C}_j$ has global coefficients, in the sense that there are functions $c_{j\gamma}$ on Γ satisfying $c^{\mathrm{g}}_{j\gamma} = c_{j\gamma} \circ \mathrm{g}$ in all coordinates g. In contrast to that, the coefficients of $\mathcal{B}$ are globally defined on Γ. We write $\mathcal{C}_j(D_\Gamma)$ with $D_\Gamma = -\mathrm{i}\nabla_\Gamma$, since for $\mathcal{C}_j$

we think of an operator in terms of the surface gradient ∇_Γ. We refer to Appendix A.5 for more informations on the surface gradient and general differential operators on a boundary acting on vector-valued functions.
Finally, it is assumed that each of the operators $\mathcal{B}_j$ and at least one operator $\mathcal{C}_j$ are nontrivial. If an operator $\mathcal{C}_j$ is trivial, i.e., $\mathcal{C}_j \equiv 0$, then we set $k_j := -\infty$ for its order.

We consider three problems that fit into the above framework. Further examples are listed in [26, Section 3].

Example 3.1.1. A linearized reaction-diffusion system with surface diffusion,

$$\begin{aligned} \partial_t u - \Delta u &= f(t,x), && x \in \Omega, && t > 0, \\ \partial_t u + \partial_\nu u - \Delta_\Gamma u &= g(t,x), && x \in \Gamma, && t > 0, \\ u(0,x) &= u_0(x), && x \in \Omega, \end{aligned} \tag{3.1.2}$$

where $\partial_\nu = \nu(x) \cdot \mathrm{tr}_\Omega \nabla_\Omega$ denotes the outer normal derivative and $-\Delta_\Gamma$ is the Laplace-Beltrami operator on Γ. The latter is in local coordinates g given by

$$(\Delta_\Gamma \rho) \circ \mathrm{g} = \frac{1}{\sqrt{|\mathrm{G}|}} \sum_{k,\,l=1}^{n-1} \partial_{x_k} \big(\sqrt{|\mathrm{G}|} \mathrm{g}^{kl} \partial_{x_l} (\rho \circ \mathrm{g}) \big),$$

where G is the first fundamental form corresponding to g and g^{kl} are the components of G^{-1}, cf. Appendix A.1. The problem (3.1.2) can be cast in the form (3.1.1) as follows,

$$\begin{aligned} \partial_t u - \Delta u &= f(t,x), && x \in \Omega, && t > 0, \\ \partial_t \rho + \partial_\nu u - \Delta_\Gamma \rho &= g(t,x), && x \in \Gamma, && t > 0, \\ \mathrm{tr}_\Omega u - \rho &= 0, && x \in \Gamma, && t > 0, \\ u(0,x) &= u_0(x), && x \in \Omega, \\ \rho(0,x) &= \mathrm{tr}_\Omega u_0(x), && x \in \Gamma. \end{aligned}$$

Hence the unknown ρ is simply the trace of u on Γ. The operator $\mathcal{A}(D) = -\Delta$ is of order 2, thus $m = 1$. We further have $\mathcal{B}_0(x,D) = \partial_\nu$, $\mathcal{C}_0(x, D_\Gamma) = -\Delta_\Gamma$, $\mathcal{B}_1 = \mathrm{tr}_\Omega$, $\mathcal{C}_1 = -\mathrm{id}$, such that $m_0 = 1$, $k_0 = 2$, and $m_1 = k_1 = 0$.

Neglecting the Laplace-Beltrami operator, we obtain

$$\begin{aligned} \partial_t u - \Delta u &= f(t,x), && x \in \Omega, && t > 0, \\ \partial_t u + \partial_\nu u &= g(t,x), && x \in \Gamma, && t > 0, \\ u(0,x) &= u_0(x), && x \in \Omega. \end{aligned} \tag{3.1.3}$$

As above, this problem can be cast in the form (3.1.1) by taking $\mathrm{tr}_\Omega u = \rho$ as static boundary condition. The only difference to (3.1.2) is that $\mathcal{C}_0 \equiv 0$, hence $k_0 = -\infty$.

Transforming the Stefan problem with surface tension to a fixed domain, the linearization of the resulting problem is of the form

$$\begin{aligned} \partial_t u - \Delta u &= f(t,x), && x \in \Omega, && t > 0, \\ \partial_t \rho + \partial_\nu u &= g_0(t,x), && x \in \Gamma, && t > 0, \\ u + \Delta_\Gamma \rho &= g_1(t,x), && x \in \Gamma, && t > 0, \\ u(0,x) &= u_0(x), && x \in \Omega, && \\ \rho(0,x) &= \rho_0(x), && x \in \Gamma. && \end{aligned} \tag{3.1.4}$$

Here the graph of $\rho(t,\cdot)$ over Γ describes the free boundary at time t. The maximal L_p-regularity for (3.1.4) is the basic tool in [37] to show analyticity of the free boundary. This problem structurally differs from (3.1.2) and (3.1.3), since ρ is not simply the trace of u, and the static coupling of these unknowns is nontrivial. It holds $\mathcal{B}_1 = \mathrm{tr}_\Omega$, $m_1 = 0$, $\mathcal{C}_1(x, D_\Gamma) = \Delta_\Gamma$, $k_1 = 2$, and further $m_0 = 1$ and $k_0 = -\infty$. ■

The Approach in the $L_{p,\mu}$-Spaces

The maximal $L_{p,\mu}$-regularity approach for (3.1.1) is as follows. Let

$$p \in (1,\infty), \qquad \mu \in (1/p, 1].$$

We look for solutions (u,ρ) so that the first component u satisfies

$$u \in \mathbb{E}_{u,\mu} := W^1_{p,\mu}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W^{2m}_p(\Omega; E)).$$

As in the static case, the results of Section 1.3 show that this regularity assumption on u necessarily implies

$$f \in \mathbb{E}_{0,\mu} := L_{p,\mu}(J; L_p(\Omega; E)), \qquad u_0 \in X_{u,\mu} := B^{2m(\mu-1/p)}_{p,p}(\Omega; E),$$

and further that

$$g_0 \in \mathbb{F}_{0,\mu} := W^{\kappa_0}_{p,\mu}(J; L_p(\Gamma; F)) \cap L_{p,\mu}(J; W^{2m\kappa_0}_p(\Gamma; F)),$$

and

$$g_j \in \mathbb{F}_{j,\mu} := W^{\kappa_j}_{p,\mu}(J; L_p(\Gamma; E)) \cap L_{p,\mu}(J; W^{2m\kappa_j}_p(\Gamma; E)), \qquad j = 1, ..., m,$$

where we set

$$\kappa_j := 1 - \frac{m_j}{2m} - \frac{1}{2mp}, \qquad j = 0, ..., m.$$

For convenience we write

$$\mathbb{F}_\mu := \mathbb{F}_{0,\mu} \times ... \times \mathbb{F}_{m,\mu}, \qquad g = (g_0, ..., g_m) \in \mathbb{F}_\mu.$$

Therefore, in (3.1.1) the dynamic equation for u takes place in $\mathbb{E}_{0,\mu}$, the dynamic equation for ρ in $\mathbb{F}_{0,\mu}$, and the static boundary conditions in $\mathbb{F}_{j,\mu}$, $j = 1, ..., m$, respectively.

Looking for optimal regularity, the space $\mathbb{E}_{\rho,\mu}$ for ρ should now be such that, assuming smoothness of the coefficients, all the summands in the terms in (3.1.1) where ρ is involved belong to the space where the respective equation takes place. It can be seen as in [26, Section 2] that

$$\mathbb{E}_{\rho,\mu} = W^{1+\kappa_0}_{p,\mu}\big(J; L_p(\Gamma;F)\big) \cap L_{p,\mu}\big(J; W^{l+2m\kappa_0}_p(\Gamma;F)\big)$$
$$\cap\, W^1_{p,\mu}\big(J; W^{2m\kappa_0}_p(\Gamma;F)\big) \cap \bigcap_{j\in\widetilde{\mathcal{J}}} W^{\kappa_j}_{p,\mu}\big(J; W^{k_j}_p(\Gamma;F)\big)$$

satisfies these requirements. Here we have used the abbreviations

$$\widetilde{\mathcal{J}} := \big\{ j \in \{0,...,m\} \;:\; k_j \neq -\infty \big\}, \qquad l_j := k_j - m_j + m_0, \qquad l := \max_{j=0,...m} l_j.$$

Observe that $\widetilde{\mathcal{J}}$ just collects the indices j for which an operator $\mathcal{C}_j$ is nontrivial, and that with the above notations it holds

$$k_j + 2m\kappa_j = l_j + 2m\kappa_0 \leq l + 2m\kappa_0.$$

Proposition 1.3.2 shows that there is redundancy in the above definition of $\mathbb{E}_{\rho,\mu}$, depending on the relation of l and $2m$. There are three possible qualitative shapes of the Newton polygon associated to $\mathbb{E}_{\rho,\mu}$ (see Section 1.3.1). The points $(0, 1+\kappa_0)$ and $(l + 2m\kappa_0, 0)$ are always vertices of the Newton polygon. The line through the points $(0, 1+\kappa_0)$ and $(2m\kappa_0, 1)$ intersects $(2m + 2m\kappa_0, 0)$, so that $W^1_{p,\mu}(J; W^{2m\kappa_0}_p(\Gamma;F))$ is redundant for $l \geq 2m$, and the points (k_j, κ_j) determine the remaining vertices of the nontrivial part of the polygon. Moreover, the lines through the points (k_j, κ_j) and $(k_j + 2m\kappa_j, 0)$ are parallel for $j = 0, ..., m$. Thus for $l \leq 2m$ the spaces corresponding to the points (k_j, κ_j) are redundant, and $(2m\kappa_0, 1)$ is a vertex if $l < 2m$. Below we give the precise nonredundant description of $\mathbb{E}_{\rho,\mu}$.

In each case, Theorem 1.3.6 yields the temporal trace space of ρ at $t = 0$, which is denoted by

$$X_{\rho,\mu} := \mathrm{tr}_{t=0}\mathbb{E}_{\rho,\mu},$$

and, if it exists, of $\partial_t\rho$ at $t = 0$, which is denoted by

$$X_{\partial_t\rho,\mu} := \mathrm{tr}_{t=0}\partial_t\mathbb{E}_{\rho,\mu}, \qquad \kappa_0 > 1 - \mu + 1/p.$$

In the Newton polygon, these spaces can be obtained by intersecting the horizontal lines $(a, 1-\mu+1/p)$ and $\big(a, 1 + (1-\mu+1/p)\big)$, $a \in \mathbb{R}$, with its nontrivial part. More precisely, Theorem 1.3.6 is applied to the intersection of the spaces that determine the edges these horizontal lines intersect, respectively (cf. Figure 1.3.2).

The description of the spaces below follows the presentation in [26].

The nonredundant description of the spaces $\mathbb{E}_{\rho,\mu}$, $X_{\rho,\mu}$ and $X_{\partial_t\rho,\mu}$.

Case 1: $l = 2m$. One has

$$\mathbb{E}_{\rho,\mu} = W^{1+\kappa_0}_{p,\mu}(J; L_p(\Gamma;F)) \cap L_{p,\mu}(J; W^{2m(1+\kappa_0)}_p(\Gamma;F)),$$

and therefore

$$X_{\rho,\mu} = B^{2m(\kappa_0+\mu-1/p)}_{p,p}(\Gamma;F), \qquad X_{\partial_t\rho,\mu} = B^{2m(\kappa_0-(1-\mu+1/p))}_{p,p}(\Gamma;F) \quad \text{if} \ \ \kappa_0 > 1-\mu+1/p.$$

Case 2: $l < 2m$. One has

$$\mathbb{E}_{\rho,\mu} = \ W^{1+\kappa_0}_{p,\mu}(J;L_p(\Gamma;F)) \cap L_{p,\mu}(J;W^{l+2m\kappa_0}_p(\Gamma;F)) \cap W^1_{p,\mu}(J;W^{2m\kappa_0}_p(\Gamma;F)),$$

and this yields similar trace spaces as in Case 1,

$$X_{\rho,\mu} = B^{2m\kappa_0+l(\mu-1/p)}_{p,p}(\Gamma;F), \qquad X_{\partial_t\rho,\mu} = B^{2m(\kappa_0-(1-\mu+1/p))}_{p,p}(\Gamma;F) \quad \text{if} \ \ \kappa_0 > 1-\mu+1/p.$$

Case 3: $l > 2m$. This is the most complicated case. One has

$$\mathbb{E}_{\rho,\mu} = \ W^{1+\kappa_0}_{p,\mu}(J;L_p(\Gamma;F)) \cap L_{p,\mu}(J;W^{l+2m\kappa_0}_p(\Gamma;F)) \cap \bigcap_{j\in\mathcal{J}} W^{\kappa_j}_{p,\mu}(J;W^{k_j}_p(\Gamma;F)),$$

where $\mathcal{J} = \{j_1,...,j_{q_{\max}}\} \subset \widetilde{\mathcal{J}}$, $q_{\max} \in \mathbb{N}$, contains those indices $j \in \widetilde{\mathcal{J}}$ so that (k_j,κ_j) belongs to the nontrivial part of the Newton polygon, i.e., the points

$$P_0 = (0,1+\kappa_0), \qquad P_1 = (k_{j_1},\kappa_{j_1}), \qquad ..., \qquad P_{q_{\max}} = (k_{j_{q_{\max}}},\kappa_{j_{q_{\max}}}),$$

are the vertices of its nontrivial part. Note that it necessarily holds $l_{j_q} > 2m$ for $j_q \in \mathcal{J}$. It is assumed that $\mathcal{J}$ is arranged in a way such that

$$k_{j_{q_1}} < k_{j_{q_2}} \quad \text{and} \quad \kappa_{j_{q_1}} > \kappa_{j_{q_2}} \qquad \text{for } q_1 < q_2.$$

For later considerations we define

$$k_{-1} := 0, \qquad \kappa_{-1} := 1+\kappa_0, \qquad m_{-1} := m_0 - 2m, \qquad l_{-1} := 2m.$$

We further denote the edge in the Newton polygon connecting the points P_q and P_{q+1} by $\mathcal{NP}_q$, $q = 0,...,q_{\max}$, and define

$$\begin{aligned} \mathcal{J}_{2q} &:= \big\{ j \in \mathcal{J} \cup \{-1\} \ : \ (k_j,\kappa_j) = P_q \big\}, \qquad q = 0,...,q_{\max}, \\ \mathcal{J}_{2q+1} &:= \big\{ j \in \mathcal{J} \cup \{-1\} \ : \ (k_j,\kappa_j) \in \mathcal{NP}_q \big\}, \qquad q = 0,...,q_{\max}. \end{aligned}$$

The temporal trace space of $\partial_t\rho$ is obtained by Theorem 1.3.6 from the spaces corresponding to $P_0 = (0,1+\kappa_0)$ and $P_1 = (k_{j_1},\kappa_{j_1})$, i.e.,

$$X_{\partial_t\rho,\mu} = B^{k_{j_1}(\kappa_0-(1-\mu+1/p))/(1+\kappa_0-\kappa_{j_1})}_{p,p}(\Gamma;F) \quad \text{if} \ \ \kappa_0 > 1-\mu+1/p.$$

Note that Theorem 1.3.6 does not directly apply if $\kappa_{j_1} < 1-\mu+1/p$. In this case one first has to use Proposition 1.3.2 and then apply Theorem 1.3.6 to the spaces corresponding to the points $(0,1+\kappa_0)$ and $(k_{j_1}\kappa_0/(1+\kappa_0-\kappa_{j_1}),1)$, cf. Remark 1.3.7.
For $X_{\rho,\mu}$ one has to distinguish three more cases.

Case 3(i): If $\kappa_j > 1-\mu+1/p$ for all $j \in \mathcal{J}$, then

$$X_{\rho,\mu} = B^{l+2m(\kappa_0-(1-\mu+1/p))}_{p,p}(\Gamma;F).$$

Case 3(ii): Denote by $j_{q_1} \in \mathcal{J}$ be the smallest index with $\kappa_{j_{q_1}} > 1 - \mu + 1/p$, and by $j_{q_2} \in \mathcal{J}$ the largest index with $\kappa_{j_{q_2}} < 1 - \mu + 1/p$. Then

$$X_{\rho,\mu} = B_{p,p}^{k_{j_{q_1}} + (\kappa_{j_{q_1}} - (1-\mu+1/p)) \frac{k_{j_{q_2}} - k_{j_{q_1}}}{\kappa_{j_{q_2}} - \kappa_{j_{q_1}}}}(\Gamma; F).$$

Case 3(iii): If $\kappa_j < 1 - \mu + 1/p$ for all $j \in \mathcal{J}$, then

$$X_{\rho,\mu} = B_{p,p}^{k_{j_1}(\kappa_0+\mu-1/p)/(1+\kappa_0-\kappa_{j_1})}(\Gamma; F). \qquad \blacksquare$$

It can be seen from the Newton polygon that in each of the Cases 1, 2 and 3 it holds

$$X_{\rho,\mu} \hookrightarrow X_{\partial_t\rho,\mu}. \tag{3.1.5}$$

We now consider compatibility conditions at the boundary at $t = 0$, which are necessary for the solvability of (3.1.1). For the dynamic equation on the boundary, if $\kappa_0 > 1-\mu+1/p$ then by Theorem 1.3.6 it holds

$$\mathbb{F}_{0,\mu} \hookrightarrow C\big(\overline{J}; B_{p,p}^{2m(\kappa_0-(1-\mu+1/p))}(\Gamma; F)\big),$$

so that this equation has to hold up to $t = 0$ by continuity, provided the coefficients of $\mathcal{B}_0$ and $\mathcal{C}_0$ are sufficiently smooth. In this case it is therefore necessary that

$$g_0(0,\cdot) - \mathcal{B}_0(0,\cdot,D)u_0 - \mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0 \in X_{\partial_t\rho,\mu} \qquad \text{if } \kappa_0 > 1 - \mu + 1/p, \tag{3.1.6}$$

since otherwise it is for all $\rho \in \mathbb{E}_{\rho,\mu}$ impossible to satisfy the dynamic equation for the data g_0, u_0, ρ_0. Moreover, if $\kappa_j > 1 - \mu + 1/p$ for some $j = 1, ..., m$, then it holds as above

$$\mathbb{F}_{j,\mu} \hookrightarrow C\big(\overline{J}; B_{p,p}^{2m(\kappa_j-(1-\mu+1/p))}(\Gamma; E)\big),$$

and also the corresponding static boundary equations must be valid up to $t = 0$ by continuity. Hence the data necessarily satisfies

$$\mathcal{B}_j(0,\cdot,D)u_0 + \mathcal{C}_j(0,\cdot,D_\Gamma)\rho_0 = g_j(0,\cdot) \quad \text{on } \Gamma \text{ if } \kappa_j > 1 - \mu + 1/p, \quad j = 1, ..., m, \tag{3.1.7}$$

if (3.1.1) has a solution $(u,\rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$, provided the coefficients are sufficiently smooth.

We illustrate the spaces $\mathbb{E}_{\rho,\mu}$, $X_{\rho,\mu}$, $X_{\partial_t\rho,\mu}$ and the compatibility conditions by reconsidering the problems from Example 3.1.1.

Example 3.1.2. Problem (3.1.2) belongs to Case 1, since $l = l_0 = 2$. We have

$$\mathbb{E}_{\rho,\mu} = W_{p,\mu}^{3/2-1/2p}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W_p^{3-1/p}(\Omega; E)),$$

and for the trace spaces

$$X_{\rho,\mu} = B_{p,p}^{2(\mu-1/p)+1-1/p}(\Gamma; F), \qquad X_{\partial_t\rho,\mu} = B_{p,p}^{2(\mu-1/p)-1-1/p}(\Gamma; F),$$

where the trace of the derivative only exists if $2(\mu-1/p) > 1+1/p$. Concerning the compatibility condition (3.1.6), note that the trace space of $\mathbb{F}_{0,\mu}$ equals $X_{\partial_t\rho,\mu}$ for $\kappa_0 > 1-\mu+1/p$. Further, the operators ∂_ν and $-\Delta_\Gamma$ map $X_{u,\mu}$ and $X_{\rho,\mu}$ into $X_{\partial_t\rho,\mu}$, respectively. Hence (3.1.6) is always satisfied. The condition (3.1.7) requires $\mathrm{tr}_\Omega u_0 = \rho_0$ if these expressions exist, which is natural for dynamic boundary conditions.

The problem (3.1.3) belongs to Case 2, since here $l = l_1 = 1 < 2$. Therefore

$$\mathbb{E}_{\rho,\mu} = W^{3/2-1/2p}_{p,\mu}(J; L_p(\Omega; E)) \cap L_{p,\mu}(J; W^{2-1/p}_p(\Omega; E)) \cap W^1_{p,\mu}(J; W^{1-1/p}_p(\Gamma; F)),$$

and further

$$X_{\rho,\mu} = B^{(\mu-1/p)+1-1/p}_{p,p}(\Gamma; F), \qquad X_{\partial_t\rho,\mu} = B^{2(\mu-1/p)-1-1/p}_{p,p}(\Gamma; F),$$

where the latter trace only exists if $2(\mu - 1/p) > 1 + 1/p$. As above (3.1.6) and (3.1.7) are naturally satisfied.

Finally, the problem (3.1.4) belongs to Case 3, due to $l = l_1 = 3 > 2$. Since $k_0 = -\infty$ it holds

$$\mathbb{E}_{\rho,\mu} = W^{3/2-1/2p}_{p,\mu}(J; L_p(\Gamma; F)) \cap L_{p,\mu}(J; W^{4-1/p}_p(\Gamma; F)) \cap W^{1-1/p}_{p,\mu}(J; W^2_p(\Gamma; F)).$$

The trace space of $\partial_t\rho$, which exists for $2(\mu - 1/p) > 1 + 1/p$, is given by

$$X_{\partial_t\rho,\mu} = B^{4(\mu-1/p)-2-2/p}_{p,p}(\Gamma; F).$$

For the trace space of ρ, if $\mu > 3/2p$ then we are in Case 3(i) and obtain

$$X_{\rho,\mu} = B^{2(\mu-1/p)+2-1/p}_{p,p}(\Gamma; F),$$

and if $\mu < 3/2p$ then we are in Case 3(iii) with

$$X_{\rho,\mu} = B^{4(\mu-1/p)+2-2/p}_{p,p}(\Gamma; F).$$

This shows that the initial regularity for ρ can change drastically if μ varies. The Case 3(ii) cannot occur in this example, since there is only one nontrivial vertex in the Newton polygon. For $\mu > 1/2 + 3/2p$, i.e., $2(\mu - 1/p) > 1 + 1/p$, the condition (3.1.6) is not always satisfied, since for $g_0 \in \mathbb{F}_{0,\mu}$ it holds, in general, $g_0(0, \cdot) \in B^{2(\mu-1/p)-1-1/p}_{p,p}(\Gamma; F)$, and the latter space has always a lower regularity than $X_{\partial_t\rho,\mu}$. Finally, for (3.1.7) the data must satisfy $u_0(\cdot) + \Delta_\Gamma\rho_0(\cdot) = g_1(0, \cdot)$ if $\mu > 3/2p$. ■

We intend to solve (3.1.1) in the following sense.

Definition 3.1.3. *We say that (3.1.1) enjoys the property of maximal $L_{p,\mu}$-regularity on the interval $J = (0, T)$ if the regularity assumptions on the data, i.e.,*

$$f \in \mathbb{E}_{0,\mu}, \qquad g \in \mathbb{F}_\mu, \qquad u_0 \in X_{u,\mu}, \qquad \rho_0 \in X_{\rho,\mu},$$

together with the compatibility conditions (3.1.6) and (3.1.7), are not only necessary for a unique solution $(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ of (3.1.1), but also sufficient.

The Assumptions on the Operators

In the sequel, the subscript $\sharp$ denotes the principle part of a differential operator, with an important exception for the $\mathcal{C}_j$, for which we set

$$\mathcal{C}_{j\sharp} := 0 \qquad \text{if } j \notin \mathcal{J}.$$

Hence only the principle parts of the operators $\mathcal{C}_j$ corresponding to a point on the nontrivial part of the Newton polygon for $\mathbb{E}_{\rho,\mu}$ are considered.
First, the coefficients of the operators are required to be such that each summand occurring in $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ is a continuous operator on the respective underlying spaces. Moreover, for localization purposes it is required that the top order coefficients of the operators are bounded and uniformly continuous. As in the static case, the Propositions 1.3.15 and 1.3.24 show that for the coefficients of $\mathcal{A}$ and $\mathcal{B}$ the following is sufficient for our purposes, respectively.

(SD) For $|\alpha| < 2m$ one of the following two conditions is valid: either

$$2m(\mu - 1/p) > 2m - 1 + n/p \quad \text{and} \quad a_\alpha \in \mathbb{E}_{0,\mu}(J \times \Omega; \mathcal{B}(E)),$$

or there are $r_\alpha, s_\alpha \in [p, \infty)$ with $\frac{p(1-\mu)+1}{s_\alpha} + \frac{n}{2mr_\alpha} < 1 - \frac{|\alpha|}{2m}$ such that

$$a_\alpha \in L_{s_\alpha}\big(J; (L_{r_\alpha} + L_\infty)(\Omega; \mathcal{B}(E))\big).$$

For $|\alpha| = 2m$ it holds $a_\alpha \in BUC(\overline{J} \times \overline{\Omega}; \mathcal{B}(E))$, and if Ω is unbounded then in addition the limits $a_\alpha(t, \infty) := \lim_{|x|\to\infty} a_\alpha(t, x)$ exist uniformly in $t \in \overline{J}$.

(SB) Let $\mathcal{E}_0 = \mathcal{B}(E, F)$, and $\mathcal{E}_j = \mathcal{B}(E)$ for $j = 1, ..., m$. For $j = 0, ..., m$ and $|\beta| \leq m_j$ one of the following two conditions is valid: either

$$\kappa_j > 1 - \mu + 1/p + \frac{n-1}{2mp} \qquad \text{and} \qquad b_{j\beta} \in \mathbb{F}_{j,\mu},$$

or there are $r^b_{j\beta}, s^b_{j\beta} \in [p, \infty)$ with

$$\frac{p(1-\mu)+1}{s^b_{j\beta}} + \frac{n-1}{2mr^b_{j\beta}} < \kappa_j + \frac{m_j - |\beta|}{2m}, \quad \kappa_j + \frac{m_j - |\beta|}{2m} - (1 - \mu + 1/p) \notin \left(0, \frac{n-1}{2mr^b_{j\beta}}\right),$$

such that

$$b_{j\beta} \in B^{\kappa_j}_{s^b_{j\beta},p}\big(J; L_{r^b_{j\beta}}(\Gamma; \mathcal{E}_j)\big) \cap L_{s^b_{j\beta}}\big(J; B^{2m\kappa_j}_{r^b_{j\beta},p}(\Gamma; \mathcal{E}_j)\big).$$

For the local coefficients of $\mathcal{C}$, the following conditions are sufficient, as Lemma 3.2.4 shows.[1]

[1] As the proof of Lemma 3.2.4 shows, the regularity assumptions in (SC) are not as sharp as the corresponding ones in (SB). Depending on the relation between the spaces $\mathbb{E}_{\rho,\mu}$ and $\mathbb{F}_\mu$, the coefficients could be less regular in the Cases 2 and 3. For the sake of a unified presentation we do not distinguish between the three cases.

(SC) Let $\mathcal{F}_0 = \mathcal{B}(F)$ and $\mathcal{F}_j = \mathcal{B}(F,E)$ for $j = 1,...,m$, and let $\mathrm{g} : V \to \Gamma$ be any coordinates for Γ. For $j = 0,...,m$ and $|\gamma| \le k_j$ one of following two conditions is valid: either

$$\kappa_j > 1 - \mu + 1/p + \frac{n-1}{2mp} \qquad \text{and} \qquad c^{\mathrm{g}}_{j\gamma} \in \mathbb{F}_{j,\mu}(J \times V; \mathcal{F}_j),$$

or there are $r^c_{j\gamma}, s^c_{j\gamma} \in [p,\infty)$ that satisfy

$$\frac{p(1-\mu)+1}{s^c_{j\gamma}} < \Big(1 - \frac{n-1}{(2m\kappa_j + k_j - |\gamma|)r^c_{j\gamma}}\Big)\Big(\kappa_j + \frac{k_j - |\gamma|}{k_j}(1 + \kappa_0 - \kappa_j)\Big)$$

and, in case $\kappa_j > 1 - \mu + 1/p$,

$$1 - \frac{1-\mu+1/p}{\kappa_j + \frac{k_j - |\gamma|}{k_j}(1+\kappa_0 - \kappa_j)} > \frac{n-1}{(2m\kappa_j + k_j - |\gamma|)r^c_{j\gamma}},$$

such that

$$c^{\mathrm{g}}_{j\gamma} \in B^{\kappa_j}_{s^c_{j\gamma},p}(J; L_{r^c_{j\gamma}}(V;\mathcal{F}_j)) \cap L_{s^c_{j\gamma}}(J; B^{2m\kappa_j}_{r^c_{j\gamma},p}(V;\mathcal{F}_j)).$$

Note that the second condition on the top order coefficients is equivalent to $\frac{p(1-\mu)+1}{s^c_{j\gamma}} + \frac{n-1}{2mr^c_{j\gamma}} < \kappa_j$, which is the same as in (SB). Proposition 1.3.24 and Lemma 3.2.4 show that (SB) and (SC) imply the continuity of the (local) top order coefficients, i.e.,

$$b_{j\beta} \in BUC(\overline{J} \times \Gamma; \mathcal{E}_j), \quad |\beta| = m_j, \qquad c^{\mathrm{g}}_{j\gamma} \in BUC(\overline{J} \times V; \mathcal{F}_j), \quad |\gamma| = k_j, \qquad j = 0,...,m.$$

We next describe the structural assumptions on $(\mathcal{A},\mathcal{B},\mathcal{C})$. As in the static case, for $\mathcal{A}$ we assume normal ellipticity.

(E) For all $t \in \overline{J}$, $x \in \overline{\Omega}$ and $|\xi| = 1$ it holds $\sigma(\mathcal{A}_\sharp(t,x,\xi)) \subset \mathbb{C}_+$. If Ω is unbounded then it holds in addition $\sigma(\mathcal{A}_\sharp(t,\infty,\xi)) \subset \mathbb{C}_+$ for all $t \in \overline{J}$ and $|\xi| = 1$.

Also conditions of Lopatinskii-Shapiro type are required. We call local coordinates g associated to $x \in \Gamma$ if the corresponding chart (U,φ) satisfies

$$\varphi(x) = 0, \qquad \varphi'(x) = \mathcal{O}_{\nu(x)}, \qquad \varphi(U \cap \Omega) \subset \mathbb{R}^n_+, \qquad \varphi(U \cap \Gamma) \subset \mathbb{R}^{n-1} \times \{0\},$$

where $\mathcal{O}_{\nu(x)}$ is a fixed orthogonal matrix that rotates $\nu(x)$ to $(0,...,0,-1) \in \mathbb{R}^n$. Such a chart (U,φ) exists by Lemma A.1.1. For such coordinates we define the rotated operators $\mathcal{A}^\nu$ and $\mathcal{B}^\nu$ by

$$\mathcal{A}^\nu(t,x,D) := \mathcal{A}(t,x,\mathcal{O}^T_{\nu(x)}D), \qquad \mathcal{B}^\nu(t,x,D) := \mathcal{B}(t,x,\mathcal{O}^T_{\nu(x)}D).$$

Moreover, we define the local operator $\mathcal{C}^g$ with respect to g by

$$\mathcal{C}^{\mathrm{g}}_j(t,x,D_{n-1}) := \sum_{|\gamma| \le k_j} c^{\mathrm{g}}_{j\gamma}(t,\mathrm{g}^{-1}(x))D^\gamma_{n-1}, \qquad j = 0,...,m,$$

where $c^{\mathrm{g}}_{j\gamma}$ are the local coefficients of $\mathcal{C}_j$. With these notations, in each of the Cases 1, 2 and 3 we assume the following.[2] Recall that $\mathcal{C}^{\mathrm{g}}_{j\sharp} := 0$ for $j \notin \mathcal{J}$.

[2]The reader should not be terrified of the Lopatinskii-Shapiro conditions. It is not too hard to verify them in applications, cf. [26, Section 3] and Section 5.2.

(LS) For each fixed $x \in \Gamma$, choose coordinates g associated to x. Then for all $t \in \overline{J}$, all $\lambda \in \overline{\mathbb{C}_+}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$, all $h_0 \in F$ and all $h_j \in E$, $j = 1, ..., m$, the ordinary initial value problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^\nu_\sharp(t,x,\xi',D_y)\big)v(y) &= 0, \qquad y > 0, \\ \mathcal{B}^\nu_{0\sharp}(t,x,\xi',D_y)v|_{y=0} + \big(\lambda + \mathcal{C}^{\mathrm{g}}_{0\sharp}(t,x,\xi')\big)\sigma &= h_0, \\ \mathcal{B}^\nu_{j\sharp}(t,x,\xi',D_y)v|_{y=0} + \mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j = 1, ..., m, \end{aligned}$$

has a unique solution $(v, \sigma) \in C_0([0,\infty); E) \times F$.

For problems from Case 2 and 3, in addition the following asymptotic Lopatinskii-Shapiro conditions (LS^-_∞) and (LS^+_∞) are required, respectively.

($\mathbf{LS}^-_\infty$) Let $l < 2m$. For each fixed $x \in \Gamma$, choose coordinates g associated to x. Then for all $t \in \overline{J}$, all $h_0 \in F$, all $h_j \in E$, $j = 1, ..., m$, and all $\lambda \in \overline{\mathbb{C}_+}$, $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$, the ordinary initial value problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^\nu_\sharp(t,x,\xi',D_y)\big)v(y) &= 0, \qquad y > 0, \\ \mathcal{B}^\nu_{j\sharp}(t,x,\xi',D_y)v|_{y=0} &= h_j, \qquad j = 1, ..., m, \end{aligned}$$

and for all $\lambda \in \overline{\mathbb{C}_+}$ and $|\xi'| = 1$ the problem

$$\begin{aligned} \mathcal{A}^\nu_\sharp(t,x,\xi',D_y)v(y) &= 0, \qquad y > 0, \\ \mathcal{B}^\nu_{0\sharp}(t,x,\xi',D_y)v|_{y=0} + \big(\lambda + \mathcal{C}^{\mathrm{g}}_{0\sharp}(t,x,\xi')\big)\sigma &= h_0, \\ \mathcal{B}^\nu_{j\sharp}(t,x,\xi',D_y)v|_{y=0} + \mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j = 1, ..., m, \end{aligned}$$

has a unique solution $(v, \sigma) \in C_0([0,\infty); E) \times F$, respectively.

($\mathbf{LS}^+_\infty$) Let $l > 2m$. For each fixed $x \in \Gamma$, choose coordinates g associated to x. Then for all $t \in \overline{J}$, all $h_0 \in F$, all $h_j \in E$, $j = 1, ..., m$, and all $\lambda \in \overline{\mathbb{C}_+}$ and $\xi \in \mathbb{R}^{n-1} \backslash \{0\}$, the ordinary initial value problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^\nu_\sharp(t,x,\xi',D_y)\big)v(y) &= 0, \qquad y > 0, \\ \mathcal{B}^\nu_{j\sharp}(t,x,\xi',D_y)v|_{y=0} + \delta_{j,\mathcal{J}_{2q_{\max}+1}}\mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j = 0, ..., m, \end{aligned}$$

and further for all $\lambda \in \overline{\mathbb{C}_+} \backslash \{0\}$, $|\xi'| = 1$ and $q = 1, ..., 2q_{\max}$, the problem

$$\begin{aligned} \big(\lambda + \mathcal{A}^\nu_\sharp(t,x,0,D_y)\big)v(y) &= 0, \qquad y > 0, \\ \mathcal{B}^\nu_{0\sharp}(t,x,0,D_y)v|_{y=0} + \delta_{-1,\mathcal{J}_q}\lambda\sigma + \delta_{0,\mathcal{J}_q}\mathcal{C}^{\mathrm{g}}_{0\sharp}(t,x,\xi')\sigma &= h_0, \\ \mathcal{B}^\nu_{j\sharp}(t,x,0,D_y)v|_{y=0} + \delta_{j,\mathcal{J}_q}\mathcal{C}^{\mathrm{g}}_{j\sharp}(t,x,\xi')\sigma &= h_j, \qquad j = 1, ..., m, \end{aligned}$$

has a unique solution $(v, \sigma) \in C_0([0,\infty); E) \times F$, respectively. Here $\delta_{j,\mathcal{J}_q} = 1$ if $j \in \mathcal{J}_q$ and $\delta_{j,\mathcal{J}_q} = 0$ otherwise.

If E and F are finite dimensional, it suffices to show that the above problems with $h_j = 0$, $j = 0, ..., m$, admit only the trivial solutions, respectively. In [26] it is shown that these conditions are necessary for uniform maximal L_p-regularity of (3.1.1) on finite intervals. The Lopatinskii-Shapiro conditions are verified in [26, Section 3] for the problems of Example 3.1.1. We also refer to Section 5.2, where we verify (LS) for a more general version of (3.1.2).

The Main Theorem

We state the main result of this chapter.

Theorem 3.1.4. *Let E and F be Banach spaces of class $\mathcal{HT}$, $p \in (1,\infty)$ and $\mu \in (1/p, 1]$. Let $J = (0,T)$ be a finite interval, $T > 0$, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$. Assume that $(\mathcal{A},\mathcal{B},\mathcal{C})$ satisfies* (E), (LS), (SD), (SB) *and* (SC), *and, in addition, for $l < 2m$ condition* (LS_∞^-), *and for $l > 2m$ condition* (LS_∞^+). *Assume further that $\kappa_j \neq 1 - \mu + 1/p$ for $j = 0, ..., m$. Then the problem*

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, \quad t \in J, \\ \partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x), && x \in \Gamma, \quad t \in J, \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x), && x \in \Gamma, \quad t \in J, \qquad j = 1,...,m, \\ u(0,x) &= u_0(x), && x \in \Omega, \\ \rho(0,x) &= \rho_0(x), && x \in \Gamma, \end{aligned} \tag{3.1.8}$$

enjoys maximal $L_{p,\mu}$-regularity, i.e., (3.1.8) has a unique solution $(u,\rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ if and only if $(f, g, u_0, \rho_0) \in \mathcal{D}$, where

$$\begin{aligned} \mathcal{D} := \big\{(f,g,u_0,\rho_0) \in \mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu} \times X_{\rho,\mu} \ : \quad & \text{for } j = 1,...,m \text{ it holds} \\ \mathcal{B}_j(0,\cdot,D)u_0 + \mathcal{C}_j(0,\cdot,D_\Gamma)\rho_0 = g_j(0,\cdot) \ \text{ on } \Gamma \quad & \text{if } \kappa_j > 1-\mu+1/p; \\ g_0(0,\cdot) - \mathcal{B}_0(0,\cdot,D)u_0 - \mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0 \in X_{\partial_t\rho,\mu} \qquad & \text{if } \kappa_0 > 1-\mu+1/p\big\}. \end{aligned}$$

The corresponding solution operator $\mathcal{L} : \mathcal{D} \to \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ is continuous. Restricting $\mathcal{L}$ to

$$\mathcal{D}_0 := \big\{(f,g,u_0,\rho_0) \in \mathcal{D} \ : \ g \in {}_0\mathbb{F}_\mu\big\},$$

for given $T_0 > 0$ its operator norm is uniform in $T \le T_0$. Finally, if the coefficients

$$(-\mathrm{i})^{|\alpha|}a_\alpha, \ |\alpha| \le 2m, \qquad (-\mathrm{i})^{|\beta|}b_{j\beta}, \ |\beta| \le m_j, \quad (-\mathrm{i})^{|\gamma|}c_{j\gamma}^g, \ |\gamma| \le k_j, \quad j = 0,...,m,$$

and the data are real-valued, then also the solution is real-valued.

Here the sets of compatible data $\mathcal{D}$ and $\mathcal{D}_0$ are endowed with the norms

$$\begin{aligned} |(f,g,u_0,\rho_0)|_{\mathcal{D}} := & |f|_{\mathbb{E}_{0,\mu}} + |g|_{\mathbb{F}_\mu} + |u_0|_{X_{u,\mu}} + |\rho_0|_{X_{\rho,\mu}} \\ & + |g_0(0,\cdot) - \mathcal{B}_0(0,\cdot,D)u_0 - \mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0|_{X_{\partial_t\rho,\mu}}, \end{aligned}$$

and

$$\begin{aligned} |(f,g,u_0,\rho_0)|_{\mathcal{D}_0} := & |f|_{\mathbb{E}_{0,\mu}} + |g|_{{}_0\mathbb{F}_\mu} + |u_0|_{X_{u,\mu}} + |\rho_0|_{X_{\rho,\mu}} \\ & + |g_0(0,\cdot) - \mathcal{B}_0(0,\cdot,D)u_0 - \mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0|_{X_{\partial_t\rho,\mu}}, \end{aligned}$$

respectively. The continuity of $\mathcal{L}$ must be understood with respect to these norms. Again it is important to distinguish between the norms of $\mathbb{F}_\mu$ and ${}_0\mathbb{F}_\mu$ (see Remark 1.1.15). Arguing as in the proof of Lemma 1.3.25, one can show that $(\mathcal{D}, |\cdot|_{\mathcal{D}})$ and $(\mathcal{D}_0, |\cdot|_{\mathcal{D}_0})$ are Banach spaces.

We have seen in Example 3.1.2 that it is possible that the condition

$$g_0(0,\cdot) - \mathcal{B}_0(0,\cdot,D)u_0 - \mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0 \in X_{\partial_t\rho,\mu}$$

is always satisfied. In this case we may consider $\mathcal{D}$ and $\mathcal{D}_0$ as closed subspaces of $\mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu} \times X_{\rho,\mu}$ and $\mathbb{E}_{0,\mu} \times {}_0\mathbb{F}_\mu \times X_{u,\mu} \times X_{\rho,\mu}$, respectively.

The proof of Theorem 3.1.4 follows [26] and is based on a localization and perturbation procedure, analogously to the proof of Theorem 2.1.4. For a general outline we refer to the end of Section 2.1. The main difficulty is the half-space case on the half-line, with vanishing initial value and vanishing domain inhomogeneity. Since there are no boundary conditions involved in the full space problem, this case is already covered by the Propositions 2.2.2 and 2.3.2.

3.2 Half-Space Problems with Boundary Conditions

3.2.1 Constant Coefficients

On the half-space $\Omega = \mathbb{R}^n_+$ with boundary $\Gamma = \mathbb{R}^{n-1}$ we consider the homogeneous differential operator

$$\mathcal{A}(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha,$$

together with the homogeneous boundary operators

$$\mathcal{B}_j(D) = \sum_{|\beta|=m_j} b_{j\beta}\mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad \mathcal{C}_j(D_{n-1}) = \sum_{|\gamma|=k_j} c_{j\gamma} D^\gamma_{n-1} \qquad j = 0, ..., m.$$

The coefficients of the operators are assumed to be constant, respectively,

$$a_\alpha, b_{j\beta} \in \mathcal{B}(E), \quad c_{j\gamma} \in \mathcal{B}(F,E), \quad j = 1,...,m, \quad b_{0\beta} \in \mathcal{B}(E,F), \quad c_{0\gamma} \in \mathcal{B}(F).$$

In this section, if nothing else is indicated, all spaces have to be understood over $\mathbb{R}_+ \times \mathbb{R}^n_+$, or over $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. We set

$${}_0\mathbb{F}_{j,\mu} := {}_0W^{\kappa_j}_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1};E)) \cap L_{p,\mu}(\mathbb{R}_+; W^{2m\kappa_j}_p(\mathbb{R}^{n-1};E)), \qquad j = 1,...,m,$$

and analogously for the spaces ${}_0\mathbb{F}_{0,\mu}$, ${}_0\mathbb{F}_\mu$, ${}_0\mathbb{E}_{u,\mu}$, and ${}_0\mathbb{E}_{\rho,\mu}$.[3]

We first consider inhomogeneous boundary conditions. The proof follows [26, Section 4.3].

Lemma 3.2.1. *Let E and F be Banach spaces of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and assume that $(\mathcal{A},\mathcal{B},\mathcal{C})$ satisfies* (E) *and* (LS), *and, in addition, for $l < 2m$ condition*

[3] More precisely, each $W^s_{p,\mu}$-space in the nonredundant description of ${}_0\mathbb{E}_{\rho,\mu}$ must be equipped with a left subscript 0.

(LS$_\infty^-$), *and for* $l > 2m$ *condition* (LS$_\infty^+$). *Then for* $g \in {}_0\mathbb{F}_\mu$ *there is a unique solution* $(u,\rho) \in {}_0\mathbb{E}_{u,\mu} \times {}_0\mathbb{E}_{\rho,\mu}$ *of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= 0, && x \in \mathbb{R}^n_+, \quad t>0, \\ \rho + \partial_t \rho + \mathcal{B}_0(D)u + \mathcal{C}_0(D_{n-1})\rho &= g_0(t,x), && x \in \mathbb{R}^{n-1}, \quad t>0, \\ \mathcal{B}_j(D)u + \mathcal{C}_j(D_{n-1})\rho &= g_j(t,x), && x \in \mathbb{R}^{n-1}, \quad t>0, \quad j=1,...,m, \\ u(0,x) &= 0, && x \in \mathbb{R}^n_+, \\ \rho(0,x) &= 0, && x \in \mathbb{R}^{n-1}. \end{aligned} \tag{3.2.1}$$

Proof. (I) We first show uniqueness for (3.2.1). On the space $L_p(\mathbb{R}^n_+;E) \times W^s_p(\mathbb{R}^{n-1};F)$, where $s = 2m\kappa_0$ in the Cases 1 and 2, and $s = \frac{k_{j_1}\kappa_0}{1+\kappa_0-\kappa_{j_1}}$ in Case 3,[4] we consider the operator A, defined by

$$A(u,\rho) := \big((1+\mathcal{A})u, \mathcal{B}_0 u + (1+\mathcal{C}_0)\rho\big), \qquad (u,\rho) \in D(A),$$

with domain

$$\begin{aligned} D(A) := \big\{(u,\rho) \in W^{2m}_p(\mathbb{R}^n_+;E) \times W^{l+2m\kappa_0}_p(\mathbb{R}^{n-1};F) \ : \\ \mathcal{B}_j u + (1+\mathcal{C}_j)\rho = 0, \ \ j=1,...,m; \quad \mathcal{B}_0 u + \mathcal{C}_0\rho \in W^s_p(\mathbb{R}^{n-1};F)\big\}. \end{aligned}$$

By (the proof of) [26, Theorem 2.2], A generates an analytic C_0-semigroup, and it thus follows from Lemma 1.2.1 that solutions of (3.2.1) are unique in the maximal $L_{p,\mu}$-regularity space for A, i.e., in

$$\mathbb{E}_{u,\mu}(\mathbb{R}_+) \times \big(W^1_{p,\mu}(\mathbb{R}_+; W^s_p(\mathbb{R}^{n-1};F)) \cap L_{p,\mu}(\mathbb{R}_+; W^{l+2m\kappa_0}_p(\mathbb{R}^{n-1};F))\big).$$

Since ${}_0\mathbb{E}_{u,\mu} \times {}_0\mathbb{E}_{\rho,\mu}$ embeds into this space, it follows that solutions of (3.2.1) are unique in ${}_0\mathbb{E}_{u,\mu} \times {}_0\mathbb{E}_{\rho,\mu}$.

(II) The rest of this proof is concerned with the existence of solutions of (3.2.1). We first suppose that

$$g = (g_0,...,g_m) \in C^\infty_c\big(\mathbb{R}_+; W^{2m}_p(\mathbb{R}^{n-1}; F \times E^m)\big).$$

We apply the Fourier transform $\mathcal{F}_t$ in time to (3.2.1), with covariable $\theta \in \mathbb{R}$, to arrive at the stationary problem

$$\begin{aligned} (1+\mathrm{i}\theta)v + \mathcal{A}(D)v &= 0, && x \in \mathbb{R}^n_+, \\ (1+\mathrm{i}\theta)\sigma + \mathcal{B}_0(D)v + \mathcal{C}_0(D_{n-1})\sigma &= \big(\mathcal{F}_t g_0\big)(\theta), && x \in \mathbb{R}^{n-1}, \\ \mathcal{B}_j(D)v + \mathcal{C}_j(D_{n-1})\sigma &= \big(\mathcal{F}_t g_j\big)(\theta), && x \in \mathbb{R}^{n-1}, \quad j=1,...,m. \end{aligned} \tag{3.2.2}$$

In [26, Section 4.3] it is shown that (3.2.2) has for each $\theta \in \mathbb{R}$ a unique solution

$$\big(v(\theta),\sigma(\theta)\big) \in W^{2m}_p(\mathbb{R}^n_+;E) \times W^{l+2m\kappa_0}_p(\mathbb{R}^{n-1};F),$$

[4] The number s is determined by the intersection of the nontrivial part of the Newton polygon of $\mathbb{E}_{\rho,\mu}$ with the vertical line $(a,1)$, $a \in \mathbb{R}$. The number $j_1 \in \mathcal{J}$ was defined in Section 3.1.

which may be represented as follows. We write $x = (x', y) \in \mathbb{R}^n_+$ with $x' \in \mathbb{R}^{n-1}$ and $y > 0$, and denote by $\mathcal{F}_{x'}$ the partial Fourier transform with respect to x', with covariable $\xi' \in \mathbb{R}^{n-1}$. We further define the symbols

$$\vartheta := (1 + \mathrm{i}\theta + |\xi'|^{2m})^{1/2m}, \qquad b := \frac{|\xi'|}{\vartheta}, \qquad \zeta := \frac{\xi'}{|\xi'|}, \qquad a := \frac{1 + \mathrm{i}\theta}{\vartheta^{2m}},$$

and the so-called boundary symbol $s(\theta, \xi')$ by

$$s(\theta, \xi') := 1 + \mathrm{i}\theta + |\xi'|^l \qquad \text{in the Cases 1 and 2,}$$

$$s(\theta, \xi') := 1 + \mathrm{i}\theta + \sum_{j \in \mathcal{J}} |\xi'|^{k_j} \vartheta^{m_0 - m_j} \qquad \text{in Case 3.}$$

Setting $h(\theta, \xi') := \big(\vartheta^{-m_0}(\mathcal{F}_{x'}\mathcal{F}_t g_0)(\theta, \xi'), ..., \vartheta^{-m_m}(\mathcal{F}_{x'}\mathcal{F}_t g_m)(\theta, \xi')\big) \in F \times E^m$, it holds

$$v(\theta, x', y) = \text{ first component of } \mathcal{F}^{-1}_{x'} e^{\vartheta \mathrm{i} A_0(b\zeta, a) y} P_s(b\zeta, a) M_u^0(b, \zeta, \vartheta) h(\theta, \cdot)$$

$$\sigma(\theta, x', y) = \mathcal{F}^{-1}_{x'} s(\cdot, \theta)^{-1} \vartheta^{m_0} M_\rho^0(b, \zeta, \vartheta) h(\theta, \cdot).$$

Here we have

$$A_0 : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathcal{B}(E^{2m}), \qquad P_s : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathcal{B}(E^{2m}),$$

$$M_u^0 : D_b \times D_\zeta \times \Sigma \to \mathcal{B}(F \times E^m, E^{2m}), \qquad M_\rho^0 : D_b \times D_\zeta \times \Sigma \to \mathcal{B}(F \times E^m, F),$$

with open sets

$$(\overline{B}_{1/2}(1/2))^{1/2m} \subset D_b \subset \mathbb{C}, \qquad \{\zeta \in \mathbb{R}^{n-1} \ : \ |\zeta| = 1\} \subset D_\zeta \subset \mathbb{C}^{n-1},$$

and a sector $\Sigma = \big\{z \in \mathbb{C}\backslash\{0\} \ : \ |\mathrm{arg} z| < \phi\big\}$, where $\phi \in (\pi/4m, \pi)$. The maps A_0, P_s, M_u^0 and M_ρ^0 are holomorphic in their complex arguments. The spectrum of $\mathrm{i}A_0(b\zeta, a)$ has a gap at the imaginary axis, and $P_s(b\zeta, a)$ is the spectral projection corresponding to the stable part of the spectrum. The functions M_u^0 and M_ρ^0 have the property that

$$\big\{|\xi'|^{|\alpha'|} D^{\alpha'}_{\xi'} M_u^0(\widetilde{b}, \zeta, \widetilde{\vartheta}) \ : \ \alpha' \in \{0,1\}^{n-1}, \ \ \xi' \neq 0, \ \ \theta \in \mathbb{R}, \ \ \widetilde{b} \in D_b, \ \ \widetilde{\vartheta} \in \Sigma\big\} \tag{3.2.3}$$

is an $\mathcal{R}$-bounded set of operators in $\mathcal{B}(F \times E^m, E^{2m})$, and that

$$\big\{|\xi'|^{|\alpha'|} D^{\alpha'}_{\xi'} M_\rho^0(\widetilde{b}, \zeta, \widetilde{\vartheta}) \ : \ \alpha' \in \{0,1\}^{n-1}, \ \ \xi' \neq 0, \ \ \theta \in \mathbb{R}, \ \ \widetilde{b} \in D_b, \ \ \widetilde{\vartheta} \in \Sigma\big\} \tag{3.2.4}$$

is an $\mathcal{R}$-bounded set of operators in $\mathcal{B}(F \times E^m, F)$.
The representation of the solution is obtained in [26] by applying $\mathcal{F}_{x'}$ to (3.2.2), which yields an ordinary initial value problem. By (LS), this problem has a unique decaying solution, from which the solution of (3.2.2) is obtained by applying $\mathcal{F}^{-1}_{x'}$. The asymptotic Lopatinskii-Shapiro conditions (LS^-_∞) and (LS^+_∞) are required to show the $\mathcal{R}$-boundedness of these sets, due to the unboundedness of ϑ. For $l = 2m$, the symbols M_u^0 and M_ρ^0 do not depend on ϑ, so that in this case asymptotic conditions are not needed. We refer to [26, Section 4.3] for details.

(III) We derive another representation of the solution operator for (3.2.2). For a function $\widetilde{h} \in \mathcal{S}(\mathbb{R}^{n-1}; E^m)$ we calculate for $x' \in \mathbb{R}^{n-1}$ and $y > 0$, neglecting the arguments of A_0 and P_s,

$$\begin{aligned}\big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 y} P_s \widetilde{h}\big)(x') &= \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s e^{-\widetilde{y}\vartheta}\widetilde{h}\big)(x')|_{\widetilde{y}=0} \\ &= -\int_0^\infty \partial_{\widetilde{y}} \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s e^{-\widetilde{y}\vartheta}\widetilde{h}\big)(x')\,\mathrm{d}\widetilde{y} \\ &= \int_0^\infty \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s \frac{1-\mathrm{i}A_0}{\vartheta^{2m-1}} \vartheta^{2m} e^{-\widetilde{y}\vartheta}\widetilde{h}\big)(x')\,\mathrm{d}\widetilde{y} \\ &= \int_0^\infty \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s \frac{1-\mathrm{i}A_0}{\vartheta^{2m-1}}\big) * \big((L_\theta \mathcal{E}_\theta \mathcal{F}_{x'}^{-1}\widetilde{h})(\cdot,\widetilde{y})\big)(x')\,\mathrm{d}\widetilde{y}.\end{aligned} \tag{3.2.5}$$

Here the operator L_θ, which corresponds to the symbol ϑ^{2m}, is defined by

$$L_\theta := 1 + \mathrm{i}\theta + (-\Delta_{n-1})^m,$$

and the extension operator $\mathcal{E}_\theta$, which corresponds to $y \mapsto e^{-y\vartheta}$, is for $f \in L_p(\mathbb{R}^{n-1}; E)$ given by

$$(\mathcal{E}_\theta f)(x', y) := e^{-yL_\theta^{1/2m}} f(x'), \qquad x' \in \mathbb{R}^{n-1}, \qquad y > 0.$$

In the last line of (3.2.5) we have used that

$$\mathcal{F}_{x'}^{-1} \vartheta^{2m} e^{-\cdot\vartheta}\widetilde{h} = L_\theta\, \mathcal{E}_\theta\, \mathcal{F}_{x'}^{-1}\widetilde{h}, \qquad \widetilde{h} \in \mathcal{S}(\mathbb{R}^{n-1}; E^m),$$

which is a consequence of the uniqueness of the $\mathcal{H}^\infty$-calculus for $-\Delta_{n-1}$ on $L_p(\mathbb{R}^{n-1}; E)$ (see [62, Example 10.2]). For $\theta \in \mathbb{R}$ and $f \in L_p(\mathbb{R}^n_+; E^{2m})$ we thus define the operator $\widetilde{\mathcal{T}}(\theta)$ by

$$(\widetilde{\mathcal{T}}(\theta) f)(x', y) := \text{ first component of } \int_0^\infty \big(\mathcal{F}_{x'}^{-1} e^{\mathrm{i}\vartheta A_0 (y+\widetilde{y})} P_s \frac{1-\mathrm{i}A_0}{\vartheta^{2m-1}}\big) * f(\cdot,\widetilde{y})(x')\,\mathrm{d}\widetilde{y}.$$

The proofs of the Lemmas 4.3 and 4.4 in [25] show

$$\widetilde{\mathcal{T}} \in C^1\big(\mathbb{R}; \mathcal{B}(L_p(\mathbb{R}^n_+; E^{2m}), W_p^{2m}(\mathbb{R}^n_+; E))\big),$$

and that

$$\big\{D^\alpha \widetilde{\mathcal{T}}(\theta),\ \theta \frac{\partial}{\partial\theta} D^\alpha \widetilde{\mathcal{T}}(\theta)\ :\ \theta \in \mathbb{R},\ \ |\alpha| \le 2m\big\} \tag{3.2.6}$$

is an $\mathcal{R}$-bounded set of operators in $\mathcal{B}\big(L_p(\mathbb{R}^n_+; E^{2m}), L_p(\mathbb{R}^n_+; E)\big)$. Using further that $\vartheta^{-m_j}\mathcal{F}_{x'} = \mathcal{F}_{x'} L_\theta^{-m_j/2m}$ for $j = 0, ..., m$, which can be seen as above, for the component $v(\theta)$ of the solution of (3.2.2) we therefore arrive at the representation

$$v(\theta) = \widetilde{\mathcal{T}}(\theta) L_\theta \mathcal{E}_\theta \left(\mathcal{F}_{x'}^{-1} M_u(b,\zeta,\vartheta)\mathcal{F}_{x'}\right) \left(L_\theta^{-m_j/2m} \mathcal{F}_t g_j(\theta)\right)_{j=0,\dots,m}.$$

Similarly, the second component $\sigma(\theta)$ may be represented by

$$\sigma(\theta) = S_\theta^{-1} L_\theta^{m_0/2m} \left(\mathcal{F}_{x'}^{-1} M_\rho^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\right) \left(L_\theta^{-m_j/2m} \mathcal{F}_t g_j(\theta)\right)_{j=0,\dots,m},$$

where the operator S_θ, that corresponds to the boundary symbol $s(\theta, \xi')$, is given by

$$S_\theta := 1 + \mathrm{i}\theta + (-\Delta_{n-1})^{l/2} \qquad \text{in the Cases 1 and 2,}$$

$$S_\theta := 1 + \mathrm{i}\theta + \sum_{j\in\mathcal{J}} (-\Delta_{n-1})^{k_j/2} L_\theta^{(m_0-m_j)/2m} \qquad \text{in Case 3.}$$

(IV) It follows from the boundedness of (3.2.6) that the map $\theta \mapsto \widetilde{T}(\theta)$ is bounded. It was further shown in Step II of the proof of Lemma 2.2.7 that $\theta \mapsto L_\theta \mathcal{E}_\theta$ is bounded. Moreover, the $\mathcal{R}$-boundedness of (3.2.3) and (3.2.4), the operator-valued Fourier multiplier theorem in $n-1$ dimensions ([24, Theorem 3.25], see also [62, Theorem 4.13]) and real interpolation yield that also the maps

$$\theta \mapsto \mathcal{F}_{x'}^{-1} M_u^0(b,\zeta,\vartheta)\mathcal{F}_{x'} \in \mathcal{B}\big(W_p^s(\mathbb{R}^{n-1}; F\times E^m), W_p^s(\mathbb{R}^{n-1}; E^{2m})\big),$$

$$\theta \mapsto \mathcal{F}_{x'}^{-1} M_\rho^0(b,\zeta,\vartheta)\mathcal{F}_{x'} \in \mathcal{B}\big(W_p^s(\mathbb{R}^{n-1}; F\times E^m), W_p^s(\mathbb{R}^{n-1}; F)\big),$$

are bounded for $\theta \in \mathbb{R}$. Hence for $g \in C_c^\infty\big(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; F\times E^m)\big)$ we may apply the inverse Fourier transform to $\big(v(\theta), \sigma(\theta)\big)$ and obtain that the solution (u,ρ) of (3.2.1) is given by

$$u = \mathcal{L}_u g := \mathcal{F}_t^{-1}\widetilde{T}(\theta)\, L_\theta\, \mathcal{E}_\theta\, \big(\mathcal{F}_{x'}^{-1} M_u^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\big)\, \big(L_\theta^{-m_j/2m}\mathcal{F}_t g_j\big)_{j=0,\dots,m},$$

$$\rho = \mathcal{L}_\rho g := \mathcal{F}_t^{-1} S_\theta^{-1} L_\theta^{m_0/2m}\, \big(\mathcal{F}_{x'}^{-1} M_\rho^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\big)\big(L_\theta^{-m_j/2m}\mathcal{F}_t g_j\big)_{j=0,\dots,m}.$$

Now as in Step III of the proof of Lemma 2.2.7 we may rewrite these solution operators to

$$\mathcal{L}_u g = \big(\mathcal{F}_t^{-1}\widetilde{T}(\theta)\mathcal{F}_t\big)\, L\, \mathcal{E}\, \big(\mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1} M_u^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\mathcal{F}_t\big)\, \big(L^{-m_j/2m} g_j\big)_{j=0,\dots,m},$$

$$\mathcal{L}_\rho g = S^{-1} L^{m_0/2m}\, \big(\mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1} M_\rho^0(b,\zeta,\vartheta)\mathcal{F}_{x'}\mathcal{F}_t\big)\big(L^{-m_j/2m} g_j\big)_{j=0,\dots,m},$$

with the operators

$$L := 1 + \partial_t + (-\Delta_{n-1})^m, \qquad \mathcal{E} := e^{-\cdot L^{1/2m}},$$

$$S := 1 + \partial_t + (-\Delta_{n-1})^{l/2} \qquad \text{in the Cases 1 and 2,}$$

$$S := 1 + \partial_t + \sum_{j\in\mathcal{J}} (-\Delta_{n-1})^{k_j/2} L^{(m_0-m_j)/2m} \qquad \text{in Case 3,}$$

that correspond to L_θ, $\mathcal{E}_\theta$ and S_θ, respectively. Since $C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; F\times E^m))$ is a dense subset of ${}_0\mathbb{F}_\mu$ by Lemma 1.3.14, the task is now to show that there is an estimate

$$|\mathcal{L}_u g|_{\mathbb{E}_{u,\mu}} + |\mathcal{L}_\rho g|_{\mathbb{E}_{\rho,\mu}} \lesssim |g|_{{}_0\mathbb{F}_\mu}, \qquad g \in C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; F\times E^m)), \tag{3.2.7}$$

since then the solution operator

$$\mathcal{L} := (\mathcal{L}_u, \mathcal{L}_\rho)$$

extends continuously to ${}_0\mathbb{F}_\mu$, and this yields the solution of (3.2.1).

(V) By Lemma 1.3.1, the realization of L on the space $L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$ is invertible, sectorial of angle not larger than $\pi/2$, and for $s \in (0,1]$ we have

$$D_L(s,p) = {}_0W_{p,\mu}^s(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E)) \cap L_{p,\mu}(\mathbb{R}_+; W_p^{2ms}(\mathbb{R}^{n-1}; E)).$$

Therefore $L^{-m_j/2m}$ maps for $j = 1, ..., m$ the space ${}_0\mathbb{F}_{j,\mu} = D_L(\kappa_j, p)$ continuously into

$$ {}_0\mathbb{Y}_E := D_L(1 - 1/2mp, p) = {}_0W^{1-1/2mp}_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E)) \\ \cap L_{p,\mu}(\mathbb{R}_+; W^{2m-1/p}_p(\mathbb{R}^{n-1}; E)). $$

The same arguments show that $L^{-m_0/2m}$ maps ${}_0\mathbb{F}_{0,\mu}$ continuously into ${}_0\mathbb{Y}_F$, which is defined as ${}_0\mathbb{Y}_E$ with E replaced by F.

(VI) We next show that the operator

$$ \mathcal{M}^0 := \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1} M^0(b, \zeta, \vartheta)\mathcal{F}_{x'}\mathcal{F}_t, $$

with symbol $M^0 : D_b \times D_\zeta \times \Sigma \to \mathcal{B}(F \times E^m, E^{2m} \times F)$ given by

$$ M^0(b, \zeta, \vartheta) := \big(M^0_u(b, \zeta, \vartheta), M^0_\rho(b, \zeta, \vartheta)\big), $$

extends continuously to an element of $\mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}^m_E, {}_0\mathbb{Y}^{2m}_E \times {}_0\mathbb{Y}_F\big)$. To this end we consider the approximating operators

$$ \mathcal{M}^{0,\varepsilon} := \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1} M^0(b, \zeta, \vartheta)(1 + \vartheta)^{-\varepsilon}\mathcal{F}_{x'}\mathcal{F}_t, \qquad \varepsilon \in (0, 1). $$

Cauchy's formula yields the representation

$$ \mathcal{M}^{0,\varepsilon} = -\frac{1}{4\pi^2}\int_{\Xi_\vartheta}\int_{\Xi_b} \mathcal{F}_t^{-1}\mathcal{F}_{x'}^{-1} M^0(\widetilde{b}, \zeta, \widetilde{\vartheta})(1 + \widetilde{\vartheta})^{-\varepsilon}(\widetilde{b} - b)^{-1}(\widetilde{\vartheta} - \vartheta)^{-1}\mathcal{F}_{x'}\mathcal{F}_t \,\mathrm{d}\widetilde{b}\,\mathrm{d}\widetilde{\vartheta}, $$

with $\Xi_\vartheta = (-\infty, 0]e^{-\mathrm{i}\phi_*} \cup [0, \infty)e^{\mathrm{i}\phi_*}$ for some $\phi_* \in (\pi/4m, \phi)$, and where Ξ_b is a closed curve in D_b surrounding $(\overline{B}_{1/2}(1/2))^{1/2m}$. Since $\zeta = \xi'/|\xi'|$ is independent of θ, we may rewrite this to

$$ \mathcal{M}^{0,\varepsilon} = -\frac{1}{4\pi^2}\int_{\Xi_\vartheta}\int_{\Xi_b} \mathcal{F}_{x'}^{-1} M^0(\widetilde{b}, \zeta, \widetilde{\vartheta})\mathcal{F}_{x'}(1 + \widetilde{\vartheta})^{-\varepsilon}(\widetilde{b} - B)^{-1}(\widetilde{\vartheta} - L^{1/2m})^{-1} \,\mathrm{d}\widetilde{b}\,\mathrm{d}\widetilde{\vartheta}, $$

where $B := (-\Delta_{n-1})^{1/2}L^{-1/2m}$ corresponds to the symbol $b = \frac{|\xi'|}{\vartheta}$. The realization of B on $L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$ is a bounded operator, and its spectrum is contained in $(\overline{B}_{1/2}(1/2))^{1/2m}$. This can be seen using the joint functional calculus for ∂_t and $(-\Delta_{n-1})^m$ [57, Theorem 4.5].

As above it follows from the $\mathcal{R}$-boundedness of the sets (3.2.3) and (3.2.4), and the operator-valued Fourier-multiplier theorem in $\mathbb{R}^{n-1}$ that the operators

$$ M^1(\widetilde{b}, \widetilde{\vartheta}) := \mathcal{F}_{x'}^{-1} M^0(\widetilde{b}, \cdot, \widetilde{\vartheta})\mathcal{F}_{x'}, \qquad \widetilde{b} \in D_b, \qquad \widetilde{\vartheta} \in \Sigma, $$

extend continuously to elements of $\mathcal{B}\big(W^s_p(\mathbb{R}^{n-1}; F \times E^m), W^s_p(\mathbb{R}^{n-1}; E^{2m} \times F)\big)$, $s \geq 0$, with uniformly bounded operators norms. Since M^0 is holomorphic, also M^1 is holomorphic in its arguments. By canonical pointwise extension we thus obtain that

$$ M^1 : D_b \times \Sigma \to \mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}^m_E, {}_0\mathbb{Y}^{2m}_E \times {}_0\mathbb{Y}_F\big) $$

is bounded and holomorphic. Hence we may rewrite $\mathcal{M}^{0,\varepsilon}$ into

$$ \mathcal{M}^{0,\varepsilon} = -\frac{1}{4\pi^2}\int_{\Xi_\vartheta}\int_{\Xi_b} M^1(\widetilde{b}, \widetilde{\vartheta})(1 + \widetilde{\vartheta})^{-\varepsilon}(\widetilde{b} - B)^{-1}(\widetilde{\vartheta} - L^{1/2m})^{-1} \,\mathrm{d}\widetilde{b}\,\mathrm{d}\widetilde{\vartheta}, $$

where the curve integrals are now in $\mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big)$. Using $L^{1-1/2mp}$ as an isomorphism ${}_0\mathbb{Y}_E \to L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$ that commutes with B, we see that the spectrum of the realization of B on ${}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m$ is also contained in $(\overline{B}_{1/2}(1/2))^{1/2m}$. Now the Dunford calculus for B yields

$$\mathcal{M}^{0,\varepsilon} = \frac{1}{2\pi \mathrm{i}} \int_{\Xi_\vartheta} M^2(\widetilde{\vartheta})(1+\widetilde{\vartheta})^{-\varepsilon}(\widetilde{\vartheta} - L^{1/2m})^{-1}\, \mathrm{d}\widetilde{\vartheta},$$

with a bounded holomorphic map

$$M^2 : \Sigma \to \mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big).$$

Since the realization of $L^{1/2m}$ on $L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))$ is sectorial with angle not larger than $\pi/4m$, it follows from [18, Corollary 1] that $L^{1/2m}$ admits an operator-valued bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle not larger than $\pi/4m$ on the real interpolation spaces ${}_0\mathbb{Y}_E^m$ and ${}_0\mathbb{Y}_F^m$. From this fact and the boundedness of M^2 on Σ we infer

$$|\mathcal{M}^{0,\varepsilon}|_{\mathcal{B}({}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m,{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F)} \lesssim \sup_{\widetilde{\vartheta}\in\Sigma} |M^2(\widetilde{\vartheta})(1+\widetilde{\vartheta})^{-\varepsilon}|_{\mathcal{B}({}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m,{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F)} \le C, \qquad (3.2.8)$$

where C does not depend on $\varepsilon \in (0,1)$. Due to [24, Proposition 2.2], for $h \in D(L^2)$ the map $\varepsilon \mapsto (1 + L^{1/2m})^\varepsilon h$ is continuous with values in $D_L(1 - 1/2mp, p)$. Together with (3.2.8), this fact yields

$$\begin{aligned}|\mathcal{M}^0 h|_{{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F} &\lesssim \limsup_{\varepsilon\to 0} |\mathcal{M}^{0,\varepsilon}|_{\mathcal{B}({}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m,{}_0\mathbb{Y}_E^{2m}\times{}_0\mathbb{Y}_F)}\, |(1+L^{1/2m})^\varepsilon h|_{{}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m} \\ &\lesssim |h|_{{}_0\mathbb{Y}_F\times{}_0\mathbb{Y}_E^m}.\end{aligned}$$

Since $D(L^2)$ is dense in $D_L(1 - 1/2mp, p)$, we obtain that $\mathcal{M}^0$ extends to an element of $\mathcal{B}\big({}_0\mathbb{Y}_F \times {}_0\mathbb{Y}_E^m, {}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F\big)$, as asserted.

(VII) Now we can show the required estimate for $\mathcal{L}_u$, i.e.,

$$|\mathcal{L}_u g|_{\mathbb{E}_{u,\mu}} \lesssim |g|_{{}_0\mathbb{F}_\mu}, \qquad g \in C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; F \times E^m)). \qquad (3.2.9)$$

By Lemma 1.3.8, the extension operator $\mathcal{E} = e^{-\cdot L^{1/2m}}$ maps continuously

$$D_L(1 - 1/2mp, p) = D_L^{1/2m}(2m - 1/p, p) \to L_p(\mathbb{R}_+; D(L)),$$

and L maps the space $L_p(\mathbb{R}_+; D(L))$ continuously into

$$L_p\big(\mathbb{R}_+; L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; E))\big) = L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E)).$$

Of course, here E may be replaced by F. Thus $L\,\mathcal{E}$ maps continuously

$${}_0\mathbb{Y}_E^{2m} \times {}_0\mathbb{Y}_F \to L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E^{2m} \times F)).$$

The $\mathcal{R}$-boundedness of (3.2.6) and the operator-valued Fourier multiplier theorem in $L_{p,\mu}$ (Theorem 1.2.4) show that $\mathcal{F}_t^{-1}\widetilde{\mathcal{T}}\mathcal{F}_t$ extends to a continuous operator

$$L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^n_+; E^{2m})) \to L_{p,\mu}(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^n_+; E)).^5$$

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[5] As in the static case, following the methods of the proof of [25, Lemma 4.4], one can show that for $|\alpha| \le 2m$ it holds $D^\alpha \widetilde{\mathcal{T}} \in C^2(\mathbb{R}; \mathcal{B}(L_p(\mathbb{R}^n_+; E^{2m} \times F))$, and that $\partial_\theta^2 |D^\alpha \mathcal{T}_j(\theta)| \lesssim \frac{1}{\theta^2}$. Hence also Proposition 1.2.5 yields that $D^\alpha \widetilde{\mathcal{T}}$ is a Fourier multiplier on $L_{p,\mu}$.

Now the equation for u shows that the $\mathbb{E}_{u,\mu}$ norm of u can be controlled by its $L_{p,\mu}(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^n_+; E))$-norm, and this yields (3.2.9).

(VIII) We finally consider the estimate for $\mathcal{L}_\rho$,

$$|\mathcal{L}_\rho g|_{\mathbb{E}_{u,\mu}} \lesssim |g|_{{}_0\mathbb{F}_\mu}, \qquad g \in C_c^\infty(\mathbb{R}_+; W_p^{2m}(\mathbb{R}^{n-1}; F \times E^m)).$$

As above we obtain that the operator $L^{m_0/2m}$ maps continuously

$${}_0\mathbb{Y}_F = D_L(1-1/p,p) \to D_L(\kappa_0, p) = {}_0\mathbb{F}_{0,\mu}.$$

It is thus left to show that S is an isomorphism ${}_0\mathbb{E}_{\rho,\mu} \to {}_0\mathbb{F}_{0,\mu}$. Using that ∂_t admits a bounded $\mathcal{H}^\infty$-calculus on $L_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; F))$ with domain ${}_0W^1_{p,\mu}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; F))$ (Theorem 1.1.7), this can be done literally in the same way as in [26, Section 4.2]. ■

Before we turn to the general half-space case we consider a right-inverse for the temporal trace on $\mathbb{E}_{\rho,\mu}$.

Lemma 3.2.2. *Let F be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$, or $\Omega = \mathbb{R}^n_+$. Assume that $\kappa_0 \neq 1-\mu+1/p$, and let $\rho_0 \in X_{\rho,\mu}(\Gamma)$ and further $\rho_1 \in X_{\partial_t\rho,\mu}(\Gamma)$ in case $\kappa_0 > 1-\mu+1/p$ be given. Then there is $\rho \in \mathbb{E}_{\rho,\mu}(\mathbb{R}_+ \times \Gamma)$ with*

$$\rho|_{t=0} = \rho_0, \qquad \text{and} \quad \partial_t\rho|_{t=0} = \rho_1 \quad \text{if } \kappa_0 > 1-\mu+1/p.$$

Proof. (I) We set

$$\rho_1 := 0 \qquad \text{for} \quad \kappa_0 < 1-\mu+1/p.$$

First suppose that $\Gamma = \mathbb{R}^{n-1}$. Let B_0 and B_1 be negative generators of exponentially stable analytic C_0-semigroups on $L_p(\mathbb{R}^{n-1}; F)$. Then we define ρ by

$$\rho(t) = \big(2e^{-tB_0} - e^{-2tB_0}\big)\rho_0 + \big(e^{-tB_1} - e^{-2tB_1}\big)B_1^{-1}\rho_1, \qquad t>0,$$

so that we have

$$\rho|_{t=0} = \rho_0, \qquad \partial_t\rho|_{t=0} = \rho_1.$$

We choose $B_i = (1-\Delta_{n-1})^{s_i/2}$, with exponents $s_i > 0$ as in [26, Section 4.1] according to the Cases 1, 2 and 3. Using Lemma 1.3.8 and arguing as in [26] one can show that $\rho \in \mathbb{E}_{\rho,\mu}(\mathbb{R}_+ \times \mathbb{R}^{n-1})$ as desired.

(II) In the general case, we describe Γ by a finite collection of charts (U_i, φ_i) and a corresponding partition of unity ψ_i for Γ, and denote by Φ_i the push-forward with respect to φ_i, i.e., $\Phi_i\rho_0 = \rho_0 \circ \varphi_i^{-1}$. We further take cut-off functions $\phi_i \in C_c^\infty(\mathbb{R}^{n-1})$ with

$$\phi_i \equiv 1 \ \text{ on } \ \operatorname{supp} \Phi_i\psi_i, \qquad \operatorname{supp}\phi_i \subset \varphi_i(U_i).$$

It follows from Lemma A.4.1 that $\Phi_i(\psi_i\rho_0) \in X_{\rho,\mu}(\mathbb{R}^{n-1})$, and also $\Phi_i(\psi_i\rho_1) \in X_{\partial_t\rho,\mu}(\mathbb{R}^{n-1})$ in case $\kappa_0 > 1-\mu+1/p$. We define

$$\rho(t) = \sum\nolimits_i \Phi_i^{-1}\phi_i\Big(\big(2e^{-tB_0} - e^{-2tB_0}\big)\Phi_i(\psi_i\rho_0) + \big(e^{-tB_1} - e^{-2tB_1}\big)B_1^{-1}\Phi_i(\psi_i\rho_1)\Big), \quad t>0,$$

where B_0 and B_1 are chosen as above according to the Cases 1, 2 and 3. Then $\rho|_{t=0} = \rho_0$ and $\partial_t\rho|_{t=0} = \rho_1$, and further $\rho \in \mathbb{E}_{\rho,\mu}(\mathbb{R}_+ \times \Gamma)$ by Lemma A.4.1. ■

We now consider the general half-space case with constant coefficients. The Banach space of compatible data is here given by

$$\begin{aligned}\mathcal{D} = \big\{(f,g,u_0,\rho_0) \in \mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu} \times X_{\rho,\mu} \ : \ &\text{for } j = 1,...,m \text{ it holds}\\ \mathcal{B}_j(D)u_0(\cdot) + \mathcal{C}_j(D_{n-1})\rho_0(\cdot) = g_j(0,\cdot) \quad &\text{on } \mathbb{R}^{n-1} \text{ if } \kappa_j > 1-\mu+1/p;\\ g_0(0,\cdot) - \mathcal{B}_0(D)u_0 - \mathcal{C}_0(D_{n-1})\rho_0 \in X_{\partial_t\rho,\mu} \quad &\text{if } \kappa_0 > 1-\mu+1/p\big\},\end{aligned}$$

and it is equipped with the norm

$$\begin{aligned}|(f,g,u_0,\rho_0)|_{\mathcal{D}} = &|f|_{\mathbb{E}_{0,\mu}} + |g|_{\mathbb{F}_\mu} + |u_0|_{X_{u,\mu}} + |\rho_0|_{X_{\rho,\mu}}\\ &+ |g_0(0,\cdot) - \mathcal{B}_0(D)u_0 - \mathcal{C}_0(D_{n-1})\rho_0|_{X_{\partial_t\rho,\mu}}.\end{aligned}$$

Proposition 3.2.3. *Let E and F be Banach spaces of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p,1]$, and assume that $(\mathcal{A},\mathcal{B},\mathcal{C})$ satisfies* (E) *and* (LS)*, and, in addition, for $l < 2m$ condition* (LS$^-_\infty$)*, and for $l > 2m$ condition* (LS$^+_\infty$)*. Let further $\kappa_j \neq 1-\mu+1/p$ for $j = 0,...,m$. Then there is a unique solution $(u,\rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ of*

$$\begin{aligned} u + \partial_t u + \mathcal{A}(D)u &= f(t,x), && x \in \mathbb{R}^n_+, && t>0,\\ \rho + \partial_t\rho + \mathcal{B}_0(D)u + \mathcal{C}_0(D_{n-1})\rho &= g_0(t,x), && x \in \mathbb{R}^{n-1}, && t>0,\\ \mathcal{B}_j(D)u + \mathcal{C}_j(D_{n-1})\rho &= g_j(t,x), && x \in \mathbb{R}^{n-1}, && t>0, \quad j=1,...,m,\\ u(0,x) &= u_0(x), && x \in \mathbb{R}^n_+,\\ \rho(0,x) &= \rho_0(x), && x \in \mathbb{R}^{n-1}, \end{aligned} \tag{3.2.10}$$

if and only if the data satisfies $(f,g,u_0,\rho_0) \in \mathcal{D}$. The corresponding solution operator $\mathcal{S}_H : \mathcal{D} \to \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ is continuous.

Proof. The necessary conditions on the data were derived in Section 3.1. If a solution operator exists, then it is continuous due to the open mapping theorem.
Uniqueness of solutions of (3.2.10) follows from Lemma 3.2.1. We are going to reduce the existence of a solution of the full problem (3.2.10) to the problem (3.2.1). Let the data $(f,g,u_0,\rho_0) \in \mathcal{D}$ be given. We extend f and u_0 to $\mathcal{E}_{\mathbb{R}^n_+}f \in \mathbb{E}_{0,\mu}(\mathbb{R}_+ \times \mathbb{R}^n)$ and $\mathcal{E}_{\mathbb{R}^n_+}u_0 \in X_{u,\mu}(\mathbb{R}^n)$, using the extension operator $\mathcal{E}_{\mathbb{R}^n_+}$ from (1.3.3). Proposition 2.2.2 yields a solution $v \in \mathbb{E}_{u,\mu}(\mathbb{R}_+ \times \mathbb{R}^n)$ of the full-space problem

$$\begin{aligned} v + \partial_t v + \mathcal{A}(D)v &= (\mathcal{E}_{\mathbb{R}^n_+}f)(t,x), && x \in \mathbb{R}^n, && t>0,\\ v(0,x) &= (\mathcal{E}_{\mathbb{R}^n_+}u_0)(x), && x \in \mathbb{R}^n, \end{aligned}$$

which we use to define $\widetilde{u} := v|_{\mathbb{R}^n_+} \in \mathbb{E}_{u,\mu}$. Moreover, the compatibility condition for $j=0$ and (3.1.5) imply

$$g_0|_{t=0} - (\rho_0 + \mathcal{B}_0(D)u_0 + \mathcal{C}_0(D_{n-1})\rho_0) \in X_{\partial_t\rho,\mu} \quad \text{if } \kappa_0 > 1-\mu+1/p.$$

It thus follows from Lemma 3.2.2 that there is $\widetilde{\rho} \in \mathbb{E}_{\rho,\mu}$ with $\widetilde{\rho}|_{t=0} = \rho_0$ and

$$\partial_t \widetilde{\rho}|_{t=0} = g_0|_{t=0} - (\rho_0 + \mathcal{B}_0(D)u_0 + \mathcal{C}_0(D_{n-1})\rho_0) \quad \text{if } \kappa_0 > 1 - \mu + 1/p.$$

Using the function $\widetilde{\rho}$, we define

$$g_0^\star := g_0 - (\widetilde{\rho} + \partial_t \widetilde{\rho} + \mathcal{B}_0(D)\widetilde{u} + \mathcal{C}_0(D_{n-1})\widetilde{\rho}) \in {}_0\mathbb{F}_{0,\mu},$$

and further

$$g_j^\star := g_j - (\mathcal{B}_j(D)\widetilde{u} + \mathcal{C}_j(D_{n-1})\widetilde{\rho}) \in {}_0\mathbb{F}_{j,\mu}, \qquad j = 1, ..., m.$$

Note that the equality $g_j^\star|_{t=0} = 0$ in case $\kappa_j > 1 - \mu + 1/p$ follows from the compatibility condition for $j = 1, ..., m$.[6] Now Lemma 3.2.1 yields a pair $(u^\star, \rho^\star) \in {}_0\mathbb{E}_{u,\mu} \times {}_0\mathbb{E}_{\rho,\mu}$ satisfying

$$\begin{aligned} u^\star + \partial_t u^\star + \mathcal{A}(D)u^\star &= 0, && x \in \mathbb{R}^n_+, && t > 0, \\ \rho^\star + \partial_t \rho^\star + \mathcal{B}_0(D)u^\star + \mathcal{C}_0(D_{n-1})\rho^\star &= g_0^\star(t,x), && x \in \mathbb{R}^{n-1}, && t > 0, \\ \mathcal{B}_j(D)u^\star + \mathcal{C}_j(D_{n-1})\rho^\star &= g_j^\star(t,x), && x \in \mathbb{R}^{n-1}, && t > 0, \qquad j = 1, ..., m, \\ u^\star(0,x) &= 0, && x \in \mathbb{R}^n_+, \\ \rho^\star(0,x) &= 0, && x \in \mathbb{R}^{n-1}. \end{aligned}$$

Thus $(u, \rho) = (u^\star + \widetilde{u}, \rho^\star + \widetilde{\rho}) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ solves (3.2.10) by construction. ■

3.2.2 Top Order Coefficients having Small Oscillation

We investigate the case of operators on a half-space with top order coefficients having small oscillation, and restrict our considerations to a finite interval

$$J = (0, T), \qquad T > 0.$$

Now the anisotropic spaces are understood over $J \times \mathbb{R}^n_+$ or $J \times \mathbb{R}^{n-1}$. We consider the differential operator

$$\mathcal{A}(t,x,D) = \sum_{|\alpha| \le 2m} a_\alpha(t,x) D^\alpha, \qquad x \in \mathbb{R}^n_+, \qquad t \in J,$$

and the boundary operators

$$\mathcal{B}_j(t,x,D) = \sum_{|\beta| \le m_j} b_{j\beta}(t,x) \mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad x \in \mathbb{R}^{n-1}, \qquad t \in J, \qquad j = 0, ..., m,$$

$$\mathcal{C}_j(t,x,D_{n-1}) = \sum_{|\gamma| \le k_j} c_{j\gamma}(t,x) D_{n-1}^\gamma \qquad x \in \mathbb{R}^{n-1}, \qquad t \in J, \qquad j = 0, ..., m.$$

The top order coefficients of the operators are assumed to be of the form

$$a_\alpha(t,x) = a_\alpha^0 + \widetilde{a}_\alpha(t,x), \qquad |\alpha| = 2m,$$

[6] Here it is necessary to exclude the values $\kappa_j = 1 - \mu + 1/p$, $j = 0, ..., m$, since otherwise it is not guaranteed that $g_j^\star \in {}_0\mathbb{F}_{0,\mu}$.

$$b_{j\beta}(t,x) = b^0_{j\beta} + \widetilde{b}_{j\beta}(t,x), \qquad |\beta| = m_j, \qquad j = 0,...,m, \tag{3.2.11}$$
$$c_{j\gamma}(t,x) = c^0_{j\gamma} + \widetilde{c}_{j\gamma}(t,x), \qquad |\gamma| = m_j, \qquad j = 0,...,m,$$

where a^0_α, $b^0_{j\beta}$ and $c^0_{j\gamma}$ do not depend on t and x. Using these constant coefficients, we define auxiliary operators $(\mathcal{A}^0, \mathcal{B}^0, \mathcal{C}^0)$ by

$$\mathcal{A}^0(D) := \sum_{|\alpha|=2m} a^0_\alpha D^\alpha, \tag{3.2.12}$$

$$\mathcal{B}^0_j(D) := \sum_{|\beta|=m_j} b^0_{j,\beta} \mathrm{tr}_{\mathbb{R}^n_+} D^\beta, \qquad \mathcal{C}^0_j(D_{n-1}) := \sum_{|\gamma|=k_j} c^0_{j\gamma} D^\gamma_{n-1}, \qquad j = 0,...,m.$$

Assuming (SD) and (SB) for the coefficients of $\mathcal{A} - \mathcal{A}^0$ and $\mathcal{B} - \mathcal{B}^0$, the Propositions 1.3.16 and 1.3.24 yield

$$\mathcal{A} \in \mathcal{B}(\mathbb{E}_{u,\mu}(J), \mathbb{E}_{0,\mu}(J)), \qquad \mathcal{B} \in \mathcal{B}(\mathbb{E}_{u,\mu}(J), \mathbb{F}_\mu(J))$$

We now show that the assumption (SC) is sufficient for the required continuity properties of $\mathcal{C}$.

Lemma 3.2.4. *Let F be of class $\mathcal{HT}$, let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$, or $\Omega = \mathbb{R}^n_+$, and let for almost every $t \in J$ the differential operator*

$$\mathcal{C}(t,\cdot,D_\Gamma) = (\mathcal{C}_0(t,\cdot,D_\Gamma),...,\mathcal{C}_m(t,\cdot,D_\Gamma)) : C^\infty(\Gamma;F) \to L_1(\Gamma; F \times E^m)$$

be given. Assume that for $j \in \{0,...,m\}$ and $|\gamma| \leq k_j$ the local coefficients $c^{\mathrm{g}}_{j\gamma}$ of $\mathcal{C}_j$ with respect to coordinates $\mathrm{g} : V \to \Gamma$ satisfy $c^{\mathrm{g}}_{j\gamma} \in \mathbb{Y}_{j\gamma}(J)$, where either

$$\kappa_j > 1 - \mu + 1/p + \frac{n-1}{2mp} \qquad \text{and} \qquad \mathbb{Y}_{j\gamma}(J) = \mathbb{F}_{j,\mu}(J \times V; \mathcal{F}_j),$$

or it holds

$$\mathbb{Y}_{j\gamma}(J) = B^{\kappa_j}_{s_{j\gamma},p}(J; L_{r_{j\gamma}}(V;\mathcal{F}_j)) \cap L_{s_{j\gamma}}(J; B^{2m\kappa_j}_{r_{j\gamma},p}(V;\mathcal{F}_j))$$

with numbers $r_{j\gamma}, s_{j\gamma} \in [p,\infty)$ as in (SC).[7] *Then for $|\gamma| < k_j$ there is a small number $\delta > 0$ with*

$$|c^{\mathrm{g}}_{j\gamma} D^\gamma_{n-1}\rho|_{\mathbb{F}_{j,\mu}(J\times V)} \lesssim |c^{\mathrm{g}}_{j\gamma}|_{\mathbb{Y}_{j\gamma}(J)} |\rho|_{W^{1+\kappa_0-\delta}_{p,\mu}(J;L_p(V;F)) \cap L_{p,\mu}(J;W^{l+2m\kappa_0-\delta}_p(V;F))}.$$

Moreover, for the top order coefficients, $|\gamma| = k_j$, it holds $c^{\mathrm{g}}_{j\gamma} \in BUC(\overline{J} \times V; \mathcal{F}_j)$ and there is a small $\delta > 0$ with

$$|c^{\mathrm{g}}_{j\gamma} D^\gamma_{n-1}\rho|_{\mathbb{F}_{j,\mu}(J\times V)} \lesssim |c^{\mathrm{g}}_{j\gamma}|_{BUC(\overline{J}\times V;\mathcal{F}_j)} |\rho|_{\mathbb{E}_{\rho,\mu}(J\times V)} + |c^{\mathrm{g}}_{j\gamma}|_{\mathbb{Y}_{j\gamma}(J)} |\rho|_{W^{1+\kappa_0-\delta}_{p,\mu}(J;L_p(V;F)) \cap L_{p,\mu}(J;W^{l+2m\kappa_0-\delta}_p(V;F))}.$$

In particular, under the assumptions of (SC) *the operator $\mathcal{C}$ extends to*

$$\mathcal{C} \in \mathcal{B}\big(\mathbb{E}_{\rho,\mu}(J \times \Gamma), \mathbb{F}_\mu(J \times \Gamma)\big).$$

Restricting to $\rho \in {}_0\mathbb{E}_{\rho,\mu}$, and assuming that the coefficients are defined on a larger time interval $J_0 = (0, T_0)$ for $T_0 > 0$, the above estimates, with J replaced by J_0 in the norms for the coefficients, hold with a uniform constant for $T \leq T_0$.

[7] Recall that $\mathcal{F}_0 = \mathcal{B}(F)$, and $\mathcal{F}_j = \mathcal{B}(F,E)$ for $j = 1,...,m$.

Proof. For both cases the boundedness and the continuity of the top order coefficients follows from Proposition 1.3.24. Let $j \in \{0, ..., m\}$ and a multiindex γ be given. We have that

$$\mathbb{E}_{\rho,\mu}(J \times V) \hookrightarrow W^{1+\kappa_0}_{p,\mu}(J; L_p(V;F)) \cap W^{\kappa_j}_{p,\mu}(J; H^{k_j}_p(V;F)) \cap L_{p,\mu}(J; W^{l+2m\kappa_0}_p(V;F)),$$

and Proposition 1.3.2 and the fact that $l + 2m\kappa_0 \geq k_j + 2m\kappa_j$ yield for all j that[8]

$$\mathbb{E}_{\rho,\mu}(J \times V) \hookrightarrow W_{p,\mu}^{\kappa_j + \frac{k_j - |\gamma|}{k_j}(1+\kappa_0-\kappa_j)}(J; H^{|\gamma|}_p(V;F)) \cap L_{p,\mu}(J; W_p^{2m\kappa_j + k_j - |\gamma|}(V;F)).$$

Hence D^γ_{n-1} maps continuously

$$\mathbb{E}_{\rho,\mu}(J \times V) \to W_{p,\mu}^{\kappa_j + \frac{k_j - |\gamma|}{k_j}(1+\kappa_0-\kappa_j)}(J; L_p(V;F)) \cap L_{p,\mu}(J; W_p^{2m\kappa_j + k_j - |\gamma|}(V;F)),$$

with uniform norm in the ${}_0\mathbb{E}_{\rho,\mu}$-case, and the latter space embeds into $\mathbb{F}_{j,\mu}(J \times V)$. In case $\kappa_j > 1 - \mu + 1/p + \frac{n-1}{2mp}$ and $\mathbb{Y}_{j\gamma}(J) = \mathbb{F}_{j,\mu}(J \times V; \mathcal{F}_j)$ the asserted estimates follow from Lemma 1.3.23. In the other case we can apply the Lemmas 1.3.21 and 1.3.22, with

$$\tau = \kappa_j + \frac{k_j - |\gamma|}{k_j}(1 + \kappa_0 - \kappa_j), \qquad \vartheta = 2m\kappa_j + k_j - |\gamma|. \qquad \blacksquare$$

For an interval $J' = (0, T')$ with $T' > 0$ the Banach space of compatible data is given by

$$\begin{aligned} \mathcal{D}(J') = \big\{(f, g, u_0, \rho_0) \in \mathbb{E}_{0,\mu}(J') \times \mathbb{F}_\mu(J') \times X_{u,\mu} \times X_{\rho,\mu} \;:\; &\text{for } j = 1, ..., m \text{ it holds} \\ \mathcal{B}_j(0,\cdot,D)u_0 + \mathcal{C}_j(0,\cdot,D_{n-1})\rho_0 = g_j(0,\cdot) \quad &\text{on } \mathbb{R}^{n-1} \text{ if } \kappa_j > 1 - \mu + 1/p; \\ g_0(0,\cdot) - \mathcal{B}_0(0,\cdot,D)u_0 - \mathcal{C}_0(0,\cdot,D_{n-1})\rho_0 \in X_{\partial_t\rho,\mu} \;&\text{if } \kappa_0 > 1 - \mu + 1/p\big\}, \end{aligned}$$

and we also consider the space

$$\mathcal{D}_0(J') = \big\{(f, g, u_0, \rho_0) \in \mathcal{D}(J') \;:\; g \in {}_0\mathbb{F}_\mu(J')\big\}.$$

We have the following result.

Proposition 3.2.5. *Let E and F be Banach spaces of class $\mathcal{HT}$, $p \in (1,\infty)$, $\mu \in (1/p, 1]$, and assume that $(\mathcal{A}^0, \mathcal{B}^0, \mathcal{C}^0)$ satisfy* (E), (LS), *and, in addition, for $l < 2m$ condition* (LS^-_∞) *and for $l > 2m$ condition* (LS^+_∞). *Suppose further that the coefficients of $(\mathcal{A} - \mathcal{A}^0, \mathcal{B} - \mathcal{B}^0, \mathcal{C} - \mathcal{C}^0)$ satisfy* (SD), (SB), (SC), *and that $\kappa_j \neq 1 - \mu + 1/p$ for $j = 0, ..., m$. Then there are a time $T_0 \in (0, T]$ and number $\varepsilon > 0$ such that if*

$$\sup_{(t,x)\in[0,T_0]\times\overline{\mathbb{R}^n_+}} |\widetilde{a}_\alpha(t,x)|_{\mathcal{B}(E)} < \varepsilon, \qquad |\alpha| = 2m, \tag{3.2.13}$$

$$\sup_{(t,x)\in[0,T_0]\times\mathbb{R}^{n-1}} |\widetilde{b}_{j\beta}(t,x)|_{\mathcal{E}_j} + |\widetilde{c}_{j\gamma}(t,x)|_{\mathcal{F}_j} < \varepsilon, \qquad |\beta| = m_j, \quad |\gamma| = k_j, \quad j = 0, ..., m,$$ [9]

[8] Using the detailed shape of the Newton polygon according to the Cases 1, 2 and 3, here one could obtain a sharper result.

[9] Recall that $\mathcal{E}_0 = \mathcal{B}(E,F)$, $\mathcal{F}_0 = \mathcal{B}(F)$, and further $\mathcal{E}_j = \mathcal{B}(E)$ and $\mathcal{F}_j = \mathcal{B}(F,E)$ for $j = 1, ..., m$.

then for all intervals $J = (0,T')$ with $T' \leq T_0$ there is a unique solution $(u,\rho) = \mathcal{S}_H^{sm}(f,g,u_0,\rho_0) \in \mathbb{E}_{u,\mu}(J') \times \mathbb{E}_{\rho,\mu}(J')$ of

$$\begin{aligned} \partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \mathbb{R}^n_+, && t \in J', \\ \partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_{n-1})\rho &= g_0(t,x), && x \in \mathbb{R}^{n-1}, && t \in J', \\ \mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_{n-1})\rho &= g_j(t,x), && x \in \mathbb{R}^{n-1}, && t \in J', \quad j = 1,...,m, \\ u(0,x) &= u_0(x), && x \in \mathbb{R}^n_+, \\ \rho(0,x) &= \rho_0(x), && x \in \mathbb{R}^{n-1}, \end{aligned} \tag{3.2.14}$$

if and only if $(f,g,u_0,\rho_0) \in \mathcal{D}(J')$. The corresponding solution operator

$$\mathcal{S}_H^{sm} : \mathcal{D}(J') \to \mathbb{E}_{u,\mu}(J') \times \mathbb{E}_{\rho,\mu}(J')$$

is continuous. The operator norm of $\mathcal{S}_H^{sm}$ restricted to $\mathcal{D}_0(J')$ is uniform in $T' \leq T_0$.

Proof. The proof is completely analogous to the static case, Proposition 2.3.1. Throughout, let $0 < T' \leq T_0 \leq T$. The necessity part and the dependence of the solution operator $\mathcal{S}_H^{\mathrm{sm}}$ on J' can be obtained in the same way as in the proof of Proposition 2.3.1. Thus we only have to show unique solvability of (3.2.14) for sufficiently small T_0 and ε.
For $(f,g,u_0,\rho) \in \mathcal{D}(J')$ we set

$$Z_{u_0,\rho_0}(J') := \big\{(u,\rho) \in \mathbb{E}_{u,\mu}(J') \times \mathbb{E}_{\rho,\mu}(J') \ : \ u(0,\cdot) = u_0,\ \rho(0,\cdot) = \rho_0\big\},$$

which is a nonempty set due to the Lemmas 1.3.9 and 3.2.2, and is further a closed subspace of $\mathbb{E}_{u,\mu}(J') \times \mathbb{E}_{\rho,\mu}(J')$ by Theorem 1.3.6. For given $(u,\rho) \in Z_{u_0,\rho_0}(J')$ we consider the problem

$$\begin{aligned} v + \partial_t v + \mathcal{A}^0 v &= f + (\mathcal{A}^0 - \mathcal{A} + 1)u && \text{in } \mathbb{R}^n_+, \quad t \in J', \\ \sigma + \partial_t \sigma + \mathcal{B}_0^0 v + \mathcal{C}_0^0 \sigma &= g_0 + (\mathcal{B}_0^0 - \mathcal{B}_0)u + (\mathcal{C}_0^0 - \mathcal{C}_0 + 1)\rho && \text{on } \mathbb{R}^{n-1},\ t \in J', \\ \mathcal{B}_j^0 v + \mathcal{C}_j^0 \sigma &= g_j + (\mathcal{B}_j^0 - \mathcal{B}_j)u + (\mathcal{C}_j^0 - \mathcal{C}_j)\rho && \text{on } \mathbb{R}^{n-1},\ t \in J',\ j = 1,...,m, \\ v(0,\cdot) &= u_0 && \text{in } \mathbb{R}^n_+, \\ \sigma(0,\cdot) &= \rho_0, && \text{on } \mathbb{R}^{n-1}, \end{aligned} \tag{3.2.15}$$

where $(\mathcal{A}^0,\mathcal{B}^0,\mathcal{C}^0)$ is defined in (3.2.12). Since $(u,\rho) \in Z_{u_0,\rho_0}(J')$, the right-hand sides of the boundary equations in (3.2.15) belong to $\mathcal{D}_{\mathcal{B}^0,\mathcal{C}^0}(J')$, the space of compatible data for $(\mathcal{A}^0,\mathcal{B}^0,\mathcal{C}^0)$. It can be seen as in Step I of the proof of Lemma 3.2.1 that for each (u,ρ) solutions of (3.2.15) are unique in $\mathbb{E}_{u,\mu}(J') \times \mathbb{E}_{\rho,\mu}(J')$. The solution $(v,\sigma) = \mathcal{S}(u,\rho) \in \mathbb{E}_{u,\mu}(J') \times \mathbb{E}_{\rho,\mu}(J')$ of (3.2.15) is given by

$$\mathcal{S}(u,\rho) := \mathcal{S}_H\big(\mathcal{E}_{J'}(f+(\mathcal{A}^0-\mathcal{A}+1)u), \mathcal{E}_{J'}(g+(\mathcal{B}^0-\mathcal{B})u+(\mathcal{C}^0-\mathcal{C}+1)\rho), u_0, \rho_0\big)|_{J'}. \tag{3.2.16}$$

Here $\mathcal{S}_H : \mathcal{D}_{\mathcal{B}^0,\mathcal{C}^0}(\mathbb{R}_+) \to \mathbb{E}_{u,\mu}(\mathbb{R}_+) \times \mathbb{E}_{\rho,\mu}(\mathbb{R}_+)$ is the solution operator for (3.2.15) on the half-line, which is given by Proposition 3.2.3 and defined on $\mathcal{D}_{\mathcal{B}^0,\mathcal{C}^0}(\mathbb{R}_+)$. Further, $\mathcal{E}_{J'}$ is the extension operator from J' to $\mathbb{R}_+$, given by Lemma 1.1.5.
Using the Lemmas 1.3.21, 1.3.22 and 1.3.23, one can show as in the proof of Proposition 2.3.1 that $\mathcal{S}$ has a unique fixed point $(u,\rho) \in Z_{u_0,\rho_0}(J')$, provided T_0 and ε are sufficiently

small. Note that to show the contraction property of $\mathcal{S}$ the continuity of $\mathcal{S}_H$ is only employed for vanishing initial values and boundary data from ${}_0\mathbb{F}_\mu$. By construction, this fixed point is the unique solution of (3.2.14) in $\mathbb{E}_{u,\mu}(J') \times \mathbb{E}_{\rho,\mu}(J')$. ■

3.3 The General Case on a Domain

We can now prove the main result of this chapter, Theorem 3.1.4, employing a localization procedure analogously to the proof of Theorem 2.1.4. For an outline of the proof we refer to the end of Section 2.1.

To set the scene, let E and F be Banach spaces of class $\mathcal{HT}$, let $J = (0,T)$ be a finite interval, and let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\Gamma = \partial\Omega$. We consider the problem

$$\begin{aligned}
\partial_t u + \mathcal{A}(t,x,D)u &= f(t,x), && x \in \Omega, \quad t \in J, \\
\partial_t \rho + \mathcal{B}_0(t,x,D)u + \mathcal{C}_0(t,x,D_\Gamma)\rho &= g_0(t,x), && x \in \Gamma, \quad t \in J, \\
\mathcal{B}_j(t,x,D)u + \mathcal{C}_j(t,x,D_\Gamma)\rho &= g_j(t,x), && x \in \Gamma, \quad t \in J, \qquad j = 1,...,m, \\
u(0,x) &= u_0(x), && x \in \Omega, \\
\rho(0,x) &= \rho_0(x), && x \in \Gamma,
\end{aligned} \tag{3.3.1}$$

where the differential operators $\mathcal{A}$ and $\mathcal{B} = (\mathcal{B}_0,...,\mathcal{B}_m)$ are given by

$$\mathcal{A}(t,x,D) = \sum_{|\alpha|\le 2m} a_\alpha(t,x) D^\alpha, \qquad t \in J, \qquad x \in \Omega,$$

$$\mathcal{B}_j(t,x,D) = \sum_{|\beta|\le m_j} b_{j\beta}(t,x)\operatorname{tr}_\Omega D^\beta, \qquad t \in J, \qquad x \in \Gamma, \qquad m_j \in \{0,...,2m-1\},$$

and where the operators $\mathcal{C} = (\mathcal{C}_0,...,\mathcal{C}_m)$ are in local coordinates g given by

$$\mathcal{C}_j^{\mathrm{g}}(t,x,D_\Gamma) = \sum_{|\gamma|\le k_j} c_{j\gamma}^{\mathrm{g}}(t,x) D_{n-1}^\gamma, \qquad t \in J, \qquad k_j \in \mathbb{N}_0, \qquad j = 0,...,m.$$

Assuming that the coefficients $a_\alpha, b_{j\beta}$, and $c_{j\gamma}^{\mathrm{g}}$ satisfy (SD), (SB) and (SC), it follows from the Propositions 1.3.16, 1.3.24 and Lemma 3.2.4 that

$$\mathcal{A} \in \mathcal{B}(\mathbb{E}_{u,\mu}, \mathbb{E}_{0,\mu}), \qquad \mathcal{B} \in \mathcal{B}(\mathbb{E}_{u,\mu}, \mathbb{F}_\mu), \qquad \mathcal{C} \in \mathcal{B}(\mathbb{E}_{\rho,\mu}, \mathbb{F}_\mu).$$

For the top order coefficients, it is assumed in resp. follows from (SD), (SB) and (SC) that

$$a_\alpha \in BUC(\overline{J} \times \overline{\Omega}; \mathcal{B}(E)), \qquad |\alpha| = 2m, \tag{3.3.2}$$

$$b_{j\beta} \in BUC(\overline{J} \times \Gamma; \mathcal{E}_j), \quad |\beta| = m_j, \qquad c_{j\gamma}^{\mathrm{g}} \in BUC(\overline{J} \times \Gamma; \mathcal{F}_j), \quad |\gamma| = k_j, \qquad j = 0,...,m,$$

where $\mathcal{E}_j$ and $\mathcal{F}_j$ are defined in (SC). The Banach space of compatible data is given by

$$\begin{aligned}
\mathcal{D} = \big\{ (f,g,u_0,\rho_0) \in \mathbb{E}_{0,\mu} \times \mathbb{F}_\mu \times X_{u,\mu} \times X_{\rho,\mu} \ &:\ \text{for } j = 1,...,m \text{ it holds} \\
\mathcal{B}_j(0,\cdot,D)u_0 + \mathcal{C}_j(0,\cdot,D_\Gamma)\rho_0 = g_j(0,\cdot) \ \text{ on } \Gamma \quad &\text{if } \kappa_j > 1 - \mu + 1/p; \\
g_0(0,\cdot) - \mathcal{B}_0(0,\cdot,D)u_0 - \mathcal{C}_0(0,\cdot,D_\Gamma)\rho_0 \in X_{\partial_t\rho,\mu} \qquad &\text{if } \kappa_0 > 1 - \mu + 1/p \big\},
\end{aligned}$$

and we also consider the space

$$\mathcal{D}_0 = \big\{ (f,g,u_0,\rho_0) \in \mathcal{D} \ :\ g \in {}_0\mathbb{F}_\mu \big\}.$$

Proof of Theorem 3.1.4.

The arguments of the Steps I and II of the proof of Theorem 2.1.4 concerning the necessary conditions, continuity of the solution operator, real solutions and causality carry over to the present situation. We thus only have to show that for $(f, g, u_0, \rho_0) \in \mathcal{D}$ there exists a unique solution $(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ of (3.3.1), where we may assume that T is sufficiently small.

(I) We localize in space, and use Proposition 2.3.2 to treat the local problems without boundary conditions, and Proposition 3.2.5 for the local problems with boundary conditions. As in the proof of Theorem 2.1.4 we take a finite number of points

$$x_i \in \overline{\Omega}, \qquad i = 1, ..., N_H,$$

together with $x_0 := \infty$ if Ω is unbounded, and corresponding open neighbourhoods $U_i \subset \mathbb{R}^n$ of these points (where $U_0 = \emptyset$ if Ω is bounded) satisfying

$$\overline{\Omega} \subset \bigcup_{i=0}^{N_H} U_i, \qquad U_i \cap \Gamma = \emptyset, \quad i = 0, ..., N_F, \qquad U_i \cap \Gamma \neq \emptyset, \quad i = N_F + 1, ..., N_H.$$

Further, the boundary $\Gamma \subset \bigcup_{i=N_F+1}^{N_H} U_i$ is described by charts $(\widetilde{U}_i, \varphi_i)$, $i = N_F + 1, ..., N_H$, having properties as in (2.4.9), i.e.,

$$\varphi_i(x_i) = 0, \qquad \varphi_i(\widetilde{U}_i) = B_{2r_i}(0), \qquad \varphi_i'(x_i) = \mathcal{O}_{\nu(x_i)},$$

$$\varphi_i(\widetilde{U}_i \cap \Omega) \subset \mathbb{R}^n_+, \qquad \varphi(\widetilde{U}_i \cap \Gamma) \subset \mathbb{R}^{n-1}, \qquad U_i = \varphi_i^{-1}\big(B_{r_i}(0)\big).$$

Here we have $r_i > 0$, and $\mathcal{O}_{\nu(x_i)}$ is the orthogonal matrix rotating $\nu(x_i)$ to $(0, ..., 0, -1) \in \mathbb{R}^N$, which we have fixed for the formulation of the Lopatinskii-Shapiro conditions.

(II) For $i = 0, ..., N_F$ we define extended coefficients a_α^i on $J \times \mathbb{R}^n$, $|\alpha| \leq 2m$, such that

$$a_\alpha^i|_{J \times U_i} = a_\alpha,$$

as in (2.4.4), (2.4.6) and (2.4.7), respectively. This yields operators

$$\mathcal{A}^i(t, x, D) := \sum_{|\alpha| \leq 2m} a_\alpha^i(t, x) D^\alpha$$

which satisfy (E), and whose coefficients satisfy (SD), respectively. If the diameters of the U_i are sufficiently small, by Proposition 2.3.2 there is for all sufficiently small $T = |J|$ a continuous solution operator

$$\mathcal{S}_F^{\mathrm{sm},i} : \mathbb{E}_{0,\mu}(J \times \mathbb{R}^n) \times X_{u,\mu}(\mathbb{R}^n) \to \mathbb{E}_{u,\mu}(J \times \mathbb{R}^n)$$

for the full-space problem

$$\begin{aligned} \partial_t v + \mathcal{A}^i(t, x, D) v &= f^i(t, x), \qquad x \in \mathbb{R}^n, \qquad t \in J, \\ v(0, x) &= u_0^i(x), \qquad x \in \mathbb{R}^n. \end{aligned} \tag{3.3.3}$$

(III) For $i = N_F+1, ..., N_H$ we denote by Φ_i the push-forward operator with respect to φ_i, i.e., $\Phi_i u = u \circ \varphi_i^{-1}$, and define the transformed operators $\mathcal{A}^{\Phi_i}$ and $\mathcal{B}^{\Phi_i}$ by

$$\begin{aligned}\mathcal{A}^{\Phi_i}(t,x,D) &:= \big(\Phi_i \mathcal{A}(t,\cdot,D)\Phi_i^{-1}\big)(x), \qquad t\in J, \qquad x\in \mathbb{R}^n_+ \cap B_{r_i}(0),\\ \mathcal{B}^{\Phi_i}(t,x,D) &:= \big(\Phi_i \mathcal{B}(t,\cdot,D)\Phi_i^{-1}\big)(x), \qquad t\in J, \qquad x\in \mathbb{R}^{n-1} \cap B_{r_i}(0).\end{aligned}$$

Denoting by g_i the local parametrization of Γ corresponding to (U_i,φ_i), we further define the localized operator $\mathcal{C}^{\mathrm{g}_i} = (\mathcal{C}_0^{\mathrm{g}_i}, ..., \mathcal{C}_m^{\mathrm{g}_i})$ by

$$\mathcal{C}_j^{\mathrm{g}_i}(t,x,D_{n-1}) := \sum_{|\gamma|\le k_j} c_{j\gamma}^{\mathrm{g}_i}(t,x) D_{n-1}^{\gamma}, \qquad t\in J, \qquad x\in\mathbb{R}^{n-1}\cap B_{r_i}(0), \qquad j=0,...,m.$$

Here $c_{j\gamma}^{\mathrm{g}_i}$ denote the coefficients from the local representation of $\mathcal{C}_j$ with respect to g_i. The coefficients of $\mathcal{A}^{\Phi_i}$ are extended to coefficients a_α^i on $J\times\mathbb{R}^n_+$ as in (2.4.6) and (2.4.7). Moreover, the coefficients of $\mathcal{B}^{\Phi_i}$ and $\mathcal{C}^{\mathrm{g}_i}$ are extended to coefficients $b_{j\beta}^i$ and $c_{j\gamma}^i$ on $J\times\mathbb{R}^{n-1}$ as in (2.4.11) and (2.4.12).
These extended coefficients yield operators $(\mathcal{A}^i,\mathcal{B}^i,\mathcal{C}^i)$ on the half-space. We define top order constant coefficient operators $(\mathcal{A}^{i,0},\mathcal{B}^{i,0},\mathcal{C}^{i,0})$ by

$$\mathcal{A}^{i,0}(D) := \sum_{|\alpha|=2m} a_\alpha^{i,0} D^\alpha, \qquad a_\alpha^{i,0} := a_\alpha^i(0,x_i),$$

$$\begin{aligned}\mathcal{B}_j^{i,0}(D) &:= \sum_{|\beta|=m_j} b_{j\beta}^{i,0}\operatorname{tr}_{\mathbb{R}^n_+} D^\beta, \qquad b_{j\beta}^{i,0} := b_{j\beta}^i(0,x_i), \qquad j=0,...,m,\\ \mathcal{C}_j^{i,0}(D_{n-1}) &:= \sum_{|\beta|=k_j} c_{j\gamma}^{i,0} D_{n-1}^\gamma, \qquad c_{j\gamma}^{i,0} := c_{j\gamma}^{\mathrm{g}_i}(0,x_i), \qquad j=0,...,m.\end{aligned}$$

It follows from Lemma A.1.2 that for $\xi'\in\mathbb{R}^{n-1}$ it holds

$$\mathcal{A}_\sharp^{i,0}(\xi',D_y) = \mathcal{A}_\sharp(0,x_i,\mathcal{O}^T_{\nu(x_i)}(\xi',D_y)), \qquad \mathcal{B}_\sharp^{i,0}(\xi',D_y) = \mathcal{B}_\sharp(0,x_i,\mathcal{O}^T_{\nu(x_i)}(\xi',D_y)).$$

Hence, by assumption, we have that $\mathcal{A}^{i,0}$ satisfies (E), and $(\mathcal{A}^{i,0},\mathcal{B}^{i,0},\mathcal{C}^{i,0})$ satisfies the Lopatinskii-Shapiro conditions on $\mathbb{R}^n_+$. The coefficients of $(\mathcal{A}^i-\mathcal{A}^{i,0},\mathcal{B}^i-\mathcal{B}^{i,0},\mathcal{C}^i-\mathcal{C}^{i,0})$ satisfy (SD), (SB) and (SC) by construction. Given $\varepsilon>0$, if T, r_i and the diameter of U_i are sufficiently small, then the top order coefficients of $(\mathcal{A}^i-\mathcal{A}^{i,0},\mathcal{B}^i-\mathcal{B}^{i,0},\mathcal{C}^i-\mathcal{C}^{i,0})$ have ε oscillation.
Therefore $(\mathcal{A}^i,\mathcal{B}^i,\mathcal{C}^i)$ satisfies for all $i=N_F+1,...,N_H$ the assumptions of Proposition 3.2.5, and there are continuous solution operators

$$\mathcal{S}_H^{\mathrm{sm},i} : \mathcal{D}_{\mathcal{B}^i,\mathcal{C}^i}(J) \to \mathbb{E}_{u,\mu}(J\times\mathbb{R}^n_+)\times\mathbb{E}_{\rho,\mu}(J\times\mathbb{R}^n_+)$$

for the problems

$$\begin{aligned}\partial_t v + \mathcal{A}^i(t,x,D)v &= f^i(t,x), && x\in\mathbb{R}^n_+, && t\in J,\\ \partial_t\sigma + \mathcal{B}_0^i(t,x,D)v + \mathcal{C}_0^i(t,x,D_{n-1})\sigma &= g_0^i(t,x), && x\in\mathbb{R}^{n-1}, && t\in J,\\ \mathcal{B}_j^i(t,x,D)v + \mathcal{C}_j^i(t,x,D_{n-1})\sigma &= g_j^i(t,x), && x\in\mathbb{R}^{n-1}, && t\in J, \quad j=1,...,m,\\ v(0,x) &= u_0^i(x), && x\in\mathbb{R}^n_+,\\ \sigma(0,x) &= \rho_0^i(x), && x\in\mathbb{R}^{n-1},\end{aligned} \tag{3.3.4}$$

provided T, r_i and the diameter of U_i are sufficiently small, respectively. Here $\mathcal{D}_{\mathcal{B}^i,\mathcal{C}^i}(J)$ denotes the space of compatible data for $(\mathcal{A}^i, \mathcal{B}^i, \mathcal{C}^i)$.

(IV) Denoting by $\{\psi_i\}_{i=0,...,N_H}$ the partition of unity for $\overline{\Omega}$ subordinate to the cover $\bigcup_{i=0}^{N_H} U_i$, as constructed in Step VIII of the proof of Theorem 2.1.4, we take $\phi_i \in C_c^\infty(\mathbb{R}^n)$, $i = 0, ..., N_H$, with

$$\phi_i \equiv 1 \text{ on } \operatorname{supp} \psi_i, \qquad \operatorname{supp} \phi_i \subset U_i.$$

As in the proof of Theorem 2.1.4 it then holds that if $(u, \rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu}$ solves (3.3.1) with data $(f, g, u_0, \rho_0) \in \mathcal{D}$, then (u, ρ) is a fixed point of the map $\mathcal{G}_{f,g,u_0,\rho_0}$, defined by

$$\begin{aligned}\mathcal{G}_{f,g,u_0,\rho_0}(u,\rho) := &\sum_{i=0}^{N_F} \big(\phi_i \mathcal{S}_F^{\mathrm{sm},i}(f^i, u_0^i; u)|_{U_i}, 0\big) \\ &+ \sum_{i=N_F+1}^{N_H} \phi_i \Phi_i^{-1}\big(\mathcal{S}_H^{\mathrm{sm},i}(f^i, g^i, u_0^i, \rho_0^i; u, \rho)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)}\big),\end{aligned}$$

on the Banach space

$$Z_{u_0,\rho_0}(J) := \{(u,\rho) \in \mathbb{E}_{u,\mu} \times \mathbb{E}_{\rho,\mu} \;:\; u(0,\cdot) = u_0, \;\; \rho(0,\cdot) = \rho_0\},$$

which is nontrivial due to the Lemmas 1.3.9 and 3.2.2. Here for $i = 0, ..., N_F$ we have set

$$f^i := \psi_i f + [\mathcal{A}, \psi_i]u, \qquad u_0^i := \psi_i u_0,$$

and further, for $i = N_F + 1, ..., N_H$,

$$f^i := \Phi_i(\psi_i f + [\mathcal{A}, \psi_i]u), \qquad g^i = \Phi_i(\psi_i g + [\mathcal{B}, \psi_i]u + [\mathcal{C}, \psi_i]\rho),$$

$$u_0^i := \Phi_i(\psi_i u_0), \quad \rho_0^i := \Phi_i(\psi_i \rho_0),$$

and the notations $\mathcal{S}_F^{\mathrm{sm},i}(f^i, u_0^i; u)$ and $\mathcal{S}_H^{\mathrm{sm},i}(f^i, g^i, u_0^i, \rho_0^i; u, \rho)$ indicate that f^i, g^i, u_0^i and ρ_0^i are defined with respect to the functions u and ρ, respectively. Moreover, $[\cdot,\cdot]$ denotes the commutator bracket.

Using that the commutators are of lower order, as in Step IX of the proof of Theorem 2.1.4 one can show that for all $(f, g, u_0, \rho_0) \in \mathcal{D}(J)$ the map $\mathcal{G}_{f,g,u_0,\rho_0}$ has a unique fixed point in $Z_{u_0,\rho_0}(J)$, provided T is sufficiently small. Note here that the required compatibility conditions for $\mathcal{S}_H^{\mathrm{sm},i}$ at the boundary are trivially satisfied, since $g^i|_{t=0}$, if it exists, is independent of $(u, \rho) \in Z_{u_0,\rho_0}(J)$. This yields a fixed point map

$$\mathcal{Q} : \mathcal{D}(J) \to Z_{u_0,\rho_0}(J), \qquad \mathcal{Q}(f, g, u_0, \rho_0) = \mathcal{G}_{f,g,u_0,\rho_0}(\mathcal{Q}(f, g, u_0, \rho_0)),$$

with the property that

$$\mathcal{Q} : \{(f, g, 0, 0) \in \mathcal{D}_0(J)\} \to Z_{0,0}(J)$$

is continuous with operator norm uniform in T smaller than a given length.

(V) As in the Steps X and XI of the proof of Theorem 2.1.4, for given $(f, g, u_0, \rho_0) \in \mathcal{D}(J)$ one can now find the appropriate data $(f^\star, g^\star, u_0, \rho_0) \in \mathcal{D}(J)$ such that $(u, \rho) =$

$\mathcal{Q}(f^\star, g^\star, u_0, \rho_0)$ solves (3.3.1) with data (f, g, u_0, ρ_0) by solving one more fixed point equation. Writing $\mathcal{Q} = \mathcal{Q}(f^\star, g^\star, u_0, \rho_0)$, here one obtains for the dynamic equation on the boundary

$$\begin{aligned}(\mathcal{B}_0, \partial_t + \mathcal{C}_0) \cdot \mathcal{Q} &= \sum_{i=N_F+1}^{N_H} \phi_i \Phi_i^{-1}\big(\Phi_i(\mathcal{B}_0, \partial_t + \mathcal{C}_0)\Phi_i^{-1}\big) \cdot \mathcal{S}_H^{\mathrm{sm},i}\big(f^{\star,i}, g^{\star,i}, u_0^i; \mathcal{Q}\big)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)} \\ &\quad + \sum_{i=N_F+1}^{N_H} [(\mathcal{B}_0, \mathcal{C}_0), \phi_i] \cdot \Phi_i^{-1}\mathcal{S}_H^{\mathrm{sm},i}\big(f^{\star,i}, g^{\star,i}, u_0^i; \mathcal{Q}\big)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)} \\ &= g_0^\star + \sum_{i=N_F+1}^{N_H} \phi_i[(\mathcal{B}, \mathcal{C}), \psi_i]\mathcal{Q} + \mathcal{K}_2^0(f^\star, g^\star),\end{aligned}$$

where the correction term $\mathcal{K}_2^0(f^\star, g^\star)$ is given by

$$\mathcal{K}_2^0(f^\star, g^\star) := \sum_{i=N_F+1}^{N_H} [(\mathcal{B}_0, \mathcal{C}_0), \phi_i] \cdot \Phi_i^{-1}\mathcal{S}_H^{\mathrm{sm},i}\big(f^{\star,i}, g^{\star,i}, u_0^i; \mathcal{Q}\big)|_{\mathbb{R}^n_+ \cap B_{r_i}(0)}.$$

Here all the terms containing $\mathcal{S}_F^{\mathrm{sm},i}$ vanish, since the functions ϕ_i vanish on Γ for $i = 0, ..., N_F$. Moreover, as $\{\psi_i\}$ is a partition of unity for Γ and $\phi_i \equiv 1$ on $\operatorname{supp}\psi_i$ it holds that

$$\sum_{i=N_F+1}^{N_H} \phi_i[(\mathcal{B}, \mathcal{C}), \psi_i]\mathcal{Q} = [(\mathcal{B}, \mathcal{C}), 1]\mathcal{Q} = 0.$$

Similarly, in case $\kappa_0 > 1 - \mu + 1/p$, due to $\mathcal{Q}(f^\star, g^\star, u_0, \rho_0)|_{t=0} = (u_0, \rho_0)$ we have

$$\mathcal{K}_2^0(f^\star, g^\star)|_{t=0} = \sum_{i=N_F+1}^{N_H} [(\mathcal{B}_0(0, \cdot, D), \mathcal{C}_0(0, \cdot, D_\Gamma)), \phi_i] \cdot \psi_i(u_0, \rho_0) = 0,$$

which yields that $\mathcal{K}_2^0$ maps into ${}_0\mathbb{F}_{0,\mu}(J)$. Defining the correction terms $\mathcal{K}_1$ for the dynamic equation in Ω and $(\mathcal{K}_2^1, ..., \mathcal{K}_2^m)$ for the static boundary equations as in Step X of the proof of Theorem 2.1.4, respectively, and setting $\widetilde{\mathcal{K}}_2 = (\mathcal{K}_2^0, ..., \mathcal{K}_2^m)$, the appropriate $(f^\star, g^\star)$ is the solution of

$$(f^\star, g^\star) + (\mathcal{K}_1, \widetilde{\mathcal{K}}_2)(f^\star, g^\star) = (f, g).$$

This equation can be rewritten to a fixed point problem on $\mathbb{E}_{0,\mu}(J) \times {}_0\mathbb{F}_\mu(J)$, and can be solved via the contraction principle as in Step XI of the proof of Theorem 2.1.4. ■

Chapter 4

Attractors in Stronger Norms for Robin Boundary Conditions

4.1 Introduction

In this chapter we are concerned with the long-time behaviour of semilinear and quasilinear reaction-diffusion systems in separated divergence form with Robin boundary conditions. For the unknown $u = u(t,x) \in \mathbb{R}^N$, where $N \in \mathbb{N}$, we consider the problem[1]

$$\begin{aligned} \partial_t u - \partial_i(a_{ij}(u)\partial_j u) &= f(u) \qquad \text{in } \Omega, \qquad t>0, \\ a_{ij}(u)\nu_i\partial_j u &= g(u) \qquad \text{on } \Gamma, \qquad t>0, \\ u(0,\cdot) &= u_0 \qquad\ \ \text{in } \Omega. \end{aligned} \tag{4.1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$, $n \geq 2$, and the outer normal unit field on Γ is denoted by $\nu = (\nu_1, ...\nu_n)$. It is assumed that (4.1.1) is of separated divergence form, i.e.,

$$a_{ij}(u) = a(u)\,\alpha_{ij} \in \mathcal{B}(\mathbb{R}^N), \qquad i,j \in \{1,...,n\},$$

where $a : \mathbb{R}^N \to \mathcal{B}(\mathbb{R}^N)$ and where the $\alpha_{ij} \in \mathbb{R}$ are constants, $i,j \in \{1,...,n\}$. We impose the following structural conditions on these coefficients.

$$\left.\begin{aligned} (\alpha_{ij})_{i,j=1,...,n} \text{ is symmetric and uniformly positive definite,} \\ \sigma(a(\zeta)) \subset \mathbb{C}_+ = \{\operatorname{Re} z > 0\}, \qquad \zeta \in \mathbb{R}^N. \end{aligned}\right\} \tag{4.1.2}$$

We further assume throughout that a and the reaction terms $f, g : \mathbb{R}^N \to \mathbb{R}^N$ are smooth. Note that the above assumptions allow to rewrite the boundary condition into the equivalent form

$$\alpha_{ij}\nu_i\partial_j u = a^{-1}(u)g(u) \qquad \text{on } \Gamma, \qquad t>0.$$

Thus for $g = 0$ and $\alpha_{ij} = \delta_{ij}$, the Kronecker symbol, one obtains homogeneous Neumann boundary conditions.

[1] We use sum convention, i.e., it is understood that one sums over double indices. For instance, $\partial_i(a_{ij}(u)\partial_j u)$ must be read as $\sum_{i,j=1}^n \partial_i(a_{ij}(u)\partial_j u)$.

Parabolic systems of type (4.1.1) model many different phenomena in physics, chemistry and biology. For $a \equiv \mathrm{id}$ and $\alpha_{ij} = \delta_{ij}$ one obtains a reaction diffusion system with nonlinear boundary conditions. Also the Keller-Segel model for chemotaxis and the Shigesada-Kawasaki-Teramoto cross-diffusion model for population dynamics can be cast in the form (4.1.1), cf. Section 4.5.

Local well-posedness in a scale of Slobodetskii spaces for problems of type (4.1.1) is well known and was established by Amann [4, 5, 6]. Being precise, Theorem 14.4 and Corollary 14.7 of [6] yield the following.

Theorem 4.1.1. *Let $p \in (n, \infty)$ and $s \in (n/p, 1 + 1/p)$. Assume that (4.1.2) holds, and further that $g(u) = \widetilde{g}(u)u$ with a smooth function $\widetilde{g} : \mathbb{R}^N \to \mathcal{B}(\mathbb{R}^N)$. Then for $u_0 \in W_p^s(\Omega, \mathbb{R}^N)$ there is a unique maximal solution*

$$u(\cdot, u_0) \in C\big([0, t^+(u_0)); W_p^s(\Omega, \mathbb{R}^N)\big) \cap C^\infty\big((0, t^+(u_0)) \times \overline{\Omega}, \mathbb{R}^N\big)$$

of (4.1.1), where $t^+(u_0) > 0$ denotes the maximal existence time. The solution map $u_0 \mapsto u(\cdot, u_0)$ defines a compact local semiflow on $W_p^s(\Omega, \mathbb{R}^N)$.

We refer to the beginning of Section 4.3 for the notion of a compact local semiflow. For a general boundary reaction term g the system (4.1.1) is still locally well-posed in the above scale, but then smoothness of the solutions is a more delicate issue, in general, cf. [6].

Criteria for global existence of solutions, $t^+(u_0) = +\infty$, were also established by Amann [4]. Roughly speaking, an a priori Hölder bound is sufficient for a solution to exist globally, as Theorem 15.3 of [6] shows. In many special cases, like triangular systems, it suffices to find an L_∞-bound [6, Theorem 15.4], or even weaker bounds.

Once global existence is established, one is interested in the long-time behaviour of solutions, especially in the convergence to equilibria or the existence of a global attractor. Let $p \in (1, \infty)$, $s \geq 0$ and $\mathcal{M}_p^s$ be a subset of $W_p^s(\Omega, \mathbb{R}^N)$. A nonempty compact set $\mathcal{A} \subset \mathcal{M}_p^s$ is called a global attractor of (4.1.1) if (4.1.1) generates a semiflow of global solutions in $\mathcal{M}_p^s$, if $\mathcal{A}$ is invariant under the semiflow ($u(t, \mathcal{A}) \subset \mathcal{A}$ for all $t \geq 0$) and if it attracts every bounded subset M of $\mathcal{M}_p^s$, i.e., it holds

$$d_H(u(t, M), \mathcal{A}) := \sup_{u_0 \in u(t,M)} \inf_{v_0 \in \mathcal{A}} |u_0 - v_0|_{W_p^s(\Omega, \mathbb{R}^N)} \to 0 \quad \text{as } t \to +\infty,$$

with respect to the Hausdorff distance d_H. In this sense the flow on the attractor, if it exists, determines the long-time behaviour of solutions. It is further known that an attractor is unique, and that in $\mathcal{M}_p^s$ it holds

$$\begin{aligned} \mathcal{A} &= \text{the union of the } \omega\text{-limit sets of all bounded sets} \\ &= \text{the union of all bounded complete orbits} \\ &= \text{the maximal bounded invariant set} \\ &= \text{the minimal bounded set that attracts all bounded sets.} \end{aligned}$$

We refer to [16, 49, 63] for more informations. Note that $\mathcal{A}$ contains in particular all equilibria, all periodic solutions and all heteroclinic orbits of (4.1.1). If $\mathcal{A}$ has finite Hausdorff dimension, then the global dynamics of (4.1.1) reduce to a finite dimensional process, which is of essentially less complexity than the original, infinite dimensional one. It is therefore desireable to have an attractor in a norm as strong as possible, since although the solutions contained in $\mathcal{A}$ might be smooth, the solutions approach the attractor only with respect to the norm in $\mathcal{M}_p^s$ where $\mathcal{A}$ was established.

Assume that an attractor $\mathcal{A}$ exists in $W_p^{s_*}$ for some $s_* \in (0,2)$ in the semilinear case, i.e., if a does not depend on u, and for linear boundary conditions. It is then a consequence of the variation of constants formula that the solutions approach $\mathcal{A}$ in the W_p^s-norm for all $s \in (s_*, 2)$ and that $\mathcal{A}$ is independent of s, cf. [16, Section 4.3]. Thus one automatically has convergence to $\mathcal{A}$ in stronger norms.

It is the purpose of this chapter to show that a corresponding result for attractors in stronger norms is valid also in the quasilinear case with nonlinear boundary conditions, i.e., for the full problem (4.1.1). The key to the semiflow in higher norms and the substitute for the variation of constants formula is the maximal $L_{p,\mu}$-regularity result given by Theorem 2.1.4.

Let us consider the results in detail. We first show in Section 4.3 that (4.1.1) generates a compact local semiflow in the scale of nonlinear phase spaces

$$\mathcal{M}_p^s := \big\{u_0 \in W_p^s(\Omega, \mathbb{R}^N) \;:\; a_{ij}(u_0)\nu_i \partial_j u_0 = g(u_0) \quad \text{on} \quad \Gamma\big\},$$

where $p \in (n+2, \infty)$ and $s \in (1 + n/p, 2 - 2/p]$. This range of regularity is not covered by Amann's theory. For each $t \in (0, t^+(u_0))$ the solutions belong to the weighted maximal regularity class

$$\mathbb{E}_{u,\mu}(0,\tau) := W_{p,\mu}^1(0,\tau; L_p(\Omega, \mathbb{R}^N)) \cap L_{p,\mu}(0,\tau; W_p^2(\Omega, \mathbb{R}^N))$$

where the weight $\mu \in (1/p, 1]$ is such that $s = 2(\mu - 1/p)$. Our result is based on the regularity properties of the nonlinear superposition operators corresponding to f and g, which are investigated in Section 4.2, and on maximal $L_{p,\mu}$-regularity for the linearized problem, Theorem 2.1.4. Our arguments can also be used to establish a local semiflow in a scale of nonlinear phase spaces as above for much more general systems than (4.1.1), as treated in [65] for $s = 2 - 2/p$ without weights.
In Section 4.4 we then use maximal $L_{p,\mu}$-regularity and the inherent smoothing effect of the weighted spaces to show that if (4.1.1) has an absorbant set in a Hölder space $C^\alpha(\overline{\Omega}, \mathbb{R}^N)$, $\alpha > 0$, i.e., there is $C > 0$ such that each solution satisfies

$$\limsup_{t \to t^+(u_0)} |u(t,u_0)|_{C^\alpha(\overline{\Omega}, \mathbb{R}^N)} \le C, \tag{4.1.3}$$

then (4.1.1) has a global attractor in the phase space $\mathcal{M}_p^s$. Since $s > 1 + n/p$, this in particular yields the convergence to the attractor in the C^1-norm, and as a result $\mathcal{A}$ also

determines the long-time behaviour of the spatial gradient of solutions with respect to the sup-norm.
We also consider special cases where the above result remains true if one replaces the C^α-norm in (4.1.3) by a weaker norm, like semilinear problems, cross-diffusion models and single equations.
In Section 4.5 we illustrate these results in obtaining an attractor in stronger norms for semilinear reaction-diffusion systems with nonlinear boundary conditions, for a chemotaxis model with volume filling effect, and for a population model with cross-diffusion.

Besides the maximal regularity class

$$\mathbb{E}_{u,\mu}(J) = W^1_{p,\mu}(J; L_p(\Omega, \mathbb{R}^N)) \cap L_{p,\mu}(J; W^2_p(\Omega, \mathbb{R}^N)),$$

throughout this chapter we work with the weighted space

$$\mathbb{E}_{0,\mu}(J) := L_{p,\mu}(J; L_p(\Omega, \mathbb{R}^N)),$$

where $J = (0,T)$ is a finite interval, $p \in (1,\infty)$ and $\mu \in (1/p, 1]$. Since the boundary operator in (4.1.1) is of order 1, the space for the boundary equation is

$$\mathbb{F}_\mu(J) := W^{1/2-1/2p}_{p,\mu}(J; L_p(\Gamma, \mathbb{R}^N)) \cap L_{p,\mu}(J; W^{1-1/p}_p(\Gamma, \mathbb{R}^N)).$$

It follows from Theorem 1.3.6 that

$$\mathbb{E}_{u,\mu}(J) \hookrightarrow C(\overline{J}; B^{2(\mu-1/p)}_{p,p}(\overline{\Omega}, \mathbb{R}^N)),$$

and therefore Sobolev's embeddings yield

$$\mathbb{E}_{u,\mu}(J) \hookrightarrow C(\overline{J}; C^1(\overline{\Omega}, \mathbb{R}^N)) \qquad \text{if } \ 2(\mu - 1/p) > 1 + n/p. \tag{4.1.4}$$

Similarly, it holds

$$\mathbb{F}_\mu(J) \hookrightarrow C(\overline{J}; B^{2(\mu-1/p)-1-1/p}_{p,p}(\Gamma, \mathbb{R}^N)) \qquad \text{if } \ 2(\mu - 1/p) > 1 + 1/p,$$

so that we have

$$\mathbb{F}_\mu(J) \hookrightarrow C(\overline{J}; C(\Gamma, \mathbb{R}^N)) \qquad \text{if } \ 2(\mu - 1/p) > 1 + n/p. \tag{4.1.5}$$

Restricting to ${}_0\mathbb{E}_{u,\mu}$- and ${}_0\mathbb{F}_\mu$-spaces, the constants for the above embeddings are independent of the underlying interval J.

4.2 Superposition Operators

For our purposes it is convenient to rewrite (4.1.1) into the abstract form

$$\begin{aligned} \partial_t u + A(u) &= 0 && \text{in } \Omega, && t > 0, \\ B(u) &= 0 && \text{on } \Gamma, && t > 0, \\ u(0,\cdot) &= u_0 && \text{in } \Omega, \end{aligned}$$

where the nonlinear differential operators A and B are for $u \in \mathbb{E}_{u,\mu}(J)$ given by

$$A(u) := -\big(\partial_i(a_{ij}(u)\partial_j u) + f(u)\big), \qquad B(u) := \alpha_{ij}\nu_i \mathrm{tr}_\Omega \partial_j u - a^{-1}(\mathrm{tr}_\Omega u) g(\mathrm{tr}_\Omega u).$$

The purpose of this section is to investigate the regularity properties of A and B. We start with some uniform estimates for nonlinear functions.

Lemma 4.2.1. *Let $\psi : \mathbb{R}^m \to \mathbb{R}^M$ be smooth for $m, M \in \mathbb{N}$, and let $\overline{B}_R(0) \subset \mathbb{R}^m$ be a fixed closed ball around the origin with radius R. In the sequel we denote by ε a continuous function $\varepsilon : [0,\infty) \to [0,\infty)$ with $\varepsilon(0) = 0$.*

a) *There is a function ε as above with*

$$|\psi(\xi+\eta) - \psi(\xi) - \psi'(\xi)\eta| \le \varepsilon(|\eta|)|\eta| \qquad \text{for all } \xi, \eta \in \overline{B}_R(0).$$

b) *Define $\phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^M$ by $\phi(\xi,\eta) := \psi(\xi+\eta) - \psi(\xi) - \psi'(\xi)\eta$. Then there is ε as above with*

$$|\phi(\xi_2,\eta_2) - \phi(\xi_1,\eta_1)| \le \varepsilon(|\eta_1| + |\eta_2|)\big(|\eta_2 - \eta_1| + |\eta_1||\xi_2 - \xi_1|\big)$$

for all $\xi_1, \xi_2, \eta_1, \eta_2 \in \overline{B}_R(0)$.

c) *Define $\varphi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^M$ by $\varphi(\xi,\eta) := \psi(\xi+\eta) - \psi(\xi)$. Then there are ε as above and a constant $C > 0$ with*

$$|\varphi(\xi_2,\eta_2) - \varphi(\xi_1,\eta_1)| \le \varepsilon(|\eta_2|)|\xi_2 - \xi_1| + C\,|\eta_2 - \eta_1|$$

for all $\xi_1, \xi_2, \eta_1, \eta_2 \in \overline{B}_R(0)$.

Proof. (I) Since ψ is smooth, for $\xi_0 \in \overline{B}_R(0)$ there is a function ε_{ξ_0} as above such that

$$|\psi(\xi_0+\eta) - \psi(\xi_0) - \psi'(\xi_0)\eta| < \varepsilon_{\xi_0}(|\eta|)|\eta|, \qquad \eta \in \overline{B}_R(0). \tag{4.2.1}$$

By continuity and compactness, (4.2.1) holds true with ξ_0 replaced by ξ for all ξ in a small neighbourhood of ξ_0. By compactness we find finitely many ξ_i such that these neighbourhoods cover $\overline{B}_R(0)$, with corresponding functions ε_i. Now $\varepsilon := \max_i \varepsilon_i$ satisfies the asserted inequality in a) for all $\xi, \eta \in \overline{B}_R(0)$.

(II) To show b), we estimate with the mean value theorem

$$\begin{aligned}|\phi(\xi_2,\eta_2) - \phi(\xi_1,\eta_1)| \le{} & \sup_{s\in[0,1]} |\partial_\xi \phi(s\xi_2 + (1-s)\xi_1, \eta_1)|\,|\xi_2 - \xi_1| \\ & + \sup_{s\in[0,1]} |\partial_\eta \phi(\xi_2, s\eta_2 + (1-s)\eta_1)|\,|\eta_2 - \eta_1|. \end{aligned} \tag{4.2.2}$$

For $\xi, \eta \in \overline{B}_R(0)$, $\eta \neq 0$, the terms

$$|\partial_\xi \phi(\xi,\eta)|/|\eta| = |\psi'(\xi+\eta) - \psi'(\xi) - \psi'(\xi)\eta|/|\eta|$$

and

$$|\partial_\eta \phi(\xi,\eta)| = |\psi'(\xi+\eta) - \psi'(\xi)|$$

tend to zero as $|\eta| \to 0$ uniformly in ξ, by a) and the uniform continuity of ψ' on $\overline{B}_R(0)$. Applying this to (4.2.2) shows b). Assertion c) is shown in a similar way. ■

We now consider the properties of the map

$$A(u) = -\big(\partial_i(a_{ij}(u)\partial_j u) + f(u)\big), \qquad u \in \mathbb{E}_{u,\mu}(J).$$

Lemma 4.2.2. *Let $J = (0,T)$ be finite, and let $p \in (n+2,\infty)$ and $\mu \in (1/p,1]$ be such that $2(\mu - 1/p) > 1 + n/p$. Then $A \in C^1(\mathbb{E}_{u,\mu}(J), \mathbb{E}_{0,\mu}(J))$, and for $u \in \mathbb{E}_{u,\mu}(J)$ we have*

$$A'(u)h = -\big(\partial_i(a_{ij}(u)\partial_j h + a'_{ij}(u)\partial_j u h) + f'(u)h\big), \qquad h \in \mathbb{E}_{u,\mu}(J).$$

Moreover, let $T_0, R > 0$ be given. Then there is a continuous function $\varepsilon : [0,\infty) \to [0,\infty)$ with $\varepsilon(0) = 0$ such that for $T \le T_0$ it holds

$$|A(u+h) - A(u) - A'(u)h|_{\mathbb{E}_{0,\mu}(J)} \le \varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)} \tag{4.2.3}$$

for all $u, h \in \mathbb{E}_{u,\mu}(J)$ with

$$h(0,\cdot) = 0, \qquad |u|_{C(\overline{J};C^1(\overline{\Omega},\mathbb{R}^N))}, |u|_{\mathbb{E}_{u,\mu}(J)}, |h|_{\mathbb{E}_{u,\mu}(J)} \le R. \tag{4.2.4}$$

Proof. Throughout we set

$$|\cdot|_{0,\infty} := |\cdot|_{C(\overline{J};C(\overline{\Omega},\mathbb{R}^N))}, \qquad |\cdot|_{1,\infty} := |\cdot|_{C(\overline{J};C^1(\overline{\Omega},\mathbb{R}^N))}.$$

(I) It is easy to see that the estimate

$$|vw|_{W_p^1(\Omega,\mathbb{R}^N)} \le |v|_{C(\overline{\Omega},\mathbb{R}^{N\times N})}|w|_{W_p^1(\Omega,\mathbb{R}^N)} + |v|_{W_p^1(\Omega,\mathbb{R}^{N\times N})}|w|_{C(\overline{\Omega},\mathbb{R}^N)} \tag{4.2.5}$$

is valid for all $v \in W_p^1(\Omega,\mathbb{R}^{N\times N})$ and $w \in W_p^1(\Omega,\mathbb{R}^N)$, provided $p > n$. We use this fact and the embedding (4.1.4) to estimate for $u, h \in \mathbb{E}_{u,\mu}(J)$ [2]

$$\begin{aligned}
&|A(u+h) - A(u) - A'(u)h|_{\mathbb{E}_{0,\mu}(J)} \\
&\quad\lesssim |f(u+h) - f(u) - f'(u)h|_{\mathbb{E}_{0,\mu}(J)} + |a'(u)\partial_j hh|_{L_{p,\mu}(J;W_p^1(\Omega,\mathbb{R}^N))} \\
&\qquad + |\big(a_{ij}(u+h) - a_{ij}(u) - a'_{ij}(u)h\big)\partial_j(u+h)|_{L_{p,\mu}(J;W_p^1(\Omega,\mathbb{R}^N))} \\
&\quad\lesssim |f(u+h) - f(u) - f'(u)h|_{0,\infty} + |a'_{ij}(u)|_{1,\infty}|h|^2_{\mathbb{E}_{u,\mu}(J)} \\
&\qquad + |a_{ij}(u+h) - a_{ij}(u) - a'_{ij}(u)h|_{1,\infty}(|u|_{\mathbb{E}_{u,\mu}(J)} + |h|_{\mathbb{E}_{u,\mu}(J)}).
\end{aligned} \tag{4.2.6}$$

Note that for $h(0) = 0$ these estimates are uniform in $T \le T_0$. For the first summand in (4.2.6) we have, using Lemma 4.2.1 and again (4.1.4),

$$|f(u+h) - f(u) - f'(u)h|_{0,\infty} \le \varepsilon(|h|_{0,\infty})|h|_{0,\infty} \le \varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)}. \tag{4.2.7}$$

In case (4.2.4), the images of u and h are contained in a compact subset of $\mathbb{R}^N$, which yields that ε is uniform in $T \le T_0$ and R. Further, the second summand in (4.2.6) may be estimated by $\varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)}$, where ε is again uniform for (4.2.4). For the third

[2] It is understood that one takes the maximum over single indices.

summand we have that the second factor there is bounded, and it is uniformly bounded in case (4.2.4). For the first factor there, we denote by ∇ the gradient on $\mathbb{R}^n$ and write the $|\cdot|_{1,\infty}$ norm as $|\cdot|_{0,\infty} + |\nabla\cdot|_{0,\infty}$. The $|\cdot|_{0,\infty}$ part may be estimated as in (4.2.7). For the $|\nabla\cdot|_{0,\infty}$ part we have, estimating again as in (4.2.7) and using (4.1.4),

$$\begin{aligned}|\nabla(a_{ij}(u+h) &- a_{ij}(u) - a'_{ij}(u)h)|_{0,\infty}\\ &\le |a''_{ij}(u)\nabla h h|_{0,\infty} + |a'_{ij}(u+h) - a'_{ij}(u) - a''_{ij}(u)h|_{0,\infty}(|u|_{1,\infty} + |h|_{1,\infty})\\ &\lesssim |a''_{ij}(u)|_{0,\infty}|h|^2_{\mathbb{E}_{u,\mu}(J)} + \varepsilon(|h|_{0,\infty})|h|_{0,\infty}\\ &\le \varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)},\end{aligned}$$

with the asserted dependence on T in case (4.2.4). This shows the uniform estimate (4.2.3), and further that A is differentiable in each $u \in \mathbb{E}_{u,\mu}(J)$ with derivative $A'(u)$.
(II) It remains to show that $A' : \mathbb{E}_{u,\mu}(J) \to \mathcal{B}(\mathbb{E}_{u,\mu}(J), \mathbb{E}_{0,\mu}(J))$ is continuous. For this we take $u, v, h \in \mathbb{E}_{u,\mu}(J)$ with $|h|_{\mathbb{E}_{u,\mu}(J)} \le 1$. Then it follows from (4.2.5) and (4.1.4) that

$$\begin{aligned}&|(A'(u) - A'(v))h|_{\mathbb{E}_{0,\mu}(J)}\\ &\le |(f'(u) - f'(v))h|_{\mathbb{E}_{0,\mu}(J)} + |(a_{ij}(u) - a_{ij}(v))\partial_j h|_{L_{p,\mu}(J;W^1_p(\Omega,\mathbb{R}^N))}\\ &\quad + |(a'_{ij}(u)\partial_j u - a'_{ij}(v)\partial_j v)h|_{L_{p,\mu}(J;W^1_p(\Omega,\mathbb{R}^N))}\\ &\lesssim |f'(u) - f'(v)|_{0,\infty} + |a_{ij}(u) - a_{ij}(v)|_{1,\infty} + |a'_{ij}(u)\partial_j u - a'_{ij}(v)\partial_j v|_{0,\infty}\\ &\quad + |a'_{ij}(u)(\partial_j u - \partial_j v)|_{L_{p,\mu}(J;W^1_p(\Omega,\mathbb{R}^{N\times N}))} + |(a'_{ij}(u) - a'_{ij}(v))\partial_j v)|_{L_{p,\mu}(J;W^1_p(\Omega,\mathbb{R}^{N\times N}))}\\ &\lesssim |f'(u) - f'(v)|_{0,\infty} + |a_{ij}(u) - a_{ij}(v)|_{1,\infty} + |a'_{ij}(u)\partial_j u - a'_{ij}(v)\partial_j v|_{0,\infty}\\ &\quad + |a'_{ij}(u)(\partial_j u - \partial_j v)|_{1,\infty} + |(a'_{ij}(u) - a'_{ij}(v))\partial_j v)|_{1,\infty},\end{aligned}$$

and this converges to zero as $u \to v$ in $\mathbb{E}_{u,\mu}(J)$ due to (4.1.4). ■

We next investigate the regularity of superposition operators on the boundary. The estimate in a) is useful for low values of q and μ.

Lemma 4.2.3. *For a finite interval $J = (0, T)$ and a smooth function $g : \mathbb{R}^N \to \mathbb{R}^N$ the following holds true.*

a) *Let $q \in (1, \infty)$, $\mu \in (1/q, 1]$, and $\kappa, \tau \in (0, 1)$. Then*

$$|g(u)|_{W^{\kappa,\tau}_{q,\mu}(J\times\Gamma,\mathbb{R}^N)} \lesssim \sup_{\zeta\in B_u} |g'(\zeta)|\, |u|_{W^{\kappa,\tau}_{q,\mu}(J\times\Gamma,\mathbb{R}^N)} + |g(u)|_{C(\overline{J}\times\Gamma,\mathbb{R}^N)}$$

for all $u \in W^{\kappa,\tau}_{q,\mu}(J\times\Gamma,\mathbb{R}^N) \cap C(\overline{J}\times\Gamma,\mathbb{R}^N)$, where B_u is a ball with $u(\overline{J}\times\Gamma) \subset B_u$.

b) *Let now $p \in (n + 2, \infty)$ and $\mu \in (1/p, 1]$ satisfy $2(\mu - 1/p) > 1 + n/p$. Then for the superposition operator G, given by $G(u) := g(\mathrm{tr}_\Omega u)$, we have*

$$G \in C^1(\mathbb{E}_{u,\mu}(J), \mathbb{F}_\mu(J)), \qquad G'(u) = g'(\mathrm{tr}_\Omega u)\mathrm{tr}_\Omega.$$

c) *In the situation of* b)*, let $T_0, R > 0$ be given. Then there is a continuous function $\varepsilon : [0, \infty) \to [0, \infty)$ with $\varepsilon(0) = 0$ such that for $T \le T_0$ it holds*

$$|g(u + h) - g(u) - g'(u)h|_{{}_0\mathbb{F}_\mu(J)} \le \varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)}$$

for all $u, h \in \mathbb{E}_{u,\mu}(J)$ *satisfying*

$$h(0,\cdot) = 0, \quad |u|_{C(\overline{J};C^1(\Gamma,\mathbb{R}^N))}, |u|_{\mathbb{E}_{u,\mu}(J)}, |u(0,\cdot)|_{W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N)}, |h|_{\mathbb{E}_{u,\mu}(J)} \le R. \tag{4.2.8}$$

Proof. (I) To show a), take $u \in W_{q,\mu}^{\kappa;\tau}(J \times \Gamma, \mathbb{R}^N) \cap C(\overline{J} \times \Gamma, \mathbb{R}^N)$. Then it holds

$$|g(u)|_{L_{q,\mu}(J;L_q(\Gamma,\mathbb{R}^N))} \lesssim |g(u)|_{C(\overline{J}\times\Gamma,\mathbb{R}^N)}.$$

For the intrinsic seminorm of the weighted Slobodetskii spaces, given by Proposition 1.1.13, we estimate with the mean value theorem

$$\begin{aligned}[g(u)]^q_{W^\kappa_{q,\mu}(J;L_q(\Gamma,\mathbb{R}^N))} &= \int_0^T \int_0^s \int_\Gamma \frac{t^{q(1-\mu)}}{(s-t)^{1+\kappa q}} |g(u(s,x)) - g(u(t,x))|^q \,\mathrm{d}\sigma(x)\,\mathrm{d}t\,\mathrm{d}s \\ &\le \sup_{\zeta\in B_u} |g'(\zeta)|^q \, [u]^q_{W^\kappa_{q,\mu}(J;L_q(\Gamma,\mathbb{R}^N))}.\end{aligned}$$

(II) To estimate $|g(u)|_{L_{q,\mu}(J;W_q^\tau(\Gamma,\mathbb{R}^N))}$ we describe Γ by a finite collection of charts (U_i, φ_i), choose a partition of unity $\{\psi_i\}$ for Γ subordinate to $\bigcup_i U_i$ and set $W_i := \operatorname{supp}\psi_i \subset U_i$. Then for almost every $t \in J$ we have

$$|g(u(t,\cdot))|_{W_q^\tau(\Gamma,\mathbb{R}^N)} \lesssim \sum\nolimits_i |\psi_i(\varphi_i^{-1})\, g(u(t,\varphi_i^{-1}))|_{W_q^\tau(\varphi_i(W_i),\mathbb{R}^N)}.$$

For each i it holds, as above

$$|\psi_i(\varphi_i^{-1})\, g(u(t,\varphi_i^{-1}))|_{L_q(\varphi_i(W_i),\mathbb{R}^N)} \lesssim |g(u)|_{C(\overline{J}\times\Gamma,\mathbb{R}^N)}.$$

For the seminorm corresponding to $W_q^\tau(\varphi_i(W_i), \mathbb{R}^N)$, cf. (A.4.2), we estimate

$$\begin{aligned}&[\psi_i(\varphi_i^{-1})\, g(u(t,\varphi_i^{-1}))]^q_{W_q^\tau(\varphi_i(W_i),\mathbb{R}^N)} \\ &= \int\!\!\int_{\varphi_i(W_i)^2} \frac{|\psi_i(\varphi_i^{-1}(x))\, g(u(t,\varphi_i^{-1}(x))) - \psi_i(\varphi_i^{-1}(y))\, g(u(t,\varphi_i^{-1}(y)))|^q}{|x-y|^{n-1+\tau q}} \,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \sup_{\zeta\in B_u} |g'(\zeta)|^q \int\!\!\int_{\varphi_i(W_i)^2} \frac{|u(t,\varphi_i^{-1}(x)) - u(t,\varphi_i^{-1}(y))|^q}{|x-y|^{n-1+\tau q}} \,\mathrm{d}x\,\mathrm{d}y + |g(u)|^q_{C(\overline{J}\times\Gamma,\mathbb{R}^N)} \\ &\lesssim \sup_{\zeta\in B_u} |g'(\zeta)|^q [u(t,\cdot)]^q_{W_q^\tau(\Gamma,\mathbb{R}^N)} + |g(u)|^q_{C(\overline{J}\times\Gamma,\mathbb{R}^N)},\end{aligned}$$

where we have used Lemma A.4.1 in the last line. Summing over i, using the above estimates and taking the $L_{q,\mu}$-norm leads to

$$|g(u)|_{L_{q,\mu}(J;W_q^\tau(\Gamma,\mathbb{R}^N))} \lesssim \sup_{\xi\in B_u} |g'(\xi)| |u|_{L_{q,\mu}(J;W_q^\tau(\Gamma,\mathbb{R}^N))} + |g(u)|_{C(\overline{J}\times\Gamma,\mathbb{R}^N)},$$

which implies a).

(III) We next show differentiability of G. For $u \in \mathbb{E}_{u,\mu}(J)$ it follows from $2(\mu - 1/p) > 1 + n/p$, $p > n$ and Theorem 1.3.6 that

$$\mathrm{tr}_\Omega u \in W_{p,\mu}^{1-1/2p,2-1/p}(J \times \Gamma, \mathbb{R}^N) \hookrightarrow C(\overline{J} \times \Gamma, \mathbb{R}^N) \cap L_{p,\mu}(J; C^1(\Gamma, \mathbb{R}^N)).$$

Hence a) implies $g'(\mathrm{tr}_\Omega u) \in \mathbb{F}_\mu(J) \cap C(\overline{J} \times \Gamma, \mathbb{R}^N)$, and Lemma 1.3.23 yields $g'(\mathrm{tr}_\Omega u)\mathrm{tr}_\Omega \in \mathcal{B}(\mathbb{E}_{u,\mu}(J), \mathbb{F}_\mu(J))$. To show the differentiability of G at $u \in \mathbb{E}_{u,\mu}(J)$, take $h \in \mathbb{E}_{u,\mu}(J)$.

Arguing as in Step I of the proof of Lemma 4.2.2 we obtain that there is $\varepsilon : [0,\infty) \to [0,\infty)$ with $\varepsilon(0) = 0$, which is uniform in R for (4.2.8), such that[3]

$$|g(u(t,\cdot)+h(t,\cdot)) - g(u(t,\cdot)) - g'(u(t,\cdot))\,h(t,\cdot)|_{C^1(\Gamma,\mathbb{R}^N)} \\ \leq \varepsilon(|h(t,\cdot)|_{C^1(\Gamma,\mathbb{R}^N)})|h(t,\cdot)|_{C^1(\Gamma,\mathbb{R}^N)}$$

is valid for almost all $t \in \overline{J}$. Taking the $L_{p,\mu}$-norm and using $C^1(\Gamma,\mathbb{R}^N) \hookrightarrow W_p^{1-1/p}(\Gamma,\mathbb{R}^N)$ and (4.1.4) we obtain

$$\begin{aligned} |g(u+h) - g(u) - g'(u)h|_{L_{p,\mu}(J;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} &\leq \varepsilon(|h|_{C(\overline{J};C^1(\Gamma,\mathbb{R}^N))})|h|_{C(\overline{J};C^1(\Gamma,\mathbb{R}^N))} \\ &\lesssim \varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)}. \end{aligned}$$

Observe that these estimates are always uniform in $T \leq T_0$ and R if (4.2.8) holds.
(IV) For the intrinsic seminorm of $W_{p,\mu}^{1/2-1/2p}(J, L_p(\Gamma,\mathbb{R}^N))$, which is given by Proposition 1.1.13, we set

$$\Xi(t,x) := g(u(t,x)+h(t,x)) - g(u(t,x)) - g'(u(t,x))h(t,x)$$

and estimate, using Lemma 4.2.1,

$$\begin{aligned} &[g(u+h) - g(u) - g'(u)h]^p_{W_{p,\mu}^{1/2-1/2p}(J;L_p(\Gamma,\mathbb{R}^N))} \\ &\qquad = \int_0^T\int_0^s\int_\Gamma \frac{t^{p(1-\mu)}}{(s-t)^{1+(1/2-1/2p)p}}|\Xi(s,x)-\Xi(t,x)|^p \,\mathrm{d}\sigma(x)\,\mathrm{d}t\,\mathrm{d}s \\ &\qquad \leq \varepsilon(|h|_{C(\overline{J}\times\Gamma,\mathbb{R}^N)})\big([h]^p_{W^\kappa_{p,\mu}(J;L_p(\Gamma,\mathbb{R}^N))} + |h|^p_{C(\overline{J}\times\Gamma,\mathbb{R}^N)}[u]^p_{W^\kappa_{p,\mu}(J;L_p(\Gamma,\mathbb{R}^N))}\big) \\ &\qquad \lesssim \varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)}. \end{aligned} \tag{4.2.9}$$

Note that these estimates are also valid on $\mathbb{R}_+$. Together with the estimates of Step III, we obtain that G is differentiable at each $u \in \mathbb{E}_{u,\mu}(J)$. But since we have used the intrinsic norm over J, (4.2.9) does not yield an estimate uniformly in T in the ${}_0W_{p,\mu}^{1/2-1/2p}(J;L_p(\Gamma,\mathbb{R}^N))$-case for (4.2.8) (see also the discussion in Remark 1.1.15).
To overcome this obstacle, let $u,h \in \mathbb{E}_{u,\mu}(J)$ be as in (4.2.8). Due to Lemma 1.3.9 there is $u_* \in \mathbb{E}_{u,\mu}(\mathbb{R}_+)$ with

$$u_*(0,\cdot) = u(0,\cdot), \qquad |u_*|_{\mathbb{E}_{u,\mu}(\mathbb{R}_+)} \lesssim |u(0,\cdot)|_{W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N)}.$$

Using this function we define

$$\widetilde{u} := \mathcal{E}_J^0(u-u_*) + u_* \in \mathbb{E}_{u,\mu}(\mathbb{R}_+), \qquad \widetilde{h} := \mathcal{E}_J^0 h \in {}_0\mathbb{E}_{u,\mu}(\mathbb{R}_+),$$

where $\mathcal{E}_J^0$ is the extension operator from Lemma 1.1.5 whose norm is independent of T. Observe that

$$|\widetilde{u}|_{BC([0,\infty)\times\overline{\Omega},\mathbb{R}^N)} \lesssim |\widetilde{u}|_{\mathbb{E}_{u,\mu}(\mathbb{R}_+)} \lesssim R + |u(0,\cdot)|_{W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N)},$$

[3] In the sequel we neglect the spatial trace tr_Ω for better readability.

and, due to $h(0)=0$,

$$|\widetilde{h}|_{BC([0,\infty)\times\overline{\Omega})} \lesssim |\widetilde{h}|_{\mathbb{E}_{u,\mu}(\mathbb{R}_+)} \lesssim |h|_{\mathbb{E}_{u,\mu}(J)} \leq R,$$

where these estimates are independent of T. Therefore, if we apply Lemma 4.2.1 to g with arguments from the images $\widetilde{u}(\mathbb{R}_+\times\overline{\Omega})$ and $\widetilde{h}(\mathbb{R}_+\times\overline{\Omega})$, then the resulting functions ε will depend on a multiple of $R+|u(0,\cdot)|_{W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N)}$, but not on $T\leq T_0$. Thus, using Proposition 1.1.11 and (4.2.9) on the half-line, we may estimate

$$\begin{aligned}
&|g(u+h)-g(u)-g'(u)h|_{{}_0W_{p,\mu}^{1/2-1/2p}(J;L_p(\Gamma,\mathbb{R}^N))}\\
&\quad\leq |g(\widetilde{u}+\widetilde{h})-g(\widetilde{u})-g'(\widetilde{u})\widetilde{h}|_{{}_0W_{p,\mu}^{1/2-1/2p}(\mathbb{R}_+;L_p(\Gamma,\mathbb{R}^N))}\\
&\quad\lesssim |g(\widetilde{u}+\widetilde{h})-g(\widetilde{u})-g'(\widetilde{u})\widetilde{h}|_{W_{p,\mu}^{1/2-1/2p}(\mathbb{R}_+;L_p(\Gamma,\mathbb{R}^N))}\\
&\quad\leq \varepsilon(|\widetilde{h}|_{BC([0,\infty)\times\Gamma,\mathbb{R}^N)})\big([\widetilde{h}]_{W_{p,\mu}^{1/2-1/2p}(\mathbb{R}_+;L_p(\Gamma,\mathbb{R}^N))}\\
&\qquad\qquad + |\widetilde{h}|_{BC([0,\infty)\times\Gamma,\mathbb{R}^N)}[\widetilde{u}]_{W_{p,\mu}^{1/2-1/2p}(\mathbb{R}_+;L_p(\Gamma,\mathbb{R}^N))}\big)\\
&\quad\lesssim \varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)}\big(R+|u(0)|_{W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N)}\big),
\end{aligned}$$

where the function ε is uniform in $T\leq T_0$ and R. This shows c).

(V) For b) it is left to show that $G':\mathbb{E}_{u,\mu}(J)\to\mathcal{B}(\mathbb{E}_{u,\mu}(J),\mathbb{F}_\mu(J))$ is continuous. To this end take $u,v,h\in\mathbb{E}_{u,\mu}(J)$ with $|h|_{\mathbb{E}_{u,\mu}(J)}\leq 1$. Then we estimate, using Lemma 1.3.23,

$$\begin{aligned}
|(g'(u)-g'(v))h|_{\mathbb{F}_\mu(J)} &\lesssim |g'(u)-g'(v)|_{C(\overline{J}\times\Gamma,\mathbb{R}^{N\times N})} + |g'(u)-g'(v)|_{W_{p,\mu}^{1/2-1/2p}(J;L_p(\Gamma,\mathbb{R}^{N\times N}))}\\
&\quad + |g'(u)-g'(v)|_{L_{p,\mu}(J;W_p^{1-1/p}(\Gamma,\mathbb{R}^{N\times N}))}.
\end{aligned}$$

As $u\to v$ in $\mathbb{E}_{u,\mu}(J)$, the first summand converges to zero. For the second summand we use Lemma 4.2.1 and estimate

$$\begin{aligned}
[g'(u)-g'(v)]_{W_{p,\mu}^{1/2-1/2p}(J;L_p(\Gamma,\mathbb{R}^{N\times N}))} &\lesssim \varepsilon(|u-v|_{C(\overline{J}\times\Gamma,\mathbb{R}^N)})[v]_{W_{p,\mu}^{1/2-1/2p}(J;L_p(\Gamma,\mathbb{R}^N))}\\
&\quad + |u-v|_{W_{p,\mu}^{1/2-1/2p}(J;L_p(\Gamma,\mathbb{R}^N))}.
\end{aligned}$$

Here the right-hand side converges to zero as $u\to v$. Using $C^1(\Gamma,\mathbb{R}^N)\hookrightarrow W_p^{1-1/p}(\Gamma,\mathbb{R}^N)$, we obtain the same for the third summand. Thus b) is finally proved. ■

For the nonlinear boundary operator

$$B(u)=\alpha_{ij}\nu_i\mathrm{tr}_\Omega\partial_j u - a^{-1}(\mathrm{tr}_\Omega u)g(\mathrm{tr}_\Omega u)$$

the above lemma yields the following.

Lemma 4.2.4. *Let $J=(0,T)$ be finite, and let $p\in(n+2,\infty)$ and $\mu\in(1/p,1]$ be such that $2(\mu-1/p)>1+n/p$. Then $B\in C^1(\mathbb{E}_{u,\mu}(J),\mathbb{F}_\mu(J))$, with derivative*

$$B'(u)=\alpha_{ij}\nu_i\mathrm{tr}_\Omega\partial_j - \big(a^{-1}g\big)'(\mathrm{tr}_\Omega u)\mathrm{tr}_\Omega,\qquad u\in\mathbb{E}_{u,\mu}(J).$$

Further, let $T_0,R>0$ be given. Then there is a continuous function $\varepsilon:[0,\infty)\to[0,\infty)$ with $\varepsilon(0)=0$, such that for $T\leq T_0$ it holds

$$|B(u+h)-B(u)-B'(u)h|_{{}_0\mathbb{F}_\mu(J)}\leq\varepsilon(|h|_{\mathbb{E}_{u,\mu}(J)})|h|_{\mathbb{E}_{u,\mu}(J)}$$

for all $u,h\in\mathbb{E}_{u,\mu}(J)$ as in (4.2.8).

4.3 The Local Semiflow

For $p \in (1,\infty)$ and $s \in (1+n/p, 2-2/p]$ recall the nonlinear phase space

$$\mathcal{M}_p^s = \big\{u_0 \in W_p^s(\Omega,\mathbb{R}^N) \ : \ B(u_0) = 0\big\},$$

which is equipped with the metric from $W_p^s(\Omega,\mathbb{R}^N)$. We say that (4.1.1) generates a *compact local semiflow* of $\mathbb{E}_{u,\mu}$-solutions on $\mathcal{M}_p^s$ if the following three conditions are satisfied.

1. For all $u_0 \in \mathcal{M}_p^s$ there is $t^+(u_0) > 0$ such that (4.1.1) has a unique maximal solution $u(\cdot,u_0) \in C\big([0,t^+(u_0)); W_p^s(\Omega,\mathbb{R}^N)\big)$ which belongs to $\mathbb{E}_{u,\mu}(0,\tau)$ for all $\tau \in (0,t^+(u_0))$.

2. For all $u_0 \in \mathcal{M}_p^s$ and $\tau \in (0,t^+(u_0))$ there is $r > 0$ such that $t^+(v_0) > \tau$ for all $v_0 \in B_r(u_0) \cap \mathcal{M}_p^s$, and the map $u(\tau,\cdot) : B_r(u_0) \cap \mathcal{M}_p^s \to \mathcal{M}_p^s$ is continuous.

3. If for a bounded set $M \subset \mathcal{M}_p^s$ there is $\tau > 0$ such that $t^+(v_0) > \tau$ for all $v_0 \in M$, then $u(\tau, M)$ is relatively compact in $\mathcal{M}_p^s$.

To verify the first condition for (4.1.1) we consider the linear initial-boundary value problem associated to $(A'(u), B'(u))$, and show that it enjoys maximal $L_{p,\mu}$-regularity for each $u \in \mathbb{E}_{u,\mu}(J)$.

Lemma 4.3.1. *Let $J = (0,T)$ be a finite interval, and let $p \in (n+2,\infty)$ and $\mu \in (1/p, 1]$ be such that*

$$s := 2(\mu - 1/p) > 1 + n/p.$$

Assume further that (4.1.2) is valid, and let the function $u \in \mathbb{E}_{u,\mu}(J)$ be given. Denote by

$$\mathcal{D}_u(J) := \big\{(\widetilde{f},\widetilde{g},\widetilde{v}_0) \in \mathbb{E}_{0,\mu}(J) \times \mathbb{F}_\mu(J) \times W_p^s(\Omega,\mathbb{R}^N) \ : \ B'(u(0,\cdot))\widetilde{v}_0 = \widetilde{g}_0 \quad \text{on} \quad \Gamma\big\}$$

the space of compatible data with respect to $(A'(u), B'(u))$. Then there exists a bounded linear solution operator $\mathcal{L} : \mathcal{D}_u(J) \to \mathbb{E}_{u,\mu}(J)$ for

$$\begin{aligned} \partial_t v + A'(u(t,x))v &= \widetilde{f}(t,x), && x \in \Omega, && t \in J,\\ B'(u(t,x))v &= \widetilde{g}(t,x), && x \in \Gamma, && t \in J,\\ v(0,x) &= \widetilde{v}_0(x), && x \in \Omega. \end{aligned}$$

Given $T_0 > 0$, the operator norm of $\mathcal{L}$ restricted to

$$\mathcal{D}_u^0(J) := \big\{(\widetilde{f},\widetilde{g},\widetilde{v}_0) \in \mathcal{D}_u(J) \ : \ \widetilde{g} \in {}_0\mathbb{F}_\mu(J)\big\}$$

is uniform in $T \le T_0$.

Proof. We check that $(A'(u), B'(u))$ satisfies the assumptions of Theorem 2.1.4. Since $u \in C(\overline{J}; C^1(\overline{\Omega},\mathbb{R}^N))$ by (4.1.4), the top order coefficients of $A'(u)$ belong to $BUC(\overline{J} \times \overline{\Omega}, \mathbb{R}^{N\times N})$. The lower order coefficients belong to $\mathbb{E}_{0,\mu}(J;\mathbb{R}^{N\times N})$. Moreover, Lemma 4.2.3 implies that the coefficients of $B'(u)$ belong to $\mathbb{F}_\mu(J;\mathbb{R}^{N\times N})$. Since the condition $1/2 -$

$1/2p > 1-\mu+1/p+\frac{n-1}{2p}$ is equivalent to $2(\mu-1/p) > 1+n/p$, we obtain that the coefficients satisfy the first conditions in (SD) and (SB), respectively.
It remains to check the ellipticity conditions. To this end consider the pair $(\mathcal{A}(u),\mathcal{B})$, given by

$$\mathcal{A}(u)v := \partial_i\big(a_{ij}(u)\partial_j v\big), \qquad \mathcal{B}v := \alpha_{ij}\nu_i \mathrm{tr}_\Omega \partial_j v, \qquad v\in \mathbb{E}_{u,\mu}(J).$$

It is shown in [5, Theorem 4.4] that (4.1.2) implies (E) and (LS) for $(\mathcal{A}(\zeta),\mathcal{B})$, $\zeta\in\mathbb{R}^N$. Since these conditions are independent of the lower order terms it follows that $(A'(u),B'(u))$ satisfies (E) and (LS) as well. Thus all the assumptions of Theorem 2.1.4 are satisfied, and the assertion follows. ■

Now we can prove local existence and uniqueness for solutions of (4.1.1). Our proof is based on maximal $L_{p,\mu}$-regularity for the linearized problem and the contraction principle, and follows [90] (see also [59, 65]).

Proposition 4.3.2. *Let $p\in(n+2,\infty)$ and $\mu\in(1/p,1]$ be such that $s=2(\mu-1/p) > 1+n/p$, and assume that (4.1.2) is valid. Then for each initial value $u_0\in W_p^s(\Omega,\mathbb{R}^N)$ with*

$$a_{ij}(u_0)\nu_i\partial_j u_0 = g(u_0) \qquad \text{on } \Gamma \tag{4.3.1}$$

the system (4.1.1) has a unique maximal solution

$$u(\cdot,u_0)\in C\big([0,t^+(u_0));W_p^s(\Omega,\mathbb{R}^N)\big),$$

which belongs to $\mathbb{E}_{u,\mu}(0,\tau)$ for all $\tau\in(0,t^+(u_0))$.

Proof. We rewrite (4.1.1) into the equivalent form

$$\begin{aligned} \partial_t u + A(u) &= 0 && \text{in } \Omega, && t>0,\\ B(u) &= 0 && \text{on } \Gamma, && t>0,\\ u(0,\cdot) &= u_0 && \text{in } \Omega, \end{aligned} \tag{4.3.2}$$

where A and B were defined in the beginning of the previous section. Note that the condition (4.3.1) on u_0 is equivalent to $B(u_0)=0$. Throughout the proof we fix $u_*\in\mathbb{E}_{u,\mu}(\mathbb{R}_+)$ with $u_*(0,\cdot)=u_0$, which exists by Lemma 1.3.9.
(I) We consider the linear problem

$$\begin{aligned} \partial_t w + A'(u_*)w &= A'(u_*)u_* - A(u_*) && \text{in } \Omega, && t>0,\\ B'(u_*)w &= B'(u_*)u_* - B(u_*) && \text{on } \Gamma, && t>0,\\ w(0,\cdot) &= u_0 && \text{in } \Omega. \end{aligned} \tag{4.3.3}$$

Due to the Lemmas 4.2.2 and 4.2.4 it holds

$$A'(u_*)u_* - A(u_*)\in\mathbb{E}_{0,\mu}(0,1), \qquad B'(u_*)u_* - B(u_*)\in\mathbb{F}_\mu(0,1),$$

and since $B(u_0)=0$ the compatibility condition

$$B'(u_0)u_0 = B'(u_0)u_0 - B(u_0) \qquad \text{on } \Gamma$$

is trivially satisfied. Thus Lemma 4.3.1 yields a unique solution $w_* \in \mathbb{E}_{u,\mu}(0,1)$ of (4.3.3). Using w_*, we define for $\sigma, \tau \in (0,1]$

$$\Sigma(\sigma,\tau) := \big\{ u \in \mathbb{E}_{u,\mu}(0,\tau) \ : \ |u - w_*|_{\mathbb{E}_{u,\mu}} \le \sigma, \ \ u(0,\cdot) = u_0 \big\}.$$

The set $\Sigma(\sigma,\tau)$ is closed in $\mathbb{E}_{u,\mu}(0,\tau)$. Moreover, (4.1.4) implies

$$|u|_{C([0,\tau];C^1(\overline{\Omega},\mathbb{R}^N))}, |u(0,\cdot)|_{W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N)}, |u|_{\mathbb{E}_{u,\mu}(0,\tau)} \lesssim 1 + |w_*|_{\mathbb{E}_{u,\mu}(0,1)}, \tag{4.3.4}$$

uniformly in $u \in \Sigma(\sigma,\tau)$ and $\sigma,\tau \in (0,1]$. For $u \in \Sigma(\sigma,\tau)$ we next consider

$$\begin{aligned} \partial_t w + A'(u_*)w &= A'(u_*)u - A(u) && \text{in } \Omega \times (0,\tau), \\ B'(u_*)w &= B'(u_*)u - B(u) && \text{on } \Gamma \times (0,\tau), \\ w(0,\cdot) &= u_0 && \text{in } \Omega. \end{aligned} \tag{4.3.5}$$

As above, for all $\tau \in (0,1]$ there is a unique solution $w = \mathcal{S}(u) \in \mathbb{E}_{u,\mu}(0,\tau)$ of (4.3.5) due to Lemma 4.3.1. This defines a map

$$\mathcal{S} : \Sigma(\sigma,\tau) \to \mathbb{E}_{u,\mu}(0,\tau).$$

Observe that $u \in \Sigma(\sigma,\tau)$ solves (4.3.2) on $(0,\tau)$ if and only if it is a fixed point of $\mathcal{S}$ in $\Sigma(\sigma,\tau)$. Since for given σ each solution of (4.3.2) in $\mathbb{E}_{u,\mu}(0,\tau)$ belongs to $\Sigma(\sigma,\tau)$ for sufficiently small τ, our task is thus to show that the map $\mathcal{S}$ has a unique fixed point in $\Sigma(\sigma,\tau)$, provided that σ and τ are sufficiently small. To this end we use the contraction principle. The existence of a maximal existence time and a maximal solution in $C([0,t^+(u_0)); W_p^s(\Omega,\mathbb{R}^N))$ then follows from standard arguments.
(II) First we show that $\mathcal{S}$ is a self map on $\Sigma(\sigma,\tau)$ for small σ and τ. For $u \in \Sigma(\sigma,\tau)$ the difference $z = \mathcal{S}(u) - w_*$ solves

$$\begin{aligned} \partial_t z + A'(u_*)z &= A(u_*) - A(u) - A'(u_*)(u_* - u) && \text{in } \Omega \times (0,\tau), \\ B'(u_*)z &= B(u_*) - B(u) - B'(u_*)(u_* - u) && \text{on } \Gamma \times (0,\tau), \\ z(0,\cdot) &= 0 && \text{in } \Omega. \end{aligned}$$

Note that the right-hand side of the boundary equation belongs to ${}_0\mathbb{F}_\mu(0,\tau)$. Thus by Lemma 4.3.1 there is a constant C_0, independent of τ, such that

$$\begin{aligned} |\mathcal{S}(u) - w_*|_{\mathbb{E}_{u,\mu}(0,\tau)} \le C_0\big(&|A(u_*) - A(u) - A'(u_*)(u_* - u)|_{\mathbb{E}_{0,\mu}(0,\tau)} \\ &+ |B(u_*) - B(u) - B'(u_*)(u_* - u)|_{{}_0\mathbb{F}_\mu(0,\tau)}\big). \end{aligned} \tag{4.3.6}$$

As above it holds

$$|u_* - u|_{\mathbb{E}_{u,\mu}(0,\tau)} \lesssim \sigma + |w_* - u_*|_{\mathbb{E}_{u,\mu}(0,1)}, \tag{4.3.7}$$

uniformly in $u \in \Sigma(\sigma,\tau)$ and $\tau \in (0,1]$. Using this fact together with Lemma 4.2.2 we obtain that the first summand in (4.3.6) may be estimated by

$$\begin{aligned} |A(u_*) - A(u) - A'(u_*)(u_* - u)|_{\mathbb{E}_{0,\mu}(0,\tau)} &\le \varepsilon(|u_* - u|_{\mathbb{E}_{u,\mu}(0,\tau)})|u_* - u|_{\mathbb{E}_{u,\mu}(0,\tau)} \\ &\le \varepsilon(|u_* - w_*|_{\mathbb{E}_{u,\mu}(0,\tau)} + \sigma)(|u_* - w_*|_{\mathbb{E}_{u,\mu}(0,\tau)} + \sigma), \end{aligned}$$

where $\varepsilon : [0,\infty) \to [0,\infty)$ is a continuous function with $\varepsilon(0) = 0$, which is independent of $\sigma, \tau \in (0,1]$. We first choose σ with $\varepsilon(2\sigma) \le 1/4C_0$ and then τ such that the fixed functions u_* and w_* satisfy $|u_* - w_*|_{\mathbb{E}_{u,\mu}(0,\tau)} \le \sigma$. Then we obtain

$$|A(u_*) - A(u) - A'(u_*)(u_* - u)|_{\mathbb{E}_{0,\mu}(0,\tau)} \le \sigma/2C_0.$$

Similarly, using (4.3.7) and Lemma 4.2.4 we obtain for the second summand in (4.3.6) that

$$|B(u_*) - B(u) - B'(u_*)(u_* - u)|_{{}_0\mathbb{F}_\mu(0,\tau)} \le \sigma/2C_0$$

as well, provided σ and τ are sufficiently small. This shows that $\mathcal{S}$ is a self mapping on $\Sigma(\sigma,\tau)$ if σ and τ are appropriately chosen.

(III) We show that $\mathcal{S}$ is a strict contraction on $\Sigma(\sigma,\tau)$. For $u, v \in \Sigma(\sigma,\tau)$ we have as above

$$\begin{aligned} |\mathcal{S}(u) - \mathcal{S}(v)|_{\mathbb{E}_{u,\mu}(0,\tau)} \le C_0\big(&|A'(u_*)u - A(u) - A'(u_*)v + A(v)|_{\mathbb{E}_{0,\mu}(0,\tau)} \\ &+ |B'(u_*)u - B(u) - B'(u_*)v + B(v)|_{{}_0\mathbb{F}_\mu(0,\tau)}\big). \end{aligned} \tag{4.3.8}$$

Using (4.3.4), (4.3.7) and Lemma 4.2.2, we estimate the first summand in (4.3.8) by

$$\begin{aligned} |A'(u_*)u &- A(u) - A'(u_*)v + A(v)|_{\mathbb{E}_{0,\mu}(0,\tau)} \\ &\le |A(v) - A(u) - A'(u)(v-u)|_{\mathbb{E}_{0,\mu}(0,\tau)} + |\big(A'(u_*) - A'(u)\big)(u-v)|_{\mathbb{E}_{0,\mu}(0,\tau)} \\ &\le \big(\varepsilon(|v-u|_{\mathbb{E}_{u,\mu}(0,\tau)}) + |A'(u_*) - A'(u)|_{\mathcal{B}(\mathbb{E}_{u,\mu}(0,\tau),\mathbb{E}_{0,\mu}(0,\tau))}\big)|v-u|_{\mathbb{E}_{u,\mu}(0,\tau)} \\ &\le \big(\varepsilon(2\sigma) + \varepsilon(\sigma + |w_* - u_*|_{\mathbb{E}_{u,\mu}(0,\tau)})\big)|v-u|_{\mathbb{E}_{u,\mu}(0,\tau)}, \end{aligned}$$

where ε is a function as above, independent of $\sigma, \tau \in (0,1]$. Thus if σ and τ are sufficiently small we obtain

$$|A'(u_*)u - A(u) - A'(u_*)v + A(v)|_{\mathbb{E}_{0,\mu}(0,\tau)} \le 1/4C_0|v-u|_{\mathbb{E}_{u,\mu}(0,\tau)}.$$

Using Lemma 4.2.4, in the same way we obtain for the second summand in (4.3.8) that

$$|B'(u_*)u - B(u) - B'(u_*)v + B(v)|_{{}_0\mathbb{F}_\mu(0,\tau)} \le 1/4C_0|v-u|_{\mathbb{E}_{u,\mu}(0,\tau)}.$$

This shows that $\mathcal{S}$ is a strict contraction on $\Sigma(\sigma,\tau)$ if σ and τ are sufficiently small. ■

Before we treat the continuous dependence on the initial values we need another preparatory result on the boundary operator B.

Lemma 4.3.3. *Let $p \in (n+2,\infty)$ and $\mu \in (1/p,1]$ satisfy $2(\mu - 1/p) > 1 + n/p$. Then we have*

$$B \in C^1\big(W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N), W_p^{2(\mu-1/p)-1-1/p}(\Gamma,\mathbb{R}^N)\big),$$

with derivative

$$B'(u_0) = \alpha_{ij}\nu_i \mathrm{tr}_\Omega \partial_j - \big(a^{-1}g\big)'(\mathrm{tr}_\Omega u_0)\mathrm{tr}_\Omega \qquad \text{for } u_0 \in W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N).$$

Further, if (4.1.2) is valid, then for each u_0 the map $B'(u_0)$ is surjective with bounded linear right-inverse.

Proof. By Lemma 1.3.9, for all $u_0 \in W_p^{2(\mu-1/p)}(\Omega,\mathbb{R}^N)$ there is $u_* \in \mathbb{E}_{u,\mu}(0,1)$ with $u_*(0,\cdot) = u_0$, which depends smoothly on u_0. It thus follows from $B(u_0) = \mathrm{tr}_0 B(u_*)$, Lemma 4.2.4 and Theorem 1.3.6 that B is C^1, with derivative as asserted.
For the right-inverse of $B'(u_0)$ we intend to use Proposition 2.5.1. Consider the operators

$$\mathcal{A} := \alpha_{ij}\partial_i\partial_j, \qquad \mathcal{B} := \alpha_{ij}\nu_i \mathrm{tr}_\Omega \partial_j.$$

Then (4.1.2) and [5, Theorem 4.4] yield that $(\mathcal{A},\mathcal{B})$ satisfies (E) and (LS), and thus also $(\mathcal{A}, B'(u_0))$ satisfies (E) and (LS). For the regularity of the coefficients of $B'(u_0)$, one can show as in Step II of the proof of Lemma 4.2.3 that

$$\big(a^{-1}g\big)'(\mathrm{tr}_\Omega u_0) \in W_p^{2(\mu-1/p)-1/p}(\Gamma,\mathbb{R}^{N\times N}) \hookrightarrow W_p^{2(\kappa-(1-\mu+1/p))}(\Gamma,\mathbb{R}^{N\times N}),$$

where $\kappa = 1/2 - 1/2p$. Thus $B'(u_0)$ satisfies the assumptions of Proposition 2.5.1, and the existence of a continuous right-inverse follows. It is clear that for real-valued u_0, a and g this right-inverse maps into a space of real-valued functions. ∎

The following result on the continuous dependence of solutions on the initial data is based on a combination of maximal $L_{p,\mu}$-regularity and the implicit function theorem. We follow the proof of [65, Theorem 14].

Proposition 4.3.4. *In the setting of Proposition 4.3.2, let $u = u(\cdot,u_0)$ be the maximal solution of (4.1.1) with initial value $u_0 \in \mathcal{M}_p^s$. Then for all $\tau \in (0,t^+(u_0))$ there is a ball $B_r(u_0)$ in $W_p^s(\Omega,\mathbb{R}^N)$, $r>0$, and a continuous map*

$$\Phi : B_r(u_0)\cap\mathcal{M}_p^s \to \mathbb{E}_{u,\mu}(0,\tau), \qquad \Phi(u_0) = u,$$

such that $\Phi(v_0)$ is the solution of (4.1.1) on $(0,\tau)$ with initial value $v_0 \in B_r(u_0)\cap\mathcal{M}_p^s$.

Proof. (I) Take $p \in (n+2,\infty)$ and $\mu \in (1/p,1]$ with $s = 2(\mu-1/p)$, such that $u \in \mathbb{E}_{u,\mu}(0,\tau)$. We consider the linear problem

$$\begin{aligned} \partial_t z + A'(u(t,x))z &= \widetilde{f}(t,x), && x\in\Omega, && t\in(0,\tau),\\ B'(u(t,x))z &= \widetilde{g}(t,x), && x\in\Gamma, && t\in(0,\tau),\\ z(0,x) &= \widetilde{w}_0(x), && x\in\Omega, \end{aligned} \tag{4.3.9}$$

and denote by

$$\mathcal{S} : \mathcal{D}_u(0,\tau) \to \mathbb{E}_{u,\mu}(0,\tau)$$

the bounded linear solution operator corresponding to (4.3.9) from Lemma 4.3.1. We have that $v \in \mathbb{E}_{u,\mu}(0,\tau)$ solves (4.1.1) (and the rewritten problem (4.3.2)) with initial value $v_0 \in \mathcal{M}_p^s$ if and only if

$$v = u + \mathcal{S}\big(F(v-u), G(v-u), v_0-u_0\big), \tag{4.3.10}$$

where the nonlinear functions F and G are given by

$$F(w) := -\big(A(u+w) - A(u) - A'(u)w\big), \qquad G(w) := -\big(B(u+w) - B(u) - B'(u)w\big).$$

Due to the Lemmas 4.2.2 and 4.2.4 it holds

$$F \in C^1\big(\mathbb{E}_{u,\mu}(0,\tau), \mathbb{E}_{0,\mu}(0,\tau)\big), \qquad G \in C^1\big(\mathbb{E}_{u,\mu}(0,\tau), \mathbb{F}_{\mu}(0,\tau)\big).$$

(II) We define the tangential space of $\mathcal{M}_p^s$ at u_0 by

$$T_{u_0}\mathcal{M}_p^s := \big\{z_0 \in W_p^s(\Omega, \mathbb{R}^N) \ : \ B'(u_0)z_0 = 0\big\}.$$

This is the kernel of the bounded linear operator $B'(u_0)$ in $W_p^s(\Omega, \mathbb{R}^N)$, and thus a Banach space. We further consider the nonlinear map

$$\mathcal{F} : T_{u_0}\mathcal{M}_p^s \times \mathbb{E}_{u,\mu}(0,\tau) \to \mathbb{E}_{u,\mu}(0,\tau),$$

defined by

$$\mathcal{F}(z_0, w) := w - \mathcal{S}\big(F(w), G(w), z_0 + \mathcal{N}_s \mathrm{tr}_0 G(w)\big).$$

Here $\mathcal{N}_s \in \mathcal{B}\big(W_p^{s-1-1/p}(\Gamma, \mathbb{R}^N), W_p^s(\Omega, \mathbb{R}^N)\big)$ denotes the continuous right-inverse of $B'(u_0)$ from Lemma 4.3.3, and tr_0 is the temporal trace at $t = 0$, i.e., $\mathrm{tr}_0 w = w(0,\cdot)$. The map $\mathcal{F}$ is well defined, since due to

$$B'(u_0)\big(z_0 + \mathcal{N}_s \mathrm{tr}_0 G(w))\big) = \mathrm{tr}_0 G(w)$$

only compatible data are inserted into $\mathcal{S}$. It further holds $\mathcal{F}(0,0) = 0$ and that $\mathcal{F}$ is continuously differentiable. The derivative of $\mathcal{F}$ with respect to the second argument at $(z_0, w) = (0,0)$ is given by

$$\begin{aligned} &\partial_2 \mathcal{F}(0,0) \\ &\quad = \mathcal{S}\big(A'(u+w) - A'(u), B'(u+w) - B'(u), \mathcal{N}_s \mathrm{tr}_0(B'(u+w) - B'(u))\big)|_{w=0} = \mathrm{id}, \end{aligned}$$

and is therefore invertible. Thus we can solve the nonlinear equation $\mathcal{F}(z_0, w) = 0$ locally around $(0,0)$ uniquely by $w = \Phi_*(z_0)$ with a C^1-function $\Phi_* : B_r(0) \to \mathbb{E}_{u,\mu}(0,\tau)$, where $B_r(0) \subset T_{u_0}\mathcal{M}_p^s$ and $r > 0$ is small.

(III) Now let $v_0 \in \mathcal{M}_p^s$ be given, and define

$$z_0 := \big(\mathrm{id} - \mathcal{N}_s B'(u_0)\big)(v_0 - u_0) \in T_{u_0}\mathcal{M}_p^s.$$

By continuity of $\mathrm{id} - \mathcal{N}_s B'(u_0)$, if v_0 is close to u_0 in $\mathcal{M}_p^s$ then the norm of z_0 in $W_p^s(\Omega, \mathbb{R}^N)$ is small, such that $w = \Phi_*(z_0) \in \mathbb{E}_{u,\mu}(J)$ is well-defined and satisfies

$$w = \mathcal{S}\big(F(w), G(w), v_0 - u_0 - \mathcal{N}_s(B'(u_0)(v_0 - u_0) - \mathrm{tr}_0 G(w))\big).$$

Due to $\mathrm{tr}_0 G(w) = -B(u_0 + w(0,\cdot)) + B'(u_0)(w(0,\cdot))$, the continuity of $\mathcal{N}_s$, $B(v_0) = 0$ and Lemma 4.3.3 yield

$$\begin{aligned} &|w(0,\cdot) - (v_0 - u_0)|_{W_p^s(\Omega,\mathbb{R}^N)} \\ &\qquad = |\mathcal{N}_s\big(B(u_0 + w(0,\cdot)) - B'(u_0)(w(0,\cdot) - (v_0 - u_0))\big)|_{W_p^s(\Omega,\mathbb{R}^N)} \\ &\qquad \lesssim |B(u_0 + w(0,\cdot)) - B(v_0) - B'(v_0)(w(0,\cdot) - (v_0 - u_0))|_{W_p^{s-1-1/p}(\Omega,\mathbb{R}^N)} \\ &\qquad\qquad + |\big(B'(v_0) - B'(u_0)\big)(w(0,\cdot) - (v_0 - u_0))|_{W_p^{s-1-1/p}(\Omega,\mathbb{R}^N)} \\ &\qquad \le \varepsilon\big(|w(0,\cdot) - (v_0 - u_0)|_{W_p^s(\Omega,\mathbb{R}^N)} + |v_0 - u_0|_{W_p^s(\Omega,\mathbb{R}^N)}\big)|w(0,\cdot) - (v_0 - u_0)|_{W_p^s(\Omega,\mathbb{R}^N)}, \end{aligned}$$

with $\varepsilon(0) = 0$. Since Φ_* is continuous and satisfies $\Phi_*(0) = 0$, if v_0 tends to u_0 then $|w(0,\cdot)|_{W_p^s(\Omega,\mathbb{R}^N)}$ tends to zero. Thus for v_0 sufficiently close to u_0 the above inequality is only possible if $w(0,\cdot) = v_0 - u_0$. This implies that the function $v = u + w \in \mathbb{E}_{u,\mu}(0,\tau)$ solves (4.3.10), and therefore (4.1.1) with initial value v_0. Now

$$\Phi(v_0) := u + \Phi_*\big((\mathrm{id} - \mathcal{N}_s B'(u_0))(v_0 - u_0)\big)$$

is the asserted continuous solution map for (4.1.1) on $B_r(u_0) \cap \mathcal{M}_p^s$. ■

The above result in particular shows that (4.1.1) satisfies also the second condition for a compact local semiflow. We now prove the required compactness property of the solution map, employing the inherent smoothing effect of the $L_{p,\mu}$-spaces. Our arguments are similar to those in Section 3 of the recent paper [59].

Proposition 4.3.5. *In the setting of Proposition 4.3.2, let the bounded set $M \subset \mathcal{M}_p^s$ and $\tau > 0$ satisfy $t^+(v_0) > \tau$ for all $v_0 \in M$. Then $u(\tau, M)$ is relatively compact in $\mathcal{M}_p^s$.*

Proof. (I) It follows from the compactness of the embedding $W_p^1(\Omega,\mathbb{R}^N) \hookrightarrow L_p(\Omega,\mathbb{R}^N)$, cf. [1, Theorem 6.3], and the interpolation result in [7, Section I.2.7] that for $s_* \in (1+n/p, s)$ the embedding

$$W_p^s(\Omega,\mathbb{R}^N) \hookrightarrow W_p^{s_*}(\Omega,\mathbb{R}^N)$$

is compact. Therefore M is relatively compact in $W_p^{s_*}(\Omega,\mathbb{R}^N)$. Take $\mu_* \in (1/p, 1]$ with $s_* = 2(\mu_* - 1/p)$. Due to Proposition 4.3.4, for each $v_0 \in M$ there is a ball $B_r(v_0)$ in $W_p^{s_*}(\Omega,\mathbb{R}^N)$ and a continuous map

$$\Phi : B_r(v_0) \cap \mathcal{M}_p^{s_*} \to \mathbb{E}_{u,\mu_*}(0,\tau)$$

such that $w = \Phi(w_0) \in \mathbb{E}_{u,\mu_*}(0,\tau)$ solves (4.1.1) with initial value $w_0 \in B_r(v_0) \cap \mathcal{M}_p^{s_*}$. This yields an open cover of M in $W_p^{s_*}(\Omega,\mathbb{R}^N)$, and thus, by compactness, there are finitely many balls B_k and maps Φ_k with the above property such that $\bigcup_k B_k$ covers M.
(II) Each Φ_k maps the relatively compact set $B_k \cap M$ continuously into $\mathbb{E}_{u,\mu_*}(0,\tau)$, with

$$\Phi_k(w_0) = u(\cdot, w_0)|_{(0,\tau)}, \qquad w_0 \in B_k \cap M.$$

Since the temporal trace

$$\mathrm{tr}_\tau : \mathbb{E}_{u,\mu_*}(0,\tau) \to W_p^{2-2/p}(\Omega,\mathbb{R}^N), \qquad \mathrm{tr}_\tau w = w(\tau,\cdot),$$

is continuous, we obtain that

$$u(\tau, M) = \bigcup_k \mathrm{tr}_\tau \circ \Phi_k(B_k \cap M)$$

is relatively compact in $W_p^s(\Omega,\mathbb{R}^N)$, as a continuous image of a relatively compact set. ■

We summarize the above considerations to the main result of this section.

Theorem 4.3.6. *Let $p \in (n+2, \infty)$, $s \in (1+n/p, 2-2/p]$ and $\mu \in (1/p, 1]$ satisfy $s = 2(\mu - 1/p)$, and assume that (4.1.2) holds true. Then the system*

$$\begin{aligned} \partial_t u - \partial_i(a_{ij}(u)\partial_j u) &= f(u) \qquad \text{in } \Omega, \qquad t > 0, \\ a_{ij}(u)\nu_i \partial_j u &= g(u) \qquad \text{on } \Gamma, \qquad t > 0, \\ u(0, \cdot) &= u_0 \qquad\;\; \text{in } \Omega, \end{aligned}$$

generates a compact local semiflow of $\mathbb{E}_{u,\mu}$-solutions on the phase space $\mathcal{M}_p^s$.

Remark 4.3.7. The methods in this section are independent of the concrete form of the nonlinear operators A and B, as long as they are C^1 and Theorem 2.1.4 is applicable to the corresponding linearized problem. Thus a compact local semiflow in a scale of nonlinear phase spaces can be obtain for much more general parabolic systems with nonlinear boundary conditions, as treated in [65], for instance.

4.4 Global Attractors in Stronger Norms

We now fix $p \in (n+2, \infty)$ and investigate the long-time behaviour of solutions of (4.1.1) for initial values from $\mathcal{M}_p^{2-2/p}$. Using the full strength of maximal $L_{p,\mu}$-regularity we estimate solutions of (4.1.1) at a later time in a strong norm by the solution at an earlier time in a weaker norm. This builds the bridge from lower to higher regularity, and is the key to global attractors in stronger norms.

Lemma 4.4.1. *Let $u_0 \in \mathcal{M}_p^{2-2/p}$, and denote by $u(\cdot, u_0)$ the maximal solution of (4.1.1). Let $q \in (1, p]$, $\mu \in (1/q, 1]$, set*

$$\sigma := 2(\mu - 1/q) \in (0, 2-2/q],$$

and assume that $\sigma \notin \{1, 1+1/q\}$. Let further $\tau > 0$, $0 < T_1 < T_2 < t^+(u_0)$ and $\tau = T_2 - T_1$. Then for $\alpha > 0$ there is a constant $C = C\big(\tau, \alpha, |u(\cdot, u_0)|_{C([T_1,T_2],C^\alpha(\overline{\Omega},\mathbb{R}^N))}\big)$ with

$$|u(T_2, u_0)|_{W_q^{2-2/q}(\Omega,\mathbb{R}^N)} \le C\big(1 + |u(T_1, u_0)|_{W_q^\sigma(\Omega,\mathbb{R}^N)}\big). \tag{4.4.1}$$

In the semilinear case, i.e., if (a_{ij}) does not depend on u, one may take $\alpha = 0$.

Proof. Throughout we set $J := (0, \tau)$. The spaces $\mathbb{E}_{0,\mu}$, $\mathbb{E}_{u,\mu}$ and $\mathbb{F}_\mu$ must now be understood with respect to q, e.g., $\mathbb{E}_{0,\mu}(J) = L_{q,\mu}(J; L_q(\Omega, \mathbb{R}^N))$.
(I) Define the function $v \in W_p^1(J; L_p(\Omega, \mathbb{R}^N)) \cap L_p(J; W_p^2(\Omega; \mathbb{R}^N))$ by

$$v(t) := u(t + T_1, u_0), \qquad t \in J.$$

Since the weight only has an effect at $t = 0$, we have

$$|u(T_2, u_0)|_{W_q^{2-2/q}(\Omega,\mathbb{R}^N)} = |v(\tau)|_{W_q^{2-2/q}(\Omega,\mathbb{R}^N)} \lesssim |v|_{\mathbb{E}_{u,\mu}(J)}, \tag{4.4.2}$$

in dependence on τ. Moreover, the function v solves the nonautonomous, inhomogeneous linear problem

$$\begin{aligned} \partial_t w - a_{ij}(v)\partial_i\partial_j w &= a'_{ij}(v)\partial_i v\partial_j v + f(v) && \text{in } \Omega, && t\in J,\\ \alpha_{ij}\nu_i\partial_j w &= a^{-1}(v)g(v) && \text{on } \Gamma, && t\in J,\\ w(0,\cdot) &= u(T_1,u_0) && \text{in } \Omega. \end{aligned}$$

It follows from Theorem 2.1.4, localization arguments similar to those in the proof of Proposition 2.3.1 and compactness that there is a constant C, which is uniform in $|u|_{C([T_1,T_2]\times\overline{\Omega},\mathbb{R}^N)}$ and τ, such that

$$|v|_{\mathbb{E}_{u,\mu}(J)} \le C\big(|a'_{ij}(v)\partial_i v\partial_j v|_{\mathbb{E}_{0,\mu}(J)} + |f(v)|_{\mathbb{E}_{0,\mu}(J)} + |a^{-1}(v)g(v)|_{\mathbb{F}_\mu(J)} + |u(T_1,u_0)|_{W_q^\sigma(\Omega,\mathbb{R}^N)}\big). \tag{4.4.3}$$

(II) Using Hölder's inequality we estimate for the first summand in (4.4.3)

$$\begin{aligned} |a'_{ij}(v)\partial_i v\partial_j v|^q_{\mathbb{E}_{0,\mu}(J)} &\lesssim_{|u|_{C([T_1,T_2]\times\overline{\Omega},\mathbb{R}^N)}} |\partial_i v\partial_j v|^q_{\mathbb{E}_{0,\mu}(J)}\\ &\le \big||\partial_i v|_{L_{2q}(\Omega,\mathbb{R}^N)}|\partial_j v|_{L_{2q}(\Omega,\mathbb{R}^N)}\big|^q_{L_{q,\mu}(J)} \le \int_J t^{q(1-\mu)}|v(t)|^{2q}_{W^1_{2q}(\Omega,\mathbb{R}^N)}\,\mathrm{d}t. \end{aligned}$$

By the Gagliardo-Nirenberg inequality (Proposition A.6.2) we have for all $t\in J$ that

$$|v(t)|^{2q}_{W^1_{2q}(\Omega,\mathbb{R}^N)} \lesssim |v(t)|^q_{W^\vartheta_q(\Omega,\mathbb{R}^N)}\,|v(t)|^q_{W^\tau_r(\Omega,\mathbb{R}^N)}$$

for $r\in(1,\infty)$ and $\vartheta,\tau>0$, provided $1-\frac{n}{2q} < \frac12\big(\tau-\frac{n}{r}\big) + \frac12\big(\vartheta-\frac{n}{q}\big)$. For given α it holds $C^\alpha(\overline{\Omega},\mathbb{R}^N)\hookrightarrow W^\tau_r(\Omega,\mathbb{R}^N)$ for $\tau\in(0,\alpha)$ and $r\in(1,\infty)$. Thus if $\vartheta<2$ is sufficiently close to 2 and r is large we obtain from the interpolation inequality and Young's inequality

$$\begin{aligned} |v(t)|^{2q}_{W^1_{2q}(\Omega,\mathbb{R}^N)} &\lesssim |v(t)|^q_{W^\vartheta_q(\Omega,\mathbb{R}^N)}\,|v(t)|^q_{C^\alpha(\overline{\Omega},\mathbb{R}^N)}\\ &\lesssim_{|u|_{C([T_1,T_2];C^\alpha(\overline{\Omega},\mathbb{R}^N))}} \varepsilon|v(t)|^q_{W^2_q(\Omega,\mathbb{R}^N)} + C_\varepsilon|u|_{C([T_1,T_2]\times\overline{\Omega},\mathbb{R}^N)}, \end{aligned}$$

where $\varepsilon>0$ may be chosen arbitrary small. We therefore have

$$|a'_{ij}(v)\partial_i v\partial_j v|_{\mathbb{E}_{0,\mu}(J)} \lesssim_{|u|_{C([T_1,T_2];C^\alpha(\overline{\Omega},\mathbb{R}^N))}} \varepsilon|v|_{\mathbb{E}_{u,\mu}(J)} + C_\varepsilon.$$

Observe that this term does not occur in the semilinear case.

(III) For the second summand in (4.4.3) it is easily seen that

$$|f(v)|_{\mathbb{E}_{0,\mu}(J)} \lesssim |f(u)|_{C([T_1,T_2]\times\overline{\Omega},\mathbb{R}^N)}.$$

For the third summand, Lemma 4.2.3, the interpolation inequality and Young's inequality yield

$$|g(v)|_{\mathbb{F}_\mu(J)} \lesssim_{|u|_{C([T_1,T_2];C(\overline{\Omega},\mathbb{R}^N))}} 1 + |v|_{\mathbb{F}_\mu(J)} \le \varepsilon|v|_{\mathbb{E}_{u,\mu}(J)} + C_\varepsilon,$$

where ε is arbitrary. If we combine the above estimates with (4.4.3) and choose ε sufficiently small, then we may subtract $\varepsilon|v|_{\mathbb{E}_{u,\mu}(J)}$ on both sides of the inequality, to obtain

$$|v|_{\mathbb{E}_{u,\mu}(J)} \lesssim_{|u|_{C([T_1,T_2];C^\alpha(\overline{\Omega},\mathbb{R}^N))}} 1 + |u(T_1,u_0)|_{W^\sigma_q(\Omega,\mathbb{R}^N)}.$$

Together with (4.4.2) this yields the asserted estimate. In the semilinear case the constant does not depend on the Hölder norm of the solution, since then only the terms $|f(v)|_{\mathbb{E}_{0,\mu}(J)}$ and $|g(v)|_{\mathbb{F}_\mu(J)}$ in (4.4.3) are estimated. ∎

We use the above estimate to give a sufficient condition for the existence of a global attractor of (4.1.1) in the phase space $\mathcal{M}_p^{2-2/p}$ in terms of lower norms. This is the main result of this chapter.

Theorem 4.4.2. *Suppose that there are $\alpha, C > 0$ such that for each solution $u(\cdot, u_0)$ of (4.1.1) with initial value $u_0 \in \mathcal{M}_p^{2-2/p}$ it holds*

$$\limsup_{t\to t^+(u_0)} |u(t,u_0)|_{C^\alpha(\overline{\Omega},\mathbb{R}^N)} \leq C.$$

Then (4.1.1) has a global attractor in $\mathcal{M}_p^{2-2/p}$.

Proof. We first show that $t^+(u_0) = +\infty$ for all $u_0 \in \mathcal{M}_p^{2-2/p}$. Assume the contrary, i.e., $t^+(u_0) < +\infty$. Then Lemma 4.4.1 and the embedding $C^\alpha(\overline{\Omega},\mathbb{R}^N) \hookrightarrow W_p^\sigma(\Omega,\mathbb{R}^N)$ for $\sigma \in (0,\alpha)$ yield

$$\sup_{t\in[0,t^+(u_0))} |u(t,u_0)|_{W_p^{2-2/p}(\Omega,\mathbb{R}^N)} \lesssim 1 + \sup_{t\in[0,t^+(u_0)/2)} |u(t,u_0)|_{C^\alpha(\overline{\Omega},\mathbb{R}^N)},$$

which means that the orbit $\{u(t,u_0)\}_{t\in[0,t^+(u_0))}$ is bounded in $W_p^{2-2/p}(\Omega,\mathbb{R}^N)$. It thus has a convergent subsequence in $W_p^s(\Omega,\mathbb{R}^N)$ for $s \in (1+n/p, 2-2/p)$, which leads to a contradiction to the maximal existence time, and therefore $t^+(u_0) = +\infty$. Now another application of Lemma 4.4.1 yields that there is $C_0 > 0$ with

$$\limsup_{t\to\infty} |u(t,u_0)|_{W_p^{2-2/p}(\Omega,\mathbb{R}^N)} \leq C_0$$

for all $u_0 \in \mathcal{M}_p^{2-2/p}$. Therefore the global semiflow generated by (4.1.1) has an absorbant ball in $\mathcal{M}_p^{2-2/p}$. Since the semiflow is also compact by Theorem 4.3.6, the existence of a global attractor follows from [16, Corollary 1.1.6]. ∎

We consider special cases of (4.1.1), where an absorbing set in a weaker norm is sufficient for an attractor in $\mathcal{M}_p^s$. We start with the semilinear case with nonlinear boundary conditions.

Corollary 4.4.3. *Assume that (a_{ij}) does not depend on u, and suppose that there are $q \in (1,\infty)$, $\sigma \in (0, 2-2/q]$ and a constant $C > 0$ such that for each solution $u(\cdot,u_0)$ of (4.1.1) with $u_0 \in \mathcal{M}_p^{2-2/p}$ it holds*

$$\limsup_{t\to t^+(u_0)} |u(t,u_0)|_{W_q^\sigma(\Omega,\mathbb{R}^N)\cap L_\infty(\overline{\Omega},\mathbb{R}^N)} \leq C.$$

Then (4.1.1) has a global attractor in $\mathcal{M}_p^{2-2/p}$.

Proof. Lemma 4.4.1 yields a constant C_0 such that

$$\limsup_{t\to t^+(u_0)} |u(t,u_0)|_{W_q^{2-2/q}(\Omega,\mathbb{R}^N)} \leq C_0 \tag{4.4.4}$$

for all $u_0 \in \mathcal{M}_p^{2-2/p}$. We employ a bootstrapping procedure to show that (4.4.4) remains true if one replaces $W_q^{2-2/q}(\Omega, \mathbb{R}^N)$ by $C^\alpha(\overline{\Omega}, \mathbb{R}^N)$ with some $\alpha > 0$, and C_0 by a possibly larger constant. It then follows from Theorem 4.4.2 that (4.1.1) has a global attractor in $\mathcal{M}_p^{2-2/p}$ as asserted. Sobolev's embedding yields

$$W_q^{2-2/q}(\Omega, \mathbb{R}^N) \hookrightarrow C^\alpha(\overline{\Omega}, \mathbb{R}^N)$$

for some $\alpha > 0$ if $q > n/2 + 1$, and we are done in this case. Otherwise, in case $q \in (1, n/2+1)$, we employ

$$W_q^{2-2/q}(\Omega, \mathbb{R}^N) \hookrightarrow W_{q_1}^{\tau}(\Omega, \mathbb{R}^N),$$

which is valid for some small $\tau > 0$ if $q_1 \in \big(q, \frac{nq}{n+2-2q}\big)$. Note here that $\frac{nq}{n+2-2q} > q$ for all n and $q \in (1, n/2+1)$. Another application of Lemma 4.4.1 yields (4.4.4) with $W_q^{2-2/q}(\Omega, \mathbb{R}^N)$ replaced by $W_{q_1}^{2-2/q_1}(\Omega, \mathbb{R}^N)$. Iteratively, this yields a strictly increasing sequence of numbers q_k as long as $q_k < n/2+1$. But since $q_k \geq \big(\frac{n(1-\delta)}{n+2-2q}\big)^k q$ for small $\delta > 0$ as long as $q_k < n/2+1$ and $\frac{n}{n+2-2q} > 1$, the sequence q_k becomes larger than $n/2+1$ after finitely many steps. Thus (4.4.4) holds true with a Hölder norm, and this finishes the proof. ■

For $N = 2$ we next consider for the unknown $u = (u_1, u_2)$ quasilinear cross-diffusion systems of the form

$$\begin{aligned} \partial_t u_1 &= \operatorname{div}\big(P(u)\nabla u_1 + R(u)\nabla u_2\big) + f_1(u) && \text{in } \Omega, && t > 0, \\ \partial_t u_2 &= \operatorname{div}\big(Q(u_2)\nabla u_2\big) + f_2(u) && \text{in } \Omega, && t > 0, \\ \partial_\nu u &= 0 && \text{on } \Gamma, && t > 0, \\ u(0, \cdot) &= u_0 && \text{in } \Omega. \end{aligned} \tag{4.4.5}$$

This problem fits into our setting with $a(u) = \begin{pmatrix} P(u) & R(u) \\ 0 & Q(u_2) \end{pmatrix}$, $\alpha_{ij} = \delta_{ij}$ and $g = 0$. We can use the results of Kuiper & Dung [61] to weaken the norm for the absorbing ball considerably. We assume the following on the coefficients of (4.4.5). There are nonnegative continuous functions Φ_1, Φ_2 and constants $C, d > 0$ such that for all $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$ it holds

$$P(\zeta) \geq d(1+\zeta_1), \quad \zeta_1 \geq 0, \qquad |R(\zeta)| \leq \Phi_1(\zeta_2)\zeta_1, \qquad Q(\zeta_2) \geq d;$$
the partial derivatives of P, R are majorized by some powers of ζ_1, ζ_2;
$$|f(\zeta)| \leq \Phi_2(\zeta_2)(1+\zeta_1), \quad g(\zeta)\zeta_1^r \leq \Phi_2(\zeta_2)(1+\zeta_1^{r+1}), \qquad \text{for all } \zeta_1, \zeta_2 \geq 0,\ r > 0.$$

Corollary 4.4.4. *Under the above assumptions, let the solutions of (4.4.5) be nonnegative for nonnegative initial data. Suppose that there are $r > n/2$ and $C > 0$ such that for all $u_0 \in \mathcal{M}_p^{2-2/p}$ it holds*

$$\limsup_{t\to\infty} |u_1(t, u_0)|_{L_r(\Omega, \mathbb{R}^2)} + |u_2(t, u_0)|_{L_\infty(\Omega, \mathbb{R}^2)} \leq C.$$

Then (4.4.5) has a global attractor in $\mathcal{M}_p^{2-2/p}$. If Q does not depend on u_2 one can take $r = 1$.

Proof. It is shown in the Theorems 7 and 8 of [61] that (4.4.5) has a global attractor in $W^1_p(\Omega, \mathbb{R}^2)$ for all $p \in (n+2, \infty)$, from which the existence of an absorbant set in a C^α-norm follows from Sobolev's embedding. The assertion is thus a consequence of Theorem 4.4.2. ∎

In case of a single equation, $N = 1$, the norm for the absorbant set can be weakend up to L_1, using estimates of De Giorgi - Nash - Moser type.

Corollary 4.4.5. *Consider for $u(t,x) \in \mathbb{R}$ the problem*

$$\begin{aligned} \partial_t u &= \operatorname{div}(a(u)\nabla u) + f(u) && \text{in } \Omega, && t > 0, \\ \partial_\nu u &= g(u) && \text{on } \Gamma, && t > 0, \\ u(0,\cdot) &= u_0 && \text{in } \Omega, \end{aligned} \tag{4.4.6}$$

where a, f and g are assumed to be bounded and that there exists $\delta > 0$ with $a(\zeta) \geq \delta$ for all $\zeta \in \mathbb{R}$. If there is a constant $C > 0$ such that for each $u_0 \in \mathcal{M}_p^{2-2/p}$ the solution $u(\cdot, u_0)$ of (4.4.6) satisfies

$$\limsup_{t \to t^+(u_0)} |u(t,u_0)|_{L_1(\Omega)} \leq C,$$

then (4.4.6) has a global attractor in $\mathcal{M}_p^{2-2/p}$.

Proof. It is shown in [33, Theorem 1] that the existence of an absorbant ball in $L_1(\Omega)$ implies the existence of an absorbant ball in $L_\infty(\Omega)$. This in turn yields an absorbant ball in a Hölder norm, see [28, Theorem III.1.3] or [34, Corollary 4.2], and the assertion follows from Theorem 4.4.2. ∎

4.5 Applications

We apply the results of the last section to show convergence to attractors in stronger norms for concrete models.

4.5.1 Reaction-Diffusion Systems with Nonlinear Boundary Conditions

In a series of papers, Carvalho et. al. [15] considered global attractors for semilinear reaction-diffusion systems with nonlinear boundary conditions of the form

$$\begin{aligned} \partial_t u - \Delta u &= f(u) && \text{in } \Omega, && t > 0, \\ \partial_\nu u &= g(u) && \text{on } \Gamma, && t > 0, \\ u(0,\cdot) &= u_0 && \text{in } \Omega. \end{aligned} \tag{4.5.1}$$

Here the smooth nonlinearities $f, g : \mathbb{R}^N \to \mathbb{R}^N$ are dissipative in the sense that there are real numbers c_i and d_i with

$$\limsup_{|\xi_i| \to \infty} \frac{f_i(\xi)}{\xi_i} < c_i, \qquad \limsup_{|\xi_i| \to \infty} \frac{g_i(\xi)}{\xi_i} < d_i,$$

such that the first eigenvalue λ_0 of the linear elliptic problem

$$\begin{aligned} -\Delta v - cv &= \lambda v && \text{in } \Omega, \\ \partial_\nu v - dv &= 0 && \text{on } \Gamma, \end{aligned}$$

is positive, where $c = (c_1, ..., c_N)$ and $d = (d_1, ..., d_N)$. The discussion in [15, Section 6] shows that the first eigenvalue of the above problem can be positive although c_i or d_i has the 'wrong' sign, i.e., is positive. In this sense f can compensate a possible nondissipativeness of g, and vice versa.
In [15, Theorem 4.1] it is shown that under the above assumptions (4.5.1) has a global attractor in the phase space $W_2^1(\Omega, \mathbb{R}^N) \cap C(\overline{\Omega}, \mathbb{R}^N)$. Corollary 4.4.3 improves this result as follows.

Theorem 4.5.1. *Under the above assumptions, for $p \in (n+2, \infty)$ the semiflow generated by (4.5.1) has a global attractor in the nonlinear phase space*

$$\big\{u_0 \in W_p^{2-2/p}(\Omega, \mathbb{R}^N) \;:\; \partial_\nu u_0 = g(u_0) \quad \text{on} \;\; \Gamma\big\}.$$

4.5.2 A Chemotaxis Model with Volume-Filling Effect

For $u(t,x), v(t,x) \in \mathbb{R}$ the following chemotaxis model with volume-filling effect was introduced by Hillen & Painter [53],

$$\begin{aligned} \partial_t u &= d_1 \Delta u - \operatorname{div}\big(uq(u)\chi(v)\nabla v\big) + uf(u) && \text{in } \Omega, && t > 0, \\ \partial_t v &= d_2 \Delta v + g_1(u) - v g_2(v) && \text{in } \Omega, && t > 0, \\ \partial_\nu u &= \partial_\nu v = 0 && \text{on } \Gamma, && t > 0, \\ u(0,\cdot) &= u_0, \quad v(0,\cdot) = v_0 && \text{in } \Omega, && \end{aligned} \tag{4.5.2}$$

This model may be cast in the form (4.4.5) and is thus of separated divergence form. It is assumed that q is given by

$$q(u) = 1 - u/U_M, \qquad U_M > 0,$$

and further that $d_1, d_2 > 0$ for the diffusion coefficients and

$$f|_{(U_M,\infty)} \le 0, \qquad g_1, g_2 \ge 0, \qquad g_1(0) = 0, \qquad \lim_{v \to \infty} v g_2(v) \to +\infty,$$

for the smooth reaction terms f, g_1 and g_2. Besides smoothness there is no structural assumption the sensitivity function χ. It may even change its sign. Wrozsek [87, 88] showed that under these assumptions (4.5.2) possesses a global attractor in the phase spaces

$$\big\{(u_0, v_0) \in W_p^1(\Omega, \mathbb{R}^2) \;:\; 0 \le u_0 \le U_M, \quad 0 \le v_0\big\}, \qquad p \in (n, \infty),$$

and further that the ω-limit set of each solution orbit consists entirely of equilibria which satisfy a certain nonlocal problem. Jiang & Zhang [56] showed that in fact every solution of (4.5.2) converges to an equilibrium. It is well known that for $U_M = \infty$ blow-up of solutions may occur if the initial mass of u_0 is too large, cf. the survey article [54]. For $U_M < \infty$ the

chemotactic term in the first equation becomes becomes small if u is close to U_M, which prevents solutions from blow-up.
Using Lemma 4.4.1, the same arguments as in the proof of Theorem 4.4.2 yield the following improvement of the result of [87].

Theorem 4.5.2. *Under the above assumptions, for $p \in (n+2, \infty)$ the chemotaxis model (4.5.2) has a global attractor in the phase space*

$$\big\{(u_0, v_0) \in W_p^{2-2/p}(\Omega, \mathbb{R}^2) \;:\; 0 \le u_0 \le U_M, \quad 0 \le v_0\big\}.$$

4.5.3 A Population Model with Cross-Diffusion

Our last example is the Shigesada-Kawasaki-Teramoto cross-diffusion model for population dynamics, introduced in [76], which is for $u(t,x), v(t,x) \in \mathbb{R}$ given by

$$\begin{aligned} \partial_t u &= \Delta\big(d_1 + \alpha_{11} u + \alpha_{12} v) u\big) + u(a_1 - b_1 u - c_1 v) && \text{in } \Omega, && t > 0, \\ \partial_t v &= \Delta\big(d_2 + \alpha_{21} u + \alpha_{22} v) v\big) + v(a_2 - b_2 u - c_2 v) && \text{in } \Omega, && t > 0, \\ \partial_\nu u &= \partial_\nu v = 0 && \text{on } \Gamma, && t > 0, \\ u(0, \cdot) &= u_0, \quad v(0, \cdot) = v_0 && \text{in } \Omega. && \end{aligned} \tag{4.5.3}$$

Again this model may be cast in the form (4.4.5). Here the constants a_i, b_i, c_i, d_i, $i = 1, 2$, are positive, and the constants α_{ij}, $i = 1, 2$, are nonnegative. In [61, Theorem 2] it is shown that (4.5.3) has a global attractor as a dynamical system in $W_p^1(\Omega, \mathbb{R}^2)$ for $p \in (n, \infty)$, provided $\alpha_{22} = 0$. For $n = 2$ this remains true also for $\alpha_{22} > 0$. Theorem 4.4.2 improves this as follows.

Theorem 4.5.3. *Under the above assumptions, for $p \in (n+2, \infty)$ the population model (4.5.3) has a global attractor in the phase space $W_p^{2-2/p}(\Omega, \mathbb{R}^2)$.*

Chapter 5

Boundary Conditions of Reactive-Diffusive-Convective Type

5.1 Introduction

In this chapter we investigate linear and quasilinear parabolic systems with dynamical boundary conditions of reactive-diffusive-convective type. For the unknown $u = u(t, x) \in \mathbb{R}^N$, where $N \in \mathbb{N}$, we consider the problem[1]

$$\begin{aligned} \partial_t u &= \partial_i(a_1(u)\partial_i u) + a_2(u)\nabla u + f(u) && \text{in } \Omega, && t > 0,\\ \partial_t u + b(\cdot, u)\partial_\nu u &= \operatorname{div}_\Gamma(c_1(\cdot, u)\nabla_\Gamma u) + c_2(\cdot, u)\nabla_\Gamma u + g(\cdot, u) && \text{on } \Gamma, && t > 0,\\ u(0, \cdot) &= u_0 && \text{in } \overline{\Omega}. \end{aligned} \tag{5.1.1}$$

It is assumed that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$, where $n \geq 2$. The outer normal unit field and the normal derivative on Γ are denoted by ν and $\partial_\nu = \nu \operatorname{tr}_\Omega \nabla$, respectively. The spatial trace on Ω is designated by tr_Ω. Further, ∇_Γ and $\operatorname{div}_\Gamma$ are the surface gradient and the surface divergence on Γ, respectively. We assume that the coefficients are smooth, and that for all $x \in \Gamma$ and $\zeta \in \mathbb{R}^N$ it holds

$$a_1(\zeta), b(x, \zeta) \in \mathcal{B}(\mathbb{R}^N), \qquad c_1(x, \zeta) \in \mathbb{R}^N,$$

$$a_2(\zeta), c_2(x, \zeta) \in \mathcal{B}(\mathbb{R}^{N\times n}, \mathbb{R}^N), \qquad f(\zeta), g(x, \zeta) \in \mathbb{R}^N.$$

The term $\operatorname{div}_\Gamma(c_1(\cdot, u)\nabla_\Gamma u)$ is meant in way that its k-th component is given by

$$\operatorname{div}_\Gamma(c_1(\cdot, u)\nabla_\Gamma u)_k := \operatorname{div}_\Gamma(c_1^k(\cdot, u)\nabla_\Gamma u_k), \qquad k = 1, ..., N.$$

The system (5.1.1) consists of two dynamic equation, coupled in a possibly nonlinear way by the flux term $b(\cdot, u)\partial_\nu u$. The term $\operatorname{div}_\Gamma(c_1(\cdot, u)\nabla_\Gamma u)$ takes into account surface diffusion effects on the boundary, where the tangential flux vector $J_\Gamma^k = -c_1^k(\cdot, u)\nabla_\Gamma u_k$ of u_k may depend nonlinearly on the surface gradient of u_k. For $c_1 \equiv 1$ one obtains the Laplace-Beltrami operator $\Delta_\Gamma = \operatorname{div}_\Gamma \nabla_\Gamma$ on Γ. Further, the term $c_2(\cdot, u)\nabla_\Gamma u$ describes nonlinear surface convection on the boundary.

[1] We use again sum convention.

We explain the differential operators on the boundary. Via the directional derivative at $x \in \Gamma$ a function $v \in C^\infty(\Gamma)$ induces an element of the dual space of $T_x\Gamma$. The surface gradient $\nabla_\Gamma v(x) \in \mathbb{R}^N$ of v at x is then the unique corresponding element of $T_x\Gamma$ given by the Riesz isomorphism, if one considers $T_x\Gamma$ as a Hilbert space with the scalar product induced from $\mathbb{R}^n$. In local coordinates g for Γ, with fundamental form $\mathrm{G} = (\mathrm{g}_{ij})$ and inverse $\mathrm{G}^{-1} = (\mathrm{g}^{ij})$, the components of the surface gradient with respect to the basis $\{\partial_1 \mathrm{g}, ..., \partial_{n-1}\mathrm{g}\}$ of the tangential space are given by the components of $\mathrm{G}^{-1}\nabla_{n-1}(v\circ \mathrm{g})^T$, i.e.,

$$\nabla_\Gamma v \circ \mathrm{g} = \sum_{i,j=1}^{n-1} \mathrm{g}^{ij}\partial_j(v\circ \mathrm{g})\partial_i \mathrm{g}.$$

For a tangential vector field $w \in C^\infty(\Gamma, \mathbb{R}^n)$, i.e., $w(x) \in T_x\Gamma$ for $x \in \Gamma$, the function $\mathrm{div}_\Gamma w \in C^\infty(\Gamma)$ is in coordinates g given by

$$\mathrm{div}_\Gamma w \circ \mathrm{g} = \frac{1}{\sqrt{|\mathrm{G}|}} \sum_{i=1}^{n-1} \partial_i\big(\sqrt{|\mathrm{G}|}\, w^i \circ \mathrm{g}\big),$$

where w^i are the components of w with respect to the basis $\{\partial_1 \mathrm{g}, ..., \partial_{n-1}\mathrm{g}\}$. For the components of the surface diffusion term in (5.1.1) we thus have

$$\mathrm{div}_\Gamma (c_1(\cdot,u)\nabla_\Gamma u)_k \circ \mathrm{g} = \frac{1}{\sqrt{|\mathrm{G}|}} \sum_{i,\,j=1}^{n-1} \partial_i\big(c_1^k(\cdot,u)\circ \mathrm{g}\,\sqrt{|\mathrm{G}|}\mathrm{g}^{ij}\partial_j(u_k\circ \mathrm{g})\big), \qquad k = 1,...,N.$$

These are well defined differential operators on Γ (cf. Appendix A.5), with principal parts equal to $c_1^k\Delta_\Gamma$, respectively.

We impose the following structural conditions on a_1, b and c_1, where $\delta > 0$ is independent of $x \in \Gamma$ and $\zeta \in \mathbb{R}^N$. By A^{kk} we denote the k-th diagonal entry of a matrix A.

$$\left.\begin{array}{l} a_1(\zeta), b(x,\zeta) \text{ are upper triangular matrices,} \quad c_1^k(x,\zeta) \geq \delta, \quad k = 1,...,N; \\ a_1^{kk}(\zeta) \geq \delta, \text{ and either } b^{kk}(x,\zeta) \geq \delta \text{ or } b^{kk}(x,\zeta) \leq -\delta, \quad k = 1,...,N. \end{array}\right\} \tag{5.1.2}$$

We emphasize that the sign of the diagonal entries of b may change from line to line.

Let us describe the results and the organization of this chapter. For $p \in (1,\infty)$ we let

$$X_0 = L_p(\Omega, \mathbb{R}^N) \times W_p^{1-1/p}(\Gamma, \mathbb{R}^N),$$

$$X_1 = \big\{(v, v_\Gamma) \in W_p^2(\Omega, \mathbb{R}^N) \times W_p^{3-1/p}(\Gamma, \mathbb{R}^N) \;:\; \mathrm{tr}_\Omega v = v_\Gamma\big\},$$

and look for solutions u in the maximal regularity class

$$\mathbb{E}_u(J) = W_p^1(J; X_0) \cap L_p(J; X_1),$$

where $J = (0,T)$ is a finite time interval, $T > 0$. Identifying a function u with the pair $(u, \mathrm{tr}_\Omega u)$, here we write $u \in \mathbb{E}_u(J)$, with a slight abuse of notation. In Section 5.2 we first consider the linear inhomogeneous, nonautonomous version of (5.1.1) and show that it enjoys the property of maximal $L_{p,\mu}$-regularity on finite intervals, verifying the conditions of Theorem 3.1.4. We then turn in Section 5.3 to the quasilinear case and show that for each initial value $u_0 \cong (u_0, \mathrm{tr}_\Omega u_0)$ from the linear phase space

$$\mathcal{M} = \big\{(v, v_\Gamma) \in W_p^{2-2/p}(\Omega, \mathbb{R}^N) \times W_p^{3-3/p}(\Gamma, \mathbb{R}^N) \ : \ \mathrm{tr}_\Omega v = v_\Gamma\big\}$$

there is a unique maximal solution $u(\cdot, u_0) \in C\big(0, t^+(u_0); \mathcal{M}\big)$ of (5.1.1), provided $p \in (n+2, \infty)$. Here $t^+(u_0) > 0$ denotes the maximal existence time. We obtain strong solutions, in the sense that

$$u(\cdot, u_0) \in \mathbb{E}_u(0, \tau) \qquad \text{for all } \ \tau \in (0, t^+(u_0)).$$

Moreover, the map $u_0 \mapsto u(\cdot, u_0)$ defines a compact local semiflow on $\mathcal{M}$, which has the property that bounded orbits are relatively compact. These results are based on maximal L_p-regularity for the linearization of (5.1.1), the regularity properties of the superposition operators occurring in (5.1.1) and the recent results of [59] on abstract quasilinear problems in $L_{p,\mu}$-spaces. Besides the structural conditions (5.1.2) we do not have to impose any restrictions on the nonlinearities to obtain the local semiflow. In particular, we do not have to impose any growth conditions.
We then turn to global issues and show in Section 5.4 that an a priori Hölder bound for a solution of (5.1.1) implies that it exists globally in time. We obtain this result by localizing (5.1.1) in space and time, employing again that the linearization of (5.1.1) has maximal L_p-regularity, and by performing appropriate estimates of the resulting nonlinear error terms. In Section 5.5 we specialize to a semilinear version of (5.1.1),

$$\begin{aligned} \partial_t u &= \Delta u + f(u) && \text{in } \Omega, && t > 0, \\ \partial_t u + \partial_\nu u &= \Delta_\Gamma u + g(u) && \text{on } \Gamma, && t > 0, \\ u(0, \cdot) &= u_0 && \text{in } \overline{\Omega}. \end{aligned} \tag{5.1.3}$$

Under appropriate dissipativity conditions on the reaction terms f and g we obtain a Lyapunov function for (5.1.3), that already appeared in [80], and a priori estimates in the energy spaces $W_2^1(\Omega, \mathbb{R}^N)$ and $W_2^1(\Gamma, \mathbb{R}^N)$. By a Moser-Alikakos iteration procedure we can show that this implies an a priori L_∞-bound, which in turn leads to global existence for the solutions of (5.1.3). The Lyapunov function, together with another a priori estimate for the equilibria of (5.1.3), yields the existence of a global attractor in $\mathcal{M}$, and that each solution converges to the set of equilibria as $t \to \infty$.

Problems related to (5.1.1) and (5.1.3) were considered, for instance, in [38, 39, 40, 80, 83]. We refer to the introduction of this thesis for more informations.

5.2 Maximal $L_{p,\mu}$-Regularity for the Linearized Problem

In this section we show that the linearized version of (5.1.1) enjoys maximal $L_{p,\mu}$-regularity by verifying the normal ellipticity condition (E) and the Lopatinskii-Shapiro condition (LS) and using Theorem 3.1.4. Besides the interest in its own, this linear result is the basis for our investigation of the quasilinear problems.

For the unknown $u = u(t,x) \in \mathbb{R}^N$ we consider linear inhomogeneous, nonautonomous parabolic systems of the form

$$\begin{aligned} \partial_t u &= A_1\Delta u + A_2\nabla u + A_3 u + \widetilde{f} && \text{in } \Omega, \quad t \in J, \\ \partial_t u + B\partial_\nu u &= C_1\Delta_\Gamma u + C_2\nabla_\Gamma u + C_3 u + \widetilde{g} && \text{on } \Gamma, \quad t \in J, \\ u(0,\cdot) &= u_0 && \text{in } \Omega, \\ \mathrm{tr}_\Omega u(0,\cdot) &= u_{0,\Gamma} && \text{on } \Gamma. \end{aligned} \tag{5.2.1}$$

A simplified version of (5.2.1) was considered in Example 3.1.1. Here the coefficients may depend on t and x, and are of the form

$$A_1(t,x), A_3(t,x), B(t,x), C_1(t,x), C_3(t,x) \in \mathcal{B}(\mathbb{R}^N), \quad A_2(t,x), C_2(t,x) \in \mathcal{B}(\mathbb{R}^{N\times n}, \mathbb{R}^N).$$

We further impose the following structural conditions on the coefficients, which are similar to (5.1.2). The number $\delta > 0$ is independent of t and x.

$$\left.\begin{array}{l} A_1(t,x), B(t,x), C_1(t,x) \text{ are upper triangular matrices; for } k = 1,...,N\text{:} \\ A_1^{kk}(t,x), C_1^{kk}(t,x) \geq \delta, \quad \text{and either } B^{kk}(x,\zeta) \geq \delta \text{ or } B^{kk}(x,\zeta) \leq -\delta. \end{array}\right\} \tag{5.2.2}$$

We may cast (5.2.1) in the form (3.1.1) by setting

$$\mathcal{A}(t,x,D) = -\big(A_1(t,x)\Delta + A_2(t,x)\nabla + A_3(t,x)\big), \qquad \mathcal{B}_0(t,x,D) = B(t,x)\nu(x)\mathrm{tr}_\Omega\nabla,$$

$$\mathcal{C}_0(t,x,D_\Gamma) = -\big(C_1(t,x)\Delta_\Gamma + C_2(t,x)\nabla_\Gamma + C_3(t,x)\big), \qquad \mathcal{B}_1 = \mathrm{tr}_\Omega, \qquad \mathcal{C}_1 = -1.$$

For $p \in (1,\infty)$ the nontrivial part of the Newton polygon associated to (5.2.1), cf. Section 3.1, is the line through to the points $(0, 3/2-1/2p)$ and $(3-1/p, 0)$. The point $(2, 1/2-1/2p)$ corresponding to the operator $\mathcal{C}_0$ lies on this line, the point $(0, 1-1/2p)$ corresponding to $\mathcal{C}_1$ does not.

To verify (E), note that the principal symbol of $\mathcal{A}$ is given by

$$\mathcal{A}_\sharp(t,x,\xi) = A_1(t,x)|\xi|^2, \qquad \xi \in \mathbb{R}^n, \qquad (t,x) \in \overline{J} \times \overline{\Omega}.$$

Since $A_1(t,x)$ is assumed to be positive definite, the spectrum of $A_1(t,x) \in \mathcal{B}(\mathbb{R}^N)$ is contained in the right-half plane. Hence (E) is valid.

Problem (5.2.1) belongs to Case 1, hence we do not have to consider asymptotic Lopatinskii-Shapiro conditions. Since further the unknown u takes values in a finite dimensional space, we only have to consider (LS) with trivial right-hand sides.

Let $(t,x) \in \overline{J} \times \Gamma$, and take coordinates g associated to x, cf. Lemma A.1.1. Then the chart (U,φ) corresponding to g satisfies $\varphi'(x) = \mathcal{O}_{\nu(x)}$, where $\mathcal{O}_{\nu(x)}$ is an orthogonal matrix that rotates $\nu(x)$ to $(0,...,0,-1) \in \mathbb{R}^n$. For $\xi' \in \mathbb{R}^{n-1}$ and $D_y = -\mathrm{i}\partial_y$ we thus have

$$\mathcal{A}_\sharp^\nu\big(t,x,\mathcal{O}_{\nu(x)}^T(\xi', D_y)\big) = A_1(t,x)(|\xi'|^2 - \partial_y^2), \qquad \mathcal{B}_{0\sharp}\big(t,x,\mathcal{O}_{\nu(x)}^T(\xi',\partial_y)\big) = -B(t,x)\partial_y.$$

Since $(\varphi^{-1})'(x) = \mathcal{O}^T_{\nu(x)}$ it holds $\mathrm{G}(x) = \mathrm{id}_{n-1}$ for the fundamental form G corresponding to g, i.e., the coordinates g are orthonormal at x, and the Laplace-Beltrami operator reduces to $\Delta_\Gamma u(x) = \Delta_{n-1}(u \circ \mathrm{g}) \circ \mathrm{g}^{-1}(x)$. This yields

$$\mathcal{C}^{\mathrm{g}}_{0\sharp}(t,x,\xi') = C_1(t,x)|\xi'|^2, \qquad \xi' \in \mathbb{R}^{n-1}.$$

By convention we further have $\mathcal{C}_{1\sharp} = 0$, since the point $(0, 1-1/2p)$ corresponding to $\mathcal{C}_1$ does not lie on the nontrivial part of the Newton polygon.

We thus have to show that for all $\lambda \in \overline{\mathbb{C}_+}\backslash\{0\}$ and $\xi' \in \mathbb{R}^{n-1}$ with $\lambda + |\xi'| \neq 0$ the only solution (v,σ) of the ordinary initial value problem

$$\begin{aligned} \big(\lambda + A_1(t,x)(|\xi'|^2 - \partial_y^2)\big)v(y) &= 0, \qquad y > 0, \\ -B(t,x)\partial_y v(0) + \big(\lambda + C_1(t,x)|\xi'|^2\big)\sigma &= 0, \\ v(0) &= 0, \end{aligned} \tag{5.2.3}$$

where v is decaying as $y \to \infty$ is the trivial one, i.e., $(v,\sigma) = (0,0)$. So let (v,σ) solve (5.2.3) and let v be decaying. We write $v = (v_1, ..., v_N)$ and $\sigma = (\sigma_1, ..., \sigma_N)$. We now make use of the triangular structure of A_1, B and C_1. Denoting by $A_1^{NN}(t,x) > 0$ the diagonal entry of $A_1(t,x)$ in the N-th row, we obtain that v_N solves

$$\big(\lambda/A_1^{NN}(t,x) + |\xi'|^2 - \partial_y^2\big)v_N = 0, \quad y > 0, \qquad v_N(0) = 0,$$

which implies $v_N \equiv 0$. Consequently $-B^{NN}(t,x)\partial_y v_N(0) = 0$, which shows that the sign of $B^{NN}(t,x)$ has no influence on the validity of (LS). We thus obtain that σ_N satisfies $\big(\lambda + C_1^{NN}(t,x)|\xi'|^2\big)\sigma_N = 0$. Since we assume $C_1^{NN}(t,x) > 0$ and that λ and ξ' do not vanish simultaneously, it follows that $\sigma_N = 0$. Iterating these arguments and using the diagonal structure we obtain that each component of v and σ vanishes. Here again the sign of the diagonal entries of B has no influence. This verifies (LS) for (5.2.3).

For the solvability of (5.2.1) the compatibility condition

$$\begin{aligned} \widetilde{g}(0,\cdot) + B(0,\cdot)\partial_\nu u_0, -C_1(0,\cdot)\Delta_\Gamma u_{0,\Gamma} \\ - C_2(0,\cdot)\nabla_\Gamma u_{0,\Gamma} - C_3(0,\cdot)u_{0,\Gamma} \in B^{2(\mu-1/p)-1-1/p}_{p,p}(\Gamma, \mathbb{R}^N) \end{aligned}$$

must be satisfied if $2(\mu - 1/p) > 1 + 1/p$. Since the trace space of $\mathbb{F}_\mu$ equals $B^{2(\mu-1/p)-1-1/p}_{p,p}(\Gamma, \mathbb{R}^N)$ it suffices, for instance, if

$$B(0,\cdot),\ C_1(0,\cdot),\ C_2(0,\cdot),\ C_3(0,\cdot) \text{ are pointwise multipliers on } B^{2(\mu-1/p)-1-1/p}_{p,p}(\Gamma, \mathbb{R}^N). \tag{5.2.4}$$

Note that if in case $2(\mu - 1/p) > 1 + n/p$, which is relevant for the treatment of quasilinear problems, the coefficients satisfy the first condition in (SB) and (SC), i.e.,

$$B, C_1, C_2, C_3 \in \mathbb{F}_\mu(J, \mathcal{F}),$$

where $\mathcal{F}$ stands for $\mathcal{B}(\mathbb{R}^N)$ or $\mathcal{B}(\mathbb{R}^{N\times n}, \mathbb{R}^N)$, then (5.2.4) is valid by Lemma 1.3.19 and Sobolev's embeddings.

The above considerations, Theorem 3.1.4 and [26, Theorem 2.2] yield the following result.

Theorem 5.2.1. *Let $J = (0, T)$ be a finite interval, $p \in (1, \infty)$ and $\mu \in (1/p, 1]$. Assume that the coefficients of (5.2.1) satisfy (5.2.2),* (SD), (SB) *and* (SC), *and further (5.2.4) if $2(\mu - 1/p) > 1 + 1/p$. Then (5.2.1) has a unique solution u satisfying*

$$u \in W^1_{p,\mu}(J; L_p(\Omega, \mathbb{R}^N)) \cap L_{p,\mu}(J; W^2_p(\Omega, \mathbb{R}^N))$$
$$\mathrm{tr}_\Omega u \in W^{3/2-1/2p}_{p,\mu}(J; L_p(\Gamma, \mathbb{R}^N)) \cap L_{p,\mu}(J; W^{3-1/p}_p(\Gamma, \mathbb{R}^N)),$$

if and only if the data is subject to

$$f \in L_{p,\mu}(J; L_p(\Omega, \mathbb{R}^N)), \qquad u_0 \in B^{2(\mu-1/p)}_{p,p}(\Omega, \mathbb{R}^N), \qquad u_{0,\Gamma} \in B^{2(\mu-1/p)+1-1/p}_{p,p}(\Gamma, \mathbb{R}^N),$$
$$g \in W^{1/2-1/2p}_{p,\mu}(J; L_p(\Gamma, \mathbb{R}^N)) \cap L_{p,\mu}(J; W^{1-1/p}_p(\Gamma, \mathbb{R}^N)),$$

and it holds $\mathrm{tr}_\Omega u_0 = u_{0,\Gamma}$ if $2(\mu-1/p) > 1/p$. In the autonomous case, i.e., if the coefficients do not depend on t, the realization of the operator

$$A = \begin{pmatrix} A_1\Delta + A_2\nabla + A_3 & 0 \\ -B\partial_\nu & C_1\Delta_\Gamma + C_2\nabla_\Gamma + C_3 \end{pmatrix}$$

on $L_p(\Omega, \mathbb{R}^N) \times W^{1-1/p}_p(\Gamma, \mathbb{R}^N)$ with domain

$$D(A) = \big\{(v, v_\Gamma) \in W^2_p(\Omega, \mathbb{R}^N) \times W^{3-1/p}_p(\Gamma, \mathbb{R}^N) \ : \ \mathrm{tr}_\Omega v = v_\Gamma\big\}$$

is the generator of an analytic C_0-semigroup. ■

Remark 5.2.2. a) The theorem is the basis for our investigations of quasilinear problems.
b) It should be possible to verify (LS) for (5.2.1) under more general structural assumptions on the coefficients.
c) Having verified (E) and (LS), in the autonomous case the maximal $L_{p,\mu}$-regularity result also follows from the result in the unweighted case [26, Theorem 2.1] combined with Theorem 1.2.2 on the independence of maximal $L_{p,\mu}$-regularity of $\mu \in (1/p, 1]$.
d) If μ is sufficiently small such that the spatial trace of u_0 does not necessarily exist there must be no relation between the initial values u_0 and $u_{0,\Gamma}$.
e) The sign of the diagonal entries of B can change from row to row. The reason is that $B\partial_\nu$ is of lower order with respect to $C_1\Delta_\Gamma$. The fact that the sign has no influence can also be seen in the verification of the Lopatinskii-Shapiro Condition above.
f) The theorem gives partial answers to the open questions posed in [83]. In this paper the problem

$$\begin{aligned} \partial_t u &= \Delta u && \text{in } \Omega, && t > 0, \\ \partial_t u - \partial_\nu u &= \Delta_\Gamma u && \text{on } \Gamma, && t > 0, \\ u(0, \cdot) &= u_0 && \text{in } \overline{\Omega}, \end{aligned} \tag{5.2.5}$$

is studied, and it is shown that (5.2.5) generates an analytic quasi-contractive semigroup in the energy space

$$H = \big\{(v, v_\Gamma) \in W^1_2(\Omega) \times W^1_2(\Gamma) \ : \ \mathrm{tr}_\Omega v = v_\Gamma\big\}.$$

Our setting differs from the one in [83]. For $p = 2$ Theorem 5.2.1 yields a semigroup in $L_2(\Omega) \times W^{1/2}_2(\Gamma)$ for (5.2.5).

5.3 The Local Semiflow for Quasilinear Problems

The functional analytical setting for the solutions of (5.1.1) is as follows. For $p \in (3/2, \infty)$ we consider the Banach spaces

$$X_0 := L_p(\Omega, \mathbb{R}^N) \times W_p^{1-1/p}(\Gamma, \mathbb{R}^N), \qquad \mathbb{E}_0(J) := L_p(J; X_0),$$

$$X_1 := \big\{(v, v_\Gamma) \in W_p^2(\Omega, \mathbb{R}^N) \times W_p^{3-1/p}(\Gamma, \mathbb{R}^N) \ : \ \mathrm{tr}_\Omega v = v_\Gamma\big\},$$

$$\mathbb{E}_u(J) := W_p^1(J; X_0) \cap L_p(J; X_1),$$

and we write $v \cong (v, v_\Gamma) \in \mathbb{E}_u(J)$. Recall further the phase space

$$\mathcal{M} := \big\{(v, v_\Gamma) \in W_p^{2-2/p}(\Omega, \mathbb{R}^N) \times W_p^{3-3/p}(\Gamma, \mathbb{R}^N) \ : \ \mathrm{tr}_\Omega v = v_\Gamma\big\},$$

which we consider as a closed subspace of $W_p^{2-2/p}(\Omega, \mathbb{R}^N) \times W_p^{3-3/p}(\Gamma, \mathbb{R}^N)$. Denoting by $(\cdot, \cdot)_{s,p}$ the real interpolation functor, $s \in (0, 1)$, we have the following characterization of $\mathcal{M}$.

Lemma 5.3.1. *For $p \in (3/2, \infty)$ it holds that $\mathcal{M} = (X_0, X_1)_{1-1/p,p}$.*

Proof. Define on X_0 the operator $\mathbb{A}$ by $\mathbb{A}u = (-\Delta u, \partial_\nu u - \Delta_\Gamma u_\Gamma)$, where $u = (u, u_\Gamma) \in D(\mathbb{A}) := X_1$. Combining Example 3.2 and Theorem 2.2 of [26] we obtain $\mathbb{A} \in \mathcal{MR}_p(0, 1)$ for all $p \in (1, \infty)$, which implies that the Cauchy Problem

$$\partial_t u + \mathbb{A}u = 0, \quad t \in (0, 1), \qquad u(0) = u_0,$$

has a unique solution $u \in \mathbb{E}_u(0, 1)$ if and only if $u_0 \in (X_0, X_1)_{1-1/p,p}$. It further follows from Lemma 1.3.5 that the temporal trace maps $\mathbb{E}_u(0, 1)$ continuously into $(X_0, X_1)_{1-1/p,p}$. On the other hand it is shown in [26, Corollary 2.3] that the above Cauchy problem has a unique solution in $\mathbb{E}_u(0, 1)$ if and only if $u_0 \in \mathcal{M}$, provided $p \in (3/2, \infty)$, and Theorem 1.3.6 yields that the temporal trace maps

$$\mathbb{E}_u(0, 1) \hookrightarrow W_p^1(0, 1; X_0) \cap L_p(0, 1; W_p^2(\Omega, \mathbb{R}^N) \times W_p^{3-1/p}(\Gamma, \mathbb{R}^N))$$

continuously into $W_p^{2-2/p}(\Omega, \mathbb{R}^N) \times W_p^{3-3/p}(\Gamma, \mathbb{R}^N)$. Therefore $\mathcal{M}$ and $(X_0, X_1)_{1-1/p,p}$ coincide as sets, and the maximal regularity estimates implied by Theorem 2.2, Corollary 2.3 of [26] and the continuity of the traces yield

$$|u_0|_{(X_0,X_1)_{1-1/p,p}} \lesssim |u|_{\mathbb{E}_u(0,1)} \lesssim |u_0|_{W_p^{2-2/p}(\Omega,\mathbb{R}^N)} + |u_0|_{W_p^{3-3/p}(\Gamma,\mathbb{R}^N)},$$

and vice versa. ■

We define the maps $A : \mathcal{M} \to \mathcal{B}(X_1, X_0)$ and $F : \mathcal{M} \to X_0$ by

$$A(u)v := \begin{pmatrix} -\partial_i(a_1(u)\partial_i v) & 0 \\ b(\cdot, u_\Gamma)\partial_\nu v & -\mathrm{div}_\Gamma(c_1(\cdot, u_\Gamma)\nabla_\Gamma v_\Gamma) \end{pmatrix}, \qquad u \in \mathcal{M}, \quad v \in X_1,$$

$$F(u) := \begin{pmatrix} a_2(u)\nabla u + f(u) \\ c_2(\cdot, u_\Gamma)\nabla_\Gamma u_\Gamma + g(\cdot, u_\Gamma) \end{pmatrix}, \qquad u \in \mathcal{M}.$$

As an abstract quasilinear evolution equation, the system (5.1.1) takes the form

$$\partial_t u(t) + A(u(t))u(t) = F(u(t)), \quad t>0, \qquad u(0)=u_0.$$

We show that the maps A and F are locally Lipschitz continuous for sufficiently large p. Recall the Sobolev embeddings

$$W_p^{2-2/p}(\Omega,\mathbb{R}^N) \hookrightarrow C^1(\overline{\Omega},\mathbb{R}^N), \quad W_p^{3-3/p}(\Gamma,\mathbb{R}^N) \hookrightarrow C^2(\Gamma,\mathbb{R}^N), \qquad p>n+2. \quad (5.3.1)$$

Lemma 5.3.2. *For $p\in(n+2,\infty)$ the functions*

$$A:\mathcal{M}\to\mathcal{B}(X_1,X_0), \qquad F:\mathcal{M}\to X_0,$$

are Lipschitz continuous on bounded subsets of $\mathcal{M}$. Moreover, under the structural conditions (5.1.2) we have that for all $u_0\in\mathcal{M}$ and all finite intervals $J=(0,T)$ the operator $A(u_0)$ on X_0 with domain X_1 enjoys maximal L_p-regularity on J.

Proof. (I) The embeddings (5.3.1) show that $A(u)\in\mathcal{B}(X_1,X_0)$ and $F(u)\in X_0$ for $u\in\mathcal{M}$. For the regularity of A, we estimate for $u,v\in\mathcal{M}$ and $w\in X_1$ with $|w|_{X_1}\le 1$

$$\begin{aligned}
&|(A(u)-A(v))w|_{X_0}\\
&\quad\lesssim \sum_{i=1}^n |(a_1(u)-a_1(v))\partial_i w|_{W_p^1(\Omega,\mathbb{R}^N)} + |(b(\cdot,u_\Gamma)-b(\cdot,v_\Gamma))\partial_\nu w|_{W_p^{1-1/p}(\Gamma,\mathbb{R}^N)}\\
&\qquad + |(c_1(\cdot,u_\Gamma)-c_1(\cdot,v_\Gamma))\nabla_\Gamma w_\Gamma|_{W_p^{2-1/p}(\Gamma,\mathbb{R}^n)}\\
&\quad\lesssim |a_1(u)-a_1(v)|_{C^1(\overline{\Omega},\mathbb{R}^{N\times N})}\\
&\qquad + |b(\cdot,u_\Gamma)-b(\cdot,v_\Gamma)|_{C^1(\Gamma,\mathbb{R}^{N\times N})} + |c_1(\cdot,u_\Gamma)-c_1(\cdot,v_\Gamma)|_{C^2(\Gamma,\mathbb{R}^N)}.
\end{aligned}$$

It is not hard to show that the superposition operators

$$u\mapsto a_1(u), \qquad u_\Gamma\mapsto b(\cdot,u_\Gamma), \qquad u_\Gamma\mapsto c_1(\cdot,u_\Gamma),$$

are Lipschitz continuous on bounded sets as maps

$$C^1(\overline{\Omega},\mathbb{R}^N)\to C^1(\overline{\Omega},\mathbb{R}^{N\times N}), \quad C^1(\Gamma,\mathbb{R}^N)\to C^1(\Gamma,\mathbb{R}^{N\times N}), \quad C^2(\Gamma,\mathbb{R}^N)\to C^2(\Gamma,\mathbb{R}^N),$$

respectively. Now (5.3.1) yields that A is Lipschitz continuous on bounded subsets of $\mathcal{M}$. Similar arguments show the asserted regularity of F.

(II) Let $u_0\in\mathcal{M}$ be given. The embeddings (5.3.1) yield that for $u_0\in\mathcal{M}$ the coefficients of $A(u_0)$ are continuous on $\overline{\Omega}$ and continuously differentiable on Γ, respectively. Thus (SD), (SB), (SC) and (5.2.4) are valid. The conditions (5.1.2) and Theorem 5.2.1 thus yield that the realization of $A(u_0)$ on X_0 with domain X_1 enjoys maximal L_p-regularity on finite time intervals. ■

After these preparations we obtain local well-posedness for (5.1.1) from the results in [59].

Theorem 5.3.3. *Assume that the coefficients of (5.1.1) are smooth, and that they satisfy the structural conditions (5.1.2). Let further* $p \in (n+2, \infty)$. *Then for all initial values* $u_0 \in \mathcal{M}$ *the problem (5.1.1) has a unique maximal solution*

$$u(\cdot, u_0) \in C\big([0, t^+(u_0)); \mathcal{M}\big),$$

such that $u(\cdot, u_0) \in \mathbb{E}_1(0, \tau)$ *for all* $\tau \in (0, t^+(u_0))$, *where* $t^+(u_0) > 0$ *denotes the maximal existence time. The solution map* $u_0 \mapsto u(\cdot, u_0)$ *is a local semiflow*[2] *on* $\mathcal{M}$. *If* $\{u(\cdot, u_0)\}_{t\in[0,t^+(u_0))}$ *is bounded in* $\mathcal{M}$, *then* $t^+(u_0) = +\infty$ *and the corresponding orbit is relatively compact in* $\mathcal{M}$.

Proof. Due to Lemma 5.3.2 we may apply the Theorems 2.1, 3.1 and Remark 2.3 of [59] to the abstract quasilinear problem

$$\partial_t u(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \qquad u(0) = u_0 \in \mathcal{M},$$

which is equivalent to (5.1.1). Hence all assertions follow, except the compactness property of the solution map. Using [59, Theorem 2.1], the proof of this fact is completely analogous to the proof of Proposition 4.3.5. ∎

5.4 A Priori Hölder Bounds imply Global Existence

We show how maximal L_p-regularity can be used to reduce the question of global existence of solutions to the boundedness in a Hölder norm.

Theorem 5.4.1. *Under the assumptions of Theorem 5.3.3, let* $u(\cdot, u_0)$ *be the maximal solution of (5.1.1) with initial value* $u_0 \in \mathcal{M}$. *If* $u(\cdot, u_0)$ *is uniformly continuous in* $[0, t^+(u_0)) \times \overline{\Omega}$ *and it holds*

$$\sup_{t\in[0,t^+(u_0))} |u(t, u_0)|_{C^\alpha(\overline{\Omega},\mathbb{R}^N)} < +\infty$$

for some $\alpha > 0$, *then* $u(\cdot, u_0)$ *exists globally,* $t^+(u_0) = +\infty$, *provided* p *is sufficiently large.*

Proof. The plan is to show that if $t^+(u_0) < +\infty$ then the orbit is bounded in $\mathcal{M}$, provided p is sufficiently large. This leads to a contradiction to Theorem 5.3.3, and shows that $t^+(u_0)$ cannot be finite. Assume that $t^+(u_0) < +\infty$, and denote by $u = u(\cdot, u_0)$ the solution of (5.1.1). We will show that for sufficiently small $\eta > 0$ the quantity $|u|_{\mathbb{E}_1(t_\eta, T)}$, where

$$t_\eta := t^+(u_0) - \eta,$$

is bounded by a constant independent of $T \in (t_\eta, t^+(u_0))$. Then $\sup_{T\in[0,t^+(u_0))} |u(T)|_{\mathcal{M}}$ is finite, and we are done.

(I) We localize the problem in space. Due to its uniform continuity on $[0, t^+(u_0))$ we may continue $u = u(\cdot, u_0)$ to a bounded uniformly continuous function on $[0, t^+(u_0)] \times \overline{\Omega}$. Thus for given $\varepsilon > 0$ there are $\eta, \delta > 0$ with

$$|u(t, x) - u(s, y)| < \varepsilon \quad \text{for } |x - y| < \delta,\ |t - s| < \eta, \quad x, y \in \overline{\Omega},\ \ t, s \in [0, t^+(u_0)]. \tag{5.4.1}$$

[2] We refer to Section 4.3 for a precise definition of a local semiflow.

For $\delta > 0$ we choose a finite number of points $x_l \in \overline{\Omega}$ such that $\bigcup_l B_\delta(x_l)$ covers $\overline{\Omega}$, and a partition of unity $\{\psi_l\}$ for $\overline{\Omega}$ subordinate to this cover. Given $\delta > 0$ we obtain

$$|u|_{\mathbb{E}_1(t_\eta,T)} \leq \sum\nolimits_l |\psi_l u|_{\mathbb{E}_1(t_\eta,T)}. \tag{5.4.2}$$

The function u solves the linear nonautonomous problem[3]

$$\begin{aligned} \partial_t v - \partial_i\big(a_1(u)\partial_i v\big) &= a_2(u)\nabla u + f(u) =: \widetilde{f}_1 && \text{in } \Omega \times (t_\eta, T),\\ \partial_t v + b(\cdot,u)\partial_\nu v - \mathrm{div}_\Gamma(c_1(\cdot,u)\nabla_\Gamma v) &= c_2(\cdot,u)\nabla_\Gamma u + g(\cdot,u) =: \widetilde{g}_1 && \text{on } \Gamma \times (t_\eta, T),\\ v(t_\eta,\cdot) &= u(t_\eta,\cdot) && \text{in } \overline{\Omega}. \end{aligned}$$

Thus for each l the function $w := \psi_l u$ satisfies

$$\begin{aligned} \partial_t w - \partial_i\big(a_1(u)\partial_i w\big) &= \psi_l \widetilde{f}_1 - [\partial_i(a_1(u)\partial_i), \psi_l]u\\ &=: \psi_l \widetilde{f}_1 + \widetilde{f}_2 \qquad \text{in } \Omega \times (t_\eta, T),\\ \partial_t w + b(\cdot,u)\partial_\nu w - \mathrm{div}_\Gamma(c_1(\cdot,u)\nabla_\Gamma w) &= \psi_l \widetilde{g}_1 + b(\cdot,u)[\partial_\nu, \psi_l]u - [\mathrm{div}_\Gamma(c_1(\cdot,u)\nabla_\Gamma), \psi_l]u\\ &=: \psi_l \widetilde{g}_1 + \widetilde{g}_2 \qquad \text{on } \Gamma \times (t_\eta, T),\\ w(t_\eta,\cdot) &= \psi_l u(t_\eta,\cdot) \qquad \text{in } \overline{\Omega}. \end{aligned}$$

Localizing in space and time, we obtain that w satisfies

$$\begin{aligned} \partial_t w - a_1(u(t_\eta,x_l))\Delta w &= \psi_l \widetilde{f}_1 + \widetilde{f}_2 + \partial_i\big(a_1(u) - a_1(u(t_\eta,x_l))\big)\partial_i w\\ &=: \psi_l \widetilde{f}_1 + \widetilde{f}_2 + \widetilde{f}_3 \quad \text{in } \Omega \times (t_\eta, T),\\ \partial_t w + b(x_l, u(t_\eta,x_l))\partial_\nu w &- c_1(x_l, u(t_\eta,x_l))\Delta_\Gamma w\\ &= \psi_l \widetilde{g}_1 + \widetilde{g}_2 + \big(b(\cdot,u) - b(x_l,u(t_\eta,x_l))\big)\partial_\nu w\\ &\quad + \mathrm{div}_\Gamma\big((c_1(\cdot,u) - c_1(x_l u(t_\eta,x_l)))\nabla_\Gamma w\big)\\ &=: \psi_l \widetilde{g}_1 + \widetilde{g}_2 + \widetilde{g}_3 \quad \text{on } \Gamma \times (t_\eta, T),\\ w(t_\eta,\cdot) &= \psi_l u(t_\eta,\cdot) \qquad \text{in } \overline{\Omega}. \end{aligned}$$

By the maximal regularity Theorem 1.2.3 there is a constant C, which does not depend on T, η and δ, such that

$$|w|_{\mathbb{E}_1(t_\eta,T)} \leq C\big(|\big(\psi_l \widetilde{f}_1 + \widetilde{f}_2 + \widetilde{f}_3, \psi_l \widetilde{g}_1 + \widetilde{g}_2 + \widetilde{g}_3\big)|_{\mathbb{E}_0(t_\eta,T)} + |\psi_l u(t_\eta,\cdot)|_{\mathcal{M}}\big). \tag{5.4.3}$$

A compactness argument further yields that C is uniform in $|u|_{BC([0,t^+(u_0)]\times\overline{\Omega},\mathbb{R}^N)}$. Our objective is now to show that for given $\sigma > 0$ an estimate of the form

$$\begin{aligned} |\big(\psi_l \widetilde{f}_1 + \widetilde{f}_2 + \widetilde{f}_3, \psi_l \widetilde{g}_1 + \widetilde{g}_2 + \widetilde{g}_3\big)|_{\mathbb{E}_0(t_\eta,T)} &\leq \sigma |u|_{\mathbb{E}_1(t_\eta,T)}\\ &\quad + C\big(|u(\cdot,u_0)|_{BC([0,t^+(u_0);C^\alpha(\overline{\Omega},\mathbb{R}^N))}, \delta, \eta, \sigma\big) \end{aligned} \tag{5.4.4}$$

is valid. If we then combine (5.4.3) with (5.4.2) and choose σ sufficiently small we may subtract $\frac{1}{2}|u|_{\mathbb{E}_1(t_\eta,T)}$ on both sides of (5.4.2) to obtain the boundedness of $|u|_{\mathbb{E}_1(t_\eta,T)}$ independent of T. Throughout we write $|\cdot|_\infty$ for any occurring sup-norm.

[3] Throughout we neglect the subscript Γ if u is considered on the boundary.

(II) The functions $\psi_l\widetilde{f}_1$, $\widetilde{f}_2$ and $\widetilde{f}_3$ must be estimated in the $L_p(t_\eta, T; L_p(\Omega, \mathbb{R}^N))$-norm. We start with the term

$$\psi_l\widetilde{f}_1 = \psi_l\big(a_2(u)\nabla u + f(u)\big).$$

For the first summand we have for given $\sigma > 0$, using the interpolation inequality and Young's inequality,

$$|\psi_l a_2(u)\nabla u|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} \le C(|u|_\infty)|\nabla u|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^{N\times n}))} \le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty, \sigma).$$

The next term is easily estimated by

$$|\psi_l f(u)|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} \le C(|u|_\infty).$$

We now consider the commutator term $\widetilde{f}_2 = [\partial_i(a_1(u)\partial_i), \psi_l]u$. For each i it holds

$$[\partial_i(a_1(u)\partial_i), \psi_l]u = [a_1'(u)\partial_i u\partial_i, \psi_l]u + [a_1(u)\partial_i\partial_i, \psi_l]u.$$

As above we have that

$$\begin{aligned}
|[a_1'(u)\partial_i u\partial_i, \psi_l]u|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} &= |\partial_i\psi_l a_1'(u)u\partial_i u|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} \\
&\le C(|u|_\infty, \delta)|u|_{L_p(t_\eta,T;W_p^1(\Omega,\mathbb{R}^N))} \\
&\le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty, \delta, \sigma),
\end{aligned}$$

and further

$$\begin{aligned}
|[a_1(u)\partial_i\partial_i, \psi_l]u|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} &\le C(|u|_\infty, \delta)|u|_{L_p(t_\eta,T;W_p^1(\Omega,\mathbb{R}^N))} \\
&\le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty, \delta, \sigma).
\end{aligned}$$

We next consider the term $\widetilde{f}_3 = \partial_i\big(a_1(u) - a_1(u(t_\eta, x_l))\big)\partial_i(\psi_l u)$. For each i we have

$$\begin{aligned}
\partial_i\big(a_1(u) - a_1(u(t_\eta, x_l))\big)\partial_i(\psi_l u) &= \big(a_1(u) - a_1(u(t_\eta, x_l))\big)\partial_i\partial_i(\psi_l u) \\
&\quad + \partial_i\psi_l a_1'(u)u\partial_i u + \psi_l a_1'(u)\partial_i u\partial_i u.
\end{aligned}$$

For the first summand we use (5.4.1) to obtain

$$\begin{aligned}
&|\big(a_1(u) - a_1(u(t_\eta, x_l))\big)\partial_i\partial_i(\psi_l u)|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} \\
&\qquad \le |\big(a_1(u) - a_1(u(t_\eta, x_l))\big)\partial_i\partial_i u|_{L_p(t_\eta,T;L_p(\Omega\cap B_\delta(x_l),\mathbb{R}^N))} \\
&\qquad\qquad + C(|u|_\infty, \delta)|u|_{L_p(t_\eta,T;W_p^1(\Omega,\mathbb{R}^N))} \\
&\qquad \le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty, \delta, \sigma),
\end{aligned}$$

provided δ and η are sufficiently small. We further have, as before,

$$|\partial_i\psi_l a_1'(u)u\partial_i u|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} \le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty, \delta, \sigma).$$

For the next term we observe that, by Hölder's inequality,

$$|\psi_l a_1'(u)\partial_i u\partial_i u|^p_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} \le C(|u|_\infty)\int_{t_\eta}^T |u(t)|^{2p}_{W^1_{2p}(\Omega,\mathbb{R}^N)}\,\mathrm{d}t.$$

The Gagliardo-Nirenberg inequality (Proposition A.6.2) yields a number $s \in (0,2)$, close to 2, such that

$$|u(t)|^{2p}_{W^1_{2p}(\Omega,\mathbb{R}^N)} \le C|u(t)|^p_{W^s_p(\Omega,\mathbb{R}^N)}|u(t)|^p_{C^\alpha(\overline{\Omega},\mathbb{R}^N)}, \qquad t \in (t_\eta, t^+(u_0)).$$

Therefore, again by the interpolation inequality and Young's inequality,

$$\begin{aligned}|\psi_l a_1'(u)\partial_i u \partial_i u|_{L_p(t_\eta,T;L_p(\Omega,\mathbb{R}^N))} &\le C(|u|_{BC([0,t^+(u_0));C^\alpha(\overline{\Omega},\mathbb{R}^N))})|u|_{L_p(t_\eta,T;W^s_p(\Omega,\mathbb{R}^N))}\\ &\le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_{BC([0,t^+(u_0));C^\alpha(\overline{\Omega},\mathbb{R}^N))},\sigma).\end{aligned}$$

We have thus estimated the terms $\psi_l\widetilde{f}_1$, $\widetilde{f}_2$ and $\widetilde{f}_3$ as desired for (5.4.4).
(III) We next treat the terms $\psi_l\widetilde{g}_1$, $\widetilde{g}_2$ and $\widetilde{g}_3$ in the $L_p(t_\eta,T;W^{1-1/p}_p(\Gamma,\mathbb{R}^N))$-norm. It follows from Lemma 1.3.20 that

$$|\varphi\phi|_{W^{1-1/p}_p(\Gamma,\mathbb{R}^N)} \lesssim |\varphi|_\infty|\phi|_{W^{1-1/p}_p(\Gamma,\mathbb{R}^N)} + |\varphi|_{W^{1-1/p}_p(\Gamma,\mathbb{R}^N)}|\phi|_\infty \tag{5.4.5}$$

for all $\varphi \in W^{1-1/p}_p(\Gamma,\mathcal{B}(\mathbb{R}^N))$ and $\phi \in W^{1-1/p}_p(\Gamma,\mathbb{R}^N)$. It can further be shown as in the proof of Lemma 4.2.3 that for a smooth function $h : \Gamma\times\mathbb{R}^N \to \mathbb{R}^N$ it holds

$$|h(\cdot,\phi)|_{W^{1-1/p}_p(\Gamma,\mathbb{R}^N)} \le C(|\phi|_\infty)\big(1 + |\phi|_{W^{1-1/p}_p(\Gamma,\mathbb{R}^N)}\big). \tag{5.4.6}$$

We start with

$$\psi_l\widetilde{g}_1 = \psi_l\big(c_2(\cdot,u)\nabla_\Gamma u + g(\cdot,u)\big).$$

For the first summand we have, using (5.4.5) and (5.4.6),

$$\begin{aligned}|\psi_l c_2(\cdot,u)\nabla_\Gamma u|_{L_p(t_\eta,T;W^{1-1/p}_p(\Gamma,\mathbb{R}^N))} &\le C(\delta)|c_2(\cdot,u)\nabla_\Gamma u|_{L_p(t_\eta,T;W^{1-1/p}_p(\Gamma,\mathbb{R}^{N\times n}))}\\ &\le C(|u|_\infty,\delta)|u|_{L_p(t_\eta,T;W^{2-1/p}_p(\Gamma,\mathbb{R}^N))} + C(\delta)|c_2(\cdot,u)|_{L_p(t_\eta,T;W^{1-1/p}_p(\Gamma,\mathbb{R}^N))}|\nabla_\Gamma u|_\infty\\ &\le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty,\delta,\sigma) + C(|u|_\infty,\delta)(1+|u|_{L_p(t_\eta,T;W^{1-1/p}_p(\Gamma,\mathbb{R}^N))})|\nabla_\Gamma u|_\infty.\end{aligned}$$

For sufficiently small $\tau > 0$ the Gagliardo-Nirenberg inequality yields

$$|u(t)|_{W^{1+(n-1)/p+\tau}_p(\Gamma,\mathbb{R}^N)} \lesssim |u(t)|^\theta_{W^{3-3/p}_p(\Gamma,\mathbb{R}^N)}|u(t)|^{1-\theta}_{C^\alpha(\overline{\Omega},\mathbb{R}^N)},$$

provided $\theta \in (0,1)$ satisfies $1 < \theta\big(3 - \frac{n+2}{p}\big) + (1-\theta)\alpha$. This inequality can be fulfilled by some $\theta < \frac{1}{3-1/p}$, provided p is sufficiently large compared to $\frac{1}{\alpha}$. We use this fact together with the embedding $\mathbb{E}_1(t_\eta,T) \hookrightarrow L_\infty(t_\eta,T;\mathcal{M})$ to obtain

$$\begin{aligned}|\nabla_\Gamma u|_\infty &\le C\,|u|_{L_\infty(t_\eta,T;W^{1+(n-1)/p+\tau}_p(\Gamma,\mathbb{R}^N))}\\ &\le C(|u|_{BC([0,t^+(u_0));C^\alpha(\overline{\Omega},\mathbb{R}^N))})|u|^\theta_{L_\infty(t_\eta,T;W^{3-3/p}_p(\Gamma,\mathbb{R}^N))}\\ &\le C(|u|_{BC([0,t^+(u_0));C^\alpha(\overline{\Omega},\mathbb{R}^N))},\eta)|u|^\theta_{\mathbb{E}_1(t_\eta,T)}\end{aligned} \tag{5.4.7}$$

for some $\theta < \frac{1}{3-1/p}$. By Young's inequality we thus have

$$C(|u|_\infty,\delta)|\nabla_\Gamma u|_\infty \le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_{BC([0,t^+(u_0));C^\alpha(\overline{\Omega},\mathbb{R}^N))},\delta,\eta,\sigma).$$

The interpolation inequality implies that

$$|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} \le C(|u|_\infty)\,|u|_{\mathbb{E}_1(t_\eta,T)}^{\frac{1-1/p}{3-1/p}}. \tag{5.4.8}$$

Combining this estimate with (5.4.7), we obtain

$$\begin{aligned} &C(|u|_\infty,\delta)|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}|\nabla_\Gamma u|_\infty \\ &\qquad\qquad \le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_{BC([0,t^+(u_0));C^\alpha(\overline{\Omega},\mathbb{R}^N))},\delta,\eta,\sigma), \end{aligned}$$

which finishes the estimates for the first summand of $\psi_l\widetilde{g}_1$. Using (5.4.5) and (5.4.6), the second summand of $\psi_l\widetilde{g}_1$ is estimated by

$$\begin{aligned} |\psi_l g(\cdot,u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} &\le C(|u|_\infty,\delta)\big(1+|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}\big) \\ &\le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty,\delta,\sigma). \end{aligned}$$

We continue with the commutator term

$$\widetilde{g}_2 = b(\cdot,u)[\partial_\nu,\psi_l]u - [\mathrm{div}_\Gamma(c_1(\cdot,u)\nabla_\Gamma),\psi_l]u.$$

For the first summand we use again (5.4.5) and (5.4.6) to obtain

$$\begin{aligned} |b(\cdot,u)[\partial_\nu,\psi_l]u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} &\le C(\delta)|b(\cdot,u)u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} \\ &\le C(|u|_\infty,\delta)\big(1+|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}\big) \\ &\le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_\infty,\delta,\sigma). \end{aligned}$$

For the second summand of $\widetilde{g}_2$ we have

$$\begin{aligned} &|[\mathrm{div}_\Gamma(c_1(\cdot,u)\nabla_\Gamma),\psi_l]u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} \\ &\quad\lesssim |(\Delta_\Gamma\psi_l)c_1(\cdot,u)u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} + |c_1(\cdot,u)\nabla_\Gamma\psi_l\nabla_\Gamma u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} \\ &\qquad + \big|\big(\nabla_\Gamma c_1(\cdot,u)+\partial_u c_1(\cdot,u)\nabla_\Gamma u\big)\big(\nabla_\Gamma(\psi_l u)+\psi_l\nabla_\Gamma u\big)\big|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}. \end{aligned} \tag{5.4.9}$$

Here the first and the second summand may be treated as above. For the third summand we concentrate on the term involving $\nabla_\Gamma(\psi_l u)$. We estimate

$$\begin{aligned} &|\nabla_\Gamma c_1(\cdot,u)\nabla_\Gamma(\psi_l u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))} \\ &\qquad \le C(|u|_\infty)|\nabla_\Gamma(\psi_l u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^{N\times n}))} \\ &\qquad\qquad + C(1+|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))})|\nabla_\Gamma(\psi_l u)|_\infty \\ &\qquad \le C(|u|_\infty,\delta)|u|_{L_p(t_\eta,T;W_p^{2-1/p}(\Gamma,\mathbb{R}^N))} \\ &\qquad\qquad + C(\delta)(1+|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))})|u|_{L_\infty(t_\eta,T;W^1_\infty(\Gamma,\mathbb{R}^N))}. \end{aligned}$$

Note that the first summand is of lower order. Using (5.4.7) and (5.4.8) we obtain that also the second term is of lower order. We further have

$$\begin{aligned}
&|\partial_u c_1(\cdot,u)\nabla_\Gamma u \nabla_\Gamma(\psi_l u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}\\
&\qquad\le C(|u|_\infty)(1+|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))})|\nabla_\Gamma u\nabla_\Gamma(\psi_l u)|_\infty\\
&\qquad\quad + C(|u|_\infty)|\nabla_\Gamma u\nabla_\Gamma(\psi_l u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^{N\times n}))}\\
&\qquad\le C(|u|_\infty,\delta)(1+|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))})|u|^2_{L_\infty(t_\eta,T;W^1_\infty(\Gamma,\mathbb{R}^N))}\\
&\qquad\quad + C(|u|_\infty,\delta)|u|_{L_\infty(t_\eta,T;W^1_\infty(\Gamma,\mathbb{R}^N))}|u|_{L_p(t_\eta,T;W_p^{2-1/p}(\Gamma,\mathbb{R}^N))}.
\end{aligned}$$

Since $\theta < \frac{1}{3-1/p}$ it follows from (5.4.7), (5.4.8) and Young's inequality that here the first term may be estimated as desired. Moreover, the interpolation inequality yields

$$|u|_{L_p(t_\eta,T;W_p^{2-1/p}(\Gamma,\mathbb{R}^N))} \le C(|u|_\infty)|u|_{\mathbb{E}_1(t_\eta,T)}^{\frac{2-1/p}{3-1/p}}.$$

Combining this with (5.4.7) we obtain the desired estimate also for the second term of $\widetilde{g}_2$, and this finishes the estimates for this term. We finally consider

$$\widetilde{g}_3 = \big(b(\cdot,u)-b(x_l,u(t_\eta,x_l))\big)\partial_\nu(\psi_l u) + \mathrm{div}_\Gamma\big((c_1(\cdot,u)-c_1(x_l,u(t_\eta,x_l)))\nabla_\Gamma(\psi_l u)\big).$$

For the first summand, choosing δ and η sufficiently small and using (5.4.1), (5.4.5), (5.4.7) and (5.4.8) we obtain

$$\begin{aligned}
&|\big(b(\cdot,u)-b(x_l,u(t_\eta,x_l))\big)\partial_\nu(\psi_l u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}\\
&\qquad\le C\,|\psi_l(b(\cdot,u)-b(x_l,u(t_\eta,x_l)))|_\infty|\partial_\nu u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}\\
&\qquad\quad + C(|u|_\infty,\delta)(1+|u|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))})|u|_{L_\infty(t_\eta,T;W^1_\infty(\Gamma,\mathbb{R}^N))}\\
&\qquad\le \sigma|u|_{\mathbb{E}_1(t_\eta,T)} + C(|u|_{BC([0,t^+(u_0));C^\alpha(\overline{\Omega},\mathbb{R}^N))},\delta,\sigma).
\end{aligned}$$

We further have

$$\begin{aligned}
&|\mathrm{div}_\Gamma\big((c_1(\cdot,u)-c_1(x_l,u(t_\eta,x_l)))\nabla_\Gamma(\psi_l u)\big)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}\\
&\qquad\le |(c_1(\cdot,u)-c_1(x_l,u(t_\eta,x_l)))\Delta_\Gamma(\psi_l u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}\\
&\qquad\quad + |(\nabla_\Gamma c_1(\cdot,u)+\partial_u c_1(\cdot,u)\nabla_\Gamma u)\nabla_\Gamma(\psi_l u)|_{L_p(t_\eta,T;W_p^{1-1/p}(\Gamma,\mathbb{R}^N))}.
\end{aligned}$$

Choosing δ and η sufficiently small, using (5.4.1), (5.4.5) and the treating lower order terms as before, here the first term may be estimated as desired. The second term is of the same type is the one in (5.4.9). This finishes the proof. ■

It is now natural to ask for sufficient conditions on the nonlinearities in (5.1.1) that guarantee an a priori Hölder bound of solutions. This will be subject to future work.

5.5 The Global Attractor for Semilinear Dissipative Systems

We now fix

$$p \in (n+2,\infty)$$

and investigate the long-time behaviour of the following semilinear version of (5.1.1),

$$\begin{aligned} \partial_t u &= \Delta u + f(u) && \text{on } \Omega, && t>0,\\ \partial_t u + \partial_\nu u &= \Delta_\Gamma u + g(u) && \text{on } \Gamma, && t>0,\\ u(0,\cdot) &= u_0 && \text{on } \overline{\Omega}, \end{aligned} \tag{5.5.1}$$

where the reaction terms $f, g : \mathbb{R}^N \to \mathbb{R}^N$ are assumed to be smooth. Theorem 5.3.3 implies that (5.5.1) generates a compact local semiflow of solutions in the phase space

$$\mathcal{M} = \big\{(v, v_\Gamma) \in W_p^{2-2/p}(\Omega, \mathbb{R}^N) \times W_p^{3-3/p}(\Gamma, \mathbb{R}^N) \;:\; \mathrm{tr}_\Omega v = v_\Gamma\big\},$$

However, due to [26, Theorem 2.2], the realization of the operator

$$\mathbb{A} = \begin{pmatrix} -\Delta & 0 \\ \partial_\nu & -\Delta_\Gamma \end{pmatrix}$$

on $X_0 = L_p(\Omega, \mathbb{R}^N) \times W_p^{1-1/p}(\Gamma, \mathbb{R}^N)$ with domain

$$D(\mathbb{A}) = X_1 = \big\{(v, v_\Gamma) \in W_p^2(\Omega, \mathbb{R}^N) \times W_p^{3-1/p}(\Gamma, \mathbb{R}^N) \;:\; \mathrm{tr}_\Omega v = v_\Gamma\big\}$$

enjoys maximal L_p-regularity on each finite interval $J = (0, T)$, and $-\mathbb{A}$ generates an analytic C_0-semigroup on X_0. Thus local well-posedness of (5.5.1) also follows from semilinear theory [51, Chapter 3]. It is now a simple consequence of the variation of constants formula that for $u_0 \in \mathcal{M}$ the corresponding maximal solution $u(\cdot, u_0)$ of (5.5.1) has the additional regularity properties

$$u(\cdot, u_0) \in C\big(0, t^+(u_0); X_1\big) \cap C^1\big(0, t^+(u_0); \mathcal{M}\big), \tag{5.5.2}$$

see [16, Corollary 2.3.1]. Interested in the long-time behaviour of solutions, we may thus assume that $u_0 \in X_1$ for the initial values of (5.5.1).

For $q \in (1, \infty)$ and $s > 0$ with $s \neq 1/q$ we introduce the Banach spaces

$$\mathcal{M}_q^s := \begin{cases} B_{q,q}^s(\Omega, \mathbb{R}^N) \times B_{q,q}^{s+1-1/q}(\Gamma, \mathbb{R}^N), & s < 1/q,\\ \big\{(u, u_\Gamma) \in B_{q,q}^s(\Omega, \mathbb{R}^N) \times B_{q,q}^{s+1-1/q}(\Gamma, \mathbb{R}^N) \;:\; \mathrm{tr}_\Omega u = u_\Gamma\big\}, & s > 1/q, \end{cases}$$

equipped with the norm of $B_{q,q}^s(\Omega, \mathbb{R}^N) \times B_{q,q}^{s+1-1/q}(\Gamma, \mathbb{R}^N)$, respectively. Observe that

$$\mathcal{M} = \mathcal{M}_p^{2-2/p}.$$

Using the maximal $L_{p,\mu}$-regularity Theorem 5.2.1 for $\mathcal{A}$, one can argue in the same way as in the proof of Lemma 4.4.1 to obtain the following result. It shows that the solution in a strong norm can be controlled by the solution in weaker norm.

Lemma 5.5.1. *Let $q \in (1, p]$ and $\mu \in (1/q, 1]$, set $s = 2(\mu - 1/q) \in (0, 2 - 2/q]$, and assume that $s \notin \{1/q, 1+1/q\}$. Then for $\tau > 0$ there is a constant $C = C(|u(\cdot, u_0)|_{C([T_1,T_2]\times\overline{\Omega},\mathbb{R}^N)}, \tau)$ such that*

$$|u(T_2, u_0)|_{\mathcal{M}_q^{2-2/q}} \le C\big(1 + |u(T_1, u_0)|_{\mathcal{M}_q^s}\big)$$

is valid for all $0 < T_1 < T_2 < +\infty$ with $\tau = T_2 - T_1$ and all $u_0 \in X_1$ with $t^+(u_0) < T_2$, where $u(\cdot, u_0)$ denotes the corresponding solution of (5.5.1).

As a consequence we have the following sufficient conditions for global existence and relatively compact orbits.

Proposition 5.5.2. *Let $u_0 \in X_1$, and suppose that the corresponding solution of (5.5.1) satisfies $u(\cdot, u_0) \in BC([0, t^+(u_0)) \times \overline{\Omega}, \mathbb{R}^N)$. Then $t^+(u_0) = +\infty$. If it additionally holds that $\{u(t, u_0)\}_{t\in[0,\infty)}$ is bounded in $\mathcal{M}_p^s$ for some $s > 0$, then $\{u(t, u_0)\}_{t\in[0,\infty)}$ is relatively compact in $\mathcal{M}$.*

Proof. **(I)** Suppose that $t^+(u_0) < \infty$. It then follows from Lemma 5.5.1 that

$$|u(T, u_0)|_{\mathcal{M}} \lesssim \sup_{t\in[0,t^+(u_0)/2)} \big(1 + |u(t, u_0)|_{\mathcal{M}}\big)$$

for all $T \in (t^+(u_0)/2, t^+(u_0))$. Thus the orbit is bounded in $\mathcal{M}$, which contradicts Theorem 5.3.3 and yields $t^+(u_0) = +\infty$.
(II) Now suppose in addition that $\{u(t, u_0)\}_{t\in[0,\infty)}$ is bounded in $\mathcal{M}_p^s$ for some $s > 0$. Then another application of Lemma 5.5.1 yields

$$|u(T + 1, u_0)|_{\mathcal{M}} \lesssim 1 + |u(T, u_0)|_{\mathcal{M}_p^s}$$

for all $T \geq 1$. Thus $\{u(t, u_0)\}_{t\in[1,\infty)}$ is bounded in $\mathcal{M}$, and the relative compactness of the orbit follows again from Theorem 5.3.3. ■

We next want to establish an L_∞ a priori estimate for (5.5.1) for a class of reaction terms $f, g : \mathbb{R}^N \to \mathbb{R}^N$. We first show that if

$$\zeta f(\zeta) \leq C(1 + |\zeta|^2), \qquad \zeta g(\zeta) \leq C(1 + |\zeta|^2), \qquad \zeta \in \mathbb{R}^N, \tag{5.5.3}$$

is valid for a constant $C > 0$, then the L_∞-norm of a solution can be controlled by its L_1-norm. Observe that (5.5.3) is a sign condition for large $|\zeta|$. We further derive an L_∞-estimate for the equilibria of (5.5.1) under the above assumption. Note that due to (5.5.2), each equilibrium of (5.5.1) must belong to $X_1 \hookrightarrow W_p^2(\Omega, \mathbb{R}^N) \times W_p^{3-1/p}(\Gamma, \mathbb{R}^N)$.

Lemma 5.5.3. *Assume that (5.5.3) holds true. Then for each $u_0 \in X_1$ there is a constant C_1 such that the corresponding solution $u(\cdot, u_0)$ of (5.5.1) satisfies*

$$|u(\cdot, u_0)|_{BC([0,t^+(u_0))\times\overline{\Omega},\mathbb{R}^N)} \leq C_1 \max\big\{|u(\cdot, u_0)|_{BC([0,t^+(u_0)),L_1(\Omega,\mathbb{R}^N)\times L_1(\Gamma,\mathbb{R}^N))}, 1\big\}.$$

Moreover, there is a constant $C_2 > 0$ such that for each equilibrium $u_0 \in X_1$ of (5.5.1) it holds

$$|u_0|_{BC(\overline{\Omega},\mathbb{R}^N)} \leq C_2 \max\big\{|u_0|_{L_1(\Omega,\mathbb{R}^N)\times L_1(\Gamma,\mathbb{R}^N)}, 1\big\}.$$

Proof. We use a Moser-Alikakos iteration procedure, presented in [16, Section 9.3] for scalar problems with static boundary conditions. Given $t \in (0, t^+(u_0))$ and $k \in \mathbb{N}$, the plan is to find an upper bound for the $L_{2^k}(\Omega, \mathbb{R}^N) \times L_{2^k}(\Gamma, \mathbb{R}^N)$-norm of $u(t, u_0)$, which is independent of t and k. With a slight abuse of notation we write $u = u(t, u_0)$ for fixed t. Recall that it holds $u \in X_1$ by (5.5.2).

(I) We take the scalar product in $\mathbb{R}^N$ of the domain equation $\partial_t u = \Delta u + f(u)$ at time t with $|u|^{2^k-2}u$, integrate over Ω, and integrate by parts, to obtain

$$\begin{aligned}\frac{1}{2^k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{2^k}\,\mathrm{d}x &= \int_\Omega \Delta u\cdot|u|^{2^k-2}u\,\mathrm{d}x + \int_\Omega f(u)\cdot|u|^{2^k-2}u\,\mathrm{d}x\\ &= -\int_\Omega\sum_{i=1}^N \nabla u_i\cdot\nabla(|u|^{2^k-2}u_i)\,\mathrm{d}x + \int_\Gamma \partial_\nu u\cdot|u|^{2^k-2}u\,\mathrm{d}\sigma(x) + \int_\Omega f(u)\cdot|u|^{2^k-2}u\,\mathrm{d}x.\end{aligned}$$

This manipulation is justified due to $u\in X_1$. Now suppose that $k\geq 2$. For the integrand of the first summand we have

$$\begin{aligned}\sum_{i=1}^N \nabla u_i\cdot\nabla(|u|^{2^k-2}u_i) &= \sum_{i=1}^N\sum_{j=1}^n \partial_j u_i\partial_j(|u|^{2^k-2}u_i)\\ &= (2^k-2)\sum_{j=1}^n |u|^{2^k-4}|\partial_j u\cdot u|^2 + \sum_{j=1}^n |u|^{2^k-2}|\partial_j u|^2\\ &\geq (2^k-2)\sum_{j=1}^n |u|^{2^k-4}|\partial_j u\cdot u|^2 + \sum_{j=1}^n |u|^{2^k-4}|\partial_j u|^2|u|^2(\cos(\partial_j u,u))^2\\ &= (2^k-1)\sum_{j=1}^n |u|^{2^k-4}|\partial_j u\cdot u|^2.\end{aligned}$$

On the other hand it holds

$$(\partial_j|u|^{2^{k-1}})^2 = |2^{k-1}|u|^{2^{k-1}-2}\partial_j u\cdot u|^2 = 2^{2k-2}|u|^{2^k-4}|\partial_j u\cdot u|^2,$$

so that we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{2^k}\,\mathrm{d}x &\leq -(2^k-1)\,2^{2-k}\int_\Omega |\nabla|u|^{2^{k-1}}|^2\,\mathrm{d}x\\ &\quad + 2^k\int_\Gamma \partial_\nu u\cdot|u|^{2^k-2}u\,\mathrm{d}\sigma(x) + 2^k\int_\Omega f(u)\cdot|u|^{2^k-2}u\,\mathrm{d}x.\end{aligned}$$

Note that this estimate is also true for $k=1$, with $|\nabla|u|^{2^{k-1}}|^2$ replaced by $|\nabla u|^2$. Similarly, taking the scalar product in $\mathbb{R}^N$ of the boundary equation $\partial_t u = \Delta_\Gamma u - \partial_\nu u + g(u)$ at time t with $|u|^{2^k-2}u$ and applying the surface divergence theorem on Γ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma |u|^{2^k}\,\mathrm{d}\sigma(x) &\leq -(2^k-1)\,2^{2-k}\int_\Gamma |\nabla_\Gamma|u|^{2^{k-1}}|^2\,\mathrm{d}\sigma(x)\\ &\quad - 2^k\int_\Gamma \partial_\nu u\cdot|u|^{2^k-2}u\,\mathrm{d}\sigma(x) + 2^k\int_\Gamma g(u)\cdot|u|^{2^k-2}u\,\mathrm{d}\sigma(x).\end{aligned}$$

Again this is justified due to $u\in X_1$. Adding these estimates and observing that for each k it holds $-(2^k-1)\,2^{2-k}\leq -2$, we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big) &\leq -C\big(|\nabla|u|^{2^{k-1}}|^2_{L_2(\Omega,\mathbb{R}^N)} + |\nabla_\Gamma|u|^{2^{k-1}}|^2_{L_2(\Gamma,\mathbb{R}^N)}\big)\\ &\quad + 2^k\int_\Omega f(u)\cdot|u|^{2^k-2}u\,\mathrm{d}x + 2^k\int_\Gamma g(u)\cdot|u|^{2^k-2}u\,\mathrm{d}\sigma(x).\end{aligned}\tag{5.5.4}$$

(II) Using the sign condition (5.5.3) and that $|\zeta|^{2^k-2} \le |\zeta|^{2^k} + 1$ for $\zeta \in \mathbb{R}^N$, we estimate the integral terms in (5.5.4) by a constant multiple of

$$2^k\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big) + 2^k.$$

It follows from the Gagliardo-Nirenberg inequality (Proposition A.6.1) and Young's inequality that for $\varepsilon \in (0,1)$ it holds

$$|v|_{L_2(\Omega,\mathbb{R}^N)} \le C\,|v|^{\frac{n}{n+2}}_{W^1_2(\Omega,\mathbb{R}^N)}|v|^{1-\frac{n}{n+2}}_{L_1(\Omega,\mathbb{R}^N)} \le \varepsilon|v|_{W^1_2(\Omega,\mathbb{R}^N)} + C\varepsilon^{-n/2}|v|_{L_1(\Omega,\mathbb{R}^N)}.$$

From this inequality we obtain

$$-|\nabla v|^2_{L_2(\Omega,\mathbb{R}^N)} \le -\frac{1-\varepsilon}{\varepsilon}|v|^2_{L_2(\Omega,\mathbb{R}^N)} + C\varepsilon^{-n/2-1}|v|^2_{L_1(\Omega,\mathbb{R}^N)}.$$

Note that this estimate remains valid if one replaces Ω by Γ and n by $n-1$, respectively. Using that $\varepsilon^{-(n-1)/2-1} \le \varepsilon^{-n/2-1}$ for $\varepsilon \in (0,1)$ we may estimate the gradient terms in (5.5.4) by

$$-C\,\frac{1-\varepsilon}{\varepsilon}\,\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big) + C\,\varepsilon^{-n/2-1}\big(||u|^{2^{k-1}}|^2_{L_1(\Omega)} + ||u|^{2^{k-1}}|^2_{L_1(\Gamma)}\big).$$

We therefore obtain from (5.5.4) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big) &\le C\,\left(-\frac{1-\varepsilon}{\varepsilon} + 2^k\right)\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big)\\ &\quad + C\,\varepsilon^{-n/2-1}\big(||u|^{2^{k-1}}|^2_{L_1(\Omega)} + ||u|^{2^{k-1}}|^2_{L_1(\Gamma)}\big) + C\,2^k.\end{aligned}$$

Now we choose $\varepsilon = \delta\,2^{-k}$ with small $\delta > 0$ such that

$$C\,\left(-\frac{1-\varepsilon}{\varepsilon} + 2^k\right) \le -2^k.$$

We further observe that

$$||u|^{2^{k-1}}|^2_{L_1(\Omega)} + ||u|^{2^{k-1}}|^2_{L_1(\Gamma)} \le \big(|u|_{L_{2^{k-1}}(\Omega,\mathbb{R}^N)} + |u|_{L_{2^{k-1}}(\Gamma,\mathbb{R}^N)}\big)^{2^k}.$$

Therefore, setting

$$m_k := \sup_{t\in[0,t^+(u_0))}\big(|u|_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big), \qquad k \in \mathbb{N}_0,$$

we arrive at the estimate

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big)\\ &\qquad \le -2^k\,\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big) + C\,(2^k)^{n/2+1}m^{2^k}_{k-1} + C\,2^k. \qquad (5.5.5)\end{aligned}$$

(III) Now suppose that m_0 is finite. Then the Gronwall's lemma yields, inductively,

$$\big(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\big) \le C\,\max\big\{|u_0|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u_0|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}, 2^{kn/2}m^{2^k}_{k-1} + 1\big\},$$

and in particular that each number m_k is finite. Taking the 2^k-th roots on both sides and the supremum over t on the left-hand side we obtain

$$\begin{aligned} m_k &\le 2 \sup_{t\in[0,t^+(u_0))} \left(|u|^{2^k}_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u|^{2^k}_{L_{2^k}(\Gamma,\mathbb{R}^N)}\right)^{1/2^k} \\ &\le C \max\left\{|u_0|_{L^{2^k}(\Omega,\mathbb{R}^N)} + |u_0|_{L^{2^k}(\Gamma,\mathbb{R}^N)}, (2^{kn/2} m_{k-1}^{2^k} + 1)^{1/2^k}\right\}. \end{aligned}$$

There is a constant C, independent of k, such that

$$|u_0|_{L^{2^k}(\Omega,\mathbb{R}^N)} + |u_0|_{L^{2^k}(\Gamma,\mathbb{R}^N)} \le C\,|u_0|_{BC(\overline{\Omega},\mathbb{R}^N)}.$$

Thus the sequence $(m_k)_{k\in\mathbb{N}}$ satisfies the recursive estimate

$$m_k \le C \max\left\{1, (2^{kn/2} m_{k-1}^{2^k} + 1)^{1/2^k}\right\},$$

with $C \ge 1$, and is therefore dominated by the sequence $(x_k)_{k\in\mathbb{N}}$, defined by

$$x_0 = C\max\{m_0, 1\}, \qquad x_k = (2^{kn/2})^{1/2^k}\, x_{k-1}, \qquad k \in \mathbb{N}.$$

Since

$$\lim_{k\to\infty} x_k = C\, x_0 \prod_{k=1}^{\infty} (2^{kn/2})^{1/2^k} = C\, 2^n \max\{m_0, 1\},$$

we obtain

$$|u(\cdot, u_0)|_{BC([0,t^+(u_0))\times\overline{\Omega},\mathbb{R}^N)} \le 2 \limsup_{k\to\infty} m_k \le C\max\{m_0, 1\}.$$

This shows the asserted estimate for an arbitrary initial value $u_0 \in X_1$.

(IV) Now suppose that $u_0 \in X_1$ is an equilibrium of (5.1.1). Using (5.5.5) directly yields

$$m_k \le C(2^{kn/2} m_{k-1}^{2^k} + 1)^{1/2^k} \le C\max\left\{2^{kn/2^{k+1}} m_{k-1}, 1\right\},$$

where now simply $m_k = |u_0|_{L_{2^k}(\Omega,\mathbb{R}^N)} + |u_0|_{L_{2^k}(\Gamma,\mathbb{R}^N)}$. As above we conclude that

$$|u_0|_{BC(\overline{\Omega},\mathbb{R}^N)} \le C_2 \max\left\{|u_0|_{L_1(\Omega,\mathbb{R}^N)} + |u_0|_{L_1(\Gamma,\mathbb{R}^N)}, 1\right\},$$

and C_2 is independent of u_0 since the constant arising in (5.5.5) is independent of it. ∎

The above lemma and Proposition 5.5.2 show that for the global existence of a solution of (5.5.1) it suffices to find an a priori L_1 bound, provided the reaction terms satisfy (5.5.3). We now consider a class of reaction terms where such an L_1 bound can in particular be obtained.

We assume that (5.5.1) is *conservative*, i.e., there are potentials $F, G : \mathbb{R}^N \to \mathbb{R}$ with

$$-\nabla F = f, \qquad -\nabla G = g, \qquad F(0) = G(0) = 0.$$

We further assume that (5.5.1) is *dissipative*, in the following sense. There are numbers $c_i, d_i \in \mathbb{R}$, $i = 1, ..., N$, such that

$$\limsup_{|\zeta_i|\to\infty} \frac{f_i(\zeta)}{\zeta_i} < c_i, \qquad \limsup_{|\zeta_i|\to\infty} \frac{g_i(\zeta)}{\zeta_i} < d_i, \qquad i = 1, ..., N, \tag{5.5.6}$$

and there is $\eta > 0$ such that for $i = 1, ..., N$ it holds

$$\frac{|\nabla\psi|^2_{L_2(\Omega,\mathbb{R}^{N\times n})} + |\nabla_\Gamma\psi|^2_{L_2(\Gamma,\mathbb{R}^{N\times n})} - 2c_i|\psi|^2_{L_2(\Omega,\mathbb{R}^N)} - 2d_i|\psi|^2_{L_2(\Gamma,\mathbb{R}^N)}}{|\psi|^2_{L_2(\Omega,\mathbb{R}^N)} + |\psi|^2_{L_2(\Gamma,\mathbb{R}^N)}} \geq \eta \tag{5.5.7}$$

for all $\psi \in W^1_2(\Omega, \mathbb{R}^N) \cap W^1_2(\Gamma, \mathbb{R}^N)$. Observe that (5.5.7) is always satisfied for $c_i, d_i < 0$. But it may happen that (5.5.7) is valid although $c_i > 0$ and $d_i < 0$, or vice versa. In this sense the interplay between the reaction terms in Ω and on Γ determines if (5.5.1) is dissipative, and the non-dissipativeness of one reaction term can be compensated by the other. This is analogous to the dissipativity condition in [15] for nonlinear Robin boundary conditions.

We record some simple consequences of the above assumptions.

Lemma 5.5.4. *Assume that f and g are conservative and dissipative. Then there is a constant $c_0 \in \mathbb{R}$ such that*

$$f_i(\zeta)\zeta_i \leq c_i\zeta_i^2 + c_0, \qquad g_i(\zeta)\zeta_i \leq d_i\zeta_i^2 + c_0, \qquad i = 1, ..., N.$$

In particular, f and g satisfy (5.5.3). Moreover, for $\zeta \in \mathbb{R}^N$ it holds

$$F(\zeta) \geq -\sum_{i=1}^N \frac{c_i}{2}\zeta_i^2 - Nc_0, \qquad G(\zeta) \geq -\sum_{i=1}^N \frac{d_i}{2}\zeta_i^2 - Nc_0.$$

Proof. The first assertion is clear. For $\zeta \in \mathbb{R}^N$ we set $\zeta' = (0, \zeta_2, ..., \zeta_N)$ and calculate

$$F(\zeta) = F(\zeta') - \int_0^1 f_1(s\zeta_1, \zeta_2, ..., \zeta_N)\zeta_1 \,\mathrm{d}s \geq F(\zeta') - \frac{c_1}{2}\zeta_1^2 - c_0.$$

Iterating this argument with the remaining $N-1$ variables yields second assertion. ■

The assumption that f and g are conservative allow to construct a Lyapunov function for (5.5.1), which already appeared in [80]. We define $\mathcal{V} : \mathcal{M} \to \mathbb{R}$ by

$$\mathcal{V}(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2 \,\mathrm{d}x + \int_\Omega F(\phi)\,\mathrm{d}x + \frac{1}{2}\int_\Gamma |\nabla_\Gamma\phi|^2 \,\mathrm{d}\sigma(x) + \int_\Gamma G(\phi)\,\mathrm{d}\sigma(x).$$

Note that $\mathcal{V}$ is well-defined and continuous, due to

$$\mathcal{M} \hookrightarrow W^1_p(\Omega, \mathbb{R}^N) \cap W^1_p(\Gamma, \mathbb{R}^N) \cap BC(\overline{\Omega}, \mathbb{R}^N).$$

Let $u = u(\cdot, u_0)$ be the solution of (5.5.1) with initial value $u_0 \in \mathcal{M}$, and $t \in (0, t^+(u_0))$. Since $u(t) \in X_1$ we may integrate by by parts to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(u(t)) &= -\int_\Omega (\Delta u(t) + f(u(t))) \cdot \partial_t u(t)\,\mathrm{d}x \\ &\quad - \int_\Gamma (\Delta_\Gamma u(t) - \partial_\nu u(t) + g(u(t))) \cdot \partial_t u(t)\,\mathrm{d}\sigma(x) \\ &= -|\partial_t u(t)|^2_{L_2(\Omega,\mathbb{R}^N)} - |\partial_t u(t)|^2_{L_2(\Gamma,\mathbb{R}^N)}.\end{aligned} \tag{5.5.8}$$

Hence $\mathcal{V}$ is nonincreasing along solutions of (5.5.1), and it is constant only along equilibria. Therefore $\mathcal{V}$ is a strict Lyapunov function for (5.5.1).[4]
Lemma 5.5.4 and assumption (5.5.7) also allow to obtain an energy estimate for (5.5.1), as follows. From (5.5.8) we obtain that for $u_0 \in \mathcal{M}$ and $t \in (0, t^+(u_0))$ it holds $\mathcal{V}(u(t)) \leq \mathcal{V}(u_0)$, and further the estimate

$$\begin{aligned}\mathcal{V}(\phi) &\gtrsim \sum_{i=1}^{N} \big(|\nabla\phi|^2_{L_2(\Omega,\mathbb{R}^{N\times n})} + |\nabla_\Gamma\phi|^2_{L_2(\Gamma,\mathbb{R}^{N\times n})} - 2c_i|\phi|^2_{L_2(\Omega,\mathbb{R}^N)} - 2d_i|\phi|^2_{L_2(\Gamma,\mathbb{R}^N)}\big)\\ &\qquad + |\nabla\phi|^2_{L_2(\Omega,\mathbb{R}^{N\times n})} + |\nabla_\Gamma\phi|^2_{L_2(\Gamma,\mathbb{R}^{N\times n})} - 1\\ &\gtrsim |\phi|^2_{W_2^1(\Omega,\mathbb{R}^N)} + |\phi|^2_{W_2^1(\Gamma,\mathbb{R}^N)} - 1\end{aligned}$$

holds true. The above considerations may be summarized as follows.

Lemma 5.5.5. *Suppose that (5.5.1) is conservative and dissipative, and let $p \in (n+2, \infty)$. Then $\mathcal{V} : \mathcal{M} \to \mathbb{R}$ is a strict Lyapunov function for (5.5.1), and there is $C > 0$ such that for each $u_0 \in \mathcal{M}$ the solution $u(\cdot, u_0)$ of (5.5.1) satisfies*

$$\sup_{t\in[0,t^+(u_0))} |u(t,u_0)|^2_{W_2^1(\Omega,\mathbb{R}^N)\cap W_2^1(\Gamma,\mathbb{R}^N)} \leq C\,(1 + \mathcal{V}(u_0)).$$

We use the above a priori estimate to show that (5.5.1) generates a compact global semiflow in $\mathcal{M}$, with relatively compact orbits.

Proposition 5.5.6. *Suppose that (5.5.1) is conservative and dissipative, and let $p \in (n+2, \infty)$. Then for $u_0 \in \mathcal{M}$ the corresponding solution $u(\cdot, u_0)$ exists globally, $t^+(u_0) = +\infty$, and the orbit $\{u(t,u_0)\}_{t\in[0,\infty)}$ is relatively compact in $\mathcal{M}$. Moreover, for each $t > 0$ the solution map $u(t,\cdot) : \mathcal{M} \to \mathcal{M}$ is compact.*

Proof. **(I)** The Lemmas 5.5.3 and 5.5.5 yield that $u(\cdot, u_0)$ is bounded in $BC(\overline{\Omega}, \mathbb{R}^N)$, and thus Proposition 5.5.2 yields $t^+(u_0) = +\infty$. The compactness of the time-t-map $u(t,\cdot) : \mathcal{M} \to \mathcal{M}$ for all $t > 0$ follows from Theorem 5.3.3.
(II) For the relative compactness of orbits we also want to apply Proposition 5.5.2, and therefore have to show that $\{u(t,u_0)\}_{t\in[0,\infty)}$ is bounded in $\mathcal{M}_p^s$ for some $s > 0$. Due to Sobolev's embeddings it holds

$$W_r^\sigma(\Omega,\mathbb{R}^N) \hookrightarrow W_q^s(\Omega,\mathbb{R}^N) \qquad \text{for} \quad \sigma - \frac{n}{r} \geq s - \frac{n}{q}, \quad \sigma \geq s, \quad r \geq q,$$

and this remains true if one replaces Ω by Γ and n by $n-1$, respectively. By Lemma 5.5.5 the solution is bounded in the energy spaces $W_2^1(\Omega,\mathbb{R}^N)$ and $W_2^1(\Gamma,\mathbb{R}^N)$. We therefore obtain the boundedness in

$$W_q^s(\Omega,\mathbb{R}^N), \quad s = 1 - n/2 + n/q, \quad \text{and} \quad W_q^{s+1-1/q}(\Gamma,\mathbb{R}^N), \quad s = 1/2 - n/2 + n/q,$$

where $q < \frac{2n}{n-1}$. Thus for these q there is a small $s > 0$ such that the orbit is bounded in

$$\mathcal{M}_q^s = W_q^s(\Omega,\mathbb{R}^N) \cap W_q^{s+1-1/q}(\Gamma,\mathbb{R}^N).$$

[4] In the literature it is sometimes required that a Lyapunov function is bounded from below. In the context of compact semiflows this property is not necessary, cf. [16, Remark 1.1.4].

By Lemma 5.5.1, the orbit is therefore bounded in $\mathcal{M}_q^{2-2/q}$. If $q > \frac{n}{2} + 1$ then Sobolev's embeddings yield $\mathcal{M}_q^{2-2/q} \hookrightarrow \mathcal{M}_p^s$ for some $s > 0$, and we are done. Otherwise we iterate the application of Sobolev's embeddings and Lemma 5.5.5 as in the proof of Proposition 4.4.2, to obtain the boundedness of the orbit in $\mathcal{M}_p^s$ for some $s > 0$ after finitely many steps. ■

The last step towards the global attractor for (5.5.1) is the boundedness of the set of its equilibria.

Lemma 5.5.7. *Suppose that (5.5.1) is conservative and dissipative, and let $p \in (1, \infty)$. Then the set of its equilibria is bounded in X_1.*

Proof. (I) An equilibrium $v \in X_1 \hookrightarrow W_p^2(\Omega, \mathbb{R}^N) \cap W_p^{3-1/p}(\Gamma, \mathbb{R}^N)$ of (5.5.1) solves

$$\begin{aligned} \Delta v + f(v) &= 0 \qquad \text{in } \Omega, \\ \Delta_\Gamma v - \partial_\nu v + g(v) &= 0 \qquad \text{on } \Gamma. \end{aligned}$$

Multiplying the domain equation with v, integrating over Ω, integrating by parts and using Lemma 5.5.4 we obtain

$$|\nabla v|^2_{L_2(\Omega, \mathbb{R}^{N \times n})} - \int_\Gamma \partial_\nu v \cdot v \, \mathrm{d}\sigma(x) - \sum_{i=1}^N c_i |v_i|^2_{L_2(\Omega, \mathbb{R}^N)} - N c_0 \leq 0.$$

Employing the boundary equation yields in a similar way that

$$|\nabla_\Gamma v|^2_{L_2(\Gamma, \mathbb{R}^{N \times n})} + \int_\Gamma \partial_\nu v \cdot v \, \mathrm{d}\sigma(x) - \sum_{i=1}^N d_i |v_i|^2_{L_2(\Gamma, \mathbb{R}^N)} - N c_0 \leq 0.$$

Adding these estimates we obtain, using (5.5.7),

$$\begin{aligned} 2Nc_0 &\geq |\nabla v|^2_{L_2(\Omega, \mathbb{R}^{N \times n})} + |\nabla_\Gamma v|^2_{L_2(\Gamma, \mathbb{R}^{N \times n})} - \sum_{i=1}^N \Big(c_i |v_i|^2_{L_2(\Omega, \mathbb{R}^N)} + d_i |v_i|^2_{L_2(\Gamma, \mathbb{R}^N)} \Big) \\ &\gtrsim |v|^2_{W_2^1(\Omega, \mathbb{R}^N)} + |v|^2_{W_2^1(\Gamma, \mathbb{R}^N)}. \end{aligned}$$

This estimate, together with Lemma 5.5.3, leads to

$$\sup\{|v|_{BC(\overline{\Omega}, \mathbb{R}^N)} \,:\, v \in X_1 \text{ is equilibrium of (5.5.1)}\} < \infty. \tag{5.5.9}$$

(II) We have seen that the realization of

$$-\mathbb{A} = \begin{pmatrix} \Delta & 0 \\ -\partial_\nu & \Delta_\Gamma \end{pmatrix}, \qquad D(-\mathbb{A}) = X_1,$$

on $X_0 = L_p(\Omega, \mathbb{R}^N) \times W_p^{1-1/p}(\Gamma, \mathbb{R}^N)$ is the generator of an analytic C_0-semigroup. In particular, there is $\lambda > 0$ such that $\mathbb{A} + \lambda$ is an isomorphism $X_1 \to X_0$. For an equilibrium v of (5.5.1) we may therefore estimate

$$\begin{aligned} |v|_{X_1} &= |(\mathbb{A} + \lambda)^{-1}(f(v) + \lambda v, g(v) + \lambda v)|_{X_1} \\ &\lesssim |f(v) + \lambda v|_{L_p(\Omega, \mathbb{R}^N)} + |g(v) + \lambda v|_{W_p^{1-1/p}(\Gamma, \mathbb{R}^N)} \end{aligned}$$

It follows from (5.4.6) that

$$|g(v)|_{W_p^{1-1/p}(\Gamma,\mathbb{R}^N)} \le C\big(|v|_{BC(\overline{\Omega},\mathbb{R}^N)}\big)\big(1+|v|_{W_p^{1-1/p}(\Gamma,\mathbb{R}^N)}\big).$$

Using (5.5.9), the interpolation inequality and Young's inequality we thus obtain

$$|v|_{X_1} \lesssim 1+|v|_{W_p^{1-1/p}(\Gamma,\mathbb{R}^N)} \le \varepsilon|v|_{W_p^{3-1/p}(\Gamma,\mathbb{R}^N)} + C_\varepsilon,$$

where $\varepsilon > 0$ is arbitrary. Choosing ε appropriately, we may subtract $\frac{1}{2}|v|_{W_p^{3-1/p}(\Gamma,\mathbb{R}^N)}$ on both sides of the above inequality, which yields a universal X_1-bound for the equilibria of (5.5.1). ■

The considerations in this section, together with [63, Theorem 2.3], yield the following result on the long-time behaviour of (5.5.1).

Theorem 5.5.8. *Suppose that (5.5.1) is conservative and dissipative. Then it generates a compact global semiflow of solutions in the phase space $\mathcal{M}$, and the set $\mathcal{E}$ of its equilibria is nonempty. The semiflow possesses a connected global attractor $\mathcal{A} \subset X_1$, the ω-limit set of each orbit is contained in $\mathcal{E}$, and also the α-limit set of each complete orbit is contained in $\mathcal{E}$. If $\mathcal{E}$ is discrete, then $\mathcal{A}$ consists precisely of equilibria and complete orbits connecting them. If in addition f and g do not have a common zero, then (5.5.1) has at least one nonconstant equilibrium, and each solution converges in $\mathcal{M}$ to such a pattern.*

Appendix A

Appendix

A.1 Boundaries of Domains in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary $\partial\Omega$, $n \geq 2$. We say that $\partial\Omega$ is *smooth*, if for each $x \in \partial\Omega$ there are a bounded open set $U \subset \mathbb{R}^n$ with $x \in U$ and a smooth diffeomorphism $\varphi : U \to \mathbb{R}^n$ with

$$\varphi(U \cap \Omega) \subset \mathbb{R}^n_+, \qquad \varphi(U \cap \partial\Omega) \subset \mathbb{R}^{n-1} \times \{0\} \cong \mathbb{R}^{n-1}.$$

The pair (U, φ) is called a *chart* for $\partial\Omega$ around x. The *parametrization* $\mathrm{g} : \mathbb{R}^{n-1} \cap \varphi(U) \to \partial\Omega$ of $\partial\Omega$ around x with respect to (U, φ), also called *local coordinates*, is defined by

$$\mathrm{g}(y) := \varphi^{-1}(y, 0), \qquad y \in \mathbb{R}^{n-1} \cap \varphi(U).$$

It holds that $\mathrm{g} : \mathbb{R}^{n-1} \cap \varphi(U) \to \partial\Omega \cap U$ is a homeomorphism, and that the derivative $\mathrm{g}'(y) \in \mathcal{B}(\mathbb{R}^{n-1}, \mathbb{R}^n)$ has maximal rank $n - 1$ for each $y \in \mathbb{R}^{n-1} \cap \varphi(U)$.

The *tangential space* $T_x\partial\Omega$ on $\partial\Omega$ in x is given by the image of the matrix $\mathrm{g}'(\varphi(x))$, and it has the dimension $n-1$. It becomes a Hilbert space when considering it as a closed subspace of $\mathbb{R}^n$. A canonical basis of $T_x\partial\Omega$ is given by $\{\partial_1\mathrm{g}(\varphi(x)), ..., \partial_{n-1}\mathrm{g}(\varphi(x))\}$. The *outer unit normal* $\nu(x) \in \mathbb{R}^n$ on $\partial\Omega$ at x is given a normalized element of the orthogonal complement of $T_x\partial\Omega$ in $\mathbb{R}^n$. The tangential spaces and the outer unit normals are independent of the chart and the corresponding parametrization, and the outer unit normal field is smooth.

The *fundamental form* $\mathrm{G} = (\mathrm{g}_{ij})_{i,j=1,...,n-1}$ with respect to (U, φ) is defined by

$$\mathrm{g}_{ij} := \partial_i\mathrm{g} \cdot \partial_j\mathrm{g}, \qquad i, j = 1, ..., n - 1,$$

where the scalar product is taken in $\mathbb{R}^n$, and $\mathrm{G}(y)$ is for all $y \in \mathbb{R}^{n-1} \cap \varphi(U)$ a symmetric positive definite matrix. Its inverse $\mathrm{G}^{-1} := (g^{ij})_{i,j=1,...,n-1}$ is also symmetric and positive definite. The determinant $|\mathrm{G}|$ of G is called the *Gramian determinant*.

It is useful to have charts and parametrizations with the following special properties.

Lemma A.1.1. *Let $\Omega \subset \mathbb{R}^N$ have a smooth boundary $\partial\Omega$, let $x_* \in \partial\Omega$, $r > 0$ and let $\mathcal{O}_{\nu(x_*)}$ be any orthogonal matrix that rotates $\nu(x_*)$ to $(0,...,0,-1) \in \mathbb{R}^n$. Then there is a chart (U, φ) for $\partial\Omega$ around x_* with the properties*

$$\varphi(x_*) = 0, \quad \varphi'(x_*) = \mathcal{O}_{\nu(x_*)}, \quad \varphi(U) = B_r(0), \quad \varphi(U \cap \Omega) \subset \mathbb{R}^n_+, \quad \varphi(U \cap \partial\Omega) \subset \mathbb{R}^{n-1},$$

and the corresponding first fundamental form satisfies $\mathrm{G}(x_*) = \mathrm{id}_{\mathbb{R}^{n-1}}$*. The coordinates* g *to such a chart are called* associated to the point $x_* \in \partial\Omega$.

Proof. The diffeomorphism $x \mapsto \mathcal{O}_{\nu(x_*)}(x - x_*)$ translates x_* into the origin and rotates $T_x\partial\Omega$ to $\mathbb{R}^{n-1}\times\{0\}$. The implicit function theorem implies that $\mathcal{O}_{\nu(x_*)}(\partial\Omega - x_*)$ may locally around the origin be represented as a graph of smooth function $h : \widetilde{U} \to \mathbb{R}$ with $\nabla h(0) = 0$, where $\widetilde{U} \subset \mathbb{R}^{n-1}$ is open, such that $\partial\Omega$ lies locally in the set $\{y = (y', y_n) \ : \ y_n = h(y')\}$. Setting $y = \mathcal{O}_{\nu(x_*)}(x - x_*)$, we obtain that $\varphi(x) := (y', y_n - h(y'))$ defines a chart for $\partial\Omega$ around x_*, which has the desired properties after restriction to the preimage of $B_r(0)$. ■

If E is a Banach space and (U, φ) a chart for $\partial\Omega$ one defines the *push-forward* operator Φ for functions $u : \Omega \cap U \to E$ by

$$\Phi u : \mathbb{R}^n_+ \cap \varphi(U) \to E, \qquad \Phi u := u \circ \varphi^{-1}.$$

Similarly, one defines the *pull-back* operator Φ^{-1} for functions $v : \mathbb{R}^n_+ \cap \varphi(U) \to E$ by

$$\Phi^{-1} v : \Omega \cap U \to E, \qquad \Phi^{-1} v = v \circ \varphi.$$

It is shown in [86, Thm. 10.3] that the principal part of a differential operator fortunately transforms in a simple way.

Lemma A.1.2. *Let E be a Banach space, let $\mathcal{P}(x, \nabla) = \sum_{|\alpha| \le k} p_\alpha(x)\nabla^\alpha$ be a differential operator of order $k \in \mathbb{N}_0$ on $\overline{\Omega}$ with $p_\alpha(x) \in \mathcal{B}(E)$, and let (U, φ) be a chart for $\partial\Omega$. Then for the principal part of the transformed operator $\mathcal{P}^\Phi$, which is for $v : \mathbb{R}^n_+ \cap \varphi(U) \to E$ given by*

$$\mathcal{P}^\Phi(x, \nabla)v := \big(\Phi\mathcal{P}(\cdot, \nabla)\Phi^{-1}v\big)(x), \qquad x \in \mathbb{R}^n_+ \cap \varphi(U),$$

it holds $\mathcal{P}^\Phi_\sharp(x, \nabla) = \mathcal{P}_\sharp(\varphi^{-1}(x), \varphi'(x)^T\nabla)$. ■

Finally, assume that $\partial\Omega$ is compact. Then there is finite collection of charts (U_i, φ_i) with

$$\partial\Omega \subset \bigcup\nolimits_i U_i.$$

There is further a smooth *partition of unity* $\{\psi_i\}$ for $\partial\Omega$ *subordinate to the cover* $\bigcup_i U_i$, i.e., $\operatorname{supp}\psi_i \subset U_i$ for all i.

A.2 Interpolation Theory

Let $(E_0, |\cdot|_{E_0})$ and $(E_1, |\cdot|_{E_1})$ be Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$. For $\theta \in (0,1)$ and $p \in [1, \infty]$ the *real interpolation spaces* $(E_0, E_1)_{\theta,p}$ and the *complex interpolation spaces* $[E_0, E_1]_\theta$ are defined and investigated in [13, 68, 82]. We list some well-known properties of these spaces. Recall that by the equality of Banach spaces we mean they coincide as sets and have equivalent norms.

Throughout, let

$$0<\theta<1, \qquad 0<\theta_1<\theta_2<1, \qquad p\in[1,\infty].$$

Then the following holds true.

a) For $1\le p_1\le p_2\le\infty$: $E_1 \overset{d}{\hookrightarrow} (E_0,E_1)_{\theta,p_1} \overset{d}{\hookrightarrow} (E_0,E_1)_{\theta,p_2} \overset{d}{\hookrightarrow} E_0$, see [68, Prop. 1.3, 1.17].

b) $E_1 \overset{d}{\hookrightarrow} [E_0,E_1]_{\theta_2} \overset{d}{\hookrightarrow} [E_0,E_1]_{\theta_1} \overset{d}{\hookrightarrow} E_0$, see [82, Thm. 1.9.3].

c) For $q\in[1,\infty]$ (see [13, Thm. 4.2.1, 4.7.1]):

$$(E_0,E_1)_{\theta_2,p} \overset{d}{\hookrightarrow} (E_0,E_1)_{\theta_1,q}, \quad (E_0,E_1)_{\theta_2,p} \overset{d}{\hookrightarrow} [E_0,E_1]_{\theta_1}, \quad [E_0,E_1]_{\theta_2} \overset{d}{\hookrightarrow} (E_0,E_1)_{\theta_1,p}.$$

d) If F_1 is a Banach space with $E_1 \overset{d}{\hookrightarrow} F_1 \overset{d}{\hookrightarrow} E_0$:

$$(E_0,E_1)_{\theta,p} \hookrightarrow (E_0,F_1)_{\theta,p}, \qquad [E_0,E_1]_\theta \hookrightarrow [E_0,F_1]_\theta.$$

e) For $q_1,q_2\in[1,\infty]$ (see [13, Thm. 3.5.3]):

$$\big((E_0,E_1)_{\theta_1,q_1},(E_0,E_1)_{\theta_2,q_2}\big)_{\theta,p} = (E_0,E_1)_{(1-\theta)\theta_1+\theta\theta_2,p}.$$

f) $\big[[E_0,E_1]_{\theta_1},[E_0,E_1]_{\theta_2}\big]_\theta = [E_0,E_1]_{(1-\theta)\theta_1+\theta\theta_2}$,
and this assertion remains valid if $[E_0,E_1]_{\theta_1}$ is replaced by E_0 and $\theta_1=0$ or $[E_0,E_1]_{\theta_2}$ is replaced by E_1 and $\theta_2=0$ (see [82, Rem. 1.9.3/1]).

g) $\big([E_0,E_1]_{\theta_1},[E_0,E_1]_{\theta_2}\big)_{\theta,p} = (E_0,E_1)_{(1-\theta)\theta_1+\theta\theta_2,p}$,
and this assertion remains valid for $\theta_1=0$ or $\theta_2=0$ as in e) (see [82, Thm. 1.10.3/2]).

h) If E_0 and E_1 are reflexive (see [82, Rem. 1.10.3/2]):

$$\big[(E_0,E_1)_{\theta_1,p},(E_0,E_1)_{\theta_2,p}\big]_\theta = (E_0,E_1)_{(1-\theta)\theta_1+\theta\theta_2,p}.$$

i) If $F_1 \overset{d}{\hookrightarrow} F_0$ with $F_1 \overset{d}{\hookrightarrow} E_1$, $F_0 \overset{d}{\hookrightarrow} E_0$, are Banach spaces (see [68, Thm. 1.6, 2.6]):

$$\mathcal{B}(E_0,F_0)\cap\mathcal{B}(E_1,F_1) \hookrightarrow \mathcal{B}\big((E_0,F_0)_{\theta,p},(E_1,F_1)_{\theta,p}\big)\cap\mathcal{B}\big([E_0,F_0]_\theta,[E_1,F_1]_\theta\big).$$

More precisely, for $A\in\mathcal{B}(E_0,F_0)\cap\mathcal{B}(E_1,F_1)$ it holds

$$|A|_{\mathcal{B}((E_0,F_0)_{\theta,p},(E_1,F_1)_{\theta,p})} \le |A|^{1-\theta}_{\mathcal{B}(E_0,F_0)}|A|^{\theta}_{\mathcal{B}(E_1,F_1)},$$

and analogously for $|A|_{\mathcal{B}([E_0,F_0]_\theta,[E_1,F_1]_\theta)}$.

j) By the *interpolation inequality* (see [68, Cor. 1.7, 2.8]):

$$|x|_{(E_0,E_1)_{\theta,p}} \le C(\theta,p)|x|^{1-\theta}_{E_0}|x|^{\theta}_{E_1}, \qquad |x|_{[E_0,E_1]_\theta} \le |x|^{1-\theta}_{E_0}|x|^{\theta}_{E_1} \qquad x\in E_1.$$

k) If $\big(A, D(A)\big)$ is the generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E_0, and $D(A)$ is equipped with the graph norm (see [68, Prop. 5.7]):

$$\big(E_0, D(A)\big)_{\theta,p} = \Big\{u \in E_0 \ : \ [u]_*^p := \int_0^\infty t^{-\theta p} |T(t)u - u|_{E_0}^p \frac{\mathrm{d}t}{t} < \infty\Big\},$$

where the space on the right-hand side is equipped with the norm $|u|_{E_0} + [u]_*$.

l) If $\big(A, D(A)\big)$ is the generator of a bounded analytic C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E_0, and $D(A)$ is equipped with the graph norm (see [82, Thm. 1.14.5]):

$$\big(E_0, D(A)\big)_{\theta,p} = \Big\{u \in E_0 \ : \ [u]_{**}^p := \int_0^\infty t^{p(1-\theta)} |AT(t)u|_{E_0}^p \frac{\mathrm{d}t}{t} < \infty\Big\},$$

where the space on the right-hand side is equipped with the norm $|u|_{E_0} + [u]_{**}$. In $[u]_{**}$ one may restrict the integration over t to $(0, \delta)$, $\delta > 0$. If $\{T(t)\}_{t\geq 0}$ is in addition exponentially stable then it suffices to take $[u]_{**}$ as a norm.

m) If (Ω, ν) is a σ-finite measure space and $\theta \in (0,1)$, $p \in [1,\infty)$ (see [82, Thm. 1.18.4]):

$$\big(L_p(\Omega; E_0), L_p(\Omega; E_1)\big)_{\theta,p} = L_p\big(\Omega; (E_0, E_1)_{\theta,p}\big),$$

$$\big[L_p(\Omega; E_0), L_p(\Omega; E_1)\big]_\theta = L_p\big(\Omega; [E_0, E_1]_\theta\big).$$

n) If (Ω, ν) is a σ-finite measure space and $1 \leq p_1 < p_2 \leq \infty$ (see [82, Thm. 1.18.4]):

$$\big[L_{p_1}(\Omega; E), L_{p_2}(\Omega; E)\big]_\theta = L_q(\Omega; E), \qquad \text{where } \frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Here one interpolates in fact between an interpolation couple, cf. [13, 68, 82].

The following Hardy-Young inequalities are useful for interpolation theory. It holds

$$\int_0^T \Big(t^{-\alpha} \int_0^t u(s) \frac{\mathrm{d}s}{s}\Big)^p \frac{\mathrm{d}t}{t} \leq \frac{1}{\alpha^p} \int_0^T \big(t^{-\alpha} u(t)\big)^p \frac{\mathrm{d}t}{t}, \tag{A.2.1}$$

$$\int_0^T \Big(t^{\alpha} \int_t^T u(s) \frac{\mathrm{d}s}{s}\Big)^p \frac{\mathrm{d}t}{t} \leq \frac{1}{\alpha^p} \int_0^T \big(t^{\alpha} u(t)\big)^p \frac{\mathrm{d}t}{t}, \tag{A.2.2}$$

for all nonnegative measurable functions $u : (0,T) \to \mathbb{R}$, $T \in (0,\infty]$, all $\alpha > 0$ and all $p \in [1,\infty)$, cf. [50, p. 245-246].

A.3 Sectorial Operators

For detailed informations on the concepts described in this section we refer to [7, 24, 48, 55, 62, 68, 82] and the references therein. Throughout, let E be a complex Banach space.

The space E is said to be *of class $\mathcal{HT}$* if the Hilbert transform on $L_2(\mathbb{R}; E)$ is bounded, i.e., if the densely defined operator H, given by

$$(H\varphi)(s) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t|\geq\varepsilon} \frac{\varphi(s-t)}{t}\,\mathrm{d}t, \qquad \varphi \in \mathcal{S}(\mathbb{R}; E),$$

may uniquely be extended to $L_2(\mathbb{R};E)$. This property is equivalent to the boundedness of the operator corresponding to the symbol $-\mathrm{i}\,\mathrm{sign}$. It is further equivalent for E to have the property of unconditional martingale differences, an important concept in stochastic analysis. Therefore spaces of class $\mathcal{HT}$ are also called UMD-spaces in the literature. Since we have a purely analytic point of view in this work, we prefer the first notion.

Banach spaces of class $\mathcal{HT}$ are always reflexive. Many spaces in applications are of class $\mathcal{HT}$, like finite dimensional spaces, Hilbert spaces, and further L_p-, Sobolev, Slobodetskii, Besov and Bessel potential spaces in the reflexive range, provided they take values in a space of class $\mathcal{HT}$ (see Appendix A.4). L_1-, L_∞ and C^k-spaces are not of class $\mathcal{HT}$. The $\mathcal{HT}$-property is stable under real and complex interpolation. For more information and proofs we refer to [7, Sections III.4.3-4.5] and [55, Chapters 6-8].

A family of bounded operators $\mathcal{T} \subset \mathcal{B}(E_0, E_1)$ between Banach spaces E_0, E_1 is called *$\mathcal{R}$-bounded* if there is $C > 0$ such that, for all $T_1, ..., T_m \in \mathcal{T}$ and $x_1, ..., x_m \in E_0$ with $m \in \mathbb{N}$, it holds

$$|\sum_{n=1}^{m} r_n T_n x_n|_{L_2(0,1;E_1)} \leq C\,|\sum_{n=1}^{m} r_n x_n|_{L_2(0,1;E_0)},$$

where $r_n(t) := \mathrm{sign}\,\sin(2^n\pi t)$ denote the Rademacher functions on $[0,1]$. We stress that the norms are *outside* the sums. If E_0, E_1 are Hilbert spaces this notion is equivalent to the uniform boundedness of $\mathcal{T}$. The infimum of all C satisfying the above estimate is called the *$\mathcal{R}$-bound* of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$. We refer to [24, Chapter 3], [62, Chapter 2] and [55, Chapter 4] for detailed informations.

Roughly speaking, spaces of class $\mathcal{HT}$ and the concept of $\mathcal{R}$-boundedness can be used to show the boundedness of operators for which standard norm estimates do actually not lead to boundedness. The advantage is that one can avoid the application of the triangle inequality at certain points, and 'leave the norm outside a sum' while estimating. Besides the interest in their own, the combination of these concepts leads to important results for the applications to partial differential equations, like the operator-valued Fourier-multiplier theorem due to Weis [85], the joint functional calculus due to Kalton and Weis [62], the characterization of maximal L_p-regularity [85], the Dore-Venni theorem [31], and many more (see below for more details on the latter two results).

Let A be an operator on E with domain $D(A)$. We call A *sectorial* if A is closed, densely defined, has dense range and if it holds $(-\infty, 0) \subset \rho(A)$ with the resolvent estimate

$$|t(t+A)^{-1}|_{\mathcal{B}(E)} \leq C, \qquad t > 0,$$

for some $C > 0$. Define the open sector

$$\Sigma_\theta := \big\{\lambda \in \mathbb{C}\backslash\{0\} \ : \ |\arg\lambda| < \theta\big\}.$$

If A is sectorial, then the resolvent estimate and a Neumann series yield an angle $\phi \in [0,\pi)$ such that $\Sigma_{\pi-\phi} \subset \rho(-A)$. One may thus define the *spectral angle* of A by

$$\phi_A := \inf\big\{\phi \in [0,\pi) \ : \ \Sigma_{\pi-\phi} \subset \rho(-A), \ \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}| < \infty\big\}.$$

A sectorial operator A is the generator of a bounded analytic C_0-semigroup if and only if $\phi_A < \pi/2$ ([35, Theorem II.4.6]). For $\phi_A = \pi/2$ the operator A does not necessarily generate a semigroup. Note that sometimes in the literature only generators of analytic semigroups are called sectorial.

The operator A is called *$\mathcal{R}$-sectorial* if it is sectorial and if the family of operators

$$\big\{t(t+A)^{-1} \ :\ t>0\big\}$$

is $\mathcal{R}$-bounded in $\mathcal{B}(E)$. As above one may define the $\mathcal{R}$-angle $\phi_A^{\mathcal{R}}$ of A by

$$\phi_A^{\mathcal{R}} := \inf\big\{\phi\in(\phi_A,\pi) \ :\ \mathcal{R}\{\lambda(\lambda+A)^{-1} \ :\ \lambda\in\Sigma_{\pi-\phi}\}<\infty\big\}.$$

It is clear from the definitions that $\phi_A \le \phi_A^{\mathcal{R}}$. The importance of this concept lies in the fact that on a Banach space of class $\mathcal{HT}$ the generator of an exponentially stable C_0-semigroup A is $\mathcal{R}$-sectorial with $\phi_A^{\mathcal{R}} < \pi/2$ if and only if it enjoys the property of maximal L_p-regularity on the half line for $p\in(1,\infty)$, i.e., if for each $f\in L_p(\mathbb{R}_+;E)$ there is a unique solution $u\in W_p^1(\mathbb{R}_+;E)\cap L_p(\mathbb{R}_+;D(A))$ of the problem

$$u'+Au=f,\quad t>0,\qquad u(0)=0.$$

This result is due to Weis [85]. For more informations on ($\mathcal{R}$-)sectorial operators we refer to [24, 62], and to [30] for a survey on maximal L_p-regularity (see also Section 1.2.1).

We now consider the functional calculus for sectorial operators. For $\phi\in(0,\pi)$ one defines the function algebras

$$\mathcal{H}(\Sigma_\phi) := \big\{f:\Sigma_\phi\to\mathbb{C} \ :\ f \text{ is holomorphic}\big\},$$

$$\mathcal{H}^\infty(\Sigma_\phi) := \big\{f:\Sigma_\phi\to\mathbb{C} \ :\ f \text{ is holomorphic and bounded}\big\},$$

$$\mathcal{H}_0(\Sigma_\phi) := \big\{f\in\mathcal{H}(\Sigma_\phi) \ :\ \text{there are } \alpha,\beta>0 \text{ with } \sup_{|\lambda|\le1}|\lambda^{-\alpha}f(\lambda)| + \sup_{|\lambda|\ge1}|\lambda^{\beta}f(\lambda)|<\infty\big\},$$

$$\mathcal{H}_1(\Sigma_\phi) := \big\{f\in\mathcal{H}(\Sigma_\phi) \ :\ \text{there are } \alpha,\beta\in\mathbb{R} \text{ with } \sup_{|\lambda|\le1}|\lambda^{-\alpha}f(\lambda)| + \sup_{|\lambda|\ge1}|\lambda^{\beta}f(\lambda)|<\infty\big\}.$$

Now fix a sectorial operator A, and let $\phi\in(\phi_A,\pi]$. For a curve $\Gamma=(\infty,0]e^{\mathrm{i}\psi}\cup[0,\infty)e^{-\mathrm{i}\psi}$ with $\psi\in(\phi_A,\phi)$ the map

$$\Phi_A:\mathcal{H}_0(\Sigma_\phi)\to\mathcal{B}(E),\qquad f(A):=\Phi_A(f):=\frac{1}{2\pi\mathrm{i}}\int_\Gamma f(\lambda)(\lambda+A)^{-1}\,\mathrm{d}\lambda,$$

defines a *functional calculus* for A, i.e., an algebra homomorphism. If there are $\phi\in(\phi_A,\pi)$ and a constant K_ϕ such that

$$|f(A)|_{\mathcal{B}(E)} \le K_\phi|f|_\infty,\qquad f\in\mathcal{H}_0(\Sigma_\phi),$$

then the functional calculus for A may uniquely be extended from $\mathcal{H}_0(\Sigma_\phi)$ to $\mathcal{H}(\Sigma_\phi)$. In this case A is said to admit a *bounded $\mathcal{H}^\infty$-calculus*, and the *$\mathcal{H}^\infty$-angle* ϕ_A^∞ of A is defined as the infimum of all $\phi\in(\phi_A,\pi)$ that admit an estimate as above.

For $f \in \mathcal{H}_1(\Sigma_\phi)$ and $\psi(\lambda) := \frac{\lambda}{1+\lambda^2}$, choose $k \in \mathbb{N}$ such that $\psi^k f \in \mathcal{H}_0(\Sigma_\phi)$. Then the operator

$$f(A) := \psi(A)^{-k}(\psi^k f)(A), \qquad D(f(A)) = \big\{x \in E \ : \ (\psi^k f)(A)x \in D(A^k) \cap R(A^k)\big\},$$

is closed and densely defined on E, and it is independent of the chosen number k. The map $f \mapsto f(A)$ is called the *extended functional calculus* for A. We refer to [24, 48] for proofs and more informations on the functional calculus for sectorial operators.

For $z \in \mathbb{C}$ the extended functional calculus in particular allows to define the fractional powers A^z of a sectorial operator A. In general, A^z is a closed and densely defined operator on X, and the domains satisfy

$$D(A^{z_1}) \overset{d}{\hookrightarrow} D(A^{z_2}) \overset{d}{\hookrightarrow} E, \qquad \operatorname{Re} z_1 > \operatorname{Re} z_2 > 0,$$

when equipped with the graph norm, cf. [24, Thm. 2.1], [68, Lem. 4.11] and [7, Thm. III.4.6.5]. If A is invertible then A^z is a bounded operator for $\operatorname{Re} z < 0$, and for $\operatorname{Re} z, \operatorname{Re}\omega > 0$ the operator A^z is an isomorphism

$$D(A^{z+\omega}) \to D(A^\omega), \qquad D(A^z) \to E,$$

see again [7, Thm. III.4.6.5]. For $p \in (1,\infty)$ and $\theta, \operatorname{Re} z > 0$ with $\theta + \operatorname{Re} z < 1$, the operator A^z is further an isomorphism

$$(E, D(A))_{\operatorname{Re} z+\theta,p} \to (E, D(A))_{\theta,p},$$

cf. [82, Thm. 1.15.2]. The *reiteration theorem* [48, Prop. 6.6.7] states that for $\theta \in (0,1)$ and $p \in (1,\infty)$ it holds

$$(E, D(A^{z_1}))_{\theta\frac{\operatorname{Re} z_2}{\operatorname{Re} z_1},p} = (E, D(A^{z_2}))_{\theta,p}, \qquad 0 < \operatorname{Re} z_2 < \operatorname{Re} z_1,$$

and moreover

$$(E, D(A^{z_1}))_{(1-\theta)\frac{\operatorname{Re}\omega}{\operatorname{Re} z_1}+\theta\frac{\operatorname{Re} z_2}{\operatorname{Re} z_1},p} = (D(A^\omega), D(A^{z_2}))_{\theta,p}, \qquad 0 < \operatorname{Re}\omega < \operatorname{Re} z_2 < \operatorname{Re} z_1.$$

For $s = k+\theta \geq 0$ with $k \in \mathbb{N}_0$, $p \in [1,\infty]$ and $\theta \in [0,1)$ it is convenient to define

$$D_A(s,p) := D(A^s) \quad \text{ if } \ s \in \mathbb{N}_0,$$

$$D_A(s,p) := \big\{x \in D(A^k) \ : \ A^k x \in (E, D(A))_{\theta,p}\big\} \quad \text{ if } \ s \notin \mathbb{N}_0,$$

where these spaces are equipped with the norm $|x|_{E_0} + |A^k x|_{(E,D(A))_{\theta,p}}$, respectively (with $E = (E, D(A))_{0,p}$).

There are rules analogously to the ones for powers of scalars. It holds that

$$A^z A^\omega \subset A^{z+\omega}, \quad D(A^\omega) \cap D(A^{z+\omega}) = D(A^z A^\omega), \qquad z, \omega \in \mathbb{C},$$

see [48, Prop. 3.2.1], and for $\operatorname{Re} z, \operatorname{Re}\omega > 0$ we have $A^z A^\omega = A^{z+\omega}$ by [48, Prop. 3.1.1]. For $\alpha \in \mathbb{R}$ with $|\alpha| < \pi/\phi_A$ the operator A^α is sectorial, and it holds $\phi_{A^\alpha} = |\alpha|\phi_A$ and

$$(A^\alpha)^z = A^{\alpha z}, \qquad z \in \mathbb{C},$$

cf. [48, Prop. 3.1.4, Cor. 3.1.5].

Closed linear operators A, B on E are called *resolvent commuting* if there are $\lambda \in \rho(A)$, $\mu \in \rho(B)$ such that

$$(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}.$$

For such operators real interpolation commutes with the intersection of the domains [47].

Lemma A.3.1. *Let A and B be resolvent commuting sectorial operators on a Banach space E. Then for $\theta \in (0,1)$ and $p \in (1,\infty)$ it holds*

$$(E, D(A) \cap D(B))_{\theta,p} = (E, D(A))_{\theta,p} \cap (E, D(B))_{\theta,p}.$$

A sectorial operator A is said to admit *bounded imaginary powers* if $A^{\mathrm{i}s} \in \mathcal{B}(E)$ for all $s \in \mathbb{R}$ and if there are $\varepsilon > 0$ and $C > 0$ with $|A^{\mathrm{i}s}|_{\mathcal{B}(E)} \leq C$ for $|s| \leq \varepsilon$. In this case $\{A^{\mathrm{i}s}\}_{s\in\mathbb{R}}$ forms a C_0-group of bounded operators on E, and the growth bound θ_A of this group, i.e.,

$$\theta_A = \limsup_{|s|\to\infty} \frac{1}{|s|} \log |A^{\mathrm{i}s}|_{\mathcal{B}(E)},$$

is called the *power angle* of A.

Operators with bounded imaginary powers enjoy very good properties. If A is invertible and admits bounded imaginary powers then Yagi's theorem states that for $0 \leq \operatorname{Re}\omega < \operatorname{Re} z$ and $\theta \in (0,1)$ it holds

$$[D(A^\omega), D(A^z)]_\theta = D(A^{(1-\theta)\omega+\theta z}), \tag{A.3.1}$$

where the domains are again equipped with the graph norm, respectively [82, Thm. 1.15.3]. The above identity is useful to determine a complex interpolation space in concrete situations. The following result is a variant of the Dore-Venni theorem [31, 72].

Theorem A.3.2. *Let E be of class $\mathcal{HT}$, and suppose that the operators A, B are resolvent commuting and admit bounded imaginary powers with $\theta_A + \theta_B < \pi$. Then for all $\rho > 0$ the following holds true.*

a) *$A + \rho B$ with $D(A + \rho B) = D(A) \cap D(B)$ is closed and sectorial;*

b) *$A + \rho B$ admits bounded imaginary powers with $\theta_{A+\rho B} \leq \max\{\theta_A, \theta_B\}$;*

c) *there is a constant $C > 0$, independent of ρ, such that*

$$|Ax|_E + \rho|Bx|_E \leq C\,|Ax + \rho Bx|_E, \qquad x \in D(A) \cap D(B).$$

If A or B is invertible, then $A + \rho B$ is invertible as well.

The next result is also called the mixed derivative theorem, see [78] and also [27].

Lemma A.3.3. *In the situation of the Dore-Venni Theorem A.3.2, for each $\alpha \in [0,1]$ it holds $B^{1-\alpha}x \in D(A^\alpha)$ for $x \in D(A) \cap D(B)$, and there is a constant $C > 0$ such that*

$$|A^\alpha B^{1-\alpha} x|_E \leq C\,|Ax + Bx|_E \qquad \text{for all } x \in D(A) \cap D(B).$$

In particular, $B^{1-\alpha} \in \mathcal{B}(D(A) \cap D(B), D(A^\alpha))$.

We next state a variant of Lemma A.3.1 for complex interpolation, see [37, Lem. 9.5].

Lemma A.3.4. *Let E be of class $\mathcal{HT}$ and $\theta \in (0,1)$. Suppose that the operators A, B are resolvent commuting, that they admit boundary imaginary powers with $\theta_A + \theta_B < \pi$, and that A or B is invertible. Then*

$$[E, D(A+B)]_\theta = [E, D(A)]_\theta \cap [E, D(B)]_\theta, \qquad \theta \in (0,1),$$

i.e., $D((A+B)^\theta) = D(A^\theta) \cap D(B^\theta)$.

For more properties of operators with bounded imaginary powers we refer to [7, Sec. III.4.7], [68, Sec. 4.2] and [24, Sec. 2.3].

The above properties of a sectorial operator A on a Banach space E of class $\mathcal{HT}$ are related as follows. If A admits a bounded $\mathcal{H}^\infty$-calculus then A admits bounded imaginary powers, and the latter property implies that A is $\mathcal{R}$-sectorial [24, Sec. 2.4, Thm. 4.5]. The angles satisfy

$$\phi_A^\infty \geq \theta_A \geq \phi_A^{\mathcal{R}} \geq \phi_A. \tag{A.3.2}$$

In particular, if E is of class $\mathcal{HT}$ and A admits a bounded $\mathcal{H}^\infty$-calculus or bounded imaginary powers with angles strictly smaller than $\pi/2$, then A enjoys maximal L_p-regularity on the half-line for all $p \in (1,\infty)$. The converse of the above assertions is false, in general.

The standard examples for operators with a bounded $\mathcal{H}^\infty$-calculus are for $p \in (1,\infty)$ and a Banach space of class $\mathcal{HT}$ the derivative ∂_t on $L_p(\mathbb{R}; E)$ with domain $W_p^1(\mathbb{R}; E)$, and the negative Laplacian $-\Delta_n$ on $L_p(\mathbb{R}^n; E)$ with domain $W_p^2(\mathbb{R}^n; E)$. For the angles we have $\phi_{\partial_t}^\infty = \pi/2$ and $\phi_{-\Delta_n}^\infty = 0$. For a proof we refer to [24, Thm. 5.5] and [48, Ch. 8] (see also Theorem 1.1.7 and Lemma 1.3.1).

We now consider further properties of sectorial operators. We already saw that a real fractional power of a sectorial operator remains sectorial if the power and the sectoriality angle are appropriate. A similar result is true for other properties of an operator.

Lemma A.3.5. *Assume that A admits a bounded $\mathcal{H}^\infty$-calculus or bounded imaginary powers, and let $\alpha > 0$ satisfy*

$$\alpha < \pi/\phi_A^\infty, \qquad \text{or} \quad \alpha < \pi/\theta_A.$$

Then A^α enjoys the same property, with

$$\phi_{A^\alpha}^\infty \leq \alpha \phi_A^\infty, \qquad \text{or} \quad \theta_{A^\alpha} \leq \alpha\theta_A.$$

Proof. (I) First suppose that A admits a bounded $\mathcal{H}^\infty$-calculus. Take a small $\varepsilon > 0$ with $\alpha(\phi_A^\infty + \varepsilon) < \pi$ and $\phi_A^\infty + \varepsilon < \pi$. Then for $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$ with $\phi \in \big(\alpha\phi_A^\infty, \alpha(\phi_A^\infty + \varepsilon)\big)$ the function $\lambda \mapsto \widetilde{f}(\lambda) := f(\lambda^\alpha)$ belongs to $\mathcal{H}_0^\infty(\Sigma_{\phi/\alpha})$, and $\phi/\alpha \in (\phi_A^\infty, \pi)$. Using that A^α is sectorial and the composition rule for the functional calculus of sectorial operators [48, Theorem 2.4.2] we obtain

$$|f(A^\alpha)|_{\mathcal{B}(E)} = |\widetilde{f}(A)|_{\mathcal{B}(E)} \leq K_{\phi/\alpha}|\widetilde{f}|_\infty = K_{\phi/\alpha}|f|_\infty.$$

Hence A^α admits a bounded $\mathcal{H}^\infty$-calculus of angle not larger than $\alpha\phi_A^\infty$.
(II) Now assume that A admits bounded imaginary powers. Then $(A^\alpha)^{\mathrm{i}s} = A^{\mathrm{i}s\alpha} \in \mathcal{B}(E)$ for all $s \in \mathbb{R}$, and $|(A^\alpha)^{\mathrm{i}s}| \leq C$ for all $|s| \leq \varepsilon/\alpha$. Thus A^α also admits bounded imaginary powers. Moreover, due to [7, Corollary III.4.7.2], for all $\theta > \theta_A$ we have $|A^{\mathrm{i}s\alpha}|_{\mathcal{B}(E)} \leq Ce^{\theta\alpha|s|}$. Taking the logarithm yields that $\theta_{A^\alpha} \leq \alpha\theta$, and the assertion follows. ■

For a σ-finite measure space (Ω, ν) and $p \in (1, \infty)$ we may define the pointwise realization of A_p on $L_p(\Omega; E)$ by

$$(A_p u)(t) := Au(t), \qquad \text{a.e.} \quad t \in \Omega, \qquad u \in D(A_p) := L_p(\Omega; D(A)),$$

where $D(A)$ is endowed with the graph norm. We show that A_p enjoys the same properties as A.

Lemma A.3.6. *Let (Ω, ν) be a σ-finite measure space, and suppose that the operator A is sectorial, admits a bounded $\mathcal{H}^\infty$-calculus or admits bounded imaginary powers. Then for $p \in (1, \infty)$ the pointwise realization A_p of A on $L_p(\Omega, E)$ enjoys the same property, with*

$$\phi_{A_p} \leq \phi_A, \qquad \phi_{A_p}^\infty \leq \phi_A^\infty, \qquad \text{or} \quad \theta_{A_p} \leq \theta_A.$$

In addition, if A is sectorial and $f \in \mathcal{H}_1^\infty(\Sigma_\phi)$ with $\phi \in (\phi_A, \pi)$, then $f(A_p) = f(A)_p$.

Proof. (I) Suppose that A is sectorial. We first show that A_p is densely defined. Let $\varepsilon > 0$ be given, and let $\sum_{i=1}^m \alpha_i x_i \in L_p(\Omega; E)$ be a step function, with $m \in \mathbb{N}$, $\alpha_i \in L_p(\Omega)$, $\alpha_i \neq 0$ and $x_i \in E$. Since $D(A)$ is dense in E we find $y_i \in D(A)$ with $|x_i - y_i|_E \leq \varepsilon/(m|\alpha_i|_{L_p(\Omega)})^{-1}$. It then holds $\sum_{i=1}^m \alpha_i y_i \in D(A_p) = L_p(\Omega; D(A))$, and further $|\sum_{i=1}^m \alpha_i x_i - \sum_{i=1}^m \alpha_i y_i|_{L_p(\Omega;E)} < \varepsilon$. Since the step functions are dense in $L_p(\Omega; E)$, it follows that $D(A_p)$ is dense in $L_p(\Omega; E)$. The density of the range of A_p in $L_p(\Omega; E)$ is shown in a similar way.
Now let $\lambda \in \rho(A)$. Then for $h \in L_p(\Omega; E)$ the unique solution $u \in L_p(\Omega; E)$ of $(\lambda + A_p)u = h$ is for almost every $t \in \Omega$ given by $u(t) = (\lambda + A)^{-1}h(t)$. Hence A_p is closed, and it holds

$$\rho(A) \subset \rho(A_p), \qquad (\lambda + A_p)^{-1} = \big((\lambda + A)^{-1}\big)_p \quad \text{for} \ \ \lambda \in \rho(A). \tag{A.3.3}$$

This yields for $\lambda \in \rho(A)$ the estimate

$$|\lambda(\lambda + A_p)^{-1}|_{\mathcal{B}(L_p(\Omega;E))} = \sup_{|h|_{L_p(\Omega;E)}=1} |\lambda(\lambda + A_p)^{-1}h|_{L_p(\Omega;E)} \leq |\lambda(\lambda + A)^{-1}|_{\mathcal{B}(E)},$$

which shows that A_p is sectorial with $\phi_{A_p} \leq \phi_A$.

(II) If A is sectorial then we infer from (A.3.3) that for $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$ with $\phi \in (\phi_A, \pi)$, $h \in L_p(\Omega; E)$ and almost every $t \in \Omega$ it holds

$$\big(f(A_p)h\big)(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(\lambda)\big((\lambda + A_p)^{-1}h\big)(t)\,\mathrm{d}\lambda = f(A)_p h(t),$$

which yields $f(A_p) = f(A)_p$. Similarly one obtains this identity for $f \in \mathcal{H}_1^\infty(\Sigma_\phi)$. Using this fact and estimating as above it is straight forward to check that the other properties of A carry over to A_p as asserted. ■

A.4 Function Spaces on Domains and Boundaries

We first consider function spaces with values in a Banach space of class $\mathcal{HT}$, and refer to [8, 9, 75, 91] for more details and proofs. For scalar-valued function spaces we refer to [82]. The $\mathcal{HT}$-valued function spaces share many properties with the scalar-valued spaces, due to the fact that appropriate Fourier multiplier theorems are available.

Let E be a complex Banach space of class $\mathcal{HT}$, and let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary. We denote by $\nabla = (\partial_1, ..., \partial_n)$ the euclidian gradient on Ω, and $\alpha \in \mathbb{N}_0^n$ denotes a multiindex. For $k \in \mathbb{N}_0$ the Banach space of the E-valued k-times bounded uniformly continuously differentiable functions on $\overline{\Omega}$ is denoted by

$$BUC^k(\overline{\Omega}; E),$$

equipped with its canonical norm. For $s = [s] + s_* \in \mathbb{R}_+ \backslash \mathbb{N}$ with $[s] \in \mathbb{N}_0$ and $s_* \in [0, 1)$ the Banach space of bounded Hölder continuous functions of order s on $\overline{\Omega}$ is given by

$$BUC^s(\overline{\Omega}; E) := \big\{u \in BUC^{[s]}(\overline{\Omega}; E) \;:\; \text{for } |\alpha| = [s] \text{ it holds } [\nabla^\alpha u]_{BUC^{s-[s]}(\overline{\Omega};E)} < \infty\big\},$$

where for $\tau \in (0, 1)$ the seminorm $[\cdot]_{BUC^\tau(\overline{\Omega};E)}$ is defined by

$$[u]_{BUC^\tau(\overline{\Omega};E)} := \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\tau},$$

and $BUC^s(\overline{\Omega}; E)$ is equipped with the norm $|u|_{BUC^{[s]}(\overline{\Omega};E)} + \sum_{|\alpha| \le [s]} [\nabla^\alpha u]_{BUC^{s-[s]}(\overline{\Omega};E)}$. For $k \in \mathbb{N}_0$ we further denote by

$$C^k(\Omega; E) \qquad \text{and} \qquad C^k(\overline{\Omega}; E)$$

the space of k-times continuously differentiable functions on Ω and $\overline{\Omega}$, respectively. For $s \geq 0$ the space of the locally Hölder continuous functions of order s on Ω and $\overline{\Omega}$ are denoted by $C^s(\Omega; E)$ and $C^s(\overline{\Omega}; E)$, respectively. If Ω is bounded, it holds

$$C^s(\overline{\Omega}; E) = BUC^s(\overline{\Omega}; E), \qquad s \geq 0.$$

We further set

$$C^\infty(\Omega; E) := \bigcap_{k \in \mathbb{N}_0} C^k(\Omega; E), \qquad C_c^\infty(\Omega; E) := \big\{u \in C^\infty(\Omega; E) \;:\; \operatorname{supp} u \subset \Omega\big\},$$

and analogously one defines $C^\infty(\overline{\Omega};E)$ and $C_c^\infty(\overline{\Omega};E)$.

For $p \in [1,\infty)$ the Banach space of the E-valued L_p-space on Ω is defined by

$$L_p(\Omega;E) := \big\{ u : \Omega \to E \text{ strongly measurable } : |u|^p_{L_p(\Omega;E)} := \int_\Omega |u(x)|_E^p \, \mathrm{d}x < \infty \big\},$$

and is endowed with the norm $|u|_{L_p(\Omega;E)}$. The space $L_\infty(\Omega;E)$ is defined with the usual modification. Since E is assumed to be reflexive, for $p \in (1,\infty)$ these spaces are also reflexive, with $L_p(\Omega;E)^* = L_q(\Omega;E^*)$ and $\frac{1}{p} + \frac{1}{q} = 1$. The set $C_c^\infty(\Omega;E)$ is dense in $L_p(\Omega;E)$ for $p \in [1,\infty)$. For the general theory of vector-valued L_p-spaces we refer to the Chapters III and IV of [32].

For $k \in \mathbb{N}_0$ and $p \in [1,\infty]$ the E-valued *Sobolev space* over Ω is defined by

$$W_p^k(\Omega;E) := \big\{ u \in L_p(\Omega;E) : \nabla^\alpha u \text{ exists weakly}, \ \nabla^\alpha u \in L_p(\Omega;E) \text{ for } |\alpha| \le k \big\},$$

and is endowed with the norm $|u|_{W_p^k(\Omega;E)} := \big(\sum_{|\alpha|\le k} |\nabla^\alpha u|^p_{L_p(\Omega;E)} \big)^{1/p}$, which turns it into a Banach space.

We further define the following E-valued function spaces: for $p,q \in [1,\infty)$ and $s > 0$ the *Besov space*

$$B^s_{p,q}(\Omega;E) := \big(L_p(\Omega;E), W_p^{[s]+1}(\Omega;E) \big)_{\frac{s}{[s]+1},q},$$

for $p \in [1,\infty)$ and $s > 0$ the *Bessel potential space*

$$H_p^s(\Omega;E) := \big[L_p(\Omega;E), W_p^{[s]+1}(\Omega;E) \big]_{\frac{s}{[s]+1}},$$

and for $p \in [1,\infty)$ and $s > 0$ the *Slobodetskii space*

$$W_p^s(\Omega;E) := \begin{cases} W_p^k(\Omega;E), & s = k \in \mathbb{N}, \\ B^s_{p,p}(\Omega;E), & s \notin \mathbb{N}. \end{cases}$$

These Banach spaces form scales according to the general properties of interpolation spaces listed in Appendix A.2. Since E is of class $\mathcal{HT}$ it further holds that

$$W_p^k(\Omega;E) = H_p^k(\Omega;E), \qquad k \in \mathbb{N}, \tag{A.4.1}$$

see [91, Satz 3.6]. Usually the Besov spaces over $\mathbb{R}^n$ are defined by a Littlewood-Paley decomposition and the Bessel potential spaces are defined using the Fourier transform [91, Def. 3.1], [75, Def. 4.3], and then the spaces over domains are defined via restriction [9, Sec 4]. But since E is assumed to be of class $\mathcal{HT}$ and we assume that $\partial\Omega$ is smooth, it is equivalent to define them via interpolation, as in [45]. This can be seen using (A.4.1), the characterizations [75, Thm. 4.2, 4.3/3] and the interpolation results [75, Thm. 4.3/2], [91, Satz 3.21].

The Slobodetskii spaces admit for $s \notin \mathbb{N}_0$ the intrinsic representation

$$W_p^s(\Omega;E) = \big\{ u \in W_p^{[s]}(\Omega,E) : \text{ for } |\alpha| = [s] \text{ it holds } [D^\alpha u]_{W_p^{s-[s]}(\Omega;E)} < \infty \big\},$$

where for $\tau \in (0,1)$ the seminorm $[\cdot]_{W_p^\tau(\Omega;E)}$ is given by

$$[u]^p_{W_p^\tau(\Omega;E)} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|_E^p}{|x-y|^{n+\tau p}}\,\mathrm{d}x\,\mathrm{d}y, \tag{A.4.2}$$

and the space on the right-hand side above is equipped with the norm

$$\Big(|u|^p_{W_p^{[s]}(\Omega;E)} + \sum_{|\alpha|=[s]} [D^\alpha u]^p_{W_p^{s-[s]}(\Omega;E)}\Big)^{1/p},$$

cf. [8, Sec. 1, 5]. As in the scalar case we have the Sobolev embeddings [8, Eq. 5.4]

$$W_p^s(\Omega;E) \hookrightarrow W_q^\tau(\Omega;E), \qquad s-\frac{n}{p} \geq \tau - \frac{n}{q}, \qquad s \geq \tau, \quad p \geq q,$$

and further

$$W_p^s(\Omega;E) \hookrightarrow C^\tau(\Omega;E), \qquad s - \frac{n}{p} \geq \tau.$$

If E is finite dimensional, Ω is bounded and the above inequalities are strict, then these embeddings are compact by [1, Thm. 6.3] and [7, Sec. I.2.7].

Now suppose that $\partial\Omega$ is compact. Then there are a finite collection of charts (U_i,φ_i) for $\partial\Omega$ with corresponding parametrisations g_i and a partition of unity $\{\psi_i\}$ subordinate to $\bigcup_i U_i$. For $p \in [1,\infty]$ the spaces $L_p(\partial\Omega;E)$ are defined in a standard way with respect to the surface measure on $\partial\Omega$. Moreover, for $s>0$ and $p,q \in [1,\infty)$ we define as in [82, Def. 3.6.1]

$$\begin{aligned}
B^s_{p,q}(\partial\Omega;E) &:= \big\{u \in L_p(\partial\Omega;E) \;:\; (\psi_i u)\circ \mathrm{g}_i \in B^s_{p,q}(\mathbb{R}^{n-1};E) \;\text{ for all } i\,\big\},\\
|u|_{B^s_{p,q}(\partial\Omega;E)} &:= \textstyle\sum_i |(\psi_i u)\circ \mathrm{g}_i|_{B^s_{p,q}(\mathbb{R}^{n-1};E)},\\
H^s_p(\partial\Omega;E) &:= \big\{u \in L_p(\partial\Omega;E) \;:\; (\psi_i u)\circ \mathrm{g}_i \in H^s_p(\mathbb{R}^{n-1};E) \;\text{ for all } i\,\big\},\\
|u|_{H^s_p(\partial\Omega;E)} &:= \textstyle\sum_i |(\psi_i u)\circ \mathrm{g}_i|_{H^s_p(\mathbb{R}^{n-1};E)},\\
C^s(\partial\Omega;E) &:= \big\{u \in C(\partial\Omega;E) \;:\; (\psi_i u)\circ \mathrm{g}_i \in C^s(\mathbb{R}^{n-1};E) \;\text{ for all } i\,\big\},\\
|u|_{C^s(\partial\Omega;E)} &:= \textstyle\sum_i |(\psi_i u)\circ \mathrm{g}_i|_{C^s(\mathbb{R}^{n-1};E)},
\end{aligned}$$

which are all Banach spaces with their respective norms. Note that here we identify the functions $(\psi_i u)\circ \mathrm{g}_i$ with their trivial extension to $\mathbb{R}^{n-1}$. If one chooses another collection of charts and another partition of unity for $\partial\Omega$, one obtains the same spaces with equivalent norms, respectively.

It follows from the definitions that the basic embeddings obtained from interpolation as well as the Sobolev embeddings for spaces over domains carry over to the corresponding spaces over a boundary, with dimension n replaced by $n-1$. Moreover, as in [86, Thm. 4.3] it can be seen that

$$C^\infty(\partial\Omega;E) \overset{d}{\hookrightarrow} B^s_{p,q}(\partial\Omega;E), H^s_p(\partial\Omega;E),$$

for $s>0$ and $p,q\in[1,\infty)$.

We consider local properties of the above function spaces on and near the boundary.

Lemma A.4.1. *Let $s > 0$ and $p, q \in [1, \infty)$, and let (U, ϕ) be a chart for $\partial\Omega$, with corresponding push forward operator Φ. Then Φ induces a continuous isomorphism*

$$B^s_{p,q}(\Omega \cap U; E) \to B^s_{p,q}(\mathbb{R}^n_+ \cap \varphi(U); E),$$

with inverse Φ^{-1}. Moreover, for $\phi \in C^\infty_c(U)$ the map $u \mapsto \Phi(\phi u)$ is continuous

$$B^s_{p,q}(\partial\Omega; E) \to B^s_{p,q}(\mathbb{R}^{n-1} \cap \varphi(U); E),$$

and for $\widetilde{\phi} \in C^\infty_c(\mathbb{R}^{n-1} \cap \varphi(U))$ the map $u \mapsto \Phi^{-1}(\widetilde{\phi} u)$ is continuous

$$B^s_{p,q}(\mathbb{R}^{n-1} \cap \varphi(U); E) \to B^s_{p,q}(\partial\Omega; E).$$

All these assertions remain true if one replaces the $B^s_{p,q}$-spaces by the H^s_p-spaces, $s \geq 0$.

Proof. In the scalar-valued case and for $p = q = 2$, first assertion is shown in [86, Thm. 4.1]. The proof for W^s_p-spaces with $s \in \mathbb{N}_0$ and $p \in (1, \infty)$ carries over to the vector-valued case, from which the general case follows from interpolation. The remaining two assertions follow immediately from the definitions of the spaces over $\partial\Omega$. ■

We finish this section with a general interpolation result for the H- and the B-spaces.

Proposition A.4.2. *Let E be of class $\mathcal{HT}$, let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary, and let $p \in [1, \infty)$, $0 \leq s_1 < s_2$, $\theta \in (0, 1)$ and $s = (1-\theta)s_1 + \theta s_2$. Then it holds*

$$\big[H^{s_1}_p(\Omega; E), H^{s_2}_p(\Omega; E)\big]_\theta = H^s_p(\Omega; E), \qquad \big(H^{s_1}_p(\Omega; E), H^{s_2}_p(\Omega; E)\big)_{\theta,p} = B^s_{p,p}(\Omega; E),$$

$$\big(B^{s_1}_{p,p}(\Omega; E), B^{s_2}_{p,p}(\Omega; E)\big)_{\theta,p} = B^s_{p,p}(\Omega; E), \qquad \big[B^{s_1}_{p,p}(\Omega; E), B^{s_2}_{p,p}(\Omega; E)\big]_\theta = B^s_{p,p}(\Omega; E),$$

where the case $s_1 = 0$ is excluded for the B-spaces. These identities remain true if one replaces $\Omega \neq \mathbb{R}^n$ by its boundary $\partial\Omega$.

Proof. (I) First let $\Omega = \mathbb{R}^n$. The complex interpolation result for the H-spaces is shown in [91, Satz 3.21]. For the real interpolation of the H-spaces we consider the realization of the shifted Laplacian $A := 1 - \Delta_n$ on $L_p(\mathbb{R}^n; E)$ with domain $D(A) = W^2_p(\mathbb{R}^n; E)$, which is invertible and admits a bounded $\mathcal{H}^\infty$-calculus of $\mathcal{H}^\infty$-angle equal to zero, due to [24, Theorem 5.5]. It thus follows from complex interpolation of the H-spaces and (A.3.1) that $D(A^{\tau/2}) = H^\tau_p$, $\tau \geq 0$. From the reiteration theorem we infer

$$\big(H^{s_1}_p, H^{s_2}_p\big)_{\theta,p} = \big(D(A^{s_1/2}), D(A^{s_2/2})\big)_{\theta,p} = \big(L_p, D(A^{([s_2]+1)/2})\big)_{\frac{s}{[s_2]+1},p} = B^s_{p,p},$$

as asserted. The real interpolation result for the B-spaces is shown in [75, Thm. 4.2]. Taking powers of A as isomorphisms we obtain from [24, Prop. 2.11] and the interpolation results that were already shown that the realization of $A^{\tau/2}$ on $B^{\tau_1}_{p,p}$ with domain $B^{\tau_1+\tau}_{p,p}$ admits a bounded $\mathcal{H}^\infty$-calculus of $\mathcal{H}^\infty$-angle equal to zero as well, $\tau, \tau_1 > 0$. Thus the complex interpolation result follows from Yagi's theorem (A.3.1).

(II) Now suppose that Ω is a domain with smooth boundary. For given $k \in \mathbb{N}$ there is a continuous extension operator $\mathcal{E}_\Omega$ from $W^l_p(\Omega; E)$ to $W^l_p(\mathbb{R}^n; E)$ for all $l \in \{0, ..., k\}$, which may be extended to the H- and the B-scale by interpolation. It also follows from interpolation that the restriction of functions on $\mathbb{R}^n$ to Ω is continuous on both scales. From the result on $\mathbb{R}^n$ we thus infer that $\mathcal{E}_\Omega$ is continuous

$$\big[H_p^{s_1}(\Omega;E), H_p^{s_2}(\Omega;E)\big]_\theta \to H_p^s(\mathbb{R}^n;E),$$

and combined with the restriction to Ω this yields that $\big[H_p^{s_1}(\Omega;E), H_p^{s_2}(\Omega;E)\big]_\theta$ embeds continuously into $H_p^s(\Omega;E)$. Conversely, $\mathcal{E}_\Omega$ maps continuously

$$H_p^s(\Omega;E) \to \big[H_p^{s_1}(\mathbb{R}^n;E), H_p^{s_2}(\mathbb{R}^n;E)\big]_\theta,$$

and the restriction maps the latter space into $\big[H_p^{s_1}(\Omega;E), H_p^{s_2}(\Omega;E)\big]_\theta$. We thus obtain the asserted complex interpolation result for the H-spaces. The remaining identities follow from the same arguments.

(III) We finally consider the spaces over $\partial\Omega$. Describe the boundary by a finite collection of charts (U_i, φ_i) with corresponding push-forward operators Φ_i, and let $\{\psi_i\}$ be a partition of unity subordinate to $\bigcup_i U_i$. Choose further $\phi_i \in C_c^\infty(\mathbb{R}^{n-1} \cap \varphi(U_i))$ with $\phi_i \equiv 1$ on $\operatorname{supp} \Phi_i\psi_i \cap \mathbb{R}^{n-1}$. We decompose the identity on $H_p^\tau(\partial\Omega;E)$, $\tau \geq 0$, according to $\mathrm{id} = \sum_i \Phi_i^{-1}\phi_i\big(\Phi_i\psi_i\big)$. For each i, Lemma A.4.1 shows that the map $u \mapsto \Phi_i\big(\psi_i u\big)$ is continuous $H_p^\tau(\partial\Omega;E) \to H_p^\tau(\mathbb{R}^{n-1} \cap \varphi_i(U_i);E)$ for all τ, and from interpolation and the result on domains we obtain that it is continuous

$$\big[H_p^{s_1}(\partial\Omega;E), H_p^{s_2}(\partial\Omega;E)\big]_\theta \to H_p^s(\mathbb{R}^{n-1} \cap \varphi_i(U_i);E).$$

Lemma A.4.1 also yields that $u \mapsto \Phi_i^{-1}(\phi_i u)$ is continuous $H_p^s(\mathbb{R}^{n-1} \cap \varphi_i(U_i);E) \to H_p^s(\partial\Omega;E)$, which implies that $\big[H_p^{s_1}(\partial\Omega;E), H_p^{s_2}(\partial\Omega;E)\big]_\theta \hookrightarrow H_p^s(\partial\Omega;E)$. The converse embedding and the remaining identities are shown in the same way. ∎

A.5 Differential Operators on a Boundary

Let $\Omega \subset \mathbb{R}^n$ be a domain with compact smooth boundary $\partial\Omega$, and let E be a Banach space of class $\mathcal{HT}$. We call a linear map

$$\mathcal{C} : C^\infty(\partial\Omega;E) \to L_1(\partial\Omega;E)$$

a *linear differential operator on* $\partial\Omega$ *of order* $k \in \mathbb{N}_0$, if for all local coordinates g for $\partial\Omega$ there are coefficients $c_\gamma^{\mathrm{g}} \in L_1\big(\mathbb{R}^{n-1} \cap \varphi(U);\mathcal{B}(E)\big)$, $\gamma \in \mathbb{N}_0^{n-1}$, $|\gamma| \leq k$ such that

$$\big(\mathcal{C}u\big) \circ \mathrm{g} = \sum_{|\gamma| \leq k} c_\gamma^{\mathrm{g}}(x) \nabla_{n-1}^\gamma (u \circ \mathrm{g})(x), \qquad x \in \mathrm{g}^{-1}(U \cap \partial\Omega), \tag{A.5.1}$$

for all $u \in C^\infty(\partial\Omega;E)$. Here $\nabla_{n-1} = (\partial_1, ..., \partial_{n-1})$ is the euclidian gradient on $\mathbb{R}^{n-1}$. Of course it is understood that at least one top order coefficient is nontrivial. The local coefficients c_γ^{g} may depend on the chosen coordinates g. We do not assume that $\mathcal{C}$ has global coefficients, in the sense that there are $c_\gamma \in L_1\big(\Gamma;\mathcal{B}(E)\big)$ with $c_\gamma^{\mathrm{g}} = c_\gamma \circ \mathrm{g}$.

The regularity of the local coefficients determines whether $\mathcal{C}$ can be continuously extended to other function spaces, or even to a whole scale. Suppose that for all coordinates g and all $\phi \in C_c^\infty(U)$ there is an estimate of the form

$$|(\phi\,\mathcal{C}u)\circ \mathrm{g}|_{W_p^s(\mathbb{R}^{n-1}\cap\varphi(U);E)} \lesssim |u|_{W_p^{s+k}(\partial\Omega;E)}, \qquad u\in C^\infty(\partial\Omega;E),$$

where $s \geq 0$. Then by density of $C^\infty(\partial\Omega;E)$ in $W_p^{s+k}(\partial\Omega;E)$ the operator $\mathcal{C}$ may be uniquely extended to a bounded linear map

$$W_p^{s+k}(\partial\Omega;E) \to W_p^s(\partial\Omega;E).$$

In local coordinates the extended operator is of the same form as in (A.5.1) for smooth functions, as a density argument shows. This reasoning remains valid for the extension of $\mathcal{C}$ to Besov and Bessel potential spaces.

A sufficient condition for the extendability of $\mathcal{C}$ to $W_p^{s+k}(\partial\Omega;E)$ is that for all $l\in\{0,...,k\}$ and all coordinates g the coefficients c_γ^{g} with $|\gamma| = l$ are pointwise multipliers from W_p^{s+l} to W_p^s. In particular, if the local coordinates are smooth then $\mathcal{C}$ extends to the whole scale of Slododetskii, Besov and Bessel potential spaces, respectively.

We consider examples for differential operators on boundaries. For $x\in\partial\Omega$ a scalar-valued function $u\in C^\infty(\partial\Omega)$ induces an element of the dual space of $T_x\partial\Omega$ via the directional derivative of tangential vectors at $x\in\partial\Omega$. Considering $T_x\partial\Omega$ as a Hilbert space with the scalar product induced from $\mathbb{R}^n$, the surface gradient $\nabla_\Gamma u(x)$ of u at x is then the unique element of $T_x\partial\Omega$ corresponding to this dual space element via the Riesz isomorphism. In local coordinates g for $\partial\Omega$, with fundamental form $\mathrm{G} = (\mathrm{g}_{ij})$ and inverse $\mathrm{G}^{-1} = (\mathrm{g}^{ij})$, the components of the surface gradient with respect to the canonical basis $\{\partial_1\mathrm{g},...,\partial_{n-1}\mathrm{g}\}$ of $T_x\partial\Omega$ are given by the components of $\mathrm{G}^{-1}\nabla_{n-1}(u\circ \mathrm{g})^T$, i.e.,

$$\nabla_{\partial\Omega}u\circ \mathrm{g} = \sum_{i,j=1}^{n-1} \mathrm{g}^{ij}\partial_j(u\circ \mathrm{g})\partial_i \mathrm{g}. \tag{A.5.2}$$

Now let E be a Banach space. We define the surface gradient for a function $u\in C^\infty(\partial\Omega;E)$ in coordinates by the formula (A.5.2), which yields that

$$\nabla_{\partial\Omega}u \in C^\infty(\partial\Omega;E^n).$$

The application of functionals and the Hahn-Banach theorem show that this definition is independent of the chosen coordinates, since it is independent of them in the scalar valued case. Moreover, for $\alpha^*\in E^*$ and $u\in C^\infty(\partial\Omega;E)$ it holds

$$\alpha^*\big(\nabla_{\partial\Omega}u\big) = \nabla_{\partial\Omega}\big(\alpha^* u\big),$$

where on the left-hand side the functional is applied componentwise to elements of E^n. In this sense the definition of the surface gradient for E-valued functions is consistent with the definition in the scalar case, and because of this we still speak of a gradient although a Riesz isomorphism is only indirectly involved.

For a multiindex $\gamma \in \mathbb{N}_0^{n-1}$ the operator $\nabla_{\partial\Omega}^{\gamma}$ is defined by taking iteratively the components of $\nabla_{\partial\Omega} u$. This yields a linear map from $C^\infty(\partial\Omega; E)$ into itself, and is thus a boundary differential operator in the above sense. In particular $\nabla_{\partial\Omega}^{\gamma}$ extends to a bounded linear map

$$W_p^{s+|\gamma|}(\partial\Omega; E) \to W_p^s(\partial\Omega; E), \qquad s \geq 0, \qquad p \in [1, \infty],$$

and analogously for the Besov and Bessel potential scale.

The surface divergence $\mathrm{div}_{\partial\Omega} v$ of a tangential vector field $v \in C^\infty(\partial\Omega, \mathbb{R}^n)$, i.e., $v(x) \in T_x\partial\Omega$ for $x \in \partial\Omega$, is in coordinates g given by

$$\mathrm{div}_{\partial\Omega} v \circ \mathrm{g} = \frac{1}{\sqrt{|\mathrm{G}|}} \sum_{i=1}^{n-1} \partial_i \big(\sqrt{|\mathrm{G}|}\, v^i \circ \mathrm{g} \big),$$

where v^i are the components of v with respect to the basis $\{\partial_1 \mathrm{g}, ..., \partial_{n-1}\mathrm{g}\}$ of $T_x\partial\Omega$. It can be shown the $\mathrm{div}_{\partial\Omega}$ is independent of the coordinates. The Laplace-Beltrami operator

$$\Delta_{\partial\Omega} := \mathrm{div}_{\partial\Omega} \nabla_{\partial\Omega}$$

is then for $u \in C^\infty(\partial\Omega)$ in local coordinates of the form

$$\big(\Delta_{\partial\Omega} u\big) \circ \mathrm{g} = \frac{1}{\sqrt{|\mathrm{G}|}} \sum_{i,j=1}^{n-1} \partial_i \big(\sqrt{|\mathrm{G}|} \mathrm{g}^{ij} \partial_j (u \circ \mathrm{g}) \big).$$

The Laplace-Beltrami operator of a vector-valued function $u \in C^\infty(\partial\Omega; E)$ is defined in coordinates by the above formula. In the same way as for the surface gradient we see that this definition is independent of the coordinates, and it holds

$$\alpha^*\big(\Delta_{\partial\Omega} u\big) = \Delta_{\partial\Omega}\big(\alpha^* u\big), \qquad \alpha^* \in E^*,$$

which shows consistency to the scalar-valued case as above.

With the Laplace-Beltrami operator and the surface gradient one can define the boundary analogon to general 'elliptic differential operators' acting on vector-valued functions. We finally remark that the considerations of this section carry over to a general Riemannian manifold.

A.6 Gagliardo-Nirenberg Inequalities

The first version of the Gagliardo-Nirenberg inequality for integer differentiabilities is taken from [41, Thm. 10.1].

Proposition A.6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth domain $\partial\Omega$, and let the integers $k \in \mathbb{N}_0$, $m \in \mathbb{N}$, with $k < m$, and the numbers $p, q, r \in [1, \infty]$ satisfy*

$$k - \frac{n}{p} = \theta\Big(m - \frac{n}{q}\Big) - (1-\theta)\frac{n}{r},$$

where $\theta \in [k/m, 1]$ *if* $m - k - n/r \notin \mathbb{N}_0$, *and* $\theta = k/m$ *if* $m - k - n/r \in \mathbb{N}_0$. *Then there is a constant* $C > 0$ *such that*

$$|u|_{W_p^k(\Omega)} \le C|u|^{\theta}_{W_q^m(\Omega)}|u|^{1-\theta}_{L_r(\Omega)} \qquad \text{for all } \; u \in W_q^m(\Omega) \cap L_r(\Omega). \qquad \blacksquare$$

In [2, Prop. 4.1] an partial extension to fractional order Slobodetskii spaces is given. Observe that for integer differentiabilities Proposition A.6.1 may lead to a stronger result, for instance for $p = q = 2$, $r = 1$, $k = 0$, $m = 1$ and $\theta = \frac{n}{n+2}$.

Proposition A.6.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain with compact smooth boundary* $\partial\Omega$, *and let the numbers* $\theta \in [0, 1]$, $s, s_0, s_1 \ge 0$ *with* $s_0 \ne s_1$ *and* $p, p_1 \in (1, \infty)$, $p_0 \in [1, \infty)$, *satisfy*

$$\frac{1}{p} \le \frac{\theta}{p_1} + \frac{1-\theta}{p_0}, \tag{A.6.1}$$

and

$$s - \frac{n}{p} < \theta\left(s_1 - \frac{n}{p_1}\right) + (1-\theta)\left(s_0 - \frac{n}{p_0}\right), \tag{A.6.2}$$

with the following exceptions: it holds $s_0 = 0$ *if* $p_0 = 1$, *it holds* $\theta > 0$ *if* $s_0 = 0$, *and it holds* $\theta < 1$ *if* $s_1 = 0$. *Then there is a constant* $C > 0$ *such that*

$$|u|_{W_p^s(\Omega)} \le C|u|^{\theta}_{W_{p_1}^{s_1}(\Omega)}|u|^{1-\theta}_{W_{p_0}^{s_0}(\Omega)} \qquad \text{for all } \; u \in W_{p_1}^{s_1}(\Omega) \cap W_{p_0}^{s_0}(\Omega).$$

If Ω *is bounded and* $s < \theta s_1 + (1-\theta)s_0$ *then (A.6.1) is not necessary. Further, the equality sign in (A.6.2) is permitted if* $p_0 > 1$ *and either* $s_0, s_1 \in \mathbb{N}$ *or* $\theta s_1 + (1-\theta)s_0 \notin \mathbb{N}$. $\blacksquare$

By definition these inequalities carry over to the spaces over $\partial\Omega$, with n replaced by $n-1$, respectively.

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Erklärung

Hiermit erkläre ich, dass ich die Dissertation selbstständig verfasst, nur die angegebenen Quellen und Hilfsmittel benutzt, wörtlich oder inhaltlich übernommene Stellen als solche gekennzeichnet und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis beachtet habe.

Karlsruhe, den 8. November 2010

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