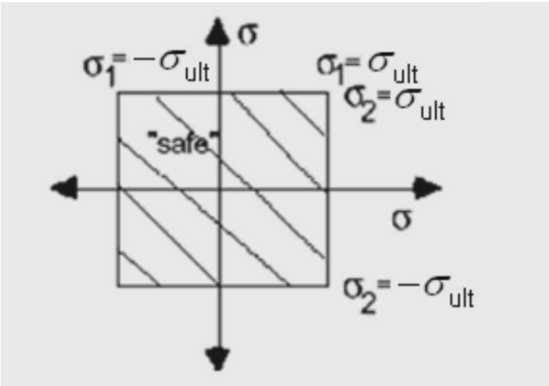
Maximum Principal Stress Theory (W. Rankin's Theory- 1850) – Brittle Material

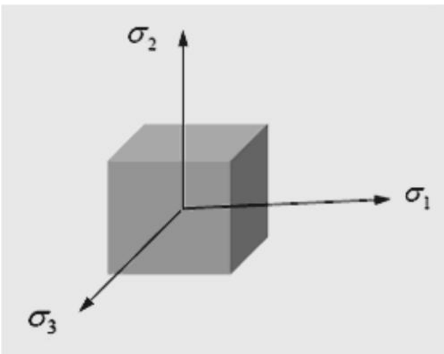
The maximum principal stress criterion:

- Rankin stated max principal stress theory as follows- a material fails by fracturing when the largest principal stress exceeds the ultimate strength σ_u in a simple tension test. That is, at the onset of facture, $|\sigma_1| = \sigma_u$ OR $|\sigma_3| = \sigma_u$
- Crack will start at the most highly stressed point in a brittle material when the largest Principal stress at that point reaches σ_u
- Criterion has good experimental verification, even though it assumes ultimate strength is same in compression and tension



Failure surface according to maximum principal stress theory

- This theory of yielding has very poor agreement with experiment. However, the theory has been used successfully for brittle materials.
- Used to describe fracture of **brittle materials** such as cast iron
- ***Limitations***
  - Doesn't distinguish between tension or compression
  - Doesn't depend on orientation of principal planes so only applicable to isotropic materials
- Generalization to 3-D stress case is easy:

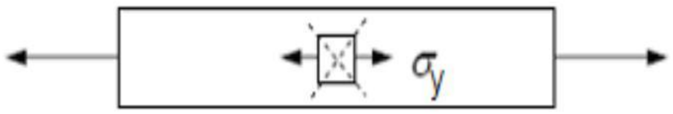


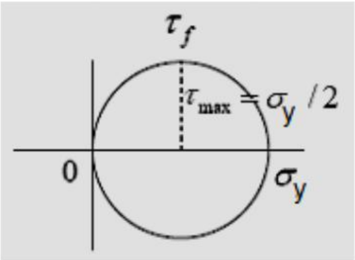
Maximum Shear Stress or Stress difference theory (Guest's or Tresca's Theory-1868)- Ductile Material

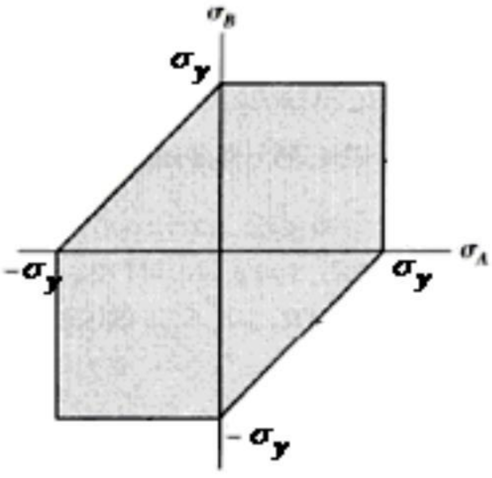
***The Tresca Criterion*:**

- Also known as the *Maximum Shear Stress* criterion.
- Yielding will occur when the maximum shear stress reaches that which caused yielding in a simple tension test.
- Recall that yielding of a material occurred by slippage between planes oriented at 45° to principal stresses. This should indicate to you that yielding of a material depends on the maximum shear stress in the material rather than the maximum normal stress.

  If $\sigma_1 > \sigma_2 > \sigma_3$ Then $\sigma_1 - \sigma_3 = \sigma_y$
- Failure by slip (yielding) occurs when the maximum shearing stress, τ_{max} exceeds the yield stress τ_f as determined in a uniaxial tension test.
- This theory gives *satisfactory* result for ***ductile material***.







Failure surface according to maximum shear stress theory

Strain Energy Theory (Haigh's Theory)

This theory is based on the assumption that strains are recoverable up to the elastic limit, and the energy absorbed by the material at failure up to this point is a single valued function independent of the stress system causing it. The strain energy per unit volume causing failure is equal to the strain energy at the elastic limit in simple tension.

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] = \frac{\sigma_y^2}{2E}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) = \sigma_y^2$$ For 3D- stress

$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \sigma_y^2$$ For 2D- stress

Shear Strain Energy Theory (Distortion Energy Theory or Mises-Henky Theory or Von-Misses Theory)-Ductile Material

- Also known as the Maximum Energy of Distortion criterion
  - Based on a more complex view of the role of the principal stress differences.
- In simple terms, the Von-Mises criterion considers the diameters of all three Mohr's circles as contributing to the characterization of yield onset in isotropic materials.
- When the criterion is applied, its relationship to the uniaxial tensile yield strength is:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

- F or a state of plane stress ($\sigma_3 = 0$)

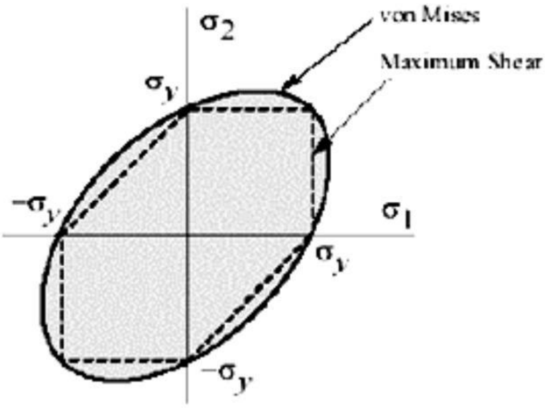
$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_y^2$$

- It is often convenient to express this as an equivalent stress, σ_e:

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$or\, \sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}$$

- In formulating this failure theory we used generalized Hooke's law for an isotropic material so the theory given is only applicable to those materials but it can be generalized to anisotropic materials.
- The von Mises theory is a little less conservative than the Tresca theory but in most cases there is little difference in their predictions of failure. Most experimental results tend to fall on or between these two theories.
- It gives very good result in **ductile material.**



Maximum Principal Strain Theory (St. Venant Theory)

According to this theory, yielding will occur when the maximum principal strain just exceeds the strain at the tensile yield point in either simple tension or compression. If ε1 and ε2 are maximum and mini mum principal strains corresponding to σ1 and σ2, in the limiting case

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) \qquad |\sigma_1| \geq |\sigma_2|$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) \qquad |\sigma_2| \geq |\sigma_1|$$

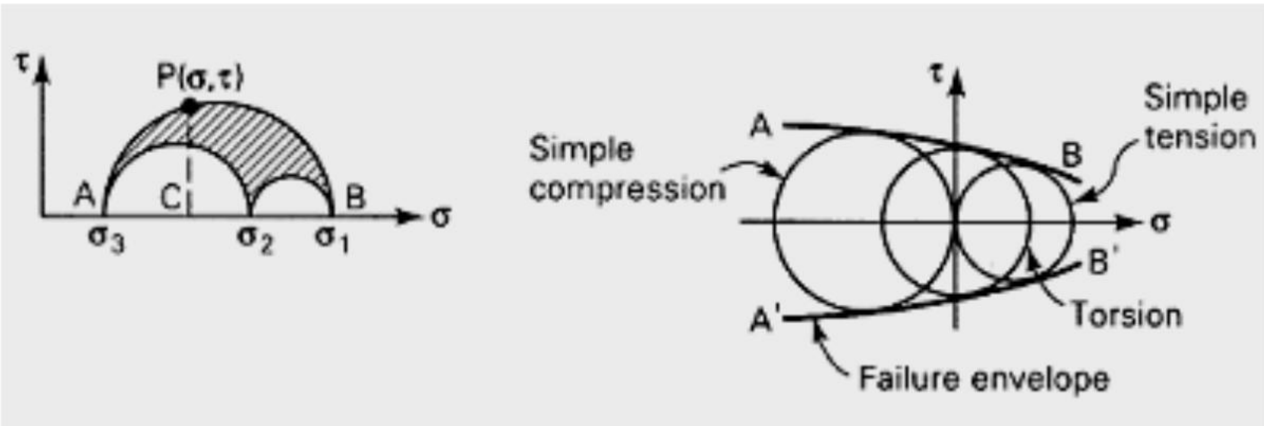
This gives, $E\varepsilon_1 = \sigma_1 - \nu\sigma_2 = \pm\sigma_y$

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_1 = \pm\sigma_y$$

<u>Mohr's theory- Brittle Material</u>

Mohr's Theory

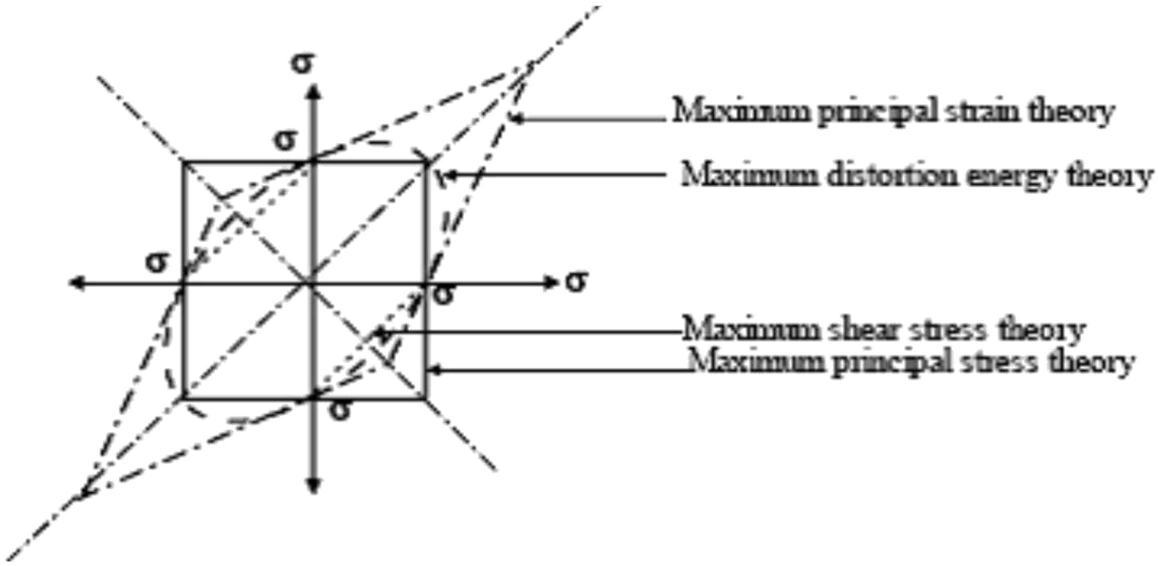
- Mohr's theory is used to predict the fracture of a material having different properties in tension and compression. Criterion makes use of Mohr's circle
- In Mohr's circle, we note that τ depends on σ, or *τ = f(σ).* Note the vertical line *PC* Represents states of stress on planes with same σ but differing *τ*, which means the weakest plane is the one with maxim um *τ* , point ***P***.
- Points on the outer circle are the weakest planes. On these planes the maximum and minimum principal stresses are sufficient to decide whether or not failure will occur.
- Experiments are done on a given material to determine the states of stress that result in failure. Each state defines a Mohr's circle. If the data are obtained from simple tension, simple compression, and pure shear, the three resulting circles are adequate to construct an ***envelope*** (AB & A'B')
- Mohr's envelope thus represents the locus of all possible failure states.



Higher shear stresses are to the left of origin, since most brittle materials have higher strength in compression

Comparison

A comparison among the different failure theories can be made by superposing the yield surfaces as shown in figure



Match 4 correct pairs between list I and List II for the questions **[GATE-1994]**

List-I	**List-II**
(a) Hooke's law	**1.** Planetary motion
(b) St. Venant's law	**2.** Conservation Energy
(c) Kepler's laws	**3.** Elasticity
(d) Tresca's criterion	**4.** Plasticity
(e) Coulomb's laws	**5.** Fracture
(f) Griffith's law	**6.** Inertia

Ans. (a) - 3, (c) -1, (d) -5, (e) -2

St. Venant's law: Maximum principal strain theory

Which theory of failure will you use for aluminium components under steady loading? **[GATE-1999]**

(a) Principal stress theory
(b) Principal strain theory
(c) Strain energy theory
(d) Maximum shear stress theory

Ans. (d) Aluminium is a ductile material so use maximum shear stress theory

According to Von-Mises' distortion energy theory, the distortion energy under three dimensional stress state is represented by [GATE-2006]

(a) $\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2v(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3\right]$

(b) $\frac{1-2v}{6E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

(c) $\frac{1+v}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

(d) $\frac{1}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-v(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

Ans. (c)

$V_s=\frac{1}{12G}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}$ Where $E=2G(1+\mu)$ simplify and get result.

A small element at the critical section of a component is in a bi-axial state of stress with the two principal stresses being 360 MPa and 140 MPa. The maximum working stress according to Distortion Energy Theory is: [GATE-1997]

(a) 220 MPa (b) 110 MPa (c) 314 MPa (d) 330 MPa

Ans. (c) According to distortion energy theory if maximum stress (σ_t) then

or $\sigma_t^2=\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2$

or $\sigma_t^2=360^2+140^2-360\times 140$

or $\sigma_t=314$ MPa

Match List-I (Theory of Failure) with List-II (Predicted Ratio of Shear Stress to Direct Stress at Yield Condition for Steel Specimen) and select the correct answer using the code given below the Lists: [IES-2006]

List-I	List-II
A. Maximum shear stress theory	1. 1·0
B. Maximum distortion energy theory	2. 0·577
C. Maximum principal stress theory	3. 0·62
D. Maximum principal strain theory	4. 0·50

Codes:	A	B	C	D		A	B	C	D
(a)	1	2	4	3	(b)	4	3	1	2
(c)	1	3	4	2	(d)	4	2	1	3

Ans. (d)

From a tension test, the yield strength of steel is found to be 200 N/mm². Using a factor of safety of 2 and applying maximum principal stress theory of failure, the permissible stress in the steel shaft subjected to torque will be: [IES-2000]

(a) 50 N/mm² (b) 57.7 N/mm² (c) 86.6. N/mm² (d) 100 N/mm²

Ans. (d) For pure shear $\tau=\pm\sigma_x$

A circular solid shaft is subjected to a bending moment of 400 kNm and a twisting moment of 300 kNm. On the basis of the maximum principal stress theory, the direct stress is σ and according to the maximum shear stress theory, the shear stress is τ. The ratio σ/ τ is: [IES-2000]

(a) $\frac{1}{5}$ (b) $\frac{3}{9}$ (c) $\frac{9}{5}$ (d) $\frac{11}{6}$

Ans. (c) $\sigma = \frac{16}{\pi d^3}\left(M + \sqrt{M^2 + T^2}\right)$ and $\tau = \frac{16}{\pi d^3}\left(\sqrt{M^2 + T^2}\right)$

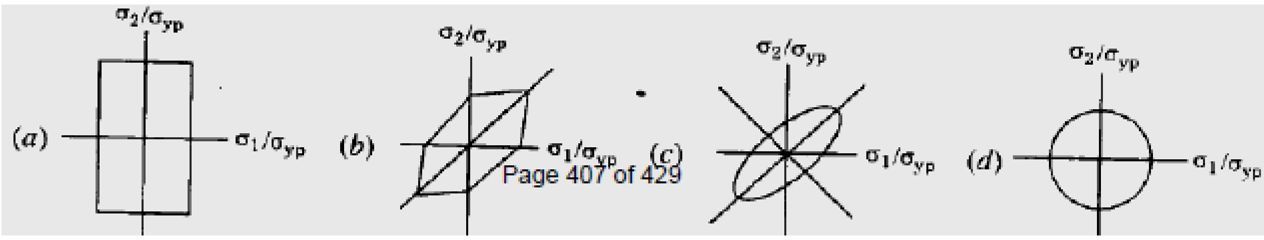
Therefore $\frac{\sigma}{\tau} = \frac{M + \sqrt{M^2 + T^2}}{\sqrt{M^2 + T^2}} = \frac{4 + \sqrt{4^2 + 3^2}}{\sqrt{4^2 + 3^2}} = \frac{9}{5}$

A transmission shaft subjected to bending loads must be designed on the basis of [IES-1996]

(a) Maximum normal stress theory
(b) Maximum shear stress theory
(c) Maximum normal stress and maximum shear stress theories
(d) Fatigue strength

Ans. (a)

Which one of the following figures represents the maximum shear stress theory or Tresca criterion? [IES-1999]



Design of shafts made of brittle materials is based on [IES-1993]

(a) Guest's theory (b) Rankine's theory (c) St. Venant's theory (d) Von Mises theory

Ans. (b) Rankine's theory or maximum principle stress theory is most commonly used for brittle materials.

According to the maximum shear stress theory of failure, permissible twisting moment in a circular shaft is 'T'. The permissible twisting moment will the same shaft as per the maximum principal stress theory of failure will be: [IES-1998: ISRO-2008]

(a) T/2 (b) T (c) $\sqrt{2T}$ (d) 2T

Ans. (d) Given $\tau = \frac{16T}{\pi d^3} = \frac{\sigma_{yt}}{2}$ principal stresses for only this shear stress are

$\sigma_{1,2} = \sqrt{\tau^2} = \pm\tau$ maximum principal stress theory of failure gives

$$\max[\sigma_1, \sigma_2] = \sigma_{yt} = \frac{16(2T)}{\pi d^3}$$

Permissible bending moment in a circular shaft under pure bending is M according to maximum principal stress theory of failure. According to maximum shear stress theory of failure, the permissible bending moment in the same shaft is: [IES-1995]

(a) 1/2 M (b) M (c) $\sqrt{2}$ M (d) 2M

Ans. (b) $\sigma = \frac{16}{\pi d^3}\left(M + \sqrt{M^2 + T^2}\right)$ and $\tau = \frac{16}{\pi d^3}\left(\sqrt{M^2 + T^2}\right)$ put T = 0

$$\text{or } \sigma_{yt} = \frac{32M}{\pi d^3} \text{ and } \tau = \frac{16M'}{\pi d^3} = \frac{\sigma_{yt}}{2} = \frac{\left(\frac{32M}{\pi d^3}\right)}{2} = \frac{16M}{\pi d^3} \text{ Therefore } M' = M$$

A rod having cross-sectional area 100 x 10^{-6} m^2 is subjected to a tensile load. Based on the Tresca failure criterion, if the uniaxial yield stress of the material is 200 MPa, the failure load is: [IES-2001]

(a) 10 kN (b) 20 kN (c) 100 kN (d) 200 kN

Ans. (b) Tresca failure criterion is maximum shear stress theory.

$$\text{We know that, } \tau = \frac{P}{A}\frac{\sin 2\theta}{2} \text{ or } \tau_{max} = \frac{P}{2A} = \frac{\sigma_{yt}}{2} \text{ or } P = \sigma_{yt} \times A$$

A cold roller steel shaft is designed on the basis of maximum shear stress theory. The principal stresses induced at its critical section are 60 MPa and - 60 MPa respectively. If the yield stress for the shaft material is 360 MPa, the factor of safety of the design is: [IES-2002]

(a) 2 (b) 3 (c) 4 (d) 6

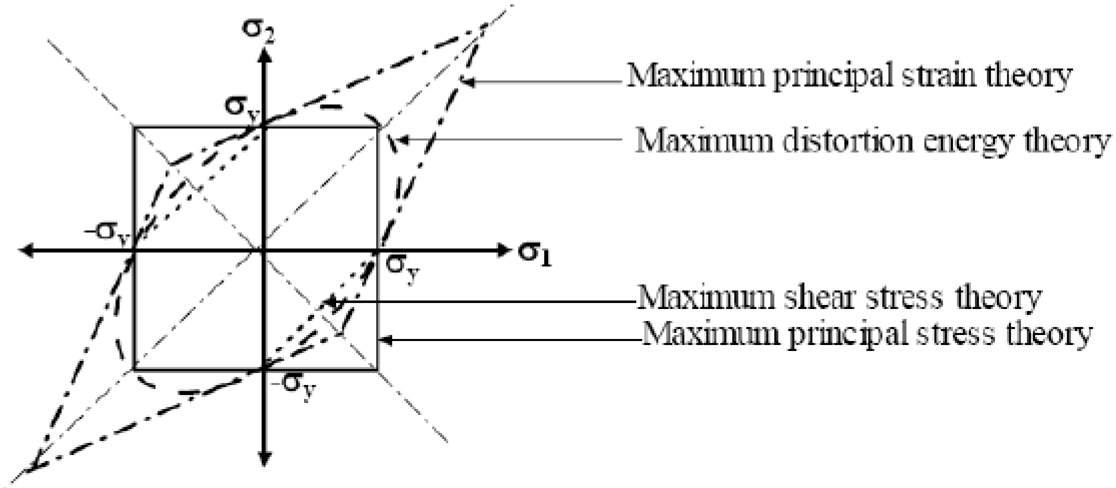
Ans. (b)

A shaft is subjected to a maximum bending stress of 80 N/mm² and maximum shearing stress equal to 30 N/mm² at a particular section. If the yield point in tension of the material is 280 N/mm², and the maximum shear stress theory of failure is used, then the factor of safety obtained will be: [IES-1994]

(a) 2.5 (b) 2.8 (c) 3.0 (d) 3.5

Ans. (b) Maximum shear stress = $\sqrt{\left(\frac{80-0}{2}\right)^2 + 30^2} = 50 \text{ N/mm}^2$

According to maximum shear stress theory, $\tau = \frac{\sigma_y}{2}; \ \therefore F.S. = \frac{280}{2 \times 50} = 2.8$



Graphical comparison of different failure theories

Above diagram shows that $\sigma_1 > 0, \sigma_2 < 0$ will occur at 4th quadrant and most conservative design will be maximum shear stress theory.

Who postulated the maximum distortion energy theory? [IES-2008]

(a) Tresca (b) Rankine (c) St. Venant (d) Mises-Henky

Ans. (d)

Maximum shear stress theory	→	Tresca
Maximum principal stress theory	→	Rankine
Maximum principal strain theory	→	St. Venant
Maximum shear strain energy theory	→	Mises – Henky

The maximum distortion energy theory of failure is suitable to predict the failure of which one of the following types of materials? [IES-2004]

(a) Brittle materials (b) Ductile materials (c) Plastics (d) Composite materials

Ans. (b)

If σ_y is the yield strength of a particular material, then the distortion energy theory is expressed as [IES-1994]

(a) $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$

(b) $(\sigma_1^2 - \sigma_2^2 + \sigma_3^2) - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$

(c) $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 3\sigma_y^2$

(d) $(1 - 2\mu)(\sigma_1 + \sigma_2 + \sigma_3)^2 = 2(1 + \mu)\sigma_y^2$

Ans. (a)

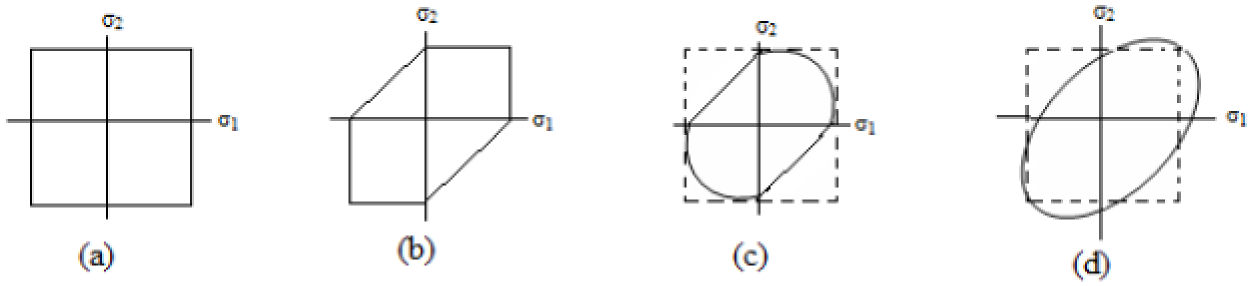
If a shaft made from ductile material is subjected to combined bending and twisting moments, calculations based on which one of the following failure theories would give the most conservative value? [IES-1996]

(a) Maximum principal stress theory (b) Maximum shear stress theory.

(d Maximum strain energy theory (d) Maximum distortion energy theory.

Ans. (b)

Which one of the following graphs represents Mises yield criterion? [IAS-1996]



Ans. (d)

Consider the following statements: [IAS-2003]

1. Distortion-energy theory is in better agreement for predicting the failure of ductile materials.
2. Maximum normal stress theory gives good prediction for the failure of brittle materials.
3. Module of elasticity in tension and compression are assumed to be different stress analysis of curved beams.

Which of these statements is/are correct?

(a) 1, 2 and 3 (b) 1 and 2 (c) 3 only (d) 1 and 3

Ans. (b)

-

Q. **The stress state at a point in a body is plane with**

$\sigma_1 = 60\,N/mm^2$ **&** $\sigma_2 = -36\,N/mm^2$

If the allowable stress for the material in simple tension or compression is 100 N/mm² calculate the value of factor of safety with each of the following criteria for failure

(i) Max Stress Criteria
(ii) Max Shear Stress Criteria
(iii) Max strain criteria
(iv) Max Distortion energy criteria **[10 Marks]**

Ans. The stress at a point in a body is plane

$\sigma_1 = 60\ N/mm^2 \qquad \sigma_2 = -36\ N/mm^2$

Allowable stress for the material in simple tension or compression is 100 N/mm²

Find out factor of safety for

(i) Maximum stress Criteria : - In this failure point occurs when max principal stress reaches the limiting strength of material.

Therefore. Let F.S factor of safety

$$\sigma_1 = \frac{\sigma\ (\text{allowable})}{F.S}$$

$$F.S = \frac{100\ N/mm^2}{60\ N/mm^2} = \underline{1.67} \qquad \text{Ans.}$$

(ii) Maximum Shear stress criteria : - According to this failure point occurs at a point in a member when maximum shear stress reaches to shear at yield point

$$\gamma_{max} = \frac{\sigma_{yt}}{2\ F.S} \qquad \sigma_{yt} = 100\ N/mm^2$$

$$\gamma_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{60+36}{2} = \frac{96}{2} = \underline{48}\ N/mm^2$$

$$48 = \frac{100}{2 \times F.S}$$

$$F.S = \frac{100}{\underline{2 \times 48}} = \frac{100}{96} = \underline{1.042}$$

$$F.S = 1.042 \qquad \text{Ans.}$$

(iii) Maximum Strain Criteria ! – In this failure point occurs at a point in a member when maximum strain in a bi – axial stress system reaches the limiting value of strain (i.e strain at yield point)

$$\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 = \left(\frac{\sigma_{\text{allowable}}}{FOS}\right)^2$$

$$FOS = 1.27$$

$$(\mu = 0.3\,\text{assume})$$

(iv) Maximum Distortion energy criteria ! – In this failure point occurs at a point in a member when distortion strain energy per unit volume in a bi – axial system reaches the limiting distortion strain energy at the of yield

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \times \sigma_2 = \left(\frac{\sigma_{yt}}{F.S}\right)^2$$

$$60^2 + (36)^2 - \times 60 \times -36 = \left(\frac{100}{F.S}\right)^2$$

$$F.S = 1.19$$

Question: **A mild steel shaft of 50 mm diameter is subjected to a beading moment of 1.5 kNm and torque T. If the yield point of steel in tension is 210 MPa, find the maximum value of the torque without causing yielding of the shaft material according to**

(i) Maximum principal stress theory

(ii) Maximum shear stress theory.

Answer: We know that, Maximum bending stress $(\sigma_b) = \frac{32M}{\pi d^3}$

and Maximum shear stress $(\tau) = \frac{16T}{\pi d^3}$

Principal stresses are given by:

$$\sigma_{1,2} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} = \frac{16}{\pi d^3}\left[M \pm \sqrt{M^2 + T^2}\right]$$

(i) According to Maximum principal stress theory

Maximum principal stress=Maximum stress at elastic limit (σ_y)

$$or\ \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right] = 210 \times 10^6$$

$$\text{or } \frac{16}{\pi (0.050)^3}\left[1500 + \sqrt{1500^2 + T^2}\right] = 210 \times 10^6$$

or T = 3332 Nm = 3.332 kNm

(ii) According to Maximum shear stress theory

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_y}{2}$$

$$or,\ \sigma_1 - \sigma_2 = \sigma_y$$

$$or,\ 2 \times \frac{16}{\pi d^3}\sqrt{M^2 + T^2} = 210 \times 10^6$$

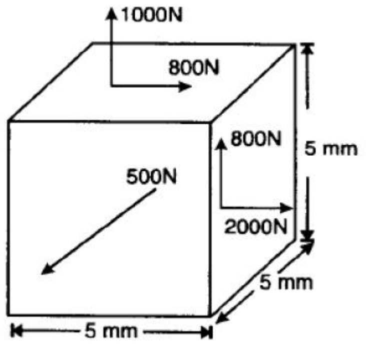
or, T = 2096 N m = 2.096 kNm

Question: **A cube of 5mm side is loaded as shown in figure below.**

(i) Determine the principal stresses $\sigma_1, \sigma_2, \sigma_3$.

(ii) Will the cube yield if the yield strength of the material is 70 MPa? Use Von-Mises theory.

Answer: Yield strength of the material σ_{et} = 70 MPa = 70 MN/m² or 70 N/mm².



(i) Principal stress $\sigma_1, \sigma_2, \sigma_3$:

$$\sigma_x = \frac{2000}{5 \times 5} = 80 \text{ N/mm}^2; \qquad \sigma_y = \frac{1000}{5 \times 5} = 40 \text{ N/mm}^2$$

$$\sigma_z = \frac{500}{5 \times 5} = 20 \text{ N/mm}^2; \qquad \tau_{xy} = \frac{800}{5 \times 5} = 32 \text{ N/mm}^2$$

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right) + \tau_{xy}^2} = \frac{80 + 40}{2} \pm \sqrt{\left(\frac{80 - 40}{2}\right)^2 + (32)^2}$$

$$= 60 \pm \sqrt{(20)^2 + (32)^2} = 97.74, 22.26$$

$\therefore$ $\sigma_1 = 97.74 \text{N/mm}^2$, or 97.74 MPa

and $\sigma_2 = 22.96 \text{N/mm}^2$ or 22.96 MPa

$\sigma_3 = \sigma_z = 20 \text{N/mm}^2$ or 22 MPa

(ii) Will the cube yield or not?

According to Von-Mises yield criteria, yielding will occur if

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_{yt}^2$$

Now $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$

$$= (97.74 - 22.96)^2 + (22.96 - 20)^2 + (20 - 97.74)^2$$

$$= 11745.8 \qquad ---(i)$$

and, $2\sigma_{yt}^2 = 2 \times (70)^2 = 9800 \qquad ---(ii)$

Since 11745.8 > 9800 so yielding will occur.

Question: **A thin-walled circular tube of wall thickness t and mean radius r is subjected to an axial load P and torque T in a combined tension-torsion experiment.**
(i) Determine the state of stress existing in the tube in terms of P and T.
(ii) Using Von-Mises - Henky failure criteria show that failure takes place when $\sqrt{\sigma^2 + 3\tau^2} = \sigma_0$, *where* σ_0 *is the yield stress in uniaxial tension,* σ *and* τ *are respectively the axial and torsional stresses in the tube.*

Answer: Mean radius of the tube = r,
Wall thickness of the tube = t,
Axial load = P, and
Torque = T.

(i) The state of stress in the tube:

Due to axial load, the axial stress in the tube $\sigma x = \frac{P}{2\pi rt}$

Due to torque, shear stress,

$$\tau_{xy} = \frac{Tr}{J} = \frac{Tr}{2\pi r^3 t} = \frac{T}{2\pi r^3 t}$$

$J = \frac{\pi}{2}\left\{(r+t)^4 - r^4\right\} = 2\pi r^3 t$-neglecting t^2 higher power of t.

$\therefore$The state of stress in the tube is, $\sigma_x = \frac{P}{2\pi rt}, \sigma_y = 0, \tau_{xy} = \frac{T}{2\pi r^3 t}$

(ii) Von Mises-Henky failure in tension for 2-dimensional stress is

$$\sigma_0^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

In this case, $\sigma_1 = \frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}$, and

$$\sigma_2 = \frac{\sigma_x}{2} - \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2} \qquad (\because \sigma_y = 0)$$

$$\therefore \sigma_0^2 = \left[\frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]^2 + \left[\frac{\sigma_x}{2} - \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]^2 - \left[\frac{\sigma_x}{2} + \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]\left[\frac{\sigma_x}{2} - \sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]^2$$

$$= \left[\frac{\sigma_x^2}{4} + \frac{\sigma_x^2}{4} + \tau_{xy}^2 + 2.\frac{\sigma_x}{2}.\sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right] + \left[\frac{\sigma_x^2}{4} + \frac{\sigma_x^2}{4} + \tau_{xy}^2 + 2.\frac{\sigma_x}{2}.\sqrt{\frac{\sigma_x^2}{4} + \tau_{xy}^2}\right]$$

$$-\left[\frac{\sigma_x^2}{4} - \frac{\sigma_x^2}{4} - \tau_{xy}^2\right]$$

$$= \sigma_x^2 + 3\tau_{xy}^2$$

$$\sigma_0 = \sqrt{\sigma_x^2 + 3\tau_{xy}^2}$$

Find the maximum principal stress developed in a cylindrical shaft. 8 cm in diameter and subjected to a bending moment of 2.5 kNm and a twisting moment of 4.2 kNm. If the yield stress of the shaft material is 300 MPa. Determine the factor of safety of the shaft according to the maximum shearing stress theory of failure.

Given: d = 8 cm = 0.08 m; M = 2.5 kNm = 2500 Nm; T = 4.2 kNm = 4200 Nm

$\sigma_{yield}\left(\sigma_{yt}\right) = 300\ MPa = 300\ \text{MN/m}^2$

Equivalent torque, $T_e = \sqrt{M^2 + T^2} = \sqrt{(2.5)^2 + (4.2)^2} = 4.888\,\text{kNm}$

Maximum shear stress developed in the shaft,

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16 \times 4.888 \times 10^3}{\pi \times (0.08)^3} \times 10^{-6}\ \text{MN/m}^2 = 48.62\,\text{MN/m}^2$$

$$\text{Permissible shear stress} = \frac{300}{2} = 150\,\text{MN/m}^2$$

$\therefore$ Factor of safety $= \dfrac{150}{48.62} = 3.085$

Problem 1: A machine element is subjected to the following stresses $\sigma_x = 60$ MPa, $\sigma_y = 45$ MPa, $\tau_{xy} = 30$ MPa. Find the factor of safety if it is made of C45 steal having yield stress as 353 MPa, using the following theories of failure.

(i) Maximum principal stress theory,

(ii) Maximum shear stress theory,

(iii) Shear energy theory, and

(iv) Maximum strain theory taking Poisson ratio as 0.3

Given data: $\sigma_x = 60$ MPa, $\sigma_y = 45$ MPa, $\tau_{xy} = 30$ MPa yield stress, $\sigma_{ys} = 353$ MPa

Poisson ratio $\upsilon = 0.3$.

(i) According to maximum principal stress or Rankine's theory of equivalent stress

$$\sigma_e = \frac{1}{2} = \left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right] \qquad ...(5\text{-}20)$$

$$\sigma_e = \frac{1}{2} = \left[(60 + 45) + \sqrt{(60 - 45)^2 + 4(30)^2}\right] = 83.42 \text{ MPa}$$

$$\therefore \qquad FOS = \frac{\sigma_{ys}}{\sigma_e} = \frac{353}{83.42} = 4.23$$

(ii) According to max. shear stress theory or Guest's theory equivalent stress

$$\tau_e = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \qquad ...(5\text{-}21)$$

or

$$\sigma_e = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \qquad \left(\because \tau_e = \frac{\sigma_e}{2}\right)$$

$$= \sqrt{(60 - 45)^2 + 4(30)^2} = 61.85 \text{ MPa}$$

$$\therefore \qquad FOS = \frac{\sigma_{ys}}{\sigma_e} = 353/61.85 = 5.71$$

(iii) According to shear energy theory or Hencky-Von-Mises theory, equivalent stress

$$\sigma_e = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2} \qquad ...(5\text{-}22)$$

$$\sigma_e = \sqrt{60^2 + 45^2 - 60 \times 45 + 3 \times 30^2} = 75 \text{ MPa}$$

$$\therefore \qquad FOS = \frac{\sigma_{ys}}{\sigma_e} = \frac{353}{75} = 4.71$$

(iv) According to Max-Strain theory or Saint-Venant theory. Equivalent stress

$$\sigma_e = \frac{1}{2}\left[(1-v)(\sigma_x+\sigma_y)+(1+v)\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right]$$

$$\sigma_e = \frac{1}{2}\left[(1-0.3)(60+45)+(1+0.3)\sqrt{(60-45)^2+4(30)^2}\right]$$

$$= 76.95 \text{ MPa}$$

$$\text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{353}{76.95} = 4.59.$$

Problem 2: A M.S. shaft having yield stress as 232 MPa is subjected to the following stresses $\sigma_x = 120$ MPa, $\sigma_y = -60$ MPa and $\tau_{xy} = 36$ MPa. Find the factor of safety using:

(i) Rankine's theory of failure,

(ii) Guest's theory of failure and

(iii) Von-Mises theory of failure.

Given data: Yield stress, $\sigma_{ys} = 232$ MPa

$\sigma_x = 120$ MPa, $\sigma_y = -60$ MPa and $\tau_{xy} = 36$ MPa.

According to Rankine's theory or maximum normal stress theory of failure

$$\sigma_e = \frac{1}{2}\left[(\sigma_x+\sigma_y)+\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right]$$

$$\sigma_e = \frac{1}{2}\left[(120-60)+\sqrt{[120-(-60)]^2+4(36)^2}\right] = 126.93 \text{ MPa}$$

$$\text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{232}{126.93} = 1.828$$

(ii) According to Guest's theory or max shear stress theory of failure

$$\tau_e = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$$

or $$\sigma_e = \sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2} = \sqrt{[120-(-60)]^4+4(36)^2}$$

$$\sigma_e = 193.87 \text{ MPa}$$

$\therefore$ $$\text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{232}{193.87} = 1.197$$

(ii) According to Hencky-Von-Mises theory or shear energy theory of failure

$$\sigma_e = \sqrt{\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2}$$

$$= \sqrt{120^2+(-60)^2-120\times(-60)+3(36)^2} = 170.55 \text{ MPa}$$

$\therefore$ $$\text{FOS} = \frac{\sigma_{ys}}{\sigma_e} = \frac{232}{170.55} = 1.36$$

Problem 3: A machine member is subjected to the following stresses $\sigma_x = 150$ MPa, $\tau_{xy} = 24$ MPa. Find the equivalent stress as per the following theories of failure.

(i) Shear stress theory,

(ii) Normal stress theory,

(iii) Von-Mises theory.

Given data: $\sigma_x = 150$ MPa, $\tau_{xy} = 24$ MPa

(σ_y = Not given) ($\sigma_y = 0$, Not given)

(i) According to maximum shear stress theory, equivalent stress $s_e = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$

$$\sigma_e = \sqrt{150^2 + 4 \times 24^2} = 157.49 \text{ MPa}$$

(ii) According to maximum normal stress theory, equivalent stress

$$\sigma_e = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

$$\sigma_e = \frac{1}{2}\left[150 + \sqrt{150^2 + 4(24)^2}\right] = 153.75 \text{ MPa}$$

(iii) According to Von-Mises theory, equivalent stress

$$\sigma_e = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 4\tau_{xy}^2}$$

$$\sigma_e = \sqrt{150^2 + 3(24)^2} = 155.65 \text{ MPa.}$$

Problem 4: Find the diameter of a rod subjected to a bending moment of 3 kNm and a twisting moment of 1.8 kNm according to the following theories of failure, taking normal yield stress as 420 MPa and factor of safety as 3.

(i) Normal stress theory, (ii) Shear stress theory.

Given data: Bending moment, $M_b = 3$ kNm $= 3 \times 10^6$ N-mm

Twisting moment, $M_t = 1.8$ kNm $= 1.8 \times 10^6$ N-mm

Yield stress, $\sigma_{ys} = 420$ MPa FOS = 3

$\therefore$ Allowable stress, $\sigma = \sigma_e = \dfrac{\sigma_{ys}}{\text{FOS}} = \dfrac{420}{3} = 140$ MPa

$$\text{Bending stress, } \sigma = \frac{M_b \cdot C}{I} = \frac{3 \times 10^6 \times d/2}{(\pi d^4/64)} = \frac{30.56 \times 10^6}{d^3}$$

$$\sigma = \frac{30.56 \times 10^6}{d^3} = \sigma_x$$

$$\text{Shear stress, } \tau = \frac{M_t r}{J} = \frac{1.8 \times 10^6 \times d/2}{(\pi d^4/32)} = \frac{9.167 \times 10^6}{d^3} = \tau_{xy}$$

(i) According to maximum normal stress theory,

$$\sigma_e = \frac{1}{2}\left[(\sigma_x + \sigma_y + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

(Here $\sigma_y = 0$, no stress in $\perp lr$ direction)

$$140 = \frac{1}{2}\left[\frac{30.56 \times 10^6}{d^3} + \sqrt{\left(\frac{30.56 \times 10^6}{d^3}\right)^2 + 4\left(\frac{9.167 \times 10^6}{d^3}\right)^2}\right]$$

$\therefore$ $d = 61.834$ mm

(ii) According to maximum shear stress theory

$$\sigma_e = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$140 = \sqrt{\left(\frac{30.56 \times 10^6}{d^3}\right)^2 + 4\left(\frac{9.167 \times 10^6}{d^3}\right)^2}$$

$\therefore$ $d = 63.376$ mm

$\therefore$ Recommended diameter, $d = 63.376 \simeq 64$ mm. (Take bigger one always).

Problem 5: A bolt is subjected to a tensile load of 18 kN and a shear load of 12 kN. The material has an yield stress of 328.6 MPa. Taking factor of safety as 2.5, determine the core diameter of bolt according to the following theories of failure.

(i) Rankine's theory,
(ii) Shear stress theory,
(iii) Shear energy theory and
(iv) Saint Venant's theory. Take Possion ratio = 0.298

Given data: Tensile load, $F_T = 18$ kN $= 18 \times 10^3$ N

Shear load, $F_s = 12$ kN $= 12 \times 10^3$ N

Yield stress, $\sigma_{ys} = 328.6$ MPa FOS = 2.5

$\therefore$ Allowable stress, $\sigma_e = \frac{\sigma_{ys}}{\text{FOS}} = \frac{328.6}{2.5} = 131.44$ MPa.

Tensile stress, $\sigma = \frac{F_T}{A} = \frac{18 \times 10^3}{A} = \sigma_x$

Shear stress, $\tau = \frac{F_s}{A} = \frac{12 \times 10^3}{A} = \tau_{xy}$

($\sigma_y = 0$, not given)

(i) According to Rankine's theory of failure

$$\sigma_e = \frac{1}{2}\left[\sigma_x + \sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right]$$

$$131.44 = \frac{1}{2}\left[\frac{18 \times 10^3}{A} + \sqrt{\left(\frac{18 \times 10^3}{A}\right)^2 + 4\left(\frac{12 \times 10^3}{A}\right)^2}\right]$$

$$A = 182.59 = \frac{\pi d_c^2}{4}$$

∴ Core dia, dc = 15.25 mm

Problem 6: A SAE 1045 steel rod (σ_{ys} = 309.9 MPa) of 80 mm diameter is subjected to a bending moment of 3 kNm and torque T. Taking Factor of safety as 2.5, find the maximum value of torque T that can be safely carried by rod according to:

(i) Maximum normal stress theory,

(ii) Maximum shear stress theory.

Given data: Material SAE 1045.

Yield stress, $\sigma_{ys} = 309.9$ MPa

FOS = 2.5 diameter d = 80 mm

$$\therefore \quad \text{Allowable stress, } \sigma_e = \frac{\sigma_{ys}}{\text{FOS}} = \frac{309.9}{2.5} = 123.96 \text{ MPa}$$

Bending moment, M_b = 3 kNm = 3×10^6 N-mm.

$$\therefore \quad \text{Bending stress, } \sigma = \frac{M_b \cdot C}{I} = \frac{3 \times 10^6 (80/2)}{(\pi/64 \times 80^4)} = 59.68 \text{ MPa} = \sigma_x$$

Torque, $M_t = T$

$$\therefore \quad \text{Shear stress, } \tau = \frac{M_t \cdot r}{J} = \frac{T \cdot (80/2)}{(\pi/32 \times 80)^4} = (9.95 \times 10^{-6}) \text{ MPa}$$

$$\therefore \quad \tau = \tau_{xy} = (9.95\ \ 10^{-6})\ T$$

(σ_y = 0, not given)

(i) According to maximum normal stress theory

$$\sigma_e = \frac{1}{2}\left[\sigma_x + \sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right]$$

$$123.96 = \frac{1}{2}\left[59.68 + \sqrt{59.68^2 + 4(9.95 \times 10^{-6} T)^2}\right]$$

$\therefore$ Torque, $T = 8.971 \times 10^6$ N-mm = 8.971 kNm

(ii) According to maximum shear stress theory

$$\tau_e = \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

Assuming, $\tau_e = 0.5\ \sigma_e = 0.5 \times 123.96 = 61.98$ MPa

$$61.98 = \frac{1}{2}\sqrt{59.68^2 + 4(9.95 \times 10^{-6} T)^2}$$

Torque, $T = 5.46 \times 10^6$ N-mm = 5.46 kNm.

Problem 7: A stressed element is loaded as shown in Fig. 2.3. Determine the following:

(i) Von-Mises stress,

(ii) Maximum shear stress,

(iii) Maximum normal stress,

(iv) Octahedral shear stress.

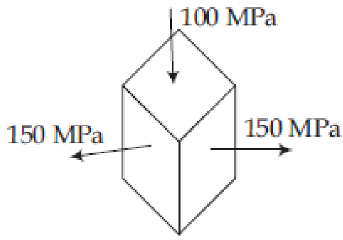


Fig. 2.3

Given data: Arranging in descending order $150 \geq 150 > -100$

$\therefore$ $\sigma_1 = 150$ MPa,

$\sigma_2 = 150$ MPa and $\sigma_3 = -100$ MPa (compressive)

(i) Von-Mises stress

$$\tau_e = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}$$

$$= \sqrt{\frac{(150-150)^2+(150+100)^2+(-100-150)^2}{2}} = 250 \text{ MPa}$$

(ii) Maximum shear stress

$$\tau_{12} = \frac{\sigma_1-\sigma_2}{2} = \frac{150-150}{2} = 0$$

$$\tau_{23} = \frac{\sigma_2-\sigma_3}{2} = \frac{150-(-100)}{2} = 125 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1-\sigma_3}{2} = \frac{150-(-100)}{2} = 125 \text{ MPa}$$

$\therefore$ $\tau_{max} = 125$ MPa (max of these 3 values)

(iii) Maximum normal stress

$$\sigma_1 > \sigma_2 > \sigma_3$$

then $\sigma_{max} = \sigma_1 = 150$ MPa.

(iv) Octahedral shear stress

$$\tau_e = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

$$= \frac{1}{3}\sqrt{(150-150)^2+(150-100)^2+(-100-150)^2} = 117.85 \text{ MPa.}$$

Problem 8: A material has a yield strength of 600 MPa. Compute the factor of safety for each of the failure theories for the each of the following stresses:

(i) $\sigma_1 = 420$ MPa, $\sigma_2 = 410$ MPa, $\sigma_3 = 0$,

(ii) $\sigma_1 = 420$ MPa, $\sigma_2 = 180$ MPa, $\sigma_3 = 0$,

(a) Von-mises theory, $\sigma_e = \sqrt{\dfrac{(\sigma_1-\sigma_2)^{-2}+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}}$

$$\sigma_e = \sqrt{\frac{(420-180)^2+(180-0)^2+(420-0)^2}{2}} = 364.97 \text{ MPa}$$

$$\therefore \quad FOS = \frac{\tau_{ys}}{\tau_e} = \frac{600}{364.97} = 1.644$$

(b) Max. normal stress theory, $\sigma_e = \sigma_1 = 420$ MPa

$$\therefore \quad FOS = \frac{\sigma_{ys}}{\sigma_e} = \frac{600}{420} = 1.4286$$

(c) Max. shear stress theory

$$\tau_{12} = \frac{\sigma_1-\sigma_2}{2} = \frac{420-180}{2} = 120 \text{ MPa}$$

$$\tau_{23} = \frac{\sigma_2-\sigma_3}{2} = \frac{180}{2} = 90 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1-\sigma_3}{2} = \frac{420}{2} = 210 \text{ MPa}$$

$$\therefore \quad \tau_{max} = 210 \text{ MPa} = \tau_e$$

$$\therefore \quad FOS = \frac{\sigma_{ys}}{2\tau_{max}} = \frac{600}{2\times 210} = 1.4286$$

(iii) $\sigma_1 = 0$, $\sigma_2 = -180$ MPa, $\sigma_3 = -420$ MPa

(a) Von-Mises theory, $\sigma_e = \sqrt{\dfrac{(0+180)^2+(-180+420)^2+(0+420)^2}{2}} = 364.96$

$$\therefore \quad FOS = \frac{\sigma_{ys}}{\sigma_e} = \frac{600}{364.93} = 1.644$$

(b) Max. normal stress theory, $\sigma_e = \sigma_1 = 0$

$$\therefore \quad FOS = \frac{600}{0} = \infty$$

But, in compression $FOS = \dfrac{600}{420} = 1.4286$

(c) Max. shear stress theory

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{0 + 180}{2} = 90 \text{ MPa}$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{-180 + 4.20}{2} = +120 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{0 + 420}{2} = 210 \text{ MPa} \quad \therefore \; \tau_{max} = 210 \text{ MPa}$$

$$\text{FOS} = \frac{\sigma_{ys}}{2\tau_{max}} = \frac{600}{2 \times 210} = 1.4286.$$

Problem 9: A hot rolled bar has yield stress of 390 MPa. Compute the factor of safety for the following theories of failure:

(i) Maximum normal stress theory,

(ii) Maximum shear stress theory and

(iii) Distortion energy theory for the following states of stress.

(a) $\sigma_1 = 225$ MPa, $\sigma_2 = 225$ MPa, $\sigma_3 = 0$

(b) $\sigma_1 = 225$ MPa, $\sigma_2 = 120$ MPa, $\sigma_3 = 0$

(c) $\sigma_1 = 225$ MPa, $\sigma_2 = 0$, $\sigma_3 = -120$ MPa.

Given data: Yield stress, $\sigma_{ys} = 390$ MPa

$$\text{FOS} = \frac{\sigma_{ys}}{\sigma_e}$$

(a) $\sigma_1 = 225$ MPa, $\sigma_2 = 225$ MPa, $\sigma_3 = 0$

$\sigma_1 > \sigma_2 > \sigma_3$

(i) Maximum normal stress theory, $\sigma_e = \sigma_1 = 225$ MPa

$$\therefore \quad \text{FOS} = \frac{390}{225} = 1.733$$

(ii) Maximum shear stress theory

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2} = \frac{225 - 225}{2} = 0$$

$$\tau_{23} = \frac{\sigma_2 - \sigma_3}{2} = \frac{225 - 0}{2} = 112.5 \text{ MPa}$$

$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2} = \frac{225 - 0}{2} = 112.5 \text{ MPa}$$

$$\therefore \quad \tau_e = \tau_{max} = 112.5 \text{ MPa}$$

and

$$\text{FOS} = \frac{\sigma_{ys}}{2\tau_{max}} = \frac{390}{2 \times 112.5} = 1.733$$