Maxwell's Equations (integral form)

Name	Equation	Description
Gauss' Law for Electricity	$\int \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$	Charge and electric fields
Gauss' Law for Magnetism	$\int \vec{B} \cdot d\vec{A} = 0$	Magnetic fields
Faraday's Law	$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$	Electrical effects from changing B field
Ampere's Law	$\int \vec{B} \cdot dl = \mu_0 i$ Needs to be modified.	Magnetic effects from current + ?

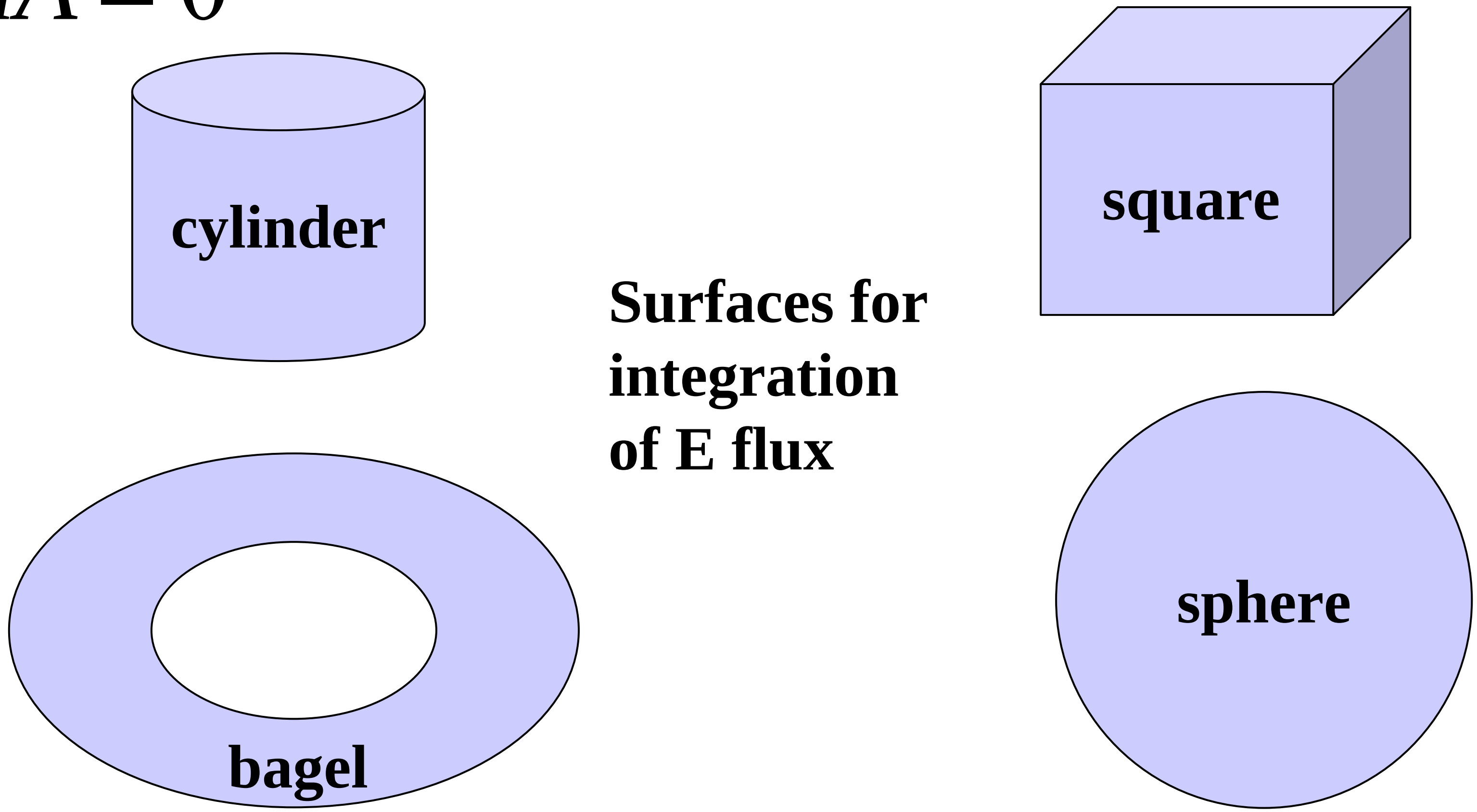
There is a serious asymmetry.

Remarks on Gauss Law's with different closed surfaces

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

Gauss Law's works for <u>ANY</u> CLOSED SURFACE

$$\oint \vec{B} \cdot d\vec{A} = 0$$



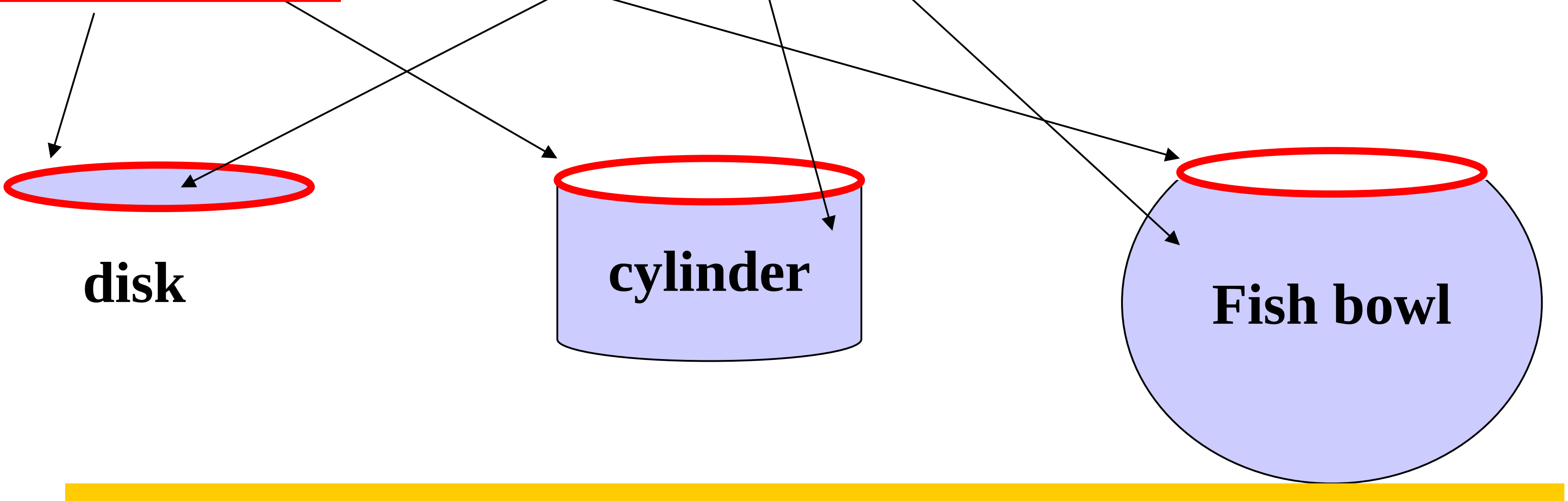
Remarks on Faraday's Law with different attached surfaces

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d \int \vec{B} \cdot d\vec{A}}{dt}$$

Faraday's Law works for any closed Loop and <u>ANY</u> attached surface area

Line integral defines the Closed loop

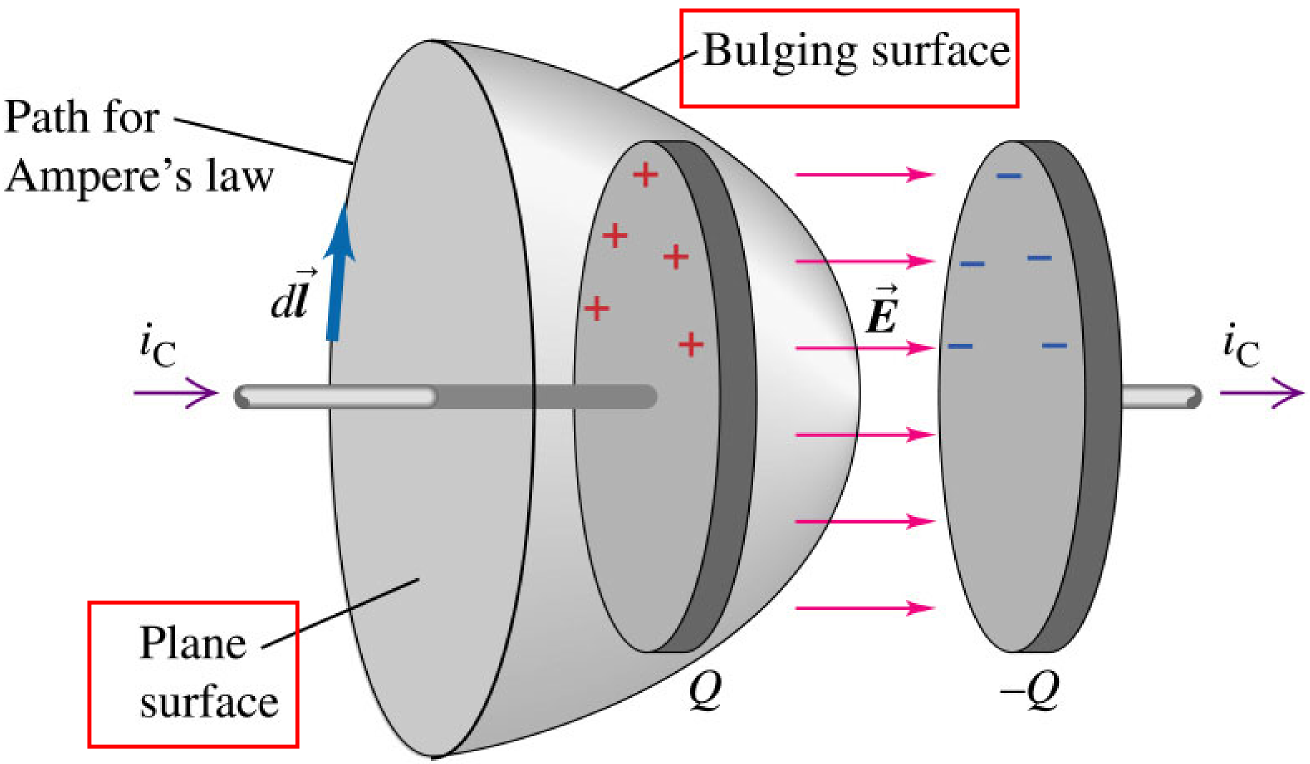
Surface area integration for B flux



This is proven in Vector Calculus with Stoke's Theorem

Generalized Ampere's Law and displacement current

Ampere's original law, $\int \vec{B} \cdot dl = \mu_0 I_{enclose}$, is incomplete. Consider the parallel plate capacitor and suppose a current i_c is flowing charging up the plate. If Ampere's law is applied for the given path in either the plane surface or the bulging surface we we should get the same results, but the bulging surface has $i_c=0$, so something is missing.



Generalized Ampere's Law and displacement current

Maxwell solved dilemma by adding an addition term called displacement current, $\boldsymbol{i_D} = \varepsilon\, d\Phi_E/dt$, in analogy to Faraday's Law.

$$\int \vec{B} \cdot dl = \mu_0 \left(i_c + i_D\right) = \mu_0 \left(i_c + \varepsilon_0 \frac{d\Phi_E}{dt}\right)$$

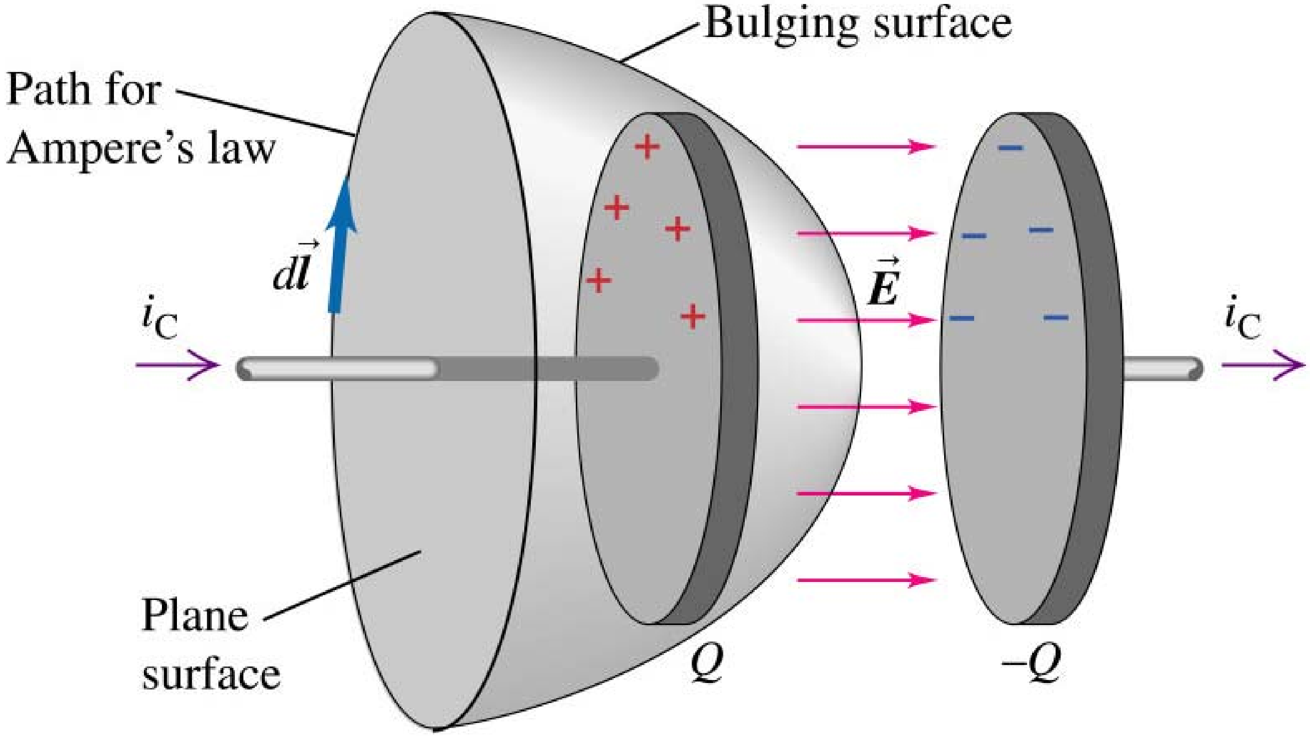
Current is once more continuous: i_D between the plates = i_C in the wire.

$$q = CV$$

$$= \frac{\varepsilon A}{d}(Ed)$$

$$= \varepsilon E A = \varepsilon \Phi_E$$

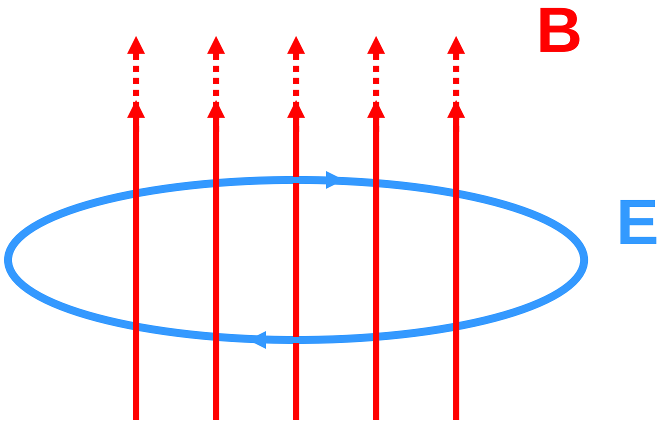
$$\frac{dq}{dt} = \varepsilon \frac{d\Phi_E}{dt} = i_c$$



Summary of Faraday's Law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

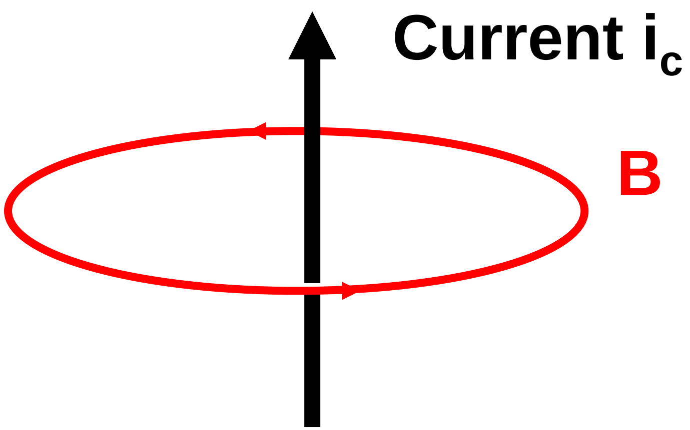
If we form any closed loop, the line integral of the electric field equals the time rate change of magnetic flux through the surface enclosed by the loop.



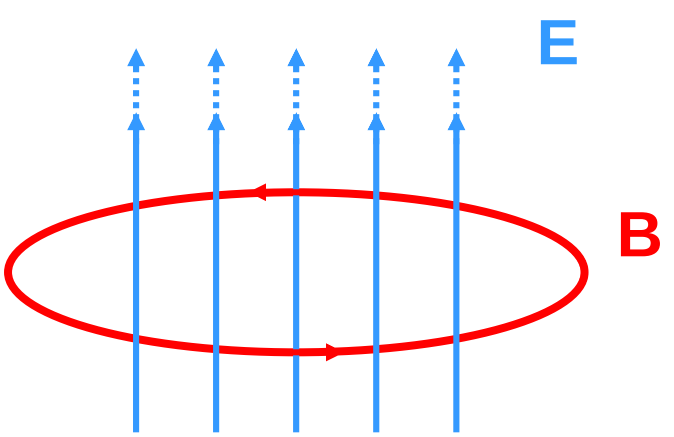
If there is a changing magnetic field, then there will be electric fields induced in closed paths. The electric fields direction will tend to reduce the changing B field.

Summary of Ampere's Generalized Law

$$\oint \vec{B} \cdot dl = \mu_0 \left(i_c + \varepsilon_0 \frac{d\Phi_E}{dt} \right)$$



If we form any closed loop, the line integral of the B field is nonzero if there is (constant or changing) current through the loop.



If there is a changing electric field through the loop, then there will be magnetic fields induced about a closed loop path.

Maxwell's Equations

James Clerk Maxwell (1831-1879)

- generalized Ampere's Law
- made equations symmetric:
  - a changing magnetic field produces an electric field
  - a changing electric field produces a magnetic field
- Showed that Maxwell's equations predicted electromagnetic waves and $c = 1/\sqrt{\varepsilon_0 \mu_0}$
- Unified electricity and magnetism and light.

All of electricity and magnetism can be summarized by Maxwell's Equations.

Maxwell's Equations (integral form)

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Ampere's Law (modified by Maxwell)	$\oint \vec{B} \cdot dl = \mu_0 \left(i_c + \varepsilon_0 \frac{d\Phi_E}{dt} \right)$	Magnetic effects from current and Changing E field

Electromagnetic Waves in free space

A remarkable prediction of Maxwell's eqns is electric & magnetic fields can propagate in vacuum.
Examples of electromagnetic waves include; radio/TV waves, light, x-rays, and microwaves. Wireless, blue tooth, cell phones, etc.

1860's – James Clerk Maxwell predicted radio waves.
1886 - Heinrich Hertz demonstrated rapid variations of electric current could produce radio waves.
1895 - Guglielmo Marconi sent and received his first radio signal in Italy.

James Clerk Maxwell

Guglielmo Marconi

On to Waves!!

- Note the symmetry now of Maxwell's Equations in free space, meaning when no charges or currents are present

$$\int \vec{E} \cdot d\vec{A} = 0 \qquad \int \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \oint \vec{B} \cdot dl = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

- Combining these equations leads to wave equations for E and B, e.g.,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

- Do you remember the wave equation???

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 h}{\partial t^2}$$

h is the variable that is changing in space (x) and time (t). v is the velocity of the wave.

Review of Waves from Physics 170

- **The one-dimensional wave equation:**

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2}$$

has a general solution of the form:

$$h(x,t) = h_1(x - vt) + h_2(x + vt)$$

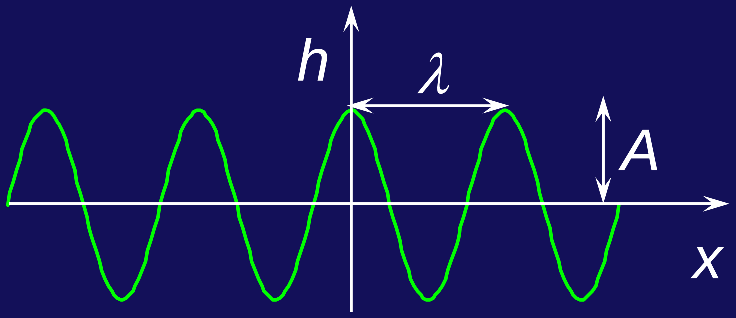
where h_1 represents a wave traveling in the +x direction and h_2 represents a wave traveling in the -x direction.

- **A specific solution for harmonic waves traveling in the +x direction is:**

$$h(x,t) = A\cos(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda} \quad \omega = 2\pi f = \frac{2\pi}{T}$$

$$v = \lambda f = \frac{\omega}{k}$$

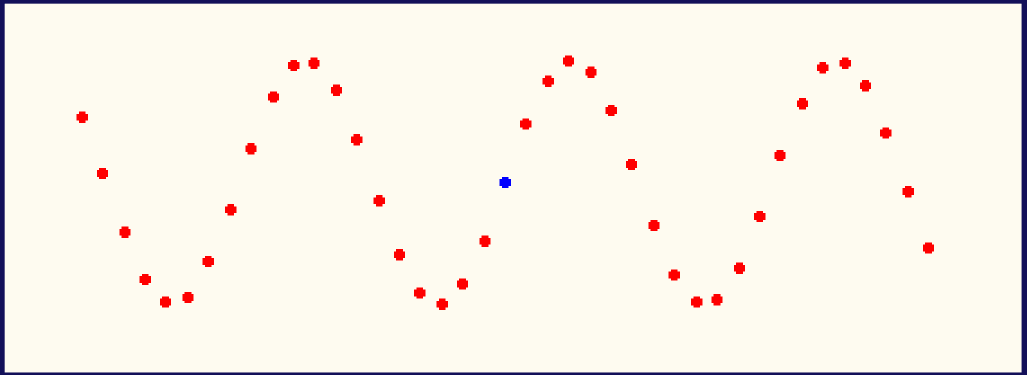


A = amplitude
λ = wavelength
f = frequency
v = speed
k = wave number

Waves

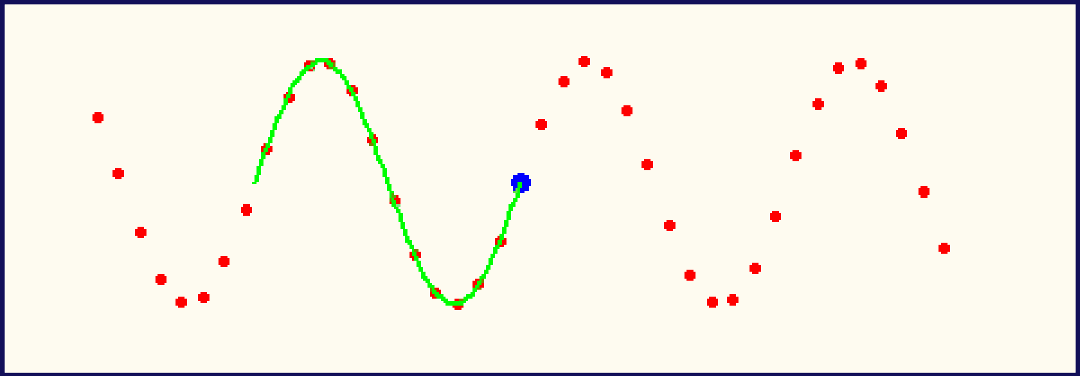
Transverse Wave:

• The wave pattern moves to the right.
• However any particular point (look at the blue one) just moves transversely (i.e., up and down) to the direction of the wave.



Wave Velocity:

• The wave velocity is defined as the wavelength divided by the time it takes a wavelength (green) to pass by a fixed point (blue).



Velocity of Electromagnetic Waves

- We derived the wave equation for E_x (Maxwell did it first, in ~1865!):

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}$$

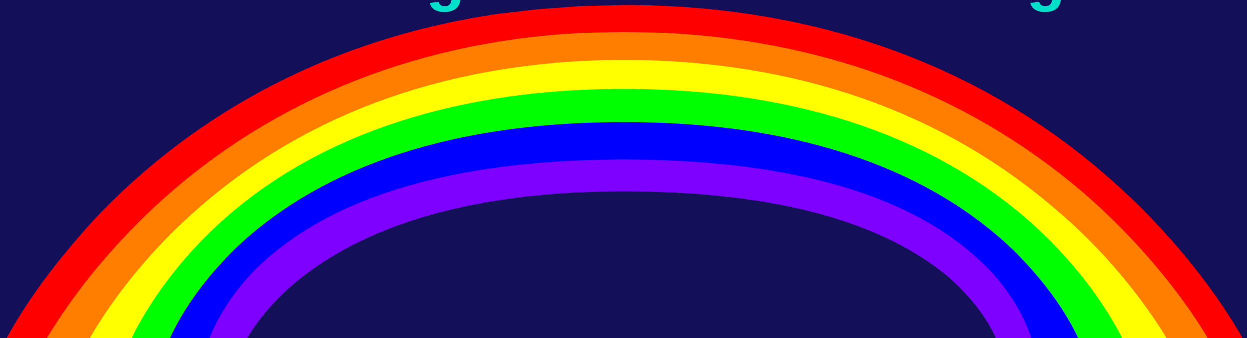
- Comparing to the general wave equation:

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 h}{\partial t^2}$$

we have the velocity of electromagnetic waves in free space:

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3.00 \times 10^8 \text{ m/s} \equiv c$$

- This value is essentially identical to the speed of light measured by Foucault in 1860!
  - Maxwell identified light as an electromagnetic wave.

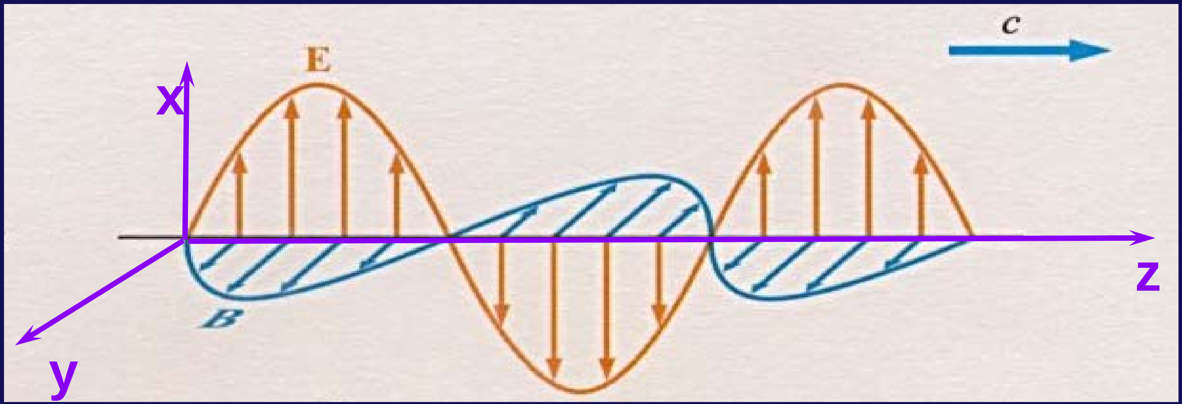


E & *B* in Electromagnetic Wave

- **Plane Harmonic Wave:**

$$E_x = E_0 \sin(kz - \omega t)$$
$$B_y = B_0 \sin(kz - \omega t)$$

where $\omega = kc$



➢ B_y is in phase with E_x

➢ $B_0 = E_0 / c$

➢The direction of propagation $\hat{s}$ is given by the cross product

$$\hat{s} = \hat{e} \times \hat{b}$$

where $(\hat{e}, \hat{b})$ are the unit vectors in the (E,B) directions.

Nothing special about (E_x, B_y); e.g., could have (E_y, $-B_x$)

Lecture 21, ACT 3

- Suppose the electric field in an e-m wave is given by:

$$\vec{E} = -\hat{y}\, E_0 \cos(-kz + \omega t)$$

3A In what direction is this wave traveling ?

(a) + z direction (b) - z direction

3B Which of the following expressions describes the magnetic field associated with this wave?

(a) $B_x = -(E_o/c)\cos(kz + \omega t)$

(b) $B_x = +(E_o/c)\cos(kz - \omega t)$

(c) $B_x = +(E_o/c)\sin(kz - \omega t)$

Lecture 21, ACT 3

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$$\vec{E} = -\hat{y}\ E_0 \cos(-kz + \omega t)$$

3A – In what direction is this wave traveling ?

(a) + z direction **(b)** - z direction

- To determine the direction, set phase = 0: $-kz + \omega t = 0 \Rightarrow z = +\frac{\omega}{k} t$
- Therefore wave moves in + z direction!
- Another way: Relative signs opposite means + direction

Lecture 21, ACT 3

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- Which of the following expressions describes the magnetic field associated with this wave?

(a) $B_x = -(E_o/c)\cos(kz + \omega t)$

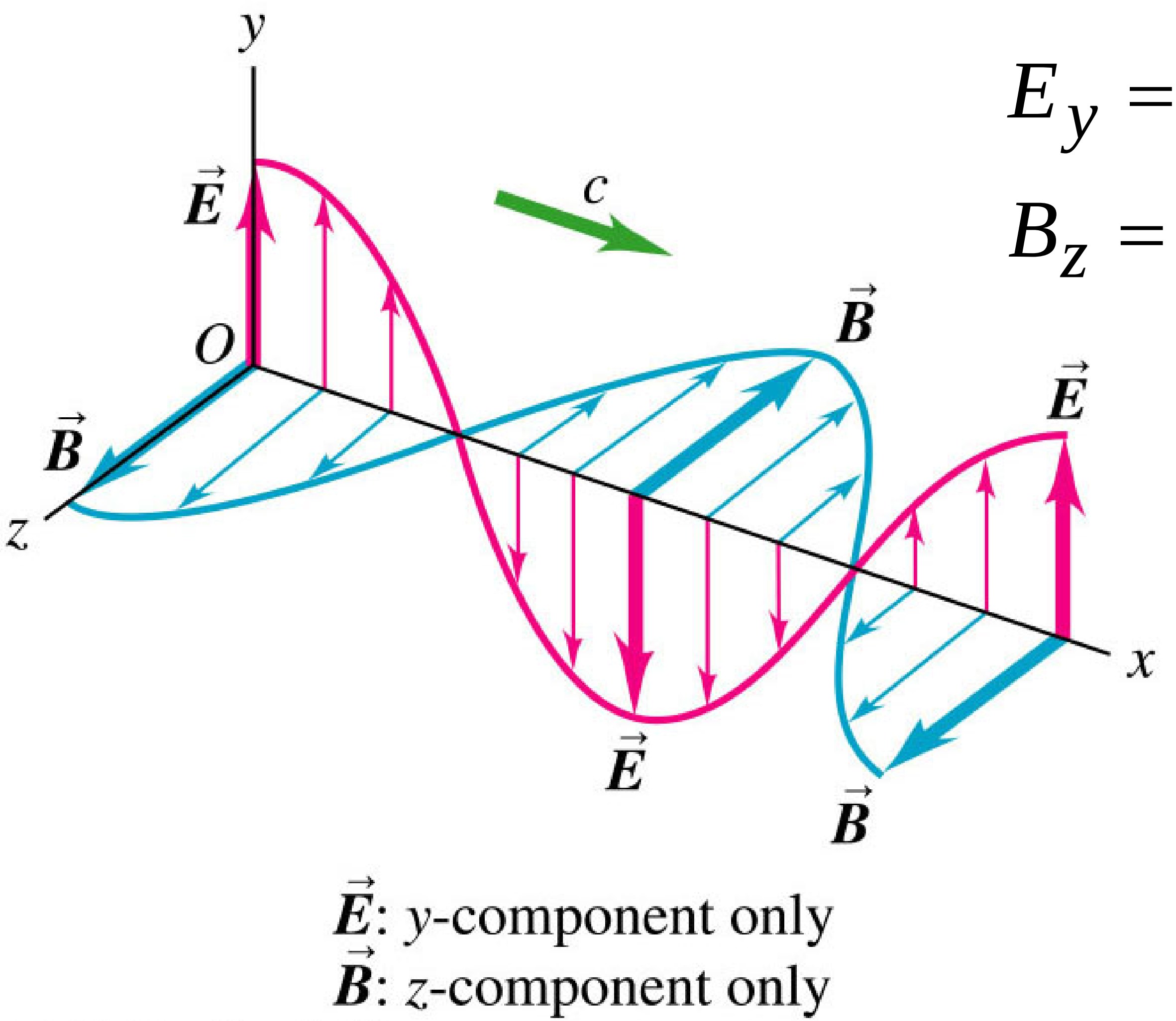
(b) $B_x = +(E_o/c)\cos(kz - \omega t)$)

(c) $B_x = +(E_o/c)\sin(kz - \omega t)$

- B is in phase with E and has direction determined from: $\hat{b} = \hat{s} \times \hat{e}$
- At t=0, z=0, $E_y = -E_o$
- Therefore at t=0, z=0, $\hat{b} = \hat{s} \times \hat{e} = \hat{k} \times (-\hat{j}) = \hat{i}$

$$\vec{B} = +\hat{i}\, \frac{E_0}{c} \cos(kz - \omega t)$$

sinusoidal EM wave solutions; moving in +x



$$E_y = E_{max}\cos(kx - \omega t)$$

$$B_z = B_{max}\cos(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f$$

$$\lambda f = c = \frac{\omega}{k}$$

Properties of electromagnetic waves (e.g., light)

Speed: in vacuum, always $3 \cdot 10^8$ m/s, no matter how fast the source is moving (there is no "aether"!). In material, the speed can be reduced, usually only by ~1.5, but in 1999 to 17 m/s!

Direction: The wave described by $\cos(kx-\omega t)$ is traveling in the $+\hat{x}$ direction. This is a "plane" wave—extends infinitely in $\hat{y}$ and $\hat{z}$.

In reality, light is often somewhat localized transversely (e.g., a laser) or spreading in a spherical wave (e.g., a star).

A plane wave can often be a good approximation (e.g., the wavefronts hitting us from the sun are nearly flat).

Plane Waves

- For any given value of z, the magnitude of the electric field is uniform everywhere in the x-y plane with that z value.