ME323 LECTURE 28

Alex Chortos

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Summary of Topics

1. Normal and shear stress distributions in beams
2. Deflections in beams – Focus on 2nd order method
3. Deflections in beams using superposition
4. Energy methods – Castigiliano's Theorems
   - Find reactions in indeterminate system
   - Find deflections

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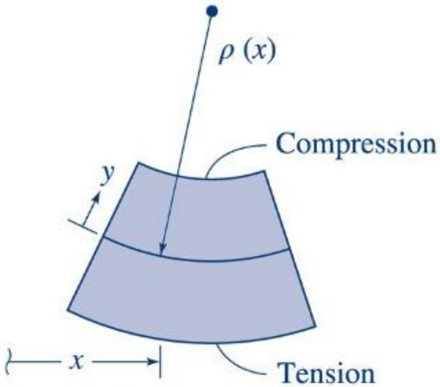
Distribution of Strains and Stresses



(a)

$$\varepsilon_x = -\frac{y}{\rho(x)}$$

$$\sigma_x = -\frac{My}{I}$$

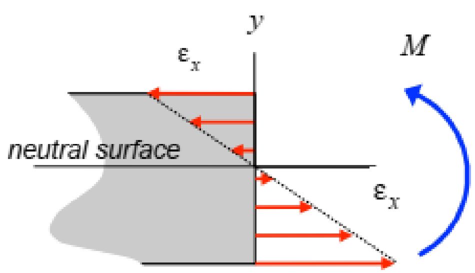
Flexure Formula

$$M(x) = \frac{E(x)I(x)}{\rho(x)}$$

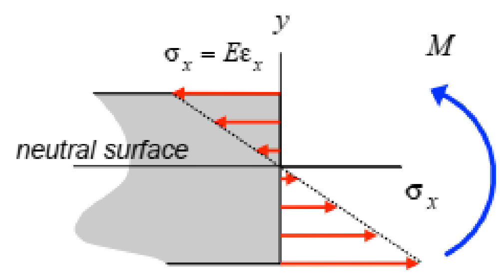
Moment-curvature

$$I_z = \int_A y^2 dA$$

Second Area Moment



strain distribution across cut



stress distribution across cut

https://www.youtube.com/watch?v=f08Y39UiC-o

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Distribution of Strains and Stresses

$$\varepsilon_x = -\frac{y}{\rho(x)}$$

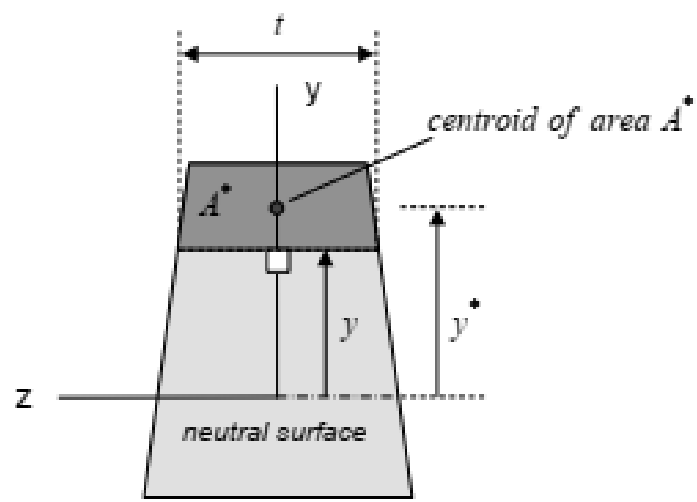
$$\sigma_x = -\frac{My}{I}$$

Flexure Formula

$$M(x) = \frac{E(x)I(x)}{\rho(x)}$$

Moment-curvature

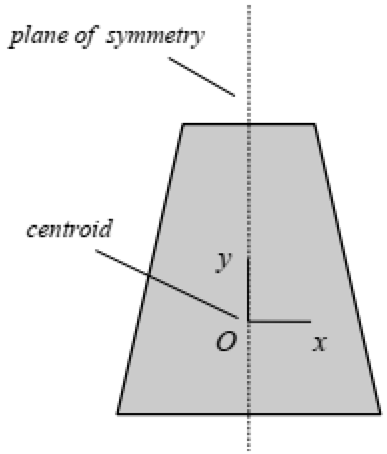
$$\tau(y) = \frac{V(x)Q(y)}{It(y)} \qquad Q = A^*\bar{y}^*$$

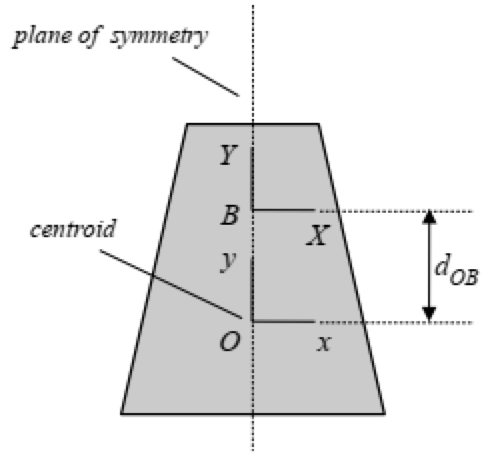


$$I_z = \int_A y^2 dA$$

Second Area Moment

Parallel Axis Theorem





$$X = x$$

$$Y = y - d_{OB}$$

$$I_B = \int_A Y^2\, dA = \int_A \left(y - d_{OB}\right)^2 dA$$

$$= \int_A \left(y^2 - 2d_{OB}y + d_{OB}^2\right) dA$$

$$= \int_A y^2\, dA - 2d_{OB}\int_A y\, dA + d_{OB}^2 \int_A dA$$

$$= I_O - 2d_{OB}\bar{y}A + Ad_{OB}^2$$

$$I_B = I_O + Ad_{OB}^2$$

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Finding Angles and Deflections

Equation (1):

$$\frac{dV}{dx} = p(x) \quad \Rightarrow \quad V(x) = V(x_1) + \int_{x_1}^{x} p(s)\,ds$$

Equation (2):

$$\frac{dM}{dx} = V(x) \quad \Rightarrow \quad M(x) = M(x_1) + \int_{x_1}^{x} V(s)\,ds$$

Equation (7):

$$EI\frac{d\theta}{dx} = M(x) \quad \Rightarrow \quad \theta(x) = \theta(x_1) + \frac{1}{EI}\int_{x_1}^{x} M(s)\,ds$$

Equation (5):

$$\frac{dv}{dx} = \theta(x) \quad \Rightarrow \quad v(x) = v(x_1) + \int_{x_1}^{x} \theta(s)\,ds$$

2nd order approach

4th order approach

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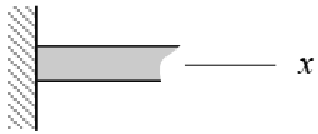
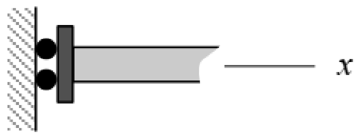
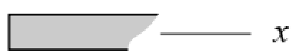
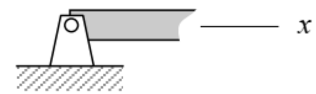
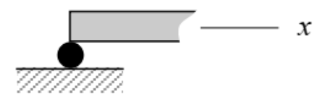
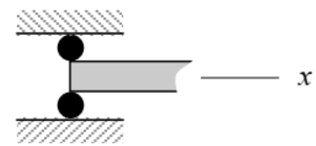
Bending in Indeterminate Beams

1. Draw FBD
2. Equilibrium for external forces and couples
3. Find external moment M(x) for each section
4. Integrate moment-curvature equation
5. Apply boundary and continuity equations
6. Solve for unknowns

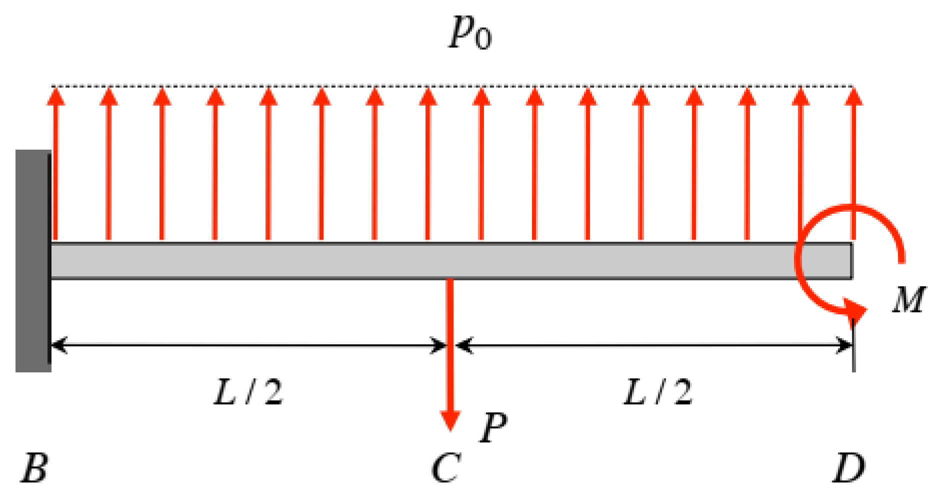
Boundary Conditions

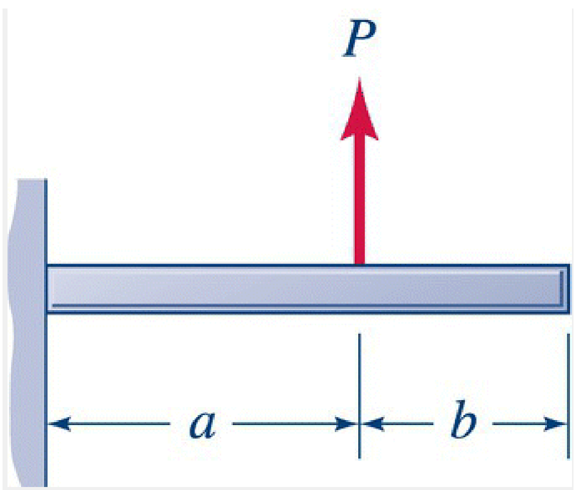
fixed end		$v = 0$ $\theta = \frac{dv}{dx} = 0$
constrained rotation end		$\theta = \frac{dv}{dx} = 0$ $V = 0$
free end		$V = 0$ $M = 0$

simple support end	*pin joint*  *roller*  *double roller* 	$v = 0$ $M = 0$

Superposition Method

Calculate the beam deflections.

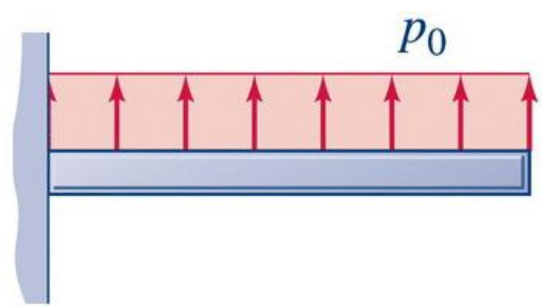




$$v = \frac{Px^2}{6EI}(3a - x) \quad v' = \frac{Px}{2EI}(2a - x) \quad 0 \le x \le a$$

$$v = \frac{Pa^2}{6EI}(3x - a) \quad v' = \frac{Pa^2}{2EI} \quad a \le x \le L$$

$$\delta_B = \frac{Pa^2}{6EI}(3L - a) \quad \theta_B = \frac{Pa^2}{2EI}$$



$$v = \frac{p_0 x^2}{24EI}(6L^2 - 4Lx + x^2)$$

$$v' = \frac{p_0 x}{6EI}(3L^2 - 3Lx + x^2)$$

$$\delta_B = \frac{p_0 L^4}{8EI} \quad \theta_B = \frac{p_0 L^3}{6EI}$$

$$v = \frac{M_0 x^2}{2EI} \quad v' = \frac{M_0 x}{EI}$$

$$\delta_B = \frac{M_0 L^2}{2EI} \quad \theta_B = \frac{M_0 L}{EI}$$

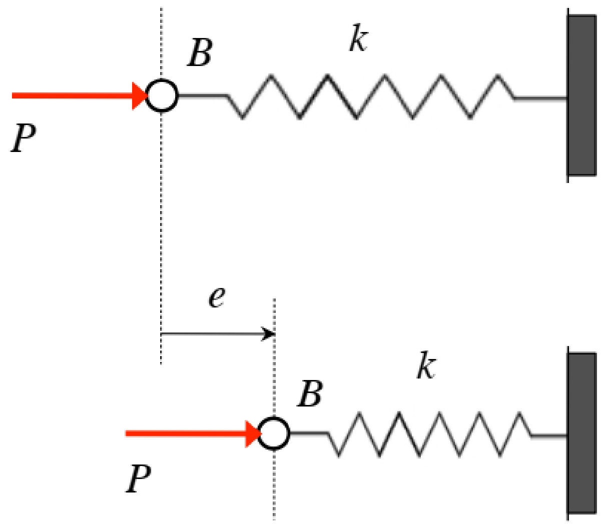
Statically Indeterminate Beams Using Superposition Method

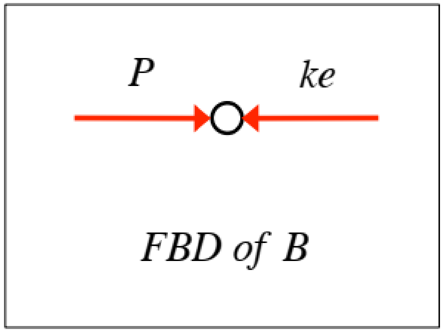
1. Determine the degree of static indeterminacy
2. Break the problem into statically determinate subproblems
3. Write compatibility equations
4. Write force deformation equations
5. Substitute force-deformation equations into the compatibility equations and solve for unknown reactants.
6. Write superposition equations.

Energy Methods

- From thermodynamics: $$W^{(nc)} = \Delta T + \Delta U$$

$$\text{W} = (\text{Force}) * (\text{Displacement}) = \Delta U$$





$$P = ke$$

$$W^{(P)} = \int_0^e P\,de = k\int_0^e e\,de = \frac{1}{2}ke^2 = \frac{1}{2}Pe$$

$$U = \frac{1}{2}ke^2 = \frac{1}{2}k\left(\frac{P}{k}\right)^2 = \frac{P^2}{2k}$$

Energy Methods

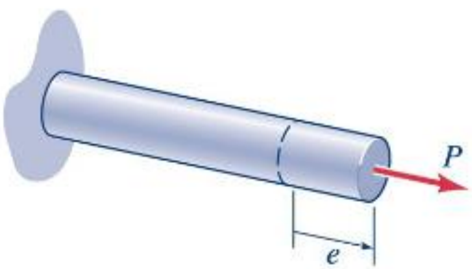
$$W = \Delta U$$

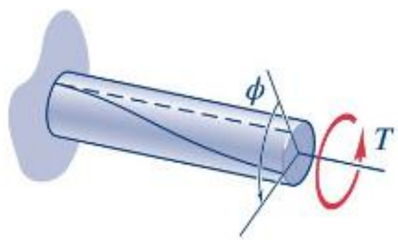
$$\bar{u} = \left(\frac{1}{2}\right)\left[\sigma_x(\varepsilon_x - \alpha\Delta T) + \sigma_y\left(\varepsilon_y - \alpha\Delta T\right) + \sigma_z(\varepsilon_z - \alpha\Delta T) + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right]$$

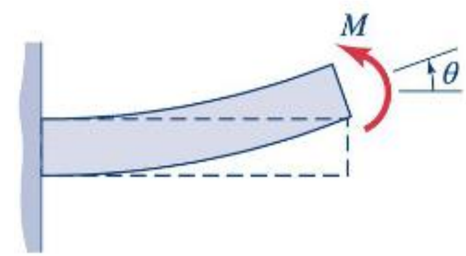
$$U = \frac{1}{2}\int_0^L \frac{F^2 dx}{EA}$$

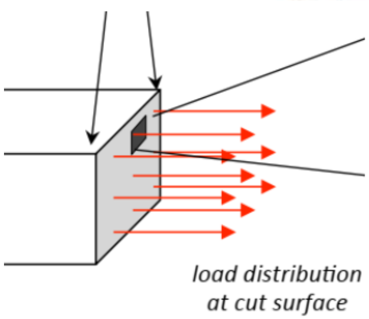
$$U = \frac{1}{2}\int_0^L \frac{T^2}{GI_p} dx$$

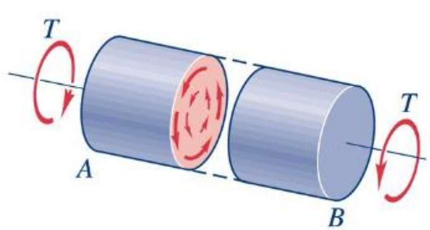
$$U_\sigma = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx \qquad U_\tau = \frac{1}{2}\int_0^L \frac{f_s V^2}{GA} dx$$

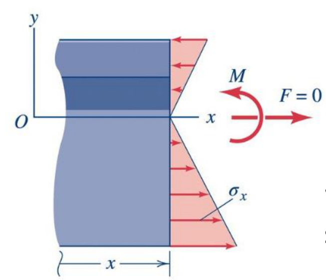


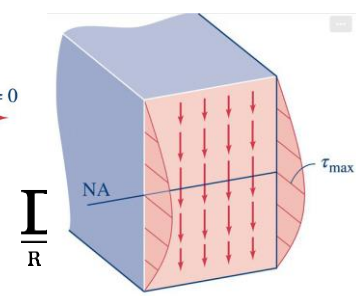












Energy Methods

If we want to find the displacements at the location of an applied force:

$$\Delta(A) = \frac{\delta U}{\delta P_A} \qquad \theta(A) = \frac{\delta U}{\delta M_A}$$

If we want to find the displacements where no force or couple is applied:

$$\Delta = \left[\frac{\delta U}{\delta P_D}\right]_{P_D=0} \qquad \theta = \left[\frac{\delta U}{\delta P_D}\right]_{P_D=0}$$

If we have an indeterminate system and want to find the reactions:

$$v(A) = 0 \quad \frac{\delta U}{\delta A_y} = 0 \qquad v'(A) = 0 \quad \frac{\delta U}{\delta M_A} = 0$$

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F2019 Problem 1

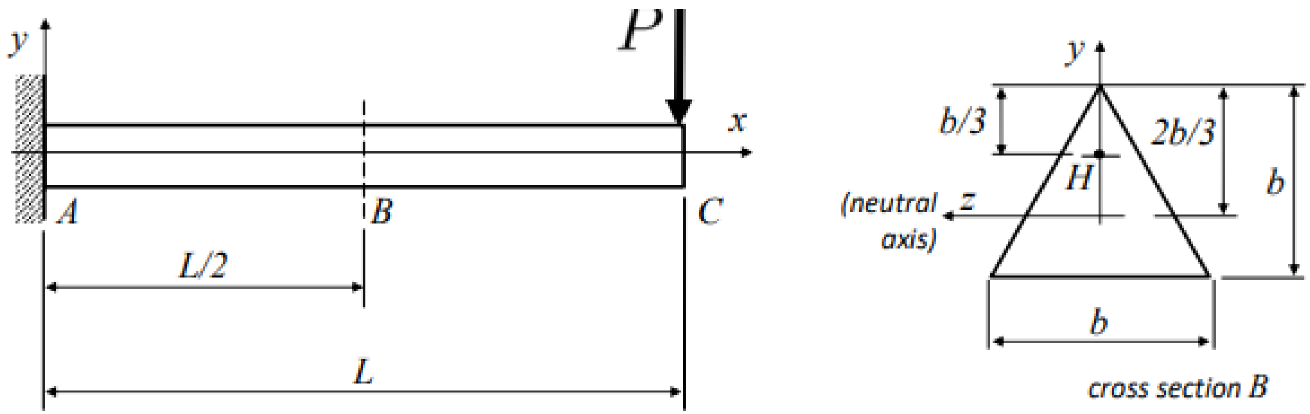

Figure 1

For the cantilever beam shown in the above figure:

a) Determine the reactions at the wall.

b) Determine the normal stress at point H of cross section B.

c) Determine the shear stress at point H of cross section B.

d) Show the state of stress at point H on the differential stress element shown below.

Note: $P = 25$ N, $L = 1$ m, $b = 0.03$ m, $I_{zz} = b^4/36$

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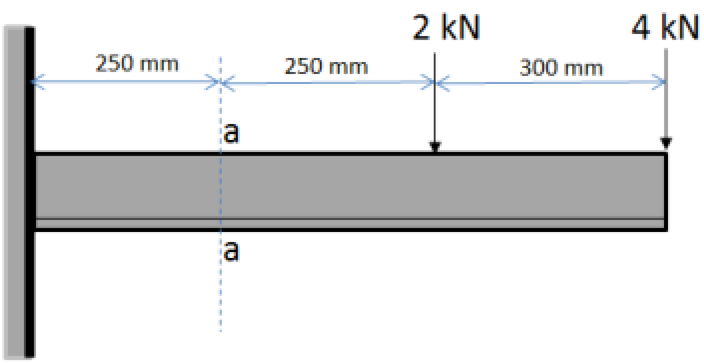
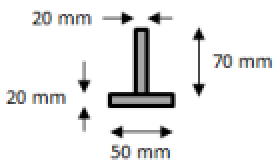
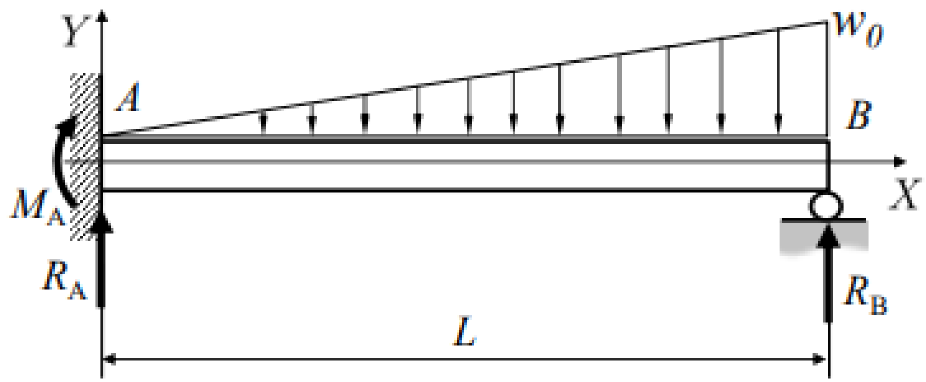
Example 10.14

Determine the maximum shear stress acting on the section a-a of the cantilevered beam below.





S2019 Problem 3



The uniform, linearly elastic beam shown in the figure supports a triangularly distributed load.

1) Write the equations of equilibrium and identify whether the structure is statically determinate or indeterminate.

Using the second-order method:

2) Determine the bending moment $M(x)$ of the beam (as a function of the reactions at A).
3) Determine the slope $v'(x)$ and deflection $v(x)$ of the beam.
4) Indicate the boundary conditions at supports A and B.
5) Solve for any constants of integration.
6) Write a system of three equations to determine the reactions at A and B, that is R_A, M_A, and R_B. Note: do not solve for the reactions.
7) Sketch the deflection curve.

F2019 – Problem 2

The beam BCD is fixed to the wall at B and supported by a roller at C. An external moment $\boldsymbol{M_0}$ is applied at C. The beam has Young's modulus $\boldsymbol{E}$ and second moment of area $\boldsymbol{I}$.

a) Draw a free-body diagram of the entire beam, and write down the equilibrium equations.

b) Use the second-order (or fourth-order) integration method to find the slope $v'(x)$ and deflection $v(x)$ of each segment of the beam. These can be left in terms of the unknown support reactions.

c) Write down the relevant boundary conditions and continuity conditions for the beam.

d) Use the boundary/continuity conditions to determine the reactions at B and C in terms of $\boldsymbol{M_0}$ and $\boldsymbol{L}$.

e) Determine the deflection at the free end D. Sketch the deflection curve over the length of the beam. The sketch does not need to be exact; show enough detail to indicate the boundary conditions.

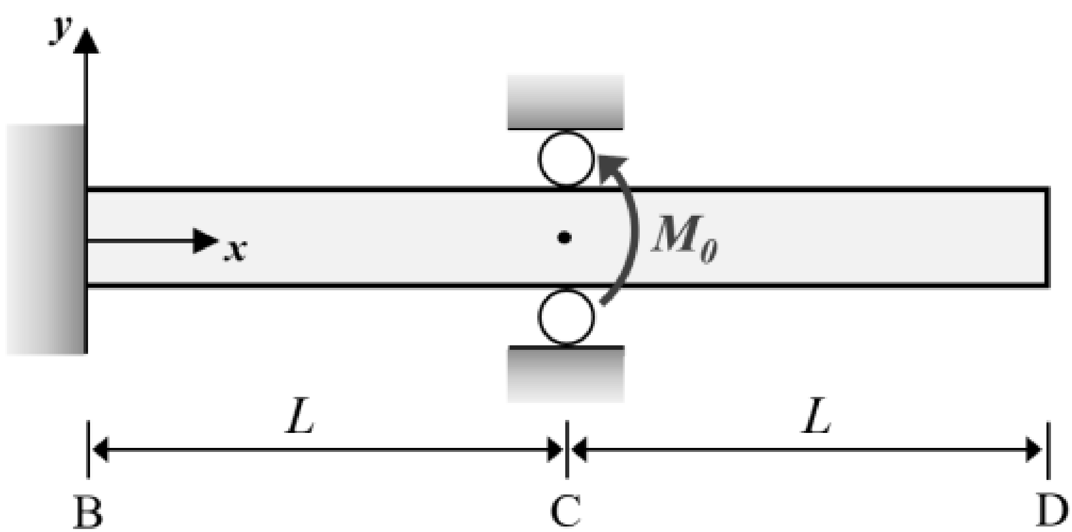


Figure 2

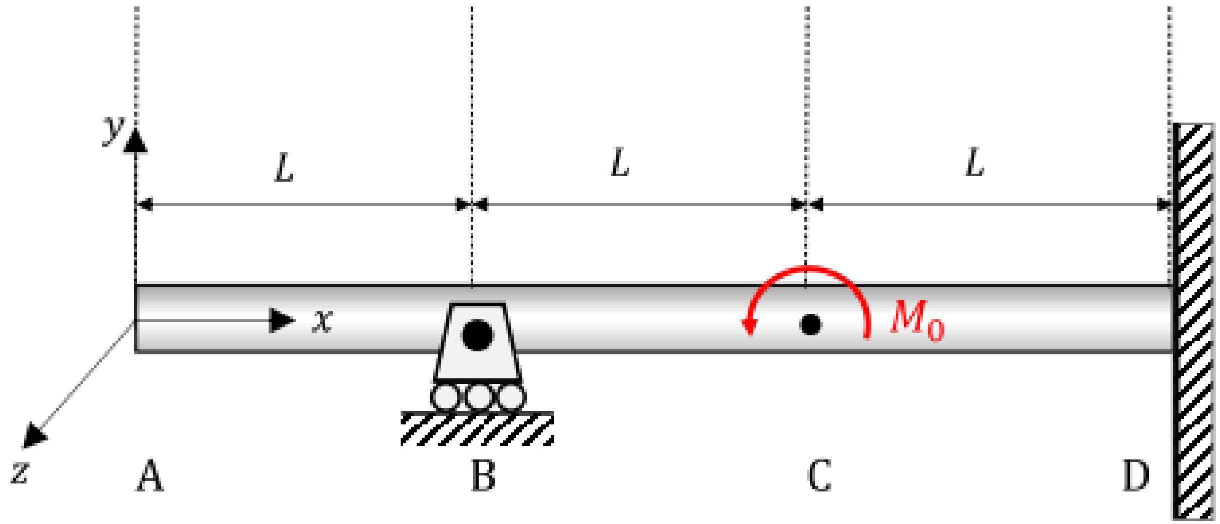
F2017 – Problem 1

The cantilever beam AD of the bending stiffness EI is subjected to a concentrated moment M_0 at C. The beam is also supported by a roller at B. Using Castigliano's theorem:

a) Determine the reaction force at the roller B.

b) Determine the rotation angle of the beam about z axis at the end A.

Ignore the shear energy due to bending. Express your answers in terms of M_0, E, and I.



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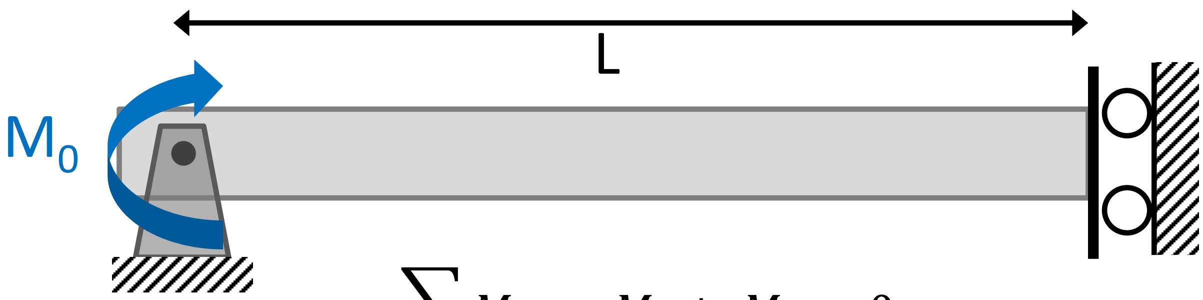
Practice Problem 1

Find the vertical displacement at the middle of the bar using Castigliano's method. Neglect shear bending energy due to shear forces.



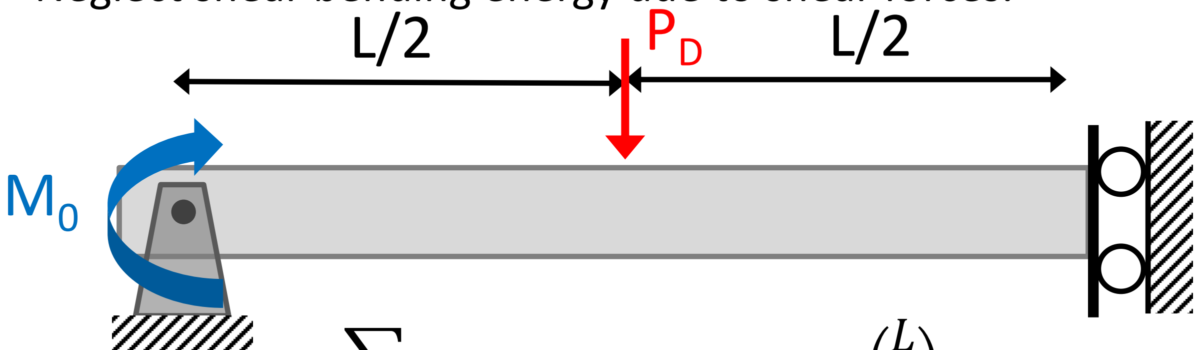
$$\sum M = -M_0 + M_B = 0$$

$$\sum F = F_A = 0$$

→ Determinate

Practice Problem 1

Find the vertical displacement at the middle of the bar using Castigliano's method. Neglect shear bending energy due to shear forces.



$$\sum M = -M_0 + M_B - P_D\left(\frac{L}{2}\right) = 0$$

$$\sum F = A_y - P_D = 0$$

$$M_1(x) = M_0 + P_D x \qquad M_2(x) = M_0 + P_D x - P_D\left(x - \frac{L}{2}\right) = M_0 + P_D\left(\frac{L}{2}\right)$$

$$\frac{\delta M_1}{\delta P_D} = x \qquad \frac{\delta M_1}{\delta P_D} = \frac{L}{2}$$

$$U = U_1 + U_2 = \left(\frac{1}{2EI}\right)\int_0^{\frac{L}{2}} M_1^2\,dx + \left(\frac{1}{2EI}\right)\int_{\frac{L}{2}}^{L} M_2^2\,dx$$

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Practice Problem 1

$$\frac{\delta U}{\delta P_D} = \left(\frac{1}{EI}\right)\int_0^{\frac{L}{2}} M_1\,(\frac{\delta M_1}{\delta P_D})dx + \left(\frac{1}{EI}\right)\int_{\frac{L}{2}}^{L} M_2\,(\frac{\delta M_2}{\delta P_D})dx$$

$$\frac{\delta U}{\delta P_D} = \left(\frac{1}{EI}\right)\int_0^{\frac{L}{2}} (M_0 + P_D x)\,(x)dx + \left(\frac{1}{EI}\right)\int_{\frac{L}{2}}^{L} \left(M_0 \; + P_D\left(\frac{L}{2}\right)\right)\left(\frac{L}{2}\right)dx$$

$$\left[\frac{\delta U}{\delta P_D}\right]_{P_D=0} = \left(\frac{1}{EI}\right)\int_0^{\frac{L}{2}} M_0\,x dx + \left(\frac{1}{EI}\right)\int_{\frac{L}{2}}^{L} M_0\left(\frac{L}{2}\right)dx$$

$$\left[\frac{\delta U}{\delta P_D}\right]_{P_D=0} = \left(\frac{3}{8}\right)\left(\frac{1}{EI}\right)M_0 L^2$$

In the negative direction

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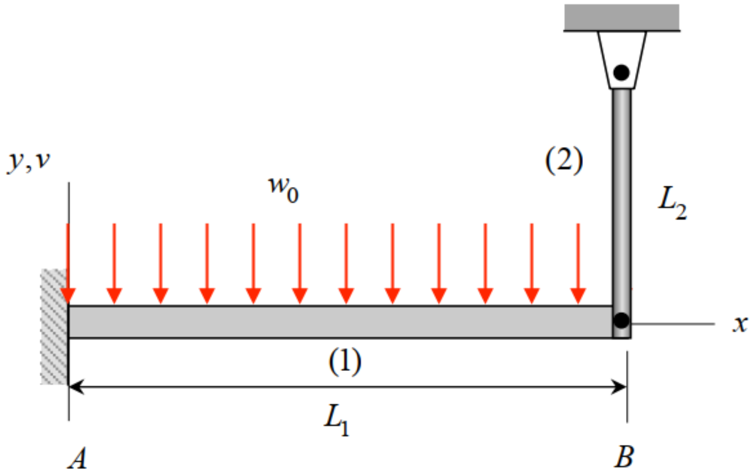
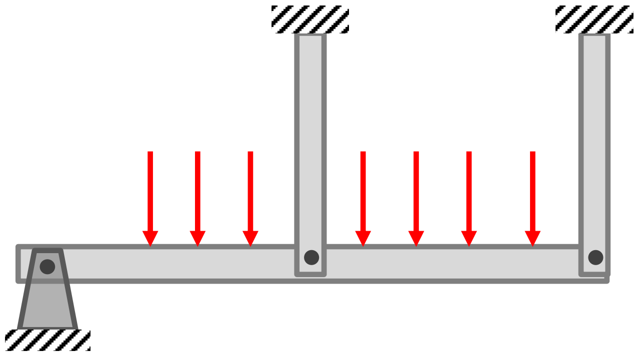
Practice Problem



$$U = U_1 + U_2$$

$$U = \left(\frac{1}{2EI}\right)\int_0^{L_1} M_1^2\,dx + \left(\frac{1}{2GA}\right)\int_0^{L_1} f_s V_1^2\,dx + \frac{F_2^2 L_2}{EA}$$



$$U = ?$$

$$U = U_1 + U_2 + U_3 + U_4$$

$$U = \left(\frac{1}{2EI}\right)\int_0^{L_1/2} M_1^2\,dx + \left(\frac{1}{2EI}\right)\int_{L_1/2}^{L_1} M_2^2\,dx + \frac{F_3^2 L_3}{EA} + \frac{F_4^2 L_4}{EA}$$