ME338A
CONTINUUM MECHANICS

lecture notes 17

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5.2.7 Incompressible Elasticity

5.2.7.1 Nearly incompressible elasticity

Many engineering materials exhibit distinct behavior under volumetric and isochoric deformations. Common examples include the nearly incompressible elasticity of rubber-like materials and soft biological tissues, shear-dominated ductile viscoplasticity of amorphous glassy polymers, and volume-preserving plasticity of metals. To this end, we decompose the deformation gradient $\boldsymbol{F}$ multiplicatively into the volumetric (spherical) $\boldsymbol{F}_{vol}$ and isochoric (unimodular) $\bar{\boldsymbol{F}}$ parts

$$\boldsymbol{F} = \bar{\boldsymbol{F}} \cdot \boldsymbol{F}_{vol} \tag{5.2.29}$$

where $\boldsymbol{F}_{vol} := J^{1/3}\,\mathbf{1}$ and $\bar{\boldsymbol{F}} := J^{-1/3}\,\boldsymbol{F}$.

This implies that the volume map is solely described by the volumetric part $\boldsymbol{F}_{vol}$ so that the identity $\det(\boldsymbol{F}_{vol}) = J$ holds. Therefore, the deformation under the unimodular part $\bar{\boldsymbol{F}}$ with $\det\bar{\boldsymbol{F}} = 1$ does not include any volume change but purely isochoric deformations. Having this multiplicative kinematic split at hand, the free energy of a hyperelastic material can be additively decomposed into the volumetric $U(J)$ and isochoric $\bar{\psi}$ parts

$$\psi = U(J) + \bar{\psi}(\bar{\boldsymbol{F}})\,. \tag{5.2.30}$$

Owing to the material frame objectivity requirement, the isochoric part of the free energy can also be expressed as $\bar{\psi} = \hat{\psi}(\bar{\boldsymbol{F}}^T \cdot \bar{\boldsymbol{F}}) = \hat{\psi}(\bar{\boldsymbol{C}})$, i.e.

$$\psi = U(J) + \hat{\psi}(\bar{\boldsymbol{C}}) \tag{5.2.31}$$

in terms of $\bar{\boldsymbol{C}} := \bar{\boldsymbol{F}}^T \cdot \bar{\boldsymbol{F}} = J^{-2/3}\boldsymbol{C}$.

The additive decompostion of the free energy results in also the additively split 'stress response

$$\boldsymbol{S} = 2\partial_{\boldsymbol{C}}\psi = \boldsymbol{S}_{vol} + \boldsymbol{S}_{iso} \tag{5.2.32}$$

in terms of the volumetric part

$$\boldsymbol{S}_{vol} := 2\partial_{\boldsymbol{C}}U(J) = U'(J)2\partial_{\boldsymbol{C}}J = JU'(J)\boldsymbol{C}^{-1} \tag{5.2.33}$$

and the isochoric part

$$\boldsymbol{S}_{iso} := 2\partial_{\boldsymbol{C}}\hat{\psi}(\bar{\boldsymbol{C}}) = 2\partial_{\bar{\boldsymbol{C}}}\hat{\psi}(\bar{\boldsymbol{C}}) : \partial_{\boldsymbol{C}}\bar{\boldsymbol{C}} = \bar{\boldsymbol{S}} : \mathbb{Q} \tag{5.2.34}$$

of the second Piola-Kirchhoff stress tensor. In (5.2.34), the *chain-rule* stresses $\bar{\boldsymbol{S}}$ and the fourth order *Lagrangean* deviatoric projection tensor $\mathbb{Q}$ are defined as

$$\bar{\boldsymbol{S}} := 2\partial_{\bar{\boldsymbol{C}}}\hat{\psi}(\bar{\boldsymbol{C}})$$

and

$$\begin{aligned} \mathbb{Q} &:= \partial_{\boldsymbol{C}}\bar{\boldsymbol{C}} = \partial_{\boldsymbol{C}}\big(J^{-2/3}\boldsymbol{C}\big) = J^{-2/3}\partial_{\boldsymbol{C}}\boldsymbol{C} + \boldsymbol{C}\otimes\partial_{\boldsymbol{C}}\big(J^{-2/3}\big) \\ &= J^{-2/3}\mathbb{I} + \boldsymbol{C}\otimes\big(-\tfrac{2}{3}\,J^{-2/3-1}\,\tfrac{1}{2}\,J\boldsymbol{C}^{-1}\big) \\ &= J^{-2/3}\left[\mathbb{I} - \tfrac{1}{3}\,\boldsymbol{C}\otimes\boldsymbol{C}^{-1}\right] \end{aligned}$$

respectively. The overall expression of the second Piola-Kirchhoff stresses can then be obtained as

$$\boldsymbol{S} = JU'(J)\boldsymbol{C}^{-1} + \bar{\boldsymbol{S}} : J^{-2/3}\left[\mathbb{I} - \frac{1}{3}\,\boldsymbol{C}\otimes\boldsymbol{C}^{-1}\right] . \tag{5.2.35}$$

The complete push forward of the second Piola-Kirchhoff stress tensor yields the Kirchhoff stress tensor

$$\boldsymbol{\tau} = \boldsymbol{F}\cdot\boldsymbol{S}\cdot\boldsymbol{F}^T = JU'(J)\boldsymbol{1} + \bar{\boldsymbol{\tau}} : \mathbb{P} \tag{5.2.36}$$

where $\bar{\boldsymbol{\tau}} := \bar{\boldsymbol{F}} \cdot \bar{\boldsymbol{S}} \cdot \bar{\boldsymbol{F}}^T$ and $\mathbb{P} := \mathbb{I} - \frac{1}{3}\,\mathbf{1} \otimes \mathbf{1}$ denotes the *spatial* deviatoric projection tensor such that $\mathrm{tr}(\bar{\tau} : \mathbb{P}) = 0$.
Then, the Cauchy stress tensor reads

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{\tau} = U'(J)\mathbf{1} + J^{-1}\bar{\boldsymbol{\tau}} : \mathbb{P}\,. \tag{5.2.37}$$

Employing the definition of the pressure, we obtain

$$p(J) := -\frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma}) = -U'(J)\,. \tag{5.2.38}$$

This result leads us to the following decomposed representation of the Cauchy stresses

$$\boldsymbol{\sigma} = -p(J)\mathbf{1} + \boldsymbol{\sigma}_{dev} \quad \text{with} \quad \boldsymbol{\sigma}_{dev} := J^{-1}\bar{\boldsymbol{\tau}} : \mathbb{P}\,. \tag{5.2.39}$$

In modeling of *nearly* incompressible response of materials in three-dimensional boundary-value problems, the pressure $p(J)$ serves as a *penalty* parameter employed to enforce the incompressibility condition. However, straightforward implementation of this penalty formulation in standard displacement finite elements often brings about undesired *volumetric locking* effects. In order to overcome this drawback, one of the remedies is the use of a three-field Hu-Washizu functional-based mixed finite element formulation that considers the pressure $\bar{p}$, the Jacobian Θ, and the deformation map $\boldsymbol{\varphi}$ as independent fields, albeit with different continuity requirements. These numerical treatments, however, are beyond the scope of this class.

5.2.7.2 Strictly incompressible elasticity

The *strictly* incompressible materials can be conceived as substances that are capable of carrying loads by undergoing solely volume conserving (isochoric) deformations; that is, the value of the volume map J is always restricted to unity

$$J := \det(\boldsymbol{F}) \doteq 1 \,. \tag{5.2.40}$$

If the type of loading has a purely spherical character, i.e. pure pressure or pure suction, an incompressible material behaves rigidly and hence does not store energy. This means, the pressure cannot be determined from the free energy function, e.g. by evaluating the function $U'(J)$ as we did in the preceding section. Therefore, pressure should be considered as unknown in the stress expressions

$$\boldsymbol{S} = -pJ\boldsymbol{C}^{-1} + 2\partial_{\boldsymbol{C}}\psi \quad \text{or} \quad \boldsymbol{P} = -pJ\boldsymbol{F}^{-T} + \partial_{\boldsymbol{F}}\psi \tag{5.2.41}$$

This formula holds only for the cases where the constraint (5.2.40) is strictly fulfilled. This implies that for a given deformation $\boldsymbol{F}$, one of the principal stretches, say λ_3, can be expressed in terms of the other principal stretches λ_1, λ_2; that is,

$$\det(\boldsymbol{F}) = \lambda_1\lambda_2\lambda_3 \doteq 1 \quad \rightsquigarrow \quad \lambda_\beta = \prod_{\alpha\neq\beta}^{3} \lambda_\alpha^{-1} \,. \tag{5.2.42}$$

For $\beta = 3$, we have $\lambda_3 = \lambda_1^{-1}\lambda_2^{-1}$. For an isotropic, incompressible material undergoing purely homogeneous deformations such as uniaxial and biaxial tests, components of the deformation gradient $\boldsymbol{F}$ are completely known. The following are the examples for the uniaxial, pure shear, and equi-biaxial cases with corresponding traction boundary conditions:

Uniaxial	$\boldsymbol{F} = \text{diag}[\lambda, \lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}}]$,	$\boldsymbol{P} = \text{diag}[P_{11}, 0, 0]$
Pure Shear	$\boldsymbol{F} = \text{diag}[\lambda, \lambda^{-1}, 1]$,	$\boldsymbol{P} = \text{diag}[P_{11}, 0, P_{33}]$
Equi-biaxial	$\boldsymbol{F} = \text{diag}[\lambda, \lambda, \lambda^{-2}]$,	$\boldsymbol{P} = \text{diag}[P_{11}, P_{22}, 0]$

The stress boundary conditions provide us with an additional equation that is solved to compute the unknown pressure.

5.2.7.3 Stretch-Based Elasticity Models

In Section 5.2.6, we concluded that the free energy of an isotropic material can be expressed as a function of principal stretches

$$\psi = \psi(\lambda_1, \lambda_2, \lambda_3)\ .$$

In order to compute the second Piola-Kirchhoff stresses from the stretch-based free energy function, we need to determine the derivative of the principal stretches with respect to $\boldsymbol{C}$. To this end, we consider the third invariant

$$I_3 = J^2 = \det(\boldsymbol{C}) = \lambda_1^2 \lambda_2^2 \lambda_3^2\ .$$

The derivative of I_3 with respect to $\boldsymbol{C}$ can be shown to be

$$\partial_{\boldsymbol{C}} I_3 = \sum_{\alpha=1}^{3} \partial_{\lambda_\alpha^2} I_3\, \partial_{\boldsymbol{C}} \lambda_\alpha^2 = I_3 \sum_{\alpha=1}^{3} \lambda_\alpha^{-2}\, \partial_{\boldsymbol{C}} \lambda_\alpha^2\ .$$

We also know that $\text{cof}(\boldsymbol{C}) = \partial_{\boldsymbol{C}} I_3$ whose spectral representation is given by

$$\partial_{\boldsymbol{C}} I_3 = I_3 \boldsymbol{C}^{-1} = I_3 \sum_{\alpha=1}^{3} \lambda_\alpha^{-2}\, \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha\ .$$

Comparing these two results, we deduce that

$$\partial_{\boldsymbol{C}}\lambda_\alpha^2 = \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \quad \text{and} \quad 2\partial_{\boldsymbol{C}}\lambda_\alpha = \lambda_\alpha^{-1}\boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \, .$$

We can then compute the second Piola-Kirchhoff stress tensor in the spectral form

$$\boldsymbol{S} = 2\partial_{\boldsymbol{C}}\psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{\alpha=1}^{3} \partial_{\lambda_\alpha^2}\psi \, 2\partial_{\boldsymbol{C}}\lambda_\alpha^2 = \sum_{\alpha=1}^{3} 2\partial_{\lambda_\alpha^2}\psi \, \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \, .$$

Substituting $S_\alpha := 2\partial_{\lambda_\alpha^2}\psi = \lambda_\alpha^{-1}\partial_{\lambda_\alpha}\psi$, we end up with the compact spectral representation of $\boldsymbol{S}$

$$\boldsymbol{S} = \sum_{\alpha=1}^{3} S_\alpha \, \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \quad \text{with} \quad S_\alpha := \lambda_\alpha^{-1}\partial_{\lambda_\alpha}\psi$$

standing for the eigenvalues of the second Piola-Kirchhoff stress tensor. From this result, we immediately observe that the second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ possesses the same eigenvectors as $\boldsymbol{C} = \sum_{\alpha=1}^{3} \lambda_\alpha^2 \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha$. This is a typical result obtained from an isotropic free energy function. When two second order tensors have the same eigenvectors, they are said to be *co-axial* and therefore they *commute*, i.e.

$$\boldsymbol{S} \cdot \boldsymbol{C} = \boldsymbol{C} \cdot \boldsymbol{S} \, .$$

The first Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the Kirchhoff stress tensor $\boldsymbol{\tau}$ are then readily computed through the push forward operation

$$\boldsymbol{P} = \boldsymbol{F} \cdot \boldsymbol{S} = \sum_{\alpha=1}^{3} P_\alpha \, \boldsymbol{n}_\alpha \otimes \boldsymbol{N}_\alpha \quad \text{with} \quad P_\alpha := \lambda_\alpha S_\alpha = \partial_{\lambda_\alpha}\psi$$

$$\boldsymbol{\tau} = \boldsymbol{P} \cdot \boldsymbol{F}^T = \sum_{\alpha=1}^{3} \tau_\alpha \, \boldsymbol{n}_\alpha \otimes \boldsymbol{n}_\alpha \quad \text{with} \quad \tau_\alpha := \lambda_\alpha P_\alpha = \lambda_\alpha \partial_{\lambda_\alpha}\psi$$

Example: The Ogden Model of Incompressible Elasticity
We now consider the well-known Ogden model of incompressible elasticity

$$\psi = \psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{n=1}^{3} \frac{\mu_n}{\beta_n}(\lambda_1^{\beta_n} + \lambda_2^{\beta_n} + \lambda_3^{\beta_n} - 3) \, .$$

For exact incompressibility $(J=1)$, we recall the expression for $\boldsymbol{P}$ from (5.2.41)

$$\boldsymbol{P} = -p\boldsymbol{F}^{-T} + \partial_{\boldsymbol{F}}\psi \, ,$$

or in the spectral form

$$\boldsymbol{P} = \sum_{\alpha=1}^{3} P_\alpha \, \boldsymbol{n}_\alpha \otimes \boldsymbol{N}_\alpha = \sum_{\alpha=1}^{3} (-p \, \lambda_\alpha^{-1} + \partial_{\lambda_\alpha}\psi) \, \boldsymbol{n}_\alpha \otimes \boldsymbol{N}_\alpha \, .$$

The eigenvalues of $\boldsymbol{P}$ are then

$$P_\alpha = -p \, \lambda_\alpha^{-1} + \partial_{\lambda_\alpha}\psi \quad \text{for} \quad \alpha = 1, 2, 3 \, .$$

For the Ogden model, the derivative of the free energy with respect to each principal stretch has the following form

$$\partial_{\lambda_\alpha}\psi = \sum_{n=1}^{3} \mu_n \lambda_\alpha^{\beta_n - 1} \quad \text{for} \quad \alpha = 1, 2, 3 \, .$$

This leads us to the model-specific expressions for the principal stresses

$$P_\alpha = -p \, \lambda_\alpha^{-1} + \sum_{n=1}^{3} \mu_n \lambda_\alpha^{\beta_n - 1} \quad \text{for} \quad \alpha = 1, 2, 3 \, .$$

We can now derive the analytic stress expressions for three distinct cases of homogeneous deformations:

- **Uniaxial** $\boldsymbol{F} = \mathrm{diag}[\lambda, \lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}}], \quad \boldsymbol{P} = \mathrm{diag}[P_{11}, 0, 0]$

For the stress-free lateral faces, we compute the pressure

$$P_{\{2,3\}} = -p\,\lambda_{\{2,3\}}^{-1} + \sum_{n=1}^{3} \mu_n \lambda_{\{2,3\}}^{\beta_n - 1} = 0 \rightsquigarrow p = \sum_{n=1}^{3} \mu_n \lambda_{\{2,3\}}^{\beta_n} .$$

Insertion of this result into P_1 yields

$$P_1 = \sum_{n=1}^{3} \mu_n (\lambda_1^{\beta_n - 1} - \lambda_{\{2,3\}}^{\beta_n}\,\lambda_1^{-1}) .$$

For $\lambda_1 = \lambda,\ \lambda_2 = \lambda_3 = \lambda^{-\frac{1}{2}}$, we obtain

$$P_1 = \sum_{n=1}^{3} \mu_n (\lambda^{\beta_n - 1} - \lambda^{-\beta_n/2 - 1}) .$$

- **Pure Shear** $\boldsymbol{F} = \mathrm{diag}[\lambda, \lambda^{-1}, 1], \quad \boldsymbol{P} = \mathrm{diag}[P_{11}, 0, P_{33}]$

Following the analogous steps

$$P_2 = -p\,\lambda_2^{-1} + \sum_{n=1}^{3} \mu_n \lambda_2^{\beta_n - 1} = 0 \rightsquigarrow p = \sum_{n=1}^{3} \mu_n \lambda_2^{\beta_n} .$$

$$P_1 = \sum_{n=1}^{3} \mu_n (\lambda_1^{\beta_n - 1} - \lambda_2^{\beta_n}\,\lambda_1^{-1}) = \sum_{n=1}^{3} \mu_n (\lambda^{\beta_n - 1} - \lambda^{-(\beta_n + 1)})$$

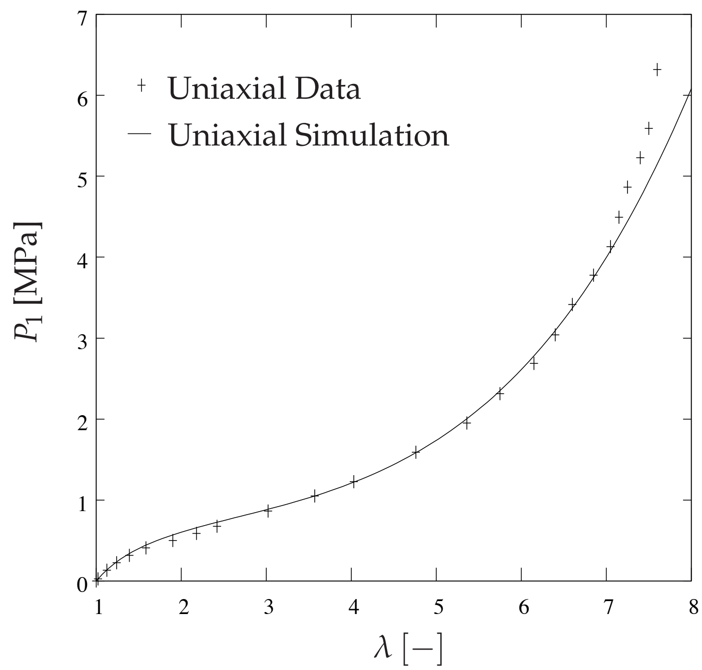
- **Equi-biaxial** $\boldsymbol{F} = \mathrm{diag}[\lambda, \lambda, \lambda^{-2}], \boldsymbol{P} = \mathrm{diag}[P_{11}, P_{22}, 0]$

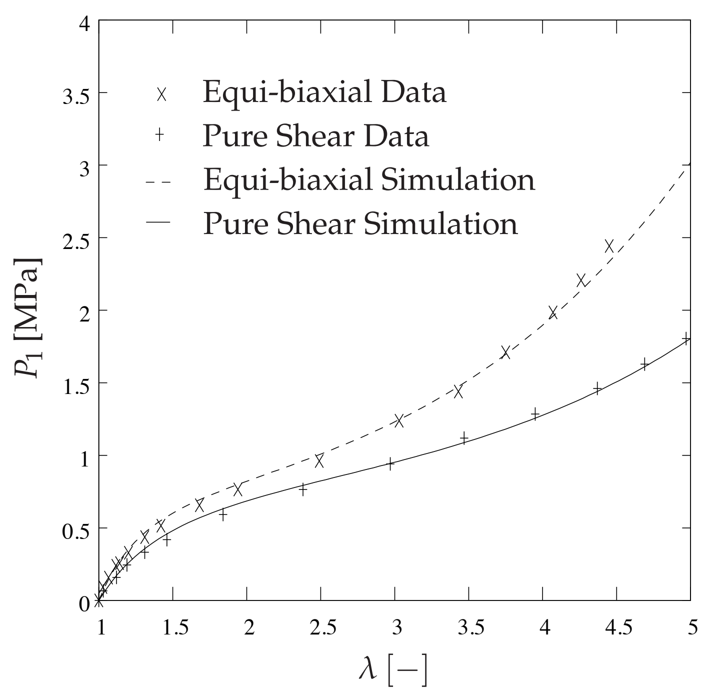
Similarly,

$$P_3 = -p\,\lambda_3^{-1} + \sum_{n=1}^{3} \mu_n \lambda_3^{\beta_n - 1} = 0 \rightsquigarrow p = \sum_{n=1}^{3} \mu_n \lambda_3^{\beta_n} .$$

$$P_{\{1,2\}} = \sum_{n=1}^{3} \mu_n (\lambda_{\{1,2\}}^{\beta_n - 1} - \lambda_3^{\beta_n}\,\lambda_{\{1,2\}}^{-1}) = \sum_{n=1}^{3} \mu_n (\lambda^{\beta_n - 1} - \lambda^{-(2\beta_n + 1)})$$

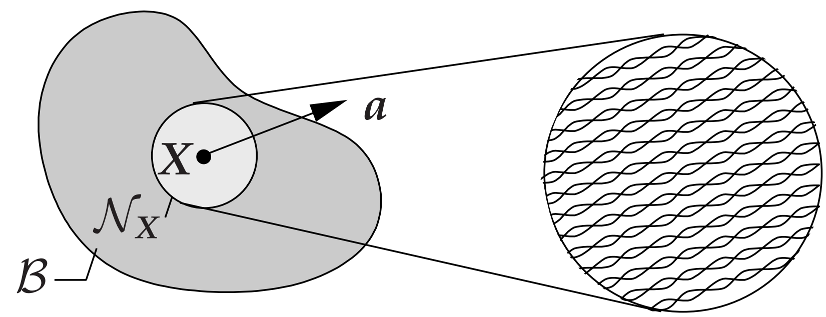
For the material parameters $\mu_1 = 0.63\,\text{MPa}$, $\mu_2 = 0.0012\,\text{MPa}$, $\mu_3 = -0.01$ MPa and $\beta_1 = 1.3$, $\beta_2 = 5$, $\beta_3 = -2$, the Ogden model captures the classical data of Treloar (1944) acquired from vulcanized unfilled natural rubber.





5.2.8 Transversely Isotropic Elasticity

This section is devoted to the coordinate-free representation of *transversely isotropic* elasticity, which is probably the simplest kind of anisotropy one can conceive. The coordinate-independent framework is obtained in terms of *structural tensors* that describe the microstructure of the material under consideration.



The microstructure of a transversely isotropic material can be considered as a continuum reinforced by micro-fibers. Common examples cover man-made fiber-reinforced composites such as carbon nanotube loaded materials, automobile tires, and natural collagenous biological tissues, to mention a few.

The fibrous microstructure of a transversely isotropic material possesses a well-defined *preferred* direction, which we denote with the unit vector $\boldsymbol{a}$, $||\boldsymbol{a}||=1$. This is the key information that helps us construct the symmetry group of a transversely isotropic matter. As we mentioned in Section 5.2.5, a symmetry group is defined by a set of rotations that preserve the structural characteristics of the material on micro-level. In the case of transverse isotropy, the symmetry group contains all rotations that do not alter the preferred orientation $\boldsymbol{a}$. These include the rotations of an arbitrary amount about the axis $\boldsymbol{a}$, i.e. $\boldsymbol{Q}_{\|\boldsymbol{a}}$ and the rotations about the axis perpendicular to $\boldsymbol{a}$ by an amount of π, i.e. $\boldsymbol{Q}^{\pi}_{\perp\boldsymbol{a}}$, that flip $\boldsymbol{a}$ horizontally. We then define the transversely isotropic symmetry group defined as

$$G_{\text{transiso}} := \left\{ \boldsymbol{Q}_{\|\boldsymbol{a}}, \boldsymbol{Q}^{\pi}_{\perp\boldsymbol{a}} \right\} \; . \tag{5.2.43}$$

The underlying key idea of coordinate-free representation of anisotropic materials is the *isotropic tensor functions* with extended set of arguments such that they remain invariant under arbitrary rotations $\boldsymbol{Q} \in \mathcal{SO}(3)$. This opens up a possibility to recast an anisotropic constitutive function into an isotropic one by means of the so-called *structural tensors*. For the case of transverse isotropy, we have

$$\psi = \hat{\psi}(\boldsymbol{C}, \boldsymbol{M}) \quad \text{with} \quad \boldsymbol{M} := \boldsymbol{a} \otimes \boldsymbol{a} \; . \tag{5.2.44}$$

denoting the *constant* rank-one structural tensor. The free energy function with the extended list of arguments is then required to be an isotropic function, i.e. invariant under arbitrary rotations $\boldsymbol{Q} \in \mathcal{SO}(3)$

$$\hat{\psi}(\boldsymbol{Q} \cdot \boldsymbol{C} \cdot \boldsymbol{Q}^T, \boldsymbol{Q} \cdot \boldsymbol{M} \cdot \boldsymbol{Q}^T) = \hat{\psi}(\boldsymbol{C}, \boldsymbol{M}) \quad \forall \boldsymbol{Q} \in \mathcal{SO}(3) \; . \tag{5.2.45}$$

Since the rotations belonging to the symmetric group G_{transiso} preserve the direction $\boldsymbol{a}$, the structural tensor $\boldsymbol{M}$ remains invariant under these rotations

$$\boldsymbol{Q} \cdot \boldsymbol{M} \cdot \boldsymbol{Q}^T = \boldsymbol{M} \quad \forall \boldsymbol{Q} \in G_{\text{transiso}} \; . \tag{5.2.46}$$

Therefore, the new isotropic free energy also fulfills

$$\hat{\psi}(\boldsymbol{Q} \cdot \boldsymbol{C} \cdot \boldsymbol{Q}^T, \boldsymbol{M}) = \hat{\psi}(\boldsymbol{C}, \boldsymbol{M}) \quad \forall \boldsymbol{Q} \in G_{\text{transiso}} \subset \mathcal{SO}(3) \ . \tag{5.2.47}$$

The representation theorem of general isotropic tensor functions of several arguments is based on the concept of *integrity bases*, which defines a minimum number of invariants for a particular set of arguments of the energy storage function. For two generic *symmetric* second order tensors, say $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, the irreducible integrity bases are given by

$$\begin{aligned} \mathcal{I} = \{ \ & J_1(\boldsymbol{A}_1), J_2(\boldsymbol{A}_1), J_3(\boldsymbol{A}_1), \ J_1(\boldsymbol{A}_2), J_2(\boldsymbol{A}_2), J_3(\boldsymbol{A}_2) \\ & J_4(\boldsymbol{A}_1, \boldsymbol{A}_2), J_5(\boldsymbol{A}_1, \boldsymbol{A}_2), J_6(\boldsymbol{A}_1, \boldsymbol{A}_2), J_7(\boldsymbol{A}_1, \boldsymbol{A}_2) \} \end{aligned} \tag{5.2.48}$$

where

$$J_1(\boldsymbol{A}_i) := \operatorname{tr}(\boldsymbol{A}_i), \ J_2(\boldsymbol{A}_i) := \operatorname{tr}(\boldsymbol{A}_i^2), \ J_3(\boldsymbol{A}_i) := \operatorname{tr}(\boldsymbol{A}_i^3)$$

are the *basic invariants* of respective tensors for $i = 1, 2$ and

$$\begin{aligned} J_4 &:= \operatorname{tr}(\boldsymbol{A}_1 \cdot \boldsymbol{A}_2), \qquad J_5 := \operatorname{tr}(\boldsymbol{A}_1^2 \cdot \boldsymbol{A}_2), \\ J_6 &:= \operatorname{tr}(\boldsymbol{A}_1 \cdot \boldsymbol{A}_2^2), \qquad J_7 := \operatorname{tr}(\boldsymbol{A}_1^2 \cdot \boldsymbol{A}_2^2) \end{aligned}$$

denote the *mixed* invariants.

In transversely isotropic elasticity, the free energy function depends on the symmetric tensors $\boldsymbol{A}_1 = \boldsymbol{C}$ and $\boldsymbol{A}_2 = \boldsymbol{M}$. Since the structural tensor $\boldsymbol{M}$ is also constant and $\boldsymbol{M}^n = \boldsymbol{M}$, the following identities

$$\begin{aligned} J_3(\boldsymbol{M}) &= \operatorname{tr}(\boldsymbol{M}^3) = \operatorname{tr}(\boldsymbol{M}^2) = \operatorname{tr}(\boldsymbol{M}) && \rightsquigarrow \ J_3(\boldsymbol{M}) = J_2(\boldsymbol{M}) = J_1(\boldsymbol{M}) \\ J_6 &= \operatorname{tr}(\boldsymbol{C} \cdot \boldsymbol{M}^2) = \operatorname{tr}(\boldsymbol{C} \cdot \boldsymbol{M}) && \rightsquigarrow \ J_6 = J_4 \\ J_7 &= \operatorname{tr}(\boldsymbol{C}^2 \cdot \boldsymbol{M}^2) = \operatorname{tr}(\boldsymbol{C}^2 \cdot \boldsymbol{M}) && \rightsquigarrow \ J_7 = J_5 \end{aligned}$$

reduce the integrity bases of transverse isotropy to

$$\mathcal{I} = \{J_1(\boldsymbol{C}), J_2(\boldsymbol{C}), J_3(\boldsymbol{C}), J_4(\boldsymbol{C},\boldsymbol{M}), J_5(\boldsymbol{C},\boldsymbol{M}); J_1(\boldsymbol{M})\} \quad (5.2.49)$$

Since we also know that $||\boldsymbol{a}|| = 1$, the basic invariant(s) of the structural tensor are none other than $J_1(\boldsymbol{M}) = 1$. Therefore, it has no effect on the energy storage function. The effective list of deformation-dependent invariants then boils down to

$$\mathcal{I} = \{J_1(\boldsymbol{C}), J_2(\boldsymbol{C}), J_3(\boldsymbol{C}), J_4(\boldsymbol{C},\boldsymbol{M}), J_5(\boldsymbol{C},\boldsymbol{M})\} .a \quad (5.2.50)$$

Since the basic invariants of $\boldsymbol{C}$ can be expressed in terms of the principal invariants

$$J_1 = I_1, \quad J_2 = I_1^2 - 2I_2, \quad J_3 = I_1^3 - 3I_1I_2 + 3I_3 ,$$

the set in (5.2.50) may be recast as

$$\mathcal{I} = \{I_1(\boldsymbol{C}), I_2(\boldsymbol{C}), I_3(\boldsymbol{C}), J_4(\boldsymbol{C},\boldsymbol{M}), J_5(\boldsymbol{C},\boldsymbol{M})\} . \quad (5.2.51)$$

5.2.8.1 Extended Neo-Hookean Model of Transverse Isotropy

We consider a model problem of transverse isotropy. The free energy is assumed to be given as

$$\psi = \hat{\psi}_I(I_1, J) + \hat{\psi}_A(I_1, I_2, J_4, J_5) , \quad (5.2.52)$$

which is split into the purely isotropic ψ_I and the anisotropic ψ_A parts. This leads us to the additive form of the stresses

$$\boldsymbol{S} = 2\partial_{\boldsymbol{C}}\psi = \boldsymbol{S}_I + \boldsymbol{S}_A \quad \text{and} \quad \boldsymbol{P} = \boldsymbol{P}_I + \boldsymbol{P}_A , \quad (5.2.53)$$

where $\boldsymbol{S}_I := 2\partial_{\boldsymbol{C}}\hat{\psi}_I, \boldsymbol{S}_A := 2\partial_{\boldsymbol{C}}\hat{\psi}_A$ and $\boldsymbol{P}_I := \boldsymbol{F}\cdot\boldsymbol{S}_I, \boldsymbol{P}_A := \boldsymbol{F}\cdot\boldsymbol{S}_A$.

The isotropic free energy function is taken to be identical to the compressible neo-Hookean model we discussed before

$$\hat{\psi}_I(I_1, J) = \frac{\Lambda}{4}(J^2 - 1) - (\mu + \frac{\Lambda}{2}) \ln J + \frac{\mu}{2}(I_1 - 3) \; . \quad (5.2.54)$$

We then retrieve the stress expressions for the isotropic part

$$\begin{aligned} \boldsymbol{S}_I &= 2\frac{\partial \hat{\psi}_I}{\partial \boldsymbol{C}} = \mu(\mathbf{1} - \boldsymbol{C}^{-1}) + \frac{\Lambda}{2}\left(J^2 - 1\right)\boldsymbol{C}^{-1} \; , \\ \boldsymbol{P}_I &= \boldsymbol{F} \cdot \boldsymbol{S}_I = \mu(\boldsymbol{F} - \boldsymbol{F}^{-T}) + \frac{\Lambda}{2}\left(J^2 - 1\right)\boldsymbol{F}^{-T} \; . \end{aligned} \quad (5.2.55)$$

For the anisotropic part of the free energy we consider the following form

$$\begin{aligned} \hat{\psi}_A(I_1, I_2, J_4, J_5) &= \alpha_1(J_4 - 1)^2 + \alpha_2[J_5 - I_1 J_4 + I_2 - 1] \\ &+ \alpha_2(J_4 - I_1 + 2) \; , \end{aligned} \quad (5.2.56)$$

where α_1 and α_2 are additional material parameters. Geometrical interpretation of the kinematic terms appearing in this free energy will make the meaning of these additional material parameters more transparent. The square of the stretch in the direction $\boldsymbol{a}$ is given by J_4, i.e.

$$\lambda_a^2 := ||\boldsymbol{F} \cdot \boldsymbol{a}||^2 = (\boldsymbol{F} \cdot \boldsymbol{a}) \cdot (\boldsymbol{F} \cdot \boldsymbol{a}) = \boldsymbol{C} : (\boldsymbol{a} \otimes \boldsymbol{a}) = \boldsymbol{C} : \boldsymbol{M} \equiv J_4 \; .$$

Thus, the first non-linear term of the free energy governs the amount energy stored due to the stretching of fibers, and the material parameter α_1 is closely related to the stiffness of these fibers. The second term in square brackets, however, measures the area stretch ν_a in the direction $\boldsymbol{a}$. We compute the square of the area stretch through the area map $\mathrm{cof}(\boldsymbol{F}) := J\boldsymbol{F}^{-T}$

$$\nu_a^2 := ||\,\mathrm{cof}(\boldsymbol{F})\cdot\boldsymbol{a}||^2 = J^2(\boldsymbol{F}^{-T}\cdot\boldsymbol{a})\cdot(\boldsymbol{F}^{-T}\cdot\boldsymbol{a}) = J^2\boldsymbol{C}^{-1} : \boldsymbol{M}\ .$$

In order to see the relation between this result and the second term of $\hat{\psi}_A$, we recall the Cayley-Hamilton theorem

$$\boldsymbol{C}^3 - I_1\boldsymbol{C}^2 + I_2\boldsymbol{C} - I_3\boldsymbol{1} = \boldsymbol{0}\ ,$$

Multiplying this equation with $\boldsymbol{C}^{-1}$ and solving the result for $I_3\boldsymbol{C}^{-1}$, we obtain

$$I_3\boldsymbol{C}^{-1} = J^2\boldsymbol{C}^{-1} = \boldsymbol{C}^2 - I_1\boldsymbol{C} + I_2\boldsymbol{1}\ .$$

Contraction of this result with the structural tensor $\boldsymbol{M}$ then gives the square of the area stretch in the preferred direction

$$\begin{aligned}\nu_a^2 = J^2\boldsymbol{C}^{-1} : \boldsymbol{M} &= \boldsymbol{C}^2 : \boldsymbol{M} - I_1\boldsymbol{C} : \boldsymbol{M} + I_2\boldsymbol{1} : \boldsymbol{M} \\ &= J_5 - I_1 J_4 + I_2\ .\end{aligned}$$

Therefore, the material parameter α_2 can be conceived as the stiffness of the isotropic matrix that surrounds the fibers. The third term $\alpha_2(J_4 - I_1 + 2)$ in $\hat{\psi}_A$ is introduced to have a stress-free state in the undeformed configuration. Computation of the stress contributions from the anisotropic part requires the knowledge of the following derivatives

$$\begin{aligned}2\partial_{\boldsymbol{C}} I_1 &= 2\partial_{\boldsymbol{C}}(\boldsymbol{C} : \boldsymbol{1}) = 2\,\boldsymbol{1}\ , \\ 2\partial_{\boldsymbol{C}} I_2 &= 2\partial_{\boldsymbol{C}}\tfrac{1}{2}\,(I_1^2 - \boldsymbol{C}^2 : \boldsymbol{1}) = 2\,(I_1\boldsymbol{1} - \boldsymbol{C})\ , \\ 2\partial_{\boldsymbol{C}} J_4 &= 2\partial_{\boldsymbol{C}}(\boldsymbol{C} : \boldsymbol{M}) = 2\,\boldsymbol{M}\ , \\ 2\partial_{\boldsymbol{C}} J_5 &= 2\partial_{\boldsymbol{C}}(\boldsymbol{C}^2 : \boldsymbol{M}) = 2\,(\boldsymbol{M}\cdot\boldsymbol{C} + \boldsymbol{C}\cdot\boldsymbol{M})\ .\end{aligned}$$

The anisotropic part of the second Piola-Kirchhoff stresses is shown to be

$$\begin{aligned} \boldsymbol{S}_A \; = \; & 4\,\alpha_1(J_4 - 1)\boldsymbol{M} \\ + \; & 2\,\alpha_2[(\boldsymbol{M}\cdot\boldsymbol{C} + \boldsymbol{C}\cdot\boldsymbol{M}) - \boldsymbol{C} + (I_1 - J_4 - 1)\boldsymbol{1} - (I_1 - 1)\boldsymbol{M}]\;, \end{aligned}$$

from which the first Piola-Kirchhoff stresses can readily be computed through the push forward operation $\boldsymbol{P}_A = \boldsymbol{F}\cdot\boldsymbol{S}_A$.

Uniaxial stress-stretch response of the transversely isotropic model in two perpendicular directions is depicted below. Observe the fairly stiffer behavior in the direction coinciding with the preferred direction $\boldsymbol{a}$.