ME751
Advanced Computational Multibody Dynamics

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Quote of the Day

If a cluttered desk is a sign of a cluttered mind, of what, then, is an empty desk a sign?

Albert Einstein

Before we get started…

- You have learned so far:
  - Rigid multibody dynamics and mechanical joints
  - Principles of Mechanics, generalized coordinates, equations of Lagrange
  - Numerical methods
  - Numerical integration of DAEs

- During next few lectures
  - We will learn about how to introduce flexibility in multibody system applications
  - Small deformation formulations
  - Large deformation formulations
  - Finite elements and their implementation in software (Chrono)

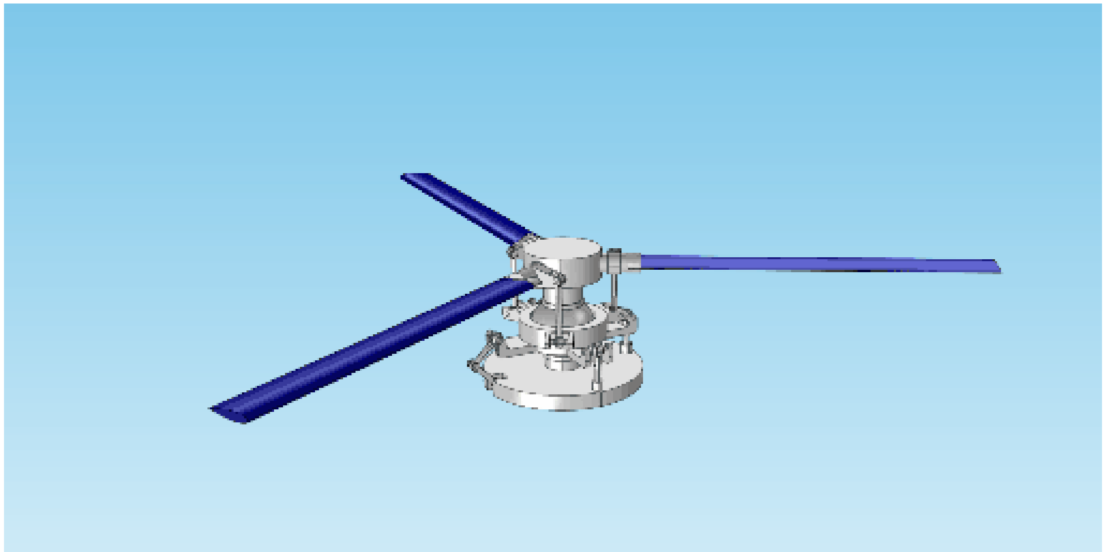
- Things that we are going to look into…
  - Co-rotational formulation –small deformation
  - Floating Frame of Reference (FFR) formulation
  - Model order reduction techniques
  - Geometrically exact beam theory
  - Absolute Nodal Coordinate Formulation

Disclaimer

- First version of this part of the course
  - Material obtained from here and there; much from papers
  - Notation is **NOT** completely consistent on all the slides
  - Not much time is spent reviewing ideas from previous lectures; basic knowledge on finite elements and elasticity is assumed
  - There will be a need to take notes
  - There will be a need for questions
  - There might be a need for side explanations/discussions
  - Additional material will be provided (mostly freely available online)

Flexible Multibody Systems

- How to deal with bodies that deform, in addition to rotate and translate?
  - We'll use methods of flexible multibody system dynamics
  - Close relation to finite element formulations
  - Implemented in commercial multibody software: COMSOL, Altair, MSC Software, SIMPACK, etc.



Source:

https://www.comsol.com/blogs/modeling-a-helicopter-swashplate-mechanism/

TOC – Flexible Multibody System

- Lecture 1:Introduction
  - Quick approach to continuum mechanics
  - Small deformation methods
- Lecture 2: Small Deformation - Kinematics of the FFR
  - Methods for small deformation in FMBD
  - Kinematics of FFR, reference conditions, modes of deformation
- Lecture 3: Full Equations of the FFR
  - Derivation of equations of motion
  - Inertia, and inertia shape integrals
  - Model order reduction

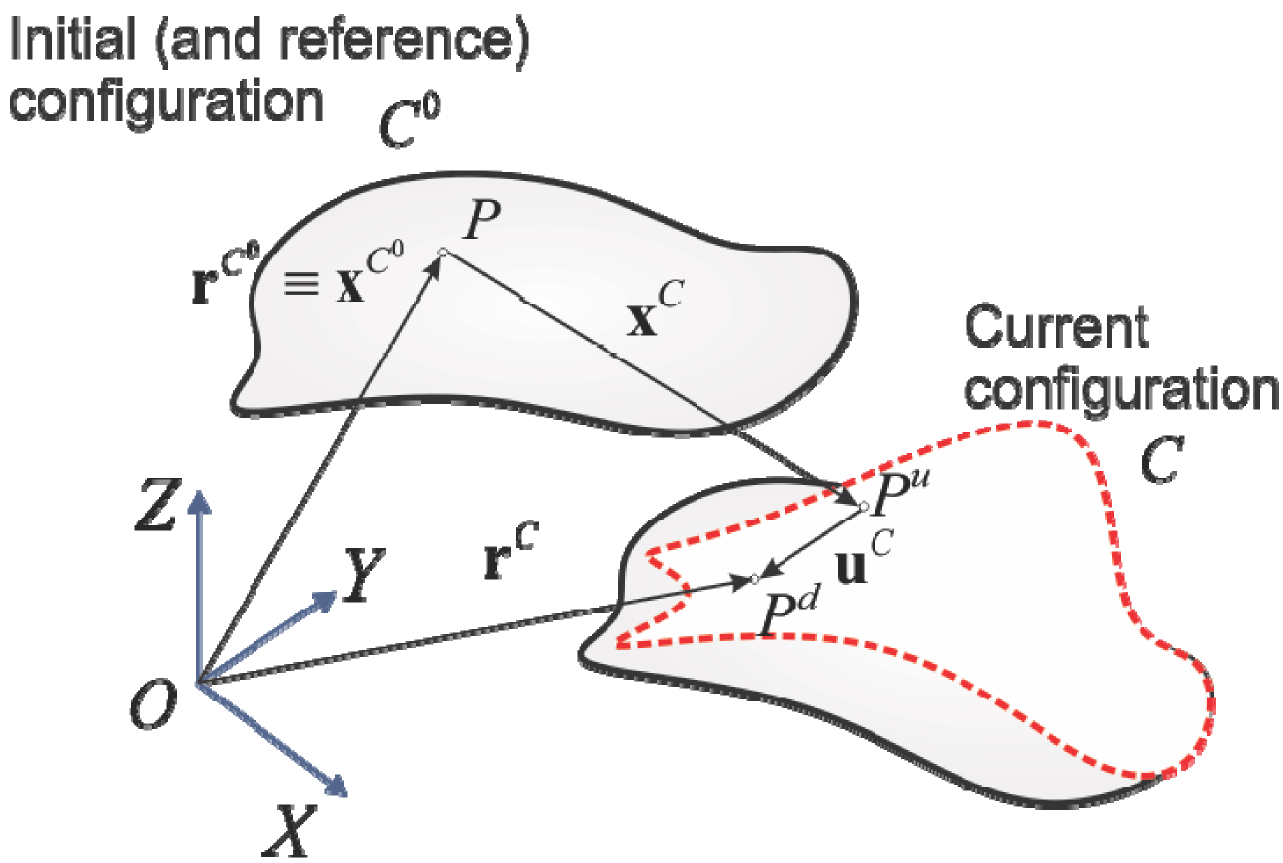
TOC – Flexible Multibody System

- Lecture 4: Finite Element FFR
  - Use of FE in the equations of motion of FFR
  - Intermediate coordinate system
  - Present kinematics of FFR including finite elements
- Lecture 5: Applications of FFR
  - Strain measures
  - Applications: How to approach problem solving
- Lecture 6: Large Deformation Formulations
  - Geometrically exact beam theory
  - Isoparametric elements
  - Absolute nodal coordinate formulation kinematics

TOC – Flexible Multibody System

- Lecture 7: Absolute Nodal Coordinate Formulation
  - ANCF beam element strains
  - Generalized, inertia forces
  - Examples with Chrono
- Lecture 8: ANCF bilinear shell element
  - Definition of strains for initially distorted geometry
  - Kinematics
  - Generalized forces
- Lecture 9: ANCF bilinear shell element
  - Locking in finite elements
  - Applications – Chrono

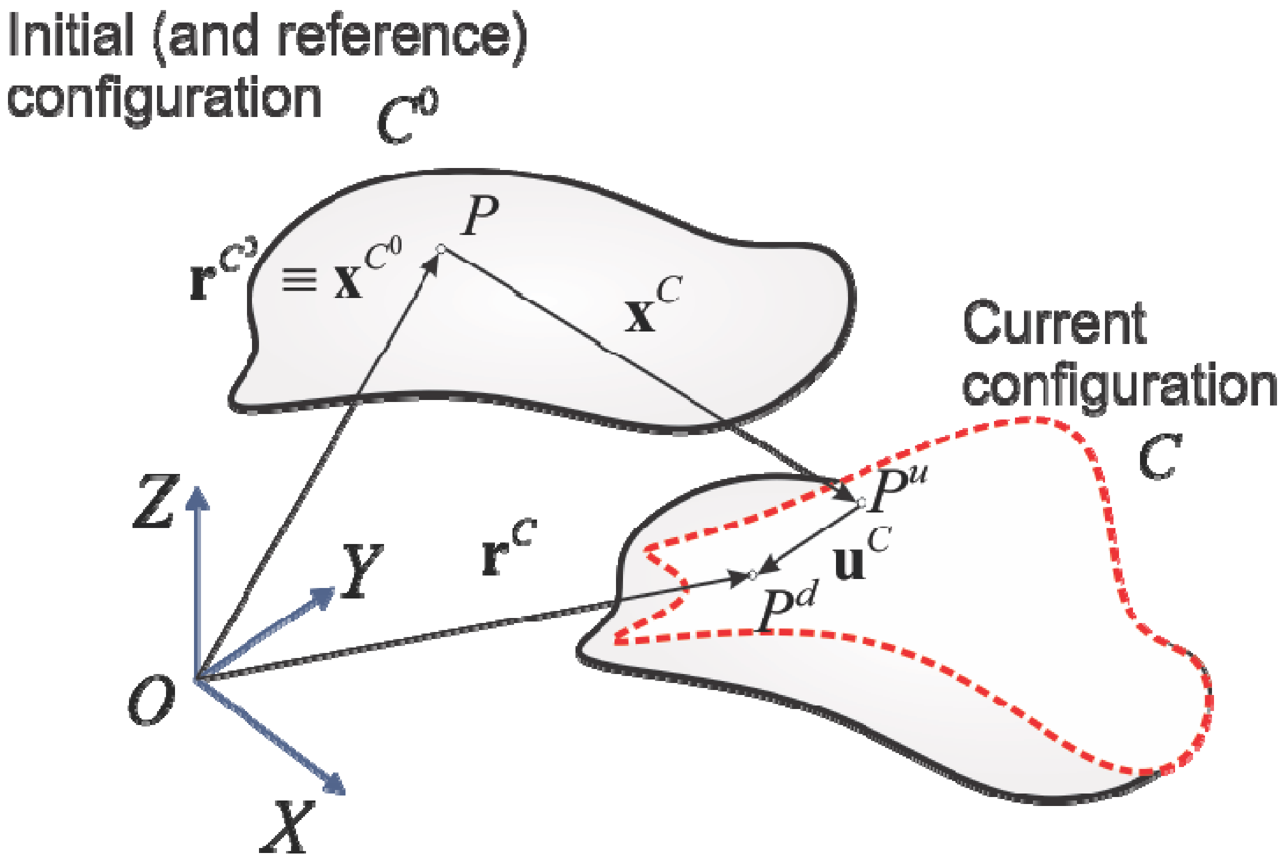
Mechanics of Deformable Bodies



Mechanics of Deformable Bodies

$$\mathbf{r}^C = \mathbf{x}^{C^0} + \mathbf{x}^C + \mathbf{u}^C$$

Global deformed position | Initial position | Rigid body displacement | Deformation displacement vector



Configurations of a Solid

Name	Definition	Identification
Initial	Configuration of body/solid at the onset of the simulation – typically undeformed	C^0
Reference	Configuration at which no deformation is considered to occur. Used in incremental/non-incremental methods	C^R
Co-rotated (or shadow)	Body- or element- attached configuration from C^0 describing rigid body motion	C^{CR}
Current	Admissible configuration taken by the body through dynamic analysis	C

Kinematic Descriptions

Name	Definition	Applicability
Total Lagrangian	Initial and reference configuration are equivalent and remain fixed throughout the simulation	**Finite** but moderate **strains** (we'll call it large deformation)
Co-rotational	Reference configuration split into initial and corotated. Strains are measured from corotated to current, whereas the initial configuration is used to measure rigid body motion	Solid and structural dynamics with arbitrarily translation and rotation and **small strains** – usually elastic behavior
Updated Lagrangian	Initial configuration remains fixed, but reference configuration is updated periodically. Reference configuration is updated to a recently computed configuration	Can capture large displacements and massive strains. Handles flow-like behavior (metal processing) and fracture

More details on solid configurations and their kinematics may be found in Chapter 7: REVIEW OF CONTINUUM MECHANICS: KINEMATICS by Carlos Felippa (Univ. of Colorado) - available online

Kinematic Descriptions

Name	In this course…
Total Lagrangian	Deformation of bodies referred to inertial frame (no intermediate reference configuration). They describe nonlinear measures (large) deformation: - Absolute Nodal Coordinate Formulation. - Geometrically Exact Beam Theory
Co-rotational	The use of a corotated frame usually implies linearization of strains. As such, these formulations can only deal with small deformation: - Floating Frame of Reference Formulation: We'll go in depth - Corotational formulation: We'll outline its kinematics
Updated Lagrangian	Will not be addressed. Not mainstream in FMBD

A word on Solid Mechanics approaches

- Two main approaches
  - Continuum Mechanics or Solid Mechanics: We do not make any assumption as to the geometry of the bodies we are analyzing
  - Structural Mechanics: Geometric particularities of the bodies under study are leveraged to develop more efficient/accurate formulations. E.g.
    - Beams
    - Shells
    - Solid shells

Deformation gradient

- How to use the kinematics to compute strains (deformation measures)?
  - The derivatives of $\mathbf{r}$ with respect to $\mathbf{x}$ constitute a fundamental tensor in Computational Mechanics, the deformation gradient:

$$\mathbf{F} = \frac{\partial(r_1,r_2,r_3)}{\partial(x_1,x_2,x_3)} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \frac{\partial r_1}{\partial x_3} \\ \frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \frac{\partial r_2}{\partial x_3} \\ \frac{\partial r_3}{\partial x_1} & \frac{\partial r_3}{\partial x_2} & \frac{\partial r_3}{\partial x_3} \end{bmatrix} \equiv \text{Deformation gradient tensor}$$

  - Determinant of $\mathbf{F}$ must never vanished. Allows coordinate transformation

$$\mathrm{d}\mathbf{r} = \begin{bmatrix} \mathrm{d}r_1 \\ \mathrm{d}r_2 \\ \mathrm{d}r_3 \end{bmatrix} = \mathbf{F}\mathrm{d}\mathbf{x} = \mathbf{F}\begin{bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{bmatrix}, \ \mathrm{d}\mathbf{x} = \mathbf{F}^{-1}\mathrm{d}\mathbf{r}$$

Gradient Transformation

- Let rigid body position of a material point in a body be defined by $\mathbf{x}$ in the coordinate system $\mathbf{XYZ}$, and by the vector $\bar{\mathbf{x}}$ in another coordinate system $\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{Z}}$. If $\mathbf{A}$ is an orientation matrix that transforms vectors in $\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{Z}}$ to $\mathbf{XYZ}$, we get

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \frac{\partial \mathbf{r}}{\partial \bar{\mathbf{x}}} \boxed{\frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{x}}} = \frac{\partial \mathbf{r}}{\partial \bar{\mathbf{x}}} \mathbf{A}^T$$

Allows us to introduce distorted configuration

- Or, analogously

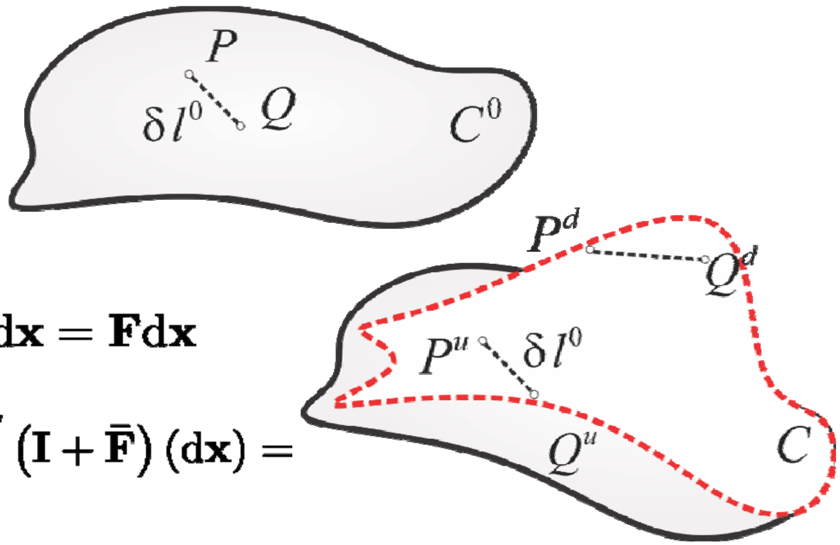
$$\frac{\partial \mathbf{r}}{\partial \bar{\mathbf{x}}} = \mathbf{F}\mathbf{A} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}} \mathbf{A}$$

- Gradient transformation is key to define local directions in flexible bodies/solids* and allows having distorted initial configurations. This will be used in later lectures to define orthotropy in materials, i.e. directions along which material properties differ

*Terms ‘flexible bodies’ and ‘solids’ will be used interchangeably in this course with the understanding we are describing bodies that can translate, rotate, and deform

Strain Components

- Here, we use a continuum mechanics approach (not structural)
- Let's draw a small vector in the undeformed body and see how it deforms



$$(\delta l^0)^2 = (\mathbf{dx})^T(\mathbf{dx}) = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2$$

$$(\delta l)^2 = (\mathbf{dr})^T(\mathbf{dr}) = (\mathrm{d}r_1)^2 + (\mathrm{d}r_2)^2 + (\mathrm{d}r_3)^2$$

$$\mathbf{dr} = \mathbf{dx} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{dx} = \left(\mathbf{I} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\mathbf{dx} \Rightarrow \mathbf{dr} = \left(\mathbf{I} + \bar{\mathbf{F}}\right)\mathbf{dx} = \mathbf{F}\mathbf{dx}$$

$$(\delta \mathrm{l})^2 = (\mathbf{dr})^T(\mathbf{dr}) = (\mathbf{dx})^T\mathbf{F}^T\mathbf{F}(\mathbf{dx}) = (\mathbf{dx})^T\left(\mathbf{I}+\bar{\mathbf{F}}\right)^T\left(\mathbf{I}+\bar{\mathbf{F}}\right)(\mathbf{dx}) =$$
$$(\mathbf{dx})^T\left(\mathbf{I} + \bar{\mathbf{F}}^T\bar{\mathbf{F}} + \bar{\mathbf{F}}^T + \bar{\mathbf{F}}\right)(\mathbf{dx})$$

$$(\delta \mathrm{l})^2 - (\delta \mathrm{l}^0)^2 = 2(\mathbf{dx})^T\left(\bar{\mathbf{F}}^T + \bar{\mathbf{F}} + \bar{\mathbf{F}}^T\bar{\mathbf{F}}\right)(\mathbf{dx}) = 2(\mathbf{dx})^T\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)(\mathbf{dx}) = 2(\mathbf{dx})^T\boldsymbol{\epsilon}_{GL}(\mathbf{dx})$$

Measure of how much differential length deformed

Strain Components

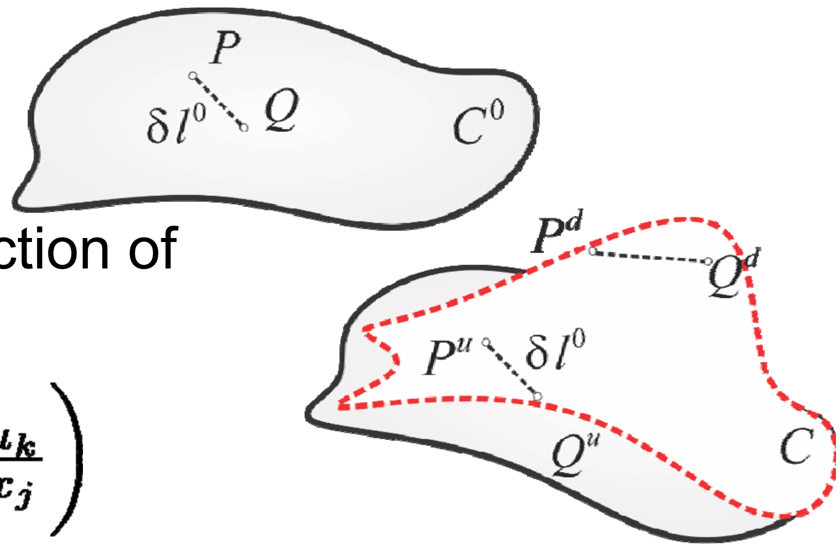
$$(\delta l)^2 - (\delta l^0)^2 = 2(\mathrm{d}\mathbf{x})^T \left(\bar{\mathbf{F}}^T + \bar{\mathbf{F}} + \bar{\mathbf{F}}^T\bar{\mathbf{F}}\right)(\mathrm{d}\mathbf{x}) = 2(\mathrm{d}\mathbf{x})^T \left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)(\mathrm{d}\mathbf{x}) = 2(\mathrm{d}\mathbf{x})^T \varepsilon_{GL}\,(\mathrm{d}\mathbf{x})$$

- Green-Lagrange strain tensor

$$\varepsilon_{GL} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$$

- Components of the tensor as a function of displacement gradients

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$



What can we tell from the expression above?

Strain Components

- Green-Lagrange strain tensor

$$\varepsilon_{GL} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right), \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

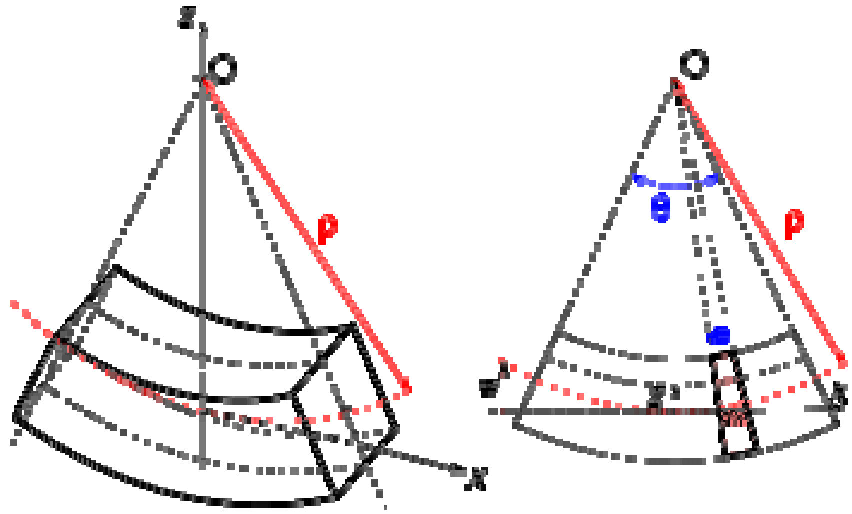
- Features
  - It is nonlinear in terms of displacement gradients
    - Can capture moderate deformation: Finite strain
  - Can be truncated (linearized), small deformation, infinitesimal strain
  - Defines three normal strains (one direction) and three shear strains (angle between two directions)

$$\varepsilon_{GL} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

  - It can be rearranged in vector form using 3 stretches (normal strains) and 3 shear strains. Note matrix symmetry

Strain Components: Structural Mechanics

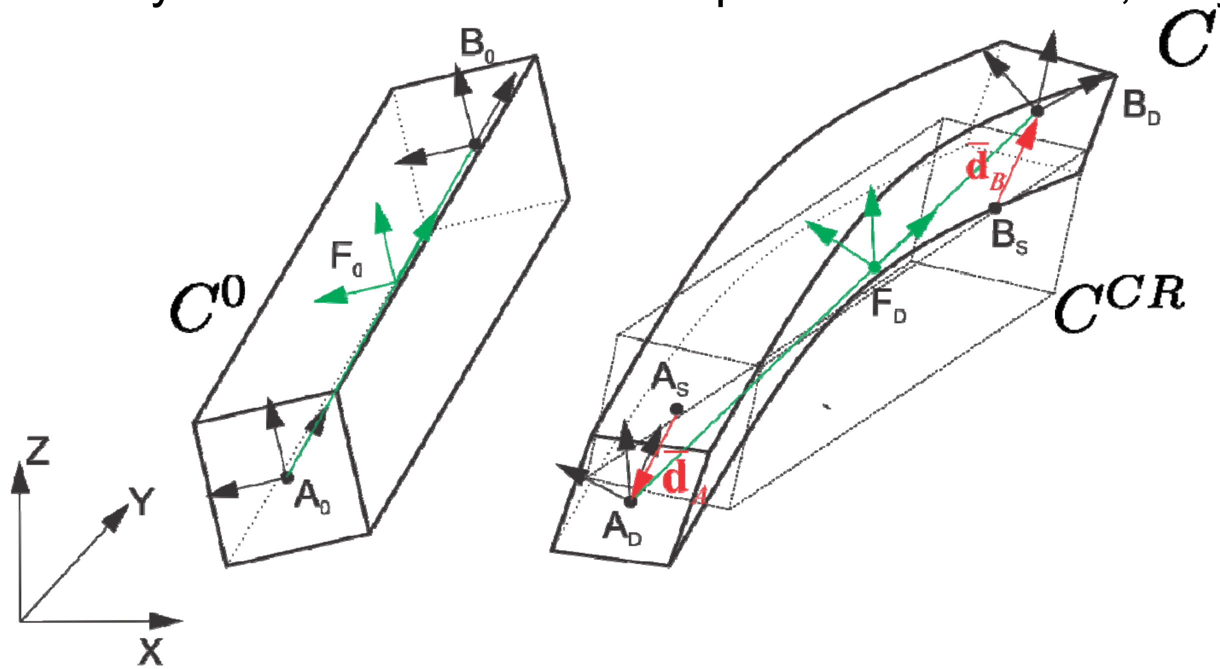
- Green-Lagrange strain tensor
  - does not consider simplifications in the geometry of the solid
  - uses 6 general strain components
  - In other words, **continuum mechanics approach**
- Structural mechanics
  - Simplifies solid's kinematics to take advantage of characteristic geometries
  - Beams, shells, and plates are paradigms of these simplifications – **structural approach**
  - Structural mechanics often involves the use of curvature



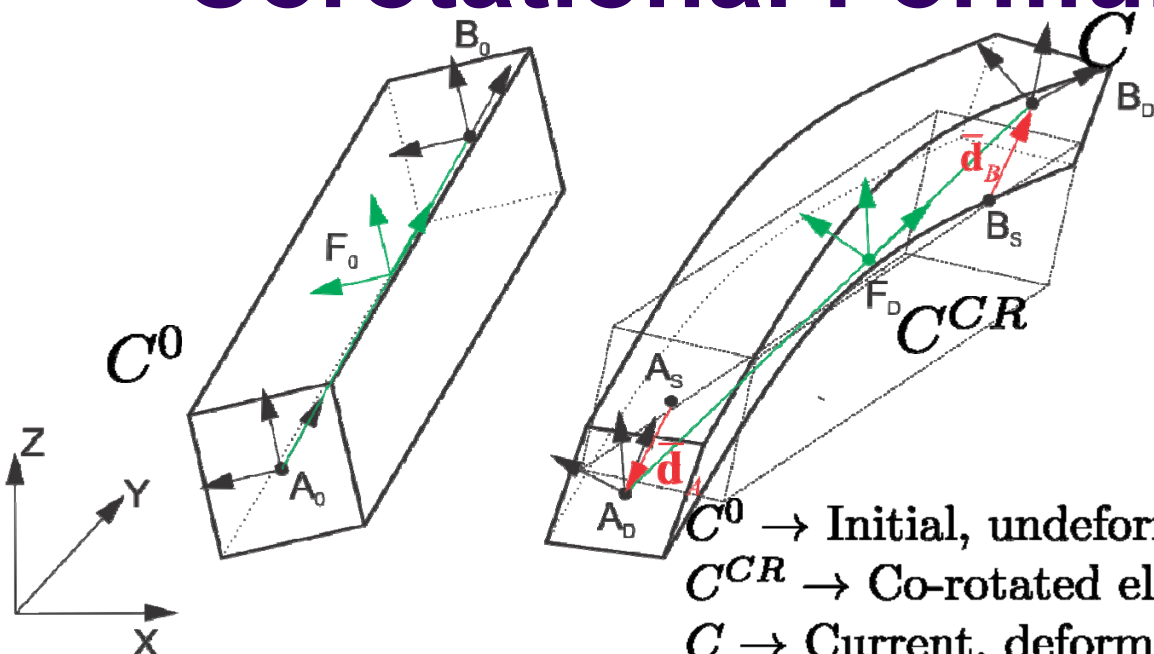
Deformed beam –structural mechanics. Source: Wikipedia

Corotational Formulation

- In a nutshell:
  - Each finite element has a frame of reference associated with it
    - This frame describes base rotations and translations –rigid body-style
  - Based on linear finite elements –infinitesimal deformation
  - The element frame of reference absorbs rigid body motion and allows defining infinitesimal deformation with respect to the element
  - We will analyze this formulation on a per-element basis, only a beam elem.



Corotational Formulation



$C^0 \rightarrow$ Initial, undeformed configuration
$C^{CR} \rightarrow$ Co-rotated element frame (rigid body motion)
$C \rightarrow$ Current, deformed configuration

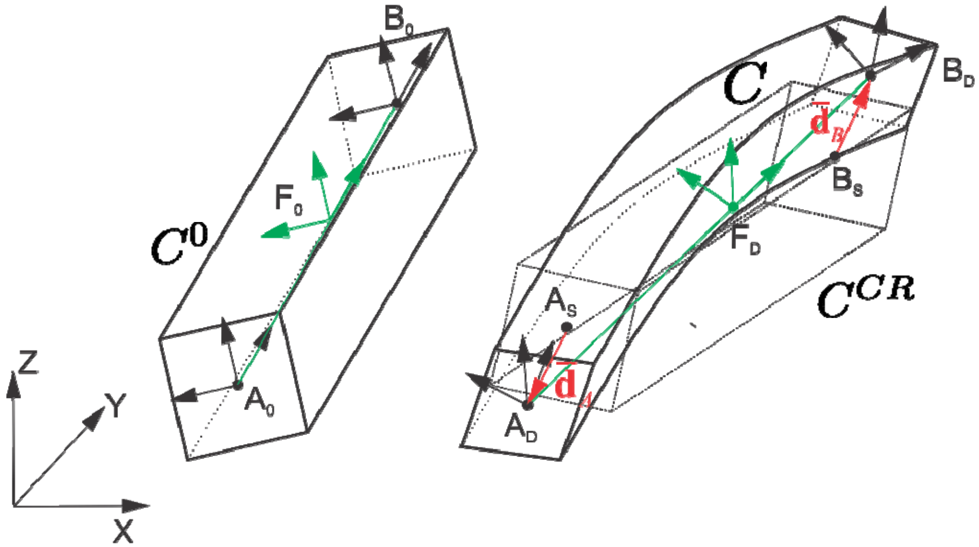
- The motion of a node i of a beam element is defined by the position vector $\boldsymbol{x}_i$ and a set of quaternions $\boldsymbol{\rho}_i$ of a reference frame.

- The vector $\boldsymbol{\rho}_i$ captures the rigid body rotation and deformation rotation of the beam cross section at node i. The state of a system with n nodes is, therefore, $\boldsymbol{s} = [\boldsymbol{q}, \boldsymbol{v}]$

- $\boldsymbol{q} = [\boldsymbol{x}_1,\ \boldsymbol{\rho}_1, \boldsymbol{x}_2,\ \boldsymbol{\rho}_2, ..., \boldsymbol{x}_3,\ \boldsymbol{\rho}_3] \in \mathbb{R}^{(3+4)n}$ and $\boldsymbol{v} = [\boldsymbol{v}_1,\ \bar{\boldsymbol{\omega}}_1, \boldsymbol{v}_2,\ \bar{\boldsymbol{\omega}}_2, ..., \boldsymbol{v}_3,\ \bar{\boldsymbol{\omega}}_3] \in \mathbb{R}^{(3+3)n}$.

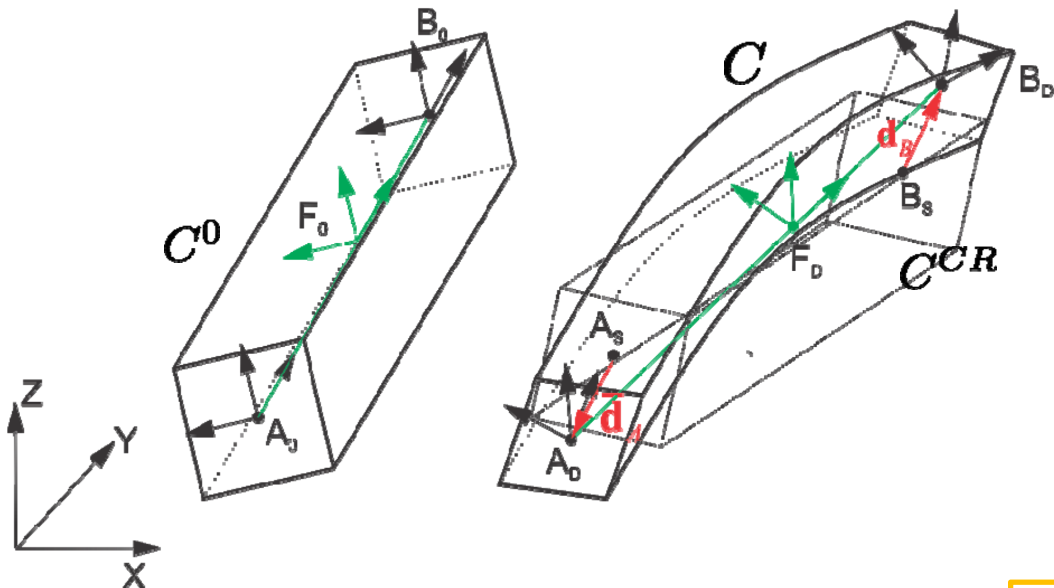
Global coordinates

Corotational Formulation



- When an element j moves, the position and rotation of the floating frame, $\langle \mathbf{F} \rangle$, are updated. The origin of $\langle \mathbf{F} \rangle$ is placed at the element's midpoint $\boldsymbol{x}_F = 1/2\,(\boldsymbol{x}_B - \boldsymbol{x}_A)$.
- The floating (shadow) frame's longitudinal axis $\boldsymbol{X}$ is aligned with the vector $\boldsymbol{x}_B - \boldsymbol{x}_A$, whereas the $\boldsymbol{Y}$ and $\boldsymbol{Z}$ axes are obtained via a Gram-Schmidt orthogonalization.
- The rotation matrix and unit quaternion of $\langle \mathbf{F} \rangle$ will be denoted as $\boldsymbol{R_F}$ and $\boldsymbol{\rho_F}$, respectively.

Corotational Formulation



- One can compute the local displacements of a node i as $\bar{\boldsymbol{d}}_i = \bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_{i_0} = \boldsymbol{R}_{\boldsymbol{F}}^{\mathrm{T}}(\boldsymbol{x}_{i_0} - \boldsymbol{x}_{\boldsymbol{F}_0}) - \boldsymbol{R}_{\boldsymbol{F}}^{\mathrm{T}}(\boldsymbol{x}_i - \boldsymbol{x}_{\boldsymbol{F}})$, where the bar over displacement quantities describes locality, and the subscript 0 refers to the initial configuration.

Local displacements

- The local rotation of the nodes can be obtained in terms of rotation matrices for nodes A and B as $\bar{\mathbf{R}}_A = \mathbf{R}_F^T \mathbf{R}_A \mathbf{R}_{A0}^T$ and $\bar{\mathbf{R}}_B = \mathbf{R}_F^T \mathbf{R}_B \mathbf{R}_{B0}^T$

Possible initial relative rotation between node and reference frame

- The local infinitesimal angles may be obtained as $\bar{\boldsymbol{\theta}}_A = \bar{\theta}_A \mathbf{u}_A$ and $\bar{\boldsymbol{\theta}}_B = \bar{\theta}_B \mathbf{u}_B$. We know that a quaternion can be written as $\boldsymbol{\rho} = [\cos(\theta/2), \sin(\theta/2)\mathbf{u}]$. To compute the local rotation vectors, $\bar{\theta}_A = 2\arccos(\Re(\bar{\boldsymbol{\rho}}_A))$, $\mathbf{u}_A = \frac{1}{\sin(\theta_a/2)}\Im(\bar{\boldsymbol{\rho}}_A)$

Infinitesimal rotations for elastic forces

Corotational Formulation

Nodal quaternions can be used to describe linearized angles for the definition of local strains. A unit quaternion $(e_0, \mathbf{e})$, where $\mathbf{e} = (e_1, e_2, e_3)$, may be written as

$$\boldsymbol{\rho} = \left[\cos\left(\theta/2\right), \quad \mathbf{n}\sin\left(\theta/2\right)\right], \tag{1}$$

with the rotation angle θ being $\theta = \arccos\left(2e_0^2 - 1\right)$, and the vector defining the axis of rotation as

$$\mathbf{n} = 2\mathbf{e}e_0 / \sin\theta. \tag{2}$$

Note that Equations (1)-(2) link Euler parameters to Euler's Rotation Theorem, which expresses any three-dimensional rotation as a finite rotation about a single axis $\mathbf{n}$. At this point, we can compute the linearized angles as $\bar{\boldsymbol{\theta}}_i = \theta_i \mathbf{n}_i$. For nodes A and B of a beam element, the vector of local deformations is $\bar{\mathbf{d}}_{12\times1} = \left[\bar{\mathbf{d}}_A, \bar{\boldsymbol{\theta}}_A, \bar{\mathbf{d}}_B, \bar{\boldsymbol{\theta}}_B\right]$. The local stiffness matrix, $\bar{\mathbf{K}}_{12\times12}(\bar{\mathbf{d}})$, and the local internal force vector, $\bar{\mathbf{f}}_{\text{in}} = \bar{\mathbf{K}}(\bar{\mathbf{d}})$, are then mapped onto the global frame of reference by introducing new matrices and building projectors.