Mean Value Theorems - GATE Study Material in PDF

The Mean Value Theorems are some of the most important theoretical tools in **Calculus** and they are classified into various types. In these **free GATE Study Notes**, we will learn about the important **Mean Value Theorems** like Rolle's Theorem, Lagrange's Mean Value Theorem, Cauchy's Mean Value Theorem and Taylor's Theorem.

This GATE study material can be **downloaded as PDF** so that your GATE preparation is made easy and you can ace your exam. These study notes are important for GATE EC, GATE EE, GATE ME, GATE CE and GATE CS. They are also important for IES, BARC, BSNL, DRDO and the rest.

Before you get started though, go through some of the other Engineering Mathematics articles in the reading list.

Recommended Reading –

Types of Matrices

Properties of Matrices

Rank of a Matrix & Its Properties

Solution of a System of Linear Equations

Eigen Values & Eigen Vectors

Linear Algebra Revision Test 1

Laplace Transforms

Limits, Continuity & Differentiability

Rolle's Theorem

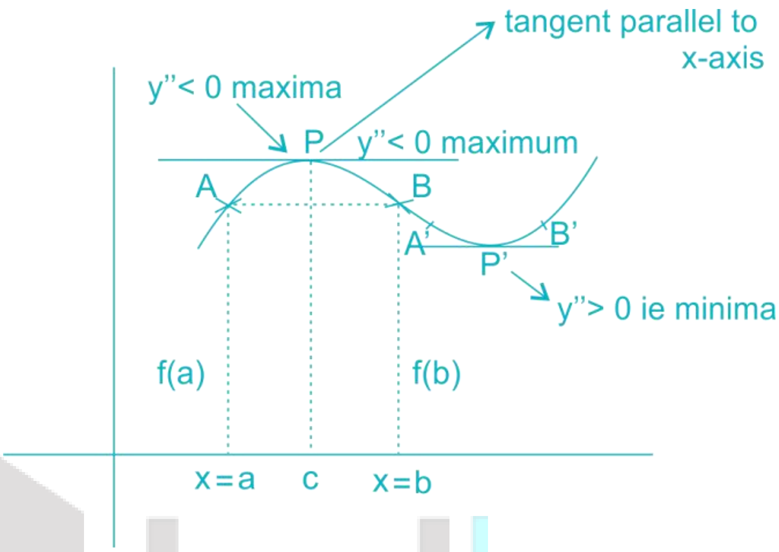
Statement: If a real valued function f(x) is

1. Continuous on [a,b]
2. Derivable on (a,b) and f(a) = f(b)

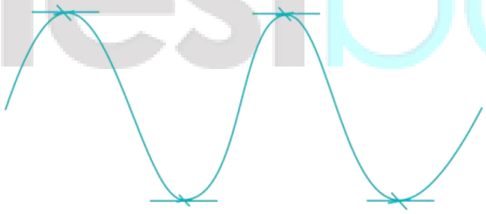
Then there exists at least one value of x say c ϵ (a,b) such that $f'(c) = 0$.

Note:

1. Geometrically, Rolle's Theorem gives the tangent is parallel to x-axis.



2. For a continuous curve maxima and minima exists alternatively.



3. Geometrically y'' gives concaveness i.e.

i. $y'' < 0 \Rightarrow$ Concave downwards and indicates maxima.

ii. $y'' > 0 \Rightarrow$ Concave upwards and indicates minima.

To know the maxima and minima of the function of single variable Rolle's Theorem is useful.

5. $y''=0$ at the point is called point of inflection where the tangent cross the curve is **4.** called point of inflection and

6. Rolle's Theorem is fundamental theorem for all Different Mean Value Theorems.

Example 1:

The function is given as $f(x) = (x-1)^2(x-2)^3$ and $x \in [1,2]$. By Rolle's Theorem find the value of c is -

Solution:

$f(x) = (x-1)^2(x-2)^3$

$f(x)$ is continuous on $[1,2]$ i.e. $f(x)$ = finite on $[1,2]$

$f'(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2$

$f'(x)$is finite in $(1,2)$ hence differentiable then $c \in (1,2)$

$\therefore f'(c) = 0$

$2(c-1)(c-2)^3 + 3(c-1)^2(c-2)^2 = 0$

$(c-1)(c-2)^2[2c - 4 + 3c - 3] = 0$

$(c-1)(c-2)^2[5c-7] = 0$

$\therefore c = \frac{7}{5} = 1.4 \in (1,2)$

Lagrange's Mean Value Theorem

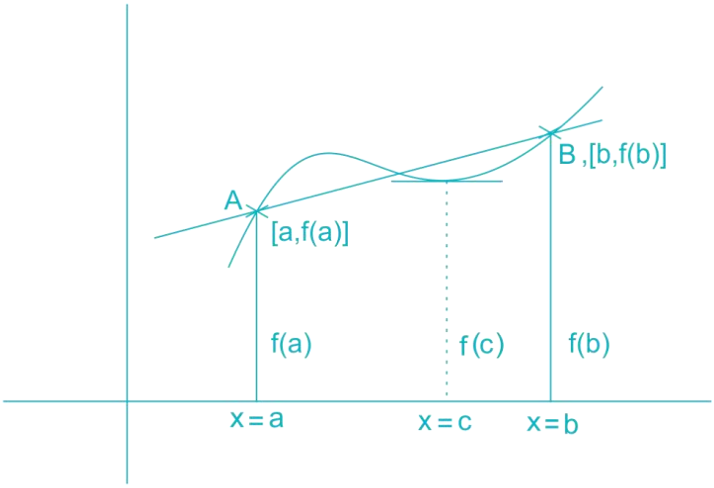
Statement: If a Real valued function f(x) is

1. Continuous on [a,b]
2. Derivable on (a,b)

Then there exists at least one value $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Note:

Geometrically, slope of chord AB = slope of tangent



Application:

1. To know the approximation of algebraic equation, trigonometric equations etc.
2. To know whether the function is increasing (or) decreasing in the given interval.

Example 2:

Find the value of c is by using Lagrange's Mean Value Theorem of the function

$f(x) = x(x-1)(x-2)\ x \in \left[0, \frac{1}{2}\right]$.

Solution:

f(x) is continuous in [0, 1/2] and it is differentiable in (0, 1/2)

$f'(x) = (x^2 - x)[1] + (x - 2)(2x - 1)$

$= x^2 - x + 2x^2 - x - 4x + 2 = 3x^2 - 6x + 2$

From Lagrange's Mean Value Theorem we have,

$f'(c) = 3c^2 - 6c + 2 = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2}} = \frac{3}{4}$

$12c^2 - 24c + 8 - 3 = 0$

$12c^2 - 24c + 5 = 0$

$\Rightarrow c = \frac{24 \pm \sqrt{576 - 240}}{24}$

$c = 1 \pm \frac{\sqrt{21}}{6}$

$\therefore c = 1 - \frac{\sqrt{21}}{6} \epsilon \left(0, \frac{1}{2}\right)$

Cauchy's Mean Value Theorem

Statement: If two functions f(x) and g(x) are

1. Continuous on [a,b]
2. Differentiable on (a,b) and g'(x) ≠ 0 then there exists at least one value of x such that c ∈ (a,b)

$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

Generally, Lagrange's mean value theorem is the particular case of Cauchy's mean value theorem.

Example 3:

If $f(x) = e^x$ and $g(x) = e^{-x}$, $x \epsilon [a,b]$. Then by the Cauchy's Mean Value Theorem the value of c is

Solution:

Here both $f(x) = e^x$ and $g(x) = e^{-x}$ are continuous on [a,b] and differentiable in (a,b)

From Cauchy's Mean Value theorem,

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$

$e^{2c} = e^{a+b} \Rightarrow \quad c = \frac{a+b}{2}$

Therefore, c is the arithmetic mean of a and b.

Taylor's Theorem

It is also called as higher order mean value theorem.

Statement: If $f^n(x)$ is

1. Continuous on [a, a + x] where x = b – a

2. Derivable on (a, a + x)

Then there exists at least one number $\theta \in (0,1)$ $(1-\theta \neq 0)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n \quad \text{\_\_\_(1)}$$

Where R_n = Lagrange's form of remainder $= \frac{h^n}{n!}f^n(a+\theta h)$

Also Cauchy's form of remainder $R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$

Note:

Substituting a = 0 and h = x in equation (1) (Taylor's series equation) we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + R_n$$

This is known as Maclaurin's series.

Here $R_n = \frac{x^n}{n!}f^n(\theta x)$ is called Lagrange's form of remainder.

$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^n(\theta x)$ is called Cauchy's form of remainder

Example 4:

Find the Maclaurin's Series expansion of e^x

Solution:

Let, $f(x) = e^x$

$f'(x) = e^x$, $f''(x) = f'''(x) = \ldots f^x(x) = e^x$

By Maclaurin's Series expansion,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \frac{x^n}{n!}f^n(x)$$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

Note

The Maclaurin's series expansion for various functions is given as

1. $\sin x = x - \frac{x^2}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^a + \cdots$
4. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots.$
4. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots$
5. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$
6. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$
7. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$
8. $\log(1 + \sin x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$
9. $\log(1 - \sin x) = -x - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \cdots$

Did you like this article on Mean Value Theorems? Let us know in the comments? You may also like the following articles –

Try out Calculus on Official GATE 2017 Virtual Calculator

Recommended Books for Engineering Mathematics

40+ PSUs Recruiting through GATE 2017

Differentiation