Mean-Variance Portfolio Optimization: Challenging the role of traditional covariance estimation

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Master of Science Thesis Stockholm, Sweden 2016 Effektiv portföljförvaltning: en utvärdering av metoder för kovariansskattning

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Abstract

Ever since its introduction in 1952, the Mean-Variance (MV) portfolio selection theory has remained a centerpiece within the realm of efficient asset allocation. However, in scientific circles, the theory has stirred controversy. A strand of criticism has emerged that points to the phenomenon that Mean-Variance Optimization suffers from the severe drawback of estimation errors contained in the expected return vector and the covariance matrix, resulting in portfolios that may significantly deviate from the true optimal portfolio.

While a substantial amount of effort has been devoted to estimating the expected return vector in this context, much less is written about the covariance matrix input. In recent times, however, research that points to the importance of the covariance matrix in MV optimization has emerged. As a result, there has been a growing interest whether MV optimization can be enhanced by improving the estimate of the covariance matrix.

Hence, this thesis was set forth by the purpose to investigate whether financial practitioners and institutions can allocate portfolios consisting of assets in a more efficient manner by changing the covariance matrix input in mean-variance optimization. In the quest of achieving this purpose, an out-of-sample analysis of MV optimized portfolios was performed, where the performance of five prominent covariance matrix estimators were compared, holding all other things equal in the MV optimization. The optimization was performed under realistic investment constraints, taking incurred transaction costs into account, and for an investment asset universe ranging from equity to bonds.

The empirical findings in this study suggest one dominant estimator: the covariance matrix estimator implied by the Gerber Statistic (GS). Specifically, by using this covariance matrix estimator in lieu of the traditional sample covariance matrix, the MV optimization rendered more efficient portfolios in terms of higher Sharpe ratios, higher risk-adjusted returns and lower maximum drawdowns. The outperformance was protruding during recessionary times. This suggests that an investor that employs traditional MVO in quantitative asset allocation can improve their asset picking abilities by changing to the, in theory, more robust GS covariance matrix estimator in times of volatile financial markets.

Keywords: portfolio allocation, mean-variance optimization, efficient frontier, covariance matrix, estimation error, optimization enigma, random matrix theory, shrinking, robust statistics

Mean-Variance Portfolio Optimization: Challenging the role of traditional covariance estimation

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Preface

This study was conducted during the spring of 2016 at the Royal Institute of Technology (KTH). I would like to express my sincere gratitude to Tomas Sörensson at the Royal Institute of Technology for providing valuable insights and support during the process of writing this thesis. In addition, I would like to thank my fellow students at KTH Royal Institute of Technology who have reviewed my work and contributed to the process of writing this thesis.

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Chapter 1 Introduction

This chapter aims to provide a background of the phenomenon under study as well as to introduce the reader to the problem that constitutes the focal point of this thesis.

1.1 Background

Financial researchers and practitioners have long been interested in ways of allocating various assets in an efficient manner. In this setting, an efficient portfolio refers to a portfolio that yields the highest possible return given a certain level of risk that the investor is willing to take. Naturally, such portfolios are appealing to portfolio managers around the world and the existing body of knowledge includes a significant amount of research on this matter, which to a large extent is dominated by quantitative models.

Out of these quantitative models, one particular approach is protruding and prominent: Mean-Variance Optimization (MVO), introduced in a groundbreaking article published in 1952 by Markowitz (1952) for which he later was awarded the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (KVA 1990). The publication proved to become a cornerstone in Modern Portfolio Theory (MPT) and an important stepping stone towards the creation of further financial models such as the Capital Asset Pricing Model (CAPM), developed by Sharpe (1964). Some purists of modern financial economics go as far as claiming that the publication by Markowitz was "the moment of birth of modern financial economics", as exemplified by Rubinstein (2003).

The article by Markowitz (1952) can be seen as a reaction to previous and existing research at that time, which to a large extent employed the law of large numbers theorem by Bernoulli, leading to conclusions that all risk could be diversified away. Markowitz, however, did not share this assertion. Instead, he claimed that the law of large numbers was not applicable to a portfolio of securities, partly due to the prevalent interdependency and complexity in financial markets. In other words, the inter-correlation between financial securities implies that diversification can not entirely eliminate risk according to Markowitz. It is from this assertion that the essence of Markowitz's revolutionary theory stems, i.e. the existence of a trade-off relationship between return and variance where variance is perceived as a measure of risk.

An underlying assumption of MVO is that investors are risk averse and rational. Thus, an investor will always select a portfolio associated with less variance, ceteris paribus, and the choice of the portfolio is solely based on the relationship between expected return and variance. According to Markowitz (1952), the portfolio selection process is divided into two stages:

- 1. Parameter estimation: from historical observations and beliefs, one forms estimations of future performance (in terms of return and variance) of the specified universe of securities.
- 2. Portfolio selection: employing the estimated parameters in the first stage, one choose an efficient portfolio of securities. The security weights of the portfolio is obtained by solving an optimization problem that is in line with the investor's preferences.

In Markowitz's pivotal publication in 1952, he was primarily interested in the portfolio selection stage. In greater detail, this stage is related to his main proposal that an investor solely should consider efficient portfolios. Recall that an efficient portfolio refers to a portfolio with a maximum expected return for a given variance or less, or conversely, a portfolio with minimum variance for a given expected return or more. In order to obtain efficient portfolios, Markowitz (1952) presents a corresponding optimization problem, with the following mean-variance (MV) objective function:

$$w^{\top}\mu - \frac{\delta}{2}w^{\top}\Sigma w \tag{1.1.1}$$

where w is a vector of portfolio weights, μ is a vector of expected returns for a set of assets and Σ is a corresponding covariance matrix of asset returns. Furthermore, δ denotes the coefficient of risk aversion, which controls the extent of how much additional risk is penalized. In short, the objective function represents a trade-off between the expected return of a portfolio and its expected variance. Markowitz (1952) showed that by solving this problem for different values of the risk aversion coefficient, a set of efficient portfolios is obtained. These portfolios constitute what Markowitz's refers to as the efficient frontier and are the only portfolios that an investor should consider in the context of portfolio selection.

During the years following the publication of Markowitz's original work in 1952, his contribution to the modern portfolio theory has been eminent. Despite decades of research and debate, Markowitz's mean-variance MPT has remained the cornerstone of portfolio selection methods. It is up to this state of time still topical and widely employed by financial practitioners, in particular by portfolio managers (Tu and Zhou 2011). However, since the emergence of the MPT in 1952 and as economic and mathematical theory have progressed, several critics have questioned the original work of Markowitz. While many agree that Markowitz's mean-variance MPT is an important theoretical advance, questions have been raised regarding the first stage of Markowitz's portfolio selection process, i.e. the stage where one estimates the expected return vector, μ , and the covariance matrix, Σ , both serving as essential inputs for the MVO problem. This was, and still is, a universally encountered problem regarding the application of traditional MPT.

There is, however, an extensive literature devoted to the above mentioned critique. In greater detail, within the realm of modern portfolio theory, the importance of the covariance matrix has historically been overshadowed by the expected return parameter. This has led to several proposals put forward regarding how one should estimate the expected return vector, some of which are illuminated below:

• In 1992, Black and Litterman (1992) published their newly developed model: the Black-Litterman model. This model stems from an equilibrium assumption that the global market portfolio is well diversified and efficient, serving as the neutral initial stage of the approach. Thereafter, through the use of a reverse optimization process, the model derives returns of assets implied by the market portfolio. In addition, an investor may convey agreement or disagreement with the returns implied by the market, as they may claim expertise in value investment that differs from the market consensus. Following the procedure of Black and Litterman (1992), a vector consisting of the forecasted expected returns for a considered asset universe can then be obtained. • In addition to the Black-Litterman model, Sharpe (1964) and Fama and French (1993) developed the CAPM and the Fama-French three-factor model respectively. In short, these are models that attempt to describe asset returns and aid in estimating expected returns.

The above models merely capture a glimpse of the existing body of literature on this matter. In recent times, however, research that point to the importance of the covariance matrix estimation in MVO has emerged. Thus, a demand for the relative performance between different methods in estimating the covariance matrix has surged among financial practitioners. While the existing body of knowledge regarding the estimation of expected returns is extensive, the covariance matrix (Σ) estimation, which is an essential input in MVO, has arguably been overshadowed. Thus, in comparison to the return parameter, less is known regarding the performance of MVO when the way of estimating the covariance matrix is varied, ceteris paribus. In a sense, there exists a gap in the existing literature regarding this phenomenon, a gap which this thesis aims to reduce.

1.2 Problematization

In the search for optimal portfolio allocations, investment managers have traditionally relied on MPT as introduced by Markowitz in 1952. However, finding optimal allocations in MPT requires estimation of covariances as well as expected returns of the assets included in the portfolio optimization. Additionally, since portfolio optimization is dependent on expectations about stochastic phenomena, the process of optimizing portfolios is prone to estimation error (Elton, Gruber, Brown, et al. 2014).

It is well known that estimation errors of the input parameters, the expected return vector and the covariance matrix, has a vital impact on the output of MV optimization. In other words, the resulting portfolios obtained from the solution of the MVO problem are sensitive to the choice of inputs. Hence, the out-of-sample performance of these optimized portfolios is strongly dependent on the estimation accuracy of the input parameters. Michaud (1989) coined this phenomenon as *"Markowitz optimization enigma"*, where he claims that MV optimized portfolios are unfeasible in practice due to their susceptibility to estimation errors.

Despite these drawbacks, MV optimization continues to be a prominent method for portfolio selection among investment managers as more accurate estimation techniques regarding expected returns have emerged since the assertion by Michaud (1989). These include the Fama and French (1993) response to CAPM and the Black and Litterman (1992) procedure that combines CAPM with unique investor views.

However, research regarding estimation techiques for the covariance matrix input in MVO has not been subject to the same level of attention until recently where Gerber, Markowitz, and Pujara (2015) introduced a, supposedly, more robust co-movement measure as a replacement for historical correlation in the estimation of the covariance matrix. Gerber et al. (2015) coin this measure as the Gerber Statistic (GS) and their main findings suggest that MV optimized portfolios using GS as a substitute to historical correlation in the covariance matrix estimation consistently outperformed portfolios using the traditional estimation technique. More specifically, their results indicate that the entire efficient frontier can be raised upward, thus resulting in higher Sharpe ratio portfolios being attainable.

The findings by Gerber et al. (2015) are indeed interesting and can result in significant monetary benefits for investment managers if deemed valid and reliable. More importantly, the research poses a problem (or an opportunity) regarding whether financial practitioners that today employ the traditional covariance matrix estimation technique should alter their quantitative models. Clearly, the choice of estimation technique for the covariance matrix affect the MV optimized portfolios. The problem is that there is an inadequate body of research regarding the effect of out-of-sample performance when solely varying the estimation technique for the covariance matrix. In other words, there is not a strong foundation regarding this field and investment managers are most likely not willing to risk capital based on frontier research that has not yet proven to stand up under close scrutiny.

1.3 Purpose, research questions and expected contribution

This thesis addresses investors that apply Markowitz's mean-variance modern portfolio theory in their quantiative models.

The purpose of this thesis is to investigate if portfolio managers that employ traditional mean-variance optimization in practice can improve their asset-picking abilities by altering the estimation technique regarding the covariance matrix. In addition, this thesis attempts to examine whether the relative forecasting performance of various covariance matrix estimators can be tied to the prevailing market regime.

The study will attempt to achieve the purpose by answering the following research questions:

RQ 1: How does the choice of technique on how to estimate the covariance matrix affect the out-of-sample performance of mean-variance optimization?

RQ 2: How does the prediction accuracy differ between the traditional sample covariance matrix and alternative estimators, and can the relative performance be tied to the prevailing market dynamics?

In addition to providing insight for financial practitioners that employ MVO in their quantitative models, I expect this thesis to add to the current knowledge regarding the robustness of different covariance estimation techniques. This knowledge can be applied to fields that are not necessarily strictly related to modern portfolio theory. Examples of such fields are the insurance industry and machine learning.

1.4 Disposition of the thesis

- Chapter 2, Literature Review, provides a literature review of relevant research within the field of portfolio optimization. This chapter further motivates the theoretical framework and methodology employed in this thesis.
- Chapter 3, Theoretical Framework, introduces the underlying theory that underlie this thesis. It starts off by presenting some preliminaries, followed by a derivation of the portfolio optimization problem considered in this thesis. Lastly, theory regarding the estimation process of the expected return vector and the covariance matrix is presented.
- Chapter 4, Methodology, presents the general methodology employed for investigating the research questions in this thesis.
- Chapter 5, Results, provides the empirical findings in this study.
- Chapter 6, Discussion, consists of a discussion regarding the empirical findings. In addition, the validity of the thesis is challenged here.
- Chapter 7, Conclusion, concludes the thesis and presents some suggestions for further research.

Chapter 2

Literature Review

This chapter will provide an extensive review of literature associated with portfolio optimization. Essentially, this chapter motivates the theoretical framework and methodology employed in this thesis.

2.1 Portfolio allocation

Almost 65 years have passed since Markowitz (1952) pioneered the use of mean-variance optimization in the context of portfolio management, where he quantified the concept of diversification by employing the notions of return, volatility and covariance. Markowitz's (1952) seminal work has played a prominent role in modern portfolio theory and has been widely debated in the literature. Despite the fact that Markowitz became a Nobel laureate for the aforementioned work, the framework has stirred controversy in scientific circles and has been subject to criticism which challenges the validity of his proposed portfolio allocation model. Prior to illuminating the criticism in greater detail, it is important to understand the concept of mean-variance optimization and how it is composed of several tethered components. In doing so, the literature review can be divided into several parts, addressing different aspects of the mean-variance framework. The following Figure 2.1 provides an overview of the portfolio selection process:



Figure 2.1: The MPT investment process. Illustration inspired by Fabozzi, Gupta, and Markowitz (2002).

The node regarding model applicability is an extension of the original representation of the MPT investment process described by Fabozzi, Gupta, and Markowitz (2002). Its appearance in this thesis may be well motivated by tracing back to events where ignorance towards model applicability resulted in catastrophic outcomes. Such an event can be found by recollecting the work by Li (1999) who pioneered the use of Gaussian copulas for predicting the performance of collateralized debt obligations (CDOs). In the years following his work, Li's model became deeply entrenched within the financial industry - in fact, it became so deeply entrenched that warnings about its limitations and applicability were ignored by the practitioners. Moving forward to the crisis of 2008, when the financial system's foundation was severely ruptured, the financial environment had altered in such magnitude that Li's model could not anticipate. Hence, the model became a recipe for disaster and has been partly credited to blame for laying the global banking system in serious peril (Jones 2009). Already in 2005, Li warned the practitioners that employed the model without being aware of the underlying assumptions of the formula. In Whitehouse (2005, p.2), Li stated that:

'The most dangerous part is when people believe everything coming out of it.'

In this context, Derman and Wilmott (2009) discuss the concept of model awareness within the field of finance. They appreciate the simplicity in financial models, but hasten to assert that while models are simple, reality is not. In other words, in the essence of models lies that they do not perfectly mirror the reality. Confusing an illusion, which is what a model essentially is, with reality can be a recipe for disaster. Moreover, Derman and Wilmott (2009, p.2) claim:

'The most important question about any financial model is how wrong it is likely to be, and how useful it is despite its assumptions.'

They further argue that, in stark contrast to true laws that may be found in the field of physics, financial models are more fragile systems. The motivation behind this assertion is that the world of finance is profoundly connected to human behavior which is too complex to entirely capture in a simplified model. Thus, in contrast to true laws such as Newton's law of gravity found in the field of physics, there are no fundamental laws of finance - and even if there were, it would not be possible to verify them through repeated experiments. Hence, Derman and Wilmott (2009) argue that it is of utmost importance to be aware of the subtleties associated with a quantitative model and not to confuse its illusion with reality. Knowing what is assumed and what is swept out of view in a model is crucial.

Having illuminated the importance of being aware of the assumptions underpinning a model, the focus of the literature review is now shifted towards the cornerstone model in this thesis: the mean-variance framework. Fabozzi, Kolm, et al. (2007) note that a common misunderstanding that is prevalent in the literature is that Markowitz's mean-variance framework relies on the assumption that security returns are jointly normally distributed. Fabozzi, Kolm, et al. (2007) argue, however, that the mean-variance approach is consistent with the assumption of joint normality and that the misconception may stem from this relationship. The basic assumption and principle of the mean-variance framework may be found in a wide range of textbooks and articles, such as inter alia Markowitz (1959) and Fabozzi, Kolm, et al. (2007) and follows as:

- The underlying assumption for the MPT mean-variance model is that an investor's preferences can be captured by a utility function of the following two moments of portfolio returns: the expected return and the variance of the portfolio.
- The principle underpinning mean-variance optimization is that for a given level of expected return, a rational investor will choose a portfolio associated with a minimum amount of variance amongst the feasible set of portfolios.

Markowitz (1959) argue that these aspects underpin the mean-variance framework under the umbrella of portfolio allocation. Following the introduction of MPT mean-variance optimization, decades of debate and research on the subject have lead to an ambiguous academic support for the framework. While the approach has found a widespread acceptance among financial practitioners (Tu and Zhou 2011), the framework has been subject to a large extent of criticism in the academic circles for not matching the real world in many ways (Feldstein (1969); Rockafellar and Uryasev (2000)).

The first strand of criticism that will be illuminated relates to the concept of employing variance as a proxy for risk. Hult et al. (2012) provide a lucid example that shed light on one of the shortcomings with employing variance as a risk measure. Noting that the following description may be considered as a parsimonious version of the mentioned example, Hult et al. (2012) remark that location (mean) and dispersion (variance) are reasonable measures of probable reward and risk, respectively, under the condition that the return is approximately normally distributed. They further assert that since variance is a full domain measure that quantifies a range of likely deviations from the mean, it may inaccurately describe the riskiness of a position if the return is e.g. asymmetrically distributed. This is illustrated in Figure 2.2:



Figure 2.2: Density functions for a random variable R with E[R] = 1.1 and $Var(R) = 0.3^2$. For the left plot, R is N(1.1,0.3²)-distributed. For the right plot, the distribution comes from a two point mixture of normal distributions.

Having the same mean and variance, both profiles in Figure 2.2 are identical from a meanvariance perspective. However, from a downside risk perspective, the riskiness of the positions is inherently nonequivalent due to the asymmetric display of the density function to the right. As the variance symmetrically accounts for deviations from the mean, it fails to adequately discriminate between return distributions (Grootveld and Hallerbach 1999).

Research by, inter alia, Post and Van Vliet (2005) and Ang, Chen, and Xing (2006) suggest that investors assign greater importance to downside risk as opposed to upside risk which they view favorably. To cope with the shortcomings of employing variance as a risk measure in this regard, several downside risk measures have emerged throughout the literature, leading to several offsprings of the MPT mean-variance framework. Arguably, Value-at-Risk (VaR), popularized by J.P. Morgan's RiskMetrics in 1996, is the downside measure that has gained most traction over the recent years within the field of finance (Glasserman, Heidelberger, and Shahabuddin 2002). VaR allows the investor to measure the maximum predicted loss at a certain confidence level (typically 95%). Consequently, empirical findings suggest that VaR enables one to better account for downside risk in comparison to variance (Litterman (1996); Hendricks (1996)). However, a general consensus prevails in the body of literature that VaR has its own drawbacks. Artzner and Delbaen (1997) display that VaR has undesirable mathematical properties such as a lack of sub-additivity and convexity. Thus, VaR does not necessarily reward diversification and can exhibit multiple local extrema, making it computationally intractable to find the global optimal point in the optimization process for portfolio allocation (Beder (1995); Mausser and Rosen (1999)). In addition, in the presence of fat left tails, VaR, like variance, has been criticized for not accomodating for the magnitude of losses beyond the VaR value (Fabozzi, Kolm, et al. 2007).

Conditional Value-at-Risk (CVaR) is a risk measure that has surfaced the academic field of finance in recent years. It attempts to rectify the mentioned shortcomings of VaR (Artzner, Delbaen, et al. 1999). In fact, Pflug (2000) proved that CVaR is a coherent risk measure, which was later supported by Krokhmal, Palmquist, and Uryasev (2002) who concluded that CVaR is indeed a more reliable risk measure than VaR as it is sub-additive and convex.

While the above risk measures have been hailed in scientific circles and led to extensions of the mean-variance framework (see e.g. mean-VaR, mean-CVaR; Artzner, Delbaen, et al. (1999)), it comes at the cost of simplicity and computational tractability. Fantazzini (2004) argue that this is why the vast majority of applied professionals prefer to rely on more traditional models such as the mean-variance framework. He supports this assertion by drawing a parallel to the financial field of option pricing where Black & Scholes still is the most practised model despite the fact that more sophisticated models have emerged within the academia.

Following the aftermath of the financial tumult in 2008, downside risk measures have regained an increased attention due to its ability to better consider for black swan events¹. However, despite their theoretical appeal, Fabozzi, Kolm, et al. (2007) argue that downside risk measures are difficult to implement in a portfolio allocation setting. They remark that this is partly due to the fact that downside risk measures often entails computational intractability.

In light of the above review, it should be apparent that the definition of risk is ambiguous in the literature. While some prefer the simplicity and interpretability of the mean-variance approach, others argue for the use of downside risk measures. As the purpose of this thesis does not lie in evaluating different risk measures, but rather how they can be estimated, it should exist no ambiguity in how risk is measured. Hence, with the motivation that the mean-variance framework is seemingly most employed in practice and in line with Markowitz (1952), this thesis will restrain to variance as a proxy for risk, bearing its limitations in mind.

2.2 Parameter estimation

More importantly for this thesis is the strand of criticism that refers to the phenomenon that mean-variance optimization suffers from the problem of estimation error. In the pivotal work by Markowitz (1952), the focus lied on the theoretical soundness of the suggested portfolio selection approach. Consequently, less emphasis was placed on how to implement MVO in practice. In order to practically implement MVO, one needs to estimate the means and covariances of asset returns as these moments are not known. These estimates are then employed to obtain a solution for the investor's optimization problem (Elton, Gruber, Brown, et al. 2014). There is an abundance of literature that conclude that this leads to one of the most important drawbacks of the mean-variance approach: i.e. the estimation error of the plug-in moments (see, inter alia, Michaud (1989); Chopra and Ziemba (1993)). The drawback arises from the fact that the optimizer is not aware that the inputs are statistical estimates and not known with certainty.

When estimating asset means and covariances of returns, the classical statistical procedure has been to gather a history of past returns and compute their respective sample estimates. This procedure relies on the assumption that historical data has some predictive power for

¹Coined by Taleb (2007), a black swan event an event that significantly deviates from the expected normal case but has a major impact, i.e. an outlier that lives in the utmost point of a heavy tail.

future performance. However, throughout the literature, several deficiencies of employing sample estimates in a portfolio setting have been well documented. For instance, Frankfurter, Phillips, and Seagle (1971) assert that MV optimized portfolios obtained by using sample estimates as plug-in parameters do not necessarily outperform an equally weighted portfolio (also known as the naïve portfolio). This was later supported by Jobson and Korkie (1980) who obtained similar results. Moreover, Best and Grauer (1991) show that the estimation error of the parameter estimates is transferred to the obtained portfolio weights from the portfolio optimization. Hence, the estimated optimal weights will almost certainly always deviate from the true optimal weights.

In this context, Ledoit and Wolf (2003a) contribute to the discussion by explaining why sample estimates may come with severe problems in a portfolio setting. They argue that the poor performance of MV optimized portfolios stem from sample estimates that contain a lot of error: in this case, the most extreme sample coefficients tend to take extreme values not because it is the truth, but rather due to the associated error. Consequently, they argue that the MV optimizer will, consistently, latch onto these extreme coefficients and place the biggest portfolio weights accordingly. It is from this phenomenon that the critique by Michaud (1989) stems, claiming that the portfolio optimizers introduced by Markowitz (1952) are in fact "estimation error maximizers". Michaud (1989) further coined this puzzle as "Markowitz's optimization enigma".

The critique by Michaud (1989) implies that in the presence of inaccurate plug-in estimates in MVO, asset managers will underrepresent their true asset-picking abilities, leading to suboptimal portfolios. To cope with this prevalent problem, a proliferation of studies regarding the subject have emerged. However, as pointed out by Ledoit and Wolf (2014), while a substantial amount of effort has been devoted to estimating the expected return vector, much less has been written about the covariance matrix. To exemplify this assertion, they refer to Green, Hand, and Zhang (2013) who list over 300 articles that relate to methods of estimating expected returns.

The disparity in attention devoted to the expected return vector versus the covariance matrix input may potentially be explained by tracing back to the findings by Chopra and Ziemba (1993), suggesting that error in the expected return has more impact on the optimized portfolio's out-of-sample performance than error contained in the covariance matrix. In addition, the concept of forecasting returns has always been subject to a great deal of attention ever since the introduction of financial markets, beyond the field of portfolio allocation. However, recently Michaud, Esch, and Michaud (2012) challenged the aforementioned claim by Chopra and Ziemba (1993). Michaud et al. (2012) remark that there is a persistent widespread error in the literature regarding the relative importance of estimation error in return relative to the covariance matrix. They argue that the paper by Chopra and Ziemba (1993) is highly flawed and unreliable: their underlying argument for this assertion is that the analysis in Chopra and Ziemba (1993) is based on an in-sample specific study, thus having no bearing regarding the impact of estimation error in rigorous out-of-sample MV optimization. Furthermore, in stark contrast, they claim that the estimation error in the covariance matrix may in fact overwhelm the optimization process as the size of the asset universe increases.

In light of the above review, it should be apparent that estimation error is one of the most important aspects of MV optimization. Reducing such error has the potential to increase out-ofsample performance of MV optimized portfolios. In other words, there is a potential to obtain more efficient portfolios, leading to significant monetary benefits for asset managers that employ MVO. Historically, the general consensus has been to focus on forecasting returns. Consequently, the body of literature regarding this matter is extensive. However, more recently, research that point to the importance of the covariance matrix estimation in MVO has emerged. Thus, a demand for the relative performance between different methods in estimating the covariance matrix has surged among financial practitioners. The following two subsections will (i) provide an overview of different approaches in forecasting expected returns, and (ii) review the literature that include research on how estimation error in the covariance matrix can be reduced, in line with the research questions in this thesis.

2.2.1 Expected returns

Explaining future return characteristics may be considered as the grail of financial economics. It can be considered as the quest for financial practitioners that seek efficient asset allocation through portfolio optimization.

The classical approach when estimating expected returns in MVO is to rely on the sample means as predictors. Under the hypothesis of normality, the sample mean is the maximum likelihood estimate (MLE) and thus the best linear unbiased estimator for the assumed distribution. In this case, the sample mean exhibits the property that an increase in sample size, leads to an improved performance of the estimate. Fabozzi, Kolm, et al. (2007), however, note that under distributions that are heavy tailed or significantly deviate from a symmetrical unimodal distribution, the above properties of the sample mean are no longer valid. They support this claim by referring to the work of Ibragimov (2005). An example where the sample mean is a poor forecast is depicted in Figure 2.3.



Figure 2.3: A bimodal density function symmetric around the mean, with E[R] = 1.1 and $Var(R) = 0.3^2$.

As one can observe, the outcomes of the random variable (in this case the return of an arbitrary asset) are not necessarily likely to take values described by the overall mean. In this case, the sample mean is a poor predictor for future returns.

Furthermore, Fabozzi, Kolm, et al. (2007) argue that the return generating process of financial time series usually do not exhibit stationarity, but varies over time. This implies that historical data from a long past may have little explanatory power for future behavior. Hence, in such a setting, the mean is not a good forecast of expected returns - consequently leading to large estimation error in the optimization process.

To cope with the shortcomings regarding sample moments, two prevalent approaches can be found in the literature. The first approach is to impose structure on the estimator, usually by relying on some factor model to forecast expected returns. The other approach is to use Bayesian estimators such as the Black-Litterman model. These different perspectives will be further presented throughout this section.

Factor models

The capital asset pricing model (CAPM) is a single-factor model that marked the birth of asset pricing theory. Building on the foundation of MPT set forth by Markowitz (1952), CAPM was initially introduced by Sharpe (1964). Following the footsteps of Markowitz, Sharpe was awarded the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel for his pioneering contribution to financial economics in 1990 (KVA 1990). In the context of CAPM, Lintner (1965) and Mossin (1966) are worthy mentions as they independently proposed similar theories as Sharpe (1964). The CAPM can be seen as an abstraction of the real-world capital markets and is based on a strict set of equilibrium assumptions. Sharpe (1964) conclude that the following three main assumptions underlie CAPM:

- 1. Investors are rational and choose mean-variance efficient portfolios according to Markowitz (1952).
- Investors are in complete agreement: i.e. investors have homogeneous expectations regarding the volatilities, correlations and expected returns of the assets.
- 3. Investors can borrow and lend at the risk-free rate, which is the same for all investors.

Under these assumptions, CAPM states that the aggregated market portfolio is efficient. Furthermore, as investors can eliminate firm-specific risk by diversifying their portfolios, no investor should price it. It is from this assertion that the key idea of CAPM stems: i.e. that all risk originates from a single factor, the market. This is commonly referred to as the systematic risk that cannot be diversified away. The major implication of CAPM is then that the expected return of any asset is determined by its covariation to the market. In other words, the CAPM is a single-factor linear model that relates expected returns of an asset and a market portfolio.

Within the field of finance, the CAPM has found a widespread application amongst practitioners ever since its introduction. Five decades later, the model is still regarded as a centerpiece in finance literature (Da, Guo, and Jagannathan 2012) and widely employed in a portfolio allocation setting. Fama and French (2004) argue that the attraction of the CAPM lies in its simplicity: it offers a quick quantitative insight into risk-reward interplay and an intuitive tool for predicting expected returns. In addition, the model is supported by a strong theoretical background from economic theory. However, in academic circles, the empirical validity of CAPM has been widely debated and there is a prevalent consensus that, owing to its idealized assumptions, the empirical record regarding the model's validity is poor. Fama and French (2004) claim that it is in fact poor enough to invalidate the way CAPM is used in applications altogether. Levy and Roll (2010) remark that the widespread belief regarding the invalidity of CAPM stems from research that conclude that various, commonly used market proxies are inefficient (see, for example, Jobson and Korkie (1982); Shanken (1985); Gibbons, Ross, and Shanken (1989)). These findings do not coincide with the CAPM theory, consequently casting doubt on CAPM. However, Levy and Roll (2010) show that slight adjustments, well within estimation error bounds, on the sample parameters employed in evaluating the market portfolio suffice to make the market proxy efficient. Hence, in stark contrast to the belief that beta is dead, their findings suggest that market proxies may be consistent with the CAPM theory after all, thus strengthening the usefulness of employing CAPM when estimating expected returns. It is important to note, however, that Levy and Roll (2010) hasten to add that their findings do not constitute a proof of the empirical validity of CAPM, but acts as a response to the rejection of the model. They further note that the validity of the global CAPM is not empirically testable as the true market portfolio is in fact not observable (hence why proxies are employed), in accordance to the critique by Roll (1977).

Following the conception of the CAPM, several offsprings have emerged throughout the literature. These offsprings are often referred to as multi-factor models, whose birth may be motivated by the growing number of studies that found that the market alone is not sufficient in explaining asset returns (see Banz (1981); Basu (1983)). Often viewed as a counterpart to CAPM, Ross (1976) derived an asset pricing model solely based on arbitrage arguments such as the no-arbitrage condition. This model is commonly referred to as the arbitrage pricing theory (APT) model. Contrary to the CAPM, the APT postulates that the expected return of an asset

is influenced by a variety of risk factors, allowing for a more accurate explanation of asset returns (Ross 1976). In addition, supporters of the APT argue that an advantage over the CAPM is that it relies on less restrictive assumptions, e.g. by not relying on the assumption that all investors are mean-variance optimizers. However, the strength of allowing for more explanatory variables in explaining returns is also its weakness, as the APT provides no specification on which factors to include in the model. Consequently, there is no consensus on the identity of explanatory variables and no consensus on the number of factors to include, resulting in a less tractable approach than the CAPM.

Moreover, the Fama and French three-factor model is a prominent extension of the CAPM that attempts to rectify the strand of criticism related to the aspect of omitted explanatory variables in the CAPM (Fama and French 1995). Based on findings implying higher average returns on small stocks and high book-to-market stocks, Fama and French (1993) argue that there are unidentified variables that produce undiversifiable risks in returns, not captured by the market return. They support this claim by illuminating the phenomena that the returns of small firms covary more with one another compared to large firms, and that returns on high book-to-market stocks. Based on this evidence, Fama and French (1995) proposed a three-factor model that extends the CAPM with the addition of two factors.

In the academic world, the Fama and French three-factor model has been widely accepted as a CAPM empirical successor (Zabarankin, Pavlikov, and Uryasev 2014). Less so by financial practitioners, which Bartholdy and Peare (2005) argue can be explained by their findings that the additional cost in complexity associated with Fama and French is not justified by its relative performance over the CAPM. Fama and French (2004) argue that the main shortcoming of their three-factor model lies in its empirical motivation: the added explanatory variables are not intuitive from a theoretical perspective, they are rather mere 'brute force' constructs meant to capture return patterns illuminated by previous research.

Noting that the above review merely captures a glimpse of the literature on factor models in a forecasting context, these models stem from the assertion that sample moments based on historical data are likely to contain random noise and errors. Factor models attempt to smooth historic data and focus on the underlying relationships, while ignoring deviations from perceived statistical relationships that are inferred by random noise. In the literature, one can indeed find that factor models tend to outperform the plug-in sample mean in a portfolio allocation setting. For instance, Chan, Karceski, and Lakonishok (1999) show in their study that estimates based on factor models lead to improved out-of-sample performance of optimized portfolios, compared to when sample plug-in estimates are employed. However, no favorite specification emerges regarding the choice of factor model.

Ait-Sahalia and Hansen (2009) note that moving from theoretical factors such as the market portfolio, to empirical factors (see Fama and French (1995)) and potentially to statistical factors, we may by construction capture more underlying relationships. However, in exchange, the factors become more difficult to interpret, which Ait-Sahalia and Hansen (2009) argue raises concerns regarding data mining. Ultimately, choosing between factor models involves a trade-off between estimation error, bias and interpretability. In this context, invoking the following quote by Ledoit and Wolf (2003b, p.2) is appropriate:

'The art of choosing a factor model adapted to a given data set without seeing its out-of-sample fit is just that: an art'

Essentially, Ledoit and Wolf (2003b) convey the message that as there is no general consensus on the identity of factors (except for the market) to employ in factor models, choosing a specific factor model is very ad hoc (we do not know how well it works a priori). The simplicity of the CAPM does not entail that it necessarily performs worse than more complex models. In fact, a common finding in forecasting literature is that simple, parsimonious models that may suffer from severe misspecification often provide stronger forecasts than more complicated models (see inter alia Swanson and White (1997); Stock and Watson (1999)). Ultimately, when it comes to factor models, the Fama-French factor model is regarded as the front figure in the academic world. However, there is a presence of disparity between the academic world and the real world setting as financial practitioners find the theoretical motivation and the mathematical simplicity of the CAPM appealing. It provides a quick quantitative insight on the risk-reward interplay of assets. Hence, as in the working paper by Gerber et al. (2015) who analyze portfolio performance in a real-world setting, this thesis will rely on the CAPM when estimating expected returns, while still acknowledging that its application have been heavily debated within scientific circles (see e.g. Galagedera and Galagedera (2007) for a detailed capital asset pricing review).

The Black-Litterman model

Within the realm of asset allocation, the Black-Litterman model is regarded as a prominent extension of traditional MVO. Developed in the original paper by Black and Litterman (1992), the Black-Litterman model stems from an equilibrium assumption that the global portfolio is well diversified and efficient, serving as the neutral initial stage of the approach. Thereafter, through the use of a reverse optimization process, the model derives returns of assets implied by the market equilibrium. At this stage, a natural question that arises is how the model differentiates from the CAPM. The answer lies in the model's flexibility to combine the market equilibrium with additional market views of an investor. More precisely, the Black-Litterman model permits analysts to convey agreement or disagreement with the returns implied by the market (Black and Litterman 1992). The intuition behind the model is that an analyst may claim expertise in value investment that differs from the market consensus: so why not let the analyst incorporate these views when deriving the vector of expected returns? In their original paper, Black and Litterman (1992) conclude the intuition of their proposed model in the following manner:

"...our approach allows us to generate optimal portfolios that start at a set of neutral weights and then tilt in the direction of the investor's views."

If carefully used, Nikbakhtt (2011) summarize some of the advantages that the Black-Litterman model may endow on the final portfolio, in comparison to a portfolio obtained through traditional MVO. These advantages include:

- Estimation error is usually reduced.
- Portfolio weights are often, by construction, more intuitive with respect to the expressed views.
- The recommended portfolio obtained through the optimization should be more efficient and less concentrated on individual assets.

However, the Black-Litterman model has been criticzed for its ambiguity as Black and Litterman (1992) did not discuss the precise nature of how one practically applies the model. Nikbakhtt (2011) argue that the incorporation of views is in fact a major limitation when putting the Black-Litterman model into practice. He claims that only the most naive analysts are confident in their additional market views and in the presence of casually expressed views, the model may become dangerous. Hence, it is of utmost importance that analysts integrate their views with the greatest of care. How one estimates the parameter of confidence on views in the Black-Litterman model is not clear as Black and Litterman (1992) did not discuss the precise nature of this phenomenon in their original article. The criticism regarding the great deal of vagueness associated with the Black-Litterman model stems from the lack of properly described parameters underpinning the model. In particular, the most severe problem of the model concerns the vagueness of how one determines the confidence parameter often referred to as the weight-on-views or tau. Articles such as: "A demystification of the Black-Litterman model" (Satchell and Scowcroft 2000) and "A step-by-step guide to the Black-Litterman model" (Idzorek 2002) serve as strong examples regarding the difficulties of interpreting the original work by Black and Litterman (1992).

In addition, Nikbakhtt (2011) illuminates the question regarding the impact of legal risk when using the Black-Litterman model: he argues that in the absence of a reliable algorithm that incorporates investor views, clients may use the "prudent expert" principle for portfolio management in court. On the other hand, if the views are well documented, objectively defined and well justified, legal risk may decline.

In this thesis, the neutral initial stage of the Black-Litterman model will be used (i.e. the CAPM model) to estimate expected returns, without incorporating any additional market views. The reasoning behind this is simply that I deem it inappropriate to dilute the portfolio outof-sample analysis with subjective opinions and leave this additional flexibility open for asset managers to integrate, if sought.

2.2.2 The covariance matrix

Within the realm of modern portfolio theory, the importance of the covariance matrix has arguably been overshadowed by the expected return parameter. As previously mentioned, this can partly be credited to the findings of Chopra and Ziemba (1993), suggesting that the relative influence of errors in the expected return vector is higher. In recent times, this claim has been challenged by the likes of Michaud et al. (2012) who take an opposing stance. They argue that errors in the covariance matrix may in fact overwhelm the optimization process when the asset universe grows large.

Nevertheless, the disparity in attention devoted to the different areas is not to be confused by an absence of literature regarding the covariance matrix estimator. In fact, following the advancements of mathematical theory and computational power in recent times, a fair amount of consideration has been put in developing alternative methods of estimating the covariance matrix (see e.g. Laloux et al. (2000); Ledoit and Wolf (2003b); Gerber et al. (2015)). Other reasons for the gained interest regarding this phenomenon can be found in Bengtsson and Holst (2002) who argue that the notorious difficulty of estimating expected returns compared to estimating the covariance matrix implies that most of the improvement that can be made on MVO lies in the covariance matrix estimation.

The classical method of estimating the covariance matrix in the context of MV optimization is to employ the sample covariance matrix. During the years following the work by Markowitz (1952), numerous studies have shown that the sample covariance matrix may suffer from drawbacks which undermine its forecasting power of future covariances (Elton, Gruber, Brown, et al. 2014). This may come as a surprise as the sample covariance matrix has the appealing property of being the maximum likelihood estimate under normality. However, this is to forget what maximum likelihood actually means. Following Ledoit and Wolf (2003b), it means that all the trust is put in the data which clearly is a sound principle if there is enough data to trust it. Not enough data is thus a problem and while increasing the amount of data is a potential solution, it may come at the expense of employing outdated noisy data with no explanatory power regarding the future. Consequently, as exemplified in Bengtsson and Holst (2002), an important drawback of the sample covariance matrix is that it may follow noise too closely and suffer from overfitting which will undermine the out-of-sample fit, despite being the best estimate in-sample.

In other words, the sample covariance matrix has been shown to require a lot of data. This is exemplified in Bengtsson and Holst (2002) who show that the sample covariance matrix of 100 (N) assets implies that 5050 (N(N + 1)/2) parameters have to be estimated. Therefore, small sample problems may occur when the considered asset universe is large.

In the literature, the cure for the feasible drawbacks associated with the sample covariance matrix is to impose some form of structure on the estimator. The vast majority of challengers stem from the notion that there exists a bias-variance tradeoff. While imposing structure may reduce the instability (variance) of the estimator, it may come at the expense of specification error (bias). The idea is to find an estimator that prevails at the optimum balance between bias and variance. Strong challengers found in the body of research regarding this area include estimators based on factor models, shrinkage models, random matrix theory and threshold theory (Elton, Gruber, Brown, et al. 2014).

Factor models

Ledoit and Wolf (2003b) claim that a natural way to impose structure on the covariance matrix estimator is to use a low-dimensional factor structure. Within the world of finance, factor models have found a widespread traction. However, as in the discussion regarding factor models when estimating expected returns, Ledoit and Wolf (2003b) argue that two questions arise in this context: how many factors should one use and what factors should be considered in the model? According to Elton, Gruber, Brown, et al. (2014), there is no general consensus regarding the answers to these questions, except for the common understanding that a market factor should be included. The use of a market factor to explain the return generating process of asset is motivated by economic theory and was first introduced in such a setting by Sharpe (1964).

Not surprisingly, the single-index model by Sharpe (1964) is one of the most prominent structural model found in the literature. The key idea of the single-index model is to impose structure by assuming that the only reason that two securities move together is due to their common response to market changes. Effectively, all other factors (such as industry factors) beyond the market are assumed not to account for any comovement between securities. This rather strong assumption is the core of the single-index model and the validity of the model is thus strongly dependent on how well the assumption holds.

In some embodiments, empirical studies have found that the estimated covariance matrix implied by the single-index outperforms the full historical covariance matrix. More specifically, Elton, Gruber, and Urich (1978) investigated the relative ability in forecasting the correlation structure between securities for various correlation estimation techniques. Some striking results of their study was that the sample correlation matrix underperformed the correlation matrix implied by the single-index model when comparing predicted and realized correlation between financial securities. Furthermore, they showed that these results were statistically significant. This suggests that a part of the correlation structure for the full historical model represents random noise, which is not captured by the structured single-index model.

Ledoit and Wolf (2003b) argue that the sample covariance matrix and the estimated covariance matrix implied by the single-index model can be viewed as two extremes. The first one can be regarded as a full factor model that puts all the trust in the data, whereas the second one is a single factor model that makes a rather restrictive assumption regarding what data is relevant. In the presence of effects beyond the market factor that account for security comovement, the single-index model may thus come at the expense of introducing specification error. In this context, a similar discussion as in the CAPM model for expected returns can be carried out regarding the introduction of additional factors. Recall that Ait-Sahalia and Hansen (2009) argue that one may, by construction, capture more underlying relationships by moving to a multi-factor model that accounts for e.g. industry factors. However, it does not necessarily entail a better out-of-sample fit. In addition, the lack of a general consensus regarding factors apart from the market factor remains a problem as it raises concerns about data mining. Ledoit and Wolf (2003b) add that choosing between factor models thus becomes very ad hoc and calls it an art.

In this context, Chan et al. (1999) study the performance of different factor specifications in a realistic portfolio allocation setting. Their findings suggest no dominating factor specification emerges. In fact, the parsimonious single-index model performed only marginally worse than more complex specifications based on a weaker theoretical foundation. However, all considered factor models outperformed the sample covariance matrix, which once again suggests that improvements can be made on this estimate.

With respect to prediction, Elton, Gruber, Brown, et al. (2014) further conclude that parsimonious models tend to outperform more complex models in many tests. Their explanation for this is that complex models with multiple factors tend to contain more noise than real information.

Shrinking

Thus far, two extreme estimators have been reviewed: the sample covariance matrix and the covariance matrix estimator implied by Sharpe's (1964) single-index model. In addition, the drawbacks of these two estimators illuminated in the literature have been presented. To reiterate, it is well known that the sample covariance matrix may suffer from overfitting as it puts all the trust in the data which in turn may render a poor out-of-sample fit. On the other hand, the strong structure imposed by the single-index model comes at the price of potentially introducing specification error. Thus, there exists a trade-off between specification error (bias) and estimation error (variance) within the realm of estimation.

With this in mind, a recent proposal by Ledoit and Wolf (2003b) is to take a different approach in imposing structure. They suggest to take a weighted average of the sample covariance matrix and the covariance matrix estimator implied by the single-index model. This way, they let the weight assigned to the single-index estimator control how much structure that is imposed. The approach is inspired by the concept of shrinking, dating back to the work by Stein (1956) where the weight assigned to the single-index estimator is the shrinkage intensity and the shrinkage target is the estimated covariance matrix implied by the single-index model. Ledoit and Wolf (2003b) argue that this approach has the advantage of being able to account for effects beyond the market factor without the need of specifying an arbitrary multi-factor structure. This is very convenient, given that there is no general consensus regarding the identity of factors except for the market factor. The method is commonly called shrinkage to market.

The central idea is to find an optimal compromise between estimation error commonly associated with the sample covariance matrix and specification error introduced by the structured single-index model. In order to achieve this, Ledoit and Wolf (2003b) derives a formula for the optimal linear shrinkage intensity that controls the amount of structure that is imposed. The derivation is done by working under large-dimensional asymptotics.

Applying their shrinkage estimator in mean-variance optimization, they further show that for NYSE and AMEX stock returns ranging from 1972 to 1995, lower out-of-sample variance of MV optimized portfolios can be obtained compared to solely using, inter alia, the sample covariance matrix or the covariance matrix estimator implied by the single-index model. Here, Ledoit and Wolf (2003b) use an equally weighted index of the asset returns as a market proxy. In this context, Bengtsson and Holst (2002) found similar results for Swedish asset returns. Both these

studies only use the minimum variance portfolio in their evaluations. Consequently, their results provides limited information regarding the performance of the covariance matrix estimators in a portfolio selection context (only one special portfolio is considered). In practice, investors have various risk profiles and may be interested in portfolios beyond the minimum variance portfolio. However, a valid response to this critique can be found in Bengtsson and Holst (2002) where they motivate this choice by not wanting to dilute the covariance matrix performance with potential errors in the expected return vector.

It is important to note that the single-index estimator is not an exclusive shrinkage target in this setting. This is illuminated in Ledoit and Wolf (2003a) where they develop a new shrinkage estimator using the constant correlation model as the shrinkage target. However, they hasten to add that for an asset universe consisting of different asset classes, the constant correlation model is not appropriate.

Lastly, Ledoit and Wolf (2004) are very clear that the improvement that the linear shrinkage introduced in Ledoit and Wolf (2003b) has over the sample covariance matrix is dependent on the situation at hand. For a relatively large asset universe (N) compared to the number of observations per asset (T), the improvement is expected to be significant. On the contrary, if the data per estimated parameter is high (i.e. when N/T is small), the improvement may be minuscule.

Random matrix theory

Although developed in the 1950s by quantum physicists, random matrix theory (RMT) is a fairly recent area within the realm of portfolio optimization. In this context, Laloux et al. (2000) conducted a pivotal study where they seek to identify measurement noise often associated with the sample covariance matrix. Their approach stems from the idea that if one can devise a method to distinguish measurement noise that devoid useful information from signal (useful information) contained in the estimated covariance matrix, the estimate can be enhanced. Employing known results from random matrix theory, they show that approximately 94% of the eigenvalues that constitute the sample correlation matrix for S&P500 stock returns (daily data ranging from 1991-1996) agree with the theoretical prediction of RMT. This suggests that the sample correlation matrix may be considered random, to a certain extent. In other words, merely a few eigenvalues were found to contain useful information in the construction which is a remarkable finding in a quantitative portfolio selection context. Under the assumption that these results are valid, Laloux et al. (2000) as well as Plerou et al. (2002) that if one filters the noisy eigenvalues and reconstructs a cleaned correlation matrix, the forecast of realized risk is improved. These findings suggests that employing random matrix theory in estimating the covariance matrix in portfolio optimization can be beneficial in portfolio optimization. However, Laloux et al. (2000) mention that the noise filtering will in particular improve the least risky portfolios, as these diversified portfolios seem to mostly be influenced by noise.

However, no general consensus exists in how one should filter the perceived noisy eigenvalues. The only consensus lies in the fact that regardless of filtering method, the trace of the correlation matrix should be preserved to ensure that the variance of the system is preserved. Nevertheless, prominent filtering methods may be found in Laloux et al. (2000), Plerou et al. (2002) and Sharifi et al. (2004).

Robust statistics

More recently, Gerber et al. (2015) contributed to the research of enhancing the covariance matrix estimation. In their working paper, a new measure of comovement is introduced altogether, based on the field of robust statistics. They coin this measure as the Gerber Statistic which aims to be

more robust than the conventional historical correlation by accommodating for noise and outliers in the data. The claim is supported by a following evaluation test, where Gerber et al. (2015) compare out-of-sample performance of MV optimized portfolios in a realistic investment setting. The asset universe is a multi-asset universe, consisting of various asset classes such as equity indices, bonds and commodities. Furthermore, the results of the evaluation test indicated that the entire realized efficient frontier could be raised upwards by replacing the sample covariance matrix with an estimated covariance matrix based on the Gerber Statistic, implying larger Sharpe ratios. Solely by changing the estimation technique for the covariance matrix, ceteris paribus, they showed that the MV optimized portfolios obtained via their newly introduced measure consistently outperformed obtained portfolios via the sample covariance matrix over a range of different investor profiles.

Clearly, the findings by Gerber et al. (2015) are of great interest for portfolio managers that employ MVO, if deemed valid. However, the study gives rise to the question whether previously developed estimators would yield an even stronger performance (i.e. weak competition). In addition, in the article by Gerber et al. (2015), it is recognized that in computing the Gerber Statistic, a non-positive semidefinite matrix may be obtained in theory. Clearly, this poses a problem in the optimization process since a solution to the problem cannot be guaranteed to be a globally optimal solution. The authors, however, claim that this problem has not been found to occur, neither in real nor in simulated practice.

Comparing covariance matrix estimators in MVO

The most common approach found in the literature to compare various covariance matrix estimators is to analyze obtained MV optimized portfolios in backtesting procedures (see e.g. Bengtsson and Holst (2002); Ledoit and Wolf (2003b)). This enables one to study the out-of-sample performance of the obtained portfolios in a real world setting, as data from the testing period is left out in the estimation phase. However, many studies only evaluate the minimum variance portfolio. This portfolio merely captures a glimpse of the relative performance in practice, as investors have varying risk profiles. A common response to this shortcoming is that by leaving out the expected return parameter, more emphasis is made on the covariance matrix and the relative performance is not diluted by potential errors in the estimated vector of expected returns. While this is a valid response, it will nonetheless render the results to be less applicable in practice. Furthermore, many studies exclude asset classes beyond stocks while it has been shown that a portfolio may experience significant benefits if asset classes such as commodity is included in the asset universe due to the recent increase in equity volatility (Conover et al. 2010).

In light of some of the above issues regarding matching the real-world environment associated with the vast majority of studies on this matter, this thesis will attempt to complement the existing body of research by studying out-of-sample performance over a range of investor profiles. In addition, the asset universe will include different asset classes. The covariance estimators that will be investigated are the ones reviewed in this section. There are indeed more estimators to be found in the literature, but it is deemed that these are not as prominent as the ones that have been reviewed.

Chapter 3

Theoretical Framework

This chapter introduces the foundation that the thesis operates within. The framework is divided into three tethered components. The first part serves as an introduction in the form of presenting some crucial preliminaries. The following two parts will provide extensive theory on meanvariance optimization and estimation techniques regarding the expected return and the covariance matrix, with the focal point lying in the latter of these elements.

3.1 Basic preliminaries

This section serves as a brief introduction to fundamental notions and concepts with regard to mean-variance optimization.

3.1.1 Return

The return of a financial security is is the gain or loss over a certain time horizon. The return of a security in this thesis is denoted as R where the return for a particular security i is R_i . In addition, the price of a security at a particular time, t, is denoted as S_t . The return for a non-dividend security over a time period [t, T], where T > t is then calculated in the following manner:

$$R_i = \frac{S_T - S_t}{S_t} \tag{3.1.1}$$

As for a security that pays dividends over the time period [t, T], the return is calculated as:

$$R_{i} = \frac{Div_{t,T} + (S_{T} - S_{t})}{S_{t}} = \underbrace{\frac{Div_{t,T}}{S_{t}}}_{\text{Yield of dividends}} + \underbrace{\frac{S_{T} - S_{t}}{S_{t}}}_{\text{Capital gain yield}}$$
(3.1.2)

At time t, the outcome of S_T and $Div_{t,T}$ is not known, hence the return of a security over the time period is a random variable. The expected outcome of this random variable will be denoted as:

$$\mu_i = E[R_i] \tag{3.1.3}$$

In the setting of portfolio allocation, a portfolio can be composed of multiple securities where each portfolio weight, w_i , of security *i* is calculated accordingly:

$$w_i = \frac{\text{Value of the investment in security } i}{\text{Total value of the portfolio}}$$
(3.1.4)

Thus, for an asset universe of size n, we have that:

$$\sum_{i=1}^{n} w_i = 1 \tag{3.1.5}$$

The expected return of such a portfolio, $E[R_P]$ is then given by:

$$E[R_P] = E\left[\sum_{i=1}^{n} w_i R_i\right] = \sum_{i=1}^{n} w_i E[R_i] = \sum_{i=1}^{n} w_i \mu_i = w^{\top} \mu$$
(3.1.6)

Here, $w^{\top} = [w_1, \ldots, w_n]$ and $\mu = [\mu_1, \ldots, \mu_n]^{\top}$. I.e., these are vector notations where $w, \mu \in \mathbb{R}^{n \times 1}$.

3.1.2 Variance

In the setting of mean-variance optimization introduced by Markowitz (1952), the notion of variance is also fundamental. For a portfolio of assets, where the asset universe consists of n assets, its variance can be derived in the following manner:

$$Var(R_P) = E\left[\left(\sum_{i=1}^n w_i R_i - E[R_P]\right)^2\right] = E\left[\left(\sum_{i=1}^n w_i (R_i - E[R_i])\right)^2\right]$$
$$= E\left[\left(\sum_{i=1}^n w_i (R_i - E[R_i])\right)\left(\sum_{j=1}^n w_j (R_j - E[R_j])\right)\right]$$
$$= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \underbrace{E[(R_i - E[R_i])(R_j - E[R_j])]}_{:=\sigma_{ij}} = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij} = w^\top \Sigma w$$

To summarize, the variance of the portfolio is given by:

$$Var(R_P) = w^{\top} \Sigma w \tag{3.1.7}$$

where Σ denotes the covariance matrix of the asset returns, composed of all covariances between the returns defined as σ_{ij} ($\sigma_{ii} \forall i = \{1, ..., n\}$ simply is the variance of asset *i*'s return, these constitute the diagonal of the covariance matrix).

3.1.3 Optimization

In mathematics, optimization refers to the selection of a best element, with regard to certain conditions, from a set of possible alternatives. A mathematical representation of an optimization problem is presented below:

$$\begin{cases} \min_{x} & f(x) \\ \text{s.t.} & g_i(x) \le b_i, \quad i = 1, \dots, m \end{cases}$$
(3.1.8)

The solution to this general form must lie within the feasible set which is given by $\mathcal{F} = \{x \in \mathbb{R}^n : g_i(x) \leq b_i, i = 1, ..., m\}$. A graphical example of a feasible set with the form as in (3.1.8), spanned by three constraints, is illustrated in Figure 3.1. In this example, it is assumed that x_1 and x_2 (two dimensional case) cannot take negative values.



Figure 3.1: A feasible set spanned by three constraints. The shadowed area illustrates the feasible set.

Convexity

With regard to opimization, convex functions plays an important role. They have the appealing property that a local minimum is also a global minimum. For concave functions, this means that a local maximum is also a global maximum. Reconnecting to the feasible set spanned by a certain number of constraints in the general optimization problem 3.1.8, a set $\mathcal{F} \subset \mathbb{R}^n$ is referred to as convex if for all $x, y \in \mathcal{F}$ and $t \in [0, 1]$, it holds that:

$$(1-t)x + ty \in \mathcal{F} \tag{3.1.9}$$

Now, a function $f: \mathcal{F} \to \mathbb{R}$ is called convex if for all $x, y \in \mathcal{F}$, the following holds:

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y) \quad \forall t \in (0,1)$$
(3.1.10)

If the above holds when the equality sign is flipped, f is said to be concave. A graphical illustration of a convex function is shown in Figure 3.2.



Figure 3.2: An illustration of a convex function. Source: Sasane and Svanberg (2014, p.79).

Optimality

Moreover, optimality is an important concept in mean-variance optimization. Given a real-valued function of a random variable x, a point $\hat{x} \in \mathcal{F}$ is said to be a local minimizer if it holds that:

$$f(\hat{x}) \le f(x) \ \forall x \in \mathcal{F} \text{ such that } |x - \hat{x}| < \delta, \ \delta \in \mathbb{R}_+$$
 (3.1.11)

and a global minimizer of f if the following holds true:

$$f(\hat{x}) \le f(x) \ \forall x \in \mathcal{F} \tag{3.1.12}$$

An illustration for these two events is depicted in Figure 3.3



Figure 3.3: Q and all points in the interior of the segment AB are local minimizers, whereas P is a global minimizer. Source: Sasane and Svanberg (2014, p. 105).

Reconnecting to convexity, the special property of a convex (concave) problem entails that a local minimizer (maximizer) is also a global minimizer (maximizer). Clearly, this is an appealing property in the optimization process as a solution to the problem can be guaranteed to be a globally optimal solution. This renders the optimizer to produce stable solutions.

Quadratic programming

Quadratic programming (QP) refers to the problem of minimizing or maximizing a quadratic function subject to linear equality and inequality constraints. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function with the following representation:

$$f(x) = \frac{1}{2}x^{\top}Hx + c^{\top}x + c_0$$
(3.1.13)

where $x \in \mathbb{R}^n$, $H \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $c \in \mathbb{R}^n$ and $c_0 \in \mathbb{R}$. A general mathematical representation of minimizing the above function is as follows:

$$(\mathbf{QP})\begin{cases} \min_{x} & \frac{1}{2}x^{\top}Hx + c^{\top}x + c_{0}\\ \text{s.t.} & Ax = b\\ & x \ge 0 \end{cases}$$
(3.1.14)

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Note that all of these are given, including H, c and c_0 , with an exception of x, sought to be solved. The above problem (**QP**) is a general form of the optimization problem presented by Markowitz (1952) for portfolio allocation. Reconnecting to convexity, the quadratic function f(x) is convex if and only if H is positive semi-definite. Consequently, the optimization problem is perceived to be *nice* if this holds. This implication is important to remember throughout this thesis.

3.2 Portfolio optimization

This section introduces the fundamental concepts of Modern Portfolio Theory (MPT). Serving as the cornerstone of MPT, the mean-variance optimization framework for portfolio selection is presented. Moreover, this section derives the MV formulation for allocating asset portfolios in a quantitative manner. The structure of this section is largely inspired by Lundström and Svensson (2014).

In its most basic form, the problem of portfolio selection may be summarized by the following four aspects (Steuer, Qi, and Hirschberger (2008); Lundström and Svensson (2014)):

- A fixed amount of money to be invested
- An asset universe of size n, constituted of possible security investments
- A predetermined holding period for the portfolio
- A portfolio rebalance frequency, determining the length of possible sub-periods within the holding period

Recalling the notations outlined in Section 3.1.1, the portfolio weight for the *i*:th asset is denoted as w_i . As shown in equation 3.1.4, the portfolio weights are defined as proportions of the fixed sum to be invested. Therefore, the portfolio weights must sum to 1. This relation serves as the first constraint for the optimization problem and is often referred to as the condition of a fully invested portfolio. Furthermore, note that future returns of assets are unknown at the beginning of the holding period, and thus to be considered as random variables. However, in MPT, the optimizer assumes that all future asset characteristics (μ_i , σ_{ii} and σ_{ij}) are known when the optimization is initialized. As this does not hold true in reality, these parameters have to be estimated. The real performance of the optimized portfolio is thus heavily reliant on the estimation accuracy of these parameters. How these are estimated in practice will be further described in the subsequent sections, Section 3.3 and Section 3.4.

Mathematically, the random return for a portfolio is defined below:

$$R_P = \sum_{i=1}^{n} w_i R_i = w^{\top} R \tag{3.2.1}$$

where $R = (R_1, R_2, \ldots, R_n)^{\top}$. Now, assuming that an investor is solely interested in maximizing the uncertain portfolio return, the portfolio selection problem has the following stochastic programming representation:

$$(\mathbf{SP})\begin{cases} \underset{w}{\text{maximize}} & R_P = w^\top R\\ \text{subject to} & w \in \mathcal{F} \end{cases}$$

Here, \mathcal{F} defines the feasible region that is spanned by the portfolio constraints:

$$\mathcal{F} = \{ w \in \mathbb{R}^n \mid \sum_{i=1}^n w_i = 1, \alpha_i \le w_i \le \beta_i \}$$

$$(3.2.2)$$

The second constraint bounds the weights for each asset, where α_i and β_i is the lower and upper bound, respectively. Two notable cases are referred to as the unconstrained and constrained case. The unconstrained case allows for w_i to take any value (i.e. $\alpha_i \to -\infty$ and $\beta_i \to \infty$), which is equivalent to removing the weight constraint altogether. On the other hand, a
common constrained case is to bound the portfolio weights by imposing $\alpha_i = 0$. This is equal to not allowing short positions in the portfolio. Moreover, an important aspect to note is that (SP) is a stochastic programming problem. This follows from the fact that the future returns of the securities are random variables and the composition of the portfolio (i.e. the vector of portfolio weights, w) must be determined at the beginning of the holding period (Steuer et al. 2008). Hence, in similar fashion as in Steuer et al. (2008), (SP) will be referred to as the investor's initial stochastic programming problem. At this stage, (SP) is not a tractable problem to solve due to the presence of stochastic variables in the objective function.

3.2.1 MVO - an equivalent deterministic formulation

The difficulty with a stochastic programming problem is that its solution is not well defined. This renders the problem to become intractable in a mathematical sense as it is not solvable through standard optimization methods.

To cope with this difficulty and to solve (SP), one requires an interpretation and a decision (Steuer et al. 2008). A common approach taken in the literature is to formulate the stochastic problem as an equivalent deterministic problem (Lundström and Svensson 2014). Typically, these formulations involve the utilization of some statistical characteristic or characteristics of the random variables. In short, (SP) has to be transformed into a simplified deterministic problem in order to be solvable. In this context, Steuer et al. (2008) argue that it is illuminating to delve into the rationale that leads from (SP) to an equivalent deterministic formulation and reviews some historical findings as follows.

In the early 17th century, mathematicians assumed that a gambler would be indifferent in receiving the uncertain outcome of a gamble and receiving its expected outcome in cash. In the context of portfolio selection, this assumption translates to the situation where an investor is indifferent in holding a portfolio of stocks or receiving its certainly equivalent (CE), defined as follows:

$$CE = E[R_P] \tag{3.2.3}$$

Under the assumption that an investor seek to maximize the amount of cash received for certain, one arrives at the following deterministic representation:

$$\begin{cases} \underset{w}{\text{maximize}} & E[R_P] = w^\top \mu \\ \text{subject to} & w \in \mathcal{F} \end{cases}$$
(3.2.4)

Recall here that the column vector of random future returns over a time period of n assets is denoted as: $R = (R_1, R_2, \ldots, R_n)^{\top}$. The column vector of expected values is further defined as $\mu = E[R]$.

However, in 1738, Bernoulli discovered the famously known St. Petersburg paradox (a translated version of his work can be found in Bernoulli (1954)). The paradox provides an example of a gamble with an infinite expected value but where a gambler in reality would be willing to forgo the gamble in exchange for a finite/less amount of money (Steuer et al. 2008). This example contradicts the classical theory that an investor is solely interested in maximizing the expected cash outcome without taking its volatility into account. Hence, in order to better match the real world setting and in contrast to the previous theory, Bernoulli suggested not to directly compare cash outcomes, but rather to compare the utilities of cash outcomes. Now, if the utility of a cash outcome is given by a function $U : \mathbb{R} \to \mathbb{R}$, the utility of the certain equivalent can be represented as follows:

$$U(CE) = E[U(R_P)] \tag{3.2.5}$$

More specifically, the utility of the certainty equivalent equals the expected utility of the uncertain future portfolio return. Given that an investor seeks to maximize the utility of the certainty equivalent, we arrive at Bernoulli's principle of maximizing expected utility as follows:

$$(\mathbf{UP})\begin{cases} \underset{w}{\text{maximize}} & E[U(R_P)]\\ \text{subject to} & w \in \mathcal{F} \end{cases}$$

This moves us one step further towards the mean-variance optimization problem suggested by Markowitz (1952). However, in its present form, (UP) cannot be solved as the utility function and its parameters are unknown. Therefore, the next step is to find a suitable utility function that properly mirrors the utility function of an investor.

In the literature, two schools of thought have evolved for dealing with the undetermined nature of the utility function. The first one involves attempting to incorporate an investor's preference structure into (UP) and obtaining an optimal portfolio, as suggested by Roy (1952). The other one, in the spirit of Markowitz, has arguably found most traction within the field of portfolio optimization. It involves a parameterization of the utility function U: thereafter, (UP) is solved for all possible values of its unknown parameters. In this context, Markowitz (1952) considered the following parameterized quadratic and concave utility function:

$$U(x) = x - (\delta/2)x^2 \tag{3.2.6}$$

Using this quadratic utility function, Markowitz (1952) showed that an optimal portfolio for an investor with a risk-aversion coefficient δ can be obtained by solving the following deterministic problem:

$$(\mathbf{DP})\begin{cases} \underset{w}{\text{maximize}} & E[R_P] - \frac{\delta}{2} Var(R_P) \\ \text{subject to} & w \in \mathcal{F} \end{cases}$$

The set of all optimal solutions of $w \in \mathbb{R}^n$ is called the efficient set, which constitutes the efficient frontier. This frontier refers to efficient portfolios that has the highest possible expected return given a specified level of variance, or conversely, the lowest possible variance given a specified level of return. The interpretation of Markowitz's suggested utility function is that an investor is solely interested in the relationship between expected return and variance when choosing between portfolios, where variance is a proxy for risk. Here, it is assumed that all investors are risk-averse. This means that any additional risk lowers the perceived utility. The risk-aversion coefficient determines the perceived cost of risk.

In greater detail, mean-variance optimization (MVO) seeks to find optimal asset allocations when both expected risk and return is considered. Indeed, this can be formulated mathematically in a number of ways. In this study the utility-maximization problem is considered, which implies that maximal utility is the objective for the investor. This is obtained by including both the expected risk and the expected return in the objective function, where the trade-off is determined by a risk aversion parameter. Extending (DP), we finally arrive at the following MVO problem:

$$(\mathbf{MVO}) \begin{cases} \underset{w}{\operatorname{maximize}} & w^{\top} \mu - \frac{\delta}{2} w^{\top} \Sigma w \\ \text{subject to} & \sum_{i=1}^{n} w_i = 1 \\ & \alpha_i \leq w_i \leq \beta_i, \ \forall i = \{1, 2, \dots, n\}. \end{cases}$$

In this form, the problem is solvable by employing tractable optimization methods. Furthermore, recall that $w^{\top}\mu$ denotes the expected portfolio return and that $w^{\top}\Sigma w$ is the portfolio variance, where Σ is the covariance matrix of all asset returns. Solving (MVO) for different values of the risk-aversion coefficient, one obtains an efficient frontier as displayed in Figure 3.4:



Volatility (I.e. Standard Deviation)

Figure 3.4: The blue frontier illustrates the efficient frontier. According to Markowitz, an investor should only consider portfolios lying on the efficient frontier when selecting a portfolio. The choice of portfolio on the efficient frontier depends on the risk-aversion coefficient of the investor.

The unconstrained optimal solution to the above MVO, i.e. when the portfolio weight constraints are lifted, has the following representation:

$$w^{\star} = \frac{\Sigma^{-1}\mathbb{1}}{c} + \frac{1}{\delta} \left(\Sigma^{-1} \left(\mu - \frac{\mathbb{1}a}{c} \right) \right)$$

$$a = \mathbb{1}^{\top} \Sigma^{-1} \mu$$
(3.2.7)

where

 $c = \mathbb{1}^\top \Sigma^{-1} \mathbb{1}$

$$w_{MVP} = \frac{\Sigma^{-1} \mathbb{1}}{c} \tag{3.2.8}$$

The proof of these solutions can be found in Hult et al. (2012, p.88) for the interested reader.

3.2.2 Matching the real world - short selling and transaction costs

In the setting of portfolio optimization, short-selling constraints are commonly imposed. The mathematical implication of this is to require the portfolio weights w_i :s to be greater or equal to zero. In practice, this constraint is frequently imposed as many funds and institutional investors are prohibited from selling short. A natural question that one may ask is whether enforcing such a constraint leads to suboptimal solutions as the optimizer is not allowed to freely allocate portfolio weights within the set of real numbers. Assuming that the optimizer is accurate and not influenced by estimation error, this is true in a global context. However, if the investor is prohibited from taking short positions, the constraint is not viewed as a limitation, but rather an adjustment so that the optimizer properly reflects the environment where the investor prevails.

Interestingly, it has also been shown in the literature that enforcing short-selling constraints often improves out-of-sample performance of the optimized portfolios. This was shown in Jagannathan and Ma (2003) who proved that mean-variance optimizers are implicitly applying some form of shrinkage on the sample covariance matrix when short positions are not allowed, consequently leading to more stable portfolio weights. Their findings can be explained by the prevalent perception that the optimal portfolio tends to amplify large estimation errors in certain directions. This stems from the inherent behavior of the mean-variance optimizer, which will assign large weights to assets that appear to have a small variance due to a significant underestimation. Similarly, if the expected return of an asset is significantly overestimated and appears to be large, a large weight will be assigned to the corresponding asset. Thus, the portfolio risk of the optimal portfolio is typically underpredicted and the return overpredicted (Karoui 2013). However, imposing short-selling constraints does not allow the optimizer to assign extreme weights as they are then bounded between zero and one. As a result, the problem of error amplification is reduced. This finding further motivates investors that apply MVO to rule out short sale positions.

The long-only MVO problem thus becomes:

$$\begin{cases} \underset{w}{\text{maximize}} & w^{\top} \mu - \frac{\delta}{2} w^{\top} \Sigma w \\ \text{subject to} & \sum_{i=1}^{n} w_i = 1 \\ & 0 \le w_i \le 1, \ \forall i = \{1, 2, \dots, n\}. \end{cases}$$
(3.2.9)

Furthermore, the inclusion of transaction costs in the portfolio selection problem is important to consider in order to better reflect the real world capital markets. In the initial MVO problem presented by Markowitz (1952), transaction costs were ignored. However, when portfolios are frequently rebalanced, the effect of transaction costs are far from insignificant. Recall that MV optimizers are by nature prone to estimation error due to the stochastic nature of future returns and covariances. In addition, small changes in these estimates (the vector of expected returns and the covariance matrix) can result in reallocations that would not necessarily occur if transaction costs were incorporated in the model. As a result, considering the inclusion of transaction costs and incorporating it into the model is expected to reduce the amount of trading and rebalancing. This is appealing for investors, as a low portfolio turnover is preferable.

Moreover, disregarding transaction costs may render inefficient portfolios as the cost of rebalancing may overwhelm the expected monetary gain of reallocating the portfolio. Thus, the investor's ability to allocate optimal portfolios is undermined.

There are numerous ways to implement transaction costs into the problem of portfolio selection. Some of these involve complicated nonlinear functions that emulate the cost penalty function of transaction costs. However, while these functions might potentially capture the actual incurred effects of transaction costs, they come with the cost of computational intractability. Thus, common approaches in practice involve a simplification to the transaction cost function which assumes the penalty function to only be dependent on the proportional cost of portfolio weight changes. Mathematically, the net expected portfolio return can thus be represented as follows:

$$E[R_P] = w^{\top} \mu - (\underbrace{b^{\top} \max\{0, w - w_0\} + s^{\top} \max\{0, w_0 - w\}}_{\text{penalty function of transaction costs}})$$
(3.2.10)

Here, $b \in \mathbb{R}^n$ is the proportional cost to purchase assets and $s \in \mathbb{R}^n$ the proportional cost to sell assets. Furthermore, w_0 contain the weights of the current portfolio, which is held at the time as the optimizer is initialized. In practice, when incorporating transaction costs, b and sare often set to be equal. Furthermore, one can make the assumption that the proportional cost of all individual assets are homogeneous. Clearly, this is a simplified case. It has the advantage of allowing the model to accommodate for transaction costs while still being easy to implement. Under the assumption that the cost of purchasing and selling assets is the same, in addition to assuming that the proportional transaction cost is the same for every asset, we can define the term accommodating for transaction costs in the following manner (using equation 3.2.10):

$$\psi \mathbb{1}^{\top} \Lambda = b^{\top} \max\{0, w - w_0\} + s^{\top} \max\{0, w_0 - w\}$$
(3.2.11)

Here, Λ is a vector of absolute values of portfolio weight changes, ψ a fixed proportional transaction cost, and $\mathbb{1}$ is a column vector of ones. In practice, ψ typically ranges between 10 to 50 basis points (Bessler, Opfer, and Wolff (2014);Gerber et al. (2015)).

Now, combining the long-only MVO problem in 3.2.9 with equation 3.2.11, we arrive at the following long-only MVO representation, where the effect of transaction costs is incorporated in the asset allocation model:

$$(\mathbf{MVO}) \begin{cases} \max_{w} \quad w^{\top} \mu - \psi \mathbb{1}^{\top} \Lambda - \frac{\delta}{2} w^{\top} \Sigma w \\ \text{s.t.} \quad \sum_{i=1}^{n} w_{i} = 1 \\ 0 \le w_{i} \le 1, \ \forall i = \{1, 2, \dots, n\}. \end{cases}$$
(3.2.12)

To clarify once again, Λ is a vector of absolute values of portfolio weight changes, ψ a fixed proportional transaction cost, and 1 is a column vector of ones. δ is a risk aversion parameter that determines the trade-off between risk and return in the problem. The first constraint forces the portfolio to be fully invested in the included assets. The second constraint bounds the weights for each asset, where 0 and 1 is the lower and upper bound, respectively (short-selling is thus prohibited). Indeed, (3.2.12) requires both μ and Σ to be estimated before the optimization is attempted. This leaves us the important task of constructing viable estimates, which is the purpose of the subsequent sections.

Furthermore, in (3.2.12) the matrix Σ needs to be positive semidefinite for the problem to be concave and consequently have the property that a local optimum is also a global optimum (see Section 3.1.3). If the matrix is negative definite, the problem is non-concave, making it difficult to find the global optimum. In order for the matrix to be a valid covariance matrix, it should by definition be positive semidefinite.

3.3 Estimating expected returns

The expected returns need to be estimated before problem 3.2.12 can be solved. The Capital Asset Pricing Model (CAPM) proposes a method for such estimation. Under rational expectations of investors, the CAPM model offers a quick quantitative insight of risk-reward interplay of assets. Three main assumptions underlie CAPM (Berk and DeMarzo 2014):

- Investors are rational and choose mean-variance efficient portfolios according to Markowitz (1952).
- 2. Investors are in complete agreement: i.e. investors have homogeneous expectations regarding the volatilities, correlations and expected returns of the assets.
- 3. Investors can borrow and lend at the risk-free rate, which is the same for all investors.

Under these assumptions, the CAPM implies that the market portfolio of all risky securities is an efficient portfolio. Furthermore, the capital market line (CML) is spanned by the set of portfolios with the highest possible expected return for a given level of volatility (risk) (Berk and DeMarzo 2014).

At its core, CAPM is a single-factor linear model (commonly referred to as the security market line) that relates the expected return of an asset and the market portfolio. The mathematical representation of the security market line is as follows:

$$\mu_i = E[R_i] = r_f + \underbrace{\beta_{i,mkt}(E[R_{mkt}] - r_f)}_{\text{Risk premium for security }i}$$
(3.3.1)

Here, $\beta_{i,mkt}$ serves as a measure of non-diversifiable (systematic) risk. It measures the amount of risk associated with an asset, that is common to the market risk. R_i is the return of some asset, R_{mkt} the total return of the market, and r_f the risk-free interest rate available to all investors in the market. The estimate of $\beta_{i,mkt}$ can be obtained by using the sample covariance and variance:

$$\beta_{i,mkt} = \frac{Cov(R_i, R_{mkt})}{Var(R_{mkt})} \tag{3.3.2}$$

An estimate of the expected market return $E[\mu_{mkt}]$, denoted as $\hat{\mu}_{mkt}$, is obtained by taking the mean return over a lookback period for each asset, where the mean returns are denoted as $\hat{\lambda}_i$, and then weighing the means by their respective market weight m_i . In other words, $\hat{\mu}_{mkt} = m^{\top}\hat{\lambda}$ where $m = (m_1, \ldots, m_N)^{\top}$ and $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_N)^{\top}$. This is referred to as a value-weighted market index and serves merely as a proxy for the market portfolio. In reality, the global market portfolio is unknown which is why proxies are employed. Now, the CAPM estimated return vector is denoted:

$$\hat{\mu} = (\hat{\mu}_1, \dots, \hat{\mu}_N)^{\top}$$
 (3.3.3)

where each individual return estimate is obtained by the estimated CAPM returns, i.e.:

$$\hat{\mu}_i = r_f + \hat{\beta}_{i,mkt} (\hat{\mu}_{mkt} - r_f) \tag{3.3.4}$$

In Section 3.4.2, the idea behind single-index models such as CAPM is explained in greater detail, hopefully providing a better understanding of the model.

3.4 Estimating the covariance matrix

In order to estimate the risk of a portfolio, one needs to know to which degree the assets in the portfolio face common risks and how their returns move together. For this intended purpose, the covariance is a widely employed measure.

In the context of covariance matrix estimation, the conventional sample covariance matrix has the appealing property of being the maximum likelihood estimator under the assumption of normality. This means that it is the best unbiased estimator. However, it also comes with drawbacks. Being the maximum likelihood estimator, all the trust is put in the data. This is a sound principle, provided that there is enough data (Ledoit and Wolf 2003b). In small samples, however, the estimator is subject to the risk of overfitting the data (follows noise too closely). This means that the sample covariance matrix (also referred to as the historical covariance matrix) may perform poorly out-of-sample, despite the fact that it performs best in-sample. Intuitively, one might think that increasing the lookback window of the sample solves this, but in reality, it may come at the cost of trusting outdated data with little explanatory power for the future. In a sense, the estimator is a double edged sword as the low bias often comes at the cost of high variance (see top right circle in Figure 3.5). In addition, the sample covariance matrix runs the risk of becoming ill-conditioned if the number of assets under consideration is large relative to the number of historical observations. More specifically, if the number of assets exceeds the number of observations for every asset, the sample covariance matrix will not be invertible which is very alarming in a portfolio optimization context (Bengtsson and Holst 2002).



Figure 3.5: Illustration of the bias-variance tradeoff. The blue dots correspond to estimates. Estimates within the red circle are considered accurate. Source: Fortmann-Roe (2012, p. 1).

3.4.1 The sample covariance matrix

Let $r_{i,t}$ denote the historical return for asset *i* at time period *t*. Then, the average historical return over the time span [1, T] with step increment of size one for each asset *i* is given by:

$$\bar{r}_i = \frac{1}{T} \sum_{t=1}^T r_{i,t}$$
(3.4.1)

The sample covariance between two assets can then be estimated in the following manner:

$$Cov(r_i, r_j) = \frac{1}{T - 1} \sum_{t=1}^{T} (r_{i,t} - \bar{r}_i)(r_{j,t} - \bar{r}_j) \coloneqq \hat{\sigma}_{ij}$$
(3.4.2)

Performing equation 3.4.2 for all combinations i, j of assets, the historical covariance matrix for N assets is then obtained by:

$$\widehat{\Sigma}_{HC} = \begin{bmatrix} \widehat{\sigma}_{11} & \widehat{\sigma}_{12} & \cdots & \widehat{\sigma}_{1N} \\ \widehat{\sigma}_{21} & \widehat{\sigma}_{22} & \cdots & \widehat{\sigma}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\sigma}_{N1} & \widehat{\sigma}_{N2} & \cdots & \widehat{\sigma}_{NN} \end{bmatrix}$$

Note that the covariance matrix Σ can be constructed according to:

$$\Sigma = \operatorname{diag}(\sigma) C \operatorname{diag}(\sigma) \tag{3.4.3}$$

where σ is a column vector of standard deviations. diag(σ) denotes a matrix with the elements of σ on the main diagonal. C is a correlation matrix. Thus, by changing the method of calculating correlation and in turn the correlation matrix C, different covariance matrices can be obtained. Here, the estimate of σ , denoted as $\hat{\sigma}$, is obtained as the sample standard deviation of the historical asset returns. Note that $\hat{\sigma}$ will not depend on which correlation estimation method is used. Therefore, the estimated covariance matrices obtained by historical correlation are:

$$\widehat{\Sigma}_{HC} = \operatorname{diag}(\widehat{\sigma})\widehat{C}_{HC}\operatorname{diag}(\widehat{\sigma})$$

An expression of C_{HC} is given by:

$$C_{HC} = \begin{bmatrix} \rho_{1,1} & \rho_{1,2} & \cdots & \rho_{1,N} \\ \rho_{2,1} & \rho_{2,2} & \cdots & \rho_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N,1} & \rho_{N,2} & \cdots & \rho_{N,N} \end{bmatrix}$$
(3.4.4)

where $\rho_{i,j}$ is the correlation between asset returns r_i and r_j . The estimate of C_{HC} , denoted as \hat{C}_{HC} , is computed by using the pairwise sample correlations of the historical asset returns.

3.4.2 The single-index market model

A prominent competitor to the sample covariance matrix is the single-index model introduced by Sharpe (1964). The single-index model is a one factor model that attempts to cure the problem of overfitting associated with the sample covariance matrix by imposing structure on the estimator.

By observing stock prices, one can see that individual stock prices tend to move together with the aggregated market. Not surprisingly, the market factor thus usually proves to be the most important factor when explaining the return generating process of stock returns. In addition, this suggests that the comovement between stocks to some extent stems from a common response to market changes.

In Sharpe's (1964) single-index model, one attempts to capture this correlation by relating the return on an individual stock to the return of a stock market index in the following manner:

$$R_i = a_i + \beta_i R_m \tag{3.4.5}$$

where a_i is a component of the return that is independent of market changes (random variable), R_m is the return on the employed market index (random variable) and β_i is a constant that measures the expected change in R_i given a change in R_m . More specifically, beta (β) measures the sensitivity of asset *i*'s return to the market index return. A beta of 0.5 implies that the return of the asset is expected to increase (decrease) by 0.5% when the market increases (decreases) by 1%.

The term a_i can be decomposed into two components as following:

$$a_i = \alpha_i + \epsilon_i \tag{3.4.6}$$

where α_i is the expected value of a_i and ϵ_i is the random element of a_i , where $E[\epsilon_i] = 0$. The return of an asset *i* can then be written as:

$$R_i = a_i + \beta_i R_m + \epsilon_i \tag{3.4.7}$$

where the residual ϵ_i and the market return R_m are uncorrelated by construction (Elton, Gruber, Brown, et al. 2014). This means that:

$$Cov(\epsilon_i, R_m) = 0 \tag{3.4.8}$$

which can be ensured with regression analysis.

Up to this point, all equations can be made to hold by construction. To proceed in the derivation of the single-index covariance matrix estimator, an assumption now must be made. The assumption is that the residual terms between different assets are independent, i.e. that ϵ_i and ϵ_j are independent for all *i* and *j* such that:

$$E[\epsilon_i \epsilon_j] = 0 \ \forall \ i, j \ (i \neq j) \tag{3.4.9}$$

This is the core assumption of Sharpe's (1964) single-index model. It implies that the only reason that two securities move together is due to their common response to market changes. As such, the model assumes that there are no effects beyond the market (e.g. industry and firm size effects) that account for comovement between securities. This assumption can not be made to hold by regressions analysis, unlike the assumed independence between the residual and the market. It is merely an idealized assumption that represents an approximation of reality, and is not to be confused as the truth. Thus, the validity of the single-index model and how well it describes asset characteristics is to a large extent dependent on how well this assumption holds.

Having laid down the assumptions and the basic equation that underlie the single-index model, one can now show that the variance of an asset i implied by the model is given by:

$$\sigma_i^2 = \underbrace{\beta_i^2 \sigma_m^2}_{\text{Systematic risk}} + \underbrace{\sigma_{\epsilon_i}^2}_{\text{Firm specific risk}}$$
(3.4.10)

where σ_m^2 is the variance of the market returns and $\sigma_{\epsilon i}^2$ is the variance of ϵ_i . Moreover, the covariance between two assets i, j is given by:

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 \ (i \neq j) \tag{3.4.11}$$

Note here that an asset's variance has two components, a systematic risk (market risk) and a firm specific risk whereas the covariance between two assets solely depends on the systematic risk, which is a result of the above mentioned key assumption. To be parsimonious, the complete derivations of equation 3.4.10 and equation 3.4.11 are left out here. A rigorous derivation for these results can be found in e.g. Elton, Gruber, Brown, et al. (2014).

Now, let $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_N)^{\top}$ represent a vector containing estimated betas for each asset and $\hat{\sigma}_{\epsilon}^2 = (\hat{\sigma}_{\epsilon 1}^2, \dots, \hat{\sigma}_{\epsilon N}^2)^{\top}$ denote the vector of estimated residual terms. These estimates can be obtained by performing linear regressions for each asset. The estimated covariance matrix implied by the single-index model is then given by:

$$\widehat{\Sigma}_{SI} = \widehat{\beta}\widehat{\beta}^{\top}\widehat{\sigma}_m^2 + \operatorname{diag}(\widehat{\sigma}_{\epsilon}^2) \tag{3.4.12}$$

where $\hat{\sigma}_m^2$ denotes the sample variance of the market index returns.

Note that the single-index covariance matrix only requires 2N+1 parameters to be estimated, as opposed to the historical covariance matrix where one must estimate N(N+1)/2 parameters. This is a major reduction and since one has more data per estimation, the estimation error is expected to decrease (Bengtsson and Holst 2002). However, it comes at the expense of introducing specification error (bias) due to its rather restrictive assumption that there are no effects beyond the market that account for comovement between assets.

3.4.3 Shrinkage towards the single-index model

Thus far, two extreme estimators have been presented. The first one being the sample covariance estimator which puts all the trust in the data and that can be considered as a full N factor model where each individual asset is considered a factor. On the other hand, the single-index is an extreme estimator in the sense that it is a one factor model that with strong structure. Now, recall that imposing structure comes at the expense of introducing specification error due to omitted variables while the unbiased sample covariance estimator may suffer from instability. In short, there exists a trade-off between bias and variance in the realm of estimation.

This is where the concept of shrinkage comes at play. In this context, shrinkage is a Bayesian statistical procedure that strives to find an optimal compromise between the sample covariance estimator and some prior with strong structure. The idea stems from the notion that the best model lies between these two extremes.

The shrinkage estimator that is considered in this thesis is the one introduced by Ledoit and Wolf (2003b). They suggest to use the single-index covariance matrix, introduced in Section 3.4.2, as the prior. Following Ledoit and Wolf (2003b), let F denote the estimated covariance matrix implied by the single-index model of Sharpe (1964) and S denote the estimated sample covariance matrix. Furthermore, it is assumed that F converges to the true covariance matrix implied by the single-index model as $T \to \infty$, which is assumed not to be equal to the true covariance matrix. The proposed shrinkage estimator then has the following representation:

$$\alpha F + (1 - \alpha)S \tag{3.4.13}$$

where $\alpha, 0 \leq \alpha \leq 1$, denotes the shrinkage intensity. This is the weight that is assigned to the single-index estimator and controls the amount of structure that is imposed on the sample covariance estimator. Note that more structure is imposed on the shrinkage estimator as the shrinkage intensity increases. In addition, asset returns are assumed to be independent and identically distributed with finite fourth moments (Ledoit and Wolf 2003b). Now, the difficulty lies in finding the optimal shrinkage intensity, α . Ledoit and Wolf (2003b) suggest to use a quadratic measure of distance between the asymptotically true covariance matrix and the shrinkage estimator based on the Frobenius norm as the objective function in determining the optimal shrinkage intensity.

Note that the definition of the Frobenius norm of a symmetric matrix $H \in \mathbb{R}^{N \times N}$ with entries h_{ij} and eigenvalues $\lambda_i, i = 1, \ldots, N$, follows as:

$$||H||_F^2 = tr(H^2) = \sum_{i=1}^N \sum_{j=1}^N h_{ij} = \sum_{i=1}^N \lambda_i^2$$
(3.4.14)

The quadratic loss function is thus defined as:

$$L(\alpha) = \|\alpha F + (1 - \alpha)S - \Sigma\|_F^2$$
(3.4.15)

where Σ is the true covariance matrix. Now, minimizing the expected value of $L(\alpha)$ with respect to α yields the following optimal shrinkage intensity (Ledoit and Wolf 2003b):

$$\alpha^{\star} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} Var(s_{ij}) - Cov(f_{ij}, s_{ij})}{\sum_{i=1}^{N} \sum_{j=1}^{N} (Var(f_{ij} - s_{ij}) + (\phi_{ij} - \sigma_{ij})^2)}$$
(3.4.16)

where ϕ_{ij} denote the entries for the true covariance matrix implied by the single-index model, s_{ij} the entries of the estimated sample covariance matrix and f_{ij} the entries for the estimated covariance matrix implied by the single-index model.

Invoking the following theorem, the asymptotic behavior of the optimal shrinkage intensity is shown.

Theorem 1 Let π denote the sum of the asymptotic variances of the entries in the sample covariance matrix scaled by \sqrt{T} , i.e. $\pi = \sum_{i=1}^{N} \sum_{j=1}^{N} AsyVar[\sqrt{T}s_{ij}]$. Also, let ρ denote the sum of asymptotic covariances of the entries in the estimated single-index covariance matrix with the entries of the sample covariance matrix scaled by \sqrt{T} , i.e. $\rho = \sum_{i=1}^{N} \sum_{j=1}^{N} AsyCov[\sqrt{T}f_{ij}, \sqrt{T}s_{ij}]$. Lastly, let γ measure the specification error of the true covariance matrix implied by the singleindex model, i.e. $\gamma = \sum_{i=1}^{N} \sum_{j=1}^{N} (\phi_{ij} - \sigma_{ij})^2$. Then the optimal shrinkage intensity satisfies the following:

$$\alpha^{\star} = \frac{1}{T} \frac{\pi - \rho}{\gamma} + O\left(\frac{1}{T^2}\right) \tag{3.4.17}$$

See Ledoit and Wolf (2003b) for the proof.

Note that the weight placed on the shrinkage target increases as the error of the sample covariance matrix increases and decreases as the specification error of the shrinkage target (single-index model in this case) increases. Furthermore, let κ denote the constant that determines the shrinkage intensity:

$$\kappa = \frac{\pi - \rho}{\gamma} \tag{3.4.18}$$

which inserted in equation 3.4.13 gives the following optimal shrinkage estimator (asymptotically):

 $\frac{\kappa}{T}F + \left(1 - \frac{\kappa}{T}\right)S$



Figure 3.6: Geometric interpretation of Theorem 1. Source: Ledoit and Wolf (2003b).

(3.4.19)

Estimating the optimal shrinkage intensity

At this point, 3.4.19 is not practically useful as κ depends on unknown parameters (π, ρ, γ) . Hence, κ has to be estimated in order for equation 3.4.19 to be applicable. First, decompose π such that $\pi = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_{ij}$ where $\pi_{ij} = AsyVar[\sqrt{T}s_{ij}]$. Similarly, let $\rho = \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij}$ where $\rho_{ij} = AsyCov[\sqrt{T}f_{ij}, \sqrt{T}s_{ij}]$ and $\gamma = \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}$ where $\gamma_{ij} = (\phi_{ij} - \sigma_{ij})^2$. Now, following Ledoit and Wolf (2003b), we have the three following Lemmas.

Lemma 1 A consistent estimator for π_{ij} is given by:

$$\hat{\pi}_{ij} = \frac{1}{T} \sum_{t=1}^{T} ((r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j) - s_{ij})^2$$
(3.4.20)

See Ledoit and Wolf (2003b) for the proof.

Lemma 2 A consistent estimator for ρ_{ii} is given by $\hat{\rho}_{ii} = \hat{\pi}_{ii}$. For $i \neq j$, a consistent estimator for ρ_{ij} is given by $\hat{\rho}_{ij} = \frac{1}{T} = \sum_{t=1}^{T} \hat{\rho}_{ijt}$ where

$$\hat{\rho}_{ijt} = \frac{s_{jM}s_{MM}a_{it} + s_{iM}s_{MM}a_{jt} - s_{iM}s_{jM}b_{Mt}}{s_{MM}^2}b_{Mt}a_{it}a_{jt} - f_{ij}s_{ij}$$
(3.4.21)

where M denotes the market index element, $a_{it} = r_{it} - \bar{r}_i$ and $b_{Mt} = r_{Mt} - \bar{r}_M$.

See Ledoit and Wolf (2003b) for the proof.

Lemma 3 A consistent estimator for $\gamma_{ij} = (\phi_{ij} - \sigma_{ij})^2$ is given by its estimated counterpart, $\hat{\gamma}_{ij} = (f_{ij} - s_{ij})^2$.

See Ledoit and Wolf (2003b) for the proof.

Employing all three Lemmas, the consistent estimator for the optimal shrinkage constant thus follows as:

$$\hat{\kappa} = \max\left(0, \min\left(\frac{\hat{\pi} - \hat{\rho}}{\hat{\gamma}}, 1\right)\right) \tag{3.4.22}$$

where $\hat{\kappa}$ is truncated to ensure that $0 \leq \hat{\kappa} \leq 1$. Now, the Ledoit and Wolf (2003b) optimal shrinkage estimator with the single-index model as the shrinkage target becomes the following (asymptotically):

$$\widehat{\Sigma}_{SM} = \frac{\widehat{\kappa}}{T}F + \left(1 - \frac{\widehat{\kappa}}{T}\right)S \tag{3.4.23}$$

3.4.4 Random matrix theory

In the context of modern portfolio theory, random matrix theory (RMT) is considered to be a relatively new area despite its development by quantum physicists in the 1950s.

The idea of employing RMT in estimating the covariance matrix of financial returns stems from the problems that arise in the sample covariance matrix when the data per estimated parameter is low. More specifically, such an estimator runs the risk of being dominated by measurement noise. Therefore, it is interesting to distinguish between noise and signal (information). RMT enables one to achieve this by comparing the properties of the sample correlation matrix to a 'null hypothesis' completely random matrix.

Consider a random correlation matrix \tilde{C} . This can be written as $\tilde{C} = \frac{1}{T}GG^{\top}$, where $G \in \mathbb{R}^{N \times T}$. Furthermore let the variance of the elements in G equal to 1 and have zero mean. It can be shown that when N (number of assets) and T (observations per asset) tend to infinity such that $Q = T/N \geq 1$ is fixed, then the density of eigenvalues of \tilde{C} is given by (Laloux et al. 2000):

$$\rho_{\widetilde{C}}(\lambda) = \frac{Q}{2\pi} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda_{min} - \lambda)}}{\lambda}$$
(3.4.24)

where $\lambda \in [\lambda_{min}, \lambda_{max}]$. The theoretical maximum and minimum eigenvalues predicted by RMT are given by:

$$\lambda_{min}^{max} = 1 + \frac{1}{Q} \pm 2\sqrt{\frac{1}{Q}}$$
 (3.4.25)

Via the eigen decomposition theorem, the sample correlation matrix C can be decomposed in the following manner:

$$C = E\Lambda E^{-1} \tag{3.4.26}$$

where E is a square matrix whose i^{th} column is given by the i^{th} eigenvector and Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues.

Laloux et al. (2000) suggest to enhance the correlation matrix C by filtering out noisy eigenvalues in the following manner. Employing equation 3.4.25, the idea is that eigenvalues beneath the upper noise band (the maximum eigen value) are considered to be noise that does not contain information. These eigenvalues constitute the noisy part of C, not expected to contain real information. The filtering method by Laloux et al. (2000) is to assign all noisy eigenvalues to the average of these eigenvalues as they are considered equally useless. The reasoning for why the noisy eigenvalues are updated with the average of their previous values is to retain the trace of the correlation matrix such that the variance of the system is preserved.

Moreover, eigenvalues in Λ that exceed the theoretical maximum eigenvalue are preserved as they are considered to significantly deviate from the random matrix noise, thus containing signal (information).

Having filtered the noisy eigenvalues according to the above, the cleansed correlation matrix is now re-built in the following manner:

$$C_{filtered} = E\Lambda_{filtered}E^{-1} \tag{3.4.27}$$

This yields us the following covariance matrix implied by the random matrix filtering estimator suggested by Laloux et al. (2000):

$$\widehat{\Sigma}_{RMT} = \operatorname{diag}(\widehat{\sigma})\widehat{C}_{filtered}\operatorname{diag}(\widehat{\sigma}) \tag{3.4.28}$$

where, $\operatorname{diag}(\hat{\sigma})$ is a diagonal matrix whose diagonal elements are the sample standard deviations.

3.4.5 Gerber statistic based covariance matrix

Explained as a "robust co-movement measure" by Gerber et al. (2015), the Gerber Statistic (GS) introduces a new method for estimating correlation between asset returns in MPT. The statistic is applied to pairs of series observations, where each pair of observations obtains a value in $\{-1, 0, 1\}$. By introducing time-indexation n of observed pairs of returns $r_i^{(n)}$ and $r_j^{(n)}$, corresponding to observations of the asset returns R_i and R_j at time n, we create a new sequence

of observations $M_{i,j}^{(n)}$. This sequence depends on a determined threshold value denoted by T_i and T_j respectively. Then, the values $M_{i,j}^{(n)}$ are computed based on whether the observed return pairs exceed their respective thresholds or not. The procedure for calculating the sequence goes as follows:

$$M_{i,j}^{(n)} = \begin{cases} 1, & \text{if } [r_i^{(n)} \ge T_i \& r_j^{(n)} \ge T_j] \text{ or } [r_i^{(n)} \le -T_i \& r_j^{(n)} \le -T_j] \\ -1, & \text{if } [r_i^{(n)} \ge T_i \& r_j^{(n)} \le -T_j] \text{ or } [r_i^{(n)} \le -T_i \& r_j^{(n)} \ge T_j] \\ 0, & \text{otherwise.} \end{cases}$$
(3.4.29)

The threshold can be based upon a standard deviation of past variable movement, whereby a lower standard deviation represents a lower threshold which implies higher sensitivity to variable movement. For example, a threshold may be set to a multiple of the variable's standard deviations based on past behavior. Here, the thresholds T_i and T_j are defined as 0.5 standard deviations of a rolling window of asset returns $r_i^{(n)}$ and $r_j^{(n)}$ respectively, following Gerber et al. (2015). The Gerber Statistic $GS_{i,j}$ for the assets *i* and *j* is then calculated as:

$$GS_{i,j} = \frac{\sum_{n} M_{i,j}^{(n)}}{\sum_{n} |M_{i,j}^{(n)}|}.$$
(3.4.30)

Note that $-1 \leq GS_{i,j} \leq 1$. Then, for any selection of asset returns $R = (R_1, \ldots, R_N)^{\top}$, a GS-based correlation matrix can be obtained by:

$$\widehat{C}_{GS} = \begin{bmatrix} GS_{1,1} & GS_{1,2} & \cdots & GS_{1,N} \\ GS_{2,1} & GS_{2,2} & \cdots & GS_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ GS_{N,1} & GS_{N,2} & \cdots & GS_{N,N} \end{bmatrix}$$
(3.4.31)

where the GS-based covariance matrix estimate $\widehat{\Sigma}_{GS}$ is obtained by:

$$\widehat{\Sigma}_{GS} = \operatorname{diag}(\widehat{\sigma})\widehat{C}_{GS}\operatorname{diag}(\widehat{\sigma}) \tag{3.4.32}$$

Here, diag($\hat{\sigma}$) is simply a diagonal matrix holding the sample standard deviation estimates. Furthermore, \hat{C}_{GS} cannot be guaranteed to to be positive semidefinite (Gerber et al. 2015). This is a problem for any optimizer and needs correction in terms of either finding the closest positive semidefinite matrix or a change in method of calculation of Σ_{GS} . To cope with this shortcoming, GS based covariance matrix estimators may have to be modified in a way that the matrix becomes positive semidefinite. The procedure for this is explained in Section 3.4.6. Alternatively, one can alter the thresholds to not be as restrictive.

As shown in 3.4.29, the discretization process associated with the Gerber Statistic implies that data points that are beyond their thresholds are given equal weight, regardless of the magnitude of exceedance. In contrast, more conventional measures consider the magnitude of deviation which renders the measure to be sensitive to outliers that may erroneously skew the analysis. In short, the idea behind the Gerber Statistic considered in this thesis is to eliminate noise from data and to normalize any outliers: the estimator is only interested in significant coordinated movements and shrinks the effect of perceived outliers. In doing so, the estimator aims to be more robust to noise in the data and attempts to reduce relationships inferred by statistical fluctuations.

To better gain an understanding of the GS-based covariance matrix, a lucid example follows below. First, in the following Figure 3.7, 24 monthly returns have been simulated for two assets. Let us call them asset A and asset B.



Figure 3.7: Simulated asset returns in a 24-month window.

Now, we need to impose the thresholds for the two assets. Recall that the thresholds are based upon a standard deviation of past variable movement for respective asset. In this case, the lookback window for past movements is 24 months. This is depicted in Figure 3.8, following below:



Figure 3.8: Thresholds imposed on both assets, which determines the sensitivity to variable movement.

In the above Figure 3.8, month four, nine and eighteen are highlighted. This is due to the fact that they all serve as great examples of different cases. The first case, in month four, both assets exceed their thresholds in the same direction. As they both exceed their thresholds, the pairwise movement is considered to be significant. In addition, as they move in the same direction, the value of the case is assigned 1 (positive union). Furthermore, in month nine, asset B exceeds its threshold while asset A does not. Hence, their pairwise movement is not considered to be significant. Lastly, one can observe that respective asset exceed their thresholds in month eighteen. However, they move in opposing directions. Thus, this value is assigned to be -1 (negative union).

After further inspection of Figure 3.8, one can derive seven significant pairwise returns. This is shown in Figure 3.9.



Figure 3.9: An illustration of the significant pairwise returns.

Alternatively, the above is depicted in the following manner:



Figure 3.10: Significant pairwise returns. Points lying in upper right area and the lower left area correspond to significant pairwise returns in the same direction (marked blue). Points lying in the upper left and lower right area correspond to significant pairwise returns in the opposing direction (marked red).

Now, by employing the definition of the Gerber Statistic, defined in equation 3.4.30, together with Figure 3.10 we arrive at the following estimate of monthly correlation between the assets:

$$GS_{A,B} = \frac{\sum_{i=1}^{4} 1 - \sum_{i=1}^{3} 1}{\sum_{i=1}^{7} 1} = \frac{1}{7} \approx 0.1429$$

Comparing this to the sample correlation coefficient, which would amount to approximately -0.0210, a clear difference in the estimates is noted.

3.4.6 Nearest positive semidefinite covariance matrix

In mean-variance optimization, it is important that the estimated covariance matrix is positive semidefinite. If not, the definition of a covariance matrix is violated which is alarming (negative variances are undefined). In addition, it may lead to unstable solutions from the optimizer. Clearly, this poses a problem in the optimization process since a solution to the problem cannot be guaranteed to be a globally optimal solution.

In Gerber et al. (2015), it is recognized that in computing the Gerber Statistic, a non-positive semidefinite correlation matrix could theoretically be obtained. While not found to occur in their study, it is important to be aware of the problem as it would halt the optimization process.

To cope with this potential problem, Higham (2002) proposes a method of finding the nearest correlation matrix, or in other words, the nearest symmetric positive semidefinite matrix with unit diagonal. In this sense, the nearest matrix is the matrix that minimizes the Frobenius norm between the matrices. This is not a trivial task. However, Higham (2002) derived an alternatating projections algorithm for solving the optimization problem (minimizing the Frobenius norm) of finding the nearest correlation matrix, which will be used in this thesis¹. This algorithm can also be employed for finding the nearest positive semidefinite covariance matrix by relaxing the constraint that the matrix must have a unit diagonal.

3.5 Performance measures

This section will define the performance measures used to evaluate the optimized portfolios in this thesis.

3.5.1 Sharpe ratio

The Sharpe ratio, introduced by Sharpe (1964), is a commonly used measure in finance. It measures the ratio between excess return (return after substracting the risk-free rate) and volatility. In this thesis, the Sharpe ratio for a strategy i is defined as:

$$SR_i = \frac{\bar{R}_i - \bar{R}_f}{\bar{\sigma}_i} \tag{3.5.1}$$

where \bar{R}_i denotes the annualized net return (after transaction costs) for strategy *i*. \bar{R}_f is the annualized risk-free rate for the evaluated period and $\bar{\sigma}_i$ is the annualized volatility for strategy *i*. The Sharpe ratio characterizes how well one is compensated for the risk taken. Thus, a higher Sharpe ratio is sought, as this implies more return per risk taken. This is a strong measure under the assumption that volatility (standard deviaton) as a good proxy for risk holds true.

3.5.2 Maximum drawdown

To further evaluate portfolio performance, the maximum drawdown is also a prominent measure. It measures the maximum loss from a peak to a nadir over a period of time of a portfolio, and complements the notion of using volatility well as it is an indicator of downside risk. In other words, it measures the maximum accumulated loss that an investor may suffer from buying high and selling low. The maximum decline of a value series i measured in return is defined as:

¹MATLAB code that implements the algorithm of finding the nearest correlation matrix can be found at Nick Highams website, nickhigham.wordpress.com.

$$MDD_{i} = \max_{i,t^{\star} \in (0,T)} \left[\max_{i,t \in (0,t^{\star})} \left(\frac{V_{i,t} - V_{i,t^{\star}}}{V_{i,t}} \right) \right]$$
(3.5.2)

where $V_{i,t}$ denotes the value of the portfolio at period t when the portfolio of strategy i is rebalanced. Clearly, a lower maximum drawdown (MDD) reflects a less risky strategy. Thus, a low maximum drawdown is attractive.

3.5.3 Portfolio weight turnover

Following DeMiguel and Nogales (2009), portfolio turnover provides information regarding the stability of a strategy i that rebalance portfolios over an investment horizon. It measures the extent of trading that has to be done to implement the strategy. A low turnover is often preferable, as it reduces risks such as liquidity risks and implies lower transaction costs. The portfolio turnover of a strategy i is defined as:

$$PT_{i} = \frac{1}{T} \sum_{t=1}^{T} \sum_{j=1}^{N} \left(|w_{i,j,t+1} - w_{i,j,t}| \right)$$
(3.5.3)

where T is the number of rebalancing points and $w_{i,j,t+1}$ is the weight of asset j under strategy j at time t + 1. N is the size of the considered asset universe. I.e., equation 3.5.3 measures the average absolute changes of the portfolio weights over the T rebalancing points.

3.5.4 Risk-adjusted return

To compare returns for portfolios obtained via a different estimator than employing the sample covariance matrix, these returns have to be adjusted so that they are associated with the same level of volatility. This can be achieved by deleveraging the volatility associated with the competing portfolio i in the following manner:

$$RAR_i = R_f + \frac{R_i - R_f}{\sigma_i} \sigma_{HC}$$
(3.5.4)

where RAR_i is the risk-adjusted return for the competing portfolio *i*, R_f the risk-free rate and σ_{HC} the standard deviation for the portfolio obtained by using the sample covariance matrix in MVO.

Chapter 4 Methodology

This chapter will present the methodology used to answer the research questions in this thesis. First, the data included in the portfolio optimization procedure is presented. This is followed by a description of the methodology adapted to evaluate the performance of portfolio performance.

4.1 Data

In order to construct multi-asset portfolios, global stocks, bonds and commodity indices will be included in the asset universe. More specifically, the S&P500 (U.S. large cap), Russell 2000 (U.S. small cap), MSCI EAFE (developed market outside the U.S. and Canada) and MSCI Emerging Markets (captures large and mid cap representation across emerging markets) equity indices will be included to cover both developed and emerging markets as well as large cap and small cap stocks. Emerging markets usually are associated with higher stock returns compared to developed markets due to higher exposure to risk factors such as low liquidity or political conditions (Iqbal, Brooks, and Galagedera 2010). It has further been shown that including equity that allows for international diversification is beneficial in a portfolio selection context (Chiou, Lee, and Chang 2009). Hence why these different indices will be included in the optimization process.

Furthermore, bonds will be included as an asset class. More specifically, US Government bonds will be employed as a low risk investment. As bond prices are typically negatively correlated with stock prices, the inclusion of this asset class is expected to provide additional diversification benefits during e.g. stock market downturns. In addition, high yield corporate bonds will be included, adding exposure to corporate default risk. These are typically associated with higher returns, however often penalized with higher risk exposure, in comparison to government bonds. To represent US Government bonds, the Barclays U.S.-Government Bond (all maturities) index will be employed. Furthermore, the Barclays U.S. Corporate High Yield index will be employed to represent high yield bonds.

Lastly, real estate and commodities will be included in the investment universe. In this context, the FTSE U.S. Real Estate Investment Trust (REIT) index will be employed to represent real estate. Furthermore, the S&P GSCI index and gold will be included to provide diverse investments in commodities. Several studies have found that investing in commodities can be viewed against a hedge against inflation and that including commodities in the investment universe may yield more efficient portfolios in the sense that higher Sharpe ratios can be obtained (Anson (1999); Conover et al. (2010)). This justifies the inclusion of commodity as an asset class. As for real estate, the FTSE U.S. REIT index is expected to further improve diversification possibilities. In short, the data to be included in the optimization procedure are monthly series on asset prices from which monthly returns are computed for all included assets. These will be corrected for dividends and capital changes such as splits. The assets to be included in the optimization are global indices: S&P500 (U.S. large cap), Russell 2000 (U.S. small cap), FTSE U.S. Real Estate Investment Trust (REIT), MSCI EAFE (developed market outside the U.S. and Canada), MSCI Emerging Markets, Barclays U.S.-Government Bond (all maturities), Barclays U.S. Corporate HY, Gold, and S&P GSCI (diversified commodity index). The price data stretches from January 1992 to December 2013 with 264 observations in total. The data is obtained from Bloomberg and is USD denominated. Descriptive statistics are presented in Table 4.1.

Index	Arithmetic Return	Geometric Return	Standard Deviation	Skew	Kurtosis
S&P500 TR	10.5	9.22	15.22	-0.7	4.06
Russell 2000 TR	11.42	9.27	19.68	-0.53	4.1
MSCI EAFE TR	7.19	5.68	16.74	-0.67	4.39
MSCI Emerging Markets TR	8.82	5.7	23.81	-0.7	4.84
FTSE REIT TR	12.63	10.32	20.1	-0.79	11.07
Barclays US Government TR	5.52	5.42	4.26	-0.12	3.99
Barclays US Corporate High Yield TR	8.19	7.75	9.03	-1.13	12.1
S&P GSCI TR	6.8	4.26	21.82	-0.4	4.53
Gold TR	5.59	4.21	16.36	0.16	4.15

Table 4.1: Asset descriptive statistics for the nine assets to be considered in the empirical analysis. The series from which the descriptive statistics relate to constitute monthly data from the period January 1994 to December 2013. Data from the period January 1992 to January 1994 is excluded as the optimizer will require two years worth of monthly data to initialize the first portfolio. I.e., January 1994 to December 2013 is the evaluation period. TR denotes total return data, which accommodates for dividends and splits.

Moreover, in line with Gerber et al. (2015), the three months US T-Bill rate will be used as a proxy for the risk-free rate.

4.1.1 Sub-periods

In an attempt to further investigate portfolio performance, the full evaluation period ranging from January 1994 to December 2013 will be divided into sub-periods, depending on the prevailing market regime. The reasoning behind doing this is to provide additional information regarding the performance during recessionary times, typically associated with high equity volatility, and during expansionary times. The regime dependent sub-periods will be determined on an ex ante basis by studying signals from monetary policy as well as stock market patterns, in line with Bessler, Holler, and Kurmann (2012). The advantage of this method, outlined in their study, is that the number of sub-periods are reduced and that the probability for valid signals is increased. More specifically, the monetary cycle is defined as the first trend change in short term interest rate by the Federal Reserve (Jensen and Mercer 2003). As for the stock market signal, this will be determined based on the intersection of the 24-month moving average of the S&P500 index with the S&P500 index itself: an intersection where S&P500 comes from above the 24-month moving average is interpreted as a recessionary signal, whereas an intersection from below is interpreted as an expansionary signal. For a transition from one market regime to another, the monetary policy signal and the stock market signal must be consistent.

4.2 Portfolio performance evaluation methodology

To test performances of different covariance matrix estimators in a portfolio optimization context, this thesis will follow a backtesting procedure. Starting 1st January 1994, monthly historical data from two years back will be used to estimate the expected return vector and the covariance matrix in the MVO problem specified in (4.2.1). This is referred to as the in-sample period. Thereafter, MV optimized portfolios for various investor profiles will be formed, which will be held for one month. This is referred to as the out-of-sample period. In other words, the portfolios are rebalanced on a monthly basis. At each rebalancing point, the expected return vector and the covariance matrices are re-estimated. This process will be repeated until 31st December 2013, which is the last rebalancing point. To evaluate whether improvements can be made on the MV optimized portfolios by changing the covariance matrix method, the backtesting procedure will be performed for all the covariance matrix estimators evaluated in this thesis (outlined in Section 3.4), while all other things are kept equal. The employed method of estimating the expected return vector is the CAPM, outlined in Section 3.3.

To reiterate, the method of evaluation in this study follows a backtesting procedure where the considered covariance matrix estimators will be compared by their realized results over time, holding everything except the covariance estimation equal. The performance of the covariance matrices will be evaluated historically in their use in portfolio optimization. This implies that the MVO problem in (4.2.1) is solved at the beginning of one period so that realized portfolio values over the following period can be calculated. By looking at different target risk levels in the MVO over time, accumulated results are obtained by the realized portfolio values for each target in each period of evaluation. Then, realized returns as well as realized volatility can be obtained for each target so that realized efficient frontiers can be calculated for comparison. This will be implemented in MATLAB.

4.2.1 MVO Specifics

In the MVO solved in each period, the restriction on the portfolio weights are such that no short selling is permitted and that the portfolio needs to be fully invested. Different target risk levels are used, which are obtained by varying δ . Proportional transaction costs are set to 10 basis points. The specific MVO problem solved each period is then:

$$\begin{cases} \max_{w} & w^{\top} \hat{\mu} - 0.001 \mathbb{1}^{\top} \Lambda - \frac{\delta}{2} w^{\top} \widehat{\Sigma} w \\ \text{s.t.} & \sum_{i=1}^{N} w_{i} = 1 \\ & 0 \leq w_{i} \leq 1, \ \forall i = \{1, 2, \dots, N\}. \end{cases}$$

$$(4.2.1)$$

where $\hat{\Sigma}$ depends on the choice of covariance matrix estimation method. Recall also that Λ is a vector of absolute values of portfolio weight changes, see Section 3.2.2 for greater detail in how the MVO problem has been constructed. (4.2.1) is solved using estimateFrontier in MATLAB.

4.2.2 Backtesting

To compute relative performance between the covariance matrices, the optimization problem (4.2.1) will be solved at the beginning of each period using different target risk levels obtained by adjusting the risk aversion parameter δ . Specifically, (4.2.1) is solved for a large range of δ , generating the efficient frontier with corresponding weight vectors w. Then, the solutions with

expected risk closest to the target risks are selected. If a target risk is not obtainable, the nearest (in Euclidian distance) solution is chosen, i.e. an end point of the frontier is chosen if the target risk lies outside the range of obtainable expected risks.

Furthermore, this means that at the beginning of each period, expected asset returns for the period will be calculated as well as two estimates of the covariance matrix, one based on the traditional sample covariance matrix and one based on a competing estimator, by using return data from a specified period of lookback which is set to 24 months in this case. The competing estimation techniques to be considered are described in Section 3.4. Then, for each target level of risk and for both methods of covariance estimation, optimal asset weights are obtained by solving the optimization problem. These weights are used to obtain realized portfolio values over the next period. This is done for the full period of evaluation.

The above procedure yields several series of realized portfolio values over the full period of evaluation (one series for each target risk level), for each method of matrix estimation. This enables an evaluation whether the performance of MV optimized portfolios can be improved by using a different estimation technique for the covariance matrix.

To clarify, all parameters except for the covariance matrix estimation method will be kept the same when forming MV optimized portfolios. This is the ceteris paribus context, using CAPM for estimating the expected return vector. The only thing that will be varied is the choice covariance matrix estimator. The competing covariance matrix estimators are outlined in Section 3.4. A brief description of these estimators follows below:

- The sample covariance matrix (HC). This is the base case estimator.
- The single-index model (SI). This is the covariance matrix implied by the market model of Sharpe (1964). The market index is assumed to be a simple average over all considered assets.
- Shrinkage towards the single-index market model (SM). This is the estimator proposed by Ledoit and Wolf (2003b) which uses a weighted average between the sample covariance matrix and the covariance matrix estimator implied by the single-index model.
- Random matrix filtering estimator (RMT). This is the covariance matrix estimator proposed by Laloux et al. (2000) which attempts to clean the sample covariance matrix from noise.
- The Gerber Statistic estimator (GS). This covariance matrix estimator is based on the work by Gerber et al. (2015) which aims to be more robust to noise and outliers in the data.

4.3 Covariance prediction accuracy

To further investigate the relative performance of alternative covariance matrices versus HCbased covariance matrices, an analysis will be performed on how well the methods predict portfolio volatility, without the need of estimating the expected return vector. As such, the covariance matrix estimators can be compared without running the risk of diluting the relative performance by estimation error associated with the estimated return vector. This will be studied by comparing predicted volatility with realized volatility from portfolios with randomly generated weights. The same assets and evaluation period highlighted in Section 4.1 will be employed in this evaluation.

The following method will be used to perform the analysis:

- 1. Estimate HC- and competitor-based covariance matrices using return data of the previous 24 months. If a matrix is not positive semidefinite, repair it by finding the nearest one that is (see Section 3.4.6).
- 2. Generate 1000 portfolios where the portfolio weights are random. The weights are drawn from the standard uniform distribution which implies positive weights. Therafter, the weights are normalized so that the total portfolio weight sum to 1 (fully-invested portfolio). Now, calculate the predicted volatility using the portfolio weights and the covariance matrices obtained in step 1. In total, this amounts to 2000 predicted volatilities, 1000 for the HC-based covariance matrix, and 1000 for the covariance matrix based on an alternative estimation technique.
- 3. Take the return data for the coming 12 months. Calculate the realized volatility over this time period, again using the same 1000 portfolios with random weights.
- 4. Take the realized volatilities and divide them by the predicted volatilities. This yields 2000 ratios. If this ratio is equal to one, it means that the predicted volatility coincides with the realized volatility. A ratio over 1 implies that the predicted volatility underestimated the realized volatility, and a ratio under 1 implies the opposite.
- 5. Take a step of one month forward, and repeat the process from step 1. Continue as long as there is data available.

Chapter 5

Results

In this chapter, the empirical results will be presented. First, the different sub-periods are defined. Thereafter, the results from the out-of-sample portfolio performance evaluation are presented. Lastly, the results from the covariance prediction analysis are presented.

5.1 Sub-periods within the full evaluation period

To offer additional information regarding portfolio performance during different market regimes, sub-periods were defined within the full evaluation period ranging from January 1994 to December 2013. The results are obtained from employing the ex-ante double signal analysis presented in Section 4.1.1 are depicted in Figure 5.1. The shaded areas represent recessionary states, whereas the non shaded areas represent expansionary states.



Figure 5.1: Ex-ante determination of sub-periods within the full evaluation period, conditional on monetary policy signals as well as stock market signals.

More specifically, the first sub-period ranges from January 1994 to January 2001, including events such as the Asian financial crisis 1997 and the Russian financial crisis 1998. Nonetheless, it is considered as an expansionary period, characterized with increasing stock prices and high interest rates with T-Bills yielding an annualized return of 5.13% on average. Moreover, the second sub-period ranges from February 2001 to June 2004, with bearish stock markets and low average interest rates, amounting to 1.77% per annum. This period is shaded in Figure 5.1 and thus considered as a recessionary sub-period. The third sub-period is considered to be an expansionary state and ranges from July 2004 to February 2008. It is characterized by bullish stock markets and relatively high US interest rates with T-Bills yielding an annualized return of 3.63% on average. The fourth and final sub-period within the evaluation period ranges from March 2008 to December 2013, which includes the 2008 financial crisis which put the global banking system in serious peril. This is shown by the substantial decline in equity value. In addition, the sub-period is characterized by low risk-free interest rates amounting to mere 0.25% per annum, on average. This last sub-period is considered as a recessionary sub-period, as shown by the shaded area in Figure 5.1. The equity trend, however, signals for a shift. These ex-ante defined sub-periods within the evaluation period are presented in the below Table 5.1.

Time Period	Perceived market regime
Jan 1994-Jan 2001	Expansionary
Feb 2001-June 2004	Recessionary
July 2004-Feb 2008	Expansionary
Mar 2008-Dec 2013	Recessionary

Table 5.1: Ex ante defined sub-periods within the full evaluation sample ranging from 1st January 1994 to 31st December 2013.

5.2 Portfolio out-of-sample performance

In this section, the results from using the backtesting procedure outlined in Section 4.2 are presented. The monthly return data used is presented in Section 4.1. All estimations are based on returns from a moving window of the past 24 months. This moving window is used to bound the number of observations so that momentum is preserved which "offer the advantage of being more responsive to structural breaks" (Gerber et al. 2015). In addition, all portfolios are rebalanced on a monthly basis.

For each method of covariance matrix estimator, fifteen target risks are considered. These correspond to a target standard deviation of 1%, 2%,...,15% (yearly) respectively and are each associated with a separate set of portfolio weights. Five investor risk profiles are investigated in greater detail. These include an ultra-conservative investor, a conservative investor, a moderate investor, an aggressive investor and an ultra-aggressive investor, with yearly portfolio target risks of 3%, 6%, ..., 15% respectively.

5.2.1 SI versus HC

The first evaluation compares the relative out-of-sample performance of MV optimized portfolios obtained by employing the covariance matrix estimator implied by the single-index model (SI) and the sample covariance matrix (HC).

The realized risk-reward frontiers of the obtained portfolios corresponding to respective covariance estimation technique are depicted in Figure 5.2.



Figure 5.2: An illustration of realized performance in terms of annualized return and annualized volatility of portfolios corresponding to different target levels of risk. The blue frontier corresponds to ex-post performance of SI-based portfolios, whereas the orange one represents HC-based portfolios.

As one can observe in Figure 5.2, the relative performance between the covariance estimators is ambiguous at this point. At lower risk levels, SI-based portfolios perform seemingly better as they are more efficient than HC-based portfolios. In other words, they provide higher Sharpe ratios as one is better compensated in terms of return per risk taken at these levels. However, the HC-based portfolios perform seemingly better at all target risk levels beyond 5%. This is further shown in Table 5.2.

The sector 17, 1, 4114	Ultra-conservative 3%		Conse	rvative	Mod	erate	Aggr	essive	Ultra-aggressive		
Larget Volatility	3%		070		9	70	12	70	15	070	
Correlation Method	SI	HC	SI	HC	\mathbf{SI}	HC	\mathbf{SI}	HC	\mathbf{SI}	HC	
Net mean return p.a.	6.07%	5.47%	7.48%	7.59%	9.35%	9.48%	10.91%	11.15%	11.61%	11.82%	
Volatility p.a.	4.26%	4.20%	7.53%	7.49%	10.85%	10.81%	13.62%	13.39%	15.73%	15.45%	
Net MDD	7.45%	10.23%	22.86%	20.56%	34.56%	30.60%	44.14%	39.39%	50.96%	47.17%	
Net Sharpe ratio	0.75	0.62	0.61	0.63	0.60	0.61	0.59	0.62	0.56	0.58	
Skewness	-0.54	-0.65	-0.83	-0.62	-0.78	-0.63	-0.80	-0.65	-0.72	-0.62	
Kurtosis	4.63	4.88	5.84	4.74	4.96	4.14	4.75	4.02	4.46	4.09	
Avrg. turnover p.a.	1.31	1.80	2.46	2.66	3.14	3.17	3.51	3.38	3.60	3.37	

Table 5.2: Descriptive statistics for SI- and HC-based portfolios, at five different risk target levels.

Within the full evaluation period, HC-based portfolios provide higher Sharpe ratios in four out of five investor profiles. In addition, the maximum drawdown is lower for HC-based portfolios for all except one investor type: the ultra-conservative one. This reflects that HC-based portfolios are associated with less downside risk for the majority of the considered investor risk profiles.

Furthermore, the risk-adjusted returns of SI-based portfolios, adjusted to have the same risk exposure as HC-based portfolios are shown in Table 5.3. This allows for a comparison of relative return performance.

Target Volatility	Ultra	a-conserv 3%	ative	Conservative 6%			Moderate 9%				Aggressive 12%	•	Ultra-aggressive 15%		
Correlation Method	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	$\begin{array}{c} 6.03\% \\ 4.20\% \\ 0.75 \end{array}$	5.47% 4.20% 0.62	0.56%	7.46% 7.49% 0.61	7.59% 7.49% 0.63	-0.12%	9.33% 10.81% 0.60	9.48% 10.81% 0.61	-0.16%	10.78% 13.39% 0.59	11.15% 13.39% 0.62	-0.37%	11.46% 15.45% 0.56	11.82% 15.45% 0.58	-0.36%

Table 5.3: Relative return performance for the full evaluation period. The returns of SI-based portfolios are converted to risk-adjusted returns by deleveraging SI portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between SI-based portfolios and HC-based portfolios.

The only benefit of estimating the covariance matrix based on the single-index model is found for the ultra-conservative investor profile (0.56% additional return). Using the single-index model in lieu of the traditional covariance matrix, however, would result in less efficient portfolios in all other considered cases.

Sub-period analysis

In an attempt to further explain the above results, the relative performance is evaluated during the different sub-periods defined within the full evaluation period is presented in the below Table 5.4 and Table 5.5.

	Ultra-co	onservative	Conse	rvative	Mod	erate	Aggr	essive	Ultra-ag	gressive
Target Volatility		3%	6	%	9	%	12	2%	15	5%
		Jan 19	94-Jan 2	2001 (Ex)	pansiona	ry Perio	4)			
Correlation Method	SI	HC	SI	HC	SI	HC	SI	HC	SI	HC
Net mean return p.a.	6.79%	5.83%	9.87%	9.73%	12.36%	12.21%	13.77%	14.10%	14.75%	15.03%
Volatility p.a.	4.40%	4.09%	7.77%	7.92%	11.11%	11.29%	13.40%	13.54%	15.32%	15.27%
Net MDD	5.18%	5.92%	9.03%	9.49%	12.56%	13.78%	15.40%	15.35%	18.73%	19.37%
Net Sharpe ratio	0.38	0.17	0.61	0.58	0.65	0.63	0.64	0.66	0.63	0.65
Avrg. turnover p.a.	1.41	2.12	1.98	2.35	2.31	2.35	2.56	2.39	2.39	2.32
		Feb 20	01-June	2004 (R	ecessiona	ry Perio	d)			
Correlation Method	SI	HC	SI	HC	SI	HC	SI	HC	SI	HC
Net mean return p.a.	7.31%	7.28%	7.69%	6.43%	7.81%	6.12%	8.80%	7.56%	8.51%	7.85%
Volatility p.a.	4.76%	4.67%	6.57%	7.34%	10.77%	10.79%	12.44%	12.57%	13.51%	13.66%
Net MDD	2.95%	3.94%	6.79%	6.96%	10.69%	9.83%	12.78%	14.92%	17.03%	18.78%
Net Sharpe ratio	1.16	1.18	0.90	0.64	0.56	0.40	0.57	0.46	0.50	0.45
Avrg. turnover p.a.	0.83	1.03	2.66	2.52	3.98	3.59	4.30	3.82	4.11	3.53
		July 2	004-Feb	2008 (Ex	pansiona	ry Perio	d)			
Correlation Method	SI	HC	SI	HC	SI	HC	SI	HC	SI	HC
Net mean return p.a.	6.24%	6.06%	9.58%	10.13%	13.75%	13.80%	17.86%	16.59%	20.12%	19.39%
Volatility p.a.	3.33%	3.37%	6.90%	7.25%	10.45%	10.71%	14.03%	14.01%	17.07%	16.71%
Net MDD	3.23%	2.96%	5.62%	5.24%	9.88%	8.70%	13.86%	12.31%	16.61%	14.88%
Net Sharpe ratio	0.78	0.72	0.86	0.90	0.97	0.95	1.01	0.93	0.97	0.94
Avrg. turnover p.a.	1.89	2.18	3.40	3.32	3.84	3.98	4.07	3.98	3.85	4.11
		Mar 2	008-Dec	2013 (Re	ecessiona	ry Perio	d)			
Correlation Method	SI	HC	SI	HC	SI	HC	SI	HC	SI	HC
Net mean return p.a.	4.28%	3.62%	4.19%	4.15%	5.13%	5.53%	5.61%	6.35%	5.01%	5.62%
Volatility p.a.	4.27%	4.53%	7.73%	7.24%	10.96%	10.35%	14.13%	13.37%	16.55%	15.87%
Net MDD	7.45%	10.23%	20.98%	19.03%	30.23%	27.13%	38.26%	34.41%	44.31%	41.23%
Net Sharpe ratio	0.94	0.74	0.51	0.54	0.45	0.51	0.38	0.46	0.29	0.34
Avrg. turnover p.a.	1.07	1.62	2.33	2.73	3.22	3.40	3.88	3.97	4.64	4.12

Table 5.4: Descriptive statistics for the four sub-periods from 1994 to 2013 for SI- and HC-based portfolios, at five different risk target levels.

Table 5.4 suggests that HC-based portfolios, in particular, outperform SI-based portfolios during the last sub-period. This is further shown in Table 5.5, where the risk-adjusted relative performance of SI-based portfolios over the last sub-period is notably weak.

Target Volatility	Ultra-conservative 3%		C	Conservative 6%			Moderate 9%			Aggressive 12%			Ultra-aggressiv 15%		
					Jan 199	4-Jan 20	01 (Expa	ansionary	Period)					
Correlation Method	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	$\begin{array}{c} 6.67\% \\ 4.09\% \\ 0.38 \end{array}$	5.83% 4.09% 0.17	0.84%	9.96% 7.92% 0.61	9.73% 7.92% 0.58	0.23%	12.47% 11.29% 0.65	${\begin{array}{c} 12.21\%\\ 11.29\%\\ 0.63 \end{array}}$	0.26%	$13.86\% \\ 13.54\% \\ 0.64$	14.10% 13.54% 0.66	-0.24%	14.72% 15.27% 0.63	15.03% 15.27% 0.65	-0.31%
					Feb 200	1-June 2	004 (Rec	essionar	y Period)					
Correlation Method	SI	HC	Δ	SI	HC	Δ	\mathbf{SI}	HC	Δ	SI	HC	Δ	SI	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	7.31% 4.67% 1.19	7.28% 4.67% 1.18	0.03%	$\begin{array}{c} 6.03\% \\ 7.34\% \\ 0.58 \end{array}$	$\begin{array}{c} 6.43\% \\ 7.34\% \\ 0.64 \end{array}$	-0.41%	5.87% 10.79% 0.38	$\begin{array}{c} 6.12\% \\ 10.79\% \\ 0.40 \end{array}$	-0.25%	$\begin{array}{c} 6.72\% \\ 12.57\% \\ 0.39 \end{array}$	7.56% 12.57% 0.46	-0.84%	7.60% 13.66% 0.43	7.85% 13.66% 0.45	-0.26%
					July 200	04-Feb 20	008 (Exp	ansionar	y Period)					
Correlation Method	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	$\begin{array}{c} 6.27\% \\ 3.37\% \\ 0.78 \end{array}$	$\begin{array}{c} 6.06\% \\ 3.37\% \\ 0.72 \end{array}$	0.22%	9.88% 7.25% 0.86	${\begin{array}{c} 10.13\% \\ 7.25\% \\ 0.90 \end{array}}$	-0.26%	13.99% 10.71% 0.97	$13.80\% \\ 10.71\% \\ 0.95$	0.20%	17.84% 14.01% 1.01	16.59% 14.01% 0.93	1.25%	19.76% 16.71% 0.97	19.39% 16.71% 0.94	0.37%
					Mar 200	08-Dec 2	013 (Rec	essionary	Period)					
Correlation Method	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ	SI	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	4.51% 4.53% 0.94	3.62% 4.53% 0.74	0.90%	3.95% 7.24% 0.51	4.15% 7.24% 0.54	-0.20%	4.85% 10.35% 0.45	5.53% 10.35% 0.51	-0.67%	5.32% 13.37% 0.38	6.35% 13.37% 0.46	-1.03%	4.81% 15.87% 0.29	5.62% 15.87% 0.34	-0.81%

Table 5.5: Relative return performance for the sub-periods that constitute the full evaluation period. The returns of SI-based portfolios are converted to risk-adjusted returns (RAR) by deleveraging SI portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between SI-based portfolios and HC-based portfolios.

Although no clear pattern is shown in relative performance, the covariance matrix estimator implied by the single-index seems to perform particularly poor during recessionary times.

5.2.2 SM versus HC

In this section, the out-of-sample performance of MV optimized portfolios implied by the shrinkage estimator (SM) described in Section 3.4.3 is evaluated. In Figure 5.3, the realized performance of SM- and HC-based portfolios in terms of risk-reward ratios for fifteen different target levels of risk is depicted. By inspection, the difference between the realized efficient frontiers is marginal. This suggests that a low shrinkage intensity has been assigned to the shrinkage target (being the single-index estimator) when the weighted average between the sample covariance matrix and the covariance matrix estimator implied by the single-index model has been calculated. In other words, the shrinkage estimator has almost converged to the sample covariance matrix. This coincides with the results presented in Section 5.2.1, where SI-based portfolios showed poor performance, consequently indicating that the covariance matrix estimator implied by the single-index model provided a bad out-of-sample fit. Plausible explanations for this could be that the assumption implying that no factors beyond the market account for asset comovement is strongly violated, or that the employed market proxy provides an inaccurate reflection of the market portfolio, or simply a combination of these two situations.



Figure 5.3: An illustration of realized performance in terms of annualized return and annualized volatility of portfolios corresponding to different target levels of risk. The blue frontier corresponds to ex-post performance of SM-based portfolios, whereas the orange one represents HC-based portfolios.

A further inspection of Table 5.6 illuminates the minuscule difference between SM- and HCbased portfolios. The relative performance is ambiguous, neither estimator is superior for all five investor types. However, SM-based portfolios are associated with higher Sharpe ratios on average. On the other hand, the maximum drawdown are lower for HC-based portfolios on average.

Target Volatility	Ultra-co	onservative 3%	$\begin{array}{c} \text{Conservative} \\ 6\% \end{array}$		Mod 9	erate %	Aggr 12	essive 2%	Ultra-aggressive 15%		
Correlation Method	\mathbf{SM}	SM HC		HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	
Net mean return p.a.	5.61%	5.47%	7.65%	7.59%	9.60%	9.48%	11.23%	11.15%	11.76%	11.82%	
Volatility p.a.	4.27%	4.20%	7.47%	7.49%	10.84%	10.81%	13.54%	13.39%	15.66%	15.45%	
Net MDD	9.78%	10.23%	21.14%	20.56%	31.76%	30.60%	40.61%	39.39%	49.05%	47.17%	
Net Sharpe ratio	0.64	0.62	0.64	0.63	0.62	0.61	0.62	0.62	0.57	0.58	
Skewness	-0.72	-0.65	-0.66	-0.62	-0.65	-0.63	-0.66	-0.65	-0.64	-0.62	
Kurtosis	5.11	4.88	4.95	4.74	4.26	4.14	4.07	4.02	4.20	4.09	
Avrg. turnover p.a.	1.58	1.80	2.55	2.66	3.14	3.17	3.42	3.38	3.40	3.37	

Table 5.6: Full period descriptive statistics for SM- and HC-based portfolios, at five different risk target levels.

As for the risk-adjusted returns presented in Table 5.7, no dominating estimation method emerges in this case too. An average outperformance of 0.016% in return across all five investor types is noted for SM-based portfolios.

Target Volatility	Ultra	a-conserv 3%	ative	С	Conservative 6%			Moderate 9%			Aggressive 12%	9	Ultra-aggressive 15%		
Correlation Method	SM	HC	Δ	\mathbf{SM}	HC	Δ	\mathbf{SM}	HC	Δ	SM	HC	Δ	\mathbf{SM}	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	5.57% 4.20% 0.64	5.47% 4.20% 0.62	0.10%	7.66% 7.49% 0.64	7.59% 7.49% 0.63	0.07%	9.58% 10.81% 0.62	9.48% 10.81% 0.61	0.10%	11.14% 13.39% 0.62	11.15% 13.39% 0.62	-0.01%	11.63% 15.45% 0.57	11.82% 15.45% 0.58	-0.18%

Table 5.7: Relative return performance for the full evaluation period. The returns of SM-based portfolios are converted to risk-adjusted returns (RAR) by deleveraging SM portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between SM-based portfolios and HC-based portfolios.

Sub-period analysis

An inspection of the descriptive statistics for the four tethered sub-periods that constitute the full evaluation period provides little additional information regarding the relative performance between SM- and HC-based portfolios. No real difference in relative performance between recessionary and expansionary periods is recognized.

Target Volatility	Ultra-conservative 3%		Conservative 6%		$\begin{array}{c} \text{Moderate} \\ 9\% \end{array}$		Aggressive 12%		Ultra-ag 15	gressive %
		Jan 19	94-Jan 2	2001 (Ex	pansiona	ry Period	1)			
Correlation Method	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC
Net mean return p.a.	6.16%	5.83%	9.72%	9.73%	12.29%	12.21%	13.86%	14.10%	14.97%	15.03%
Volatility p.a.	4.33%	4.09%	7.85%	7.92%	11.26%	11.29%	13.63%	13.54%	15.43%	15.27%
Net MDD	5.57%	5.92%	9.38%	9.49%	13.28%	13.78%	15.64%	15.35%	19.66%	19.37%
Net Sharpe ratio	0.24	0.17	0.58	0.58	0.64	0.63	0.64	0.66	0.64	0.65
Avrg. turnover p.a.	1.69	2.12	2.17	2.35	2.31	2.35	2.50	2.39	2.31	2.32
		Feb 20	01-June	2004 (R	ecessiona	ry Perio	d)			
Correlation Method	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC
Net mean return p.a.	7.13%	7.28%	6.54%	6.43%	6.22%	6.12%	7.67%	7.56%	7.92%	7.85%
Volatility p.a.	4.71%	4.67%	7.27%	7.34%	10.75%	10.79%	12.65%	12.57%	13.62%	13.66%
Net MDD	4.05%	3.94%	7.14%	6.96%	9.53%	9.83%	14.09%	14.92%	18.51%	18.78%
Net Sharpe ratio	1.14	1.18	0.66	0.64	0.41	0.40	0.47	0.46	0.45	0.45
Avrg. turnover p.a.	1.04	1.03	2.52	2.52	3.74	3.59	3.90	3.82	3.59	3.53
		July 2	004-Feb 2	2008 (Ex	pansiona	ry Perio	d)			
Correlation Method	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC
Net mean return p.a.	6.15%	6.06%	10.17%	10.13%	13.98%	13.80%	17.40%	16.59%	19.71%	19.39%
Volatility p.a.	3.32%	3.37%	7.19%	7.25%	10.65%	10.71%	14.08%	14.01%	16.91%	16.71%
Net MDD	3.03%	2.96%	5.39%	5.24%	8.96%	8.70%	12.36%	12.31%	15.03%	14.88%
Net Sharpe ratio	0.76	0.72	0.91	0.90	0.97	0.95	0.98	0.93	0.95	0.94
Avrg. turnover p.a.	1.98	2.18	3.32	3.32	3.99	3.98	3.99	3.98	4.06	4.11
		Mar 2	008-Dec	2013 (Re	ecessiona	ry Perio	1)			
Correlation Method	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC	\mathbf{SM}	HC
Net mean return p.a.	3.75%	3.62%	4.27%	4.15%	5.66%	5.53%	6.35%	6.35%	5.25%	5.62%
Volatility p.a.	4.49%	4.53%	7.33%	7.24%	10.56%	10.35%	13.67%	13.37%	16.29%	15.87%
Net MDD	9.78%	10.23%	19.52%	19.03%	28.15%	27.13%	35.71%	34.41%	43.19%	41.23%
Net Sharpe ratio	0.78	0.74	0.55	0.54	0.51	0.51	0.45	0.46	0.31	0.34
Avrg. turnover p.a.	1.49	1.62	2.55	2.73	3.28	3.40	3.88	3.97	4.23	4.12

Table 5.8: Descriptive statistics for the four sub-periods from 1994 to 2013 for SM- and HC-based portfolios, at five different risk target levels.

Table 5.9 shows risk-adjusted returns of SM-based portfolios for the four sub-periods.

Target Volatility	Ultra-conservative 3% Conservative 6% Lap 1994 Lap 2			re		Moderate 9%		Aggressive 12%		9	Ult	sive			
				J	an 1994	Jan 200	01 (Expa	nsionary	Period)	1					
Correlation Method	\mathbf{SM}	HC	Δ	\mathbf{SM}	HC	Δ	\mathbf{SM}	HC	Δ	\mathbf{SM}	HC	Δ	SM	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	$\begin{array}{c} 6.10\% \\ 4.09\% \\ 0.24 \end{array}$	5.83% 4.09% 0.17	0.27%	9.75% 7.92% 0.58	9.73% 7.92% 0.58	0.03%	12.30% 11.29% 0.64	$\begin{array}{c} 12.21\% \\ 11.29\% \\ 0.63 \end{array}$	0.09%	${\begin{array}{c} 13.81\%\\ 13.54\%\\ 0.64\end{array}}$	14.10% 13.54% 0.66	-0.29%	14.87% 15.27% 0.64	15.03% 15.27% 0.65	-0.16%
				F	eb 2001-	June 20	04 (Rece	ssionary	Period)					
Correlation Method	\mathbf{SM}	HC	Δ	SM	HC	Δ	SM	HC	Δ	SM	HC	Δ	SM	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	7.09% 4.67% 1.14	7.28% 4.67% 1.18	-0.20%	6.58% 7.34% 0.66	6.43% 7.34% 0.64	0.15%	6.24% 10.79% 0.41	6.12% 10.79% 0.40	0.12%	7.64% 12.57% 0.47	7.56% 12.57% 0.46	0.08%	7.94% 13.66% 0.45	7.85% 13.66% 0.45	0.09%
				J	uly 2004	-Feb 200	08 (Expa	nsionary	Period)					
Correlation Method	SM	HC	Δ	SM	HC	Δ	SM	HC	Δ	\mathbf{SM}	HC	Δ	SM	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	$\begin{array}{c} 6.19\% \\ 3.37\% \\ 0.76 \end{array}$	$\begin{array}{c} 6.06\% \\ 3.37\% \\ 0.72 \end{array}$	0.14%	$\begin{array}{c} 10.22\% \\ 7.25\% \\ 0.91 \end{array}$	$\begin{array}{c} 10.13\% \\ 7.25\% \\ 0.90 \end{array}$	0.09%	14.03% 10.71% 0.97	$\begin{array}{c} 13.80\% \\ 10.71\% \\ 0.95 \end{array}$	0.24%	$17.33\% \\ 14.01\% \\ 0.98$	16.59% 14.01% 0.93	0.74%	19.52% 16.71% 0.95	19.39% 16.71% 0.94	0.13%
				N	/lar 2008	-Dec 20	13 (Rece	ssionary	Period))					
Correlation Method	SM	HC	Δ	SM	HC	Δ	SM	HC	Δ	\mathbf{SM}	HC	Δ	SM	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	3.78% 4.53% 0.78	3.62% 4.53% 0.74	0.16%	4.22% 7.24% 0.55	4.15% 7.24% 0.54	0.07%	5.56% 10.35% 0.51	5.53% 10.35% 0.51	0.03%	6.21% 13.37% 0.45	$\begin{array}{c} 6.35\% \\ 13.37\% \\ 0.46 \end{array}$	-0.14%	5.12% 15.87% 0.31	5.62% 15.87% 0.34	-0.50%

Table 5.9: Relative return performance for the sub-periods that constitute the full evaluation period. The returns of SM-based portfolios are converted to risk-adjusted returns (RAR) by deleveraging SM portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between SM-based portfolios and HC-based portfolios.

5.2.3 RMT versus HC

In this section, the out-of-sample performance of MV optimized portfolios obtained through the use of random matrix theory (RMT) in constructing the covariance matrix estimator is presented. The realized risk-reward frontiers of RMT- and HC-based portfolios are depicted in Figure 5.10.



Figure 5.4: An illustration of realized performance in terms of annualized return and annualized volatility of portfolios corresponding to different target levels of risk. The blue frontier corresponds to ex-post performance of RMT-based portfolios, whereas the orange one represents HC-based portfolios.

Figure 5.4 shows that the relative performance between the RMT method and the HC method is, yet again, ambiguous. Note that RMT seemingly outperforms HC for volatility levels below

approximately 6%. In contrast, HC seemingly outperforms RMT on higher risk levels. In between these volatility levels, the relative performance is close to indifferent. Thus there is no evidence for one method dominating the other. For higher levels of volatility, RMT-based portfolios seem to underpredict volatility. An example of this can be observed at the target risk level of 15%, where the realized portfolio volatility amounts to 16%.

These results are consistent with Table 5.10 which shows that RMT-based portfolios have higher Sharpe ratios for more conservative investors, whereas HC-based portfolios are more efficient at higher levels of target volatility. In addition, the maximum drawdown are lower for HC-based portfolios across all investor profiles except for the ultra-conservative profile.

Target Volatility	Ultra-conservative 3%		Conser 6	rvative %	Mod 9	erate %	Aggre 12	essive %	Ultra-aggressive 15%		
Correlation Method	RMT	RMT HC		HC	RMT	HC	RMT	HC	RMT	HC	
Net mean return p.a. Volatility p.a. Net MDD	5.94% 4.32% 10.10%	5.47% 4.20% 10.23%	7.78% 7.81% 22.36%	7.59% 7.49% 20.56%	9.89% 11.18% 33.50%	9.48% 10.81% 30.60%	11.11% 14.06% 43.80%	11.15% 13.39% 39.39%	11.32% 15.93% 51.92%	11.82% 15.45% 47.17%	
Net Sharpe ratio	0.71	0.62	0.63	0.63	0.63	0.61	0.59	0.62	0.53	0.58	
Kurtosis Avrg. turnover p.a.	5.01 1.36	4.88 1.80	6.31 2.38	4.74 2.66	5.32 2.96	4.14 3.17	5.35 3.38	4.02 3.38	4.53 3.49	4.09 3.37	

Table 5.10: Full period descriptive statistics for RMT- and HC-based portfolios, at five different risk target levels.

No evidence of a dominating method is found in terms of risk-adjusted returns either. However, the risk-adjusted outperformance of HC to RMT is on average 0.13%. Descriptive statistics are presented in Table 5.11.

Target Volatility	Ultra	a-conserv 3%	ative	$\begin{array}{c} \text{Conservative} \\ 6\% \end{array}$			Moderate 9%				Aggressive 12%	•	Ultra-aggressive 15%		
Correlation Method	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	5.86% 4.20% 0.71	5.47% 4.20% 0.62	0.39%	7.58% 7.49% 0.63	7.59% 7.49% 0.63	-0.01%	$9.66\%\ 10.81\%\ 0.63$	9.48% 10.81% 0.61	0.17%	10.72% 13.39% 0.59	11.15% 13.39% 0.62	-0.43%	11.07% 15.45% 0.53	11.82% 15.45% 0.58	-0.75%

Table 5.11: Relative return performance for the full evaluation period. The returns of RMT-based portfolios are converted to risk-adjusted returns (RAR) by deleveraging RMT portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between RMT-based portfolios and HC-based portfolios.

Sub-period analysis

Table 5.12 shows that HC-based portfolios generally perform better during the first and last sub-period, while RMT-based portfolios are characterized by stronger performance measures for the second and third sub-period.

	Ultra-co	nservative	Conse	rvative	Mod	erate	Aggr	essive	Ultra-aggressive			
Target Volatility	3	%	6	%	9	%	12	2%	15%			
		Jan 19	994-Jan 2	2001 (Ex	pansiona	ry Perio	1)					
Correlation Method	RMT	HC	RMT	HC	RMT	HC	RMT	HC	RMT	HC		
Net mean return p.a.	6.38%	5.83%	9.31%	9.73%	11.79%	12.21%	13.28%	14.10%	14.52%	15.03%		
Volatility p.a.	4.28%	4.09%	7.85%	7.92%	11.19%	11.29%	13.55%	13.54%	15.36%	15.27%		
Net MDD	5.51%	5.92%	9.24%	9.49%	12.82%	13.78%	15.66%	15.35%	19.05%	19.37%		
Net Sharpe ratio	0.29	0.17	0.53	0.58	0.60	0.63	0.60	0.66	0.61	0.65		
Avrg. turnover p.a.	1.48	2.12	2.08	2.35	2.37	2.35	2.52	2.39	2.32	2.32		
Feb 2001-June 2004 (Recessionary Period)												
Correlation Method	RMT	HC	RMT	HC	RMT	HC	RMT	HC	RMT	HC		
Net mean return p.a.	7.66%	7.28%	7.09%	6.43%	7.56%	6.12%	7.68%	7.56%	8.29%	7.85%		
Volatility p.a.	4.73%	4.67%	7.39%	7.34%	10.17%	10.79%	12.48%	12.57%	13.46%	13.66%		
Net MDD	3.96%	3.94%	7.58%	6.96%	9.25%	9.83%	14.35%	14.92%	17.78%	18.78%		
Net Sharpe ratio	1.25	1.18	0.72	0.64	0.57	0.40	0.47	0.46	0.48	0.45		
Avrg. turnover p.a.	0.92	1.03	2.50	2.52	3.56	3.59	4.07	3.82	3.55	3.53		
		July 2	004-Feb	2008 (Ex	pansiona	ry Perio	d)					
Correlation Method	RMT	HC	RMT	HC	RMT	HC	RMT	HC	RMT	HC		
Net mean return p.a.	6.56%	6.06%	11.12%	10.13%	15.75%	13.80%	19.11%	16.59%	20.08%	19.39%		
Volatility p.a.	3.28%	3.37%	7.28%	7.25%	11.04%	10.71%	14.39%	14.01%	16.65%	16.71%		
Net MDD	2.69%	2.96%	5.46%	5.24%	8.23%	8.70%	11.41%	12.31%	12.45%	14.88%		
Net Sharpe ratio	0.89	0.72	1.03	0.90	1.10	0.95	1.08	0.93	0.99	0.94		
Avrg. turnover p.a.	1.69	2.18	3.09	3.32	3.49	3.98	3.90	3.98	4.20	4.11		
		Mar 2	008-Dec	2013 (Re	ecessiona	ry Perio	1)					
Correlation Method	RMT	HC	RMT	HC	RMT	HC	RMT	HC	RMT	HC		
Net mean return p.a.	4.05%	3.62%	4.29%	4.15%	5.36%	5.53%	5.56%	6.35%	3.88%	5.62%		
Volatility p.a.	4.70%	4.53%	8.35%	7.24%	11.84%	10.35%	15.29%	13.37%	17.40%	15.87%		
Net MDD	10.10%	10.23%	21.64%	19.03%	30.75%	27.13%	40.02%	34.41%	47.96%	41.23%		
Net Sharpe ratio	0.81	0.74	0.48	0.54	0.43	0.51	0.35	0.46	0.21	0.34		
Avrg. turnover p.a.	1.24	1.62	2.25	2.73	2.99	3.40	3.69	3.97	4.41	4.12		

Table 5.12: Descriptive statistics for the four sub-periods from 1994 to 2013 for RMT- and HC-based portfolios, at five different risk target levels.

Furthermore, the risk-adjusted returns in Table 5.13 show similar relative performance for the different sub-periods. The nature of RMT and HC performing well during expansionary as well as recessionary phases suggests that the relative performance cannot be tied to the prevailing market regime.

Target Volatility	Ultra-conservative Conservative 3% 6%			/e	Moderate 9%				Aggressive 12%			Ultra-aggressive 15%			
Jan 1994-Jan 2001 (Expansionary Period)															
Correlation Method	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	$\begin{array}{c} 6.32\% \\ 4.09\% \\ 0.29 \end{array}$	5.83% 4.09% 0.17	0.49%	9.35% 7.92% 0.53	9.73% 7.92% 0.58	-0.38%	11.85% 11.29% 0.60	12.21% 11.29% 0.63	-0.36%	13.27% 13.54% 0.60	14.10% 13.54% 0.66	-0.83%	14.46% 15.27% 0.61	15.03% 15.27% 0.65	-0.57%
Feb 2001-June 2004 (Recessionary Period)															
Correlation Method	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	7.59% 4.67% 1.25	7.28% 4.67% 1.18	0.30%	7.06% 7.34% 0.72	6.43% 7.34% 0.64	0.62%	7.91% 10.79% 0.57	6.12% 10.79% 0.40	1.78%	7.73% 12.57% 0.47	7.56% 12.57% 0.46	0.17%	8.38% 13.66% 0.48	7.85% 13.66% 0.45	0.53%
					July 200	4-Feb 20	08 (Expa	nsionary	Period))					
Correlation Method	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	$\begin{array}{c} 6.65\% \\ 3.37\% \\ 0.89 \end{array}$	$\begin{array}{c} 6.06\% \\ 3.37\% \\ 0.72 \end{array}$	0.59%	11.09% 7.25% 1.03	10.13% 7.25% 0.90	0.95%	15.38% 10.71% 1.10	13.80% 10.71% 0.95	1.58%	18.70% 14.01% 1.08	16.59% 14.01% 0.93	2.11%	20.13% 16.71% 0.99	19.39% 16.71% 0.94	0.74%
					Mar 200	8-Dec 20	013 (Reco	essionary	Period)						
Correlation Method	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ	RMT	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	3.91% 4.53% 0.81	3.62% 4.53% 0.74	0.29%	3.76% 7.24% 0.48	4.15% 7.24% 0.54	-0.40%	4.72% 10.35% 0.43	5.53% 10.35% 0.51	-0.80%	4.89% 13.37% 0.35	$\begin{array}{c} 6.35\% \\ 13.37\% \\ 0.46 \end{array}$	-1.46%	3.56% 15.87% 0.21	5.62% 15.87% 0.34	-2.06%

Table 5.13: Relative return performance for the sub-periods that constitute the full evaluation period. The returns of RMT-based portfolios are converted to risk-adjusted returns (RAR) by deleveraging RMT portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between RMT-based portfolios and HC-based portfolios.

5.2.4 GS versus HC

The final evaluation compares the relative out-of-sample performance of MV optimized portfolios obtained by employing the covariance matrix estimator implied by the Gerber Statistic (GS) and the sample covariance matrix (HC). The realized risk-reward frontiers of the obtained portfolios corresponding to respective covariance estimation technique are depicted in Figure 5.5.



Figure 5.5: An illustration of realized performance in terms of annualized return and annualized volatility of portfolios corresponding to different target levels of risk. The blue frontier corresponds to ex-post performance of GS-based portfolios, whereas the orange one represents HC-based portfolios.

As one can observe in Figure 5.5, GS-based portfolios provide a striking out-of-sample performance in relation to HC-bases portfolios. Almost the entire frontier of GS-based portfolios is raised upwards and to the left, implying more efficient portfolios in the sense they yield higher returns per risk taken. This is further supported in Table 5.14, where GS-based portfolios are associated with higher Sharpe ratios, consistently across all considered risk profiles. In addition, GS portfolios have a lower associated downside risk in terms of maximum drawdown. Thus, GS portfolios dominate HC portfolios in this setting. However, the portfolio turnover of GS portfolios is consistently higher, which suggests that these will be more negatively affected with an increase in proportional transaction costs. Also, more trading is required when rebalancing GS portfolios, which increases exposure to liquidity risks.

	Ultra-conservative		Conservative		Mod	erate	Aggr	essive	Ultra-aggressive		
Target Volatility	3%		6%		9	%	12	2%	15%		
Correlation Method	\mathbf{GS}	HC	GS	HC	GS	HC	GS	HC	GS	HC	
Net mean return p.a.	5.77%	5.47%	7.47%	7.59%	9.46%	9.48%	11.08%	11.15%	12.34%	11.82%	
Volatility p.a.	4.15%	4.20%	6.81%	7.49%	10.09%	10.81%	12.77%	13.39%	15.14%	15.45%	
Net MDD	6.85%	10.23%	16.73%	20.56%	28.85%	30.60%	38.66%	39.39%	46.26%	47.17%	
Net Sharpe ratio	0.70	0.62	0.68	0.63	0.65	0.61	0.64	0.62	0.62	0.58	
Skewness	-0.82	-0.65	-0.68	-0.62	-0.58	-0.63	-0.57	-0.65	-0.61	-0.62	
Kurtosis	6.13	4.88	5.34	4.74	4.35	4.14	3.79	4.02	3.91	4.09	
Avrg. turnover p.a.	2.25	1.80	2.51	2.66	3.31	3.17	3.75	3.38	3.44	3.37	

Table 5.14: Full period descriptive statistics for GS- and HC-based portfolios, at five different risk target levels.

Furthermore, Table 5.15 suggests an average of 0.43% additional return across the five investor types if one uses the GS covariance matrix estimator in lieu of the traditional sample covariance matrix. Note that the outperformance is not only on average, but consistent throughout all five risk profiles.

Target Volatility	Ultra-conservative 3%			Conservative 6%			Moderate 9%			Aggressive 12%			Ultra-aggressive 15%		
Correlation Method	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	\mathbf{GS}	HC	Δ	GS	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	5.80% 4.20% 0.70	5.47% 4.20% 0.62	0.33%	7.94% 7.49% 0.68	7.59% 7.49% 0.63	0.35%	9.92% 10.81% 0.65	9.48% 10.81% 0.61	0.44%	11.48% 13.39% 0.64	11.15% 13.39% 0.62	0.33%	12.52% 15.45% 0.62	11.82% 15.45% 0.58	0.71%

Table 5.15: Relative return performance for the full evaluation period. The returns of GS-based portfolios are converted to risk-adjusted returns (RAR) by deleveraging GS portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between GS-based portfolios and HC-based portfolios.

Sub-period analysis

In an attempt to further explain the above results, the relative performance is evaluated during the different sub-periods defined within the full evaluation period is presented in the below Table 5.16 and Table 5.17. By first studying Table 5.16, it is apparent that relative performance of GS-based portfolios seems to be tied to the prevailing market regime. In recessionary times, often associated with volatile equity markets, GS-based portfolios have consistently higher Sharpe ratios and lower maximum drawdowns. However, in expansionary periods, the contrary seems to hold, albeit less significant in this direction.

Target Volatility	Ultra-conservative 3%		Conse: 6	rvative %	Mod 9	erate %	Aggr 12	essive 2%	Ultra-aggressive 15%				
		Jan 19	94-Jan 2	2001 (Exp	pansiona	ry Period	d)						
Correlation Method	GS	HC	GS	HC	GS	HC	GS	HC	GS	HC			
Net mean return p.a.	5.61%	5.83%	8.88%	9.73%	11.67%	12.21%	14.08%	14.10%	15.40%	15.03%			
Volatility p.a.	4.56%	4.09%	7.43%	7.92%	10.50%	11.29%	12.96%	13.54%	15.00%	15.27%			
Net MDD	6.85%	5.92%	8.02%	9.49%	10.86%	13.78%	13.76%	15.35%	18.09%	19.37%			
Net Sharpe ratio	0.11	0.17	0.51	0.58	0.62	0.63	0.69	0.66	0.68	0.65			
Avrg. turnover p.a.	2.50	2.12	2.48	2.35	2.71	2.35	2.66	2.39	2.48	2.32			
Feb 2001-June 2004 (Recessionary Period)													
Correlation Method	GS	HC	GS	HC	GS	HC	GS	HC	GS	HC			
Net mean return p.a.	7.00%	7.28%	7.15%	6.43%	7.68%	6.12%	8.87%	7.56%	8.83%	7.85%			
Volatility p.a.	4.40%	4.67%	7.12%	7.34%	10.77%	10.79%	12.46%	12.57%	13.53%	13.66%			
Net MDD	2.80%	3.94%	7.78%	6.96%	10.37%	9.83%	13.03%	14.92%	17.21%	18.78%			
Net Sharpe ratio	1.19	1.18	0.76	0.64	0.55	0.40	0.57	0.46	0.52	0.45			
Avrg. turnover p.a.	1.48	1.03	2.60	2.52	3.98	3.59	4.13	3.82	3.44	3.53			
		July 20	004-Feb	2008 (Ex	pansiona	ry Perio	d)						
Correlation Method	GS	HC	GS	HC	GS	HC	GS	HC	GS	HC			
Net mean return p.a.	5.89%	6.06%	9.55%	10.13%	12.30%	13.80%	13.96%	16.59%	18.67%	19.39%			
Volatility p.a.	3.31%	3.37%	6.38%	7.25%	10.33%	10.71%	13.69%	14.01%	16.98%	16.71%			
Net MDD	4.04%	2.96%	5.69%	5.24%	10.62%	8.70%	14.65%	12.31%	17.43%	14.88%			
Net Sharpe ratio	0.68	0.72	0.93	0.90	0.84	0.95	0.75	0.93	0.89	0.94			
Avrg. turnover p.a.	3.28	2.18	2.94	3.32	4.50	3.98	5.74	3.98	4.24	4.11			
		Mar 2	008-Dec	2013 (Re	ecessiona	ry Perio	d)						
Correlation Method	GS	HC	GS	HC	GS	HC	GS	HC	GS	HC			
Net mean return p.a.	5.16%	3.62%	4.71%	4.15%	6.12%	5.53%	7.03%	6.35%	6.81%	5.62%			
Volatility p.a.	4.04%	4.53%	6.14%	7.24%	9.13%	10.35%	12.28%	13.37%	15.11%	15.87%			
Net MDD	4.85%	10.23%	13.82%	19.03%	23.11%	27.13%	31.26%	34.41%	38.36%	41.23%			
Net Sharpe ratio	1.22	0.74	0.73	0.54	0.64	0.51	0.55	0.46	0.43	0.34			
Avrg. turnover p.a.	1.75	1.62	2.24	2.73	2.89	3.40	3.61	3.97	4.14	4.12			

Table 5.16: Descriptive statistics for the four sub-periods from 1994 to 2013 for GS- and HC-based portfolios, at five different risk target levels.

Furthermore, Table 5.17 coincides with the above Table 5.16 in the sense that the relative performance of risk-adjusted returns is seemingly connected to the market regimes. The outperformance of GS-based portfolios during recessionary times overweigh the underperformance shown in some instances during expansionary times. Hence why the GS-based portfolios showed consistently higher risk-adjusted returns over the full evaluation period.

Target Volatility	Ultra-conservative Conservati 3% 6%			onservativ 6%	7e		Moderate 9%		Aggressive 12%			Ultra-aggressive 15%			
Jan 1994-Jan 2001 (Expansionary Period)															
Correlation Method	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	5.56% 4.09% 0.11	5.83% 4.09% 0.17	-0.27%	9.13% 7.92% 0.51	9.73% 7.92% 0.58	-0.60%	$\begin{array}{c} 12.16\% \\ 11.29\% \\ 0.62 \end{array}$	$12.21\% \\ 11.29\% \\ 0.63$	-0.05%	$\begin{array}{c} 14.48\% \\ 13.54\% \\ 0.69 \end{array}$	$\begin{array}{c} 14.10\% \\ 13.54\% \\ 0.66 \end{array}$	0.38%	15.59% 15.27% 0.68	15.03% 15.27% 0.65	0.56%
Feb 2001-June 2004 (Recessionary Period)															
Correlation Method	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	7.33% 4.67% 1.19	7.28% 4.67% 1.18	0.05%	7.31% 7.34% 0.76	6.43% 7.34% 0.64	0.88%	7.69% 10.79% 0.55	$\begin{array}{c} 6.12\% \\ 10.79\% \\ 0.40 \end{array}$	1.56%	8.93% 12.57% 0.57	7.56% 12.57% 0.46	1.37%	8.90% 13.66% 0.52	7.85% 13.66% 0.45	1.05%
					July 2004	4-Feb 200	08 (Expa	nsionary	Period)						
Correlation Method	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	5.93% 3.37% 0.68	$\begin{array}{c} 6.06\% \\ 3.37\% \\ 0.72 \end{array}$	-0.13%	10.36% 7.25% 0.93	10.13% 7.25% 0.90	0.22%	12.61% 10.71% 0.84	13.80% 10.71% 0.95	-1.18%	14.20% 14.01% 0.75	16.59% 14.01% 0.93	-2.39%	18.43% 16.71% 0.89	19.39% 16.71% 0.94	-0.97%
				1	Mar 2008	8-Dec 20	13 (Rece	ssionary	Period)						
Correlation Method	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ	GS	HC	Δ
Net mean RAR p.a. Volatility p.a. Net Sharpe ratio	5.74% 4.53% 1.22	3.62% 4.53% 0.74	2.13%	5.51% 7.24% 0.73	4.15% 7.24% 0.54	1.36%	$\begin{array}{c} 6.91\% \\ 10.35\% \\ 0.64 \end{array}$	5.53% 10.35% 0.51	1.38%	7.63% 13.37% 0.55	$\begin{array}{c} 6.35\% \\ 13.37\% \\ 0.46 \end{array}$	1.28%	7.14% 15.87% 0.43	5.62% 15.87% 0.34	1.52%

Table 5.17: Relative return performance for the sub-periods that constitute the full evaluation period. The returns of GS-based portfolios are converted to risk-adjusted returns (RAR) by deleveraging GS portfolio volatility to HC portfolio volatility. Δ denotes the difference in return between GS-based portfolios and HC-based portfolios.

5.2.5 Summary of out-of-sample performance

Among the covariance estimators considered in this thesis, the Gerber Statistic (GS) based covariance matrix was the only estimator that consistently outperformed the sample covariance matrix in terms of out-of-sample Sharpe ratios and risk-adjusted returns of corresponding MV optimized portfolios. In addition, with exception for the ultra-conservative risk profile, GS-based portfolios outperformed all other competing methods in terms of risk-adjusted return. This is shown in Figure 5.6.



Figure 5.6: Realized risk-adjusted returns of portfolios over the full evaluation period, allocated through employing various covariance matrix estimators in MVO.
To further investigate the relative performance between the covariance matrix estimators, the full period was divided into four sub-periods based on monetary policy signals as well as stock market signals. The sub-periods could be defined as an expansionary or a recessionary period. In expansionary periods, GS-based portfolios consistently underperformed all the other portfolios in terms of risk-adjusted returns. This is depicted in 5.7.



Figure 5.7: Realized risk-adjusted returns of portfolios over expansionary periods, for five different risk profiles.

However, during recessionary times, GS-based portfolios provided a strong outperformance in terms of risk-adjusted returns. This is illustrated in Figure 5.8.



Figure 5.8: Realized risk-adjusted returns of portfolios over the full evaluation period, for five different risk profiles.

Thus, the dominance of GS-based portfolios during the full evaluation period is primarily driven by the strong relative performance in recessionary times. These results suggest that even more efficient portfolios can be obtained by combining the GS covariance estimator with e.g. the traditional sample covariance matrix (HC). As the sub-periods are determined on an ex-ante basis, a procedure can be formed that uses the GS covariance estimator during recessionary times and switches to the sample covariance matrix when a regime transition signal is triggered. Although this would improve the performance in the data set considered in this thesis, further research is warranted regarding a possible regime connection prior to delving into such procedures. Plausible explanations for the seemingly regime dependent performance of GS in particular will be discussed in Chapter 6. Furthermore, GS-based portfolios are also associated with lower maximum drawdowns which measures the maximum loss from a peak to a nadir over the evaluation period between the rebalancing points of the portfolio. This implies less risky portfolios in terms of downside risk. See Figure 5.9 below.



Figure 5.9: Maximum monthly drawdown of portfolios over the full evaluation period, for five different risk profiles.

Finally, GS-based portfolios were generally associated with higher portfolio turnover. This is shown in Figure 5.10. Recall that transaction costs are included in the MVO problem for allocating portfolios in this thesis. Thus, the costs incurred by a high portfolio turnover are already incorporated in the portfolios. However, high portfolio turnover also entails higher exposure to liquidity risks, which is not caputed in MVO as the model assumes efficient financial markets. In addition, portfolio turnover measures the sensitivity to changes in transaction costs. This means that if the proportional transaction cost of trading goes beyond 10 basis points used in this thesis, GS-based portfolios will suffer the most.



Figure 5.10: Annualized portfolio weight turnover over the full evaluation period, for five different risk profiles.

In summary, all covariance matrix estimators considered in this thesis, other than the GS covariance matrix estimator, showed ambiguous results in a portfolio allocation context. The only method that consistently outperformed the traditional sample covariance matrix was the covariance matrix estimator constructed with the Gerber Statistic comovement measure. Replacing the sample covariance matrix with the GS-based covariance matrix rendered more efficient

portfolios in terms of higher Sharpe ratios, risk-adjusted returns and lower maximum drawdowns. This suggests that an investor that employs traditional MVO in quantitative asset allocation can improve their asset picking abilities by changing to the, in theory, more robust GS covariance matrix estimator. The enhancement is particularly prominent during recessionary times, often associated with high equity volatility due to e.g. leverage effects. A plausible explanation for this phenomenon is that noise and outliers in the data is more protruding in bearish markets, which the structured GS-estimator seems to better account for, thus coinciding with the underlying justification for the development of the GS-measure.

It is important here to note that the GS-based estimator lead to non-positive semidefinite matrices in the vast majority of rebalancing points. This was unique for the GS-estimator, as all other estimators yielded positive semidefinite matrices. As illuminated in Section 3.4.6, it is important that the estimated covariance matrix is positive semidefinite in the context of mean-variance optimization. If not, the definition of a covariance matrix is violated. Consequently, unstable solutions may result in the optimizer. Clearly, this poses a problem in the optimization process since a solution to the problem cannot be guaranteed to be a globally optimal solution.

In Gerber et al. (2015), it is recognized that in computing the Gerber Statistic, a non-positive semidefinite correlation matrix is not an unlikely outcome, in theory. While not found to occur in their study, it was prevalent in this study. To tackle this problem, the GS-based covariance matrices had to be repaired in the sense that they were replaced with the nearest positive semidefinite covariance matrix in Frobenius norm, using the algorithm suggested by Higham (2002) (in Section 3.4.6). This raises concerns regarding the validity of the obtained results and will be further discussed in Chaper 6.

5.3 Covariance prediction analysis

This section provides the results obtained from the covariance prediction analysis, performed by employing the methodology outlined in Section 4.3. In short, the accuracy of predicted portfolio volatility is evaluated for portfolios whose asset weights have been randomly selected. Consequently, no optimization is performed when allocating these portfolios. This allows for a further investigation regarding the relative performance between the considered covariance estimators, without running the risk of potentially diluted results caused by estimation error in the estimated return vector. This entails that the analysis becomes less tailored for portfolio optimization, as no optimization is performed. Nevertheless, combined with the out-of-sample performance analysis, the robustness of the results connected to the relative performance between the covariance estimators is deemed to increase.

Initially, the performance of the competing covariance estimators in relation to the traditional sample covariance will be presented. Thereafter, a summary for all considered covariance estimators is given.

5.3.1 SI versus HC

In similar fashion as in Section 5.2, we begin with the covariance matrix estimator implied by the single-index model (SI) versus the traditional sample covariance matrix (HC). In Figure 5.11, the mean of ratios of realized to predicted volatility is plotted. Recall that in each month, the portfolio volatility of 1000 randomly allocated long-only portfolios (fully invested) is predicted using monthly data from two years back, for each covariance matrix estimator. This is then compared to the realized monthly volatility for these portfolios, looking one year forward. The process is reiterated until the last month of the evaluation period, i.e. December 2013.



Figure 5.11: Time series for the mean of ratios of realized to predicted volatility for HC- and SI-based estimators.

Observing Figure 5.11, one can see that there apparently exists a time dependence. That is, over time the difficulty of predicting volatility varies. In general, there seems to be no difference between the methods with respect to time dependence. Both methods show peaks in the graph at the same time. However, one can note that for the peaks in 1998 and in particular 2009, the mean is to closer to one for SI-ratios compared to the HC-ratios. This suggests that the structured SI-covariance estimator performed better, on average, during the subprime mortgage crisis, albeit still being far from one which implies an under-prediction of portfolio volatility.

In Figure 5.12, histograms for all the ratios are shown. The mean for HC-factors is approximately 1.10, whereas the mean for SI-factors is approximately 1.03. This corresponds to more accurate prediction for SI (on average). In addition, the variance of HC-factors is 0.3760 and the variance of SI-factors is 0.3182. This indicates that the the performance of SI is more consistent than the performance of HC. However, it is worth noting that an arithmetic average can be misleading here: the peaks can be seen as outliers that has a large impact on the average. The high variance of both methods is shown by the wide distributions in the histograms.



Figure 5.12: Histograms for the ratios of realized to predicted volatility for HC- and SI-based estimators.

5.3.2 SM versus HC

Similarly, for the shrinkage to market estimator (SM), Figure 5.13 shows the mean of realized to predicted volatility ratios over the evaluation period. Once again, one can see that there apparently exists a time dependence. That is, over time the difficulty of predicting volatility

varies. However, in this case, minuscule differences can be noted, even at the peaks.



Figure 5.13: Time series for the mean of ratios of realized to predicted volatility for HC- and SM-based estimators.

As shown in Figure 5.14, the SM-based covariance estimator seems to predict better than HC, on average. However, the difference is minuscule. In addition, the variance of the SM-factors is 0.3718 (compared to the 0.3760 of the HC-factors), once again suggesting a very marginal improvement in terms of consistency. A plausible explanation for the small differences could be that the shrinkage estimator has assigned a small shrinkage intensity to the shrinkage target, which implies a fit that is close to the sample covariance matrix.



Figure 5.14: Histograms for the ratios of realized to predicted volatility for HC- and SM-based estimators.

5.3.3 RMT versus HC

Moreover, Figure 5.15 illustrates the time series of mean realized/predicted volatility ratios for HC-and RMT-based covariance matrix estimators. Yet again, the difference between the estimators is marginal. However, under close inspection, one can see that the estimator (RMT) implied by the random matrix filtering technique notably underestimates risk for the peak period during the recent financial crisis. Clearly, no estimator based on historical data will perform well here as it can be considered as a black swan event. The noise filtering technique, however, performs relatively poor in this case.



Figure 5.15: Time series for the mean of ratios of realized to predicted volatility for HC- and SM-based estimators.

The relatively poor performance is further noted by studying the mean of the RMT-factors. This amounted to approximately 1.13, implying that the realized risk was 13% higher than the predicted risk over the full period (on average). In addition, the variance of the RMT-factors was 0.4184 (compared to 0.3760 of the HC-factors), consequently indicating lower stability of the estimates. This is shown in Figure 5.16.



Figure 5.16: Histograms for the ratios of realized to predicted volatility for HC- and RMT-based estimators.

5.3.4 GS versus HC

Finally, the covariance estimator implied by the Gerber Statistic (GS) was evaluated in this setting. The mean of the realized volatility to predicted volatility ratios is plotted for the HC-method as well as the GS-method in Figure 5.17.



Figure 5.17: Time series for the mean of ratios of realized to predicted volatility for HC- and GS-based estimators.

Once again, we can see that there apparently exists a time dependence. That is, over time the difficulty of predicting volatility varies. In general, there seems to be no difference between the methods with respect to time dependence. Both methods show peaks in the graph at the same time. However, further inspection of the peaks shows that the GS-estimator performs much better than the HC-estimator during the subprime mortgage crisis. Although the ratios still prevail far from one, the GS-estimator underestimates risk less than the HC-estimator in these times. A plausible explanation for this is that recessionary times are often associated with volatile markets, influenced by noise and outliers. The discretization process of the GS-measure that attempts to accommodate for these events in the estimation process is seemingly working in this case.

Furthermore, the mean of the GS-factors amounts to approximately 1.01, as opposed to the HC-factors with a mean of approximately 1.10. This implies that the GS-based estimator provides more accurate forecasts, on average. In addition, the performance of the GS-based estimator is more consistent as the variance of the realized-to-predicted volatility ratios is 0.3026 (compared to 0.3760 for the HC-factors). This is shown in Figure 5.18 by the slightly wider tail of the distribution for the HC-factors



Figure 5.18: Histograms for the ratios of realized to predicted volatility for HC- and GS-based estimators.

5.3.5 Summary of covariance prediction analysis

To summarize, the obtained results from the covariance prediction analysis are presented in Table 5.18

Full period (1994-2013)		
Covariance matrix estimator	Realized-to-predicted portfolio risk ratios (on average)	Variance of the ratios
Sample covariance matrix (HC)	1.10	0.38
Single-index model (SI)	1.03	0.32
Shrinkage to market (SM)	1.10	0.37
Random matrix theory (RMT)	1.13	0.42
Gerber statistic (GS)	1.01	0.30

Table 5.18: Portfolio risk prediction accuracy for the five covariance matrix estimators considered in this thesis. A ratio of 1 implies that the realized portfolio risk coincides with the predicted portfolio risk.

We see that all estimators tend to underestimate portfolio risk, as a ratio that exceeds one implies that the realized risk exceeds the predicted portfolio risk. In addition, the variance of the ratios seem to stem from a time dependence in forecasting portfolio risk. That is, over time the difficulty of predicting volatility varies. In general, there seems to be a marginal difference between the methods with respect to time dependence. All estimators exhibit peaks (large deviations between forecasted and realized risk) at similar time periods. However, the GS- and the SI-based perform relatively better during these times.

In particular, the GS-based estimator is seemingly the best in terms of prediction accuracy amongst the considered estimators. On average, the realized portfolio risk is approximately 1% higher than the prediction. In addition, the GS-based estimator is more consistent in this case, as shown by the lowest variance for the ratios of realized to predicted volatility.

However, it is worth to note that an average performance may render misleading results as outliers (peaks) may significantly skew the average. In addition, recall that the evaluated portfolios in this empirical test have been randomly allocated. Thus, no optimization has been performed. The underlying reason for this procedure was to evaluate the covariance matrix estimators in an isolated manner, without having to potentially dilute the performance by errors in the expected returns estimation.

Nevertheless, invoking the results from the out-of-sample performance results presented in Section 5.2 for MV optimized portfolios (using the different covariance matrix estimators), coinciding conclusions can be drawn. See Figure 5.19.



Figure 5.19: Difference between realized and predicted risk for five different investor profiles, with different target portfolio risks. Note that the y-axis is the difference in this case, and not the ratio between realized and predicted portfolio risk. The percentage shown in the parentheses on the x-axis correspond to the target portfolio risk for the particular investor profile. The closer the bar is to zero on the y-axis, the better.

Figure 5.19 shows that the estimators also underestimate portfolio risk in a portfolio optimization setting (i.e. realized risks are higher than the target risks). In addition, one can observe that the GS-based estimator performs best here too, where the ex-post volatility of the portfolios is more aligned with the sought portfolios volatility for all five risk profiles. This further suggests that the GS-based covariance matrix estimator provides more accurate forecasts of portfolio risk, in comparison to the competing estimators.

Chapter 6 Discussion

This thesis was set forth by the purpose to investigate whether financial practitioners and institutions can allocate portfolios consisting of assets in a more efficient manner by changing the covariance matrix input in mean-variance optimization. Indeed, the results presented in Chapter 5 are striking as one covariance matrix estimator showed a dominant performance: the estimator implied by the Gerber Statistic relationship.

However, prior to delving into the findings in greater detail, it is important remember the environment in which this thesis operates within. Russell (1931) once said the following:

'Although this may seem a paradox, all exact science is dominated by the idea of approximation. When a man tells you that he knows the exact truth about anything, you are safe in infering that he is an inexact man.'

This quote is deemed to be highly relevant in this thesis. While financial models such as the mean-variance framework are simple, reality is not. The mean-variance framework is merely an illusion of the reality, and confusing these two can be devastating. In this thesis, it is assumed that the mean-variance framework properly mirrors the utility functions of individuals. Therefore, it is naive, and even dangerous, to think that the findings in this thesis are generalizable over all environments. Knowing what is assumed and what is swept out of view in a model is crucial, and this thesis takes the stance that the assumptions of the mean-variance framework hold. Thus, this study is based on the idea of approximation in the sense that the validity of the findings is highly dependent on how well the assumptions that underlie MVO hold in reality.

Now, having clarified that this thesis addresses investors that employ mean-variance optimization in practice, the findings in this thesis will be further discussed.

6.1 Portfolio out-of-sample performance

In the quest of answering how asset managers that employ MVO may improve their asset-picking abilities, a backtesting procedure was performed. Holding all things equal, except the covariance matrix input, the performance of optimized portfolios for the corresponding covariance matrix estimators was investigated.

Interestingly, the traditional sample covariance matrix performed seemingly well and on par with the vast majority of the competing estimators. This was true for all except the covariance estimator implied by the Gerber Statistic, which in addition to showing higher Sharpe ratios, higher risk-adjusted returns and lower maximum drawdowns, also matched the investor risk profile the best. This is in agreement with the findings by Gerber et al. (2015), albeit they solely compared the Gerber Statistic estimator with the sample covariance matrix.

The ambiguous results for the covariance matrices implied by the single-index model (SI), the shrinkage to market model (SM) and the random matrix filtering estimator (RMT) may come as a surprise. While Chan et al. (1999) concluded that the SI-based estimator outperformed the sample covariance matrix in terms of portfolio performance, no such conclusion can be drawn in this study. A plausible explanation for this result is that the restrictive assumption that there are no other effects beyond the market factor that account for asset comovement, holds poorly in this setting. Consequently, the SI-based estimator will be penalized by the introduction of specification error. Another explanation may lie in how the market was defined in this thesis: this consisted of an equally weighted index for the considered assets, as suggested by Ledoit and Wolf (2003b). If this proxy poorly reflects the global market portfolio, the model is already violated by construction.

Furthermore, the SM-based estimator showed a minuscule improvement over the sample covariance matrix. As suggested by Ledoit and Wolf (2003b), the shrinkage estimator is based on large-dimensional asymptotics. However, in this thesis, only nine assets are considered with 24 monthly observations per asset. As the ratio of the number of assets divided by the number of observations per asset is low and far from 1, the shrinkage estimator will perceive the sample covariance matrix to be well conditioned. This suggests that the shrinkage estimator will show a marginal difference to the sample covariance matrix, which coincides with the results that showed a minuscule improvement for the shrinkage estimator.

As for the RMT-based estimator, the results were ambiguous. However, this was seemingly the worst estimator in the sense that it produced, on average, worse portfolios than the sample covariance matrix. The aspect of large-scale asymptotics can also be discussed here: the use of random matrix theory is mostly suitable for large-scale matrices, as suggested by Laloux et al. (2000). However, in this thesis, the asset universe is spanned by a relatively low dimension which might render RMT-approximations inaccurate.

The main implication of the findings is that replacing the sample covariance matrix with the GS-based covariance matrix rendered more efficient portfolios in terms of higher Sharpe ratios, higher risk-adjusted returns and lower maximum drawdowns. This suggests that an investor that employs traditional MVO in quantitative asset allocation can improve their asset picking abilities by changing to the, in theory, more robust GS covariance matrix estimator. In addition, GS-based portfolios were better aligned with the ex-ante defined risk profiles. The enhancement is particularly prominent during recessionary times, while in expansionary periods GS shows no improvement. A plausible explanation for this apparent market regime dependence is that recessionary times are often associated with high equity return volatility due to e.g. leverage effects (debt to equity ratio rises), as exemplified in Ait-Sahalia, Fan, and Li (2013). It is further expected that noise and outliers in return data is more protruding in bearish market, which the structured GS-estimator seems to better account for, thus coinciding with the underlying justification for the development of the GS-measure. The poor performance during expansionary times could be explained by how the thresholds are defined in the Gerber Statistic measure. In this thesis, the thresholds have been defined in the same manner as in Gerber et al. (2015). In more stable periods, these thresholds may be far too restrictive and perceive signal data as noise which leads to a poor fit. Gerber et al. (2015) do not discuss how the threshold when constructing the Gerber Statistic can be differently defined. However, the obtained results in this thesis indicate that further refinements could be made in the sense that the threshold may be dynamically adjusted based on recent behavior of the underlying variable, or alternatively, according to the prevailing market regime. This calls for further research regarding this measure.

Furthermore, in stark contrast to Gerber et al. (2015), the GS-based covariance matrices

were non-positive semidefinite and had to be adjusted by employing Higham's (2002) method for finding the nearest positive semidefinite covariance matrix in Frobenius norm. In Gerber et al. (2015), this problem was not found to occur, neither in real nor simulated practice. This is quite surprising as it was a persistent problem for the GS-estimator in this study. Ultimately, this raises concerns regarding the validity of the results in the sense that the GS-based covariance estimators used in the optimization is not solely based on the definition of the Gerber Statistic. Thus, it should be clear that when referring to GS-based estimators in this thesis, Higham's (2002) method for finding the nearest positive semidefinite matrix is implemented to ensure a positive semidefinite covariance matrix.

6.2 Additional validity concerns

One should be aware that the nature of the employed methodology in evaluating portfolio performance implies variability depending on how the ceteris paribus context is defined. For instance, the length of the in-sample period as well as the out-of-sample period are very much chosen arbitrarily. There is no general consensus in the literature regarding how these parameters should be defined in this setting. However, the relative performance between the covariance matrix estimators may vary depending on how these are set. For instance, increasing the number of observations (longer in-sample period) is expected to benefit the sample covariance matrix as this is the maximum likelihood estimator (assuming that the data is to be trusted). As the sample covariance matrix is the base case in this thesis, the main decision criteria has been to ensure that the number of assets does not exceed the number of observations per asset. A suggestion for further research is to investigate how the length of the in-sample estimation window affects the performance of the estimates.

The choice of estimation regarding the expected return vector, kept equal for all methods, may also dilute the relative performance between the covariance matrix estimators as estimation errors contained in this vector may favor certain covariance estimators. As a result, the validity of the results for the portfolio out-of-sample performance for various risk profiles is undermined. This is the reason for the covariance prediction analysis for randomly allocated portfolios (no optimization performed). In doing so, I was enabled to perform a further investigation regarding the relative performance between the considered covariance estimators, without running the risk of potentially diluted results caused by estimation error in the estimated return vector. Combining this analysis with the optimized portfolio evaluation, the robustness of the results connected to the relative performance between the covariance estimators was deemed to increase. Indeed, the covariance prediction analysis led to a coinciding conclusion that the GS-based estimator dominated the competing estimators in terms of prediction accuracy, implying portfolios that are better aligned with the predetermined risk profiles. Furthermore, this analysis further supported that the dominance of the GS-based estimator seemingly stems from its striking outperformance during bearish equity markets, often influenced by high market volatility.

Nevertheless, further research is warranted regarding the robustness of the results. The portfolio construction is juxtaposed by many different factors. The main question is thus whether one would obtain consistent results by changing how all other things are kept equal in the optimization process (e.g. how the asset universe is defined and how the expected returns are estimated). In this context, a suggestion is to investigate whether the results are consistent when the expected return vector is estimated via the Black-Litterman model or the Fama and French three factor model, as opposed to using the CAPM model.

Chapter 7 Conclusion

In the quest of challenging the traditional sample covariance matrix within the context of portfolio optimization, the purpose of this thesis has been to investigate whether the performance of mean-variance optimized portfolios can be improved by using an alternative covariance matrix estimator in lieu of the sample covariance matrix.

To achieve this purpose, the relative performance of prominent challengers to the sample covariance matrix has been evaluated in a realistic backtesting procedure, where all other factors in the mean-variance optimizaton were kept equal. The empirical results from the investigation pointed towards one dominant estimator: the covariance matrix estimator implied by the Gerber Statistic (GS), which was first introduced by Gerber et al. (2015). In other words, by replacing the sample covariance matrix with this estimator, more efficient portfolios were obtained in terms of being associated with higher Sharpe ratios, higher risk-adjusted returns and lower maximum drawdowns. However, these portfolios were generally connected to relatively high turnovers. This indicate that portfolio managers can achieve monetary benefits by employing GS in their covariance matrix estimation, given that the proportional transaction cost of trading is not abnormally high. In addition, GS-based portfolios performed best in terms of ex-post performance alignment with the predetermined investor risk profile.

Further investigations showed that the dominance of GS-based portfolios could be explained by its striking forecast performance during bearish equity markets in relation to the competing estimators considered in this thesis. Taking it for granted that financial markets are more volatile in times of distress, this implies more noisy return observations. Hence, the GS-based estimator is seemingly excellent in distinguishing signal from noise in historical return observations in this setting. In other words, the empirical findings suggest that the GS-based covariance matrix estimator provides more efficient diversification when it matters the most.

However, in this study, the GS-based estimator comes at the expense of producing nonpositive semidefinite covariance matrices. Clearly this poses a problem as it halts the optimization process. Thus, these estimators have required correction in the sense of replacement with the nearest positive semidefinite covariance matrices in the Frobenius norm. To a certain extent, this challenges the validity of the empirical results.

Nevertheless, by using the GS covariance matrix estimator in lieu of the traditional sample covariance matrix, the MV optimization rendered more efficient portfolios in terms of higher Sharpe ratios, higher risk-adjusted returns and lower maximum drawdowns. The outperformance was protruding during recessionary times. This suggests that an investor that employs traditional MVO in quantitative asset allocation can improve their asset picking abilities by changing to the, in theory, more robust GS covariance matrix estimator in times of volatile financial markets.

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