MEASUREMENT AND DECOMPOSITION: MAKING SENSE OF THE AREA OF A CIRCLE

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As a component of a course on geometry for preservice elementary teachers (PSTs), we derive area formulas for a variety of polygons including triangles, quadrilaterals, and both regular and irregular shapes whose areas can be measured empirically using decomposition. Decomposing a circle to justify why its area can be measured using the standard formula is more challenging as it requires both empirical and deductive reasoning involving limits. In spite of the challenge, we expected decomposition strategies to transcend work with polygons and support PSTs when thinking about the area of circles. Results show that few PSTs utilized decomposition and instead focused on finding meaning in the symbolism of the formula. Concept images related to area will be discussed.

Keywords: Geometry and Geometrical and Spatial Thinking; Mathematical Knowledge for Teaching; Teacher Education-Preservice; Reasoning and Proof

Background

As a component of a course on geometry for preservice elementary teachers (PSTs), we derive area formulas for a variety of polygons including triangles, quadrilaterals, and both regular and irregular shapes. Generalized formulas for finding the area of these types of shapes can be justified using the actions of composition and decomposition. For example, one way to derive a general formula for the area of a trapezoid is to decompose it into two triangles (with altitude equal to that of the original trapezoid and each taking one of the parallel sides as a base) and then summing the two areas.

Being able to give an informal derivation between the circumference and the area of a circle is a common middle-grades standard (CCSSO, 2010), yet the curved boundary lends challenge to students used to working with polygons. There are multiple justifications possible. Tent (2001) describes one method of figuring out the area of a circle through the action of composition and decomposition that were commonly used in the sixteenth and seventeenth centuries. A circle with radius, r is sliced into sectors, which are then rearranged to approximate a parallelogram with height equal to the radius and width equal to half of the original circumference. When the sectors are sliced infinitely thin, the parallelogram becomes a rectangle. The area of which, we deduce, can be measured as πr^2 *(See Fig. 1).*

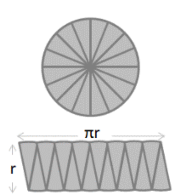


Figure 1: Decomposing a Circle to Reason About Its Area

Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. St Louis, MO: University of Missouri.

We want to better understand how PSTs might transfer strategies for conceptualizing area in the context of polygons to the circle. At the end of our instructional unit on area of polygons, and before any instruction on circles, we gave a pre-assessment where we asked 69 PSTs to justify the standard formula for the area of a circle. Through the analysis of this set of written work, we sought to gain a better understanding of the claims and supports PSTs utilized when justifying why the standard formula for area of a circle, Area = $\pi(radius)^2$. This study sought to answer the question: how can we characterize PST's justifications of the general formula for finding the area of a circle?

Theoretical Perspective

Mathematics education research community knows very little about students' (PSTs included) conceptions of the area formula of the circle beyond memorizing and applying the formula to simple routine problems. In a recent review of research on the teaching and learning of measure (Smith and Barrett, 2017), discussion focused on area in the context of rectangular regions alone. Other than the general notion that area is an attribute that measures the amount of space inside the boundary of a 2-D shape and can be quantified by counting the number of area units, research studies focusing on rectangular regions or polygons provide limited insights into the challenges students face when trying to make sense of the area formula of the circle.

First of all, the boundary of a circle is curved instead of straight which makes the idea of tiling the space inside with square units seem impossible. Second, a justification of why the area can be measured using the standard formula, Area=$\pi(radius)^2$, requires an infinite number of subdivisions. The challenge emerges in our inability to physically decompose the shape (using common tools such as grid paper or scissors) into an infinite number of subshapes that can be used to quantify the amount of surface inside a circle. To do so conceptually requires a good understanding of the concept of limit through approximation metaphor (Oehrtman, 2004). Lastly, researchers have found that the idea of a non-repeating, non-ending number is very challenging for both preservice teachers and students (Fischbein, Jehiam, & Cohen, 1995; Güven, Çekmez & Karataş, 2011). So the presence of π in the area formula can pose an additional challenge for sense-making.

In addition to this analysis of the challenges in making sense of area formula of the circle, the design of this study was guided by prior work on proof and justification. Based on Stylianides's (2007) conception of proof and proving in elementary school, we take justification to mean a viable argument that supports a new claim with previously established definitions and facts as well as a connected sequence of assertions backed by by sound reasoning. We had worked with our PSTs' previously on justifying area formulas for various special polygons and we expected that some of them might even use diagrams to augment their argument.

Theoretical Framework

Harel and Sowder's (1998) proof schema were initially helpful in framing categories for the written responses. Specifically, we used *externally-based*, *empirical* and *analytical* proof schemes to organize the data. A written response is classified as an externally-based proof scheme when either 1) the argument is based on a textbook or other authoritative figure (authoritarian proof scheme), or 2) manipulation of symbols without meaning (symbolic proof scheme). The empirical proof scheme consists of arguments based on 1) multiple numerical examples (inductive proof scheme) or 2) rudimentary mental images (perceptual proof scheme), that is, 'images that consist of perceptions and a coordination of perceptions but lack the ability to transform or to anticipate the results of a transformation," (Harel & Sowder, 1998, p. 255). When an argument is based on logical deduction, it is said to use an analytical scheme.

Methods

Setting and Participants

This study was conducted in a Midwestern university. All participants (n=69) were enrolled in one of two sections of a course on geometry for preservice elementary teachers taught by the first author. The data used in this study were collected at the end of a unit on measurement in which significant work had been done toward a deep understanding of linear and area measurement. This work included activity related to the meaning of perimeter and area in the context of simple and composed polygons. Participants in this study had previously been asked to justify the area formulas for a variety of polygons including triangles, rectangles, parallelograms, and trapezoids using decomposition as a technique with tools such as dot and grid paper, tracing paper, and geoboards. At the time of data collection, participants had not engaged in course activity involving circles, including the measurements of radii, diameters, or circumferences nor the ratio of π.

Data Collection and Analysis

Students were asked to justify individually why the area of a circle could be found using the formula $Area = \pi r^2$. This formula was explicitly given, though accompanied by no additional guidelines or specifics. PSTs had been asked to justify or derive formulas for other shapes prior to this work, though they had not had any instruction regarding circles or the meaning of π. Their work was done by hand on blank white paper.

We used the proof schemes by Harel and Sowder (1998) to sort all written justifications into three piles: those used external authority based, empirical based and analytical based proof schemes. While some data fit clearly within the framework, we did not find it was useful in all cases. We needed the flexibility to group responses that lacked the coherence of an intentionally written proof or who blended schema from across the framework; also, the writing was informal and spontaneously generated. We then turned to grounded theory (Corbin & Strauss, 1990) to categorize responses in order to bring out themes that had been visible to us, but were not emerging through the framework. Specifically, we wanted to understand how PSTs were using what they had been learning about area and decomposition to justify the area of a circle.

Our process of categorization was to first read through a subset of the data to look for similarities and differences. Once an original set of categories were identified, we used an iterative process of reading through the data and refining the categories. In all cases, the entire response was used to garner meaning. In other words, we used the figures that were drawn as a means to interpret what was written and vice versa. Through this process, it became apparent to us that the drawn figures were being used as tools for different purposes and we will speak more to that in the discussion. When a response seemed to span two categories, we refined the categories when possible. However, when the category did not warrant subdivision, we placed each response according to the most sophisticated reasoning provided. For example, if a PST used a worked example, but also provided a justification for decomposing the circle into sectors, we chose to place it within the decomposition category.

Findings

We present five different categories of responses, grounded in the work of Harel and Sowder (1998). Second, we look across our categories to identify key conceptual challenges indicated by the responses. Reasoning about π and conflicting area concept images are visible within attempts at justification.

Characterizing PSTs' Written Justifications

Non-justification. We begin by noting that about 27.5% (n=19) of PSTs in this study made no attempt at justification. In a few cases PSTs defined r as the radius of a circle or noted that π was a number, but were not able to make use of that information relative to the formula. In other cases, PSTs simply wrote explicitly that they did not know or provided a description of a circle (i.e. infinite or round) that was not related to the formula, such as Malana, who wrote, *"the circle is shaped like a pie, so the formula works!"* These responses did not supply a viable argument to support new claim based on these definitions, so we did not categorize them as a justification (Stylianides, 2007).

Externally-based and empirical schema. We found 7 cases that fit the proof scheme framework by Harel & Sowder (1988). Six of them possessed the characteristics of an externally-based proof scheme by invoking authorities such as "mathematicians who discovered it" or simply reciting the formula from memory: "*multiply radius x itself and then multiply by* π". The remaining student submitted a true "proof by example" within the induction scheme.

The framework proved limited in analyzing the remaining 43 responses. While the majority of them (n=30) were motivated by a study of symbolism, there were clear attempts to go beyond and give meaning to those symbols. So, they did not quite fit under the big umbrella of externally-based schema. We created a new category for these which we called "Dimensional Analysis" (DA). This helped us capture one way PSTs attempted to give meaning to the area formula of a circle. The remaining 13 responses were classified either as Approximation Strategies or Decomposition Strategies. We will go on to describe each of our new categories illustrated by the responses that fit within them. Table 3 provides an orienting view of the entire data set according to these categories.

Table 1: Justifications by Category

Strategy	Frequency
Non-Justification	19
Externally-based	6
Empirical	1
Dimensional Analysis	
Diameter is r^2 *(only)*	10
Circumference (π) times diameter (r^2)	8
Other	12
Approximation	6
Decomposition	
Summation of Linear Measures	2
Summation of Equal-sized Sectors	5
Total	**69**

Dimensional analysis. In our study, 43.5% (n=30) of PSTs justified the area formula through a process of symbolic deconstruction aimed at associating meaning with the symbols and operations within the general formula. In most cases, this meant finding meaning in the isolated terms r, r^2, and π. While r was always defined as the radius of the circle (and often accompanied by a drawn and labeled figure), there was more variation in the ways that PSTs found meaning in r^2 and π.

In 10 cases, PSTs associated the square of the radius with measuring the diameter of the circle and then ended their justification without connecting that meaning back to the area of the circle. Within that group, some justified the squaring action on the basis of the relationship between the radius and the diameter, "*You square it because the radius is half of the diameter*" while others explicitly said that squaring was a doubling action, or indicated that r^2 was equivalent to the measurement of the diameter. Rhoda was the opposite, finding no meaning behind r^2, except to say that "*in order to go around the circle, it takes the r^2 3.14 times.*" Rhoda, like the others, was finding a way to associate the symbolism with measurement.

In 8 more cases, PSTs went one step further to include π in their sense-making, claiming that the formula worked because it multiplied that diameter (r^2) by the circumference (π). While that group of 8 is the largest cluster of similar meaning we could find, there were 12 other PSTs who also created symbolic combinations that focused on identifying measurements, but were either less specific in their meaning or unique in their interpretation. We present three examples of justifications from that category to highlight both the level of specificity and uniqueness of the responses:

> Brandon: "Because radius is half of a circle in order to get the full circle, you must square the radius. You then multiply by π because π is a measurement used in circles,"

> Bree: "Squaring the radius will get you the diameter of the circle going up and down and π will get you a measurement of the insides."

> Katie: "We do πr to find the distance around the entire circle. Then, you must multiply that number r again to account for all the area from the edge of the circle to the center of the circle."

Seneca, whose response is also included in this category, is the only one in our sample who explicitly mentioned that area was measured in square units, although her attention to those units was limited to a justification of why we square the radius and was diluted by a comparison to finding the area of a rectangle by multiplying length times width "*but just squaring the one length.*"

Approximation strategies. Six PSTs justified the area formula for a circle by comparing it to the area of a square made up by r^2 or $(2r)^2$. Four of them supported their arguments with a diagram of either a square with an inscribed circle or circumscribed square (Figure 2 shows both). The analysis of their diagrams indicated the struggle they had with the relationship between the square they had imagined and its relationship to the radius. This tension can be seen in Heather's written below (Figure 2).

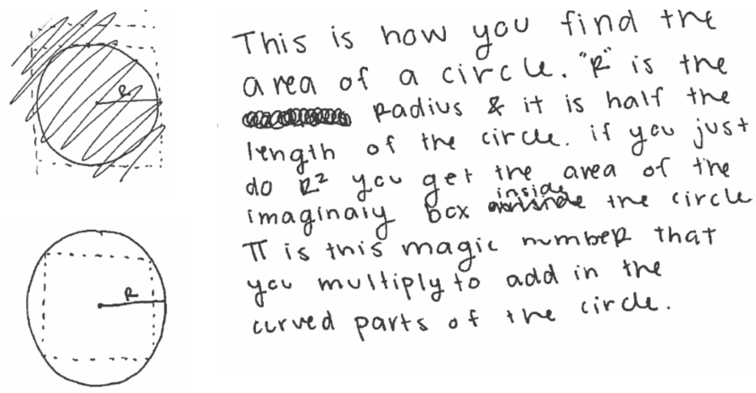


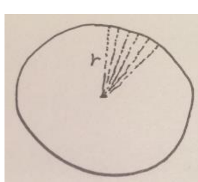
Figure 2: A PST's Attempted to Approximate the Area of a Circle with a Square

Five of these PSTs, including Heather, used π to justify the difference between the area of the circumscribed square and the area of the circle. Only one PST was able to successfully label the square with the circle inscribed (similar to the one crossed out by Heather) as $(2r)^2$, and acknowledged that the true area of the circle lied in-between those two approximations.

Decomposing strategies. Seven PSTs tried to make sense of the area formula for a circle by decomposing the circle in some fashion. Two of them imagined sweeping the region inside of the circle with a line segment (either r or $2r$) through π. For example, Chase wrote "*The radius is half the diameter of the circle, so you have to square the radius or r.*" (Note: On Hir paper, ze drew a diagram to support this). Ze went on to write, "*A circle is a symmetrical shape that is never ending. You multiply it by π to incorporate all of the degrees and angles of the circle."* At first glance, this justification seemed to have some validity, but further examination reveals a common misconception that area can be found by summing linear instead of area measures.

Another 5 PSTs decomposed the circle into equal-sized sectors. They made different associations between the decomposed shape and the formula as our class had done with polygons. Four struggled, but based reasoning on the belief that area was calculated by multiplying two length measures. For example, two PSTs (including a student named Audrey) made a distinction between the multiplied radii: they thought of one r as the length of the radius and the other as the number of sections, describing π as "*each of the sections in between the radii*", in another words, the arclength of the pie shape.

Only one PST's written justification (Figure 3) showed some glimpse of the idea of finding the area of circle through the concept of limit. The drawings and the accompanying explanation, though too brief to be classified as having an analytical proof scheme, showed evidence of finding the area of the circle by first decomposing a circle into many tiny sections and then summing the area of these tiny sections, which would resemble triangles.



I will cut a circle into a lot of pieces of triangles and get the area of one triangle and then add them together.

Figure 3: One PST's Decomposition Strategy to Justify the Area Formula of the Circle.

Discussion

This data was collected after instruction on justifying the area formulas for polygons, but prior to any instruction related to circles or π. As such, we should assume that these justifications are just quick snapshots or rough drafts and that PSTs ability to reason about the general formula and area in general would evolve and change with exposure to these concepts in class, or if this assignment had first been given to small groups to discuss.

While it was useful to utilize proof-based framework to categorize some responses, there were a number of responses that existed somewhere in between empirical and analytic schema. Those that we categorized into the new Dimensional Analysis schema were applying more analytic reasoning than empirical, though the resulting justification sought more to find meaning in this particular arrangement of symbols than in measurement concepts or imagery. In each of these 30 justifications, PSTs are searching among the symbols for cues as to which two dimensions would yield a 2-dimensional measurement. Becca sums it up when she says, *"This works because the radius squared is similar to bxh which is area of a parallelogram."*

Those we categorized within an approximating schema utilize empirical methods that are more solidly grounded by measurement concepts. However, as their arguments include some generality, they cannot be categorized under the empirical umbrella. Lastly, those that we categorized within the decomposition schema represent what we think of as pre-analytic arguments. While it can be useful to create sectors within the circle, finding the areas of individual sectors for the final sum creates a circular argument as these areas are often expressed as fractions of the whole circle. In order to make full use of the individual sectors, it's easier to rearrange them to form a new whole.

Conceptual Challenges

In the remaining space, we will identify two more conceptual challenges that were apparent when looking across the data. We will talk about concept imagery and the contradictions that created struggle. Then we explore in some depth the role π might have played in that struggle.

Concept imagery. While we did not expect PSTs to be able to fully carry out similar arguments, we were surprised to see only 7 of them even made an attempt to use the decompose/compose strategies they had learned when trying to justifying the area formula for polygons. Given that all of the participants had learned geometry in classrooms supported by the CCSS for mathematics, certainly they had previous knowledge of the geometry of circles. However, the curved boundary appeared to be a challenge for the PSTs in this study. Furthermore, only 3 PSTs acknowledged that the area of the circle could be calculated by summing up the area of an infinite number of smaller pieces in a way similar to the one by Chase discussed above. Looking across our data, we can see evidence of conflicting area concept images. In Seneca, we see her association with square units acting in contradiction to her reasoning about the curved boundary. She, like almost half of her colleagues, struggled to assimilate the formula with their image of area as the product of two one-dimensional measurements. We will share more of this in our presentation.

Meaning of π. Our analysis of PSTs' uses of π in their written justification also indicated their limited conception of this number. Just over half of our PSTs mentioned π in their written justifications. Of those that did mention it, some simply referred to it as a number attributed with important, yet indescribable power; "*an important ingredient of a circle*" or as Pat said, "*π is this magic number that you multiply to add in the curved parts of the circle."*

Measurement was another theme in the way PSTs addressed π, whether in terms of length, area, or angle. It was common for PSTs to connect π with a measure of circumference. For example, Lynn wrote, "*π is the standard circumference of a circle with a radius of 1 unit*." Often PSTs who had this conception of π drew a diagram (like the one in Figure 4) to support their justification. Rose considered it an area measure when she said, "*π gives you area because the number is exactly how many times that radius can fit in the circle."* Just five PSTs thought of π as a ratio, either as a ratio between the circumference and radius or the circumference and diameter.

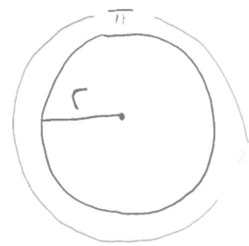


Figure 4: A PST's Diagram to Illustrate the Meaning of π and Radius

Implications

The fact that decomposition/recomposition did not factor more heavily into PSTs responses indicates that it is not enough to exclusively explore area concepts in the context of polygons. While concept imagery related to area such as *covering, iteration* and *decomposition* might be well-developed in the context of polygons, it seems necessary to carry over conceptual development to figures with curved boundaries including measuring circles with square units.

The variation that we saw in PSTs struggle to carve out dimensional measure from within the formula is of heightened interest to us. Jumping to an application-based instructional model (one that emphasizes plugging in values for π and r) earlier in their education might have increased our PSTs willingness to analyze each symbol separately and to overlook a more cohesive or conceptual justification based on existing images. It was clear that most were not cognizant of the relationship between circumference and area formulas, even if they might have recalled them on a different type of assessment. Moving away from plug-and-chug instruction and practice would better serve students of all ages.

There are some interesting concept images present in some of the approximation and decomposition justifications that might point to ways to extend the study of circles. First, approximating the area of the sectors using the radius and arc length might be an interesting activity. Second, using this context to make a connection to exponents and algebra might be a way to help students return to this concept to make additional sense outside the scope of an algebra lesson. At the very least, making the difference between doubling and squaring explicit in this context would raise awareness among PSTs about language and mathematical precision. Last, we were really compelled by those arguments that are based on accumulation. Specifically, the accumulation of length to create a measure of area. This reasoning could be developed further, specifically related to defining angle as a turn (Keiser, 2000) or as an early representation of radian measure.

One activity that models the sector strategy (Tent, 2001) is to have students use scissors to physically decompose and recompose the shape. However, Or (2012) suggests that a digital applet that uses sliders to guide the process (and incorporates far more precision) might help students bridge the epistemic gap between approximation and an exact value.

Setting Up Future Study

While the written justifications were the target for analysis in this study, we suspect that the nature of and ways in which PSTs use images or drawn figures will change after exposure to the types of activities mentioned in the previous section and that this would be an interesting line of inquiry for future studies.

A second area for future work will be to revisit and expand the proof schema to make room for the types of reasoning shown here. In particular, we think there is work to be done to expand the ways in which we reason about and with symbols. We believe that categorizing the ways in which people enact proof is something that merits more study and validation.

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