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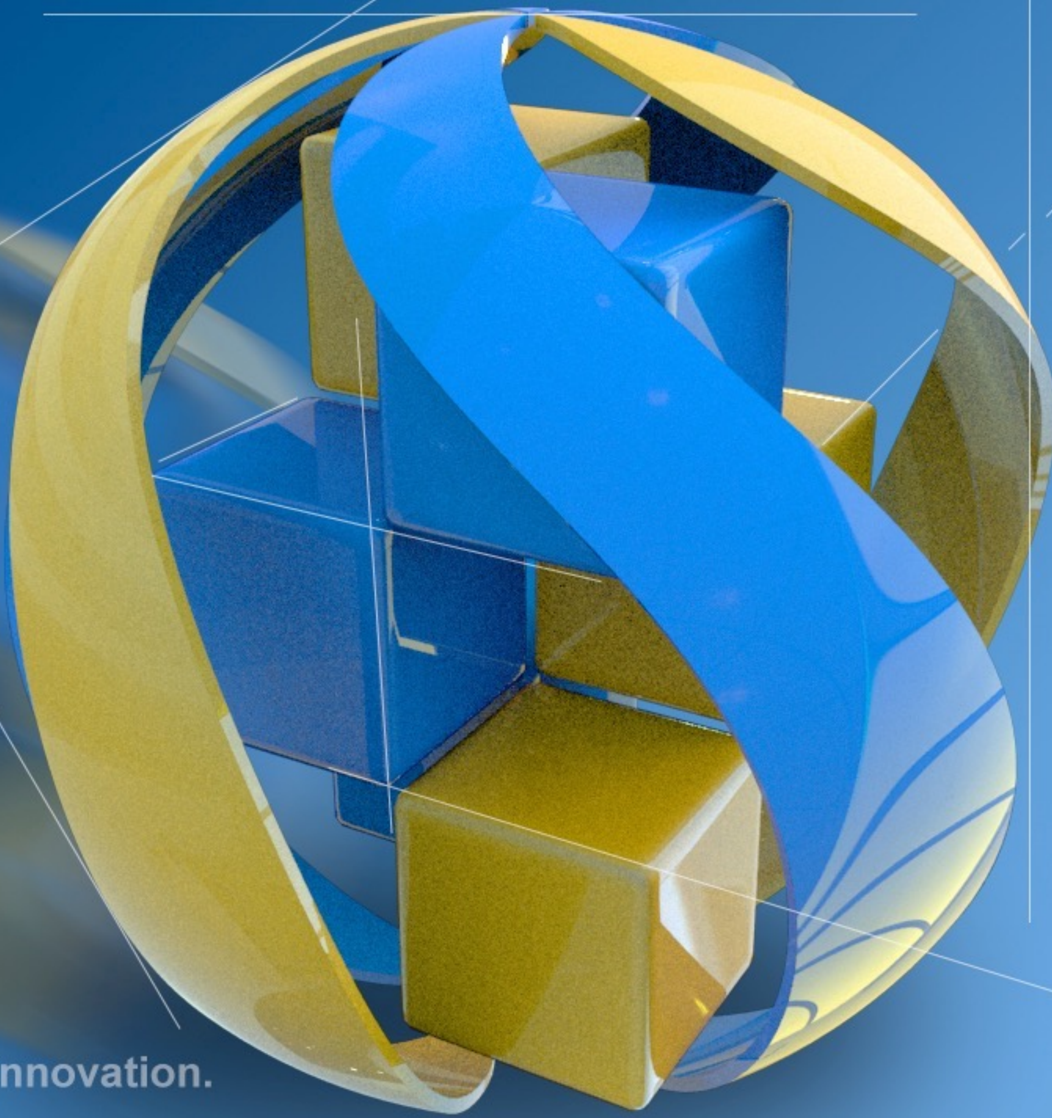

The convergence of research and innovation.

A toolbox for computing spectral properties of dynamical systems

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Outline

- Theory: “periodic approximation”
- Numerical method
- Examples
  - Cat map
  - Chirikov Standard map
- Conclusions

Theory:
“periodic approximation”

The concept: “periodic approximation”

- Approximate the dynamical system by a periodic one.
- Notion originally introduced by Halmos.
- Katok & Stepin:
  - relate the rate of approximation of an automorphism by periodic approximation to the type of spectra the automorphism has

Our goal:

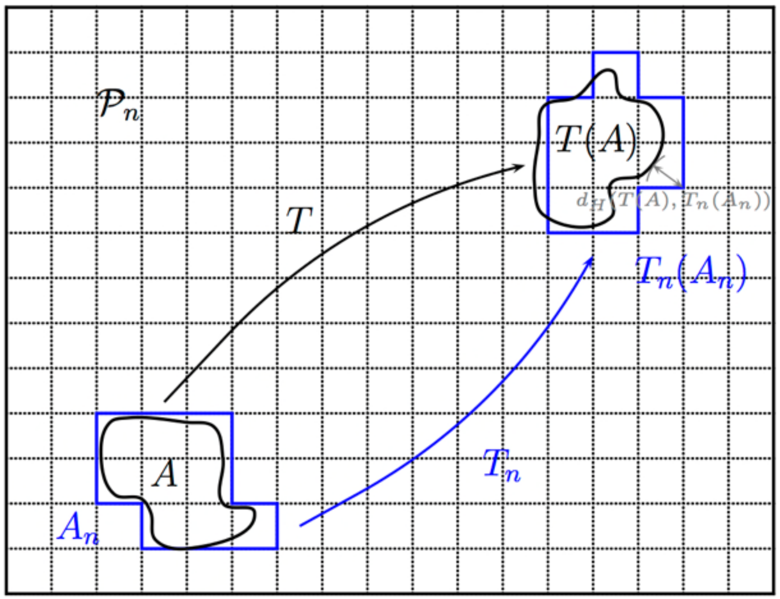
Use this concept to develop a numerical method to approximate the spectral decomposition of the Koopman operator.

Class of dynamical systems:

measure-preserving maps on compact domains.

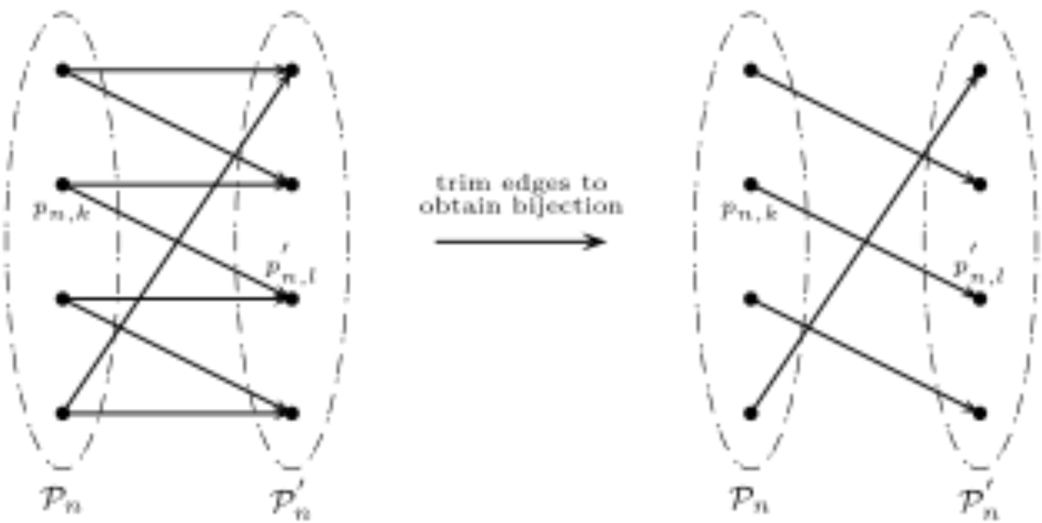
Periodic approximation: Discretization

1. Partition domain into sets
   1. Elements of equal measure
   2. Diameter shrinks to zero
   3. Each partition is a refinement (subdivision) of the previous
2. Solve bipartite matching problem to get discrete map T_n
   1. Solution guaranteed by Hall's marriage problem



$$\mu(p_{n,j}) = \frac{\mu(X)}{q(n)}$$

Periodic approximation: Hall Marriage Problem



$G_n = (\mathcal{P}_n \times \mathcal{P}'_n, E)$ $\qquad$ $\tilde{G}_n = (\mathcal{P}_n \times \mathcal{P}'_n, \tilde{E})$

<u>Hall's Marriage Problem</u>

- Perfect matching iff for any subset of nodes B on the left, its cardinality is less than the cardinality of the union of their neighbors, $N_G(B)$.

$$|B| \leq |N_G(B)|$$

$$\begin{aligned} |N_G(B)| &:= \sum_{k\in B}\left(\sum_{l:\mu(T(p_{n,k})\cap p_{n,l}>0)} 1\right) \\ &\geq \sum_{k\in B}\frac{q(n)}{\mu(X)}\sum_{l=1}^{q(n)}\mu(T(p_{n,k})\cap p_{n,l}) \\ &= \sum_{k\in B} 1 = |B| \end{aligned}$$

$$G_n = (\mathcal{P}_n \times \mathcal{P}'_n, E) \xrightarrow{\text{Bi-partite matching algorithm}} T_n : \mathcal{P}_n \mapsto \mathcal{P}_n$$

Bi-partite matching algorithm

Tn is a permutation => unitary

Discretization overview (1)

continuous case:

$$L^2(X, \mathcal{M}, \mu) := \{g : X \mapsto \mathbb{C} \quad | \quad \|g\| < \infty\}, \qquad \|g\| := \left(\int_X |g(x)|^2 d\mu\right)^{\frac{1}{2}}$$

$$\mathcal{U} : L^2(X, \mathcal{M}, \mu) \mapsto L^2(X, \mathcal{M}, \mu)$$

$$(\mathcal{U}g)(x) := g \circ T(x)$$

$$\mathcal{U}g = \int_{\mathbb{S}} e^{i\theta} \mathrm{d}\mathcal{S}_\theta g$$

$$\mathcal{S}_D g = \int_D d\mathcal{S}_\theta g$$

Discretization overview (2)

discretization of the automorphism:

$$T_n : \mathcal{P}_n \mapsto \mathcal{P}_n$$

T_n is a periodic approximation

discretization of the observable:

$$(\mathcal{W}_n g)(x) = g_n(x) := \sum_{j=1}^{q(n)} g_{n,j} \chi_{p_{n,j}}(x)$$

$$g_{n,j} := \frac{q(n)}{\mu(X)} \int_X g(x) \chi_{p_{n,j}}(x) d\mu$$

Discretization overview (3)

discrete case:

$$L_n^2(X,\mathcal{M},\mu) := \left\{ g_n : X \mapsto \mathbb{C} \quad | \quad \sum_{j=1}^{q(n)} c_j \chi_{p_{n,j}}(x), \quad c_j \in \mathbb{C} \right\}, \qquad \chi_{p_{n,j}}(x) = \begin{cases} 1 & x \in p_{n,j} \\ 0 & x \notin p_{n,j} \end{cases}$$

$$\mathcal{U}_n : L_n^2(X,\mathcal{M},\mu) \mapsto L_n^2(X,\mathcal{M},\mu)$$

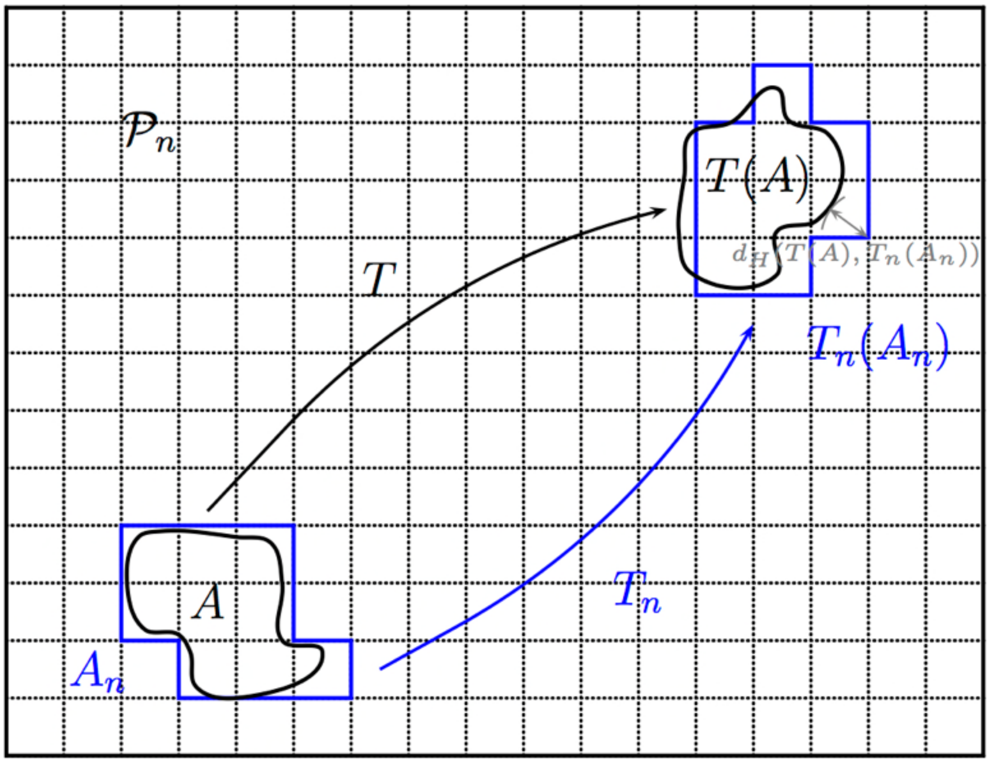
$$\left(\mathcal{U}_n g_n\right)(x) := \sum_{j=1}^{q(n)} g_{n,j} \chi_{T_n^{-1}(p_{n,j})}(x)$$

$$\mathcal{U}_n g_n = \sum_{k=1}^{q(n)} e^{i\theta_{n,k}} \mathcal{S}_{n,\theta_{n,k}} g_n$$

$$\mathcal{S}_{n,D} g_n = \sum_{\theta_{n,k} \in D} \mathcal{S}_{n,\theta_{n,k}} g_n$$

$$\mathcal{U}_n v_{n,k} = e^{i\theta_{n,k}} v_{n,k} \qquad\qquad \mathcal{S}_{n,\theta_{n,k}} g_n = v_{n,k} \left\langle v_{n,k}, g_n \right\rangle$$

Convergence of Periodic approximation



$$\lim_{n\to\infty} \sum_{l=-k}^{k} d_H(T^l(A), T_n^l(A_n)) = 0$$

Spectral convergence results

$$\mathcal{U}^k g = \int_{\mathbb{S}} e^{i\theta k} \mathrm{d}\mathcal{S}_\theta g = \sum_{l=1}^{N} a_l e^{ik\theta_l} \phi_l + \int_{\mathbb{S}} e^{ik\theta} \mathrm{d}\mathcal{S}_\theta^r g, \qquad k \in \mathbb{Z}. \tag{2}$$

(i) For some interval $D \subset \mathbb{S}$, if $g \in L^2(X, \mathcal{M}, \mu)$ has no nonzero modes a_k in (2) corresponding to eigenvalues on boundary $\partial D = \overline{D} \setminus \operatorname{int} D$, then:

$$\lim_{n \to \infty} \|\mathcal{S}_D g - \mathcal{S}_{n,D} g_n\| = 0$$

(ii) Define:

$$\rho_{\alpha,n}(\theta; g_n) := \frac{\alpha}{2\pi} \sum_{j=1}^{\alpha} \left\|\mathcal{S}_{n,D_{\alpha,j}} g_n\right\|^2 \chi_{D_{\alpha,j}}(\theta), \tag{11}$$

where

$$D_{\alpha,j} := [\theta_{\alpha,j-1}, \theta_{\alpha,j}), \quad \theta_{\alpha,j} := -\pi + \frac{2\pi}{\alpha} j.$$

It follows that:

$$\lim_{n,\alpha \to \infty} \int_{\mathbb{S}} \varphi(\theta) \rho_{\alpha,n}(\theta; g_n) \mathrm{d}\theta = \int_{\mathbb{S}} \varphi(\theta) \rho(\theta; g) \mathrm{d}\theta, \qquad \text{for every } \varphi \in \mathcal{D}(\mathbb{S}).$$

Numerical method

Basic outline of numerical method

- Class of dynamical systems:

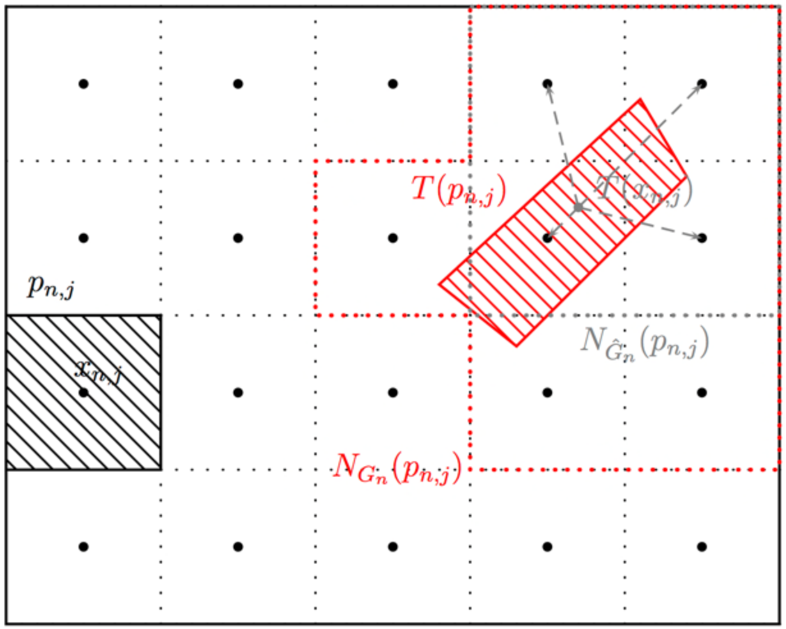
  Volume preserving maps on the d-torus

- Two steps:
  - 1. Construct periodic approximation.
  - 2. Compute spectral projections of the periodic map.

1. Construct periodic approximation

Basic idea:

- Partition the d-torus into boxes and define grid points
- Evaluate map at grid points
- Construct neighborhood graph
- Solve bipartite matching problem
- Solution of bipartite matching is a candidate periodic discrete map.



2. Compute spectral projection

Basic idea:

- Use the Koopman operator of discrete map to approximate spectral projections.
- Discrete Koopman operator has a permutation structure.
- Find cycle decomposition of the permutation.
- Once the cycle decomposition is known, spectral projections can be computed with the FFT algorithm.

Matlab routine:

- Uses David Gleich MatlabBGL graph library to find periodic approximation.
- Code will be made public simultaneously with the papers.

Examples

Cat map

- Arnold's cat map:

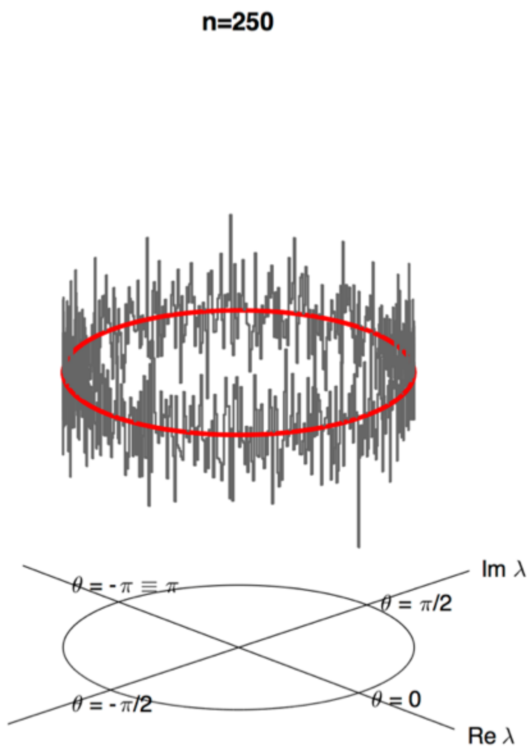
$$T(x_1, x_2) = (2x_1 + x_2, x_1 + x_2) \mod 1$$

- An example of an Anosov diffeomorphism
- Has "*Lebesgue spectrum*"
- The following observables give the following densities:

$$\begin{array}{lll} g_1(x_1,x_2) = e^{2\pi i(2x_1+x_2)} & \Rightarrow & \rho(\theta; g_1) = \frac{1}{2\pi} \\ g_2(x_1,x_2) = e^{2\pi i(2x_1+x_2)} + \frac{1}{2}e^{(2\pi i(5x_1+3x_2))} & \Rightarrow & \rho(\theta; g_2) = \frac{1}{2\pi}\left(\frac{5}{4} + \cos\theta\right) \\ g_3(x_1,x_2) = e^{2\pi i(2x_1+x_2)} + \frac{1}{2}e^{(2\pi i(5x_1+3x_2))} + \frac{1}{4}e^{2\pi i(13x_1+8x_2)} & \Rightarrow & \rho(\theta; g_3) = \frac{1}{2\pi}\left(\frac{21}{16} + \frac{10}{8}\cos\theta + \frac{1}{2}\cos 2\theta\right) \end{array}$$

Cat map: some results

$$g_1(x_1, x_2) = \exp(i2\pi(2x_1 + x_2))$$
$$\rho(\theta; g_1) = \frac{1}{2\pi}$$



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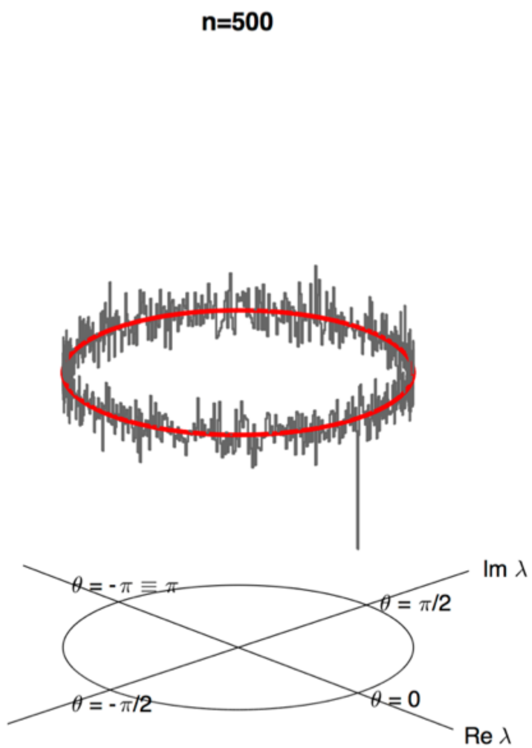

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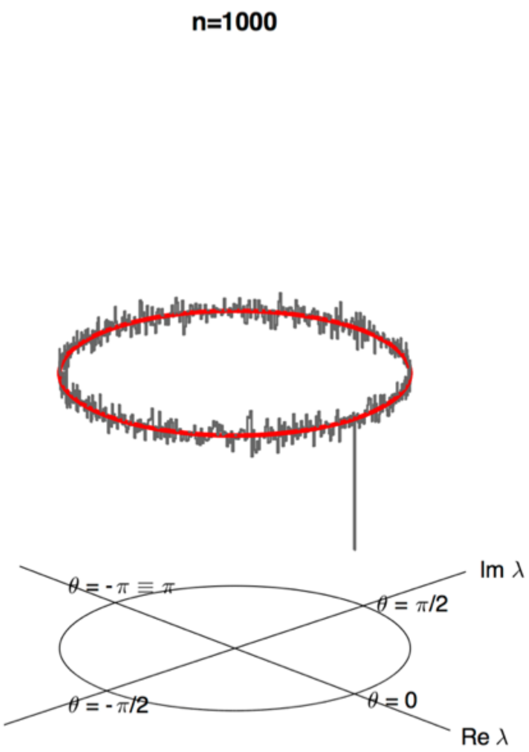

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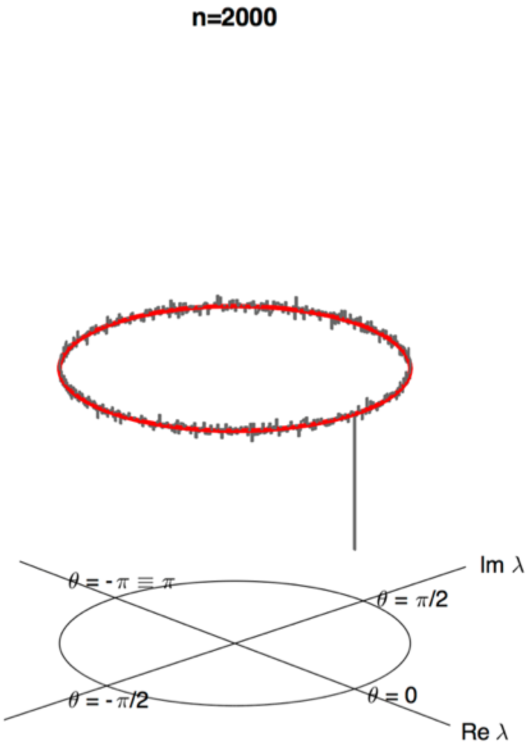
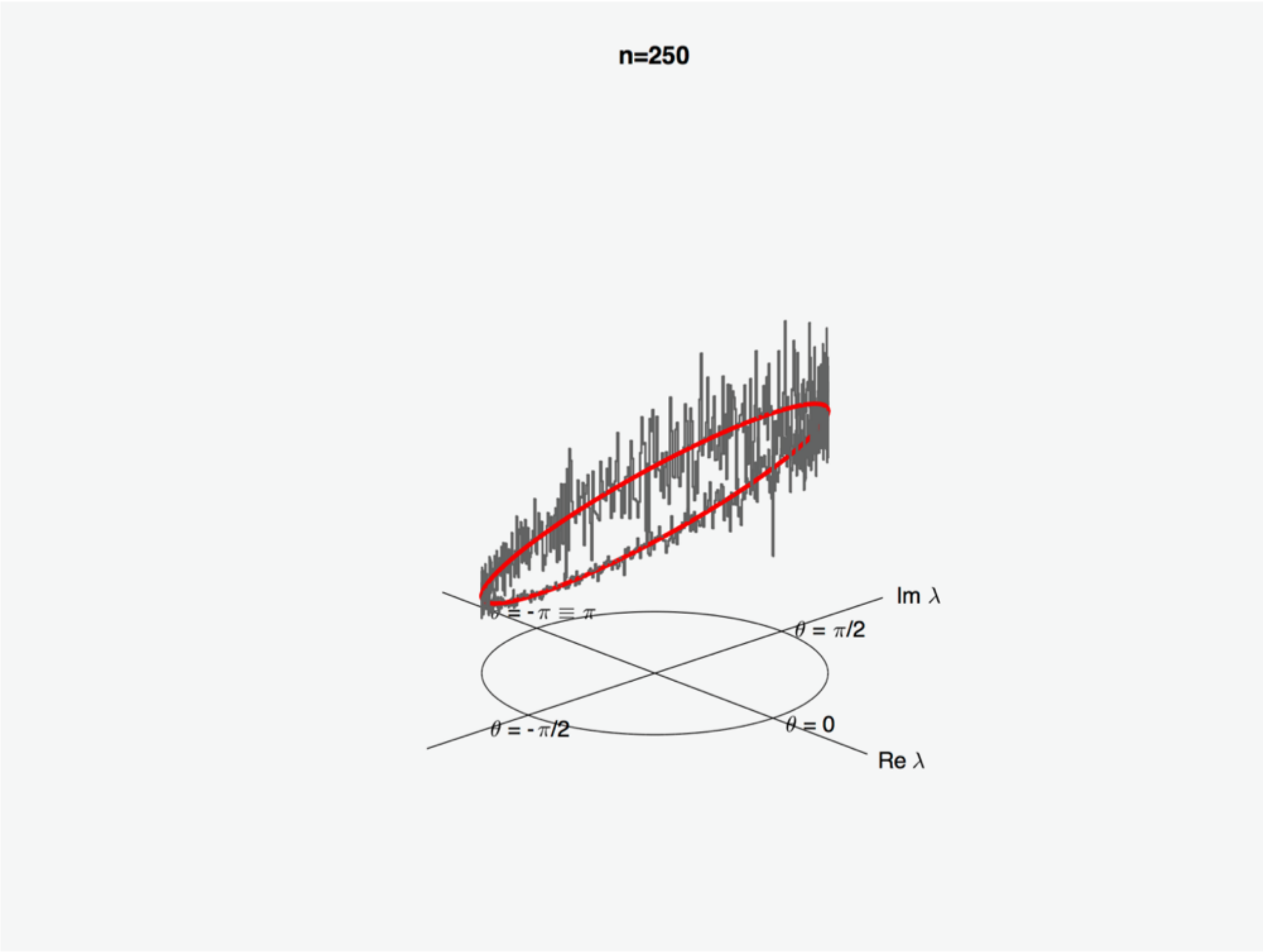


Cat map: some results

$$g_2(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) + \exp(i2\pi(5x_1 + 3x_2))$$

$$\rho(\theta; g_2) = \frac{1}{2\pi}\left(\frac{5}{4} + \cos\theta\right)$$



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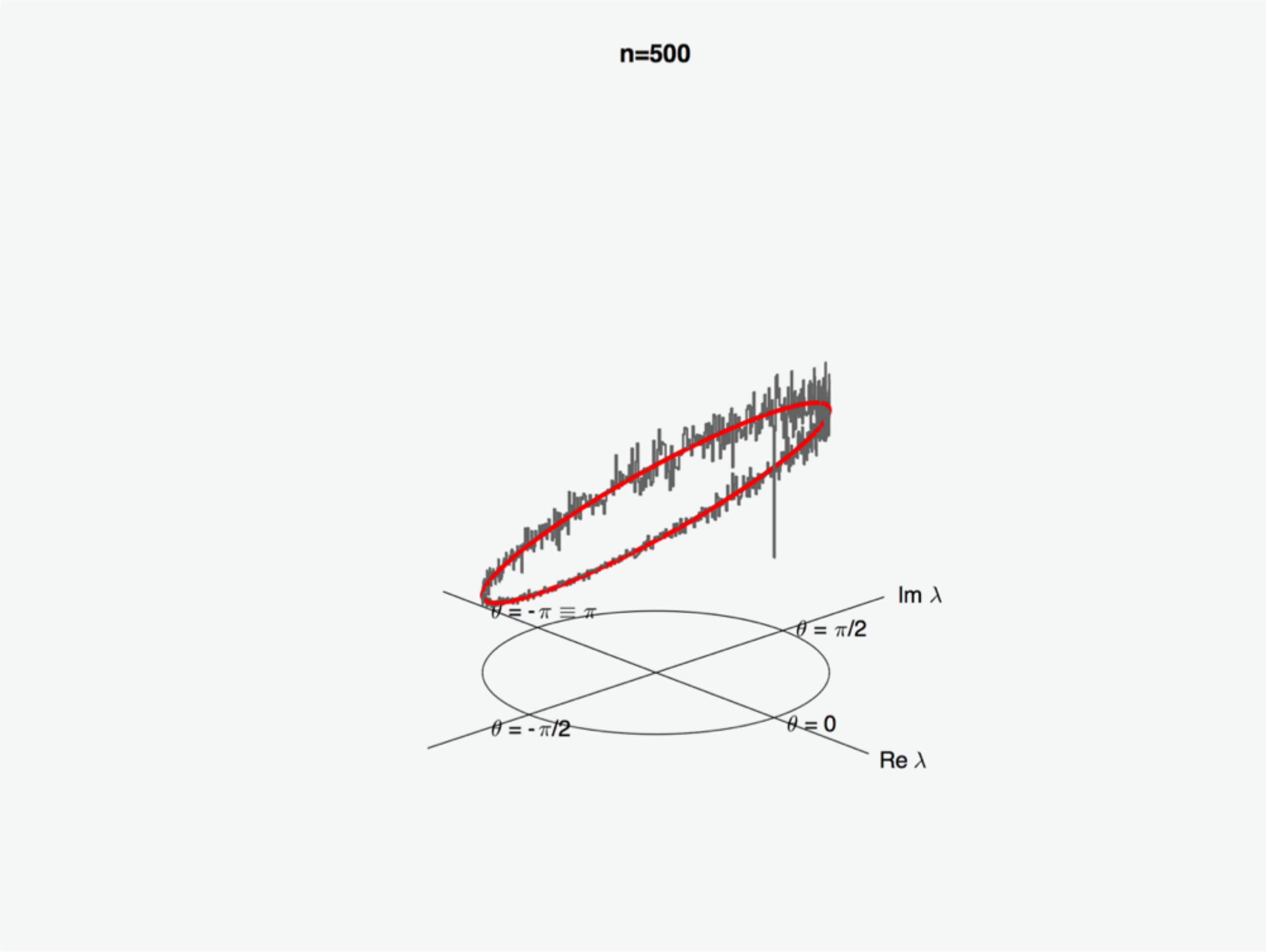

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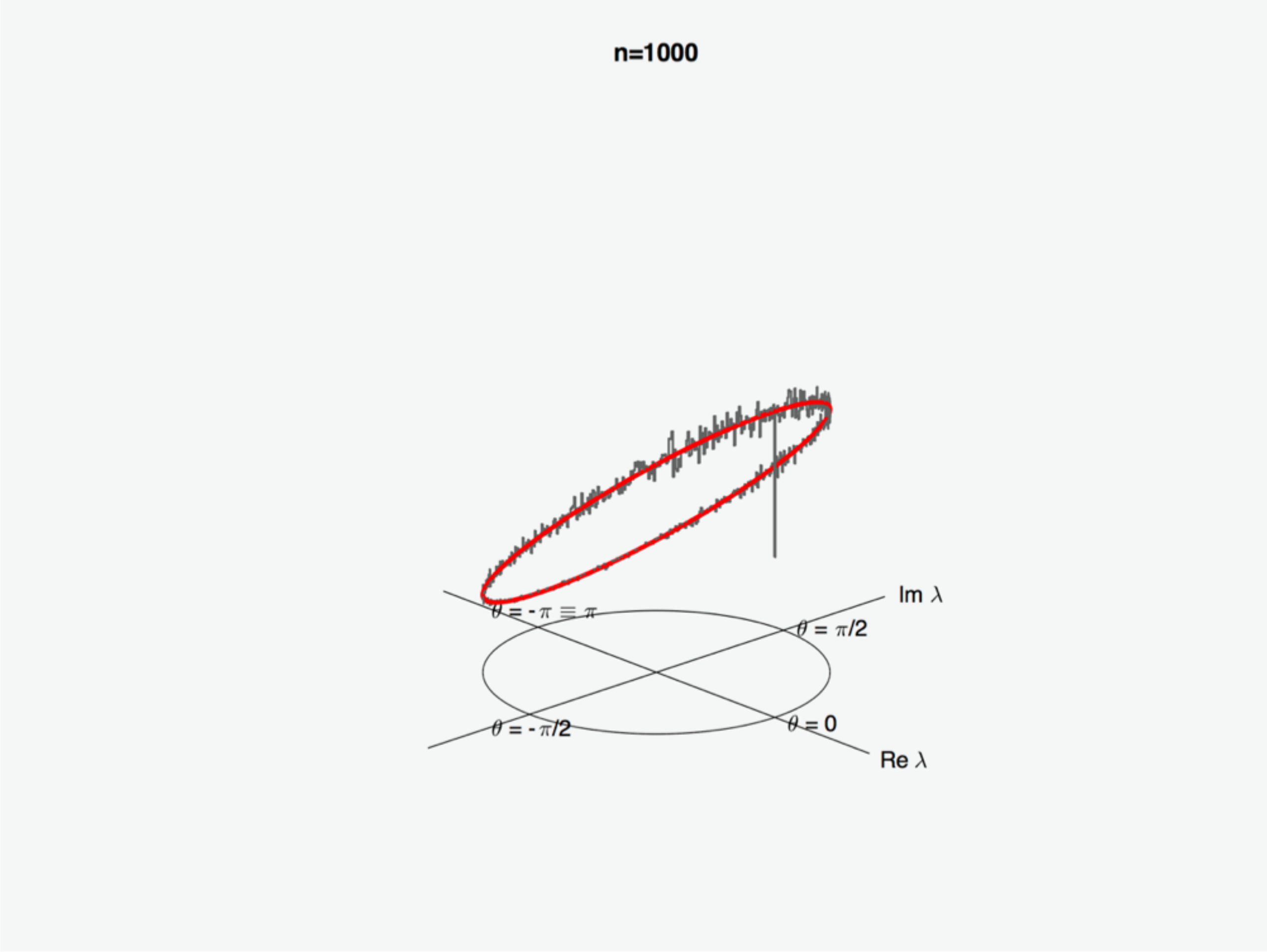

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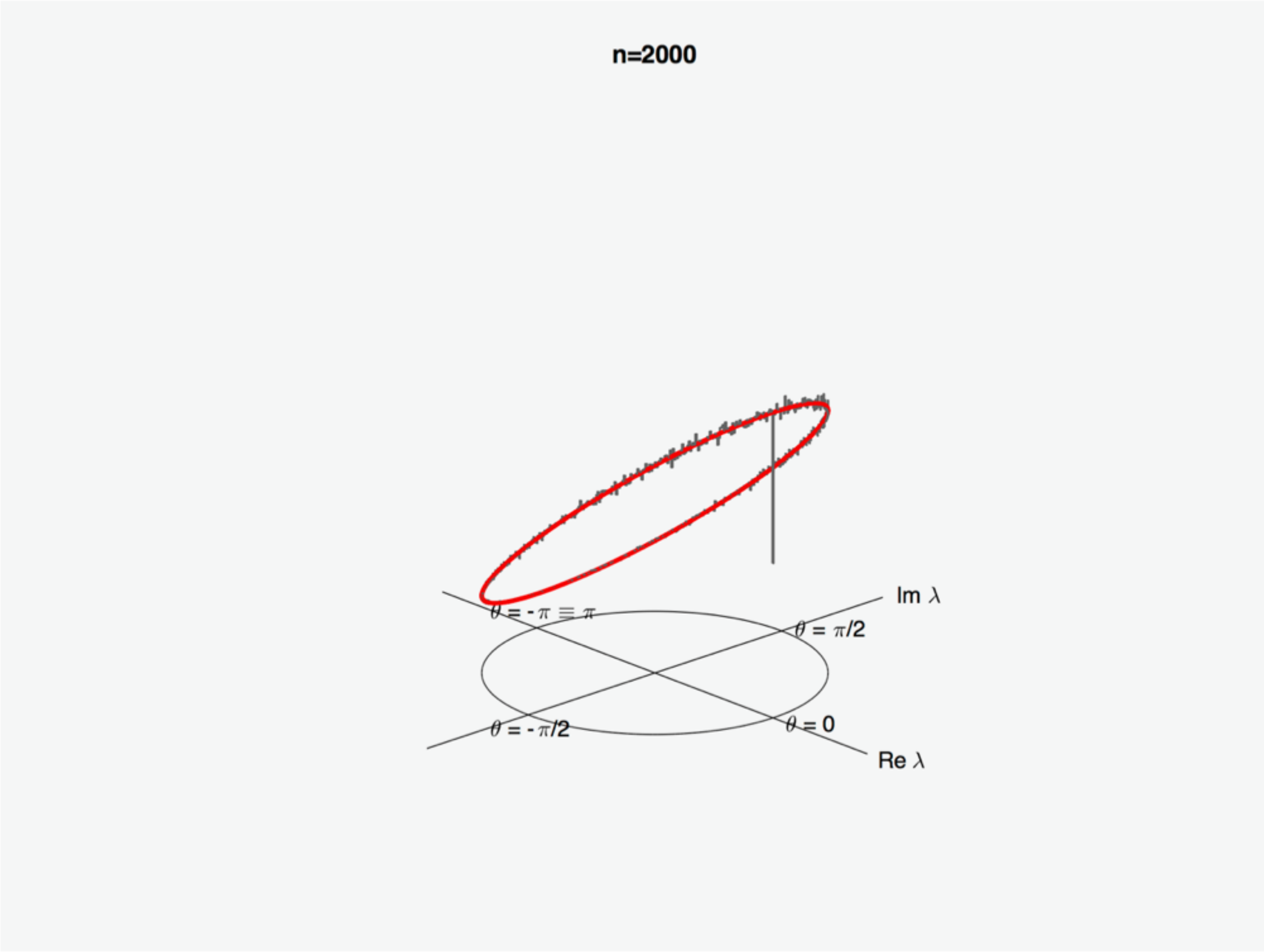
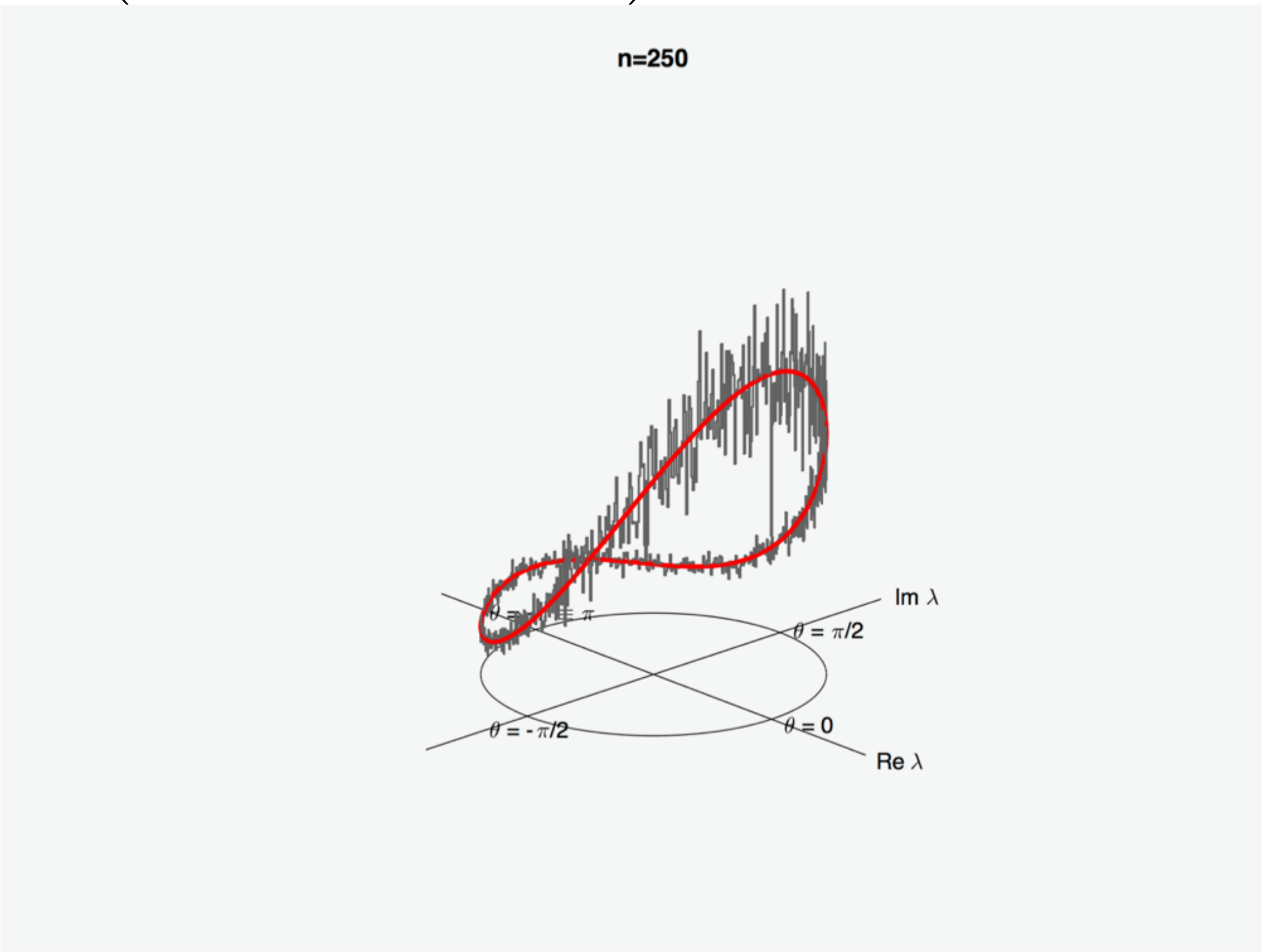


Cat map: some results

$$g_3(x_1, x_2) = \exp(i2\pi(2x_1 + x_2)) + \exp(i2\pi(5x_1 + 3x_2)) + \exp(i2\pi(13x_1 + 8x_2))$$

$$\rho(\theta; g_3) = \frac{1}{2\pi}\left(\frac{21}{16} + \frac{10}{8}\cos\theta + \frac{1}{2}\cos 2\theta\right)$$



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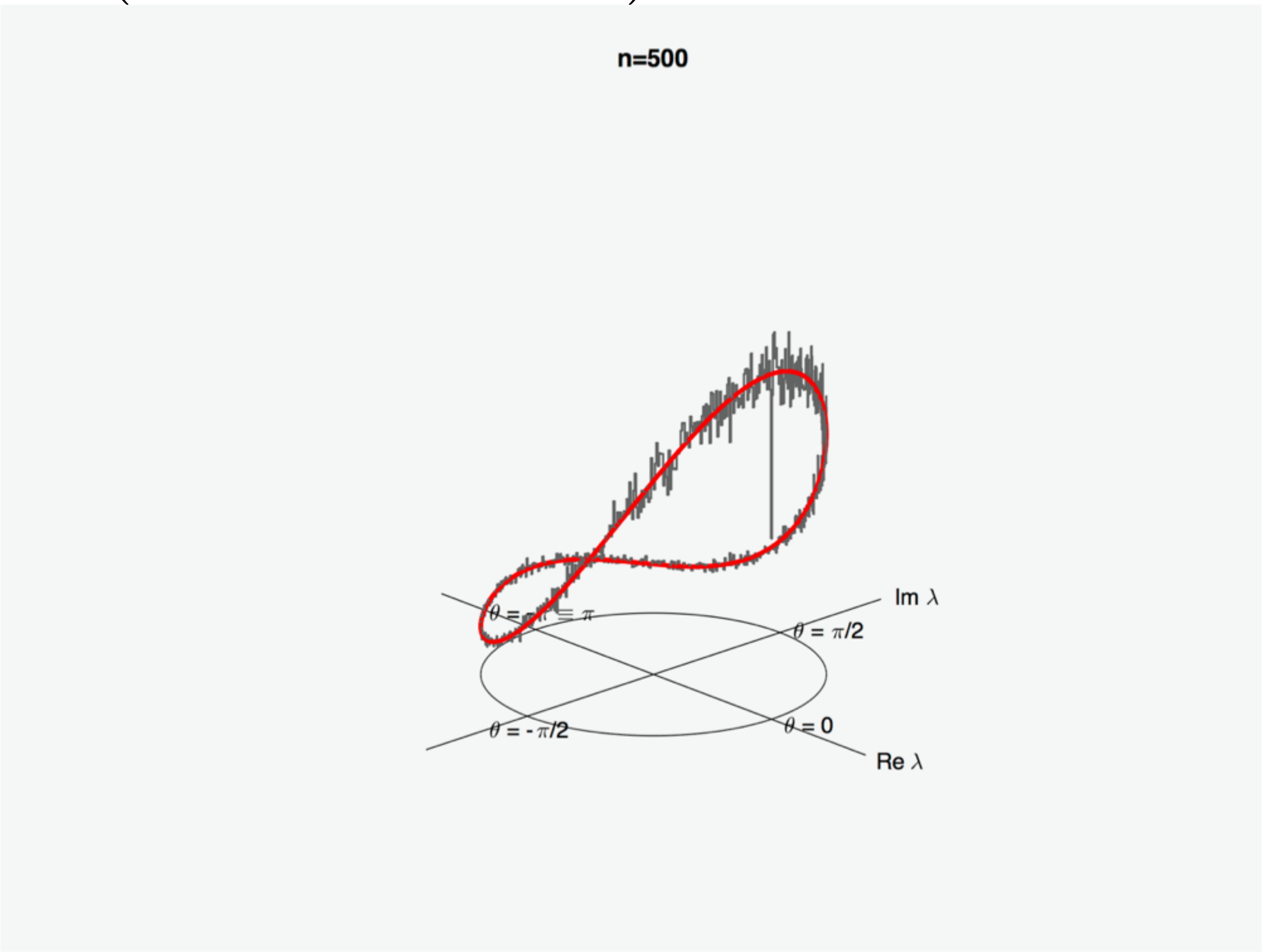

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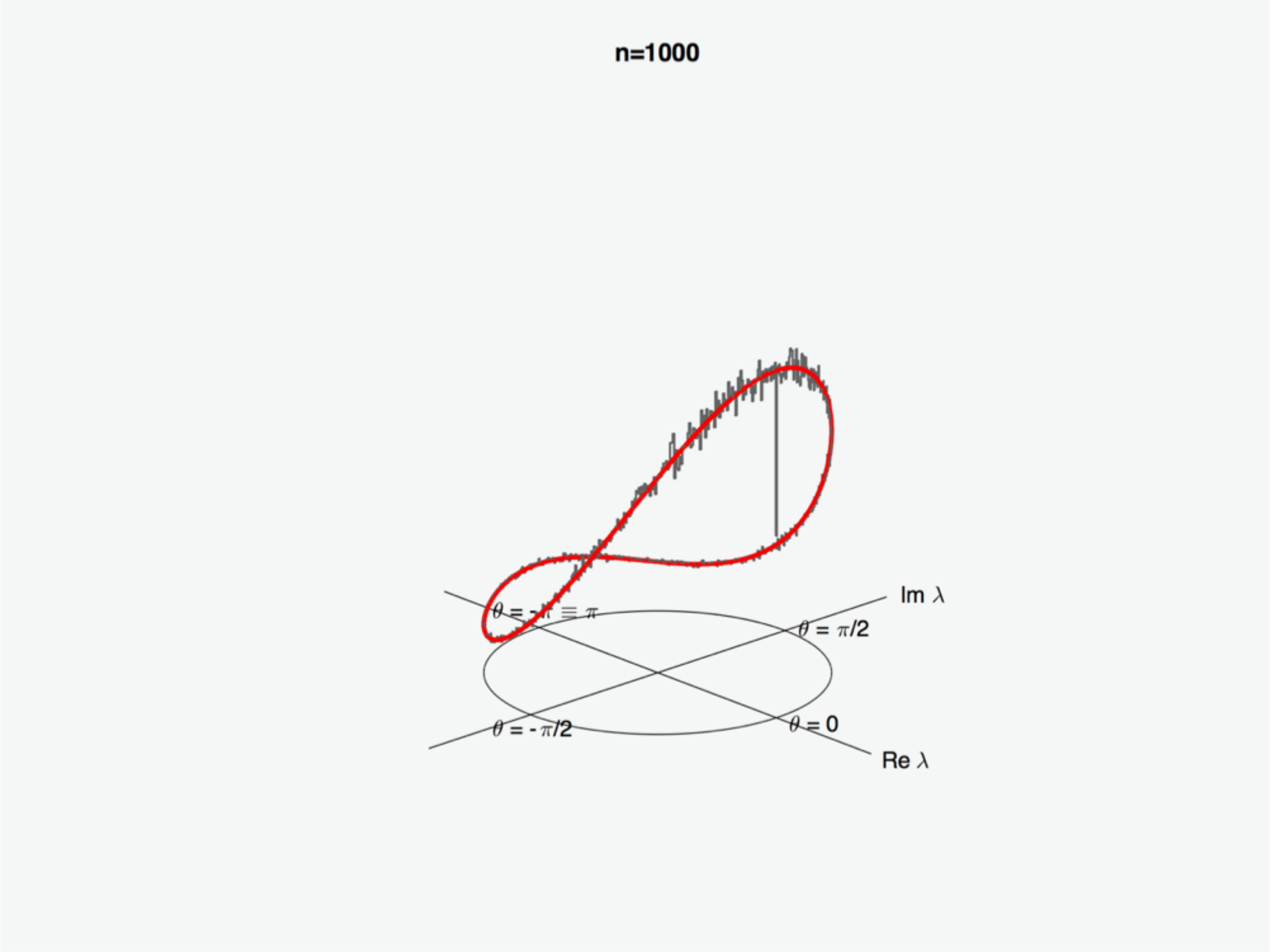

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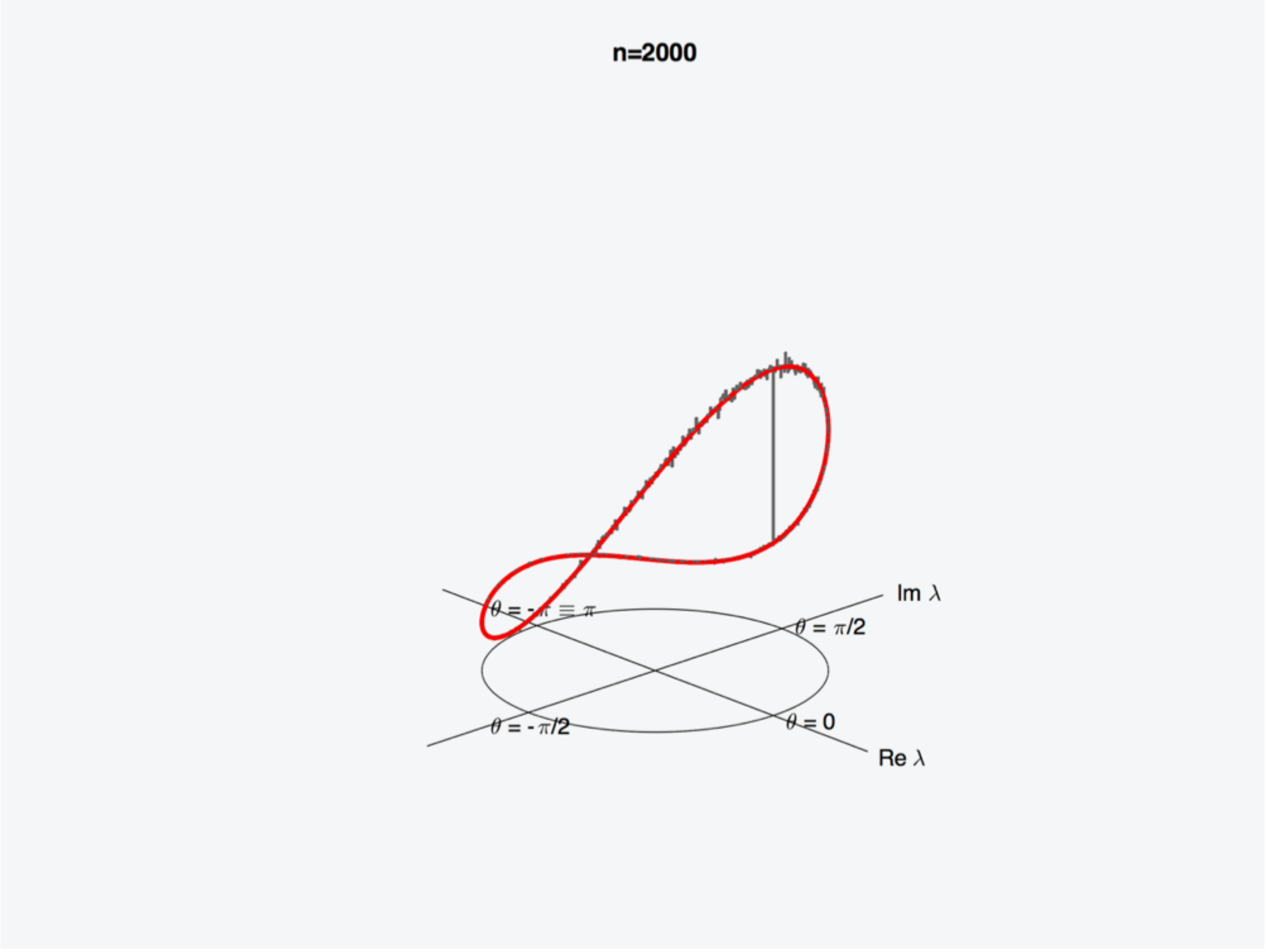


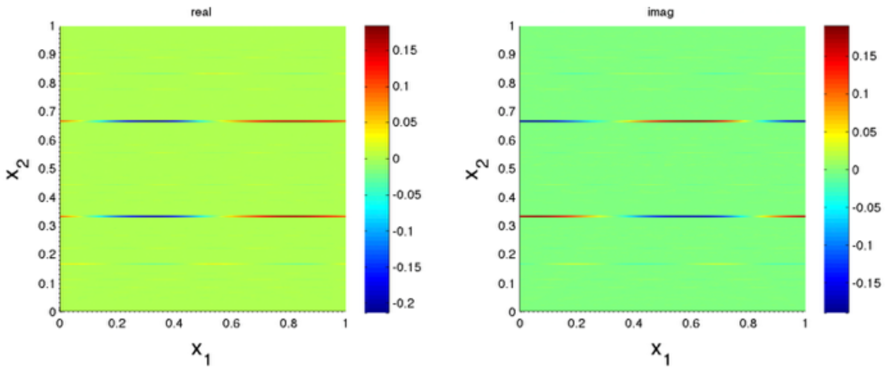
Chirikov Standard map

- Chirikov-Taylor map:

$$T(x_1, x_2) = \begin{bmatrix} x_1 + x_2 + K\sin(2\pi x_1) \\ x_2 + K\sin(2\pi x_1) \end{bmatrix} \mod 1$$

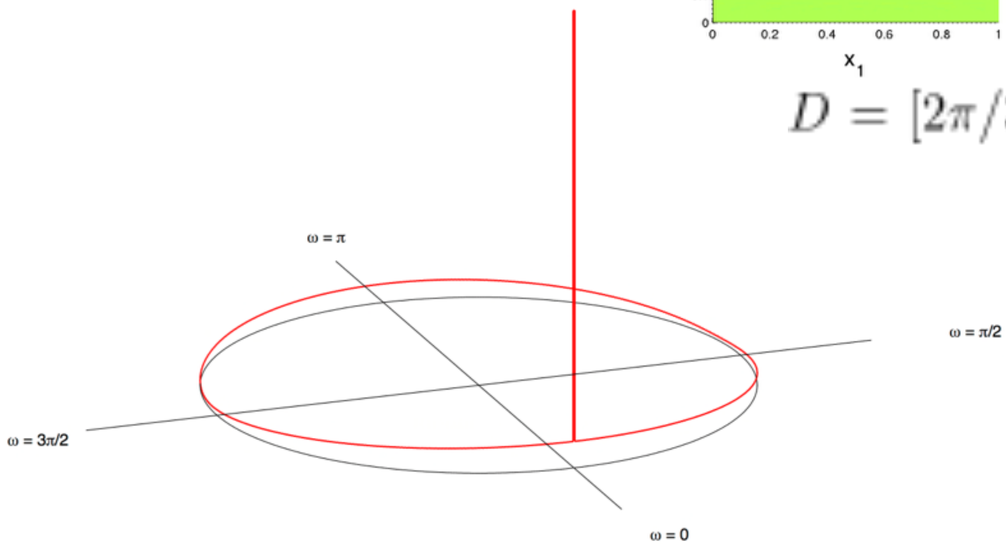
- Model of a kicked rotor.
- Has mixed spectra?

Chirikov Standard map: some results



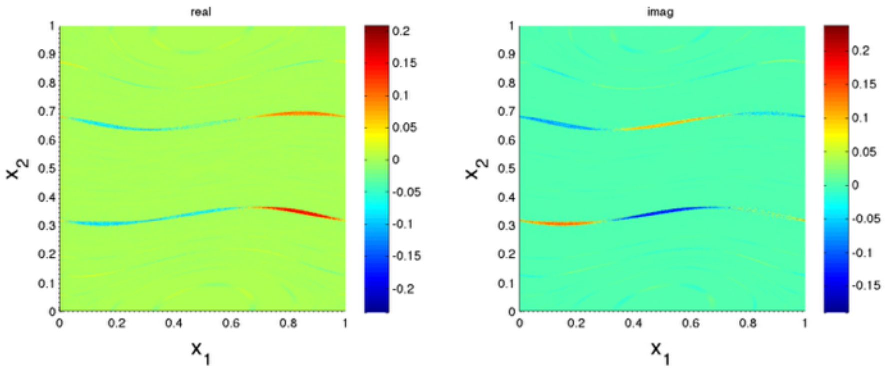
$$D = [2\pi/3 - 0.02, 2\pi/3 + 0.02]$$

K=0.00



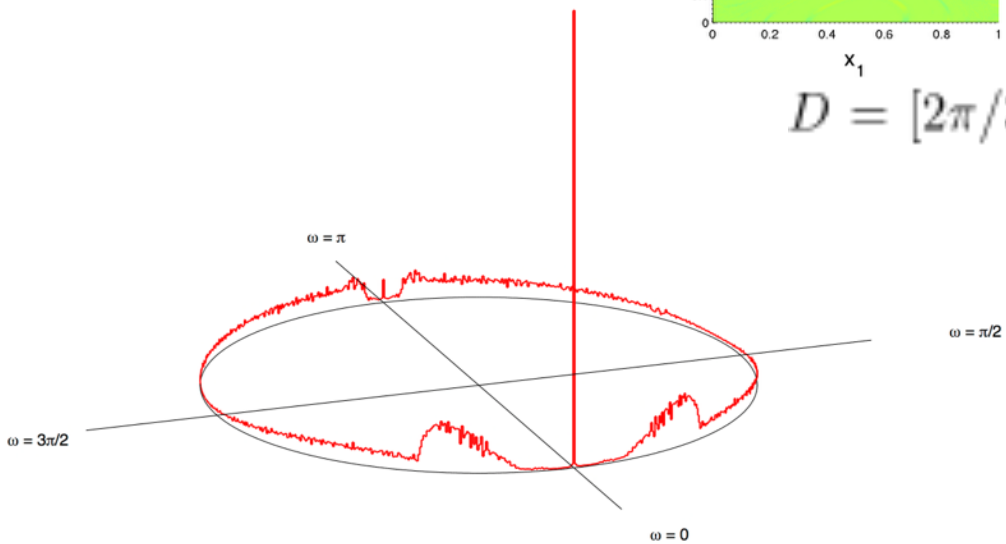
$$g(x) = \sin(2\pi x_1)\cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$

Chirikov Standard map: some results



K=0.05

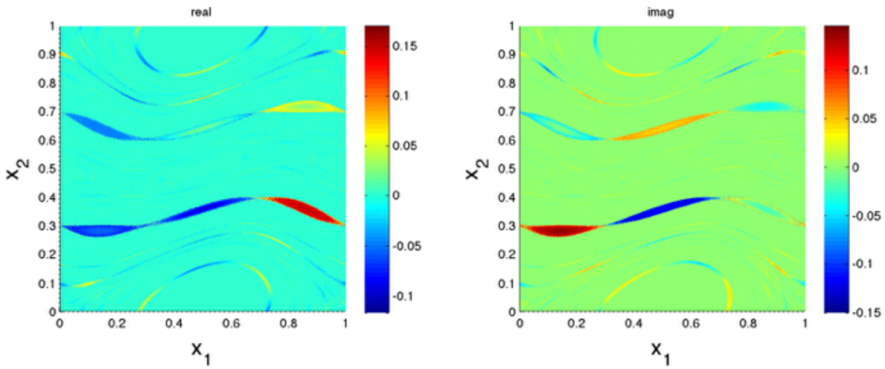
$$D = [2\pi/3 - 0.02, 2\pi/3 + 0.02]$$



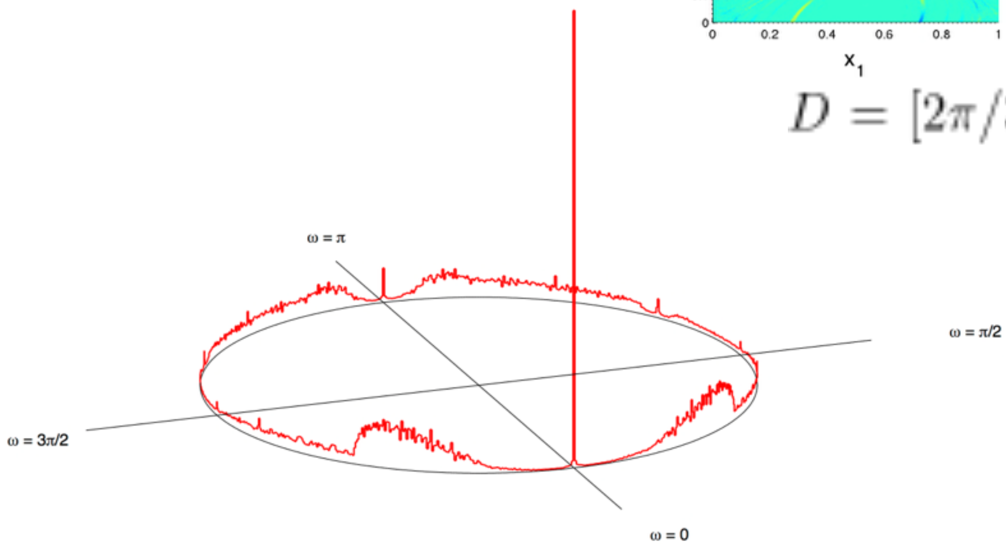
$$g(x) = \sin(2\pi x_1)\cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$

Chirikov Standard map: some results

K=0.10



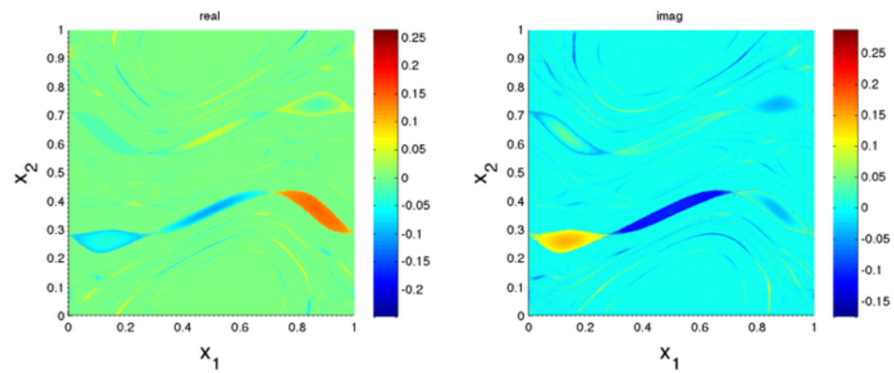
$$D = [2\pi/3 - 0.02, 2\pi/3 + 0.02]$$



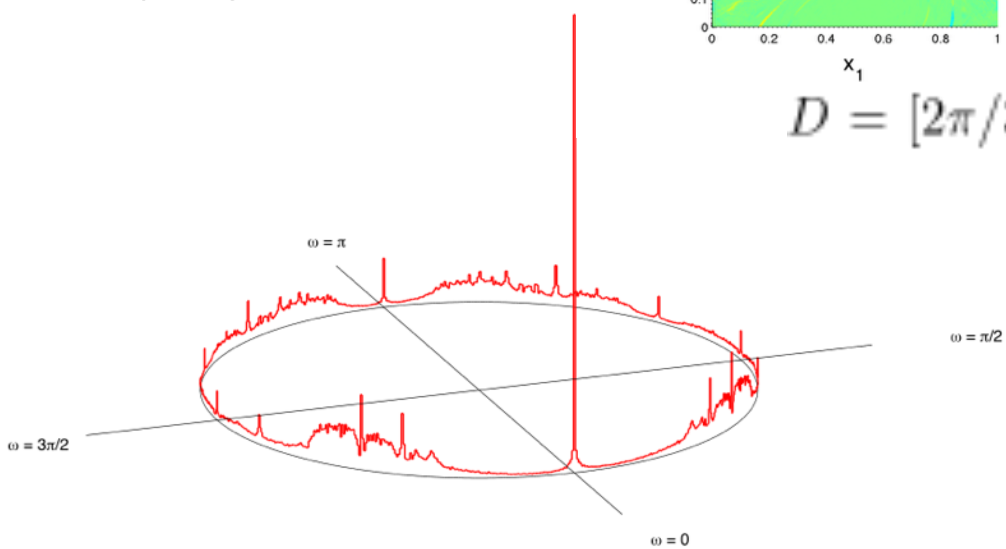
$$g(x) = \sin(2\pi x_1)\cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$

Chirikov Standard map: some results

K=0.15



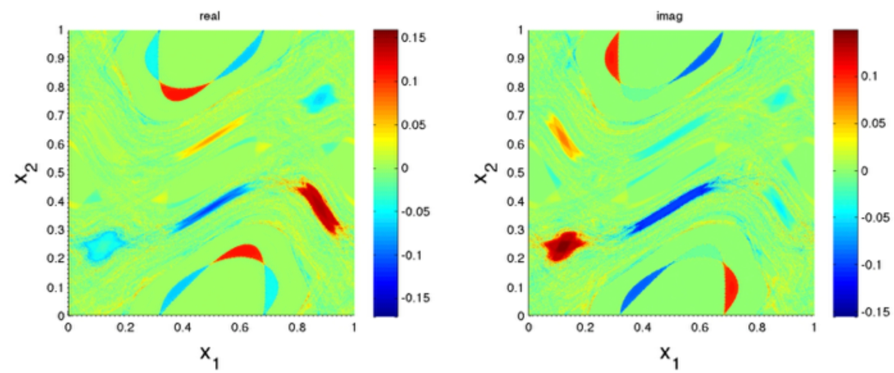
$$D = [2\pi/3 - 0.02, 2\pi/3 + 0.02]$$



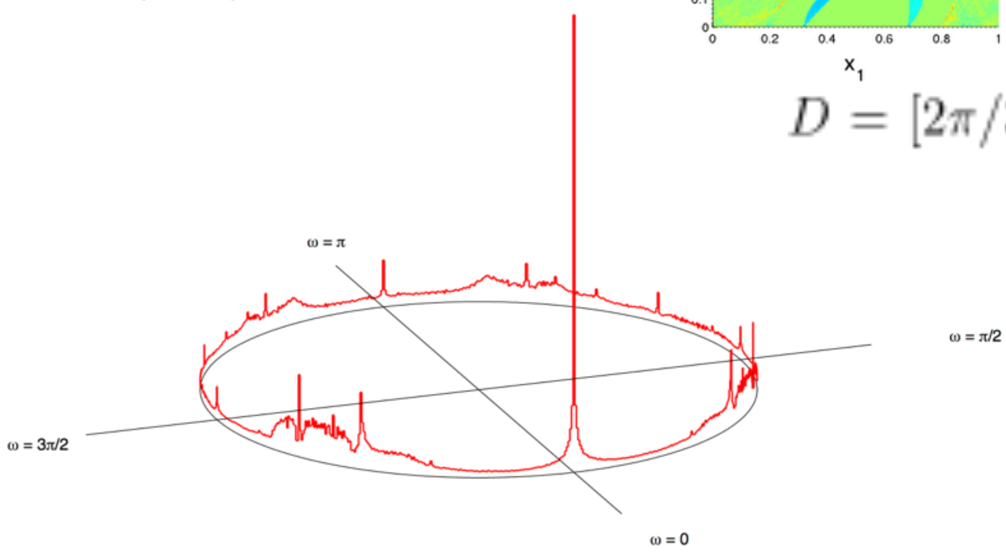
$$g(x) = \sin(2\pi x_1)\cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$

Chirikov Standard map: some results

K=0.20



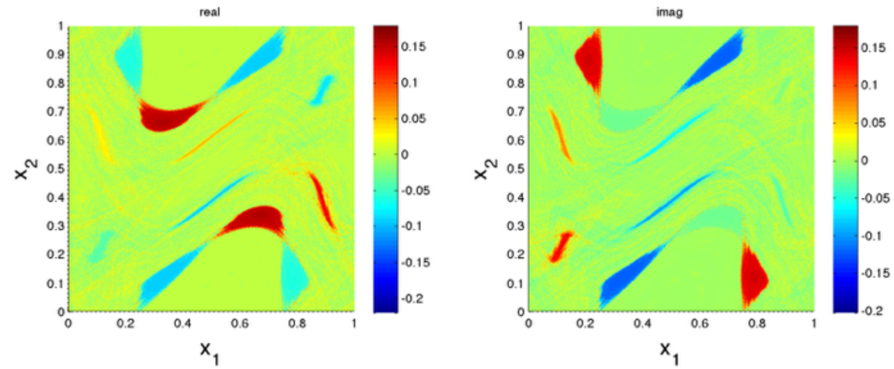
$$D = [2\pi/3 - 0.02, 2\pi/3 + 0.02]$$



$$g(x) = \sin(2\pi x_1)\cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$

Chirikov Standard map: some results

K=0.25



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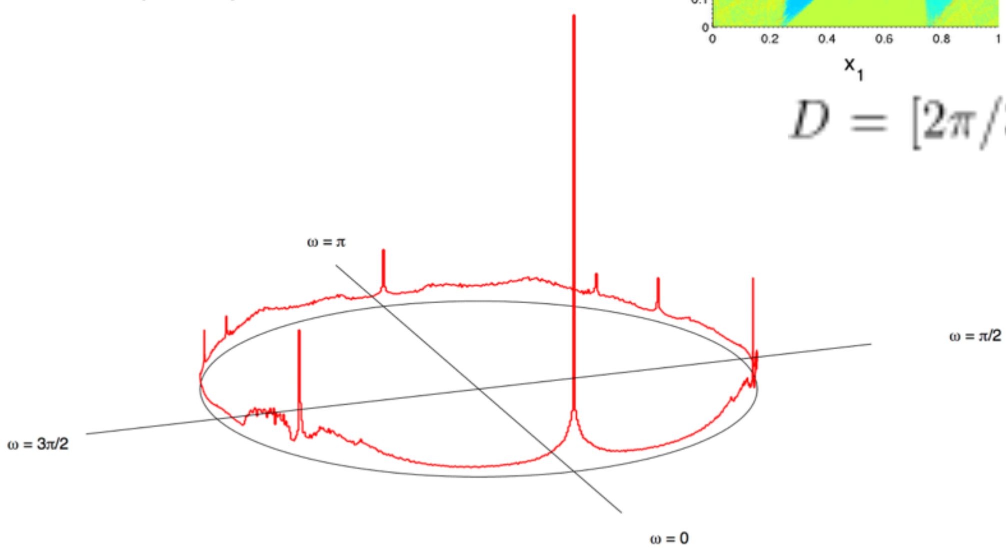
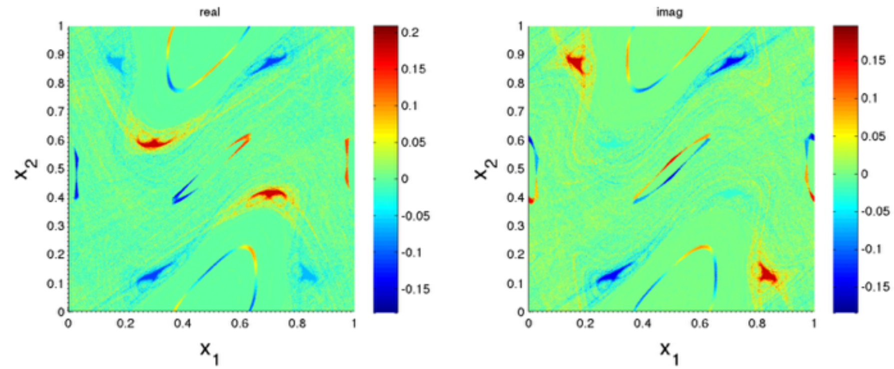


$$g(x) = \sin(2\pi x_1)\cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$

Chirikov Standard map: some results

K=0.30



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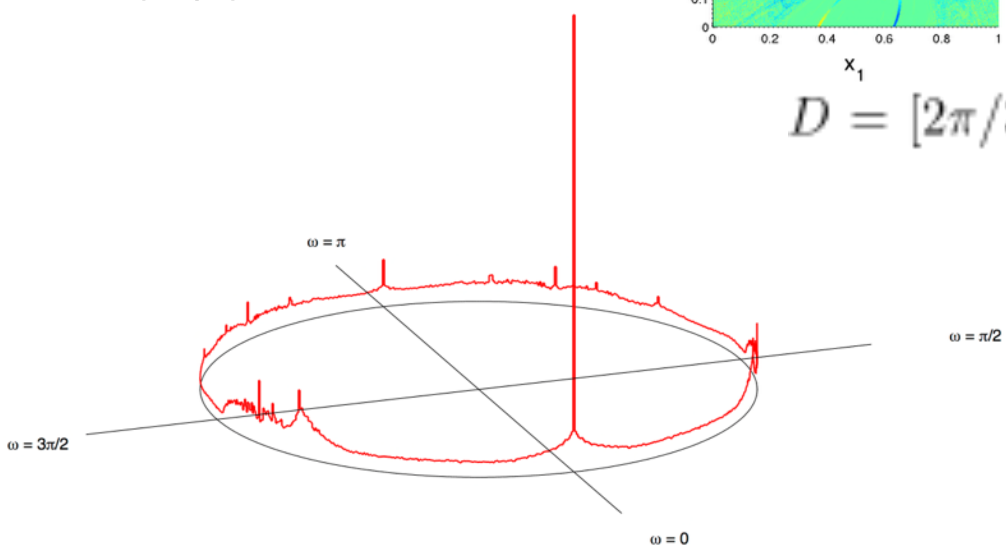

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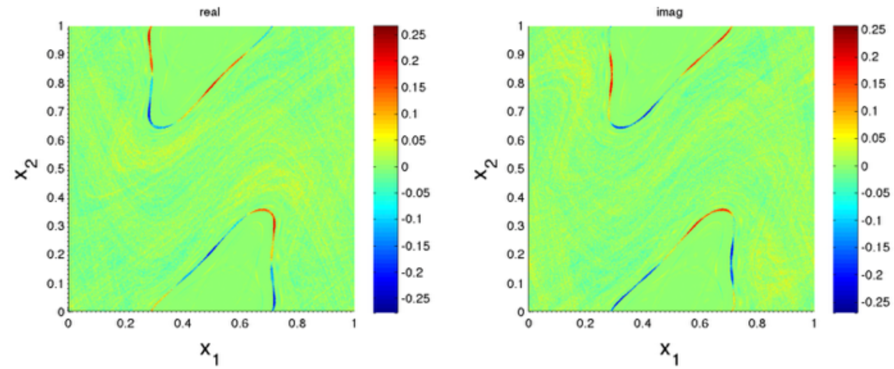
Chirikov Standard map: some results

K=0.35



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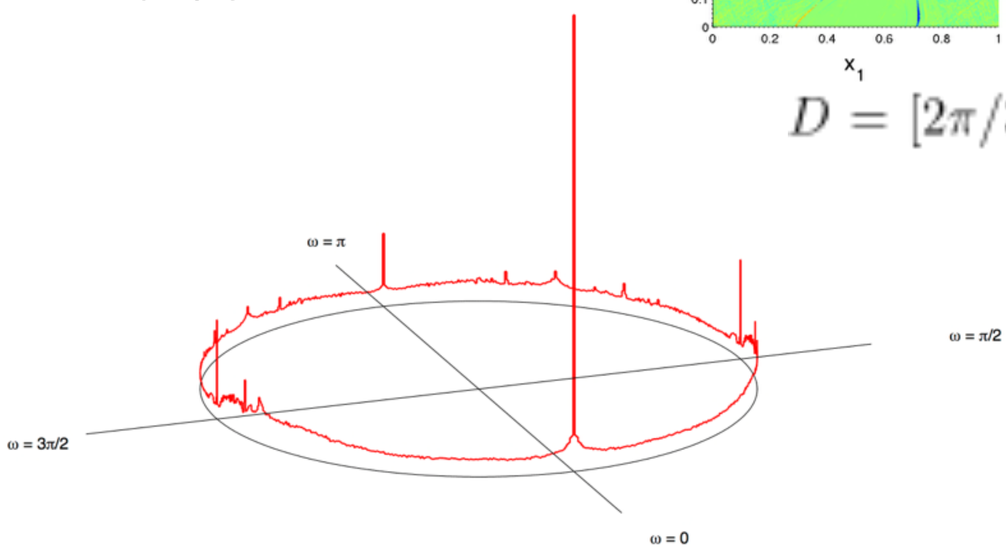

$$g(x) = \sin(2\pi x_1)\cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin(\pi x_1^2) + 1} - 1$$

Conclusions

Conclusions

- Asymptotic convergence of the spectra is guaranteed in a weak-sense.
- Method can deal with continuous spectra.
- Method is only tractable for low dimensional maps.

Associated papers (in preparation):

- **Theory:** *"A finite dimensional approximation of the Koopman operator with convergent spectral properties"* N. Govindarajan, R Mohr, S. Chandrasekaran, I. Mezić.
- **Numerical method:** *"A convergent numerical method for computing Koopman spectra of volume-preserving maps on the torus"* N. Govindarajan, R. Mohr, S. Chandrasekaran, I. Mezić.
- **Generalization to flows:** *"On the approximation of Koopman spectral properties of measure-preserving flows "* N. Govindarajan, R. Mohr, S. Chandrasekaran, I. Mezić.