Mechanical Metallurgy

INME 6016

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GENERAL INFORMATION

Course Number	INME 6016
Course Title	Mechanical Metallurgy
Credit Hours	3 (Lecture: 3hours)
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Assessment

The course will be assessed in the following manner:

❑ Partial Exam	30%
❑ Final Exam	30%
❑ Quizzes (3)*	33%
❑ Attendance and Class Participation	7%

(*) A total of three quizzes will be performed.

(**) Class Attendance (after the second absence - 1 point will be deducted for each non-authorized absence). The participation in class will be taken into account.

Attendance

Attendance and participation in the lectures **are mandatory** and will be considered in the grading. ***Students should bring calculators, rulers, pen and pencils to be used during the lectures***. Students are expected to keep up with the assigned reading and be prepared for the pop-quizzes or to answer questions on these readings during lecture.

Texbooks

G.E. Dieter; Mechanical Metallurgy; Mc Graw Hill

M.A. Meyers and K.K. Chawla; Mechanical Metallurgy: Principles and Applications; Prentice-Hall

I will also post my lecture notes in the web: **http://academic.uprm.edu/pcaceres**

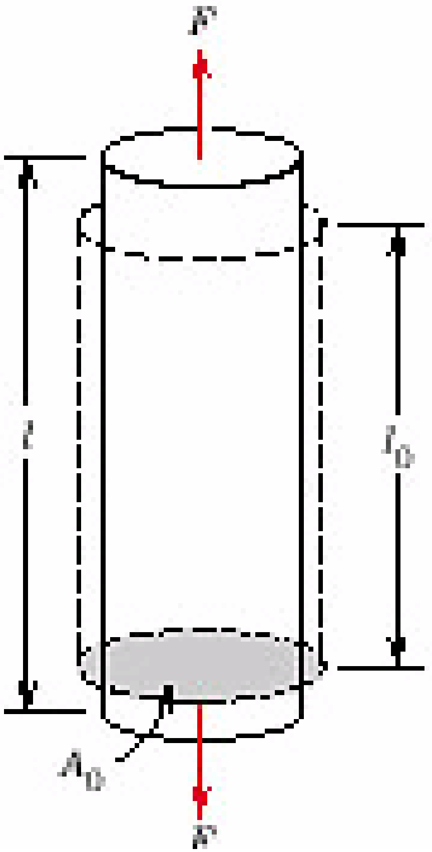
TENTATIVES DATES

Jan/9-11 **Basic Principles**	Jan/14-18 **Stress-Strain**	Jan/21-25 **Basic Elasticity**
Jan/28-Feb/01 **Basic Elasticity**	Feb/04-08 **Single Crystals**	Feb/11-15 **Dislocation Theory**
Feb/18-22 **Dislocation Theory 1st Exam**	Feb/25-29 **Strengthening Mechanisms**	March/3-7 **Strengthening Mechanisms**
March/10-14 **Strengthening Mechanisms**	Mar/17-21– No Class - **Holy Week**	Mar/24-28 **Fracture**
April/1-4 **Fracture**	**April/7-11 Mechanical Properties Exam 2**	April /14-18 **Mechanical Properties**
April/22-25 – **Mechanical Properties**	Apr/28-May02- **Mechanical Properties**	

Content

- Stress and Strain Relationships for Elastic Behavior
- Elements of the Theory of Elasticity
- Plastic Deformation of Single Crystals
- Dislocation Theory
- Strengthening Mechanisms
- Fracture
- Mechanical Properties

The Concept of Stress



Uniaxial tensile stress: A force F is applied perpendicular to the area (A). Before the application of the force, the cross section area was A_O

Engineering stress or nominal stress: Force divided by the original area.

$$\sigma = \frac{F}{A_0}$$

True stress: Force divided by the instantaneous area

$$\sigma_T = \frac{F}{A}$$

Engineering Strain or Nominal Strain: Change of length divided by the original length

$$\varepsilon = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}$$

True Strain: The rate of instantaneous increase in the instantaneous gauge length.

$$\varepsilon_T = \int \frac{d\ell}{\ell} = \ln\left(\frac{\ell_i}{\ell_o}\right)$$

Relationship between engineering and true stress and strain

We will assume that the volume remains constant.

$$A_o \ell_o = A_i \ell_i$$

$$\sigma_T = \frac{F}{A_i} = \frac{F}{A_i} * \frac{A_o}{A_o} = \frac{F}{A_o} * \frac{A_o}{A_i}$$

$$\frac{A_o}{A_i} = \frac{\ell_i}{\ell_o} = \frac{\Delta l + \ell_o}{\ell_o} = \frac{\Delta l}{\ell_o} + 1 = (1+\varepsilon)$$

$$\sigma_T = \frac{F}{A_o}(1+\varepsilon)$$

$$\varepsilon_T = \int \frac{d\ell}{\ell} = \ln\left(\frac{\ell_i}{\ell_o}\right)$$

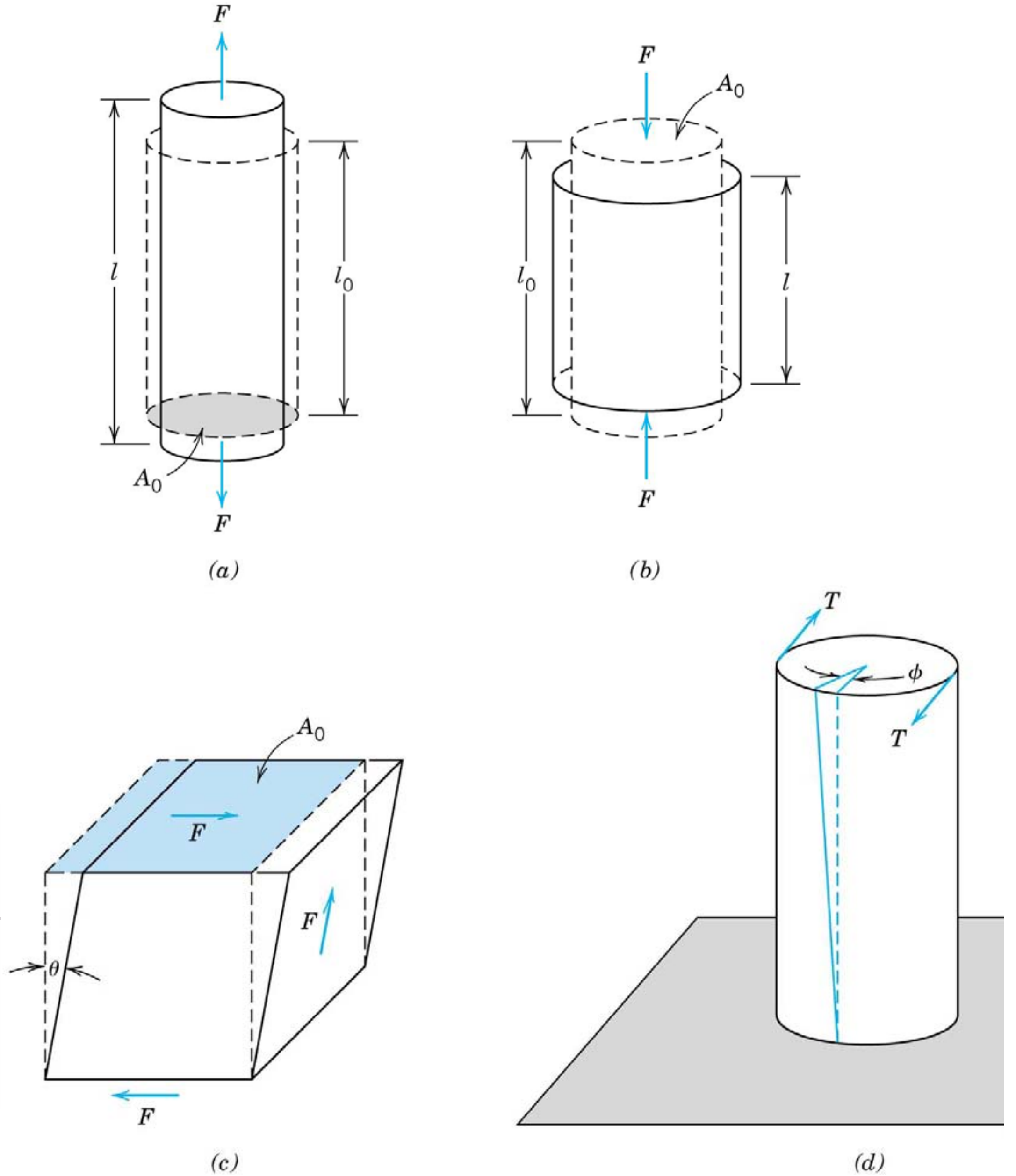
$$\boxed{\sigma_T = \sigma(1+\varepsilon)}$$

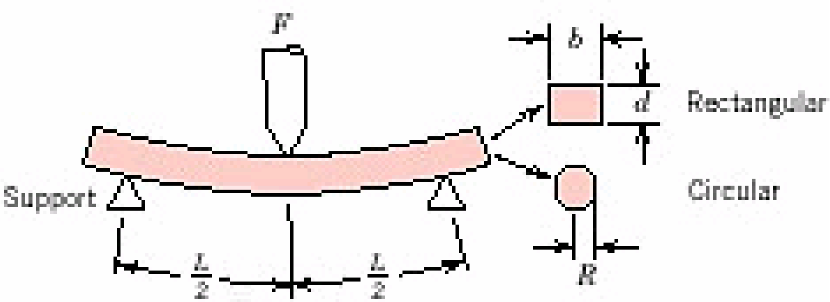
$$\varepsilon_T = \ln\left(\frac{\ell_o + \Delta\ell}{\ell_o}\right) \Rightarrow \ln\left(\frac{\ell_o}{\ell_o} + \frac{\Delta\ell}{\ell_o}\right)$$

$$\boxed{\varepsilon_T = \ln(1+\varepsilon)}$$

Primary Types of Loading

(a) Tension
(b) Compression
(c) Shear
(d) Torsion
(e) Flexion





Hooke's Law

When strains are small, most of materials are linear elastic.

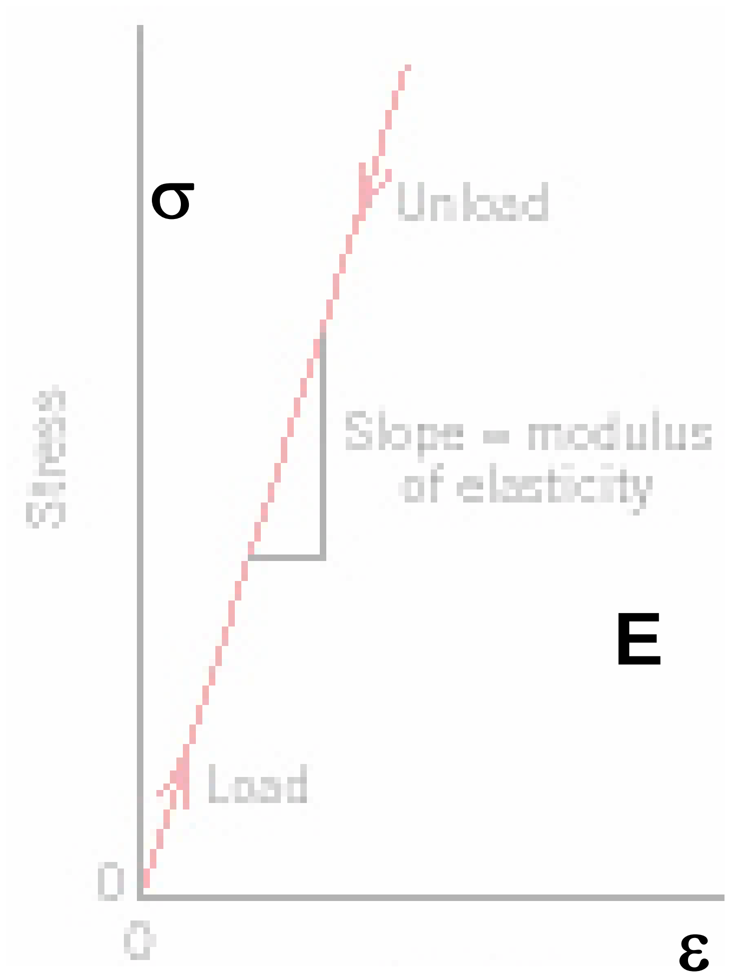
Young's modulus

Normal: $\sigma = E\,\varepsilon$

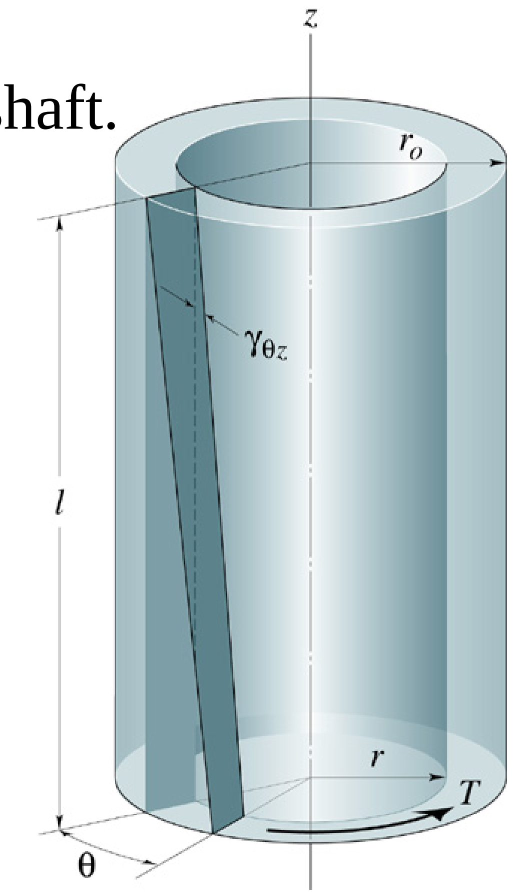
$$\Delta l = \frac{F \cdot l_o}{A_o \cdot E}$$

Springs: the spring rate

$$k = \frac{F}{\Delta l} = \frac{A_o \cdot E}{l_o}$$



Torsion Loading resulting from the twist of a shaft.



$$\gamma_{\theta,z} = r \cdot \frac{\delta\theta}{\delta z} \cong \frac{r \cdot \theta}{l}$$

Shear strain

$$\tau_{\theta,z} = G \cdot \gamma_{\theta,z} = G \cdot r \cdot \frac{\delta\theta}{\delta z} \cong \frac{G \cdot r \cdot \theta}{l}$$

G = Shear Modulus of Elasticity

Twist Moment or Torque

$$T = \int_A r \cdot \tau_{\theta,z} \cdot \delta A = \frac{G \cdot \theta}{l} \int_A r_{\theta,z}^2 \cdot \delta A$$

Area Polar Moment of Inertia $J = \int_A r^2 \cdot \delta A$

$$T = \frac{G \cdot \theta \cdot J}{l} \quad \text{or} \quad \theta = \frac{T \cdot l}{G \cdot J}$$

Thus: $\tau_{\theta,z} \cong \dfrac{T \cdot r}{J}$ $\quad \tau_{Max} \cong \dfrac{T \cdot r_o}{J}$

Angular spring rate:

$$k_a = \frac{T}{\theta} = \frac{J \cdot G}{l}$$

Stress Components

Normal Stresses $\sigma_x, \sigma_y, \sigma_z$

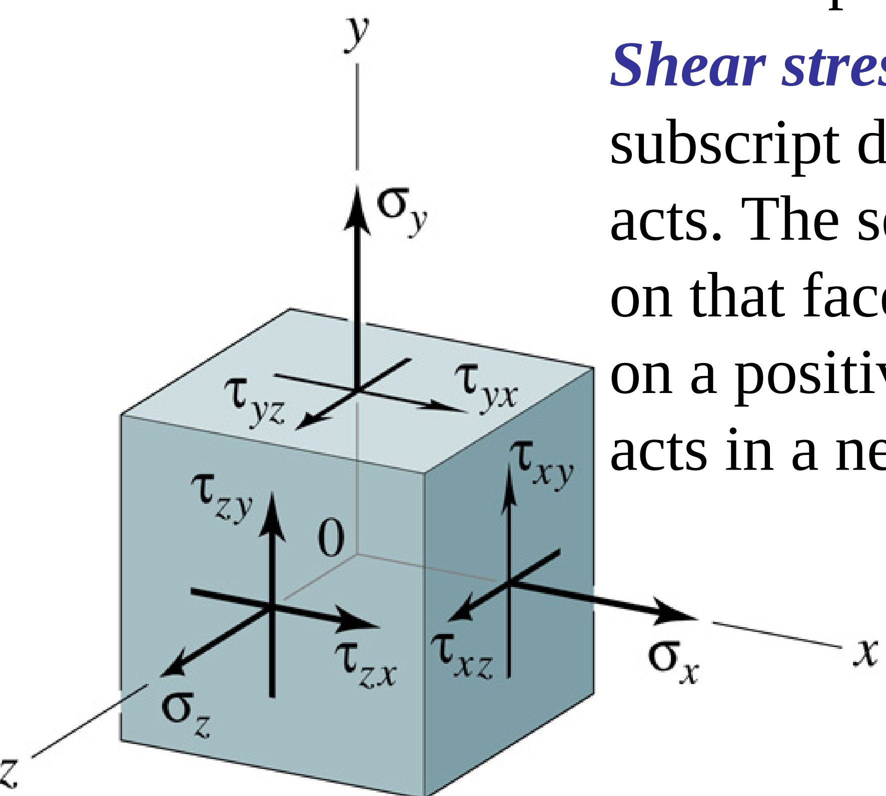
Shear Stresses $\tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx}, \tau_{zy}, \tau_{yz}$

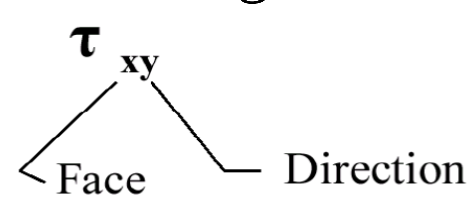
From equilibrium principles:

$$\tau_{xy} = \tau_{yx}\ , \tau_{xz} = \tau_{zx}\ , \tau_{zy} = \tau_{yz}$$

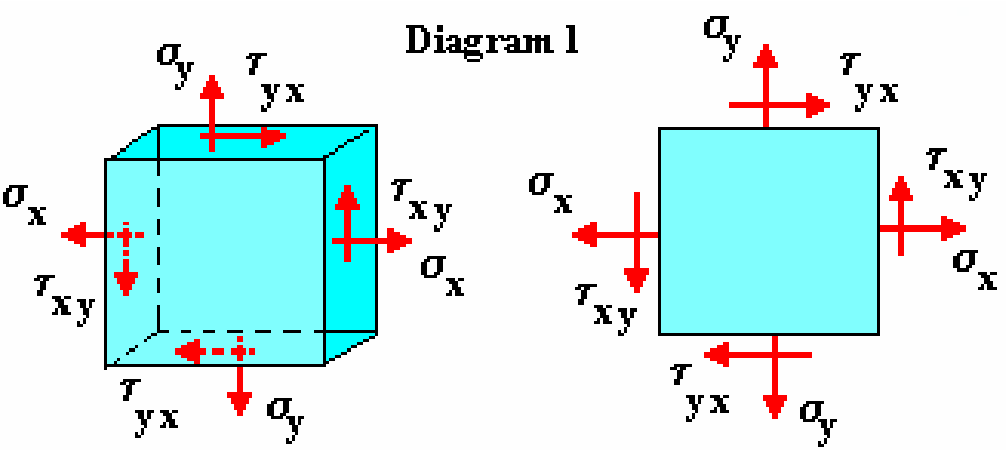
Normal stress (σ) **:** the subscript identifies the face on which the stress acts. Tension is positive and compression is negative.

Shear stress (τ) **:** it has two subscripts. The first subscript denotes the face on which the stress acts. The second subscript denotes the direction on that face. A shear stress is positive if it acts on a positive face and positive direction or if it acts in a negative face and negative direction.

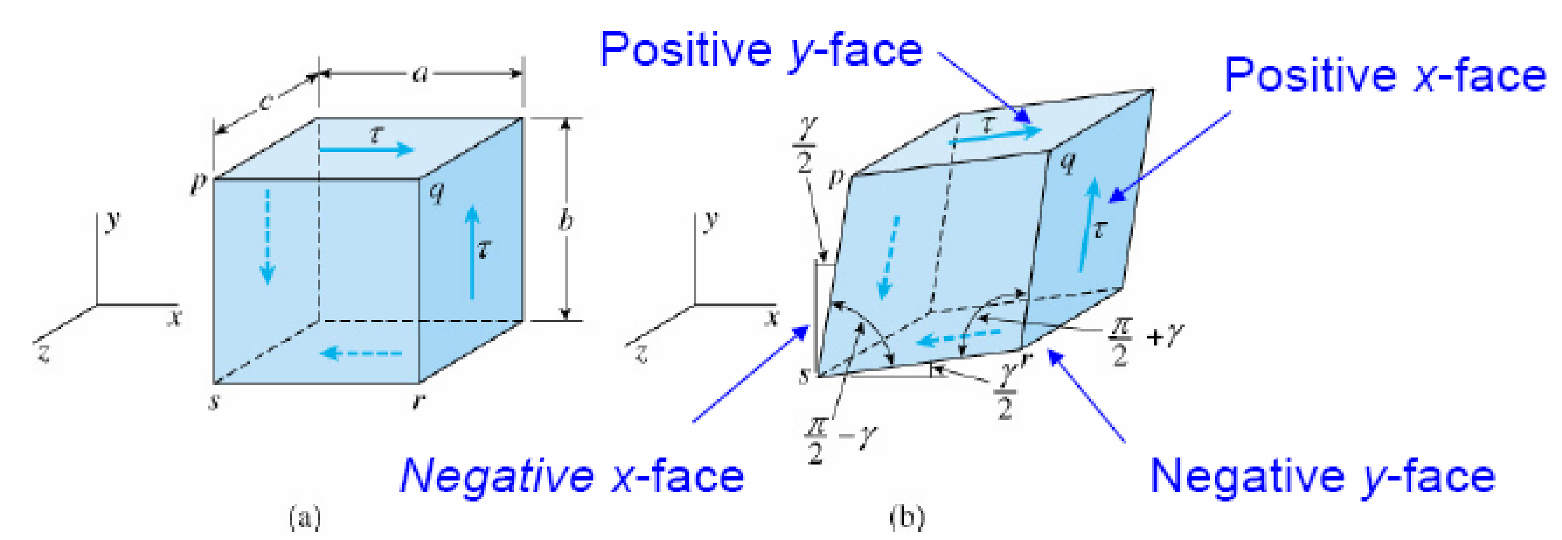




Sign Conventions for Shear Stress and Strain



The Shear Stress will be considered positive when a pair of shear stress acting on opposite sides of the element produce a counterclockwise (ccw) torque (couple).



A shear strain in an element is positive when the angle between two positive faces (or two negative faces) is reduced, and is negative if the angle is increased.

For static equilibrium $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{zy} = \tau_{yz}$ resulting in

Six independent scalar quantities. These six scalars can be arranged in a 3x3 matrix, giving us a ***stress tensor***.

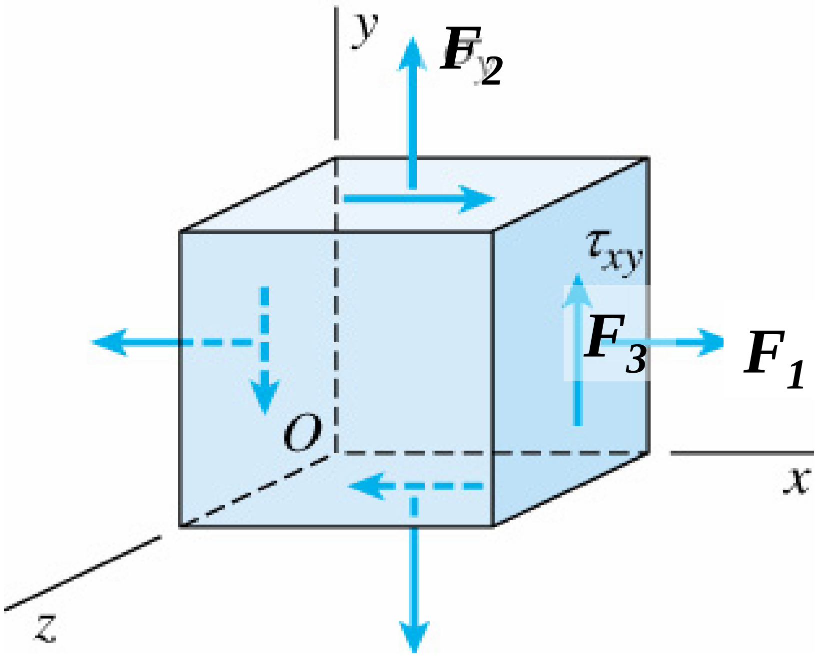
$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

$\boldsymbol{\tau}_{\mathbf{xy}}$

Face   Direction

The sign convention for the stress elements is that a positive force on a positive face or a negative force on a negative face is positive. All others are negative.

The stress state is a second order tensor since it is a quantity associated with two directions (two subscripts direction of the surface normal and direction of the stress).



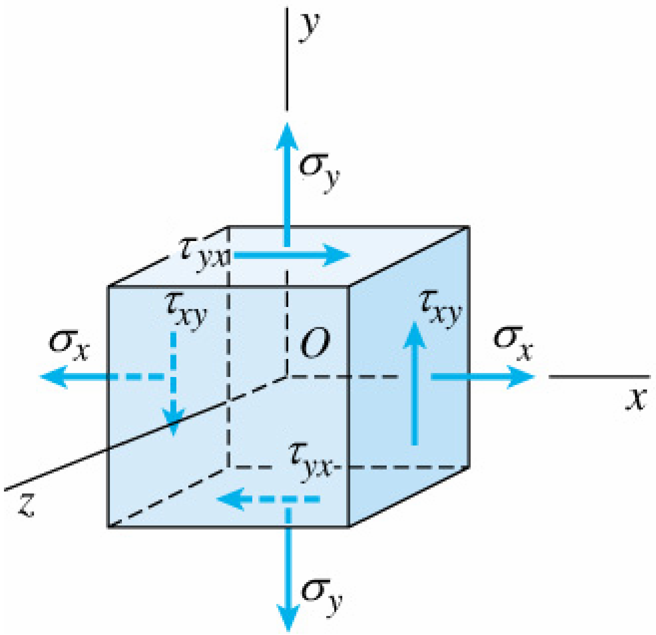
Cube with a Face area A_o

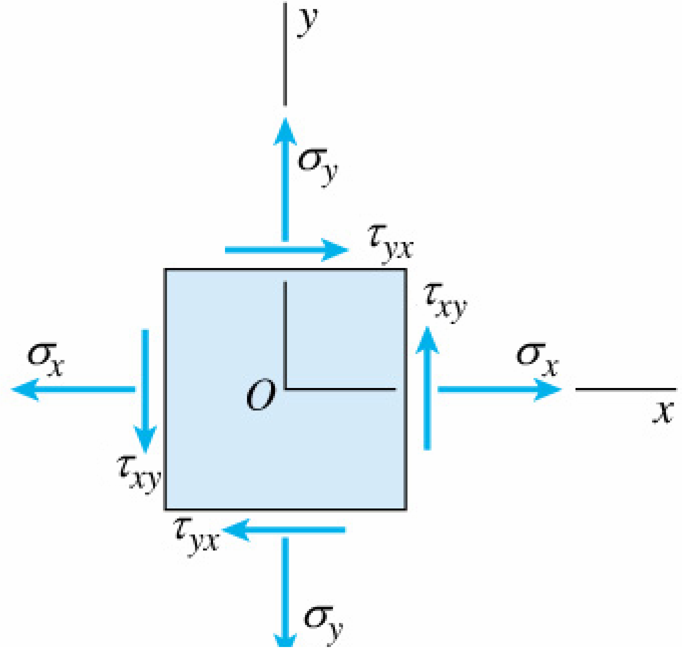
$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \frac{F_1}{A_o} & \frac{F_3}{A_o} & 0 \\ \frac{F_3}{A_o} & \frac{F_2}{A_o} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A property of a symmetric tensor is that there exists an orthogonal set of axes ***1***, ***2*** and ***3*** (called principal axes) with respect to which the tensor elements are all zero except for those in the diagonal.

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \qquad \boldsymbol{\sigma}' = \sigma_{ij}' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Plane Stress or Biaxial Stress : When the material is in plane stress in the plane ***xy***, only the ***x*** and ***y*** faces of the element are subject to stresses, and all the stresses act parallel to the *x* and *y* axes.

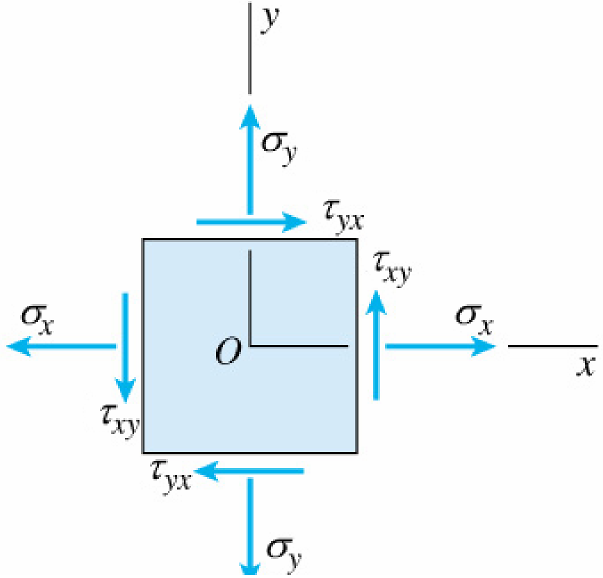


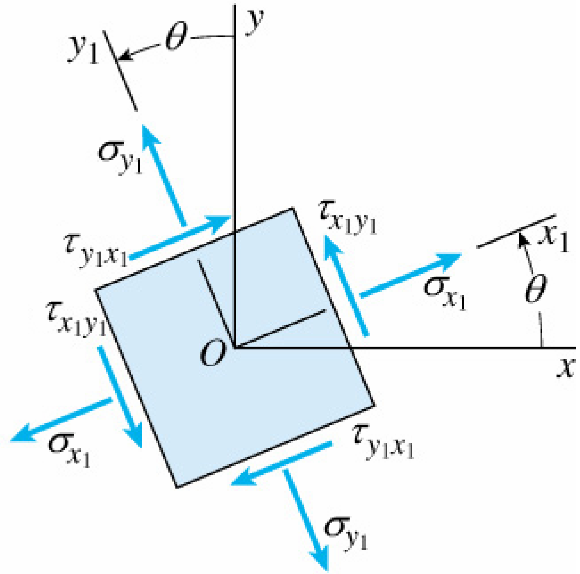


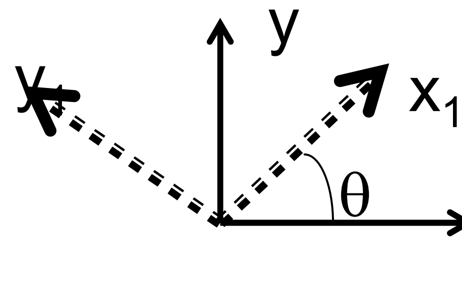
$$\begin{bmatrix} \sigma_{xx} & \tau_{yx} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

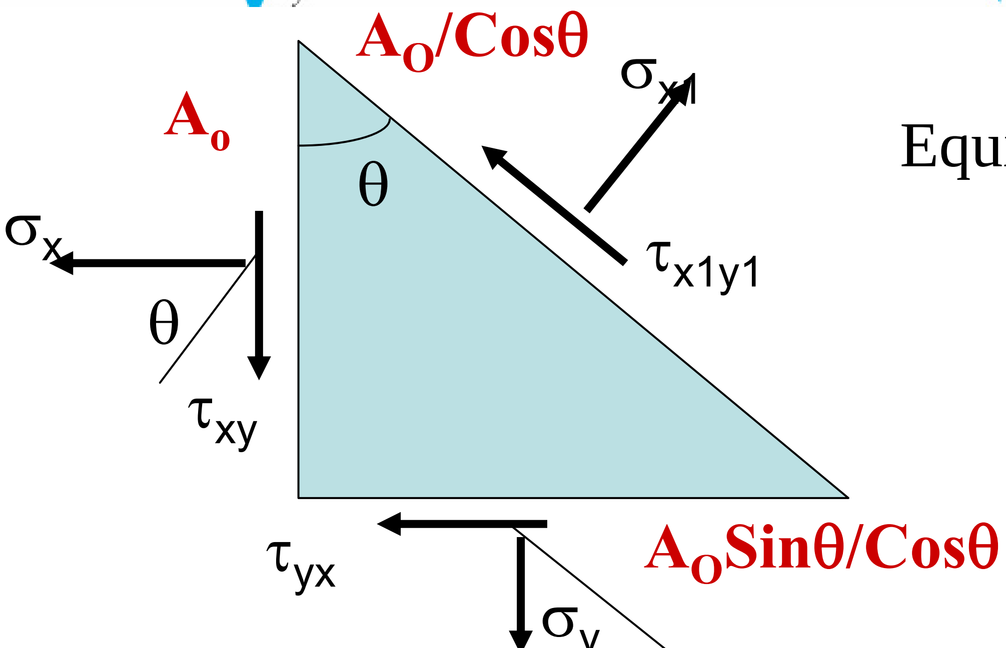
Stresses on Inclined Sections

Knowing the normal and shear stresses acting in the element denoted by the ***xy*** axis, we will calculate the normal and shear stresses acting in the element denoted by the axis $\boldsymbol{x_1y_1}$.





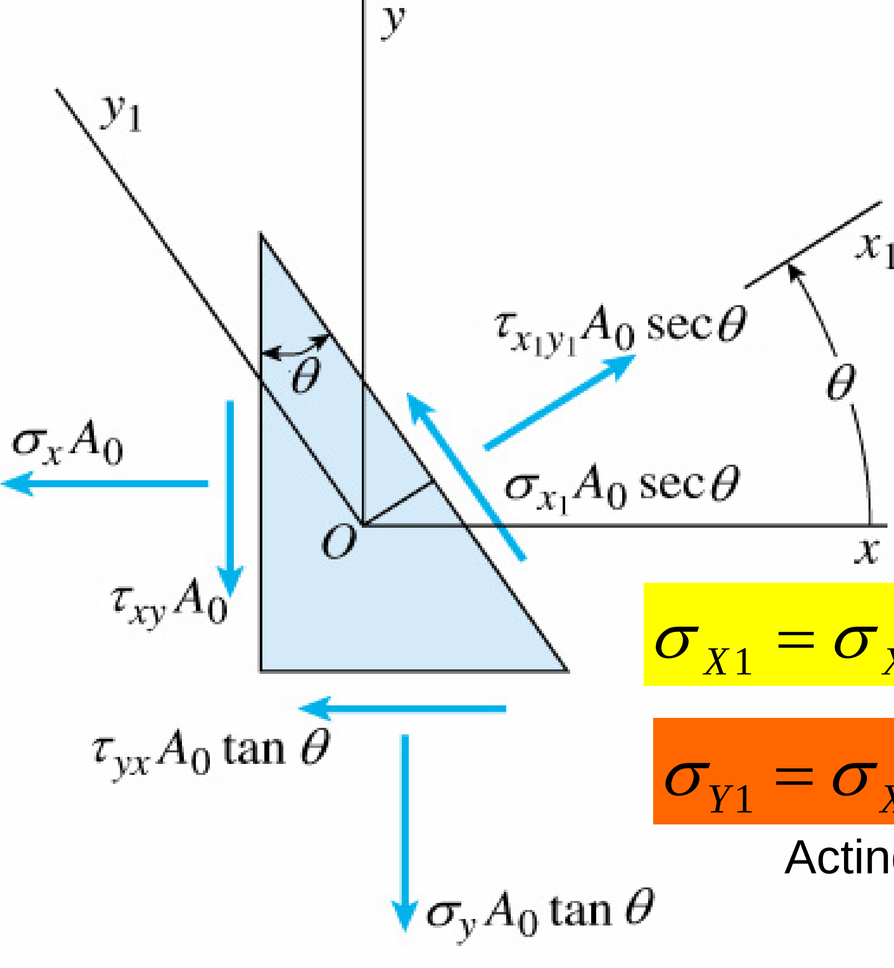




Equilibrium of forces: Acting in x_1

$$\sigma_{X1} \cdot \frac{A_O}{\cos\theta} = \sigma_X \cos\theta \cdot A_O + \tau_{XY} \sin\theta \cdot A_O + \sigma_Y Sin\theta \cdot A_O \frac{Sin\theta}{Cos\theta} + \tau_{YX} \cos\theta \cdot A_O \frac{Sin\theta}{Cos\theta}$$

Eliminating A_o , $sec\theta = 1/cos\theta$ and $\tau_{xy} = \tau_{yx}$



(b) Forces

$$\sigma_{X1} = \sigma_X \cos^2\theta + \sigma_Y \sin^2\theta + 2\tau_{XY} \sin\theta\cos\theta$$

$$\sigma_{Y1} = \sigma_X \sin^2\theta + \sigma_Y \cos^2\theta - 2\tau_{XY} \sin\theta\cos\theta$$

Acting in y_1

$$\tau_{x1y1}A_o\sec\theta = -\sigma_x A_o\sin\theta + \tau_{xy}A_o\cos\theta + \sigma_y A_o\tan\theta\cos\theta - \tau_{yx}A_o\tan\theta\sin\theta$$

Eliminating A_o , $sec\theta = 1/cos\theta$ **and** $\tau_{xy} = \tau_{yx}$

$$\tau_{x1y1} = -\sigma_x \cdot \sin\theta \cdot \cos\theta + \sigma_y \cdot \sin\theta \cdot \cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

Transformation Equations for Plane Stress

Using the following trigonometric identities:

$\cos^2\theta = ½ (1+ \cos 2\theta)$ $\sin^2\theta = ½ (1- \cos 2\theta)$ $\sin \theta \cos \theta = ½ \sin 2\theta$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

These equations are known as ***the transformation equations for plane stress.***

Special Cases

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

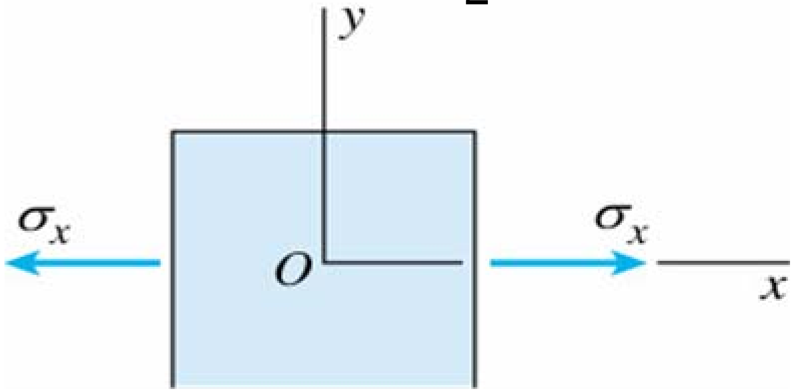
$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Case 1: Uniaxial stress

$$\sigma_y = 0 \quad \tau_{xy} = \tau_{yx} = 0$$

$$\sigma_{x1} = \sigma_x \cdot \left(\frac{1 + Cos2\theta}{2}\right)$$

$$\tau_{x1,y1} = -\sigma_x \cdot \left(\frac{Sin2\theta}{2}\right)$$

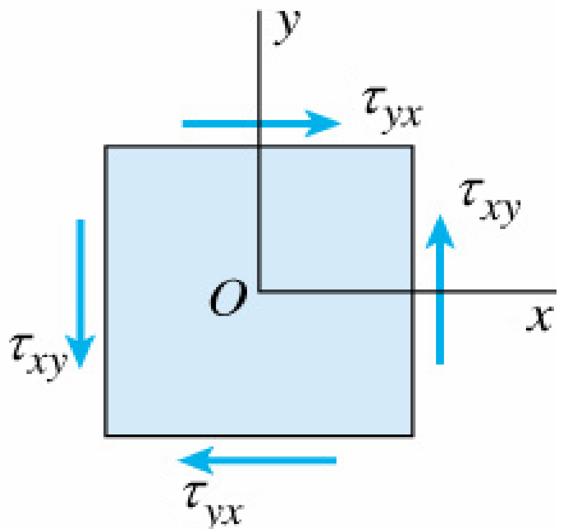


Case 2 : Pure Shear

$$\sigma_x = \sigma_y = 0$$

$$\sigma_{x1} = \tau_{xy} \cdot Sin2\theta$$

$$\tau_{x1,y1} = \tau_{xy} \cdot Cos2\theta$$

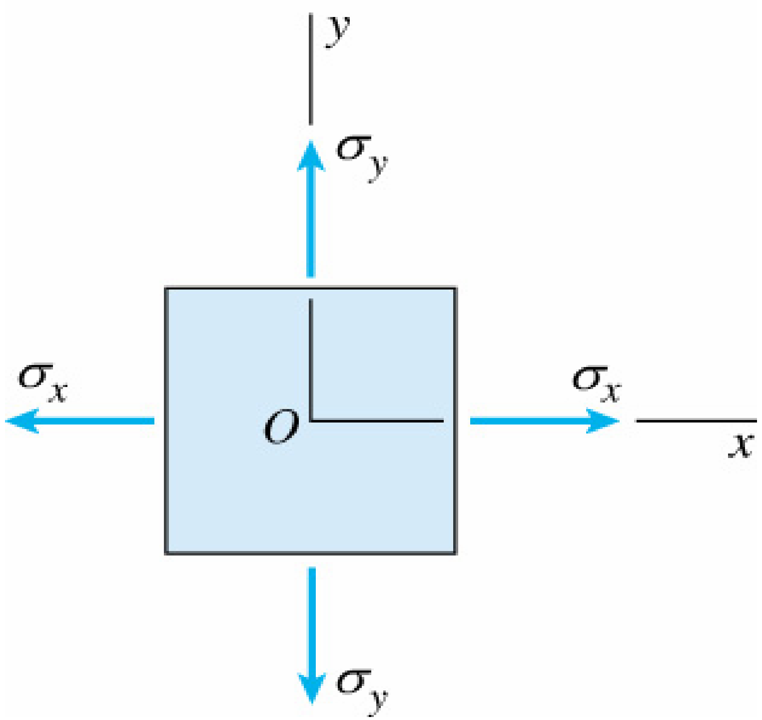


Case 3: Biaxial stress

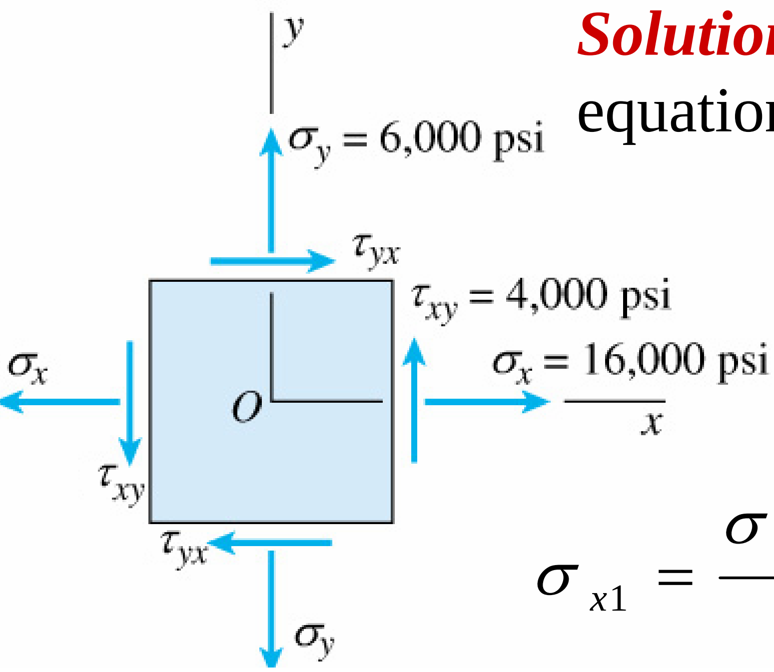
$$\tau_{xy} = 0$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot Cos2\theta$$

$$\tau_{x1,y1} = -\frac{\sigma_x - \sigma_y}{2} \cdot Sin2\theta$$



Example: An element in plane stress is subjected to stresses σ_x=16000psi, σ_y=6000psi, and $\tau_{xy}=\tau_{yx}$= 4000psi (as shown in figure below). Determine the stresses acting on an element inclined at an angle $\theta=45^o$ (counterclockwise - ccw).



Solution: We will use the following transformation equations:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Numerical substitution

$\frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(16000 + 6000) = 11000\text{psi}$

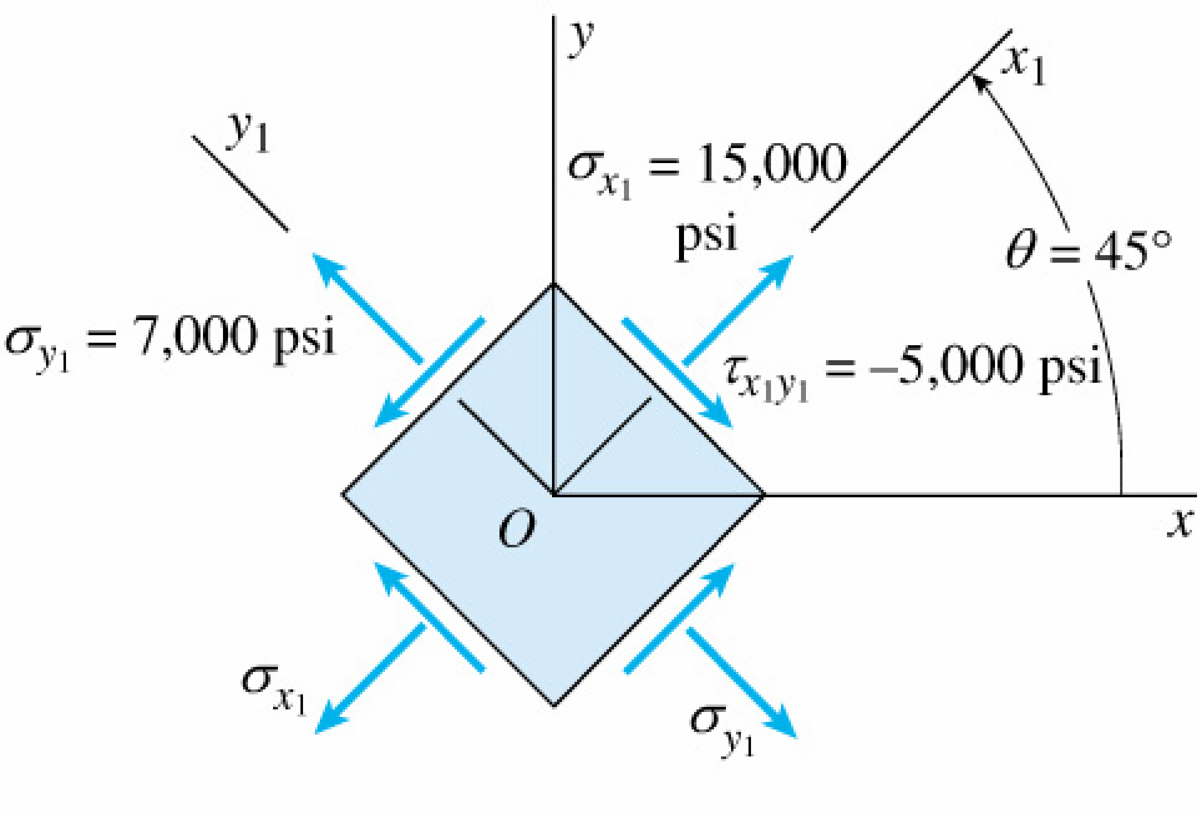
$\frac{1}{2}(\sigma_x - \sigma_y) = \frac{1}{2}(16000 - 6000) = 5000\text{psi}$ $\tau_{xy} = 4000\text{psi}$

$\sin 2\theta = \sin 90^o = 1$ $\cos 2\theta = \cos 90^o = 0$ Then

$\sigma_{x1} = 11000\text{psi} + 5000\text{psi}\,(0) + 4000\text{psi}\,(1) = 15000\text{psi}$

$\tau_{x1y1} = -(5000\text{psi})(1) + (4000\text{psi})(0) = -5000\text{psi}$

$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$ then $\sigma_{y1} = 16000 + 6000 - 15000 = 7000\text{psi}$

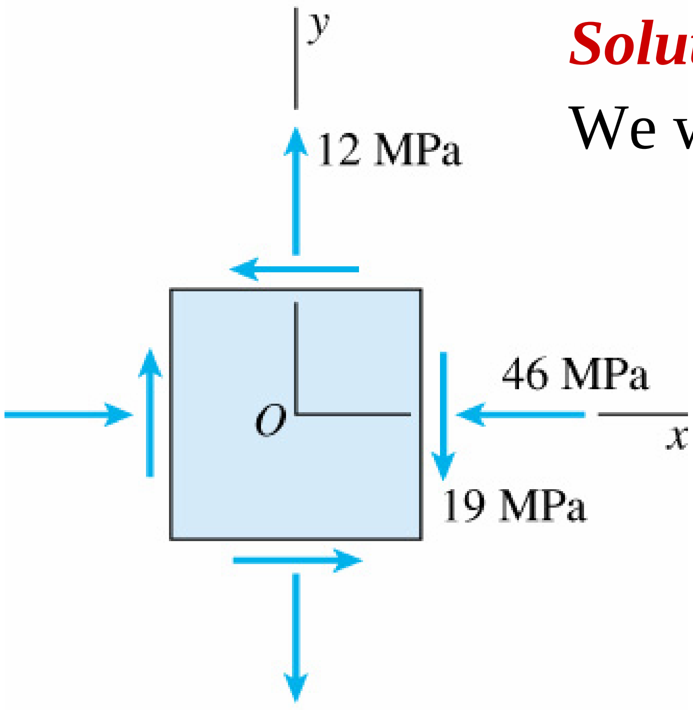


$$\sigma_1 = 17{,}403\,psi$$

$$\sigma_2 = 4{,}597\,psi$$

$$\tau_{Max} = 6{,}403\,psi$$

Example: A plane stress condition exists at a point on the surface of a loaded structure such as shown below. Determine the stresses acting on an element that is oriented at a clockwise (cw) angle of ***15º*** with respect to the original element.



Solution:

We will use the following transformation equations:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Numerical substitution

$\frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(-46 + 12) = -17\text{MPa}$

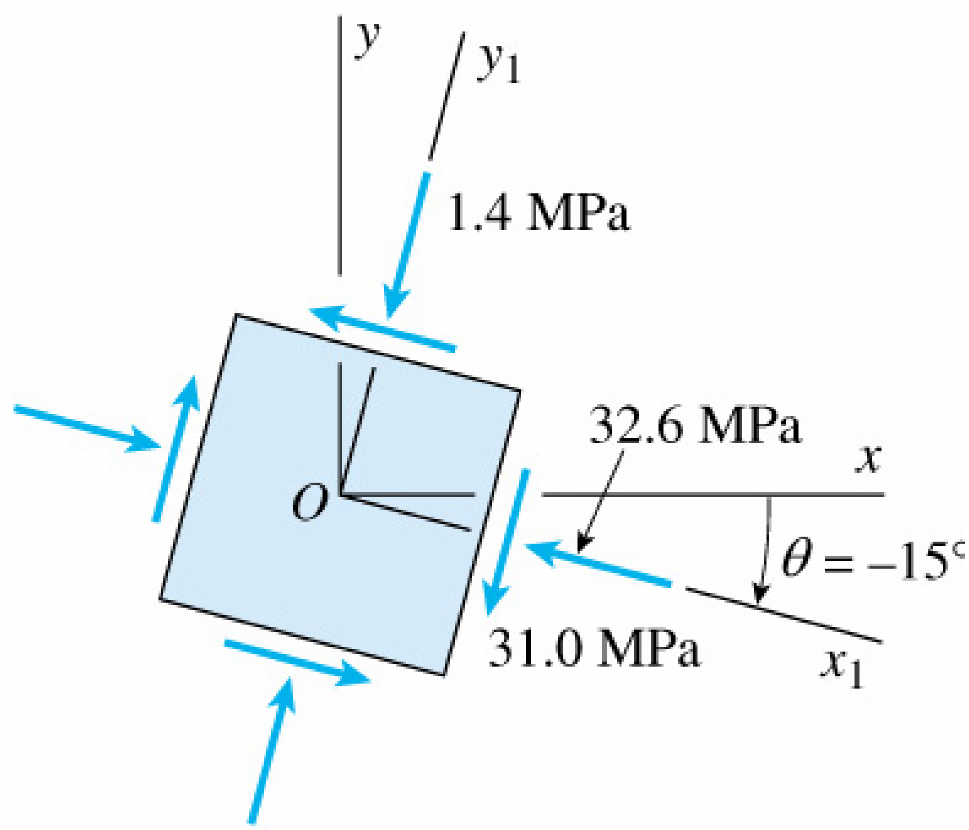
$\frac{1}{2}(\sigma_x - \sigma_y) = \frac{1}{2}(-46 - 12) = -29\text{MPa}$ $\tau_{xy} = -19\text{MPa}$

$\sin 2\theta = \sin(-30^o) = -0.5$ $\cos 2\theta = \cos(-30^o) = 0.866$ then

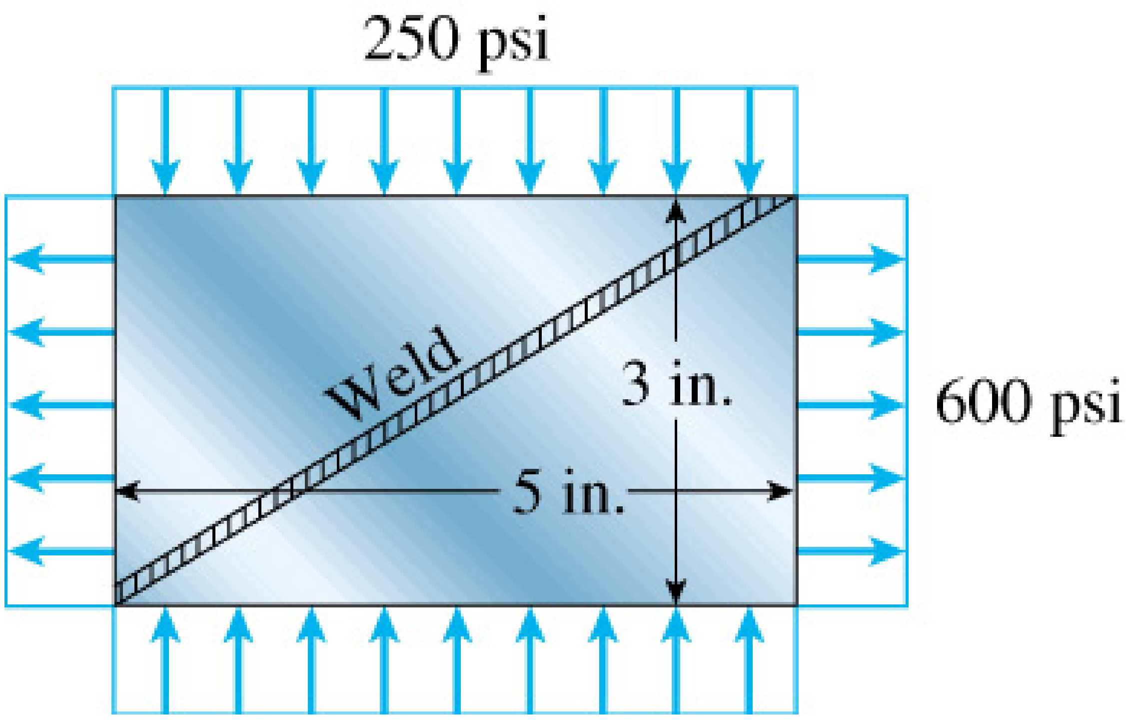
$\sigma_{x1} = -17\text{MPa} + (-29\text{MPa})(0.8660) + (-19\text{MPa})(-0.5) = -32.6\text{MPa}$

$\tau_{x1y1} = -(-29\text{MPa})(-0.5) + (-19\text{MPa})(0.8660) = -31.0\text{MPa}$

$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$ then $\sigma_{y1} = -46\text{MPa} + 12\text{MPa} - (-32.6\text{MPa}) = -1.4\text{MPa}$



Example : A rectangular plate of dimensions 3.0 in x 5.0 in is formed by welding two triangular plates (see figure). The plate is subjected to a tensile stress of 600psi in the long direction and a compressive stress of 250psi in the short direction. Determine the normal stress σ_w acting perpendicular to the line or the weld and the shear stress τ_w acting parallel to the weld. (Assume σ_w is positive when it acts in tension and τ_w is positive when it acts counterclockwise against the weld).



Solution

Biaxial stress weld joint σ_x = 600psi σ_y = -250psi τ_{xy} = 0

From the figure

tanθ = 3 / 5 θ = arctan(3/5) = arctan(0.6) = 30.96°

We will use the following transformation equations:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Numerical substitution

½ $(\sigma_x + \sigma_y)$ = 175psi ½ $(\sigma_x - \sigma_y)$ = 425psi τ_{xy} = 0psi

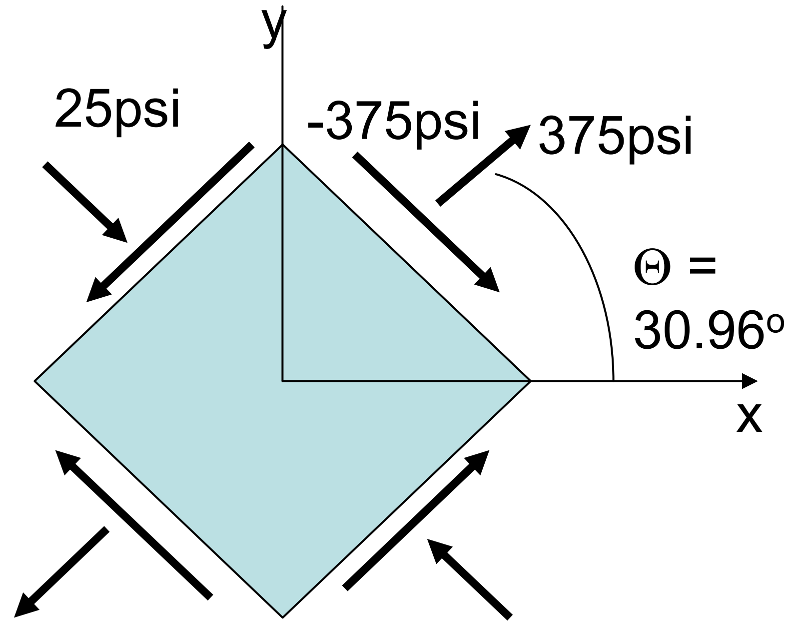
sin 2θ = sin 61.92° cos 2θ = cos 61.92°

Then

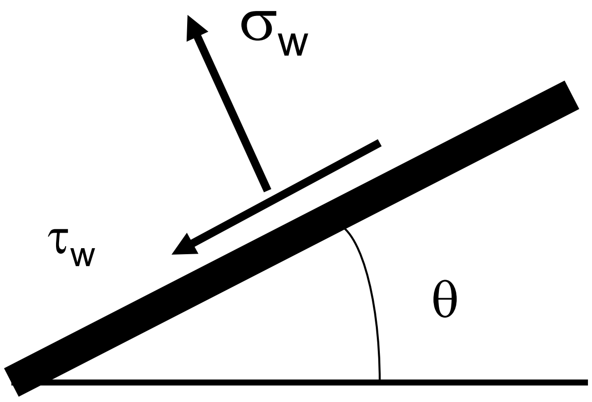
σ_{x1} = 375psi τ_{x1y1} = - 375psi

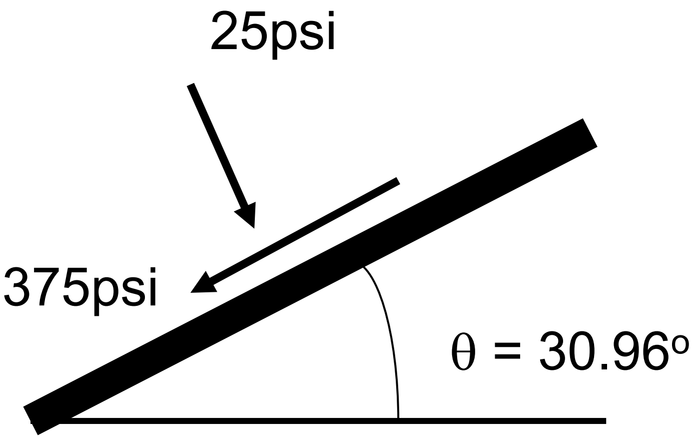
$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$

then σ_{y1} = 600 + (- 250) – 375 = -25psi



Stresses acting on the weld





σ_w = -25psi and τ_w = 375psi

Principal Stresses and Maximum Shear Stresses

The sum of the normal stresses acting on perpendicular faces of plane stress elements is constant and independent of the angle θ.

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{X1} + \sigma_{Y1} = \sigma_X + \sigma_Y$$

As we change the angle θ there will be maximum and minimum normal and shear stresses that are needed for design purposes.

The maximum and minimum normal stresses are known as the principal stresses. These stresses are found by taking the derivative of $\boldsymbol{\sigma_{x1}}$ with respect to θ and setting equal to zero.

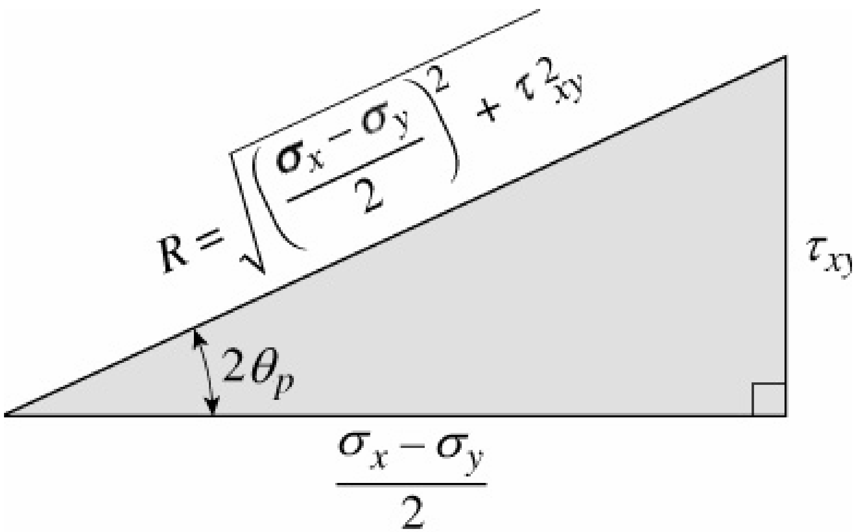
$$\frac{\delta\sigma_{x1}}{\delta\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta_P = \frac{\tau_{xy}}{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}$$

The subscript ***p*** indicates that the angle θ_p defines the orientation of the principal planes. The angle θ_p has two values that differ by ***90º***. They are known as the principal angles.

For one of these angles $\boldsymbol{\sigma}_{\mathbf{x1}}$ is a ***maximum*** principal stress and for the other a ***minimum***. The principal stresses occur in mutually perpendicular planes.

$$\tan 2\theta_P = \frac{\tau_{xy}}{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}$$



$$\sin 2\theta_P = \frac{\tau_{xy}}{R} \qquad \cos 2\theta_P = \frac{(\sigma_x - \sigma_y)}{2R}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

for the maximum stress $\theta = \theta_{\mathrm{P}}$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{R}\right) = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\left(\frac{1}{R}\right) + \left(\tau_{xy}\right)^2\left(\frac{1}{R}\right)$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{1}{R}\right)\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2\right]$$

But $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$ $\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$

Principal stresses:

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

The plus sign gives the algebraically larger principal stress and the minus sign the algebraically smaller principal stress.

Maximum Shear Stress

The location of the angle for the maximum shear stress is obtained by taking the derivative of $\boldsymbol{\tau_{x1y1}}$ with respect to $\boldsymbol{\theta}$ and setting it equal to zero.

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{\delta\tau_{x1y1}}{\delta\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_S = -\frac{\frac{\sigma_x - \sigma_y}{2}}{\tau_{xy}}$$

Therefore, $\mathbf{2\theta_s - 2\theta_p = -90^o}$ **or** $\mathbf{\theta_s = \theta_p \pm 45^o}$

The planes for maximum shear stress occurs at ***45º*** to the principal planes. The plane of the maximum positive shear stress $\boldsymbol{\tau_{max}}$ is defined by the angle $\boldsymbol{\theta_{S1}}$ for which the following equations apply:

$$\tan 2\theta_S = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}} = -\frac{1}{\tan 2\theta_P} = -\cot 2\theta_P$$

$$\frac{\sin 2\theta_S}{\cos 2\theta_S} = -\frac{\cos 2\theta_P}{\sin 2\theta_P} = \frac{-\sin\left(90^o - 2\theta_P\right)}{\cos\left(90^o - 2\theta_P\right)} = \frac{\sin\left(2\theta_P - 90^o\right)}{\cos\left(2\theta_P - 90^o\right)}$$

$$\cos 2\theta_{s1} = \frac{\tau_{xy}}{R} \qquad \sin 2\theta_{s1} = -\frac{(\sigma_x - \sigma_y)}{2R} \quad \text{and} \quad \theta_{s1} = \theta_{P1} - 45^0$$

The corresponding maximum shear is given by the equation

$$\tau_{MAX} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

Another expression for the maximum shear stress

$$\tau_{MAX} = \frac{\left(\sigma_1 - \sigma_2\right)}{2}$$

The normal stresses associated with the maximum shear stress are equal to

$$\sigma_{AVER} = \frac{\left(\sigma_x + \sigma_y\right)}{2}$$

Equations of a Circle

General equation

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Consider

$$\sigma_{x1} - \sigma_{AVER} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Equation (1)

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 = \left[\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Equation (2)

$$\left(\tau_{x1y1}\right)^2 = \left[-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2$$

Equation (1) + Equation (2)

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 + \left(\tau_{x1y1}\right)^2 = \left[\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2 + \left[-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2$$

$$\left[\frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\cos^2 2\theta + \left(\tau_{xy}\right)^2\sin^2 2\theta + 2\left(\frac{\sigma_x - \sigma_y}{2}\right)\left(\tau_{xy}\right)\sin 2\theta\cos 2\theta$$

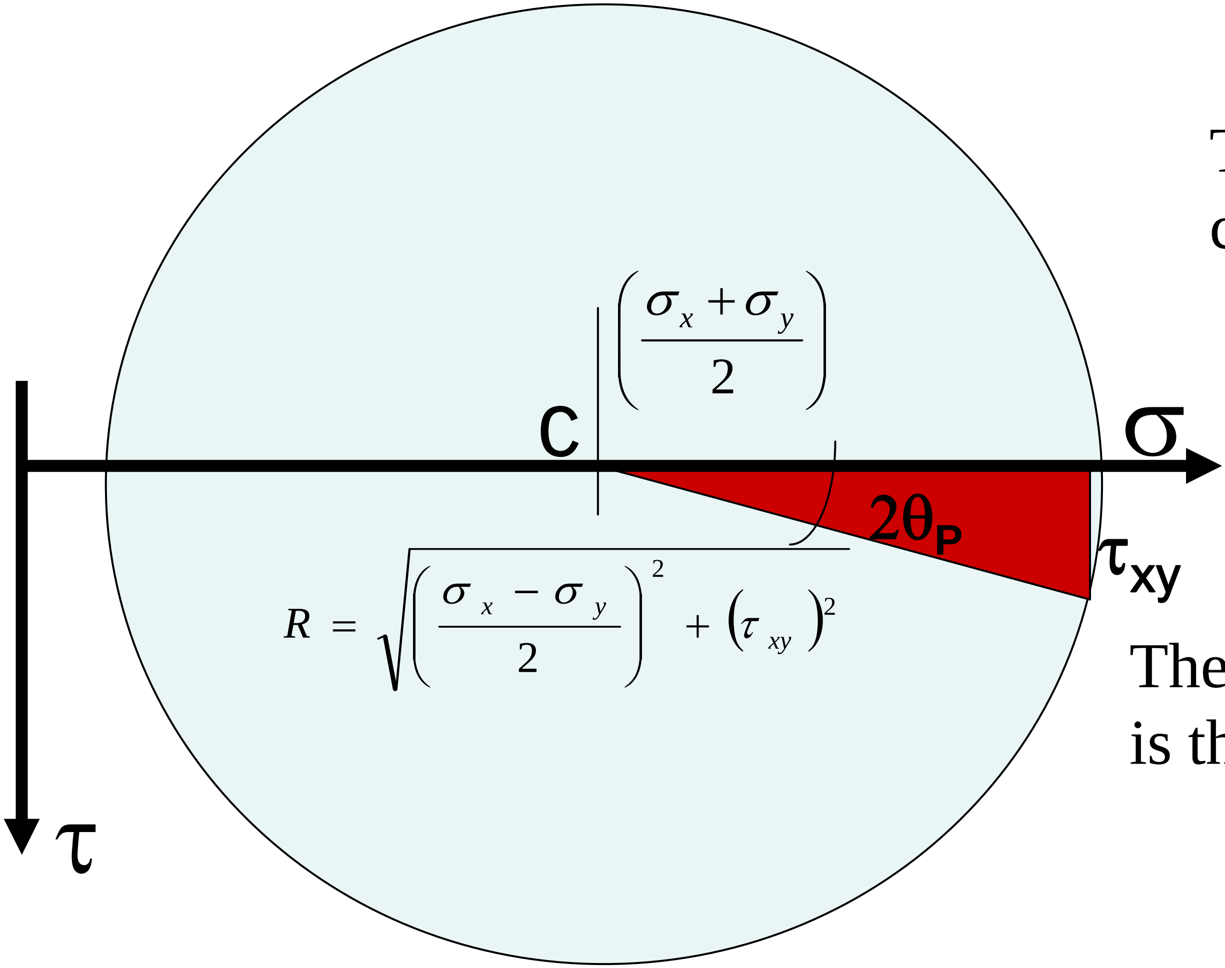
$$\left[-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2 = \left(-\frac{\sigma_x - \sigma_y}{2}\right)^2\sin^2 2\theta + \left(\tau_{xy}\right)^2\cos^2 2\theta - 2\left(\frac{\sigma_x - \sigma_y}{2}\right)\left(\tau_{xy}\right)\sin 2\theta\cos 2\theta$$

SUM

$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2 = R^2$$

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 + \left(\tau_{x1y1}\right)^2 = R^2$$

Mohr Circle



The radius of the Mohr circle is the magnitude ***R***.

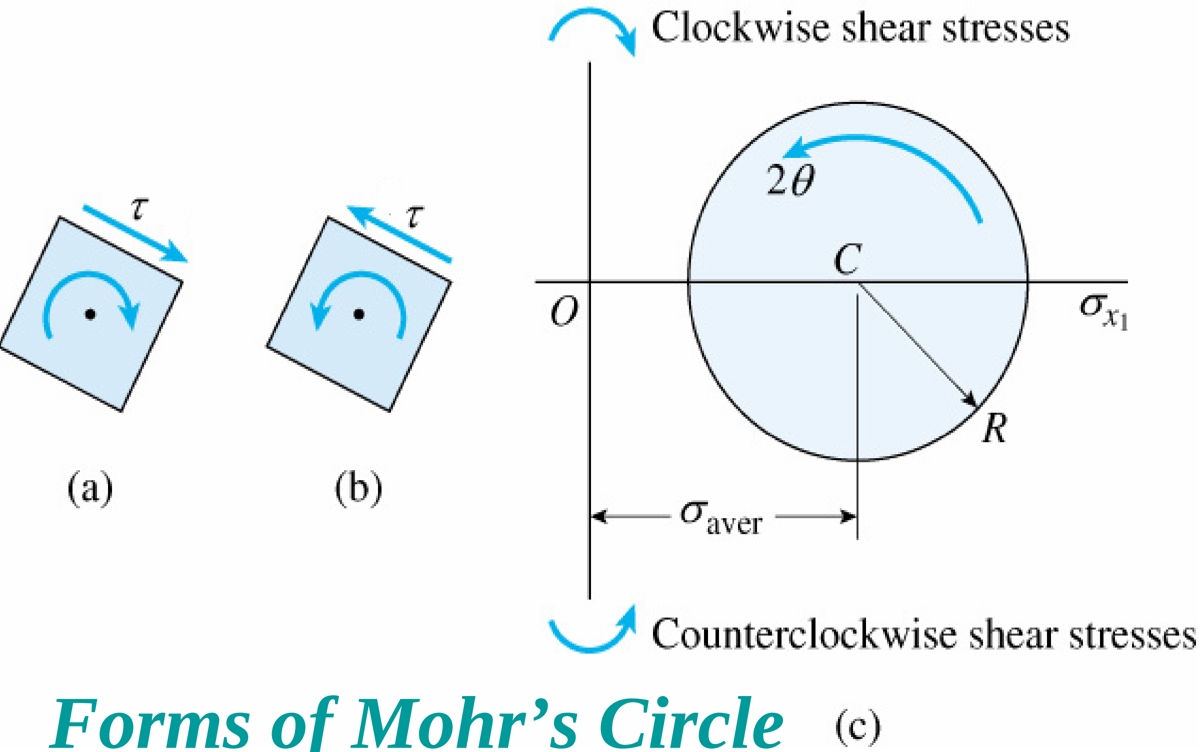
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

The center of the Mohr circle is the magnitude

$$\sigma_{AVER} = \frac{\left(\sigma_x + \sigma_y\right)}{2}$$

$$\left(\sigma_{x1} - \sigma_{AVER}\right)^2 + \left(\tau_{x1y1}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2$$

Forms of Mohr's Circle



Alternative sign conversion for shear stresses:
(a) clockwise shear stress,
(b) counterclockwise shear stress, and
(c) axes for Mohr's circle.

Note that clockwise shear stresses are plotted upward and counterclockwise shear stresses are plotted downward.

a) We can plot the normal stress σ_{x1} positive to the right and the shear stress τ_{x1y1} positive downwards, i.e. the angle 2θ will be positive when counterclockwise or

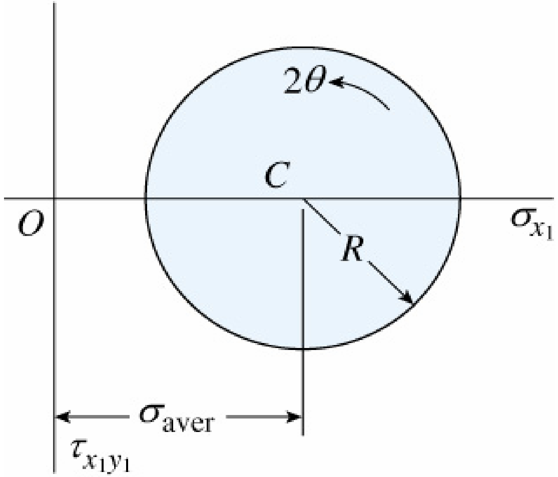
b) We can plot the normal stress σ_{x1} positive to the right and the shear stress τ_{x1y1} positive upwards, i.e. the angle 2θ will be positive when clockwise.

Both forms are mathematically correct. **We use (a)**

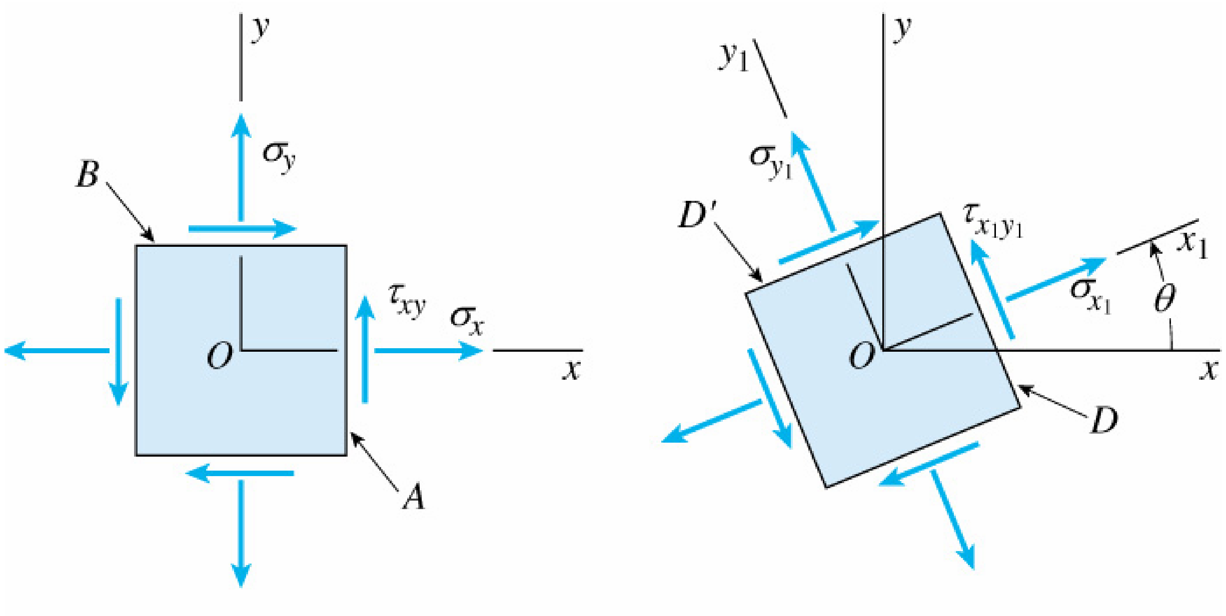
Two forms of Mohr's circle:

(a) $\boldsymbol{\tau_{x1y1}}$ is positive downward and the angle $\boldsymbol{2\theta}$ is positive counterclockwise, and

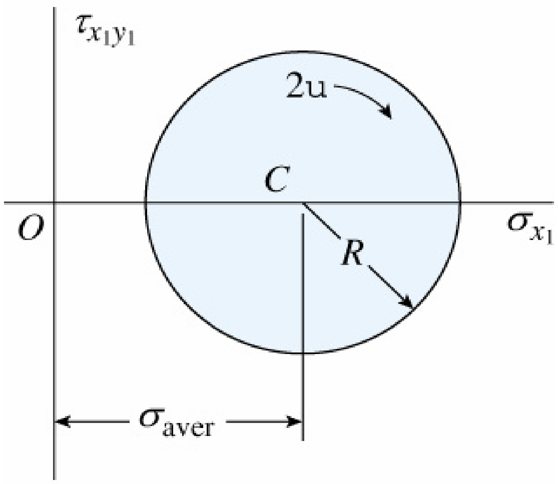
(b) $\boldsymbol{\tau_{x1y1}}$ is positive upward and the angle $\boldsymbol{2\theta}$ is positive clockwise. (*Note:* The first form is used here)



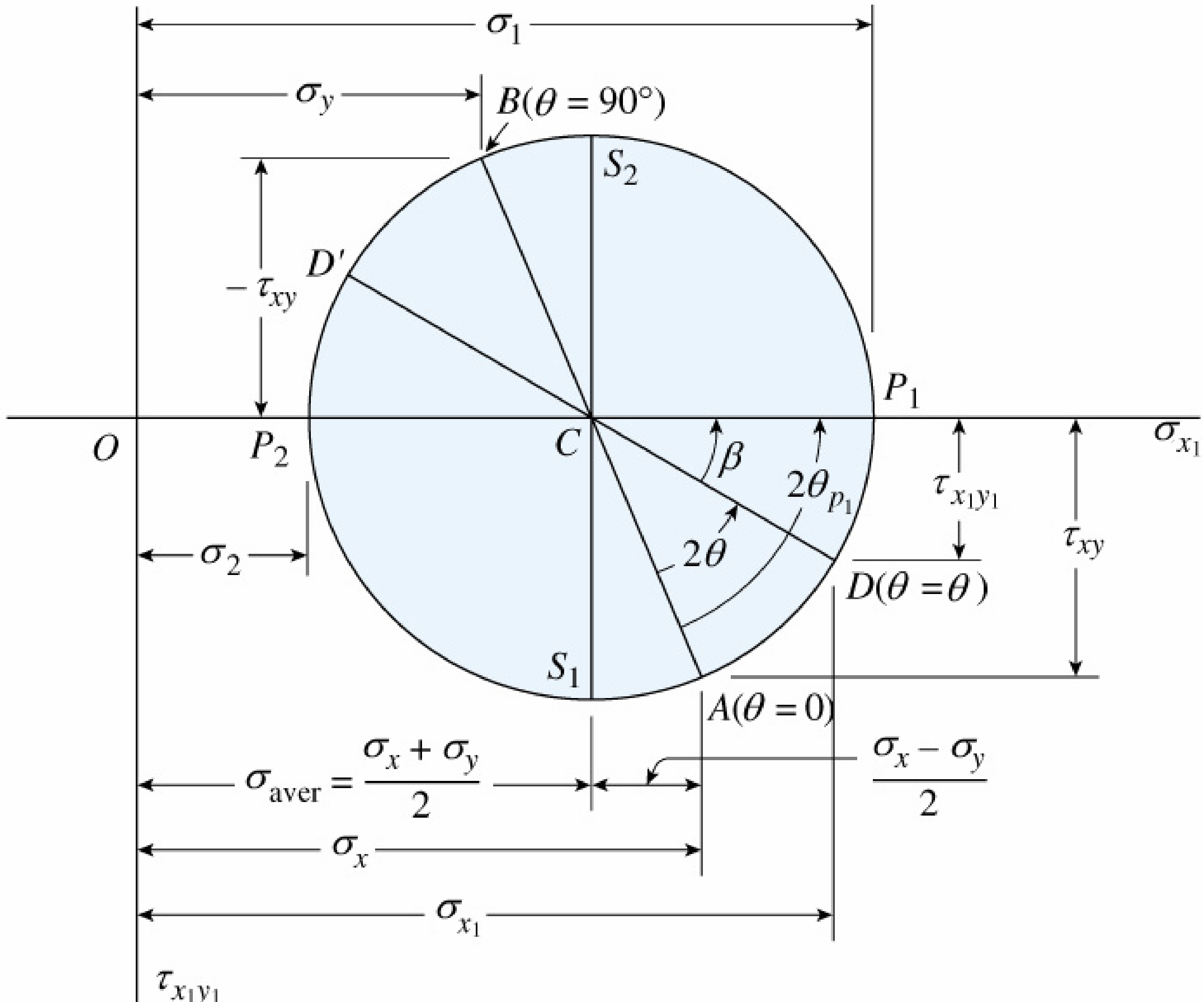
(a)



(a) (b)



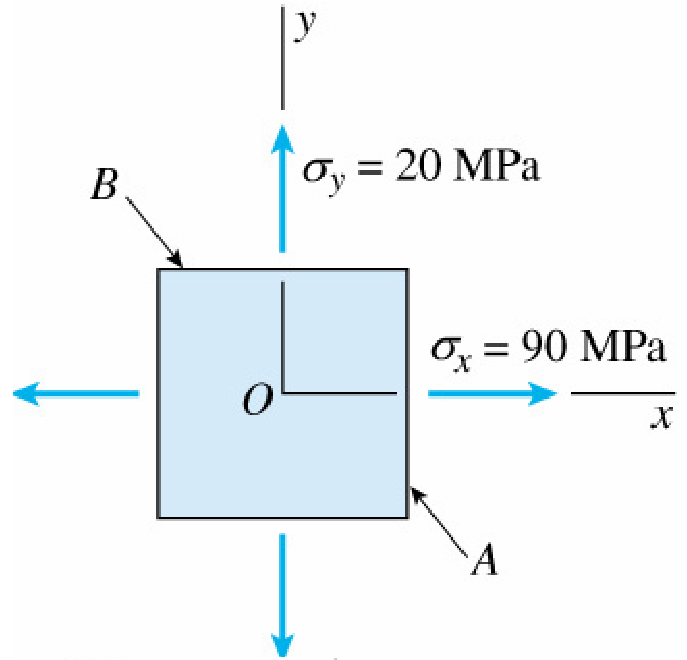
(b)



(c)

Construction of Mohr’s circle for plane stress.

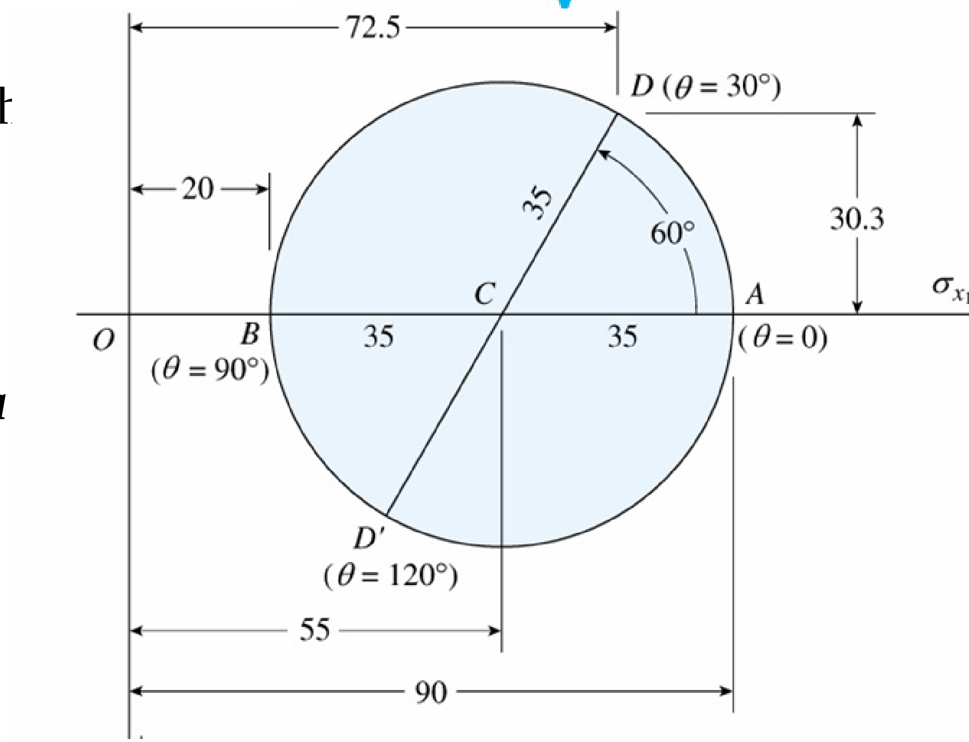
Example: At a point on the surface of a pressurized cylinder, the material is subjected to biaxial stresses $\boldsymbol{\sigma_x = 90MPa}$ and $\boldsymbol{\sigma_y = 20MPa}$ as shown in the element below.



Using the Mohr circle, determine the stresses acting on an element inclined at an angle $\boldsymbol{\theta = 30^o}$ (Sketch a properly oriented element).

Solution $\boldsymbol{(\sigma_x = 90MPa,\ \ \sigma_y = 20MPa\ \ \ and\ \ \tau_{xy} = 0MPa)}$

Because the shear stress is zero, these are th principal stresses.



Construction of the Mohr's circle

The center of the circle is

$\boldsymbol{\sigma_{aver} = ½ (\sigma_x + \sigma_y) = ½ (90 + 20) = 55MPa}$

The radius of the circle is

$\boldsymbol{R = SQR[((\sigma_x - \sigma_y)/2)^2 + (\tau_{xy})^2]}$

$\boldsymbol{R = (90 - 20)/2 = 35MPa.}$

Stresses on an element inclined at $\boldsymbol{\theta = 30^o}$

By inspection of the circle, the coordinates of point ***D*** are

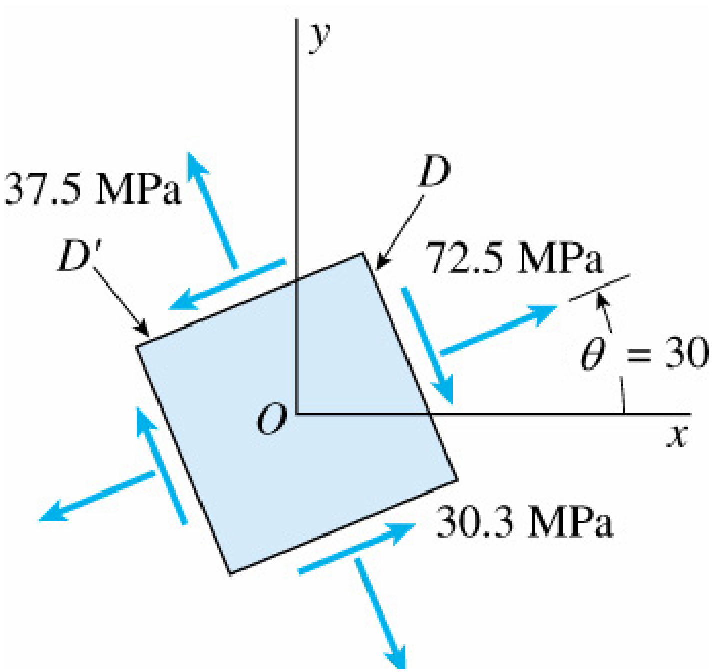
$\sigma_{x1} = \sigma_{aver} + R \cos 60^o = 55MPa + 35MPa\ (\text{Cos } 60^o) = 72.5MPa$

$\tau_{x1y1} = - R \sin 60^o = - 35MPa\ (\text{Sin } 60^o) = - 30.3MPa$

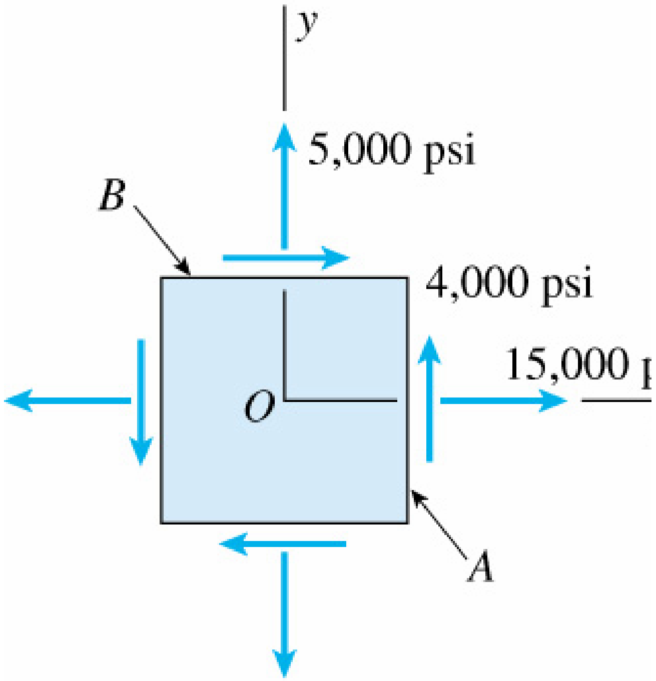
In a similar manner we can find the stresses represented by point D’, which correspond to an angle $\theta = 120^o$ ($2\theta = 240^o$)

$\sigma_{y1} = \sigma_{aver} - R \cos 60^o = 55MPa - 35MPa\ (\text{Cos } 60^o) = 37.5MPa$

$\tau_{x1y1} = R \sin 60^o = 35MPa\ (\text{Sin } 60^o) = 30.3MPa$



Example: An element in plane stress at the surface of a large machine is subjected to stresses $\boldsymbol{\sigma_x = 15000psi}$, $\boldsymbol{\sigma_y = 5000psi}$ and $\boldsymbol{\tau_{xy} = 4000psi}$, as shown in the figure.



Using the Mohr's circle determine the following:

a) The stresses acting on an element inclined at an angle $\boldsymbol{\theta = 40^o}$

b) The principal stresses and

c) The maximum shear stresses.

Solution

Construction of Mohr's circle:

Center of the circle (Point C): $\sigma_{aver} = ½ (\sigma_x + \sigma_y) = ½ (15000 + 5000) = 10000psi$

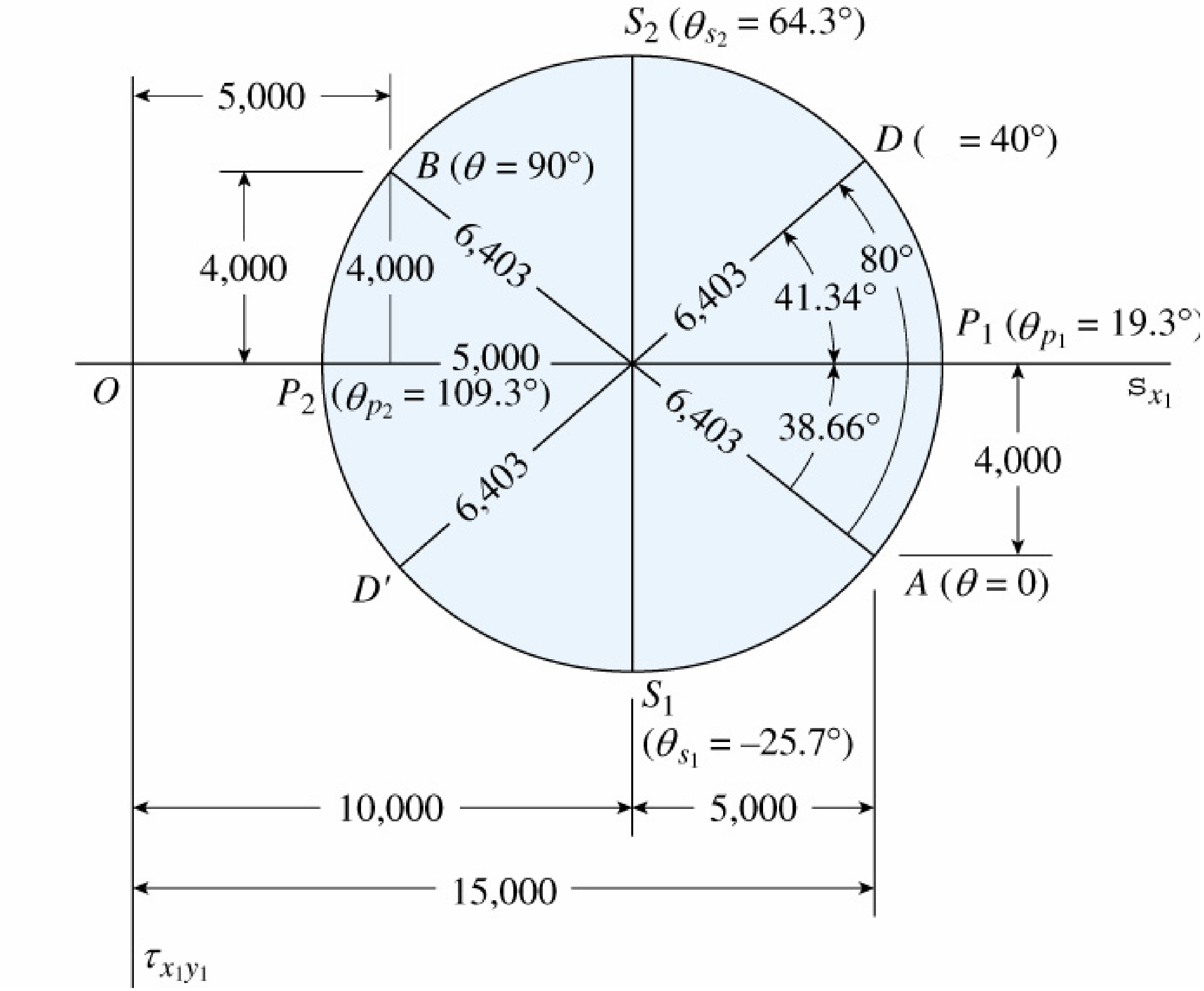
Radius of the circle: $R = SQR[((\sigma_x – \sigma_y)/2)^2 + (\tau_{xy})^2]$

$R = SQR[((15000 – 5000)/2)^2 + (4000)^2] = 6403psi.$

Point A, representing the stresses on the ***x*** face of the element ($\theta = 0^o$) has the coordinates $\sigma_{x1} = 15000psi$ and $\tau_{x1y1} = 4000psi$

Point B, representing the stresses on the ***y*** face of the element ($\theta = 90^o$) has the coordinates $\sigma_{y1} = 5000psi$ and $\tau_{y1x1} = - 4000psi$

The circle is now drawn through points A and B with center C and radius R



Stresses on an element inclined at $\theta = 40^o$

These are given by the coordinates of point ***D*** which is at an angle ***$2\theta = 80^o$*** from point A. By inspection the angle ACP_1 for the principal stresses (point P_1) is tan ACP_1 = 4000/5000 = 0.8 or 38.66^o.

Then, the angle P_1CD is $80^o - 38.66^o = 41.34^o$

Knowing this angle, we can calculate the coordinates of point D (by inspection)

$\sigma_{x1} = \sigma_{aver} + R \cos 41.34^o = 10000psi + 6403psi (\cos 41.34^o) = 14810psi$

$\tau_{x1y1} = - R \sin 41.34^o = - 6403psi (\sin 41.34^o) = - 4230psi$

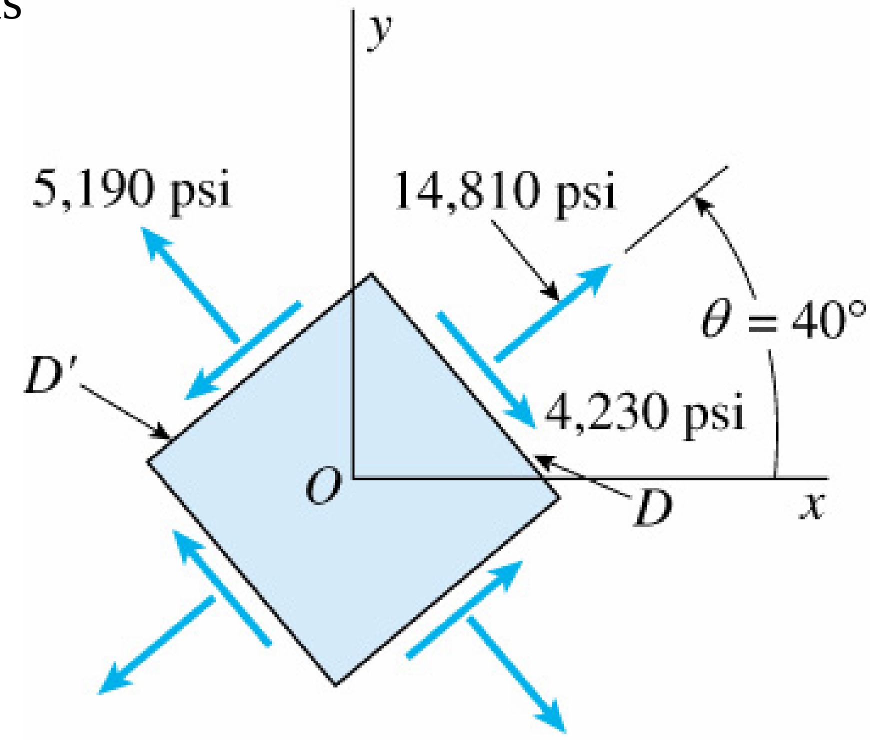
In an analogous manner, we can find the stresses represented by point D', which correspond to a plane inclined at an angle $\theta = 40^o + 90^o = 130^o$

$\sigma_{y1} = \sigma_{aver} - R \cos 41.34^o = 10000psi - 6403psi (\cos 41.34^o) = 5190psi$

$\tau_{x1y1} = R \sin 41.34^o = 6403psi (\sin 41.34^o) = 4230psi$

And of course, the sum of the normal stresses is

14810psi + 5190psi = 15000psi + 5000psi



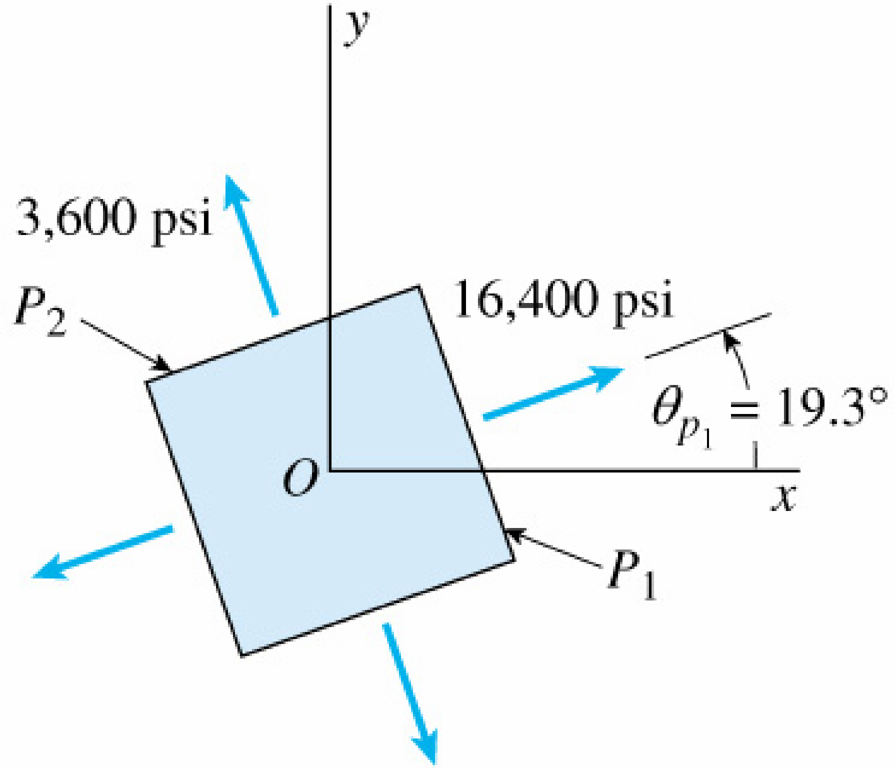
Principal Stresses

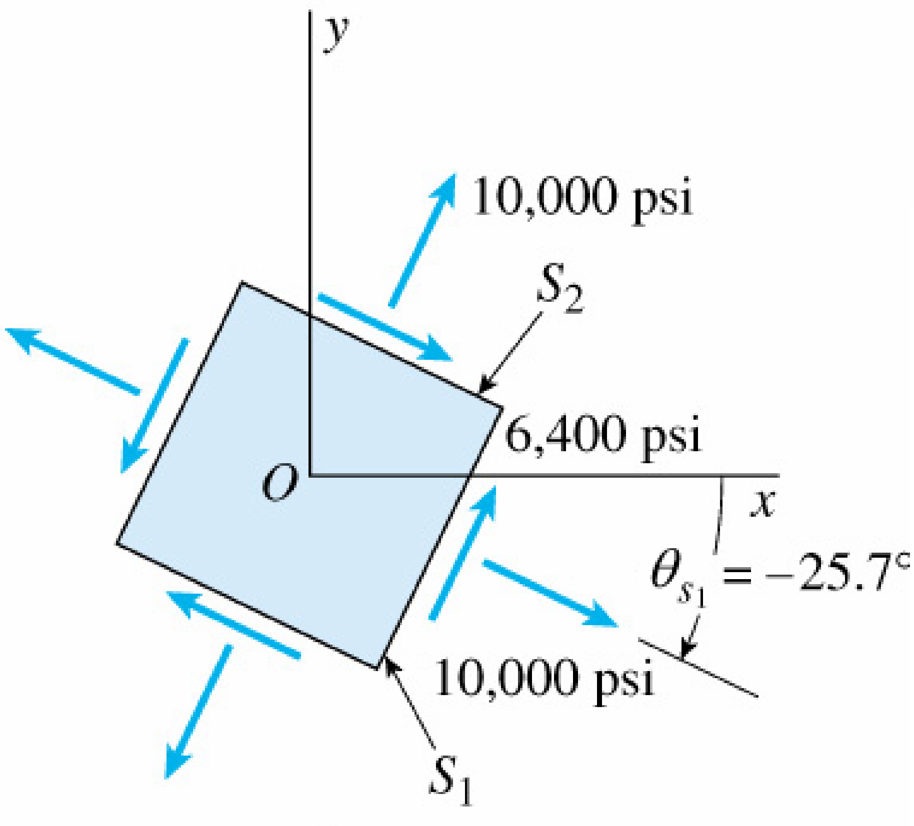
The principal stresses are represented by points P_1 and P_2 on Mohr's circle.

σ_1 = 10000psi + 6400psi = 16400psi

σ_2 = 10000psi – 6400psi = 3600psi

The angle it was found to be $2\theta = 38.66^o$ or $\theta = 19.3^o$





Maximum Shear Stresses

These are represented by point S_1 and S_2 in Mohr's circle.

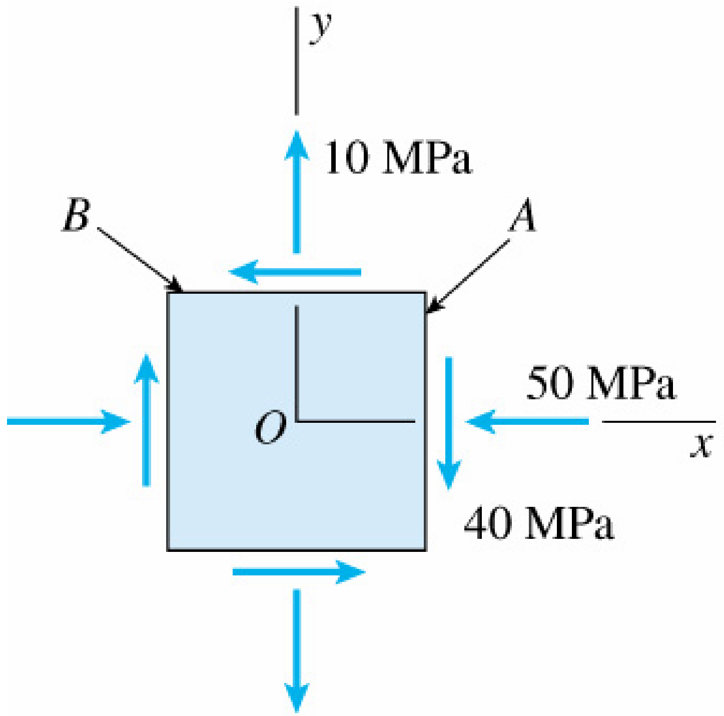
The angle ACS_1 from point A to point S_1 is $2\,\theta_{S1} = 51.34^o$. This angle is negative because is measured clockwise on the circle. Then the corresponding θ_{S1} value is -25.7^o.

Example: At a point on the surface of a generator shaft the stresses are σ_x = -50MPa, σ_y = 10MPa and τ_{xy} = - 40MPa as shown in the figure. Using Mohr's circle determine the following:

(a) Stresses acting on an element inclined at an angle $\theta = 45^o$,

(b) The principal stresses and

(c) The maximum shear stresses



Solution

Construction of Mohr's circle:

Center of the circle (Point C): $\sigma_{aver} = ½ (\sigma_x + \sigma_y) = ½ ((-50) + 10) = - 20$MPa

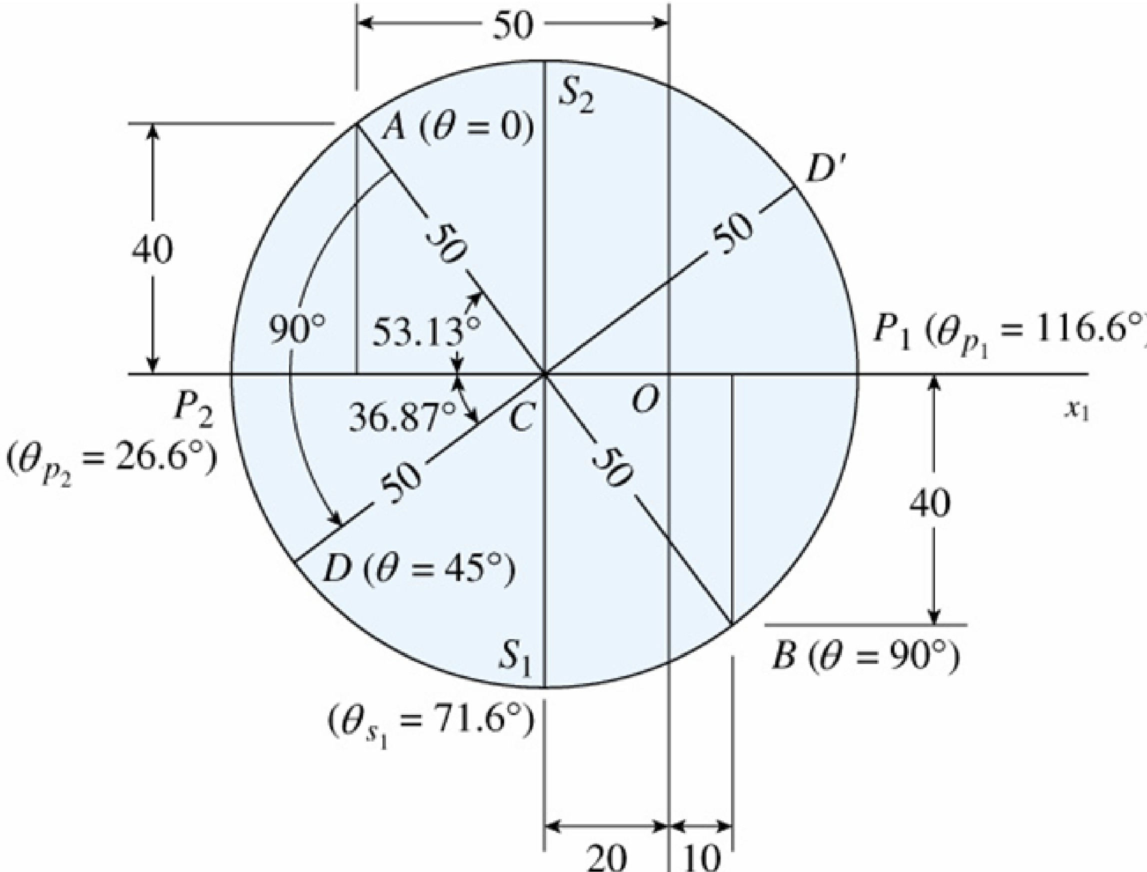
Radius of the circle: $R = SQR[((\sigma_x – \sigma_y)/2)^2 + (\tau_{xy})^2]$

$R = SQR[((- 50 – 10)/2)^2 + (- 40)^2] = 50$MPa.

Point A, representing the stresses on the ***x*** face of the element ($\theta = 0^o$) has the coordinates σ_{x1} = -50MPa and τ_{x1y1} = - 40MPa

Point B, representing the stresses on the ***y*** face of the element ($\theta = 90^o$) has the coordinates σ_{y1} = 10MPa and τ_{y1x1} = 40MPa

The circle is now drawn through points A and B with center C and radius R.



Stresses on an element inclined at $\theta = 45^o$

These stresses are given by the coordinates of point D ($2\theta = 90^o$ of point A). To calculate its magnitude we need to determine the angles ACP2 and P2CD.

tan ACP_2=40/30=4/3 ACP_2=53.13^o $P_2CD = 90^o - ACP_2 = 90^o - 53.13^o = 36.87^o$

Then, the coordinates of point D are

$\sigma_{x1} = \sigma_{aver} + R \cos 36.87^o$ = - 20MPa – 50MPa (Cos 36.87^o) = - 60MPa

$\tau_{x1y1} = R \sin 36.87^o$ = 50MPa (Sin 36.87^o) = 30MPa

In an analogous manner, the stresses represented by point D’, which correspond to a plane inclined at an angle $\theta = 135^o$ or $2\theta = 270^o$

σ_{y1} = -20MPa + 50MPa (Cos 36.87^o) = 20MPa τ_{x1y1} = -30MPa

And of course, the sum of the normal stresses is -50MPa+10MPa = -60MPa +20MPa

Principal Stresses

They are represented by points P_1 and P_2 on Mohr's circle.

σ_1 = - 20MPa + 50MPa = 30MPa

σ_2 = -20MPa – 50MPa = - 70MPa

The angle ACP_1 is $2\theta_{P1} = 180^o + 53.13^o = 233.13^o$ or $\theta_{P1} = 116.6^o$

The angle ACP_2 is $2\theta_{P2} = 53.13^o$ or $\theta_{P2} = 26.6^o$

Maximum Shear Stresses

These are represented by point S_1 and S_2 in Mohr's circle.

The angle ACS_1 is $2\theta_{S1} = 90^o + 53.13^o = 143.13^o$ or $\theta = 71.6^o$.

The magnitude of the maximum shear stress is 50MPa and the normal stresses corresponding to point S1 is -20MPa.